Supplementary material for "What Determines Volunteer Work? On the Effects of Adverse Selection and Intrinsic Motivation"

by Nadia Burani and Arsen Palestini

A The second-best contract

For the sake of clarity, let us list the equations presented in the main text. The workers' information rent (surplus) is

$$U(\theta) = w(\theta) - \frac{1}{2}(\theta+1)\, e(\theta)^2 + \gamma e(\theta) \tag{1}$$

whereby, solving for the wage,

$$w(\theta) = U(\theta) + \frac{1}{2}(\theta+1)\, e(\theta)^2 - \gamma e(\theta). \tag{2}$$

The firm's problem is

$$\max_{e} \int_0^1 \left[(1+\gamma)\, e(\theta) - U(\theta) - \frac{1}{2}(\theta+1)\, e(\theta)^2\right] d\theta, \tag{3}$$

subject, for all $\theta \in [0,1]$, to: (i) the monotonicity condition

$$\frac{\partial e(\theta)}{\partial \theta} \leq 0, \tag{4}$$

(ii) the envelope condition

$$\frac{\partial U(\theta)}{\partial \theta} = -\frac{1}{2} e(\theta)^2, \tag{5}$$

and (iii) the individual rationality condition

$$U(\theta) \geq 0. \tag{6}$$

Given the envelope condition, incentive compatibility implies that only the participation constraint of the least able type be binding, whereby the participation constraint (6) reduces to the boundary condition $U(1) = 0$.

This is an optimal control problem, where e is the control variable and U is the state variable. In order to solve this problem let us build the Hamiltonian

$$H = (1+\gamma)\, e(\theta) - U(\theta) - \frac{1}{2}(\theta+1)\, e(\theta)^2 + \lambda(\theta)\left(-\frac{1}{2} e(\theta)^2\right),$$

where multiplier λ is the co-state variable. The first order conditions are the following ones

$$\begin{aligned}
\frac{\partial H}{\partial e} &= (1+\gamma) - (\theta+1)\, e(\theta) - \lambda(\theta)\, e(\theta) = 0 && (a)\\
-\frac{\partial H}{\partial U} &= 1 = \lambda'(\theta) && (b)\\
\frac{\partial U(\theta)}{\partial \theta} &= -\frac{1}{2} e(\theta)^2 && (c)\\
\lambda(0) &= 0 && (d)
\end{aligned}$$

where (d) is the transversality condition, since there is no constraint on $U(0)$. Integrating (b) over θ, one gets

$$\lambda(\theta) = \theta + c$$

and, using (d) to compute the value of the constant c, one obtains $c = 0$ and $\lambda(\theta) = \theta$. Replacing the latter expression into (a) yields the optimal effort

$$(1+\gamma) - (\theta+1)\, e(\theta) - \theta e(\theta) = 0 \quad \Longrightarrow \quad e_{SB}(\theta) = \frac{1+\gamma}{2\theta+1}. \tag{8}$$

The optimal wage rate is obtained by the envelope condition

$$\frac{\partial U(\theta)}{\partial \theta} = -\frac{1}{2} e(\theta)^2 = -\frac{(1+\gamma)^2}{2(2\theta+1)^2}$$

and integrating it over θ, with the requirement that $U(1) = 0$, yields

$$U_{SB}(\theta) = \frac{(1+\gamma)^2}{4(2\theta+1)} - \frac{(1+\gamma)^2}{12} = \frac{(1+\gamma)^2(1-\theta)}{6(2\theta+1)} \tag{9}$$

which reaches a maximum of $U_{SB}(0) = \frac{(1+\gamma)^2}{6}$. Substituting (9) and (8) into (2) one gets

$$w_{SB}(\theta) = \frac{(1+\gamma)\left((2\theta+1)(1-2\gamma) + (1-\theta)(1+\theta)(1+\gamma)\right)}{3(2\theta+1)^2},$$

where a sufficient condition for $w_{SB}(\theta) > 0$ is that $\gamma \leq \frac{1}{2}$. Conversely, for $\gamma > \frac{1}{2}$, $w_{SB}(\theta) > 0$ holds for

$$\theta < \frac{-(2\gamma-1) + \sqrt{3(\gamma^2-\gamma+1)}}{(1+\gamma)} = \widehat{\theta}$$

where $\widehat{\theta} < 1$ if and only if $\gamma > \frac{1}{2}$.

B The limited liability contract

When a limited liability constraint is introduced, the second-best solution is no longer valid, at least for workers with low skill levels. Therefore, one might expect that some bunching or some exclusion occurs for types with low ability, i.e. high θ.

Three possible scenarios can arise, and the firm will choose the regime delivering the highest possible payoffs. Each one of them is exposed in the next Subsections.

