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# Classification of linearly compact simple Nambu-Poisson algebras

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We introduce the notion of a universal odd generalized Poisson superalgebra associated with an associative algebra  $A$ , by generalizing a construction made in the work of De Sole and Kac [Jpn. J. Math. **8**, 1–145 (2013)]. By making use of this notion we give a complete classification of simple linearly compact (generalized)  $n$ -Nambu-Poisson algebras over an algebraically closed field of characteristic zero. *Published by AIP Publishing.* [<http://dx.doi.org/10.1063/1.4948409>]

## I. INTRODUCTION

In 1973 Nambu proposed a generalization of Hamiltonian mechanics, based on the notion of an  $n$ -ary bracket in place of the usual binary Poisson bracket.<sup>10</sup> Nambu dynamics is described by the flow, given by a system of ordinary differential equations which involves  $n - 1$  Hamiltonians,

$$\frac{du}{dt} = \{u, h_1, \dots, h_{n-1}\}. \quad (1.1)$$

The (only) example, proposed by Nambu, is the following  $n$ -ary bracket on the space of functions in  $N \geq n$  variables:

$$\{f_1, \dots, f_n\} = \det \left( \frac{\partial f_i}{\partial x_j} \right)_{i,j=1}^n. \quad (1.2)$$

He pointed out that this  $n$ -ary bracket satisfies the following axioms, similar to that of a Poisson bracket:

$$\begin{aligned} \text{(Leibniz rule)} \quad \{f_1, \dots, f_i \tilde{f}_i, \dots, f_n\} &= f_i \{f_1, \dots, \tilde{f}_i, \dots, f_n\} + \tilde{f}_i \{f_1, \dots, f_i, \dots, f_n\}, \\ \text{(skewsymmetry)} \quad \{f_{\sigma(1)}, \dots, f_{\sigma(n)}\} &= (\text{sign } \sigma) \{f_1, \dots, f_n\}. \end{aligned}$$

Twelve years later this example was rediscovered by Filippov in his theory of  $n$ -Lie algebras which is a natural generalization of ordinary (binary) Lie algebras.<sup>8</sup> Namely, an  $n$ -Lie algebra is a vector space with an  $n$ -ary bracket  $[a_1, \dots, a_n]$ , which is skewsymmetric (as above) and satisfies the following Filippov-Jacobi identity:

$$\begin{aligned} [a_1, \dots, a_{n-1}, [b_1, \dots, b_n]] &= [[a_1, \dots, a_{n-1}, b_1], b_2, \dots, b_n] + [b_1, [a_1, \dots, a_{n-1}, b_2], b_3, \dots, b_n] + \dots \\ &+ [b_1, \dots, b_{n-1}, [a_1, \dots, a_{n-1}, b_n]]. \end{aligned} \quad (1.3)$$

In particular, Filippov proved that the Nambu bracket (1.2) satisfies the Filippov-Jacobi identity.

Following Takhtajan,<sup>11</sup> we call an  $n$ -Nambu-Poisson algebra a unital commutative associative algebra  $\mathcal{N}$ , endowed with an  $n$ -ary bracket, satisfying the Leibniz rule, skew-symmetry, and Filippov-Jacobi identity. Of course for  $n = 2$  this is the definition of a Poisson algebra.

In Ref. 4 we classified simple linearly compact  $n$ -Lie algebras with  $n > 2$  over a field  $\mathbb{F}$  of characteristic 0. The classification is based on a bijective correspondence between  $n$ -Lie algebras and pairs  $(L, \mu)$ , where  $L$  is a  $\mathbb{Z}$ -graded Lie superalgebra of the form  $L = \bigoplus_{j=-1}^{n-1} L_j$  satisfying certain additional properties, and  $L_{n-1} = \mathbb{F}\mu$ , thereby reducing it to the known classification of simple linearly compact Lie superalgebras and their  $\mathbb{Z}$ -gradings.<sup>9,1</sup> For this construction we used the universal  $\mathbb{Z}$ -graded Lie superalgebra, associated with a vector superspace.

In the present paper we use an analogous correspondence between linearly compact  $n$ -Nambu-Poisson algebras and certain “good” pairs  $(\mathcal{P}, \mu)$ , where  $\mathcal{P}$  is a  $\mathbb{Z}_+$ -graded odd Poisson superalgebra  $\mathcal{P} = \bigoplus_{j \geq -1} \mathcal{P}_j$  and  $\mu \in \mathcal{P}_{n-1}$  is an element of parity  $n \bmod 2$ . For this construction we use the universal  $\mathbb{Z}$ -graded odd Poisson superalgebra, associated with an associative algebra, considered in Ref. 6. As a result, using the classification of simple linearly compact odd Poisson superalgebras,<sup>3</sup> we obtain the following theorem.

**Theorem 1.1.** *For  $n > 2$ , any simple linearly compact  $n$ -Nambu-Poisson algebra is isomorphic to the algebra  $\mathbb{F}[[x_1, \dots, x_n]]$  with the  $n$ -ary bracket (1.2).*

Note the sharp difference with the Poisson case, when each algebra  $\mathbb{F}[[p_1, \dots, p_n, q_1, \dots, q_n]]$  carries a Poisson bracket

$$\{f, g\}_P = \sum_{i=1}^n \left( \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right), \tag{1.4}$$

making it a simple linearly compact Poisson algebra (and these are all up to isomorphism<sup>2</sup>).

In the present paper we treat also the case of a generalized  $n$ -Nambu-Poisson bracket, which is an  $n$ -ary analogue of the generalized Poisson bracket, called also the Lagrange bracket. For the latter bracket the Leibniz rule is modified by adding an extra term,

$$\{a, bc\} = \{a, b\}c + \{a, c\}b - \{a, 1\}bc.$$

In order to treat this case along similar lines, we construct the universal  $\mathbb{Z}$ -graded generalized odd Poisson superalgebra, associated with an associative algebra, which is a generalization of the construction in Ref. 6. Our main result in this direction is the following theorem, which uses the classification of simple linearly compact odd generalized Poisson superalgebras.<sup>3</sup>

**Theorem 1.2.** *For  $n > 2$ , any simple linearly compact generalized  $n$ -Nambu-Poisson algebra is gauge equivalent (see Remark 2.4 for the definition) either to the Nambu  $n$ -algebra from Theorem 1.1 or to the Dzhumadiladzev  $n$ -algebra,<sup>7</sup> which is  $\mathbb{F}[[x_1, \dots, x_{n-1}]]$  with the  $n$ -ary bracket*

$$\{f_1, \dots, f_n\} = \det \begin{pmatrix} f_1 & \dots & f_n \\ \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_1} \\ \dots & \dots & \dots \\ \frac{\partial f_1}{\partial x_{n-1}} & \dots & \frac{\partial f_n}{\partial x_{n-1}} \end{pmatrix}. \tag{1.5}$$

Note again the sharp difference with the generalized Poisson case, when each algebra  $\mathbb{F}[[p_1, \dots, p_n, q_1, \dots, q_n, t]]$  carries a Lagrange bracket

$$\{f, g\}_L = \{f, g\}_P + (2 - E)f \frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}(2 - E)g, \tag{1.6}$$

where  $\{f, g\}_P$  is given by (1.4) and  $E = \sum_{i=1}^n (p_i \frac{\partial}{\partial p_i} + q_i \frac{\partial}{\partial q_i})$ , making it a simple linearly compact generalized Poisson algebra (and those, along with (1.4), are all up to gauge equivalence).

Throughout the paper our base field  $\mathbb{F}$  has characteristic 0 and is algebraically closed.

## II. NAMBU-POISSON ALGEBRAS

*Definition 2.1.* A generalized  $n$ -Nambu-Poisson algebra is a triple  $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \cdot)$  such that

- $(\mathcal{N}, \cdot)$  is a unital associative commutative algebra;
- $(\mathcal{N}, \{\cdot, \dots, \cdot\})$  is an  $n$ -Lie algebra;
- the following generalized Leibniz rule holds:

$$\{a_1, \dots, a_{n-1}, bc\} = \{a_1, \dots, a_{n-1}, b\}c + b\{a_1, \dots, a_{n-1}, c\} - \{a_1, \dots, a_{n-1}, 1\}bc. \tag{2.1}$$

If  $\{a_1, \dots, a_{n-1}, 1\} = 0$ , then (2.1) is the usual Leibniz rule and  $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \cdot)$  is called simply  $n$ -Nambu-Poisson algebra.

For  $n = 2$  Definition 2.1 is the definition of a generalized Poisson algebra. Simple linearly compact generalized Poisson (super)algebras were classified in Ref. 2, [Corollary 7.1].

*Example 2.2.* Let  $\mathcal{N} = \mathbb{F}[[x_1, \dots, x_n]]$  with the usual commutative associative product and  $n$ -ary bracket defined, for  $f_1, \dots, f_n \in \mathcal{N}$ , by

$$\{f_1, \dots, f_n\} = \det \begin{pmatrix} D_1(f_1) & \dots & D_1(f_n) \\ \dots & \dots & \dots \\ D_n(f_1) & \dots & D_n(f_n) \end{pmatrix}$$

where  $D_i = \frac{\partial}{\partial x_i}$ ,  $i = 1, \dots, n$ . Then  $\mathcal{N}$  is an  $n$ -Nambu-Poisson algebra, introduced by Nambu,<sup>10</sup> that we will call the  $n$ -Nambu algebra (cf. Refs. 10, 8, and 4).

*Example 2.3.* Let  $\mathcal{N} = \mathbb{F}[[x_1, \dots, x_{n-1}]]$  with the usual commutative associative product and  $n$ -ary bracket defined, for  $f_1, \dots, f_n \in \mathcal{N}$ , by

$$\{f_1, \dots, f_n\} = \det \begin{pmatrix} f_1 & \dots & f_n \\ D_1(f_1) & \dots & D_1(f_n) \\ \dots & \dots & \dots \\ D_{n-1}(f_1) & \dots & D_{n-1}(f_n) \end{pmatrix}$$

where  $D_i = \frac{\partial}{\partial x_i}$ ,  $i = 1, \dots, n - 1$ . Then  $\mathcal{N}$  is a generalized Nambu-Poisson algebra that we will call the  $n$ -Dzhumadil'daev algebra (cf. Refs. 7 and 4).

*Remark 2.4.* Let  $\mathcal{N} = (\mathcal{N}, \{\cdot, \dots, \cdot\}, \cdot)$  be a generalized  $n$ -Nambu-Poisson algebra. For any invertible element  $\varphi \in \mathcal{N}$  define the following bracket on  $\mathcal{N}$ :

$$\{f_1, \dots, f_n\}^\varphi = \varphi^{-1} \{ \varphi f_1, \dots, \varphi f_n \}. \tag{2.2}$$

Then  $\mathcal{N}^\varphi = (\mathcal{N}, \{\cdot, \dots, \cdot\}^\varphi, \cdot)$  is another generalized  $n$ -Nambu-Poisson algebra. Indeed, the skew-symmetry of the bracket is straightforward and the Filippov-Jacobi identity for the bracket  $\{\cdot, \dots, \cdot\}^\varphi$  easily follows from the Filippov-Jacobi identity for the bracket  $\{\cdot, \dots, \cdot\}$ . Let us check that  $\{\cdot, \dots, \cdot\}^\varphi$  satisfies the generalized Leibniz rule. We have

$$\begin{aligned} \{f_1, \dots, f_{n-1}, gh\}^\varphi &= \varphi^{-1} \{ \varphi f_1, \dots, \varphi f_{n-1}, \varphi gh \} = \varphi^{-1} (\{ \varphi f_1, \dots, \varphi f_{n-1}, \varphi g \} h \\ &+ \varphi g \{ \varphi f_1, \dots, \varphi f_{n-1}, h \} - \{ \varphi f_1, \dots, \varphi f_{n-1}, 1 \} \varphi gh) = \{f_1, \dots, f_{n-1}, g\}^\varphi h \\ &+ g \{ \varphi f_1, \dots, \varphi f_{n-1}, h \} - \{ \varphi f_1, \dots, \varphi f_{n-1}, 1 \} gh = \{f_1, \dots, f_{n-1}, g\}^\varphi h \\ &+ g \{ \varphi f_1, \dots, \varphi f_{n-1}, h \} - \{ \varphi f_1, \dots, \varphi f_{n-1}, 1 \} gh + g \{f_1, \dots, f_{n-1}, h\}^\varphi \\ &- \varphi^{-1} g \{ \varphi f_1, \dots, \varphi f_{n-1}, \varphi h \} = \{f_1, \dots, f_{n-1}, g\}^\varphi h + g \{ \varphi f_1, \dots, \varphi f_{n-1}, h \} \\ &- \{ \varphi f_1, \dots, \varphi f_{n-1}, 1 \} gh + g \{f_1, \dots, f_{n-1}, h\}^\varphi - g \{ \varphi f_1, \dots, \varphi f_{n-1}, h \} \\ &- \varphi^{-1} gh \{ \varphi f_1, \dots, \varphi f_{n-1}, \varphi \} + \{ \varphi f_1, \dots, \varphi f_{n-1}, 1 \} gh \\ &= \{f_1, \dots, f_{n-1}, g\}^\varphi h + g \{f_1, \dots, f_{n-1}, h\}^\varphi - \{f_1, \dots, f_{n-1}, 1\}^\varphi gh. \end{aligned}$$

We shall say that the generalized Nambu-Poisson algebras  $\mathcal{N}$  and  $\mathcal{N}^\varphi$  are *gauge equivalent*.

