



Critical Singular Problems in Carnot Groups

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Abstract

We consider a power-type mild singular perturbation of a Dirichlet semilinear critical problem settled in an open and bounded set in a Carnot group. Here, the term critical has to be understood in the sense of the Sobolev embedding. We aim to prove the existence of two positive weak solutions: the first one is obtained by means of the variational Perron's method, while for the second one we adapt a classical argument relying on proper estimates of a family of functions which mimic the role of the classical Aubin-Talenti functions in the Euclidean setting. Our results fall in the framework of semilinear PDEs in Carnot groups but, as far as we know, are the first ones dealing with singular perturbations of power-type.

Keywords PDEs on Carnot groups · Singular equations · Critical equations

Mathematics Subject Classification 35R03 · 35B33 · 35B25 · 35J70

1 Introduction

Let $\mathbb{G}$ be a Carnot group, and let $\Omega \subset \mathbb{G}$ be a bounded and connected open set. Moreover, let $\gamma \in (0, 1)$, let $2_Q^* := \frac{2Q}{Q-2}$ be the critical Sobolev exponent related to the Sobolev inequality in $\mathbb{G}$, and let $\lambda > 0$. We consider the following singular Dirichlet problem

$$\begin{cases} -\Delta_{\mathbb{G}} u = \frac{\lambda}{u^\gamma} + u^{2_Q^*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (P_\lambda)$$

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where $\Delta_{\mathbb{G}}$ denotes a sub-Laplacian operator on $\mathbb{G}$ (see Section 2 for the relevant definitions). We immediately state the main result of this paper. In what follows, we refer to Definition 2.5 for the precise definition of *weak solution* of (P_{λ}) .

Theorem 1.1 *Let $\Omega \subset \mathbb{G}$ be a bounded and connected open set, and let $\gamma \in (0, 1)$. Then, there exists $\Lambda > 0$ such that the following assertions hold:*

- a) *problem (P_{λ}) admits at least two weak solutions for every $0 < \lambda < \Lambda$;*
- b) *problem (P_{λ}) admits at least one weak solution;*
- c) *problem (P_{λ}) does not admit weak solutions for every $\lambda > \Lambda$.*

The above theorem is the natural generalization to Carnot groups of classical results of Haitao [34] and Hirano, Saccon and Shioji [35], where the authors considered critical perturbations of mild singular terms using sub and supersolution methods ([34]) or a Nehari manifold approach ([35]). We stress that similar results have been obtained with different leading operators, see e.g. [4, 32, 41]. Before commenting on the proof of Theorem 1.1, we want to give a brief account of the existing literature concerning partial differential equations (PDEs, for short) on Carnot groups, with a focus on *critical semilinear equations*.

- *PDEs on Carnot groups.* To begin with, it is worth mentioning that, if $\mathbb{G}$ is a Carnot group and if $\Delta_{\mathbb{G}}$ is a sub-Laplacian on $\mathbb{G}$ (see Section 2 for the relevant definitions), then $\Delta_{\mathbb{G}}$ is a second-order differential operator with *non-negative characteristic form*, and thus it falls in the class of *degenerate-elliptic operators*. These degenerate operators have appeared in the literature since the early 1900s, due to their appearance in models of theoretical physics and of diffusion processes; however, the lack of *regularizing properties* (caused by the lack of ellipticity) creates several difficulties and prevents the application of several techniques.

In this perspective, a major difference between *general* degenerate-elliptic operators and the sub-Laplacians on Carnot groups is that $\Delta_{\mathbb{G}}$ is a *sum of squares* of vector fields satisfying the celebrated *Hörmander hypoellipticity condition*, see [36]. Starting from this fact, and by taking benefit of the underlying geometry attached to $\Delta_{\mathbb{G}}$ induced by $\mathbb{G}$, G.B. Folland, L.P. Rothschild and E.M. Stein developed in the 70's the *singular integral theory in nilpotent Lie groups* (see, e.g., Folland's survey [22] for a thorough discussion of the contributions of Stein in this context). More precisely, in 1975 Folland [21] accomplished a functional analytic study of sub-Laplacians on Carnot Lie groups and proved the existence of an associated *well-behaved global fundamental solution*. One year later Rothschild and Stein [60] proved their celebrated *lifting theorem* enlightening the fundamental role played by the sub-Laplacians in the theory of second order PDEs which are sum of squares of vector fields; this remarkable result paved the way for a deep study of PDEs on Carnot groups.

- *Critical PDEs on Carnot groups.* Firstly, when $\mathbb{G} = \mathbb{H}^n$ is the Heisenberg group and $\Omega = \mathbb{H}^n$, problem (P_0) (i.e., with $\lambda = 0$) coincides with the CR-Yamabe problem which has been deeply studied in a series of papers by Jerison and Lee [37–39], in connection with the existence of extremals for the associated Sobolev inequality. Further results concerning the CR-Yamabe problem can be found in e.g. [15, 26, 27]. In [38], Jerison and Lee actually provided the explicit expression of such extremals, resembling the Euclidean ones by Aubin and Talenti. After the seminal paper by Brézis and Nirenberg

[13], it became clear that the explicit knowledge of these functions was a key tool to attack the study of critical PDEs both in bounded and unbounded domains, at least in Heisenberg groups. In this perspective, we refer e.g. to [5, 6, 11, 17, 20, 29, 42, 47, 49, 51, 53, 55, 57–59, 65, 66] where several existence and non-existence results have been proved for the critical (or slightly sub-critical) equations; we also mention the papers [1, 16, 25, 28, 48] for perturbation results in the spirit of Brézis and Nirenberg, even dealing with the Grushin operator. We refer to e.g. [50, 52, 54] and the references therein for the case of sign-changing solutions.

The situation may a priori change when considering structures different from Heisenberg groups. In [30, 31] Garofalo and Vassilev exhibited a family of minimizers for the Sobolev inequality in groups of Iwasawa type. As far as we know, there are no other structures, nor Sobolev inequalities with $p \neq 2$, for which the minimizers are explicitly known. Nevertheless, it is worth mentioning that the best constant in the Sobolev inequality is achieved in all Carnot groups, see [30]. Moreover, the asymptotic behaviour at infinity of the minimizers has been found also in the case $p \neq 2$, see [45]. Going back to critical PDEs, thanks to [8], Loiudice in [43] obtained the sufficient asymptotic expansions of a family of functions naturally associated with the extremals, and this was enough to obtain a perturbation result à la Brézis-Nirenberg in a general Carnot group $\mathbb{G}$. We briefly mention that the above-mentioned papers are mainly interested in the existence, multiplicity or non-existence of positive solutions. Following this line of research, in [44, 46] Loiudice considered the case of singular perturbations of critical problems, where the term *singular* has to be considered in the sense of a suitable Hardy-type potential. See also [10] for further results in this line of research. Our interest in the present paper is to consider *mild singular perturbation* of the form $u^{-\gamma}$ with $\gamma \in (0, 1)$ (hence the term *mild*). Since the seminal paper by Crandall, Rabinowitz and Tartar [18], the literature dealing with singular problems of this kind in the Euclidean setting, even for $\gamma \geq 1$, has seen a great amount of contributions: we refer to the recent survey [56] and the references therein for a very detailed account. We mention also [2] where a wide class of possibly nonlocal operators is considered. On the other hand, to the best of our knowledge, this seems to be the first contribution which considers singular power-type perturbations in the setting of Carnot groups.