B.1 Full participation and pooling

Suppose that all types of workers are employed, even though some types (those with high effort cost) are not separated and are offered the same contract. In this case, full participation of workers' types is guaranteed, but there exists an optimal threshold $\overline{\theta}$ such that types below $\overline{\theta}$ are fully separated whereas types above $\overline{\theta}$ are pooled.

Then, by continuity of the optimal allocation, the solution is such that, for some threshold $\overline{\theta}$,

$$e(\theta) = \begin{cases} \frac{1+\gamma}{2\theta+1} & \text{for } 0 \leq \theta \leq \overline{\theta} \\ \frac{1+\gamma}{2\overline{\theta}+1} & \text{for } \overline{\theta} \leq \theta \leq 1 \end{cases} . \tag{10}$$

When workers' types belong to the range $\overline{\theta} \leq \theta \leq 1$, the schedule of information rents is given by (1) with $w = 0$ and $e = \frac{1+\gamma}{2\overline{\theta}+1}$ and it is such that

$$U(\theta) = \frac{(\gamma+1)\left(4\overline{\theta}\gamma - (1-\gamma) - \theta(1+\gamma)\right)}{2\left(2\overline{\theta}+1\right)^2}$$

Under full participation, it is optimal for the employer to leave the worst worker with zero rents, so it must be that $U(1) = 0$, which yields the optimal threshold

$$\overline{\theta} = \frac{1}{2\gamma}$$

and the optimal constant level of effort, that is required for types in the range $\overline{\theta} \leq \theta \leq 1$,

$$e = \frac{1+\gamma}{2\left(\frac{1}{2\gamma}\right)+1} = \gamma.$$

Accordingly, the optimal allocation (10) can be fully specified as

$$e_L(\theta) = \begin{cases} \frac{1+\gamma}{2\theta+1} & \text{for } 0 \leq \theta \leq \frac{1}{2\gamma} \\ \gamma & \text{for } \frac{1}{2\gamma} \leq \theta \leq 1 \end{cases} .$$

When workers' types belong to the interval $0 \leq \theta \leq \frac{1}{2\gamma}$, the function $U(\theta)$ can be recovered from the envelope condition (5) and it is equal to

$$U(\theta) = \frac{(1+\gamma)^2}{4(2\theta+1)} + c,$$

where the constant c can be computed using the continuity of the surplus function at $\overline{\theta}$ and the fact that

$$U\left(\theta, \overline{\theta} = \frac{1}{2\gamma}\right) = \frac{(1-\theta)\gamma^2}{2}.$$

This leads to

$$U_L(\theta) = \begin{cases} \frac{(1+\gamma)^2}{4(2\theta+1)} - \frac{1}{4}\gamma(2-\gamma) & \text{for } 0 \leq \theta \leq \frac{1}{2\gamma} \\ \frac{(1-\theta)\gamma^2}{2} & \text{for } \frac{1}{2\gamma} \leq \theta \leq 1 \end{cases}$$

The wage rate as a function of θ is such that

$$w_L(\theta) = \begin{cases} \frac{(\gamma+1)(4\theta(1-\gamma)+3-\gamma)}{4(2\theta+1)^2} - \frac{1}{4}\gamma(2-\gamma) & \text{for } 0 \leq \theta \leq \frac{1}{2\gamma} \\ 0 & \text{for } \frac{1}{2\gamma} \leq \theta \leq 1 \end{cases}.$$

Considering the easier case in which $\alpha = 1$, the firm's profits, computed by integrating over the possible levels of ability θ, are given by

$$\begin{aligned} \pi_1^S &= \int_0^{\frac{1}{2\gamma}} \left(\frac{(1+\gamma)}{(2\theta+1)} - \left(\frac{(\gamma+1)(4\theta(1-\gamma)+3-\gamma)}{4(2\theta+1)^2} - \frac{1}{4}\gamma(2-\gamma) \right) \right) d\theta \\ &= (\ln 2)\left(\frac{1}{2}\gamma + \frac{1}{4}\gamma^2 + \frac{1}{4}\right) - \frac{1}{4}\gamma + \left(\ln\left(\frac{1}{2\gamma} + \frac{1}{2}\right)\right)\left(\frac{1}{2}\gamma + \frac{1}{4}\gamma^2 + \frac{1}{4}\right) + \frac{1}{8} \end{aligned}$$

for the interval of ability levels in which full separation is possible, $0 \leq \theta \leq \frac{1}{2\gamma}$, and by

$$\pi_{1a}^P = \int_{\frac{1}{2\gamma}}^1 \gamma d\theta = \gamma - \frac{1}{2}$$

for the interval of ability levels in which pooling is necessary, $\frac{1}{2\gamma} \leq \theta \leq 1$. Thus, total profits in this case amount to

$$\pi_1 = \pi_1^S + \pi_1^P = \left(\ln 2 + \ln\left(\frac{(1+\gamma)}{2\gamma}\right)\right)\left(\frac{1}{4}(\gamma+1)^2\right) + \frac{3}{8}(2\gamma - 1).$$