### III. ODD GENERALIZED POISSON SUPERALGEBRAS

*Definition 3.1.* An odd generalized Poisson superalgebra  $(\mathcal{P}, [\cdot, \cdot], \wedge)$  is a triple such that

- $(\mathcal{P}, \wedge)$  is a unital associative commutative superalgebra with parity  $p$ ;
- $(\Pi\mathcal{P}, [\cdot, \cdot])$  is a Lie superalgebra (here  $\Pi\mathcal{P}$  denotes the space  $\mathcal{P}$  with parity  $\bar{p} = p + \bar{1}$ );

- the following generalized odd Leibniz rule holds:

$$[a, b \wedge c] = [a, b] \wedge c + (-1)^{p(a)+1} b \wedge [a, c] + (-1)^{p(a)+1} D(a) \wedge b \wedge c, \tag{3.1}$$

where  $D(a) = [1, a]$ . If  $D = 0$ , then relation (3.1) becomes the odd Leibniz rule; in this case  $(\mathcal{P}, [\cdot, \cdot], \wedge)$  is called an odd Poisson superalgebra (or Gerstenhaber superalgebra). Note that  $D$  is an odd derivation of the associative product and of the Lie superalgebra bracket.

*Example 3.2.* Consider the commutative associative superalgebra  $O(m, n) = \Lambda(n)[[x_1, \dots, x_m]]$ , where  $\Lambda(n)$  denotes the Grassmann algebra over  $\mathbb{F}$  on  $n$  anti-commuting indeterminates  $\xi_1, \dots, \xi_n$ , and the superalgebra parity is defined by  $p(x_i) = \bar{0}$ ,  $p(\xi_j) = \bar{1}$ .

Set  $m = n$  and define the following bracket, known as the Buttin bracket, on  $O(n, n)$  ( $f, g \in O(n, n)$ ):

$$[f, g]_{HO} = \sum_{i=1}^n \left( \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial \xi_i} + (-1)^{p(f)} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial x_i} \right). \tag{3.2}$$

Then  $O(n, n)$  with this bracket is an odd Poisson superalgebra, which we denote by  $PO(n, n)$ .

*Example 3.3.* Consider the associative superalgebra  $O(n, n + 1)$  with even indeterminates  $x_1, \dots, x_n$  and odd indeterminates  $\xi_1, \dots, \xi_n, \xi_{n+1} = \tau$ . Define on  $O(n, n + 1)$  the following bracket ( $f, g \in O(n, n + 1)$ ):

$$[f, g]_{KO} = [f, g]_{HO} + (E - 2)(f) \frac{\partial g}{\partial \tau} + (-1)^{p(f)} \frac{\partial f}{\partial \tau} (E - 2)(g), \tag{3.3}$$

where  $[\cdot, \cdot]_{HO}$  is the Buttin bracket (3.2) and  $E = \sum_{i=1}^n (x_i \frac{\partial}{\partial x_i} + \xi_i \frac{\partial}{\partial \xi_i})$  is the Euler operator. Then  $O(n, n + 1)$  with bracket  $[\cdot, \cdot]_{KO}$  is an odd generalized Poisson superalgebra with  $D = -2 \frac{\partial}{\partial \tau}$  [Ref. 1, Remark 4.1], which we denote by  $PO(n, n + 1)$ .

*Remark 3.4.* Let  $P = (\mathcal{P}, [\cdot, \cdot], \cdot)$  be an odd generalized Poisson superalgebra. For any invertible element  $\varphi \in \mathcal{P}$ , such that  $p(\varphi) = \bar{0}$  and  $[\varphi, \varphi] = 0$ , define the following bracket on  $P$ :

$$[a, b]^\varphi = \varphi^{-1} [\varphi a, \varphi b]. \tag{3.4}$$

Then  $P^\varphi = (\mathcal{P}, [\cdot, \cdot]^\varphi, \cdot)$  is another odd generalized Poisson superalgebra, with derivation

$$D_\varphi(a) = [1, a]^\varphi = [\varphi, a] - D(\varphi)a.$$

The odd generalized Poisson superalgebras  $P$  and  $P^\varphi$  are called *gauge equivalent* (cf. Ref. 3 [Example 3.4]). Note that the associative products in  $P$  and  $P^\varphi$  are the same.

**Theorem 3.5 (Ref. 3, Corollary 9.2).**

- (a) Any simple linearly compact odd generalized Poisson superalgebra is gauge equivalent to  $PO(n, n)$  or  $PO(n, n + 1)$ .
- (b) Any simple linearly compact odd Poisson superalgebra is isomorphic to  $PO(n, n)$ .

*Definition 3.6.* A  $\mathbb{Z}$ -graded (resp.  $\mathbb{Z}_+$ -graded) odd generalized Poisson superalgebra is an odd generalized Poisson superalgebra  $(\mathcal{P}, [\cdot, \cdot], \wedge)$  such that  $(\Pi\mathcal{P}, [\cdot, \cdot])$  is a  $\mathbb{Z}$ -graded Lie superalgebra:  $\Pi\mathcal{P} = \bigoplus_{j \in \mathbb{Z}} \mathcal{P}_j$  (resp. a  $\mathbb{Z}$ -graded Lie superalgebra of depth 1:  $\Pi\mathcal{P} = \bigoplus_{j \geq -1} \mathcal{P}_j$ ) and  $(\mathcal{P}, \wedge)$  is a  $\mathbb{Z}$ -graded commutative associative superalgebra:  $\mathcal{P} = \bigoplus_{k \in \mathbb{Z}} \mathcal{Q}_k$  (resp. a  $\mathbb{Z}_+$ -graded commutative associative superalgebra:  $\mathcal{P} = \bigoplus_{k \in \mathbb{Z}_+} \mathcal{Q}_k$ ) such that  $\mathcal{P}_j = \Pi\mathcal{Q}_{j+1}$ .

*Example 3.7.* Let us consider the odd Poisson superalgebra  $PO(n, n)$  (resp.  $PO(n, n + 1)$ ). Set  $\deg x_i = 0$  and  $\deg \xi_i = 1$  for every  $i = 1, \dots, n$  (resp.  $\deg x_i = 0$ ,  $\deg \xi_i = 1$  for every  $i = 1, \dots, n$  and  $\deg \tau = 1$ ). Then  $PO(n, n)$  (resp.  $PO(n, n + 1)$ ) becomes a  $\mathbb{Z}_+$ -graded odd (resp. generalized) Poisson superalgebra with

$$\mathcal{Q}_j = \{f \in O(n, n) \mid \deg(f) = j\}$$

and

$$\Pi\mathcal{P}_j = \{f \in \mathcal{O}(n, n) \mid \deg(f) = j + 1\}.$$

We will call this grading a grading of type  $(0, \dots, 0 \mid 1, \dots, 1)$  (resp.  $(0, \dots, 0 \mid 1, \dots, 1, 1)$ ). We thus have, for  $\mathcal{P} = PO(n, n)$ ,

$$\Pi\mathcal{P}_{-1} = \mathcal{Q}_0 = \mathbb{F}[[x_1, \dots, x_n]]$$

and, for  $j \geq 0$ ,

$$\Pi\mathcal{P}_j = \mathcal{Q}_{j+1} = \langle \xi_{i_1} \dots \xi_{i_{j+1}} \mid 1 \leq i_1 < \dots < i_{j+1} \leq n \rangle \otimes \mathbb{F}[[x_1, \dots, x_n]].$$

Similarly, for  $\mathcal{P} = PO(n, n + 1)$ , we have

$$\Pi\mathcal{P}_{-1} = \mathcal{Q}_0 = \mathbb{F}[[x_1, \dots, x_n]]$$

$$\Pi\mathcal{P}_j = \mathcal{Q}_{j+1} = \langle \xi_{i_1} \dots \xi_{i_{j+1}} \mid 1 \leq i_1 < \dots < i_{j+1} \leq n + 1 \rangle \otimes \mathbb{F}[[x_1, \dots, x_n]].$$

*Remark 3.8.* From the properties of the  $\mathbb{Z}$ -gradings of the Lie superalgebras  $HO(n, n)$  and  $KO(n, n + 1)$  (see, for example, Ref. 9), one can deduce that the grading of type  $(0, \dots, 0 \mid 1, \dots, 1)$  (resp.  $(0, \dots, 0 \mid 1, \dots, 1, 1)$ ) is, up to isomorphisms, the only  $\mathbb{Z}_+$ -grading of  $\mathcal{P} = PO(n, n)$  (resp.  $\mathcal{P} = PO(n, n + 1)$ ) such that  $\mathcal{P}_{-1}$  is completely odd.

*Remark 3.9.* Let  $\mathcal{P} = PO(n, n)$  or  $\mathcal{P} = PO(n, n + 1)$  and let  $\mathcal{P}^\varphi$  be an odd generalized Poisson superalgebra gauge equivalent to  $\mathcal{P}$ . Then the grading of type  $(0, \dots, 0 \mid 1, \dots, 1)$  (resp.  $(0, \dots, 0 \mid 1, \dots, 1, 1)$ ) is, up to isomorphisms, the only  $\mathbb{Z}_+$ -grading of  $\mathcal{P}^\varphi$  such that  $\mathcal{P}_{-1}^\varphi$  is completely odd. Indeed, let  $\mathcal{P}^\varphi = \bigoplus_{k \in \mathbb{Z}_+} \mathcal{Q}_k^\varphi = \bigoplus_{j \geq -1} \mathcal{P}_j^\varphi$ , with  $\mathcal{P}_j^\varphi = \Pi\mathcal{Q}_{j+1}^\varphi$  a  $\mathbb{Z}_+$ -grading of  $\mathcal{P}^\varphi$ . By the same argument as in Refs. 9 and 5 [Sec. 5.4], we can assume, up to isomorphisms, that the elements  $x_i$ 's and  $\xi_j$ 's are homogeneous with respect to the  $\mathbb{Z}_+$ -grading. Suppose that,  $x_i \in \mathcal{Q}_k^\varphi$  and  $\xi_i \in \mathcal{Q}_j^\varphi$  for some  $1 \leq i \leq n$  and some  $k, j \in \mathbb{Z}_+$ . Then

$$[x_i, \xi_i]^\varphi \in \Pi\mathcal{P}_{k+j-2}^\varphi = \mathcal{Q}_{k+j-1}^\varphi. \tag{3.5}$$

On the other hand, by (3.4), we have

$$\begin{aligned} [x_i, \xi_i]^\varphi &= \varphi^{-1}[\varphi x_i, \varphi \xi_i] = \varphi^{-1}([\varphi x_i, \varphi] \xi_i + \varphi[\varphi x_i, \xi_i] - D(\varphi x_i) \varphi \xi_i) = \\ &= [x_i, \varphi] \xi_i - D(\varphi) x_i \xi_i + [\varphi x_i, \xi_i] - D(\varphi x_i) \xi_i = \frac{\partial \varphi}{\partial \xi_i} \xi_i + \frac{1}{2} x_i \xi_i D(\varphi) - D(\varphi) x_i \xi_i + \frac{\partial \varphi}{\partial x_i} x_i + \\ &\quad + \varphi + \frac{1}{2} D(\varphi) x_i \xi_i - D(\varphi) x_i \xi_i = \frac{\partial \varphi}{\partial \xi_i} \xi_i - D(\varphi) x_i \xi_i + \frac{\partial \varphi}{\partial x_i} x_i + \varphi, \end{aligned}$$

where  $D = 0$  if  $\mathcal{P} = PO(n, n)$  and  $D = -2 \frac{\partial}{\partial \tau}$  if  $\mathcal{P} = PO(n, n + 1)$ . Note that  $[x_i, \xi_i]^\varphi$  is invertible since  $\varphi$  is invertible and, by (3.5), it is homogeneous, hence  $k + j = 1$ , i.e., either  $k = 0$  and  $j = 1$  or  $k = 1$  and  $j = 0$ . It follows that the only  $\mathbb{Z}_+$ -grading of  $\mathcal{P}^\varphi$  such that  $\mathcal{P}_{-1}^\varphi$  is completely odd is the grading of type  $(0, \dots, 0 \mid 1, \dots, 1)$ . We can thus simply denote the graded components of  $\mathcal{P}^\varphi$  with respect to this grading by  $\mathcal{P}_j = \Pi\mathcal{Q}_{j+1}$ .

Now let  $a \in \mathcal{Q}_i = \Pi\mathcal{P}_{i-1}$  and  $b \in \mathcal{Q}_k = \Pi\mathcal{P}_{k-1}$ . We have  $[a, b]^\varphi = [a, \varphi]b + [\varphi a, b] + (-1)^{p(a)+1} (D(\varphi)ab + D(\varphi a)b)$ . Suppose that  $\varphi = \sum_{j \geq 0} \varphi_j$  with  $\varphi_j \in \mathcal{Q}_j$ . Then one can show, using the fact that  $[a, b]^\varphi \in \Pi\mathcal{P}_{i+k-2} = \mathcal{Q}_{i+k-1}$ , that  $[a, b]^\varphi = [a, b]^{\varphi_0}$ . It follows that when dealing with the  $\mathbb{Z}_+$ -graded odd generalized Poisson superalgebras  $\mathcal{P}^\varphi$  we can always assume that  $\varphi \in \mathcal{Q}_0$ .

#### IV. THE UNIVERSAL ODD GENERALIZED POISSON SUPERALGEBRA

*Definition 4.1.* Let  $A$  be a unital commutative associative superalgebra with parity  $p$ . A linear map  $X : A \rightarrow A$  is called a generalized derivation of  $A$  if it satisfies the generalized Leibniz rule,

$$X(bc) = X(b)c + (-1)^{p(b)p(c)} X(c)b - X(1)bc. \tag{4.1}$$

We denote by  $GDer(A)$  the set of generalized derivations of  $A$ . If  $X(1) = 0$ , relation (4.1) becomes the usual Leibniz rule and  $X$  is called a derivation. We denote by  $Der(A)$  the set of derivations of  $A$ .

*Proposition 4.2.* The set  $GDer(A)$  is a subalgebra of the Lie superalgebra  $End(A)$ .