Let us now briefly comment on the main result stated in Theorem 1.1. The proof follows the argument performed in the Euclidean case by Haitao [34], suitably adapted to the Carnot group setting, and it consists of several technical steps that we list here below:

- we prove the existence of a first solution by means of a variational sub and supersolution scheme, adapting the approach of Struwe [62]. The subsolution is naturally provided by the unique solution of the purely singular problem, see Theorem 2.7, while the supersolution is constructed in Lemma 3.5. We notice that this scheme immediately provides a threshold Λ (see (3.1)) for the non-existence;
- we show that for $\lambda \in (0, \Lambda)$ the first solution obtained as described before is a local minimizer in the natural topology associated with problem (P_λ) , see Lemma 3.6;

- we follow an argument originally due to Tarantello [64] (see also [3]) to prove the existence of a second solution. Here we heavily employ the asymptotic expansions found in [43], taking care of the new singular term, and we adapt the Euclidean estimates of [14] exploiting the properties of the convolution in Carnot groups proved in [23].

The paper is organized as follows: in Section 2 we recall the basic facts on Carnot groups needed in what follows, like the Folland-Stein spaces which provide the natural variational framework where problem (P_λ) is set, strong maximum principle and weak Harnack inequality for the operator $-\Delta_{\mathbb{G}} + c$ which we were not able to find in the literature but is probably well known to experts in the field. We also provide the basic result for the purely singular problem whose importance has been already described. Section 3 is devoted to find the first solution while the second solution (for $\lambda \in (0, \Lambda)$) is found in Section 4.

2 Preliminaries

In this section we collect all the relevant notations, definitions and preliminaries needed in the rest of the paper.

2.1 Carnot Groups

A Carnot group $\mathbb{G} = (\mathbb{R}^N, \diamond)$ of step k is a connected, simply connected Lie group whose finite dimensional Lie algebra $\mathfrak{g}$ of left-invariant (w.r.t. $\diamond$) vector fields admits a stratification of step k , namely there exist k linear subspaces $\mathfrak{g}_1, \dots, \mathfrak{g}_k$ such that

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k, \quad [\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}, \quad \mathfrak{g}_k \neq \{0\}, \quad \mathfrak{g}_i = \{0\} \text{ for all } i > k.$$

In particular, this implies that Carnot groups are a special instance of graded groups.

We call $\mathfrak{g}_1$ the horizontal layer. We denote by $X_1, \dots, X_N$ a basis of left-invariant vector fields of $\mathfrak{g}$ such that the following holds:

- $X_1, \dots, X_{m_1}$ is a basis of $\mathfrak{g}_1$;
- for every $1 < i \leq k$, $X_{m_{i-1}+1}, \dots, X_{m_i}$ is a basis of $\mathfrak{g}_i$;
- $m_0 = 0$ and $n_i := m_i - m_{i-1} = \dim \mathfrak{g}_i$ for every $1 \leq i \leq k$;
- $m_1 + \dots + m_k = N$.

We notice that N is the topological dimension of $\mathbb{G}$, but we can also define its *homogeneous dimension* Q as follows

$$Q := \sum_{i=1}^k i n_i. \quad (2.1)$$

We notice that $N \leq Q$ and that $N = Q$ if and only if $\mathbb{G}$ is the classical Euclidean group $(\mathbb{R}^N, +)$. In particular, this is the only possible case whenever $Q \leq 3$.

Since the exponential map is a one-to-one diffeomorphism from $\mathfrak{g}$ to $\mathbb{G}$, any point $g \in \mathbb{G}$ can be uniquely written in exponential coordinates as

$$g = g_1 X_1 + \dots + g_N X_N = (g_1, \dots, g_N).$$

Being a graded group, every Carnot group $\mathbb{G}$ possesses a family of anisotropic dilations $\delta_\lambda : \mathbb{G} \rightarrow \mathbb{G}$ defined as

$$\delta_\lambda(g) = (\lambda^{\alpha_1} g_1, \dots, \lambda^{\alpha_N} g_N), \quad \text{for every } \lambda > 0, \quad (2.2)$$

where $\alpha_j = i$ if $m_{i-1} < j \leq m_i$. We notice that $Q = \alpha_1 + \dots + \alpha_N$.

An explicit expression of the group operation $\diamond$ can be then determined by means of the Campbell-Baker-Hausdorff formula, see e.g. [7] for more details. The null element of $\diamond$ is the identity $0 = (0, \dots, 0)$ and the inverse of a certain $g \neq 0$ is usually denoted by g^{-1} .

The group operation $\diamond$ can also be used to define a further family of automorphisms of $\mathbb{G}$ known as *left translations* $\tau : \mathbb{G} \rightarrow \mathbb{G}$. More precisely, given a base point $h \in \mathbb{G}$, we define

$$\tau_h(g) := h \diamond g. \quad (2.3)$$

It is possible to endow a Carnot group $\mathbb{G}$ with a richer structure. Firstly, it is possible to define a scalar product $\langle \cdot, \cdot \rangle_{\mathfrak{g}_1}$ such that the basis $\{X_1, \dots, X_{m_1}\}$ of the horizontal layer becomes orthonormal, and this provides a sub-Riemannian structure over $\mathbb{G}$. We notice that one can also provide a purely Riemannian structure defining a scalar product on the Lie algebra $\mathfrak{g}$ making the entire basis $\{X_1, \dots, X_N\}$ orthonormal. The sub-Riemannian Carnot group $\mathbb{G}$ can also be endowed with an intrinsic metric structure by means of the so called *Carnot-Carathéodory* (CC in short) distance. It is well known that with this distance, these spaces are not Riemannian at any scale, see e.g. [61].

From a more analytic point of view, we can define several differential operators modelled on the horizontal vector fields $\{X_1, \dots, X_{m_1}\}$. First, given a smooth horizontal vector field $V = v_1 X_1 + \dots + v_{m_1} X_{m_1}$, we define its horizontal divergence as

$$\operatorname{div}_{\mathbb{G}} V := X_1 v_1 + \dots + X_{m_1} v_{m_1}. \quad (2.4)$$

Moreover, given a smooth enough scalar-valued function $u : \mathbb{G} \rightarrow \mathbb{R}$, we can define the horizontal gradient of u as

$$\nabla_{\mathbb{G}} u := (X_1 u, \dots, X_{m_1} u), \quad (2.5)$$

and the sub-Laplacian of u as

$$\Delta_{\mathbb{G}} u := \operatorname{div}_{\mathbb{G}}(\nabla_{\mathbb{G}} u) = X_1^2 u + \dots + X_{m_1}^2 u, \quad (2.6)$$

We point out that both the divergence $\operatorname{div}_{\mathbb{G}}$ and the horizontal gradient $\nabla_{\mathbb{G}}$ (and *a fortiori* the sub-Laplacian $\Delta_{\mathbb{G}}$) are independent of the choice of the basis on $\mathfrak{g}_1$: in fact, they are intrinsically associated to the sub-Riemannian structure on $\mathbb{G}$.