B.2 Full separation and exclusion

Suppose now that the firm offers fully separating contracts but that it excludes workers with low ability levels. In this case, there exists an optimal threshold $\widetilde{\theta}$ such that types below $\widetilde{\theta}$ are fully separated whereas types above $\widetilde{\theta}$ are excluded. Consider the schedule of rents obtained by setting $w(\theta) = 0$ and e independent of θ and equal to $e = \frac{1+\gamma}{2\widetilde{\theta}+1}$

$$U(\theta) = \frac{(\gamma+1)\left(4\widetilde{\theta}\gamma - (1-\gamma) - \theta(1+\gamma)\right)}{2\left(2\widetilde{\theta}+1\right)^2}.$$

When is it that this function is non-negative? Consider $\theta = \widetilde{\theta}$ and substitute above. Then $U(\theta) = 0$ for $\theta = \widetilde{\theta} = \frac{1-\gamma}{3\gamma-1}$ and $U(\theta) > 0$ for $\theta < \widetilde{\theta} = \frac{1-\gamma}{3\gamma-1}$. For all θ higher than $\widetilde{\theta}$ information rents are negative and all types with $\theta > \widetilde{\theta}$ are excluded.

The effort function is

$$e^*(\theta) = \begin{cases} \frac{1+\gamma}{2\theta+1} & \text{for } 0 \leq \theta \leq \widetilde{\theta} \\ 0 & \text{for } \widetilde{\theta} \leq \theta \leq 1 \end{cases}$$

with

$$\widetilde{\theta} = \frac{1-\gamma}{(3\gamma-1)}$$

where $\widetilde{\theta} < 1$ when $\gamma > \frac{1}{2}$; moreover $e^*\left(\widetilde{\theta}\right) = 3\gamma - 1$.

Then $U(\theta)$ can be recovered from the envelope condition, and, for $0 \leq \theta \leq \widetilde{\theta}$, imposing the terminal condition $U\left(\widetilde{\theta}\right) = 0$ one gets

$$U(\theta) = \frac{(1+\gamma)^2}{4(2\theta+1)} - \frac{(3\gamma-1)(\gamma+1)}{4}.$$

So the optimal rents left to workers are

$$U^*(\theta) = \begin{cases} \frac{(1+\gamma)^2}{4(2\theta+1)} - \frac{(3\gamma-1)(\gamma+1)}{4} & \text{for } 0 \leq \theta \leq \widetilde{\theta} = \frac{1-\gamma}{(3\gamma-1)} \\ 0 & \text{for } \widetilde{\theta} = \frac{1-\gamma}{(3\gamma-1)} \leq \theta \leq 1 \end{cases}$$

and the wage rate is

$$w^*(\theta) = \begin{cases} \frac{(1+\gamma)(4\theta(1-\gamma)+3-\gamma)}{4(2\theta+1)^2} - \frac{(3\gamma-1)(\gamma+1)}{4} & \text{for } 0 \leq \theta \leq \frac{1-\gamma}{(3\gamma-1)} \\ 0 & \text{for } \frac{1-\gamma}{(3\gamma-1)} \leq \theta \leq 1 \end{cases}.$$

In this case, profits, computed by integrating over the possible levels of ability θ, are equal to

$$\begin{aligned} \pi_2 &= \int_0^{\frac{1-\gamma}{3\gamma-1}} \left(\frac{(1+\gamma)}{(2\theta+1)} - \left(\frac{(1+\gamma)(4\theta(1-\gamma)+3-\gamma)}{4(2\theta+1)^2} - \frac{(3\gamma-1)(\gamma+1)}{4} \right) \right) d\theta \\ &= (\ln 2)\left(\tfrac{1}{2}\gamma + \tfrac{1}{4}\gamma^2 + \tfrac{1}{4}\right) + \left(\ln\left(\tfrac{1-\gamma}{3\gamma-1} + \tfrac{1}{2}\right)\right)\left(\tfrac{1}{2}\gamma + \tfrac{1}{4}\gamma^2 + \tfrac{1}{4}\right) \end{aligned}.$$

It is straightforward to check that $\pi_1 > \pi_2$ is always true for $\gamma > \frac{1}{2}$, which is precisely the case at hand. So full participation and pooling always dominates full separation and exclusion. The latter solution can then be discarded.