*Proof.* This follows by direct computations. □

Our construction of the universal odd generalized Poisson superalgebra is inspired by the one of the universal odd Poisson superalgebra explained in Ref. 6. The universal odd Poisson superalgebra associated with  $A$  is the full prolongation of the subalgebra  $Der(A)$  of the Lie superalgebra  $End(A)$  (the definitions will be given below). In this section we generalize this construction when  $Der(A)$  is replaced by the subalgebra  $GDer(A)$ .

Consider the universal Lie superalgebra  $W(\Pi A)$  associated with the vector superspace  $\Pi A$ : this is the  $\mathbb{Z}$ -graded Lie superalgebra,

$$W(\Pi A) = \bigoplus_{k=-1}^{\infty} W_k(\Pi A),$$

where  $W_{-1} = \Pi A$  and for all  $k \geq 0$ ,  $W_k(V) = \text{Hom}(S^{k+1}(\Pi A), \Pi A)$  is the vector superspace of  $(k + 1)$ -linear supersymmetric functions on  $\Pi A$  with values in  $\Pi A$ . Again, we denote the parity in  $W(\Pi A)$  by  $\bar{p}$ . The Lie superalgebra structure on  $W(\Pi A)$  is defined as follows: for  $X \in W_p(\Pi A)$  and  $Y \in W_q(\Pi A)$  with  $p, q \geq -1$ , we define  $X \square Y \in W_{p+q}(\Pi A)$  by

$$X \square Y(a_0, \dots, a_{p+q}) = \sum_{\substack{i_0 < \dots < i_q \\ i_{q+1} < \dots < i_{p+q}}} \epsilon_a(i_0, \dots, i_{p+q}) X(Y(a_{i_0}, \dots, a_{i_q}), a_{i_{q+1}}, \dots, a_{i_{p+q}}). \quad (4.2)$$

Here  $\epsilon_a(i_0, \dots, i_{p+q}) = \epsilon_{a_0, \dots, a_{p+q}}(i_0, \dots, i_{p+q}) = (-1)^N$ , where  $N$  is the number of interchanges of indices of odd  $a_i$ 's in the permutation  $\sigma(s) = i_s$ ,  $s = 0, \dots, p + q$ . Then the bracket on  $W(\Pi A)$  is given by

$$[X, Y] = X \square Y - (-1)^{\bar{p}(X)\bar{p}(Y)} Y \square X.$$

As  $GDer(A)$  is a subalgebra of the Lie superalgebra  $W_0(\Pi A) = \text{End}(\Pi A)$ , we can consider its full prolongation  $\mathcal{G}W^{as}(\Pi A)$ : this is the  $\mathbb{Z}$ -graded subalgebra  $\mathcal{G}W^{as}(\Pi A) = \bigoplus_{k=-1}^{\infty} \mathcal{G}W_k^{as}(\Pi A)$  of the Lie superalgebra  $W(\Pi A)$  defined by setting  $\mathcal{G}W_{-1}^{as}(\Pi A) = \Pi A$ ,  $\mathcal{G}W_0^{as}(\Pi A) = GDer(\Pi A)$ , and inductively for  $k \geq 1$ ,

$$\mathcal{G}W_k^{as}(\Pi A) = \{X \in W_k(\Pi A) \mid [X, W_{-1}(\Pi A)] \subset \mathcal{G}W_{k-1}^{as}(\Pi A)\}.$$

*Proposition 4.3.* For  $k \geq 0$ , the superspace  $\mathcal{G}W_k^{as}(\Pi A)$  consists of linear maps  $X : S^{k+1}(\Pi A) \rightarrow \Pi A$  satisfying the following generalized Leibniz rule:

$$X(a_0, \dots, a_{k-1}, bc) = X(a_0, \dots, a_{k-1}, b)c + (-1)^{p(b)p(c)} X(a_0, \dots, a_{k-1}, c)b - X(a_0, \dots, a_{k-1}, 1)bc \quad (4.3)$$

for  $a_0, \dots, a_{k-1}, b, c \in \Pi A$ .

*Proof.* According to formula (4.2), for all  $X \in W_p(\Pi A)$  and  $Y \in W_{-1}(\Pi A) = \Pi A$ , we have

$$[X, Y](a_1, \dots, a_p) = X(Y, a_1, \dots, a_p), \quad (4.4)$$

with  $a_1, \dots, a_p \in \Pi A$ . Now we proceed by induction on  $k \geq 0$ : for  $k = 0$ ,  $\mathcal{G}W_0^{as}(\Pi A) = GDer(A)$  and equality (4.3) holds by definition of generalized derivation. Assume property (4.3) for elements in  $\mathcal{G}W_{k-1}^{as}(\Pi A)$ , and let  $X$  in  $\mathcal{G}W_k^{as}(\Pi A)$ . For any  $a_0, a_1, \dots, a_{k-1}, b, c \in \Pi A$ , we have by (4.4)

$$X(a_0, a_1, \dots, a_{k-1}, bc) = [X, a_0](a_1, \dots, a_{k-1}, bc).$$

By definition of  $\mathcal{G}W^{as}(\Pi A)$ , we have  $[X, a_0] \in \mathcal{G}W_{k-1}^{as}(\Pi A)$ . Using the inductive hypothesis on  $[X, a_0]$ , we get

$$\begin{aligned} [X, a_0](a_1, \dots, a_{k-1}, bc) &= [X, a_0](a_1, \dots, a_{k-1}, b)c + (-1)^{p(b)p(c)} [X, a_0](a_1, \dots, a_{k-1}, c)b \\ &\quad - [X, a_0](a_1, \dots, a_{k-1}, 1)bc, \end{aligned}$$

which is exactly formula (4.3) for  $X$ . □

For  $X \in \Pi W_{h-1}(\Pi A)$  and  $Y \in \Pi W_{k-1}(\Pi A)$  with  $h, k \geq 0$ , we define their concatenation product  $X \wedge Y \in \Pi W_{h+k-1}(\Pi A)$  by

$$X \wedge Y(a_1, \dots, a_{h+k}) = \sum_{\substack{i_1 < \dots < i_h \\ i_{h+1} < \dots < i_{h+k}}} \epsilon_a(i_1, \dots, i_{h+k}) (-1)^{p(Y)(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_h}))} \times X(a_{i_1}, \dots, a_{i_h}) Y(a_{i_{h+1}}, \dots, a_{i_{h+k}}), \tag{4.5}$$

where  $\epsilon_a$  is defined as in (4.2) with  $a_1, \dots, a_{h+k} \in \Pi A$ .

*Proposition 4.4.*  $(\Pi \mathcal{G}W^{as}(\Pi A), [\cdot, \cdot], \wedge)$  is a  $\mathbb{Z}_+$ -graded odd generalized Poisson superalgebra.

We will denote  $(\Pi \mathcal{G}W^{as}(\Pi A), [\cdot, \cdot], \wedge)$  by  $\mathcal{G}(A)$  and call it the universal odd generalized Poisson superalgebra associated with  $A$ . The rest of this section is devoted to the proof of Proposition 4.4.

*Lemma 4.5.*  $(\Pi \mathcal{G}W^{as}(\Pi A), \wedge)$  is a unital  $\mathbb{Z}_+$ -graded associative commutative superalgebra with parity  $p$ .

*Proof.* It is already proved in Ref. 6 that  $(\Pi W(\Pi A), \wedge)$  is a unital  $\mathbb{Z}_+$ -graded associative commutative superalgebra with parity  $p$ , therefore we only need to prove that for  $X \in \Pi \mathcal{G}W_{h-1}^{as}(\Pi A)$  and  $Y \in \Pi \mathcal{G}W_{k-1}^{as}(\Pi A)$  with  $h, k \geq 0$ ,  $X \wedge Y \in \Pi W_{h+k-1}(\Pi A)$  satisfies the generalized Leibniz rule (4.3). We have

$$\begin{aligned} & X \wedge Y(a_1, \dots, a_{h+k-1}, bc) = \\ &= \sum_{\substack{i_1 < \dots < i_h \\ i_{h+1} < \dots < i_{h+k} = h+k}} \epsilon_{a_1, \dots, a_{h+k-1}, bc}(i_1, \dots, i_{h+k}) (-1)^{p(Y)(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_h}))} \\ & \quad \times X(a_{i_1}, \dots, a_{i_h}) Y(a_{i_{h+1}}, \dots, a_{i_{h+k-1}}, bc) \\ &+ \sum_{\substack{i_1 < \dots < i_h = h+k \\ i_{h+1} < \dots < i_{h+k}}} \epsilon_{a_1, \dots, a_{h+k-1}, bc}(i_1, \dots, i_{h+k}) (-1)^{p(Y)(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_{h-1}}) + \bar{p}(bc))} \\ & \quad \times X(a_{i_1}, \dots, a_{i_{h-1}}, bc) Y(a_{i_{h+1}}, \dots, a_{i_{h+k}}). \end{aligned} \tag{4.6}$$

For the first summand in the right hand side, since  $i_{h+k} = h + k$ , we have

$$\begin{aligned} \epsilon_{a_1, \dots, a_{h+k-1}, bc}(i_1, \dots, i_{h+k}) &= \epsilon_{a_1, \dots, a_{h+k-1}, b}(i_1, \dots, i_{h+k}) \\ &= \epsilon_{a_1, \dots, a_{h+k-1}, c}(i_1, \dots, i_{h+k}) \\ &= \epsilon_{a_1, \dots, a_{h+k-1}, 1}(i_1, \dots, i_{h+k}) \end{aligned}$$

and

$$\begin{aligned} Y(a_{i_{h+1}}, \dots, a_{i_{h+k-1}}, bc) &= Y(a_{i_{h+1}}, \dots, a_{i_{h+k-1}}, b)c + (-1)^{p(b)p(c)} Y(a_{i_{h+1}}, \dots, a_{i_{h+k-1}}, c)b \\ &\quad - Y(a_{i_{h+1}}, \dots, a_{i_{h+k-1}}, 1)bc. \end{aligned}$$

In the second summand, since  $i_{h+k} = h$ , we have

$$\begin{aligned} \epsilon_{a_1, \dots, a_{h+k-1}, bc}(i_1, \dots, i_{h+k}) &= \epsilon_{a_1, \dots, a_{h+k-1}, b}(i_1, \dots, i_{h+k}) (-1)^{p(c)(\bar{p}(a_{i_{h+1}}) + \dots + \bar{p}(a_{i_{h+k}}))} \\ &= \epsilon_{a_1, \dots, a_{h+k-1}, c}(i_1, \dots, i_{h+k}) (-1)^{p(b)(\bar{p}(a_{i_{h+1}}) + \dots + \bar{p}(a_{i_{h+k}}))} \\ &= \epsilon_{a_1, \dots, a_{h+k-1}, 1}(i_1, \dots, i_{h+k}) (-1)^{p(bc)(\bar{p}(a_{i_{h+1}}) + \dots + \bar{p}(a_{i_{h+k}}))} \end{aligned}$$



and

$$\begin{aligned}
 &X(a_{i_1}, \dots, a_{i_{h-1}}, bc)Y(a_{i_{h+1}}, \dots, a_{i_{h+k}}) = \\
 &(-1)^{p(c)(p(Y)+\bar{p}(a_{i_{h+1}})+\dots+\bar{p}(a_{i_{h+k}}))}X(a_{i_1}, \dots, a_{i_{h-1}}, b)Y(a_{i_{h+1}}, \dots, a_{i_{h+k}})c \\
 &+ (-1)^{p(b)(p(c)+p(Y)+\bar{p}(a_{i_{h+1}})+\dots+\bar{p}(a_{i_{h+k}}))}X(a_{i_1}, \dots, a_{i_{h-1}}, c)Y(a_{i_{h+1}}, \dots, a_{i_{h+k}})b \\
 &- (-1)^{p(bc)(p(Y)+\bar{p}(a_{i_{h+1}})+\dots+\bar{p}(a_{i_{h+k}}))}X(a_{i_1}, \dots, a_{i_{h-1}}, 1)Y(a_{i_{h+1}}, \dots, a_{i_{h+k}})bc.
 \end{aligned}$$

The generalized Leibniz rule for  $X \wedge Y$  then follows by replacing these equalities in (4.6). □

It remains to prove that the Lie bracket on  $\Pi\mathcal{G}W^{as}(\Pi A)$  satisfies the generalized odd Leibniz rule (3.1). This follows from the following lemma.