We notice that both the horizontal gradient $\nabla_{\mathbb{G}}$ and the sub-Laplacian $\Delta_{\mathbb{G}}$ are left-invariant operators, i.e.

$$\nabla_{\mathbb{G}}(u \circ \tau_h) = (\nabla_{\mathbb{G}}u) \circ \tau_h \quad \text{and} \quad \Delta_{\mathbb{G}}(u \circ \tau_h) = (\Delta_{\mathbb{G}}u) \circ \tau_h, \quad \text{for every } h \in \mathbb{G}, \quad (2.7)$$

and they are, respectively, homogeneous of degree one and two w.r.t. the family of dilations δ_λ defined in (2.2), namely

$$\nabla_{\mathbb{G}}(u \circ \delta_\lambda) = \lambda(\nabla_{\mathbb{G}}u) \circ \delta_\lambda \quad \text{and} \quad \Delta_{\mathbb{G}}(u \circ \delta_\lambda) = \lambda^2(\Delta_{\mathbb{G}}u) \circ \delta_\lambda, \quad \text{for every } \lambda > 0. \quad (2.8)$$

The Lebesgue measure $\mathcal{L}^N$ coincides with the Haar measure of $\mathbb{G}$ and hence is left-invariant and satisfies the following scaling property:

$$\mathcal{L}^N(\delta_\lambda(E)) = \lambda^Q \mathcal{L}^N(E) \quad \text{for every measurable set } E \subset \mathbb{G}. \quad (2.9)$$

Every integral in this manuscript has to be understood with respect to the Haar measure, unless otherwise stated.

Moreover, the homogeneity of the X_i 's implies that the (formal) adjoint of X_i in the Lebesgue space $L^2(\mathbb{G})$ is precisely $-X_i$ (for $1 \leq i \leq m_1$), that is,

$$\int_{\mathbb{G}} (X_i \varphi) \psi = - \int_{\mathbb{G}} \varphi (X_i \psi) \quad \text{for every } 1 \leq i \leq m_1. \quad (2.10)$$

In particular, $-\Delta_{\mathbb{G}}$ is a *self-adjoint operator*.

Every Carnot group can be endowed with several homogeneous norms. A homogeneous (quasi)norm $\rho : \mathbb{G} \rightarrow \mathbb{R}$ is a non-negative function further satisfying the following properties:

- $\rho(g) = 0$ if and only if $g = 0$;
- $\rho(\delta_\lambda(g)) = \lambda \rho(g)$ for every $g \in \mathbb{G}$ and for every $\lambda > 0$;
- $\rho(h \diamond g) \leq C(\rho(h) + \rho(g))$ for every $g, h \in \mathbb{G}$ and for some constant $C \geq 1$.

The importance of such objects is witnessed by a famous result of Folland: in [21], he showed that there exists a homogeneous norm $|\cdot|_{\mathbb{G}}$ on $\mathbb{G}$ and a positive constant $C_Q > 0$, depending only on Q , such that the function

$$\Gamma_h(g) := \frac{C_Q}{|h^{-1} \diamond g|_{\mathbb{G}}^{Q-2}}, \quad \text{with } Q \geq 3, \quad (2.11)$$

is a fundamental solution of $-\Delta_{\mathbb{G}}$ with pole at $h \in \mathbb{G}$. Moreover, homogeneous norms can be used to define distances, different from the CC-distance, as follows:

$$d_\rho(g, h) := \rho(h^{-1} \diamond g).$$

In any case, all these norms (and the relative distances) are equivalent and they all induce on $\mathbb{G}$ the Euclidean topology. For our purposes, we prefer to work with the

homogeneous norm $|\cdot|_{\mathbb{G}}$ (and the associated distance $d_{\mathbb{G}}$) which provides the fundamental solution defined in (2.11). In particular, we will denote by

$$B_r(g_0) := \{g \in \mathbb{G} : d_{\mathbb{G}}(g, g_0) = |g_0|^{-1} \diamond g|_{\mathbb{G}} < r\},$$

the open ball of radius $r > 0$ and center $g_0 \in \mathbb{G}$.

2.2 Folland-Stein Spaces and Critical PDEs

Let $\mathcal{O} \subseteq \mathbb{G}$ be an open set. For every $f \in C_0^\infty(\mathcal{O})$ there exists a positive constant $C_Q > 0$ depending only on the homogeneous dimension Q such that the following Sobolev inequality holds true

$$\|f\|_{L^{2^*}(\mathcal{O})}^2 \leq C_Q \|\nabla_{\mathbb{G}} f\|_{L^2(\mathcal{O})}^2, \quad (2.12)$$

where

$$2^* := \frac{2Q}{Q-2}, \quad (2.13)$$

denotes the (sub-elliptic) critical Sobolev exponent, resembling the classical Euclidean one

$$2^* = \frac{2N}{N-2}.$$

Thanks to (2.12), $\|\nabla_{\mathbb{G}} f\|_{L^2(\Omega)}$ provides a norm on the space $C_0^\infty(\Omega)$. We define the Folland-Stein space $S_0^1(\mathcal{O})$ as the completion of $C_0^\infty(\mathcal{O})$ w.r.t. the above norm, and we set

$$\|u\|_{S_0^1(\mathcal{O})} = \|\nabla_{\mathbb{G}} u\|_{L^2(\mathcal{O})} \quad \text{for every } u \in S_0^1(\mathcal{O}).$$

We explicitly observe that, owing to (2.12), we have

$$S_0^1(\mathcal{O}) = \{u \in L^{2^*}(\mathcal{O}) : X_i u \in L^2(\mathcal{O}) \text{ for all } 1 \leq i \leq m_1\}, \quad (2.14)$$

where $X_1 u, \dots, X_{m_1} u$ are meant in the sense of distributions, that is (see also (2.10)),

$$\int_{\mathcal{O}} (X_j u) \varphi = - \int_{\mathcal{O}} u X_j \varphi \quad \text{for all } \varphi \in C_0^\infty(\mathcal{O}).$$

In the particular case when $\mathcal{O}$ is *bounded* (which is the case we are mainly interested in, together with the case $\mathcal{O} = \mathbb{G}$), the space $S_0^1(\mathcal{O})$ enjoys the following classical properties.

(1) $S_0^1(\mathcal{O})$ is endowed with a structure of real Hilbert space by the inner product

$$\langle u, v \rangle_{S_0^1(\mathcal{O})} = \int_{\mathcal{O}} \langle \nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}} v \rangle_{\mathfrak{g}_1} \quad (u, v \in S_0^1(\mathcal{O})),$$

whose associated norm is precisely $\|\cdot\|_{S_0^1(\mathcal{O})}$.

- (2) By density, the Sobolev inequality (2.12) actually holds for every function $u \in S_0^1(\mathcal{O})$. As a consequence, $S_0^1(\mathcal{O})$ is *continuously embedded* into $L^p(\mathcal{O})$ for every $1 \leq p \leq 2_Q^*$. Furthermore, this embedding turns out to be *compact* when $1 \leq p < 2_Q^*$.
- (3) If $u \in S_0^1(\mathcal{O})$, then $u_+ = \max\{u, 0\}$, $u_- = \max\{-u, 0\} \in S_0^1(\mathcal{O})$, and

$$\nabla_{\mathbb{G}} u_+ = \nabla_{\mathbb{G}} u \cdot \chi_{\{u>0\}} \quad \text{and} \quad \nabla_{\mathbb{G}} u_- = \nabla_{\mathbb{G}} u \cdot \chi_{\{u<0\}}.$$

In particular, for every $c \in \mathbb{R}$, we derive

$$\nabla_{\mathbb{G}} u = 0 \quad \text{a.e. on any level set } \{u = c\}. \quad (2.15)$$

We refer, e.g., to [24] for good approximation results in terms of smooth functions, allowing to prove the above facts.