B.3 Pooling and exclusion

If neither full separation nor full participation are feasible, there exist two optimal thresholds a and b, with $a < b$, such that workers' types θ below a are fully separated, types included between a and b are pooled and types above b are excluded. So the optimal allocation takes the form

$$e(\theta) = \begin{cases} \frac{1+\gamma}{2\theta+1} & \text{for } 0 \leq \theta \leq a \\ \frac{1+\gamma}{2a+1} & \text{for } a \leq \theta \leq b \\ 0 & \text{for } b \leq \theta \leq 1 \end{cases}$$

The schedule of information rents becomes $U(\theta) = 0$ for $b \leq \theta \leq 1$, whereas for $a \leq \theta \leq b$ it is such that

$$U(\theta) = -\frac{1}{2}(\theta+1)e^2 + \gamma e. \tag{11}$$

At the boundary, when $\theta = b$, rents are zero. Substituting for $\theta = b$ into (11) and equating to zero yields

$$-\frac{1}{2}(b+1)\left(\frac{1+\gamma}{2a+1}\right)^2 + \gamma\left(\frac{1+\gamma}{2a+1}\right) = 0 \quad \Longrightarrow \quad b = \frac{\gamma + 4a\gamma - 1}{1+\gamma}.$$

When θ is between a and b rents are

$$U(\theta) = \frac{(4a\gamma - \theta(1+\gamma) - (1-\gamma))(\gamma+1)}{2(2a+1)^2} \tag{12}$$

and, in particular

$$U(a) = \frac{(a(3\gamma-1) - (1-\gamma))(\gamma+1)}{2(2a+1)^2}.$$

When $\theta < a$, the schedule of information rents can be recovered by the envelope condition and, by continuity at a (as in the previous cases), we obtain

$$U(\theta) = \frac{(1+\gamma)^2}{4(2\theta+1)} - \frac{(3-\gamma+4a(1-\gamma))(\gamma+1)}{4(2a+1)^2}.$$

Summing up, information rents are given by the following expression (which still contains the threshold a, to be determined)

$$U^*(\theta) = \begin{cases} \frac{(1+\gamma)^2}{4(2\theta+1)} - \frac{(3-\gamma+4a(1-\gamma))(\gamma+1)}{4(2a+1)^2} & \text{for } 0 \le \theta \le a \\ \frac{(4a\gamma-\theta(1+\gamma)-(1-\gamma))(\gamma+1)}{2(2a+1)^2} & \text{for } a \le \theta \le b = \frac{\gamma+4a\gamma-1}{1+\gamma} \\ 0 & \text{for } \frac{\gamma+4a\gamma-1}{1+\gamma} = b \le \theta \le 1 \end{cases}.$$

It is then possible to recover the wage as a function of θ and a, which is the following

$$w^*(\theta) = \begin{cases} \frac{(3a+3\theta+4a\theta-a\gamma-\theta\gamma-4a\theta\gamma+2)(\gamma+1)(a-\theta)}{(2\theta+1)^2(2a+1)^2} & \text{for } 0 \le \theta \le a \\ 0 & \text{for } a \le \theta \le 1 \end{cases}.$$

Finally, profits to the firm as a function of θ and a are given by

$$\begin{aligned} \pi_3^S &= \int_0^a \left(\frac{(1+\gamma)}{(2\theta+1)} - \frac{(3a+3\theta+4a\theta-a\gamma-\theta\gamma-4a\theta\gamma+2)(\gamma+1)(a-\theta)}{(2\theta+1)^2(2a+1)^2} \right) d\theta \\ &= \frac{a(\gamma+1)(4a-\gamma-4a\gamma+3)}{4(2a+1)^2} - \frac{a(\gamma+1)^2}{4(2a+1)} + \left(\ln\left(a+\tfrac{1}{2}\right) + (\ln 2)\right)\left(\tfrac{1}{4}(\gamma+1)^2\right) \end{aligned}$$

when separation occurs, i.e. when $0 \le \theta \le a$, and by

$$\pi_3^P = \int_a^{\frac{\gamma+4a\gamma-1}{1+\gamma}} \frac{1+\gamma}{2a+1} d\theta = \frac{(\gamma-a+3a\gamma-1)}{(2a+1)}$$

when pooling occurs, i.e. when $a \le \theta \le b = \frac{\gamma+4a\gamma-1}{1+\gamma}$. Hence, total profits in this case amount to

$$\pi_3 = \pi_3^S + \pi_3^P = \frac{(\gamma-a+3a\gamma-1)(2+3a-a\gamma)}{2(2a+1)^2} + \left(\ln\left(a+\tfrac{1}{2}\right) + (\ln 2)\right)\left(\tfrac{1}{4}(\gamma+1)^2\right).$$

Differentiating these profits with respect to the cut-off a yields

$$\frac{\partial \pi_3}{\partial a} = \frac{2\left(2a+a^2+a^2\gamma+1\right)(\gamma+1)}{(2a+1)} > 0,$$

whereby profits to the firm are always increasing in a. This means that profits to the firm are maximal at the highest possible value of the threshold a, but this simply implies that we are back to the first case in which the firm does not exclude any type of worker and resorts to bunching for the less efficient types.