*Lemma 4.6.* *The following equalities hold for  $X, Y, Z \in \Pi\mathcal{G}W^{as}(\Pi A)$ :*

$$\begin{aligned}
 X \square (Y \wedge Z) &= (X \square Y) \wedge Z + (-1)^{\bar{p}(X)p(Y)} Y \wedge (X \square Z) - (X \square 1) \wedge Y \wedge Z, \\
 (X \wedge Y) \square Z &= X \wedge (Y \square Z) + (-1)^{p(Y)\bar{p}(Z)} (X \square Z) \wedge Y.
 \end{aligned}$$

*Proof.* An analogue result is proved in Ref. 6 [Lemma 3.5]. For  $X \in \Pi\mathcal{G}W^{as}_{l-k}(\Pi A)$ ,  $Y \in \Pi\mathcal{G}W^{as}_{h-1}(\Pi A)$ , and  $Z \in \Pi\mathcal{G}W^{as}_{k-h-1}(\Pi A)$  with  $h, k-h, l-k+1 \geq 0$ , we have

$$\begin{aligned}
 X \square (Y \wedge Z)(a_1, \dots, a_l) &= \sum_{\substack{i_1 < \dots < i_h \\ i_{h+1} < \dots < i_k \\ i_{k+1} < \dots < i_l}} \epsilon_a(i_1, \dots, i_l) (-1)^{p(Z)(\bar{p}(a_{i_1})+\dots+\bar{p}(a_{i_h}))} \\
 &\quad \times X(Y(a_{i_1}, \dots, a_{i_h})Z(a_{i_{h+1}}, \dots, a_{i_k}), a_{i_{k+1}}, \dots, a_{i_l}).
 \end{aligned} \tag{4.7}$$

The generalized Leibniz rule for  $X$  can be rewritten in the following way:

$$\begin{aligned}
 X(bc, a_{k+1}, \dots, a_l) &= (-1)^{p(c)(\bar{p}(a_{k+1})+\dots+\bar{p}(a_l))}X(b, a_{k+1}, \dots, a_l)c \\
 &+ (-1)^{p(b)\bar{p}(X)}bX(c, a_{k+1}, \dots, a_l) \\
 &- (-1)^{p(bc)(\bar{p}(a_{k+1})+\dots+\bar{p}(a_l))}X(1, a_{k+1}, \dots, a_l)bc.
 \end{aligned}$$

Using this equality in (4.7),  $X \square (Y \wedge Z)(a_1, \dots, a_l)$  is then of the form

$$X \square (Y \wedge Z)(a_1, \dots, a_l) = A + B - C.$$

The first term  $A$  is equal to

$$\begin{aligned}
 &\sum_{\substack{i_1 < \dots < i_h \\ i_{h+1} < \dots < i_k \\ i_{k+1} < \dots < i_l}} \epsilon_a(i_1, \dots, i_l) (-1)^{p(Z)(\bar{p}(a_{i_1})+\dots+\bar{p}(a_{i_h}))} (-1)^{(p(Z)+\bar{p}(a_{i_{h+1}})+\dots+\bar{p}(a_{i_k}))(\bar{p}(a_{i_{k+1}})+\dots+\bar{p}(a_{i_l}))} \\
 &\quad \times X(Y(a_{i_1}, \dots, a_{i_h}), a_{i_{k+1}}, \dots, a_{i_l})Z(a_{i_{h+1}}, \dots, a_{i_k}) = \\
 &= \sum_{\substack{i_1 < \dots < i_h \\ i_{h+1} < \dots < i_{l-k+h} \\ i_{l-k+h+1} < \dots < i_l}} \epsilon_a(i_1, \dots, i_l) (-1)^{(\bar{p}(a_{i_{h+1}})+\dots+\bar{p}(a_{i_{l-k+h}}))(\bar{p}(a_{i_{l-k+h+1}})+\dots+\bar{p}(a_{i_l}))} \\
 &\quad \times (-1)^{p(Z)(\bar{p}(a_{i_1})+\dots+\bar{p}(a_{i_h}))} (-1)^{(p(Z)+\bar{p}(a_{i_{l-k+h+1}})+\dots+\bar{p}(a_{i_l}))(\bar{p}(a_{i_{h+1}})+\dots+\bar{p}(a_{i_{l-k+h}}))} \\
 &\quad \times X(Y(a_{i_1}, \dots, a_{i_h}), a_{i_{h+1}}, \dots, a_{i_{l-k+h}})Z(a_{i_{l-k+h+1}}, \dots, a_{i_l}) = \\
 &\quad = (X \square Y) \wedge Z(a_1, \dots, a_l).
 \end{aligned}$$

The second term  $B$  is equal to

$$\begin{aligned} & \sum_{\substack{i_1 < \dots < i_h \\ i_{h+1} < \dots < i_k \\ i_{k+1} < \dots < i_l}} \epsilon_a(i_1, \dots, i_l) (-1)^{p(Z)(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_h}))} (-1)^{(p(Y) + \bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_h}))\bar{p}(X)} \\ & \quad \times Y(a_{i_1}, \dots, a_{i_h}) X(Z(a_{i_{h+1}}, \dots, a_{i_k}), a_{i_{k+1}}, \dots, a_{i_l}) \\ & = (-1)^{p(Y)\bar{p}(X)} \sum_{\substack{i_1 < \dots < i_h \\ i_{h+1} < \dots < i_k \\ i_{k+1} < \dots < i_l}} \epsilon_a(i_1, \dots, i_l) (-1)^{(p(Z) + \bar{p}(X))(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_h}))} \\ & \quad \times Y(a_{i_1}, \dots, a_{i_h}) X(Z(a_{i_{h+1}}, \dots, a_{i_k}), a_{i_{k+1}}, \dots, a_{i_l}) = (-1)^{\bar{p}(X)p(Y)} Y \wedge (X \square Z)(a_1, \dots, a_l), \end{aligned}$$

since  $p(X \square Z) = \bar{p}(X) + p(Z)$ .

Finally, the third term  $C$  is equal to

$$\begin{aligned} & \sum_{\substack{i_1 < \dots < i_h \\ i_{h+1} < \dots < i_k \\ i_{k+1} < \dots < i_l}} \epsilon_a(i_1, \dots, i_l) (-1)^{p(Z)(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_h}))} (-1)^{(p(Y) + p(Z) + \bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_k}))(\bar{p}(a_{i_{k+1}}) + \dots + \bar{p}(a_{i_l}))} \\ & \quad \times X(1, a_{i_{k+1}}, \dots, a_{i_l}) Y(a_{i_1}, \dots, a_{i_h}) Z(a_{i_{h+1}}, \dots, a_{i_k}) \\ & = \sum_{\substack{i_1 < \dots < i_h \\ i_{h+1} < \dots < i_k \\ i_{k+1} < \dots < i_l}} \epsilon_a(i_1, \dots, i_l) (-1)^{(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_k}))(\bar{p}(a_{i_{k+1}}) + \dots + \bar{p}(a_{i_l}))} \\ & \quad \times (-1)^{p(Z)(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_h}) + \bar{p}(a_{i_{k+1}}) + \dots + \bar{p}(a_{i_l}))} (-1)^{p(Y)(\bar{p}(a_{i_{k+1}}) + \dots + \bar{p}(a_{i_l}))} \\ & \quad \times X(1, a_{i_{k+1}}, \dots, a_{i_l}) Y(a_{i_1}, \dots, a_{i_h}) Z(a_{i_{h+1}}, \dots, a_{i_k}) = \\ & = \sum_{\substack{i_1 < \dots < i_{l-k} \\ i_{l-k+1} < \dots < i_{l-k+h} \\ i_{l-k+h+1} < \dots < i_l}} \epsilon_a(i_1, \dots, i_l) (-1)^{p(Z)(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_{l-k+h}}))} (-1)^{p(Y)(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_{l-k}}))} \\ & \quad \times (X \square 1)(a_{i_1}, \dots, a_{i_{l-k}}) Y(a_{i_{l-k+1}}, \dots, a_{i_{l-k+h}}) Z(a_{i_{l-k+h+1}}, \dots, a_{i_l}) = (X \square 1) \wedge Y \wedge Z(a_1, \dots, a_l). \end{aligned}$$

This proves the first equality. The second equality can be proved in the same way, using the definition of the box product (4.2) and the concatenation product (4.5).  $\square$

**V. THE MAIN CONSTRUCTION**

Recall that we denoted by  $\mathcal{G}(A)$  the universal odd generalized Poisson superalgebra associated with  $A$ , i.e.,  $\mathcal{G}(A) = (\Pi \mathcal{G}W^{as}(\Pi A), [\cdot, \cdot], \wedge)$ .

Let  $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \cdot)$  be a generalized  $n$ -Nambu-Poisson algebra and denote by  $\Pi \mathcal{N}$  the space  $\mathcal{N}$  with reversed parity. Define

$$\begin{aligned} \mu & : \Pi \mathcal{N} \otimes \dots \otimes \Pi \mathcal{N} \rightarrow \Pi \mathcal{N}, \\ \mu(f_1, \dots, f_n) & = \{f_1, \dots, f_n\}. \end{aligned} \tag{5.1}$$

Then  $\mu$  is a supersymmetric function on  $(\Pi \mathcal{N})^{\otimes n}$  [Ref. 3, Lemma 1.2]. Furthermore  $\mu$  satisfies the generalized Leibniz rule

$$\mu(f_1, \dots, f_{n-1}, gh) = \mu(f_1, \dots, f_{n-1}, g)h + g\mu(f_1, \dots, f_{n-1}, h) - \mu(f_1, \dots, f_{n-1}, 1)gh,$$

hence  $\mu$  lies in  $\mathcal{G}W_{n-1}^{as}(\Pi \mathcal{N})$ .

Let  $OP(\mathcal{N})$  be the odd Poisson subalgebra of  $\mathcal{G}(\mathcal{N})$  generated by  $\Pi \mathcal{N}$  and  $\mu$ . Then, by construction,  $OP(\mathcal{N})$  is a transitive Lie subalgebra of  $\mathcal{G}W^{as}(\Pi \mathcal{N})$ , hence it is a transitive subalgebra of  $W(\Pi \mathcal{N})$ . Furthermore  $OP(\mathcal{N})$  is a  $\mathbb{Z}_+$ -graded odd Poisson subalgebra of  $\mathcal{G}(\mathcal{N})$ . Let us denote by  $OP(\mathcal{N}) = \oplus_{j \geq -1} \mathcal{P}_j(\mathcal{N})$  its depth 1  $\mathbb{Z}$ -grading as a Lie superalgebra.

*Proposition 5.1.* If  $\mathcal{N}$  is a simple generalized  $n$ -Nambu-Poisson algebra then  $OP(\mathcal{N})$  is a simple generalized odd Poisson superalgebra.

*Proof.* Let  $I$  be a non-zero ideal of  $OP(\mathcal{N})$ . Then, by transitivity,  $I \cap \mathcal{P}_{-1}(\mathcal{N}) = I \cap \mathcal{N} \neq 0$ . Note that  $I \cap \mathcal{N}$  is a Nambu-Poisson ideal of  $\mathcal{N}$ . Indeed,  $(I \cap \mathcal{N}) \cdot \mathcal{N} = (I \cap \mathcal{N}) \wedge \mathcal{N} \subset I \cap \mathcal{N}$  and  $[I \cap \mathcal{N}, \mathcal{N}] \subset [\mathcal{N}, \mathcal{N}] = 0$ . Since  $\mathcal{N}$  is simple,  $I \cap \mathcal{N} = \mathcal{N}$ , hence  $1 \in I$ , hence  $I = OP(\mathcal{N})$ .  $\square$

*Remark 5.2.* We recall that since  $(\mathcal{N}, \{\cdot, \dots, \cdot\})$  is an  $n$ -Lie algebra, the Filippov-Jacobi identity holds, i.e., for every  $a_1, \dots, a_{n-1} \in \mathcal{N}$ , the map  $D_{a_1, \dots, a_{n-1}} : \mathcal{N} \rightarrow \mathcal{N}$ ,  $D_{a_1, \dots, a_{n-1}}(a) = \{a_1, \dots, a_{n-1}, a\}$  is a derivation of  $(\mathcal{N}, \{\cdot, \dots, \cdot\})$ . By Ref. 4 [Lemma 2.1(b)], this is equivalent to the condition  $[\mu, D_{a_1, \dots, a_{n-1}}] = 0$  in  $OP(\mathcal{N})$ . By (5.1), we have  $D_{a_1, \dots, a_{n-1}} = [[\mu, a_1], \dots, a_{n-1}]$ , therefore  $\mu$  satisfies the following condition:

$$[\mu, [[\mu, a_1], \dots, a_{n-1}]] = 0 \text{ for every } a_1, \dots, a_{n-1} \in \mathcal{N}.$$

*Definition 5.3.* We say that a pair  $(\mathcal{P}, \mu)$ , consisting of a  $\mathbb{Z}_+$ -graded generalized odd Poisson superalgebra  $\mathcal{P}$  and an element  $\mu \in \mathcal{P}_{n-1}$  of parity  $p(\mu) \equiv n \pmod{2}$ , is a good  $n$ -pair if it satisfies the following properties:

- (G1)  $\mathcal{P} = \bigoplus_{j \geq -1} \mathcal{P}_j$  is a transitive  $\mathbb{Z}$ -graded Lie superalgebra of depth 1 such that  $\mathcal{P}_{-1}$  is completely odd;
- (G2)  $\mu$  and  $\mathcal{P}_{-1}$  generate  $\mathcal{P}$  as a (generalized) odd Poisson superalgebra;
- (G3)  $[\mu, [[\mu, a_1], \dots, a_{n-1}]] = 0$  for every  $a_1, \dots, a_{n-1} \in \mathcal{P}_{-1}$ .

*Example 5.4.* Let  $\mathcal{P} = PO(2h, 2h)$ ,  $h \geq 1$ , with the grading of type  $(0, \dots, 0|1, \dots, 1)$ , and let  $\mu = \sum_{i=1}^h \xi_i \xi_{i+h}$ . Then  $(\mathcal{P}, \mu)$  is a good 2-pair. Indeed, for  $1 \leq i \leq h$ ,  $[x_i, \mu]_{HO} = \xi_{h+i}$  and  $[x_{h+i}, \mu]_{HO} = -\xi_i$ , therefore  $\mathcal{P}_{-1}$  and  $\mu$  generate  $\mathcal{P}$ . Furthermore, for  $f \in \mathcal{P}_{-1} = \mathbb{F}[[x_1, \dots, x_n]]$ , we have  $[f, \mu]_{HO} = \sum_{i=1}^h \left( \frac{\partial f}{\partial x_i} \xi_{i+h} - \frac{\partial f}{\partial x_{i+h}} \xi_i \right)$ , hence

$$\begin{aligned} [\mu, [f, \mu]_{HO}]_{HO} &= \sum_{i,j=1}^h \left[ \xi_j \xi_{j+h}, \frac{\partial f}{\partial x_i} \xi_{i+h} - \frac{\partial f}{\partial x_{i+h}} \xi_i \right]_{HO} = \\ &= \sum_{i,j=1}^h \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \xi_{j+h} \xi_{i+h} - \frac{\partial^2 f}{\partial x_{j+h} \partial x_i} \xi_j \xi_{i+h} - \frac{\partial^2 f}{\partial x_j \partial x_{i+h}} \xi_{j+h} \xi_i + \frac{\partial^2 f}{\partial x_{i+h} \partial x_{j+h}} \xi_j \xi_i \right) = 0. \end{aligned}$$

Therefore  $(\mathcal{P}, \mu)$  satisfies property (G3).