Remark 2.1 On account of the above properties of $S_0^1(\Omega)$, it is possible to prove the following *convergence result*, which will be repeatedly used in the sequel.

Assume that $\{u_k\}_k \subseteq S_0^1(\Omega)$ is a *bounded sequence*. Since $S_0^1(\Omega)$ is a real *Hilbert space*, and since we have already pointed out that the embedding

$$S_0^1(\Omega) \hookrightarrow L^p(\Omega)$$

is compact for every $1 \leq p < 2_Q^*$, as in the classical case we deduce that there exists a function $u \in S_0^1(\Omega)$ such that (up to a subsequence, and as $k \rightarrow +\infty$)

- $u_k \rightarrow u$ weakly in $S_0^1(\Omega)$;
- $u_k \rightarrow u$ strongly in $L^p(\Omega)$ for every $1 \leq p < 2_Q^*$;
- $u_k \rightarrow u$ pointwise a.e. in Ω .

On the other hand, since the embedding $S_0^1(\Omega) \hookrightarrow L^{2_Q^*}(\Omega)$ is *continuous*, we have that $u \in L^{2_Q^*}(\Omega)$, and $\{u_k\}_k$ is *bounded also in $L^{2_Q^*}(\Omega)$* ; in particular, setting

$$p_0 = \frac{2_Q^*}{2_Q^* - 1} \in (1, 2_Q^*),$$

we have that $\{v_k = u_k^{2_Q^*-1}\}_k$ is bounded in $L^{p_0}(\Omega)$. As a consequence of this fact, and since we know that $u_k \rightarrow u$ pointwise a.e. in Ω , we deduce that $\{v_k\}_k$ *converges weakly to $u^{2_Q^*-1}$* (as $k \rightarrow +\infty$, and up to a subsequence) in $L^{p_0}(\Omega)$, that is,

$$\int_{\Omega} u_k^{2_Q^*-1} \varphi \, dx \rightarrow \int_{\Omega} u^{2_Q^*-1} \varphi \, dx \quad \text{for every } \varphi \in L^{p_0'}(\Omega) = L^{2_Q^*}(\Omega) \supseteq S_0^1(\Omega).$$

In [31] Garofalo and Vassilev adapted the celebrated *concentration-compactness principle* of Lions showing that the best Sobolev constant in (2.12) can be achieved

and it is characterized as follows

$$S_{\mathbb{G}} := \inf_{f \in S_0^1(\mathbb{G})} \frac{\|\nabla_{\mathbb{G}} f\|_{L^2(\mathbb{G})}^2}{\|f\|_{L^{2^*_{\mathbb{G}}}(\mathbb{G})}^2}. \quad (2.16)$$

As in the Euclidean case, this fact has immediate consequences at the level of the critical PDE

$$-\Delta_{\mathbb{G}} u = u^{2^*_{\mathbb{G}}-1} \quad \text{in } \mathbb{G}. \quad (2.17)$$

We list here below the most important results concerning (2.16) and (2.17).

- Every minimizing sequence of (2.16) is relatively compact in $S_0^1(\mathbb{G})$, after possibly translating and dilating each of its elements. In particular, the minimum in (2.16) is achieved and (2.17) admits a non-negative and non-trivial solution $U \in S_0^1(\mathbb{G})$, see [31, Theorem 6.1].
- By Bony's maximum principle [9], every non-negative solution of (2.17) is actually positive.
- If $T \in S_0^1(\mathbb{G})$ is a positive solution of (2.17), then there exists a positive constant $M_1 > 0$ such that

$$T(g) \leq M_1 \min\{1, |g|_{\mathbb{G}}^{2^*_{\mathbb{G}}-Q}\}, \quad \text{for every } g \in \mathbb{G}, \quad (2.18)$$

see [8, Theorem 3.4].

- If $T \in S_0^1(\mathbb{G})$ is a positive solution of (2.17), then there exists a positive constant $M_2 > 0$ such that

$$T(g) \geq M_2 \frac{|B_1(0)|}{(1 + |g|_{\mathbb{G}})^{Q-2}}, \quad \text{for every } g \in \mathbb{G}, \quad (2.19)$$

see [43, Lemma 3.2].

Let now $T \in S_0^1(\mathbb{G})$ be a minimizer of (2.16). For every $\varepsilon > 0$ define the rescaled function

$$T_{\varepsilon}(g) := \varepsilon^{(2-Q)/2} T(\delta_{1/\varepsilon}(g)). \quad (2.20)$$

Let further be $R > 0$ such that $B_R(0) \subset \Omega$ and let $\varphi \in C_0^\infty(B_R(0))$ be a cut-off function such that $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ in $B_{R/2}(0)$. Define the function

$$U_{\varepsilon}(g) := \varphi(g) T_{\varepsilon}(g), \quad g \in \mathbb{G}. \quad (2.21)$$

The following holds:

- Up to multiplicative constants, T_{ε} is a solution of (2.17).
- Due to scaling invariance, we can set T as to have

$$\|\nabla_{\mathbb{G}} T_{\varepsilon}\|_{L^2(\mathbb{G})}^2 = \|T_{\varepsilon}\|_{L^{2^*_{\mathbb{G}}}(\mathbb{G})}^{2^*_{\mathbb{G}}} = S_{\mathbb{G}}^{Q/2}. \quad (2.22)$$

- The function U_ε satisfies the following estimates as $\varepsilon \rightarrow 0^+$

$$\|\nabla_{\mathbb{G}} U_\varepsilon\|_{L^2(\mathbb{G})}^2 = S_{\mathbb{G}}^{Q/2} + O(\varepsilon^{Q-2}) \quad (2.23)$$

$$\|U_\varepsilon\|_{L^{2^*_Q}(\mathbb{G})}^{2^*_Q} = S_{\mathbb{G}}^{Q/2} + O(\varepsilon^Q), \quad (2.24)$$

see [43, Lemma 3.3].

2.3 Strong Maximum Principle and Harnack Inequality

Now we have reviewed the basic concepts concerning the Carnot groups setting, and before starting our study of problem (P_λ) , for a future reference we explicitly state here below a *Weak Harnack inequality* for

$$L = -\Delta_{\mathbb{G}} + c(x) \quad (\text{with } c \geq 0),$$

from which we will also derive a *Strong Maximum Principle*. These results will be fundamental throughout the rest of the paper.