*Example 5.5.* Let  $\mathcal{P} = PO(n, n)$  with the grading of type  $(0, \dots, 0|1, \dots, 1)$ , and let  $\mu = \xi_1 \dots \xi_n$ . Then  $(\mathcal{P}, \mu)$  is a good  $n$ -pair. Indeed,  $[x_{n-1}, [\dots, [x_2, [x_1, \mu]]]]_{HO} = \xi_n$  and similarly all the  $\xi_i$ 's can be obtained by commuting  $\mu$  with different  $x_j$ 's. Therefore  $\mathcal{P}_{-1}$  and  $\mu$  generate  $\mathcal{P}$ . Furthermore, let  $f = \sum_{i=1}^n f_i \xi_i \in \mathcal{P}_0$ , with  $f_i \in \mathbb{F}[[x_1, \dots, x_n]]$ , such that

$$\sum_{i=1}^n \frac{\partial f_i}{\partial x_i} = 0. \tag{5.2}$$

Then  $[f, \mu]_{HO} = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} \xi_1 \dots \xi_n = 0$ . Notice that all elements of the form  $[[\mu, a_1], \dots, a_{n-1}]$  with  $a_1, \dots, a_{n-1} \in \mathcal{P}_{-1} = \mathbb{F}[[x_1, \dots, x_n]]$  satisfy property (5.2), hence  $(\mathcal{P}, \mu)$  satisfies property (G3).

*Example 5.6.* Let  $\mathcal{P} = PO(2h + 1, 2h + 2)$ ,  $h \geq 1$ , with the grading of type  $(0, \dots, 0|1, \dots, 1, 1)$ , and let  $\mu = \sum_{i=1}^{h+1} \xi_i \xi_{i+h+1}$  (recall that  $\xi_{2h+2} = \tau$ ). Then  $(\mathcal{P}, \mu)$  is a good 2-pair. Indeed, we have  $[1, \mu]_{KO} = 2\xi_{h+1}$  and  $[x_i, \mu]_{KO} = \xi_{i+h+1} - x_i \xi_{h+1}$  for  $1 \leq i \leq h + 1$ ,  $[x_{i+h+1}, \mu]_{KO} = -\xi_i - x_{i+h+1} \xi_{h+1}$  for  $1 \leq i \leq h$ . Hence  $\mathcal{P}_{-1}$  and  $\mu$  generate  $\mathcal{P}$ . Furthermore, if  $f \in \mathcal{P}_{-1} = \mathbb{F}[[x_1, \dots, x_n]]$ , we have

$$[f, \mu]_{KO} = \sum_{i=1}^h \left( \frac{\partial f}{\partial x_i} \xi_{i+h+1} - \frac{\partial f}{\partial x_{i+h+1}} \xi_i \right) + \frac{\partial f}{\partial x_{h+1}} \xi_{2h+2} - (E - 2)(f) \xi_{h+1},$$

hence

$$\begin{aligned}
 [\mu, [f, \mu]_{KO}]_{KO} &= \left[ \sum_{j=1}^{h+1} \xi_j \xi_{h+1+j}, \sum_{i=1}^h \left( \frac{\partial f}{\partial x_i} \xi_{i+h+1} - \frac{\partial f}{\partial x_{i+h+1}} \xi_i \right) + \frac{\partial f}{\partial x_{h+1}} \xi_{2h+2} - (E-2)(f) \xi_{h+1} \right]_{KO} \\
 &= \sum_{i=1, \dots, h; j=1, \dots, h+1} \xi_{h+1+j} \left( \frac{\partial^2 f}{\partial x_j \partial x_i} \xi_{h+i+1} - \frac{\partial^2 f}{\partial x_j \partial x_{i+h+1}} \xi_i \right) + \sum_{j=1}^{h+1} \xi_{h+1+j} \left( \frac{\partial^2 f}{\partial x_j \partial x_{h+1}} \xi_{2h+2} \right. \\
 &\quad \left. - \frac{\partial((E-2)(f))}{\partial x_j} \xi_{h+1} \right) - \sum_{i,j=1}^h \xi_j \left( \frac{\partial^2 f}{\partial x_{h+1+j} \partial x_i} \xi_{h+i+1} - \frac{\partial^2 f}{\partial x_{h+1+j} \partial x_{i+h+1}} \xi_i \right) - \sum_{j=1}^h \xi_j \left( \frac{\partial^2 f}{\partial x_{h+1+j} \partial x_{h+1}} \right. \\
 &\quad \left. \xi_{2h+2} - \frac{\partial((E-2)(f))}{\partial x_{h+1+j}} \xi_{h+1} \right) - \xi_{h+1}(E-2)([f, \mu]_{KO}) = \xi_{2h+2} \sum_{i=1}^h \left( \frac{\partial^2 f}{\partial x_{h+1} \partial x_i} \xi_{h+i+1} - \frac{\partial^2 f}{\partial x_{h+1} \partial x_{i+h+1}} \right. \\
 &\quad \left. \xi_i \right) + \sum_{j=1}^{h+1} \frac{\partial^2 f}{\partial x_j \partial x_{h+1}} \xi_{h+1+j} \xi_{2h+2} - \sum_{j=1}^{h+1} \xi_{h+1+j} \frac{\partial((E-2)(f))}{\partial x_j} \xi_{h+1} - \sum_{j=1}^h \left( \frac{\partial^2 f}{\partial x_{h+j+1} \partial x_{h+1}} \xi_j \xi_{2h+2} \right. \\
 &\quad \left. - \frac{\partial((E-2)(f))}{\partial x_{h+1+j}} \xi_j \xi_{h+1} \right) - \xi_{h+1}(E-2)([f, \mu]_{KO}) = \xi_{h+1} \left( \sum_{j=1}^{h+1} \xi_{h+1+j} \frac{\partial((E-2)(f))}{\partial x_j} \right. \\
 &\quad \left. - \sum_{j=1}^h \frac{\partial((E-2)(f))}{\partial x_{h+1+j}} \xi_j - (E-2) \left( \sum_{i=1}^h \left( \frac{\partial f}{\partial x_i} \xi_{i+h+1} - \frac{\partial f}{\partial x_{i+h+1}} \xi_i \right) + \frac{\partial f}{\partial x_{h+1}} \xi_{2h+2} \right) \right) = 0.
 \end{aligned}$$

*Example 5.7.* Let  $\mathcal{P} = PO(n, n+1)$ , with the grading of type  $(0, \dots, 0|1, \dots, 1, 1)$ , and let  $\mu = \xi_1 \dots \xi_n \tau$  (recall that  $\tau = \xi_{n+1}$ ). Then  $(\mathcal{P}, \mu)$  is a good  $(n+1)$ -pair. Indeed, we have  $[1, \mu]_{KO} = 2(-1)^{n+1} \xi_1 \dots \xi_n$ ,  $[x_i, [\dots, [x_{i_{n-1}}, \xi_1 \dots \xi_n]_{KO}]_{KO}]_{KO} = \pm \xi_i$  for  $i_1 \neq \dots \neq i_{n-1} \neq i_n$ ,  $[x_i, \xi_i \dots \xi_n \tau]_{KO} = \xi_{i+1} \dots \xi_n \tau + (-1)^{n-i} x_i \xi_i \dots \xi_n$  for  $1 \leq i \leq n$ . Hence  $\mathcal{P}_{-1}$  and  $\mu$  generate  $\mathcal{P}$ . Now let  $div_1 = \Delta + (E-n) \frac{\partial}{\partial \tau}$ , where  $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i \partial \xi_i}$  is the odd Laplacian, and let  $f = \sum_{i=1}^{n+1} f_i \xi_i \in \mathcal{P}_0$ ,  $f_i \in \mathbb{F}[[x_1, \dots, x_n]]$ , such that  $0 = div_1(f) = \sum_{i,j=1}^n \frac{\partial f_i}{\partial x_j} + (E-n)(f_{n+1})$ . Then we have

$$\begin{aligned}
 \left[ \sum_{i=1}^{n+1} f_i \xi_i, \mu \right]_{KO} &= \left[ \sum_{i=1}^{n+1} f_i \xi_i, \mu \right]_{HO} + \sum_{i=1}^{n+1} (E-2)(f_i \xi_i) (-1)^n \xi_1 \dots \xi_n - f_{n+1} (n-2) \xi_1 \dots \xi_n \tau \\
 &= \sum_{i,j=1}^n \frac{\partial f_i}{\partial x_j} \xi_1 \dots \xi_n \tau + (-1)^n (E-2)(f_{n+1} \xi_{n+1}) \xi_1 \dots \xi_n - (n-2) f_{n+1} \xi_1 \dots \xi_n \tau \\
 &= \left( \sum_{i,j=1}^n \frac{\partial f_i}{\partial x_j} + (E-2)(f_{n+1}) - (n-2) f_{n+1} \right) \xi_1 \dots \xi_n \tau = 0.
 \end{aligned}$$

Notice that, since  $div_1(\mu) = 0$  and  $div_1(f) = 0$  for every  $f \in \mathcal{P}_{-1}$ , then  $div_1([\mu, a_1], \dots, a_{n-1}) = 0$  for every  $a_1, \dots, a_{n-1} \in \mathcal{P}_{-1}$ . Hence property (G3) is satisfied.

*Remark 5.8.* Let us consider  $\mathcal{P} = PO(k, k)$  (resp.  $\mathcal{P} = PO(k, k+1)$ ) with the grading of type  $(0, \dots, 0|1, \dots, 1)$  (resp.  $(0, \dots, 0|1, \dots, 1, 1)$ ). Let  $\varphi \in \mathcal{P}_{-1}$  be an invertible element. By Remark 3.9, the grading of type  $(0, \dots, 0|1, \dots, 1)$  (resp.  $(0, \dots, 0|1, \dots, 1, 1)$ ) defines a  $\mathbb{Z}_2$ -graded structure on the odd generalized Poisson superalgebra  $\mathcal{P}^\varphi$ , such that  $\mathcal{P}_j = \mathcal{P}_j^\varphi$ . Then, by (3.4),  $(\mathcal{P}, \mu)$  is a good  $n$ -pair with respect to this grading if and only if  $(\mathcal{P}^\varphi, \varphi^{-1} \mu)$  is a good  $n$ -pair.

The map  $\mathcal{N} \mapsto (OP(\mathcal{N}), \mu)$  establishes a correspondence between (simple) generalized  $n$ -Nambu-Poisson algebras  $\mathcal{N}$  and good  $n$ -pairs  $(OP(\mathcal{N}), \mu)$ . We now want to show that this correspondence is bijective.

*Lemma 5.9.* Let  $\mathcal{N}$  be a generalized  $n$ -Nambu-Poisson algebra. Then the 0-th graded component  $\mathcal{P}_0(\mathcal{N})$  of  $OP(\mathcal{N})$  is generated, as a Lie superalgebra, by elements of the form

$$[a_1, [a_2, \dots, [a_{n-1}, \mu]]]b,$$

with  $a_i, b \in \Pi\mathcal{N}$ .

*Proof.* Let  $L_{-1} := \Pi\mathcal{N}$  and let  $L_0$  be the Lie subsuperalgebra of  $\mathcal{G}W_0^{as}(\Pi\mathcal{N}) = GDer(\Pi\mathcal{N})$  generated by the elements of the form  $[a_1, [a_2, \dots, [a_{n-1}, \mu b]]]$  with  $a_1, \dots, a_{n-1}, b \in \Pi\mathcal{N}$ . Note that, since  $\mathcal{G}W_0^{as}(\Pi\mathcal{N})$  is  $\mathbb{Z}$ -graded of depth 1, and  $1 \in \mathcal{N}$ , the restriction to  $\mathcal{N}$  of the derivation  $D$  of  $\mathcal{G}(\mathcal{N})$  is zero, hence

$$[a_1, [a_2, \dots, [a_{n-1}, \mu b]]] = [a_1, [a_2, \dots, [a_{n-1}, \mu]]]b.$$

An induction argument on the length of the commutators of the generating elements of  $L_0$  shows that  $L_0$  is stable with respect to the concatenation product by elements of  $\Pi\mathcal{N}$ .

Let  $L$  be the full prolongation of  $L_{-1} \oplus L_0$ , i.e.,  $L = L_{-1} \oplus L_0 \oplus (\oplus_{j \geq 1} L_j)$ , where  $L_j = \{\varphi \in \mathcal{G}W^{as}(\Pi\mathcal{N}) \mid [\varphi, L_{-1}] \subset L_{j-1}\}$ . Note that  $L_j$ , for  $j \geq 1$ , is stable with respect to the concatenation product by elements of  $\Pi\mathcal{N}$ . Indeed, if  $\varphi \in L_j$ , then

$$[\varphi \Pi\mathcal{N}, L_{-1}] = [\varphi, L_{-1}] \Pi\mathcal{N} \subset L_{j-1} \Pi\mathcal{N},$$

hence one can conclude by induction on  $j$  since  $L_0 \Pi\mathcal{N} \subset L_0$ . It follows that  $L$  is closed under the concatenation product, hence it is an odd generalized Poisson subsuperalgebra of  $\mathcal{G}W^{as}(\Pi\mathcal{N})$ . Indeed, using induction on  $i + j \geq 0$ , one shows that  $L_i L_j \subset L$  for every  $i, j \geq 0$ .

It follows that  $OP(\mathcal{N})$  is an odd generalized subsuperalgebra of  $L$ , since  $L$  is an odd generalized Poisson superalgebra containing  $\Pi\mathcal{N}$  and  $\mu$ . As a consequence, the 0-th graded component  $\mathcal{P}_0(\mathcal{N})$  of  $OP(\mathcal{N})$  is generated, as a Lie superalgebra, by elements of the form

$$[a_1, [a_2, \dots, [a_{n-1}, \mu]]]b,$$

with  $a_i, b \in \Pi\mathcal{N}$ . □

*Proposition 5.10.* Let  $(\mathcal{P}, \mu)$  be a good  $n$ -pair, and define on  $\mathcal{N} := \Pi\mathcal{P}_{-1}$  the following product:

$$\{x_1, \dots, x_n\} = [\dots [[\mu, x_1], \dots, x_n]].$$

Then,

- (a)  $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \wedge)$  is a generalized Nambu-Poisson algebra,  $\wedge$  being the restriction to  $\mathcal{N}$  of the commutative associative product  $\wedge$  defined on  $\mathcal{P}$ .
- (b) If  $\mathcal{P}$  is a simple odd generalized Poisson superalgebra, then  $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \wedge)$  is a simple generalized Nambu-Poisson algebra.

*Proof.* (a) By Definitions 3.1 and 3.6,  $\mathcal{N} = \mathcal{Q}_0$  is a commutative associative subalgebra of  $\mathcal{P}$ . Furthermore  $\{\cdot, \dots, \cdot\}$  is an  $n$ -Lie bracket due to Ref. 4 [Prop. 2.4] and property (G3). Finally, for  $f_1, \dots, f_{n-1}, g, h \in \Pi\mathcal{P}_{-1}$ , we have

$$\begin{aligned} \{f_1, \dots, f_{n-1}, gh\} &= [[\dots [\mu, f_1], \dots, f_{n-1}], gh] = [[\dots [\mu, f_1], \dots, f_{n-1}], g]h \\ &+ g[[\dots [\mu, f_1], \dots, f_{n-1}], h] \\ &+ (-1)^{p([\dots [\mu, f_1], \dots, f_{n-1}])+1} [1, [\dots [\mu, f_1], \dots, f_{n-1}]]gh = \{f_1, \dots, f_{n-1}, g\}h + g\{f_1, \dots, f_{n-1}, h\} \\ &- (-1)^{p([\dots [\mu, f_1], \dots, f_{n-1}])+1} (-1)^{\tilde{p}([\dots [\mu, f_1], \dots, f_{n-1}])} \{f_1, \dots, f_{n-1}, 1\}gh \\ &= \{f_1, \dots, f_{n-1}, g\}h + g\{f_1, \dots, f_{n-1}, h\} - \{f_1, \dots, f_{n-1}, 1\}gh. \end{aligned}$$

(b) Now we want to show that if  $\mathcal{P}$  is simple, then  $\mathcal{N}$  is simple. Suppose that  $I$  is a non zero ideal of  $\mathcal{N}$ , and let  $\tilde{I}$  be the ideal of  $\mathcal{P}$  generated by  $\Pi I$  and  $\mu$ :  $\tilde{I} = \oplus_{j \geq -1} \tilde{I}_j$ , with  $\tilde{I}_j \subset \mathcal{P}_j$ . We want to show that  $\tilde{I}_{-1} = \tilde{I} \cap \mathcal{P}_{-1} = \Pi I$ . In fact, the concatenation product by elements in  $\oplus_{j \geq 1} \mathcal{Q}_j$  maps  $\mathcal{Q}_0$  to  $\oplus_{j \geq 1} \mathcal{Q}_j$ , hence it does not produce any element in  $\mathcal{P}_{-1} = \mathcal{Q}_0$ . On the other hand,  $I \wedge \mathcal{Q}_0 = I \wedge \mathcal{N} \subset I$ , since  $I$  is an ideal of  $\mathcal{N}$ . The bracket between elements in  $\oplus_{j \geq 0} \mathcal{P}_j$  lies in  $\oplus_{j \geq 0} \mathcal{P}_j$  and the bracket between  $I$  and elements in  $\oplus_{j \geq 1} \mathcal{P}_j$  lies in  $\oplus_{j \geq 0} \mathcal{P}_j$ . Therefore we just need to

consider the brackets between elements in  $I$  and elements in  $\mathcal{P}_0$ . By hypothesis,  $\mathcal{P}$  is generated by  $\mathcal{P}_{-1}$  and  $\mu$ , hence, by the same argument as in Lemma 5.9,  $\mathcal{P}_0$  is generated by elements of the form  $[a_1, [a_2, \dots, [a_{n-1}, \mu]]]b$  with  $a_i, b \in \Pi\mathcal{P}_{-1}$ . We have

$$[I, [a_1, [a_2, \dots, [a_{n-1}, \mu]]]b] = [I, [a_1, [a_2, \dots, [a_{n-1}, \mu]]]]b,$$

since  $[I, b] = 0$  and  $D|_I = 0$ . Since  $[I, [a_1, [a_2, \dots, [a_{n-1}, \mu]]]] = \{I, a_1, \dots, a_{n-1}\}$  and  $I$  is an ideal of  $\mathcal{N}$ ,  $[I, \mathcal{P}_0] \subset I$ . □

*Definition 5.11.* Two good  $n$ -pairs  $(\mathcal{P}, \mu)$  and  $(\mathcal{P}', \mu')$  are called isomorphic if there exists an odd Poisson superalgebras isomorphism  $\Phi : \mathcal{P} \rightarrow \mathcal{P}'$  such that  $\Phi(\mathcal{P}_j) = \mathcal{P}'_j$ ,  $\Phi(Q_j) = Q'_j$  for all  $j$  and  $\phi(\mu) \in \mathbb{F}^\times \mu'$ .

**Theorem 5.12.** *The map*

$$\mathcal{N} \rightarrow (OP(\mathcal{N}), \mu),$$

with  $\mu$  defined as in (5.1), establishes a bijection between isomorphism classes of generalized  $n$ -Nambu-Poisson algebras and isomorphism classes of good  $n$ -pairs. Moreover,

- (i)  $\mathcal{N}$  is simple (linearly compact) if and only if  $OP(\mathcal{N})$  is simple;
- (ii)  $\mathcal{N}$  is a Nambu-Poisson algebra if and only if  $OP(\mathcal{N})$  is an odd Poisson superalgebra.

*Proof.* The proof follows immediately from Propositions 5.1 and 5.10. The fact that the linear compactness of  $\mathcal{N}$  implies that of  $OP(\mathcal{N})$  can be proved in the same way as in Ref. 4 [Proposition 2.4]. □

*Remark 5.13.* One can check (see also Ref. 4) that if  $\mathcal{N}$  is the  $n$ -Nambu algebra, then  $(OP(\mathcal{N}), \mu) = (PO(n, n), \xi_1 \dots \xi_n)$  and if  $\mathcal{N}$  is the  $n$ -Dzhumaldidaev algebra, then  $(OP(\mathcal{N}), \mu) = (PO(n - 1, n), \xi_1 \dots \xi_{n-1}\tau)$ .

## VI. CLASSIFICATION OF GOOD PAIRS

In this section we will consider the odd Poisson (resp. generalized odd Poisson) superalgebra  $PO(n, n)$  (resp.  $PO(n, n + 1)$ ) with the grading of type  $(0, \dots, 0|1, \dots, 1)$  (resp.  $(0, \dots, 0|1, \dots, 1, 1)$ ).

*Proposition 6.1.* Let  $\mathcal{P} = PO(n, n)$  or  $\mathcal{P} = PO(n, n + 1)$  and  $(\mathcal{P}, \mu)$  be a good  $k$ -pair. Then the Lie subalgebra  $\mathcal{P}_0$  of  $\mathcal{P}$  is spanned by elements of the form

$$[[\mu, a_1], \dots, a_{k-1}]b,$$

with  $a_1, \dots, a_{k-1}, b \in \mathcal{P}_{-1}$ .

*Proof.* By Theorem 5.12,  $\mathcal{P} = OP(\mathcal{N})$  for some  $k$ -Nambu-Poisson algebra  $\mathcal{N}$ . Hence, by Lemma 5.9,  $\mathcal{P}_0$  is generated as a Lie algebra by elements of the form

$$[[\mu, a_1], \dots, a_{k-1}]b,$$

with  $a_1, \dots, a_{k-1}, b \in \mathcal{P}_{-1}$ . Let  $S = \langle [[\mu, a_1], \dots, a_{k-1}] \mid a_1, \dots, a_{k-1} \in \mathcal{P}_{-1} \rangle \subset \mathcal{P}_0$ .

Let  $\mathcal{P} = PO(n, n)$ . Then, for  $z_1, z_2 \in S$ ,  $b_1, b_2 \in \mathcal{P}_{-1}$ , we have

$$\begin{aligned} [z_1 b_1, z_2 b_2] &= [z_1 b_1, z_2] b_2 + (-1)^{p(z_2)(p(z_1)+p(b_1)+1)} z_2 [z_1 b_1, b_2] = (-1)^{p(b_1)(p(z_2)+1)} [z_1, z_2] b_1 b_2 + \\ &+ z_1 [b_1, z_2] b_2 + (-1)^{p(b_1)(p(b_2)+1)+p(z_2)(p(z_1)+p(b_1)+1)} z_2 [z_1, b_2] b_1, \end{aligned}$$

since  $[b_1, b_2] = 0$ . We recall that  $[z_1, z_2]$  lies in  $S$  by Ref. 4 [Theorem 0.2]. Finally, note that  $[z_1, b_2]$  and  $[b_1, z_2]$  lie in  $\mathcal{P}_{-1}$ . It follows that  $\mathcal{P}_0 \subseteq \langle [[\mu, a_1], \dots, a_{k-1}]b \mid a_i, b \in \mathcal{P}_{-1} \rangle \subset \mathcal{P}_0$ , hence the statement holds for  $\mathcal{P} = PO(n, n)$ .

If  $\mathcal{P} = PO(n, n + 1)$ , one uses exactly the same argument and the fact that  $D|_{\mathcal{P}_{-1}} = 0$ ,  $D(S) \subset \mathcal{P}_{-1}$ . □

For any element  $f \in \mathcal{P}_{k-1} = \mathbb{F}[[x_1, \dots, x_n]] \otimes \wedge^k \mathbb{F}^n$ , we let  $f_0 = f|_{x_1=\dots=x_n=0} \in \wedge^k \mathbb{F}^n$ . We shall say that  $f$  has positive order if  $f_0 = 0$ .

*Corollary 6.2.* Let  $\mathcal{P} = PO(n, n)$  (resp.  $PO(n, n + 1)$ ) with the grading of type  $(0, \dots, 0|1, \dots, 1)$  (resp.  $(0, \dots, 0|1, \dots, 1, 1)$ ). If  $\mu \in \mathcal{P}_{k-1}$  is such that  $\mu_0$  lies in the Grassmann subalgebra of  $\wedge^k(\mathbb{F}^n)$  (resp.  $\wedge^k(\mathbb{F}^{n+1})$ ) generated by some variables  $\xi_{i_1}, \dots, \xi_{i_h}$ , for some  $h < n$  (resp.  $h < n + 1$ ), then  $\mu$  does not satisfy property (G2). In particular, if  $\mu_0 = 0$ , then  $\mu$  does not satisfy property (G2).

*Proof.* Suppose, on the contrary, that some  $\xi_i$  does not appear in the expression of  $\mu_0$ . Then, by Proposition 6.1,  $\mathcal{P}_0$  does not contain  $\xi_i$  and this is a contradiction since if  $\mathcal{P} = PO(n, n)$  (resp.  $\mathcal{P} = PO(n, n + 1)$ ),  $\mathcal{P}_0 = \langle \xi_1, \dots, \xi_n \rangle \otimes \mathbb{F}[[x_1, \dots, x_n]]$  (resp.  $\mathcal{P}_0 = \langle \xi_1, \dots, \xi_{n+1} \rangle \otimes \mathbb{F}[[x_1, \dots, x_n]]$ ).

□

**A. The case  $PO(n, n)$**

In this subsection we shall determine good  $k$ -pairs  $(\mathcal{P}, \mu)$  for  $\mathcal{P} = PO(n, n)$  with the  $\mathbb{Z}_+$ -grading of type  $(0, \dots, 0|1, \dots, 1)$ . We will denote the Lie superalgebra bracket in  $PO(n, n)$  simply by  $[\cdot, \cdot]$ . Recall the corresponding description of the  $\mathbb{Z}_+$ -grading given in Example 3.7. When writing a monomial in  $\xi_i$ 's we will assume that the indices increase; elements from  $\wedge^k \mathbb{F}^n$  will be written as linear combinations of such monomials.

*Lemma 6.3.* Let  $2 < k < n - 1$  and suppose that  $\mu \in PO(n, n)_{k-1}$  can be written in the following form:

$$\mu = \xi_1 \dots \xi_k + \xi_1 \dots \xi_h \xi_{k+1} \xi_{k+2} \xi_{i_{h+1}} \dots \xi_{i_{k-2}} + \varphi + \psi, \tag{6.1}$$

where

$$\mu_0 = \xi_1 \dots \xi_k + \xi_1 \dots \xi_h \xi_{k+1} \xi_{k+2} \xi_{i_{h+1}} \dots \xi_{i_{k-2}} + \varphi, \varphi \in \wedge^k \mathbb{F}^n, \psi_0 = 0,$$

$$h = \max\{0 \leq j \leq k - 2 \mid \frac{\partial^{j+2} \mu_0}{\partial \xi_{i_1} \dots \partial \xi_{i_j} \partial \xi_r \partial \xi_s} \neq 0, \text{ for some } i_1 < \dots < i_j \leq k, \text{ and some } r, s > k\},$$

$$\frac{\partial^{k-1} \varphi}{\partial \xi_1 \dots \partial \xi_{k-1}} = 0, \frac{\partial^k \varphi}{\partial \xi_1 \dots \partial \xi_h \partial \xi_{k+1} \partial \xi_{k+2} \partial \xi_{i_{h+1}} \dots \partial \xi_{i_{k-2}}} = 0.$$

Then  $\mu$  does not satisfy property (G3).