Proposition 2.2 (Weak Harnack inequality for $-\Delta_{\mathbb{G}} + c$) *Let $\mathcal{O} \subset \mathbb{G}$ be a bounded open set, and let $c \in L^\infty(\mathcal{O})$, $c \geq 0$. Moreover, let $u \in S_0^1(\mathcal{O})$ be a weak supersolution of the equation*

$$-\Delta_{\mathbb{G}} u + cu = 0 \text{ in } \mathcal{O}, \quad (2.25)$$

that is (see also the subsequent Definition 2.5),

$$\int_{\mathcal{O}} \langle \nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}} \varphi \rangle_{\mathfrak{g}_1} + \int_{\mathcal{O}} cu\varphi \geq 0 \quad \text{for every } \varphi \in C_0^\infty(\mathcal{O}), \varphi \geq 0.$$

If, in addition, $u \geq 0$ a.e. in $\mathcal{O}$, there exist constants $c_0 > 0$ and $p_0 \in (0, 1)$, independent of u , such that, for every $g_0 \in \mathcal{O}$, $r > 0$ with $B_{4r}(g_0) \Subset \mathcal{O}$, we have

$$\left(\int_{B_{3r}(g_0)} |u|^{p_0} \right)^{1/p_0} \leq c_0 \inf_{B_r} u. \quad (2.26)$$

Proof The proof of this proposition is contained in the proof of [33, Theorem 4.1], where the Authors establish a *full Harnack inequality* for the *non-negative weak solutions* u of

$$Lu = 0 \text{ in } \Omega, \quad (2.27)$$

and L is a general X -elliptic operator (of which $-\Delta_{\mathbb{G}} + c(\cdot)$ is a particular case). Indeed, to prove the cited Harnack inequality for a given non-negative weak solution u of (2.27), the Authors in [33] use an adaptation of the Moser iteration technique, and they show that

i) since u is, in particular, a *weak subsolution* of (2.27), then

$$\sup_{B_r(g_0)} u \leq C_1 \left(\int_{B_{2r}(g_0)} |u|^s \right)^{1/s},$$

for some constants $C_1 > 0$, $s > 0$ independent of u ;

ii) since u is, in particular, a *weak supersolution* of (2.27), then

$$\inf_{B_r(g_0)} u \geq C_2 \left(\int_{B_{3r}(g_0)} |u|^{-p_0} \right)^{-1/p_0},$$

for some constant $C_2 > 0$ and every p_0 small enough;

iii) there exists a constant $C_3 > 0$, only depending on p_0 , such that

$$\left(\int_{B_{3r}(g_0)} |u|^{p_0} \right)^{1/p_0} \leq C_3 \left(\int_{B_{3r}(g_0)} |u|^{-p_0} \right)^{-1/p_0},$$

provided that $B_{4r}(g_0) \Subset \mathcal{O}$. Thus, since the proofs of i) - iii) are mutually independent, if u is just a weak supersolution of (2.25) (which is a particular case of (2.27)), the demonstration of assertion ii) in [33] gives exactly the desired estimate (2.26). $\square$

From Proposition 2.2, and using a classical covering argument, we obtain the following

Corollary 2.3 *Let $\mathcal{O} \subseteq \mathbb{G}$ be a bounded open set, and let $c \in L^\infty(\mathcal{O})$, $c \geq 0$. Furthermore, let $u \in S_0^1(\mathcal{O})$ be a weak supersolution of equation (2.25) satisfying*

$$u > 0 \text{ a.e. on every open ball } B = B_r(g_0) \Subset \Omega.$$

Then, for every open set $\mathcal{O}' \Subset \mathcal{O}$ there exists $C = C(\mathcal{O}', u) > 0$ such that

$$u \geq C(\mathcal{O}', u) > 0 \text{ a.e. in } \mathcal{O}'.$$

As anticipated, Proposition 2.2 implies the following *Strong Maximum Principle*.

Proposition 2.4 (Strong Maximum Principle for L) *Let $\mathcal{O} \subseteq \mathbb{G}$ be a bounded and connected open set, and let $u \in S_0^1(\mathcal{O})$, $u \geq 0$ be a weak supersolution of (2.25). Then,*

$$\text{either } u \equiv 0 \text{ or } u > 0 \text{ a.e. in } \mathcal{O}. \quad (2.28)$$

Proof We assume that there exists a set $\mathcal{Z} \subseteq \mathcal{O}$ of positive Lebesgue measure such that $u \equiv 0$ (pointwise) on $\mathcal{Z}$, and we prove that in this case we have $u \equiv 0$ a.e. on $\mathcal{O}$.

To this end we first observe that, since $|\mathcal{Z}| > 0$, we can find $r_0 > 0$ such that

$$\mathcal{Z}_0 = \mathcal{Z} \cap \{g \in \mathcal{O} : d_{\mathbb{G}}(g, \partial\mathcal{O}) \geq r_0\}$$

has *positive Lebesgue measure*; moreover, since $K = \{g \in \mathcal{O} : d_{\mathbb{G}}(g, \partial\mathcal{O}) \geq r_0\} \subseteq \Omega$ is compact (recall that $d_{\mathbb{G}}$ induces the Euclidean topology), there exist $g_1, \dots, g_p \in K$ and $r_1, \dots, r_p > 0$ (for some $p \in \mathbb{N}$) such that $B_{4r_i}(g_i) \Subset \mathcal{O}$ (for $1 \leq i \leq p$) and

$$\mathcal{Z}_0 = \bigcup_{i=1}^p (\mathcal{Z}_0 \cap B_{r_i}(g_i)).$$

As a consequence, there exists a $d_{\mathbb{G}}$ -ball $B_0 = B_{r_{i_0}}(g_{i_0})$ (for some $1 \leq i_0 \leq p$) such that

$$|\mathcal{Z}_0 \cap B_{r_{i_0}}(g_{i_0})| > 0. \quad (2.29)$$

In particular, since $u \geq 0$ a.e. in $\mathcal{O}$ and $u = 0$ on $\mathcal{Z}_0$, we have

$$\inf_{B_{r_{i_0}}(g_{i_0})} u = \inf_{\Omega} u = 0. \quad (2.30)$$

With (2.29)-(2.30) at hand, we then define the set

$$S = \{g \in \mathcal{O} : \text{there exists a ball } B = B_r(g) \subseteq \mathcal{O} \text{ s.t. } u = 0 \text{ a.e. on } B\},$$

and we prove that $S = \mathcal{O}$ by a *connection argument*:

- $S \neq \emptyset$. Let $g_{i_0} \in \mathcal{O}$ be as in (2.29)-(2.30). Since u is a non-negative weak supersolution of equation (2.25), and since $B = B_{4r_{i_0}}(g_{i_0}) \Subset \mathcal{O}$, we can apply the Weak Harnack Inequality in Proposition 2.2: this gives the following estimate

$$\left(\int_{B_{3r_{i_0}}(g_{i_0})} |u|^{p_0} \right)^{1/p_0} \leq c \inf_{B_{r_{i_0}}(g_{i_0})} u = 0,$$

from which we derive that $u = 0$ a.e. in $B_{3r_{i_0}}$. Hence, $g_{i_0} \in S$.

- S is open. Assume that $g \in S$, and let $B = B_r(g) \subseteq \mathcal{O}$ be a $d_{\mathbb{G}}$ -ball such that

$$u = 0 \text{ a.e. in } B.$$

Given any $g' \in B$, if we choose $\rho > 0$ in such a way that $B_{\rho}(g') \subseteq B$, we obviously have

$$u = 0 \text{ a.e. on } B_{\rho}(g') \subseteq B,$$

and this proves that $g' \in S$. By the arbitrariness of $g' \in B_r(g)$, we conclude that $B_r(g) \subseteq S$, and hence S is open, as desired.