*Proof.* Let us first suppose that  $h \geq 1$ . We have

$$[x_{k+1}, \mu] = (-1)^h \xi_1 \dots \xi_h \xi_{k+2} \xi_{i_{h+1}} \dots \xi_{i_{k-2}} + \frac{\partial(\varphi + \psi)}{\partial \xi_{k+1}},$$

$$[x_{i_{k-2}}, \dots, [x_{i_{h+1}}, [x_h, \dots, [x_2, [x_1^2, [x_{k+1}, \mu]]]]]]$$

$$= 2(-1)^{k-2} x_1 \xi_{k+2} + 2x_1 \frac{\partial^{k-1}(\varphi + \psi)}{\partial \xi_{i_{k-2}} \dots \partial \xi_{i_{h+1}} \partial \xi_h \dots \partial \xi_1 \partial \xi_{k+1}}.$$

Therefore  $[\mu, [x_{i_{k-2}}, \dots, [x_{i_{h+1}}, [x_h, \dots, [x_2, [x_1^2, [x_{k+1}, \mu]]]]]] =$

$$= 2(-1)^k (\xi_2 \dots \xi_k ((-1)^{k-2} \xi_{k+2} + \frac{\partial^{k-1} \varphi}{\partial \xi_{i_{k-2}} \dots \partial \xi_{i_{h+1}} \partial \xi_h \dots \partial \xi_1 \partial \xi_{k+1}})) +$$

$$+ \xi_2 \dots \xi_h \xi_{k+1} \xi_{k+2} \xi_{i_{h+1}} \dots \xi_{i_{k-2}} \frac{\partial^{k-1} \varphi}{\partial \xi_{i_{k-2}} \dots \partial \xi_{i_{h+1}} \partial \xi_h \dots \partial \xi_1 \partial \xi_{k+1}} +$$

$$\frac{\partial \varphi}{\partial \xi_1} ((-1)^{k-2} \xi_{k+2} + \frac{\partial^{k-1} \varphi}{\partial \xi_{i_{k-2}} \dots \partial \xi_{i_{h+1}} \partial \xi_h \dots \partial \xi_1 \partial \xi_{k+1}})) + \omega,$$

for some  $\omega$  of positive order. Note that, the summand  $2\xi_2 \dots \xi_k \xi_{k+2}$  in the expression of  $[\mu, [x_{i_{k-2}}, \dots, [x_{i_{h+1}}, [x_h, \dots, [x_2, [x_1^2, [x_{k+1}, \mu]]]]]]$  does not cancel out. Indeed, due to the hypotheses on  $\varphi$ , the only possibility to cancel the summand  $2\xi_2 \dots \xi_k \xi_{k+2}$  is that the expression of  $\varphi$  contains the sum  $a\xi_1 \dots \xi_h \xi_{k+1} \xi_{i_{h+1}} \dots \xi_{i_{k-2}} \xi_t + b\xi_1 \dots \xi_{t-1} \xi_{t+1} \dots \xi_k \xi_{k+2}$ , for some  $t, 2 \leq t \leq k$ , and some suitable coefficients  $a, b \in \mathbb{F}^*$ . But this is impossible since it is in contradiction with the maximality

of  $h$  if  $h = k - 2$  and with the hypotheses on  $\varphi$  if  $h < k - 2$ . It follows that  $[\mu, [x_{i_{k-2}}, \dots, [x_{i_{h+1}}, [x_h, \dots, [x_2, [x_1^2, [x_{k+1}, \mu]]]]]] \neq 0$  and property (G3) is not satisfied.

If  $h = 0$ , then one can use the same argument by showing that the commutator  $[\mu, [x_1 x_{k+1}, [x_{i_1}, \dots, [x_{i_{k-2}}, \mu]]]]$  is different from zero.  $\square$

**Theorem 6.4.** *Let  $\mathcal{P} = PO(n, n)$ . Suppose that  $2 < k < n - 1$  and that  $\mu \in PO(n, n)_{k-1}$ . Then  $(\mathcal{P}, \mu)$  is not a good  $k$ -pair.*

*Proof.* By Corollary 6.2, if  $\mu_0 = 0$  then  $\mu$  does not satisfy property (G2). Now suppose  $\mu_0 \neq 0$ . Since  $\mu_0$  lies in  $\wedge^k(\mathbb{F}^n)$ , we can assume, up to a linear change of indeterminates, that  $\mu_0 = \xi_1 \dots \xi_k + f$  for some  $f \in \wedge^k(\mathbb{F}^n)$  such that  $\frac{\partial^k f}{\partial \xi_1 \dots \partial \xi_k} = 0$ . Then, either  $\mu$  does not satisfy property (G2) and  $(\mathcal{P}, \mu)$  is not a good  $k$ -pair or, again by Corollary 6.2, all  $\xi_i$ 's appear in the expression of  $\mu_0$ . Let us thus assume to be in the latter case. Then, since  $k < n - 1$ , either there exist some  $r, s > k$  such that the indeterminates  $\xi_r$  and  $\xi_s$  both appear in the expression of  $\mu_0$  in at least one monomial (case A) or all the indeterminates  $\xi_r$  and  $\xi_s$  with  $r, s > k$  appear in distinct monomials (case B).

Suppose we are in case (A), and let  $h = \max\{0 \leq j \leq k - 2 \mid \frac{\partial^{j+2} \mu_0}{\partial \xi_{i_1} \dots \partial \xi_{i_j} \partial \xi_r \partial \xi_s} \neq 0, i_1 < \dots < i_j \leq k; r, s > k\}$ . Then we can write

$$\mu_0 = \xi_1 \dots \xi_k + \xi_{i_1} \dots \xi_{i_h} \xi_r \xi_s \xi_{i_{h+1}} \dots \xi_{i_{k-2}} + \varphi$$

for some  $r, s, i_{h+1}, \dots, i_{k-2} > k, i_1, \dots, i_h \leq k$  and some  $\varphi \in \wedge^k(\mathbb{F}^n)$  such that  $\frac{\partial^k \varphi}{\partial \xi_1 \dots \partial \xi_k} = 0$  and  $\frac{\partial^k \varphi}{\partial \xi_{i_1} \dots \partial \xi_{i_h} \partial \xi_r \partial \xi_s \partial \xi_{i_1} \dots \partial \xi_{i_{k-2}}} = 0$ . Up to a permutation of indices we can assume  $r = k + 1, s = k + 2, \{i_1, \dots, i_h\} = \{1, \dots, h\}$  and up to a linear change of indeterminates we can assume  $\frac{\partial^{k-1} \varphi}{\partial \xi_1 \dots \partial \xi_{k-1}} = 0$ . Therefore  $\mu$  satisfies the hypotheses of Lemma 6.3, hence it does not satisfy property (G3).

Now suppose we are in case (B). Then

$$\mu_0 = \xi_1 \dots \xi_k + \xi_{i_1} \dots \xi_{i_{k-1}} \xi_{k+1} + \xi_{j_1} \dots \xi_{j_{k-1}} \xi_{k+2} + \psi$$

for some  $i_1 < \dots < i_{k-1} \leq k, j_1 < \dots < j_{k-1} \leq k$  and  $\psi \in \wedge^k(\mathbb{F}^n)$  such that  $\frac{\partial^k \psi}{\partial \xi_1 \dots \partial \xi_k} = 0, \frac{\partial^k \psi}{\partial \xi_{i_1} \dots \partial \xi_{i_{k-1}} \partial \xi_{k+1}} = 0, \frac{\partial^k \psi}{\partial \xi_{j_1} \dots \partial \xi_{j_{k-1}} \partial \xi_{k+2}} = 0, \frac{\partial^2 \psi}{\partial \xi_r \partial \xi_s} = 0$  for every  $r, s > k$ . Again by Corollary 6.2, we can assume that  $\{i_1, \dots, i_{k-1}\} \neq \{j_1, \dots, j_{k-1}\} \neq \{1, \dots, k - 1\}$ . Therefore there exists an index  $j_l \in \{1, \dots, k\} \cap \{j_1, \dots, j_{k-1}\}$  such that  $j_l \notin \{i_1, \dots, i_{k-1}\}$ .

Now consider the following change of indeterminates:

$$\xi'_{j_l} = \xi_{j_l} + \xi_{k+1}, \quad \xi'_j = \xi_j \quad \forall j \neq j_l.$$

Then

$$\mu_0 = \xi'_1 \dots \xi'_k + \xi'_{j_1} \dots \xi'_{j_{k-1}} \xi'_{k+2} + \xi'_{j_1} \dots \xi'_{j_{i-1}} \xi'_{j_{i+1}} \dots \xi'_{j_{k-1}} \xi'_{k+1} \xi'_{k+2} + \rho$$

for some  $\rho \in \wedge^k(\mathbb{F}^n)$  such that  $\frac{\partial^k \rho}{\partial \xi'_1 \dots \partial \xi'_k} = 0, \frac{\partial^k \rho}{\partial \xi'_{j_1} \dots \partial \xi'_{j_{k-1}} \partial \xi'_{k+2}} = 0, \frac{\partial^k \rho}{\partial \xi'_{j_1} \dots \partial \xi'_{j_{i-1}} \partial \xi'_{j_{i+1}} \dots \partial \xi'_{j_{k-1}} \partial \xi'_{k+1} \partial \xi'_{k+2}} = 0$ . We are now again in case (A), hence the proof is concluded.  $\square$

**Theorem 6.5.** *Let  $\mathcal{P} = PO(n, n)$ . If  $(\mathcal{P}, \mu)$  is a good  $k$ -pair, then, up to isomorphisms, one of the following possibilities may occur:*

- (a) If  $n = 2h$ ,
  - (a1)  $k = 2$  and  $\mu_0 = \sum_{i=1}^h \xi_i \xi_{i+h}$
  - (a2)  $k = n$  and  $\mu_0 = \xi_1 \dots \xi_n$ .
- (b) If  $n = 2h + 1$ ,
  - (b1)  $k = n$  and  $\mu_0 = \xi_1 \dots \xi_n$ .

*Proof.* By Theorem 6.4, the only possibilities for  $k$  are  $k = 2, k = n - 1, \text{ or } k = n$ .



By Corollary 6.2,  $\frac{\partial \mu_0}{\partial \xi_i} \neq 0$  for every  $i = 1, \dots, n$ . Using the classification of non-degenerate skew-symmetric bilinear forms, it thus follows that the case  $k = 2$  can occur only if  $n = 2h$  and, up to equivalence,  $\mu_0 = \sum_{i=1}^h \xi_i \xi_{i+h}$ , hence we get  $a1)$ .

If  $k = n$  then, up to rescaling the odd indeterminates,  $\mu_0 = \xi_1 \dots \xi_n$  and we get cases  $a2)$  and  $b1)$ .

Now assume  $k = n - 1$ . Assume that  $\frac{\partial^{n-2} \mu_0}{\partial \xi_{i_1} \dots \partial \xi_{i_{n-2}}} = \alpha \xi_{i_{n-1}} + \beta \xi_{i_n}$  for some  $i_1 < \dots < i_{n-2}, i_{n-1} < i_n$ , and some  $\alpha, \beta \in \mathbb{F}^*$ . Consider the following change of indeterminates:

$$\xi'_{i_{n-1}} = \alpha \xi_{i_{n-1}} + \beta \xi_{i_n}, \quad \xi'_{i_j} = \xi_{i_j} \quad \forall j \neq n - 1.$$

Then  $\frac{\partial^{n-2} \mu_0}{\partial \xi'_{i_1} \dots \partial \xi'_{i_{n-2}}} = \xi'_{i_{n-1}}$ . By using induction on the lexicographic order of the indices  $i_1 < \dots < i_{n-2}$ , one can thus show that, up to a linear change of indeterminates,  $\mu_0 = \xi_1 \dots \xi_{n-1}$ , hence  $(\mathcal{P}, \mu)$  is not a good  $k$ -pair due to Corollary 6.2.  $\square$

**B. The case  $PO(n, n + 1)$**

In this subsection we shall determine good pairs  $(\mathcal{P}, \mu)$  for  $\mathcal{P} = PO(n, n + 1)$  with the  $\mathbb{Z}$ -grading of type  $(0, \dots, 0|1, \dots, 1, 1)$ . We shall adopt the same notation as in Sec. VI A.

*Lemma 6.6.* Let  $2 \leq k < n - 1$ ,  $\mu \in PO(n, n + 1)_k$ , and suppose that  $\mu_0$  can be written in one of the following forms:

1.

$$\mu_0 = \xi_1 \dots \xi_k \tau + \xi_1 \dots \xi_h \xi_{k+1} \xi_{k+2} \xi_{i_{h+1}} \dots \xi_{i_{k-1}} + \varphi, \tag{6.2}$$

where

(a)  $h = \max\{0 \leq j \leq k \mid \frac{\partial^{j+2} \mu_0}{\partial \xi_{i_1} \dots \partial \xi_{i_j} \partial \xi_r \partial \xi_s} \neq 0, \text{ for some } i_1 < \dots < i_j \leq k, \text{ and } r, s > k\}$ ,

(b)  $\varphi \in \wedge^{k+1} \mathbb{F}^{n+1}$  is such that  $\frac{\partial^{k+1} \varphi}{\partial \xi_1 \dots \partial \xi_k \partial \tau} = 0$ .

2.

$$\mu_0 = \xi_1 \dots \xi_k \tau + \xi_1 \dots \xi_h \xi_{k+1} \tau \xi_{i_{h+1}} \dots \xi_{i_{k-1}} + \varphi, \tag{6.3}$$

where

(a)  $h = \max\{0 \leq j < k \mid \frac{\partial^{j+1} \mu_0}{\partial \xi_{i_1} \dots \partial \xi_{i_j} \partial \tau} \neq 0, \text{ for some } i_1 < \dots < i_j \leq k\}$ ,

(b)  $\varphi \in \wedge^{k+1} \mathbb{F}^{n+1}$  is such that  $\frac{\partial^{k+1} \varphi}{\partial \xi_1 \dots \partial \xi_h \partial \xi_{k+1} \partial \tau \partial \xi_{i_{h+1}} \dots \partial \xi_{i_{k-1}}} = 0$  and  $\frac{\partial^{k+1} \varphi}{\partial \xi_2 \dots \partial \xi_k \partial \tau} = 0$ .