- S is closed. Assume that $(g_k)_k$ is a sequence of points in S which converges (as $k \rightarrow +\infty$) to some $g \in \mathcal{O}$, and let $\rho > 0$ be such that $B_{4\rho}(g) \Subset \mathcal{O}$. Since $g_k \rightarrow g$ as $k \rightarrow +\infty$, there exists some $k_0 \in \mathbb{N}$ such that $g_{k_0} \in B_{\rho}(g)$; on the other hand, since $g_{k_0} \in S$, it is possible to find a suitable $d_{\mathbb{G}}$ -ball $B = B_r(g_{k_0})$ such that $B \subseteq B_{\rho}(g)$ and

$$u = 0 \text{ a.e. in } B.$$

Hence, using once again the Weak Harnack Inequality in Proposition 2.2, we get

$$\left(\int_{B_{3\rho}(g)} |u|^{p_0} \right)^{1/p_0} \leq c \inf_{B_\rho(g)} u = 0,$$

and thus $u = 0$ a.e. in $B_{3\rho}(g)$, that is, $g \in S$. Hence, S is closed.

Gathering the above facts, and recalling that $\mathcal{O}$ is connected, we then conclude that $\mathcal{O} = S$, and this obviously implies that $u = 0$ a.e. in $\mathcal{O}$, as desired. $\square$

2.4 Weak Sub/supersolutions of (P_λ)

We are now ready to properly set the definition of weak sub/supersolution of (P_λ) . However, we prefer to provide such definitions for the following slightly more general singular Dirichlet problem:

$$\begin{cases} -\Delta_{\mathbb{G}} u = \frac{\lambda}{u^\gamma} + f(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.31)$$

where $\gamma \in (0, 1)$, $\lambda > 0$ and $f : \Omega \times (0, +\infty) \rightarrow \mathbb{R}$ is a Carathéodory function satisfying the following critical growth condition: there exists a positive constant $K_f > 0$ such that

$$|f(x, t)| \leq K_f(1 + t^{2^*-1}) \quad \text{for a.e. } x \in \Omega \text{ and for every } t > 0. \quad (2.32)$$

Here and throughout, $\Omega \subseteq \mathbb{G}$ is a fixed *bounded and connected open set* (as in (P_λ)).

Definition 2.5 Let $f : \Omega \times (0, +\infty) \rightarrow \mathbb{R}$ be a Carathéodory function satisfying the growth condition (2.32). We say that a function $u \in S_0^1(\Omega)$ is a weak subsolution (resp. supersolution) of (2.31) if it satisfies the following properties:

- (i) $u > 0$ in Ω and $u^{-\gamma} \in L_{\text{loc}}^1(\Omega)$.
- (ii) For every $0 \leq \varphi \in C_0^\infty(\Omega)$, it holds that

$$\int_{\Omega} \langle \nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}} \varphi \rangle_{\mathfrak{g}_1} \leq (\text{resp. } \geq) \lambda \int_{\Omega} u^{-\gamma} \varphi + \int_{\Omega} f(x, u) \varphi. \quad (2.33)$$

Finally, we say that $u \in S_0^1(\Omega)$ is a weak solution of (2.31) if it is both a weak subsolution and a weak supersolution of (2.31) without the non-negativity condition on φ .

Remark 2.6 We now list here below some comments concerning the above Definition 2.5. In what follows, we tacitly understand that $f : \Omega \rightarrow (0, +\infty)$ is a Carathéodory function satisfying the growth assumption (2.32).

1) If $u \in S_0^1(\Omega)$ is a weak sub/supersolution of problem (2.31), all the integrals appearing in (2.33) exist and are finite. Indeed, by Hölder's inequality, for every $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$ in Ω , we obtain

$$\left| \int_{\Omega} \langle \nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}} \varphi \rangle_{\mathfrak{g}_1} \right| \leq \|u\|_{S_0^1(\Omega)} \|\varphi\|_{S_0^1(\Omega)} \quad (2.34)$$

Moreover, using (2.32) and the Sobolev inequality, we also get

$$\begin{aligned} \int_{\Omega} f(x, u) |\varphi| &\leq K_f \left(\|\varphi\|_{L^1(\Omega)} + \int_{\Omega} |u|^{2_{\mathbb{Q}}^*-1} |\varphi| dx \right) \\ &\quad (\text{using Hölder's inequality}) \\ &\leq K_f \left(\|\varphi\|_{L^1(\Omega)} + \|u\|_{L^{2_{\mathbb{Q}}^*}(\Omega)}^{2_{\mathbb{Q}}^*-1} \cdot \|\varphi\|_{L^{2_{\mathbb{Q}}^*}(\Omega)} \right) \\ &\leq K_f \left(\|\varphi\|_{L^1(\Omega)} + S_{\mathbb{G}}^{-2_{\mathbb{Q}}^*/2} \|u\|_{S_0^1(\Omega)}^{2_{\mathbb{Q}}^*-1} \cdot \|\varphi\|_{S_0^1(\Omega)} \right) \\ &\quad (\text{again by Hölder's inequality and Poincaré inequality}) \\ &\leq C \|\varphi\|_{S_0^1(\Omega)} (1 + \|u\|_{S_0^1(\Omega)}^{2_{\mathbb{Q}}^*-1}) < +\infty, \end{aligned} \quad (2.35)$$

where $S_{\mathbb{G}} > 0$ is the best Sobolev constant in $\mathbb{G}$, and $C > 0$ depends on Q , f , $|\Omega|$. Finally, since we are assuming that $u^{-\gamma} \in L_{\text{loc}}^1(\Omega)$, we obviously have

$$0 \leq \int_{\Omega} u^{-\gamma} \varphi \leq \|\varphi\|_{L^\infty(\Omega)} \int_{\text{supp}(\varphi)} u^{-\gamma} < +\infty.$$

We explicitly stress that this last estimate (which is related with the *singular term* $u^{-\gamma}$) is the unique estimate involving the L^∞ -norm of the test function φ ; on the contrary, estimates (2.34)-(2.35) only involve the S_0^1 -norm of φ .

2) If $u \in S_0^1(\Omega)$ is a weak solution of problem (2.31), and if $\varphi \in C_0^\infty(\Omega)$ is a non-negative test function (that is, $\varphi \geq 0$ in Ω), by (2.34)-(2.35) we have

$$\begin{aligned} 0 \leq \int_{\Omega} u^{-\gamma} \varphi &= \frac{1}{\lambda} \left(\int_{\Omega} \langle \nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}} \varphi \rangle_{\mathfrak{g}_1} + \int_{\Omega} f(x, u) \varphi \right) \\ &\leq C \|\varphi\|_{S_0^1(\Omega)} (1 + \|u\|_{S_0^1(\Omega)}^{2_{\mathbb{Q}}^*-1}); \end{aligned}$$

from this, by using a standard *density argument*, we can easily prove the following facts:

- a) $u^{-\gamma} \varphi \in L^1(\Omega)$ for every $\varphi \in S_0^1(\Omega)$;
- b) identity (2.33) actually holds for every $\varphi \in S_0^1(\Omega)$.

3) Let $u \in S_0^1(\Omega)$ be a weak solution of problem (2.31). Since, in particular, we know that $u > 0$ a.e. in Ω (and $u \equiv 0$ a.e. in $\mathbb{G} \setminus \Omega$), it is quite easy to recognize that

u is a weak supersolution of the equation

$$-\Delta_{\mathbb{G}} u = 0 \quad \text{in } \Omega,$$

in the sense of Definition 2.5. As a consequence, we are entitled to apply [33], ensuring that

for every $\mathcal{O} \Subset \Omega$ there exists $c(\mathcal{O}, u) > 0$ s.t. $u \geq c(\mathcal{O}, u) > 0$ a.e. in $\mathcal{O}$.

2.5 Singular Problem

The last preliminary result we aim to present concerns the *unperturbed*, purely singular version of problem (P_{λ}) , that is,

$$\begin{cases} -\Delta_{\mathbb{G}} u = \frac{\lambda}{u^{\gamma}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.36)$$

where $\gamma \in (0, 1)$ and $\lambda > 0$. We say that $u \in S_0^1(\Omega)$ is a weak solution of (2.36) according to Definition 2.5 with $f \equiv 0$. In this context, we have the following theorem.

Theorem 2.7 *Let $\Omega \subseteq \mathbb{G}$ be a bounded open set. Moreover, let $\gamma \in (0, 1)$ and $\lambda > 0$. Then, there exists a unique weak solution $w_{\lambda} \in S_0^1(\Omega) \cap L^{\infty}(\Omega)$ to (2.36). Moreover, w_{λ} is a global minimizer in the $S_0^1(\Omega)$ -topology of the functional*

$$J_{\lambda}(u) := \frac{1}{2} \int_{\Omega} |\nabla_{\mathbb{G}} u|^2 - \frac{\lambda}{1-\gamma} \int_{\Omega} |u|^{1-\gamma}. \quad (2.37)$$

Finally, we also have that $J_{\lambda}(w_{\lambda}) < 0$.

Proof The proof of this theorem is very similar to that of [63, Theorem 1.1] (for the solvability of (2.36)), and exploits a classical scheme of Stampacchia (for the global boundedness of w_{λ}); however, we present it here in detail for the sake of completeness.

To ease the readability, we split the proof into five steps.

STEP I. In this first step we prove that J_{λ} possesses a *global and strictly negative minimum*, which is attained by a function $w_{\lambda} \in S_0^1(\Omega) \setminus \{0\}$ such that $w_{\lambda} \geq 0$ a.e. in Ω .

To this we first observe that, since $S_0^1(\Omega)$ is a Hilbert space and since $0 < 1-\gamma < 1$, the functional J_{λ} is weakly lower semicontinuous on $S_0^1(\Omega)$ (notice that $\|\nabla_{\mathbb{G}} u\|_{L^2(\Omega)}$ is precisely the *norm* of $S_0^1(\Omega)$). Moreover, owing to the Sobolev inequality (2.12) and using Hölder's inequality, we have

$$J_{\lambda}(u) \geq \frac{\|u\|_{S_0^1(\Omega)}^2}{2} - \frac{\lambda C}{1-\gamma} \|u\|_{S_0^1(\Omega)}^{1-\gamma}$$

(for some positive constant C only depending on Q and Ω), and this proves that J_λ is *coercive* on $S_0^1(\Omega)$. Gathering these facts, we then derive that J_λ attains a *global minimum*, that is, there exists a function $w_\lambda \in S_0^1(\Omega)$ such that

$$J_\lambda(w_\lambda) = \min_{u \in S_0^1(\Omega)} J_\lambda(u) = m_\lambda.$$

We now observe that, if $\varphi \in C_0^\infty(\Omega)$ is a fixed non-vanishing function, we have

$$J_\lambda(t\varphi) = \frac{t^2}{2} \int_\Omega |\nabla_{\mathbb{G}} \varphi|^2 - \frac{\lambda t^{1-\gamma}}{1-\gamma} \int_\Omega |\varphi|^{1-\gamma} \rightarrow -\infty \quad \text{as } t \rightarrow 0^+$$

(recall that $0 < 1 - \gamma < 1$); as a consequence, we have $m_\lambda < 0$, and thus $w_\lambda \not\equiv 0$.

Finally, taking into account that $J_\lambda(|u|) = J_\lambda(u)$ for every $u \in S_0^1(\Omega)$, by possibly replacing the function w_λ with $|w_\lambda|$, we can assume that $w_\lambda \geq 0$ a.e. in Ω .

STEP II). In this second step we prove that the global, non-negative minimizer $w_\lambda \in S_0^1(\Omega)$ obtained in the previous step is actually strictly positive (a.e.) in Ω .

To this end, we arbitrarily fix a *non-negative* function $\varphi \in C_0^\infty(\Omega)$ and we notice that, since the function w_λ is a global minimizer of J_λ , we have

$$0 \leq \liminf_{t \rightarrow 0^+} \left\{ \frac{1}{t} (J_\lambda(w_\lambda + t\varphi) - J_\lambda(w_\lambda)) \right\} \leq \int_\Omega \langle \nabla_{\mathbb{G}} w_\lambda, \nabla_{\mathbb{G}} \varphi \rangle_{\mathfrak{g}_1},$$

and thus $-\Delta_{\mathbb{G}} u \geq 0$ in the weak sense on Ω ; from this, by using the Strong Maximum Principle in Proposition 2.4 (and since $w_\lambda \not\equiv 0$), we deduce that $w_\lambda > 0$ a.e. in Ω .

STEP III). In this third step, we prove that the function $w_\lambda \in S_0^1(\Omega)$ (which we have already proved to be a global and strictly positive minimizer of J_λ) is a weak solution of

$$-\Delta_{\mathbb{G}} w_\lambda = \frac{\lambda}{w_\lambda^\gamma} \quad \text{in } \Omega, \quad (2.38)$$

and hence of the whole problem (2.36).

To this end we first observe that, since w_λ is a global minimizer of J_λ , the point $t_0 = 0$ is a *global minimum* for the (smooth) map $h : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$h(t) = J_\lambda((1+t)w_\lambda) = \frac{(1+t)^2}{2} \int_\Omega |\nabla_{\mathbb{G}} w_\lambda|^2 - \frac{\lambda(1+t)^{1-\gamma}}{1-\gamma} \int_\Omega |w_\lambda|^{1-\gamma};$$

as a consequence, we have

$$0 = h'(0) = \int_\Omega |\nabla_{\mathbb{G}} w_\lambda|^2 - \lambda \int_\Omega |w_\lambda|^{1-\gamma}. \quad (2.39)$$

Moreover, again by the minimality of $J_\lambda(w_\lambda)$, we also get

$$\begin{aligned} 0 &\leq J_\lambda(w_\lambda + t\varphi) - J_\lambda(w_\lambda) \\ &\leq \int_\Omega \langle \nabla_{\mathbb{G}} w_\lambda, \nabla_{\mathbb{G}} \varphi \rangle_{\mathfrak{g}_1} + \frac{t^2}{2} \int_\Omega |\nabla_{\mathbb{G}} \varphi|^2 - \frac{\lambda}{1-\gamma} \int_\Omega [(w_\lambda + t\varphi)^{1-\gamma} - w_\lambda^{1-\gamma}] \end{aligned}$$

for every non-negative function $\varphi \in S_0^1(\Omega)$ and every $t > 0$; thus, by letting $t \rightarrow 0^+$ with the aid of the Fatou Lemma (and by the arbitrariness of φ), we obtain

$$\int_\Omega (\langle \nabla_{\mathbb{G}} w_\lambda, \nabla_{\mathbb{G}} \varphi \rangle_{\mathfrak{g}_1} - \lambda w_\lambda^{-\gamma} \varphi) \geq 0, \quad \text{for every } \varphi \in S_0^1(\Omega), \varphi \geq 0. \quad (2.40)$$

With estimates (2.39)-(2.40) at hand, we can easily complete the demonstration of this step.