Then  $\mu$  does not satisfy property (G3).

*Proof.* Let us first suppose that  $\mu_0$  is of the form (6.2). Then, using the same arguments as in the proof of Lemma 6.3, one can show that  $[\mu, [x_{i_{k-1}}, \dots, [x_{i_{h+1}}, [x_h, \dots, [x_2, [x_1^2, [x_{k+1}, \mu]]]]]]] \neq 0$ , since in its expression the summand  $\xi_2 \dots \xi_k \xi_{k+2} \tau$  does not cancel out.

Similarly, if  $\mu_0$  is of the form (6.3), then one can show that  $[\mu, [x_{i_{k-1}}, \dots, [x_{i_{h+1}}, [x_h, \dots, [x_2, [x_1^2, [1, \mu]]]]]]] \neq 0$ , since in its expression the summand  $\xi_2 \dots \xi_{k+1} \tau$  does not cancel out.  $\square$

**Theorem 6.7.** Let  $\mathcal{P} = PO(n, n + 1)$ . Suppose that  $2 \leq k < n - 1$  and that  $\mu \in \mathcal{P}_k$ . Then  $(\mathcal{P}, \mu)$  is not a good  $(k + 1)$ -pair.

*Proof.* Let us fix a set of odd indeterminates  $\xi_1, \dots, \xi_n, \xi_{n+1} = \tau$  and the corresponding basis of monomials of  $\wedge(\mathbb{F}^{n+1})$ . By Corollary 6.2, if  $\mu_0 = 0$  or  $\frac{\partial \mu_0}{\partial \tau} = 0$ , then  $\mu$  does not satisfy property (G2). Hence suppose that  $\frac{\partial \mu_0}{\partial \tau} \neq 0$ . Then we may assume, up to a linear change of indeterminates, that  $\mu_0 = \xi_1 \dots \xi_k \tau + \varphi$  for some  $\varphi \in \wedge^{k+1}(\mathbb{F}^{n+1})$  such that  $\frac{\partial^{k+1} \varphi}{\partial \xi_1 \dots \partial \xi_k \partial \tau} = 0$ . Then, either  $\frac{\partial \varphi}{\partial \tau} = 0$  or  $\frac{\partial \varphi}{\partial \tau} \neq 0$ .

Suppose first  $\frac{\partial \varphi}{\partial \tau} = 0$ . Then, either for every  $r, s > k$  the indeterminates  $\xi_r, \xi_s$  appear in different monomials in the expression of  $\varphi$  or there exist some  $r, s > k$  such that  $\xi_r, \xi_s$  appear in the same monomial.

In the first case  $\mu_0 = \xi_1 \dots \xi_k \tau + \xi_1 \dots \xi_k (\xi_{k+1} + \xi_{k+2}) + \rho$  for some  $\rho \in \wedge^{k+1}(\mathbb{F}^{n+1})$  such that  $\frac{\partial^{k+1} \rho}{\partial \xi_1 \dots \partial \xi_k \partial \xi_{k+1}} = 0 = \frac{\partial^{k+1} \rho}{\partial \xi_1 \dots \partial \xi_k \partial \xi_{k+2}}$ . By Corollary 6.2 such an element does not satisfy property (G2). Therefore we may assume that there exist some  $r, s > k$  such that  $\xi_r, \xi_s$  appear in the same monomial, i.e., that, up to a linear change of indeterminates,  $\mu_0$  is of the following form:

$$\mu_0 = \xi_1 \dots \xi_k \tau + \xi_1 \dots \xi_h \xi_{k+1} \xi_{k+2} \xi_{i_{h+1}} \dots \xi_{i_{k-1}} + \varphi'$$

for some  $\varphi' \in \wedge^{k+1}(\mathbb{F}^{n+1})$  such that  $\frac{\partial^{k+1} \varphi'}{\partial \xi_1 \dots \partial \xi_h \partial \xi_{k+1} \partial \xi_{k+2} \partial \xi_{i_{h+1}} \dots \partial \xi_{i_{k-1}}} = 0$  and  $\frac{\partial^{k+1} \varphi'}{\partial \xi_1 \dots \partial \xi_k \partial \tau} = 0$ , where

$h = \max\{0 \leq j \leq k \mid \frac{\partial^{j+2} \mu_0}{\partial \xi_{i_1} \dots \partial \xi_{i_j} \partial \xi_r \partial \xi_s} \neq 0, \text{ for some } i_1 < \dots < i_j \leq k, \text{ and } r, s > k\}$ . Therefore  $\mu$  satisfies hypothesis 1 of Lemma 6.6, hence it does not satisfy property (G3).

Now suppose  $\frac{\partial \varphi}{\partial \tau} \neq 0$ . Then

$$\mu_0 = \xi_1 \dots \xi_k \tau + \xi_{i_1} \dots \xi_{i_h} \tau \xi_{i_{h+1}} \dots \xi_{i_k} + \psi$$

for some  $i_1 < \dots < i_h \leq k < i_{h+1} < \dots < i_k$ , for some  $\psi \in \wedge^{k+1}(\mathbb{F}^{n+1})$  such that  $\frac{\partial^{k+1} \psi}{\partial \xi_{i_1} \dots \partial \xi_{i_k} \partial \tau} = 0$  and  $\frac{\partial^{k+1} \psi}{\partial \xi_1 \dots \partial \xi_k \partial \tau} = 0$ , where  $h = \max\{0 \leq j < k \mid \frac{\partial^{j+1} \mu_0}{\partial \xi_{i_1} \dots \partial \xi_{i_j} \partial \tau} \neq 0, \text{ for some } i_1 < \dots < i_j \leq k\}$ . Now, up to a permutation of indices, we may assume that  $\{i_1, \dots, i_h\} = \{1, \dots, h\}$  and  $i_{h+1} = k + 1$ . Then, either  $\mu$  does not satisfy property (G2) or we may also assume that  $\frac{\partial^k \psi}{\partial \xi_2 \dots \partial \xi_k \partial \tau} = 0$ . Therefore  $\mu$  satisfies hypothesis 2 of Lemma 6.6, hence it does not satisfy property (G3).  $\square$

**Theorem 6.8.** *Let  $\mathcal{P} = PO(n, n + 1)$ . If  $(\mathcal{P}, \mu)$  is a good  $(k + 1)$ -pair, then, up to isomorphisms, one of the following possibilities occur:*

- (a) If  $n = 2h + 1$ ,
  - (a1)  $k = 1$  and  $\mu_0 = \sum_{i=1}^{h+1} \xi_i \xi_{i+h+1}$ ,
  - (a2)  $k = n$  and  $\mu_0 = \xi_1 \dots \xi_{n+1}$ .
- (b) If  $n = 2h$ ,
  - (b1)  $k = n$  and  $\mu_0 = \xi_1 \dots \xi_{n+1}$ .

*Proof.* By Theorem 6.7, the only possibilities for  $k$  are  $k = 1, k = n - 1$ , or  $k = n$ .

By Corollary 6.2,  $\frac{\partial \mu_0}{\partial \xi_i} \neq 0$  for every  $i = 1, \dots, n + 1$ . It follows that, due to the classification of non-degenerate skew-symmetric bilinear forms, the case  $k = 2$  can occur only if  $n = 2h + 1$  and, up to equivalence,  $\mu_0 = \sum_{i=1}^{h+1} \xi_i \xi_{i+h+1}$ , hence we get a1).

If  $k = n$  then, up to rescaling the odd indeterminates,  $\mu_0 = \xi_1 \dots \xi_n \xi_{n+1}$  and we get cases a2) and b1).

Now assume  $k = n - 1$ . Then, using the same argument as in the proof of Theorem 6.5, one can show that, up to a linear change of indeterminates, we may assume  $\mu_0 = \xi_1 \dots \xi_{n-1} \xi_{n+1} + f$  for some  $f \in \wedge^n(\mathbb{F}^{n+1})$  such that  $\frac{\partial f}{\partial \xi_{n+1}} = 0$ . If  $f = 0$  then  $\mu$  does not satisfy property (G2) by Corollary 6.2. If  $f \neq 0$ , then, up to a linear change of indeterminates,  $\mu_0 = \xi_1 \dots \xi_{n-1} \xi_{n+1} + \xi_1 \dots \xi_n = \xi_1 \dots \xi_{n-1} (\xi_{n+1} + \xi_n)$ . Then, by Proposition 6.1,  $\mu$  does not satisfy property (G2).  $\square$

## VII. THE CLASSIFICATION THEOREM

*Remark 7.1.* For every invertible element  $\varphi \in \mathbb{F}[[x_1, \dots, x_n]]$ , the following change of indeterminates preserves the odd symplectic form, i.e., the bracket in  $HO(n, n)$ , and maps  $\varphi \xi_1 \dots \xi_n$  to  $\xi'_1 \dots \xi'_n$ :

$$x'_1 = \int_0^{x_1} \varphi^{-1}(t, x_2, \dots, x_n) dt =: \Phi, \quad \xi'_1 = \varphi \xi_1,$$

$$x'_i = x_i \quad \forall i \neq 1, \quad \xi'_i = \xi_i - \varphi \frac{\partial \Phi}{\partial x_i} \xi_1 \quad \forall i \neq 1.$$

Indeed one can check that  $\{x'_i, x'_j\}_{HO} = 0 = \{\xi'_i, \xi'_j\}_{HO}$  and  $\{x'_i, \xi'_j\}_{HO} = \delta_{ij}$  for every  $i, j = 1, \dots, n$ .

Note that the same change of variables, with the extra condition  $\tau' = \tau$ , preserves the bracket in the Lie superalgebra  $KO(n, n+1)$  and maps  $\varphi\xi_1 \dots \xi_n \tau$  to  $\xi'_1 \dots \xi'_n \tau'$ .

**Theorem 7.2.** *A complete list, up to isomorphisms, of good  $k$ -pairs with  $k > 2$ , is the following:*

- (i)  $(\mathcal{P}^\varphi, \varphi^{-1}\mu)$  with  $\mathcal{P} = PO(n, n)$ ,  $n > 2$ ,  $k = n$ ,  $\mu = \xi_1 \dots \xi_n$ ,  $\varphi \in \mathbb{F}[[x_1, \dots, x_n]]$ ,
- (ii)  $(\mathcal{P}^\varphi, \varphi^{-1}\mu)$  with  $\mathcal{P} = PO(n, n+1)$ ,  $n > 1$ ,  $k = n+1$ ,  $\mu = \xi_1 \dots \xi_n \tau$ ,  $\varphi \in \mathbb{F}[[x_1, \dots, x_n]]$ .

*Proof.* Let  $\mathcal{P} = PO(n, n)$  with the grading of type  $(0, \dots, 0|1, \dots, 1)$ , and let  $(\mathcal{P}, \mu)$  be a good  $k$ -pair for  $k > 2$ . Then, by Theorem 6.5, we have necessarily  $n > 2$ ,  $k = n$ , and  $\mu_0 = \xi_1 \dots \xi_n$ . It follows that  $\mu = \xi_1 \dots \xi_n \psi$  for some invertible element  $\psi$  in  $\mathbb{F}[[x_1, \dots, x_n]]$ . By Remark 7.1, up to a change of variables, we may assume  $\psi = 1$ . In Example 5.5 we showed that the pair  $(\mathcal{P}, \xi_1 \dots \xi_n)$  is a good  $n$ -pair. Statement (i) then follows from Theorem 3.5, Remark 3.9, and Remark 5.8.

Likewise, if  $\mathcal{P} = PO(n, n+1)$  with the grading of type  $(0, \dots, 0|1, \dots, 1, 1)$  and  $(\mathcal{P}, \mu)$  is a good  $k$ -pair for  $k > 2$ , by Theorem 6.8 we have necessarily  $n > 1$ ,  $k = n+1$  and  $\mu_0 = \xi_1 \dots \xi_n \tau$ . It follows that  $\mu = \xi_1 \dots \xi_n \tau \psi$  for some invertible element  $\psi$  in  $\mathbb{F}[[x_1, \dots, x_n]]$ . Again by Remark 7.1, we may assume  $\psi = 1$ . Furthermore in Example 5.7 we showed that  $(\mathcal{P}, \xi_1 \dots \xi_n \tau)$  is a good  $n$ -pair. Statement (ii) then follows from Theorem 3.5, Remark 3.9, and Remark 5.8.  $\square$

**Theorem 7.3.** *Let  $n > 2$ .*

- (a) *Any simple linearly compact generalized  $n$ -Nambu-Poisson algebra is gauge equivalent either to the  $n$ -Nambu algebra or to the  $n$ -Dzhumadildaev algebra.*
- (b) *Any simple linearly compact  $n$ -Nambu-Poisson algebra is isomorphic to the  $n$ -Nambu algebra.*

*Proof.* By Theorems 5.12 and 3.5, we first need to consider good  $n$ -pairs  $(\mathcal{P}^\varphi, \mu)$  where  $\mathcal{P} = PO(k, k)$  or  $\mathcal{P} = PO(k, k+1)$  and  $n > 2$ . A complete list, up to isomorphisms, of such pairs is given in Theorem 7.2. The statement then follows from the construction described in Proposition 5.10. We point out that the pair  $(\mathcal{P}^\varphi, \varphi^{-1}\xi_1 \dots \xi_n)$ , with  $\mathcal{P} = PO(n, n)$ , corresponds to  $\mathcal{N}^\varphi$  where  $\mathcal{N}$  is the  $n$ -Nambu algebra; similarly, the pair  $(\mathcal{P}^\varphi, \varphi^{-1}\xi_1 \dots \xi_n \tau)$ , with  $\mathcal{P} = PO(n, n+1)$ , corresponds to  $\mathcal{N}^\varphi$ , where  $\mathcal{N}$  is the  $n$ -Dzhumadildaev algebra (see also Remark 5.13).  $\square$

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