Indeed, given any $\psi \in S_0^1(\Omega)$, by applying (2.40) to

$$\varphi = (w_\lambda + \varepsilon\psi)_+ \in S_0^1(\Omega)$$

(and with $\varepsilon > 0$ arbitrarily fixed), and by using (2.39), we get

$$\begin{aligned} 0 &\leq \int_\Omega (\langle \nabla_{\mathbb{G}} w_\lambda, \nabla_{\mathbb{G}} \varphi \rangle_{\mathfrak{g}_1} - \lambda w_\lambda^{-\gamma} \varphi) \\ &= \int_{\{w_\lambda + \varepsilon\psi \geq 0\}} (\langle \nabla_{\mathbb{G}} w_\lambda, \nabla_{\mathbb{G}} (w_\lambda + \varepsilon\psi) \rangle_{\mathfrak{g}_1} - \lambda w_\lambda^{-\gamma} (w_\lambda + \varepsilon\psi)) \\ &= \int_\Omega (\langle \nabla_{\mathbb{G}} w_\lambda, \nabla_{\mathbb{G}} (w_\lambda + \varepsilon\psi) \rangle_{\mathfrak{g}_1} - \lambda w_\lambda^{-\gamma} (w_\lambda + \varepsilon\psi)) \\ &\quad - \int_{\{w_\lambda + \varepsilon\psi < 0\}} (\langle \nabla_{\mathbb{G}} w_\lambda, \nabla_{\mathbb{G}} (w_\lambda + \varepsilon\psi) \rangle_{\mathfrak{g}_1} - \lambda w_\lambda^{-\gamma} (w_\lambda + \varepsilon\psi)) \\ &= \left(\int_\Omega |\nabla_{\mathbb{G}} w_\lambda|^2 - \lambda \int_\Omega |w_\lambda|^{1-\gamma} \right) + \varepsilon \int_\Omega (\langle \nabla_{\mathbb{G}} w_\lambda, \nabla_{\mathbb{G}} \psi \rangle_{\mathfrak{g}_1} - \lambda w_\lambda^{-\gamma} \psi) \\ &\quad - \int_{\{w_\lambda + \varepsilon\psi < 0\}} (\langle \nabla_{\mathbb{G}} w_\lambda, \nabla_{\mathbb{G}} (w_\lambda + \varepsilon\psi) \rangle_{\mathfrak{g}_1} - \lambda w_\lambda^{-\gamma} (w_\lambda + \varepsilon\psi)) \\ &\quad \text{(here we use (2.39))} \\ &= \varepsilon \int_\Omega (\langle \nabla_{\mathbb{G}} w_\lambda, \nabla_{\mathbb{G}} \psi \rangle_{\mathfrak{g}_1} - \lambda w_\lambda^{-\gamma} \psi) \\ &\quad - \int_{\{w_\lambda + \varepsilon\psi < 0\}} (\langle \nabla_{\mathbb{G}} w_\lambda, \nabla_{\mathbb{G}} (w_\lambda + \varepsilon\psi) \rangle_{\mathfrak{g}_1} - \lambda w_\lambda^{-\gamma} (w_\lambda + \varepsilon\psi)) \\ &\leq \varepsilon \int_\Omega (\langle \nabla_{\mathbb{G}} w_\lambda, \nabla_{\mathbb{G}} \psi \rangle_{\mathfrak{g}_1} - \lambda w_\lambda^{-\gamma} \psi) - \varepsilon \int_{\{w_\lambda + \varepsilon\psi < 0\}} \langle \nabla_{\mathbb{G}} w_\lambda, \nabla_{\mathbb{G}} \psi \rangle, \end{aligned}$$

where we have used the fact that $w_\lambda > 0$ a.e. in Ω . In view of this fact, we also recognize that the Lebesgue measure of the set $A_\varepsilon = \{w_\lambda + \varepsilon\psi < 0\}$ goes to 0 as

$\varepsilon \rightarrow 0^+$: indeed, since

$$\psi < -\frac{1}{\varepsilon} w_\lambda < 0 \quad \text{on } A_\varepsilon,$$

and since $\Omega \supseteq A_\varepsilon$ has finite measure, we have

$$\lim_{\varepsilon \rightarrow 0^+} |A_\varepsilon| = \left| \bigcap_{\varepsilon > 0} A_\varepsilon \right| = |\{w_\lambda \leq 0\}| = 0.$$

Hence, from the above estimate (and by the arbitrariness of $\varepsilon > 0$), we obtain

$$\int_{\Omega} (\langle \nabla_{\mathbb{G}} w_\lambda, \nabla_{\mathbb{G}} \psi \rangle_{\mathfrak{g}_1} - \lambda w_\lambda^{-\gamma} \psi) \geq \lim_{\varepsilon \rightarrow 0^+} \int_{\{w_\lambda + \varepsilon \psi < 0\}} \langle \nabla_{\mathbb{G}} w_\lambda, \nabla_{\mathbb{G}} \psi \rangle = 0.$$

This, together with the arbitrariness of $\psi \in S_0^1(\Omega)$, allows us to finally conclude that w_λ is a weak solution of equation (2.38), as desired.

STEP IV). In this fourth step we show that $w_\lambda \in L^\infty(\Omega)$. To this end we first observe that, since we *already know* that w_λ is a weak solution of (2.36), for every $k > 1$ we have

$$\int_{\Omega} \langle \nabla_{\mathbb{G}} w_\lambda, \nabla_{\mathbb{G}} [(w_\lambda - k)_+] \rangle_{\mathfrak{g}_1} - \lambda \int_{\Omega} w_\lambda^{-\gamma} (w_\lambda - k)_+ = 0;$$

from this, by using the Hölder inequality and the weighted Young inequality, together with the continuous embedding of $S_0^1(\Omega)$ into $L^{2^*_Q}(\Omega)$, we get

$$\begin{aligned} \int_{\Omega} |\nabla_{\mathbb{G}} [(w_\lambda - k)_+]|^2 &\leq \frac{\lambda}{k^\gamma} \int_{A_k} (w_\lambda - k)_+ \leq \frac{\lambda}{k^\gamma} |A_k|^{1-1/2^*_Q} \left(\int_{\Omega} |(w_\lambda - k)_+|^{2^*_Q} \right)^{1/2^*_Q} \\ &\leq c |A_k|^{1-1/2^*_Q} \left(\int_{\Omega} |\nabla_{\mathbb{G}} [(w_\lambda - k)_+]|^2 \right)^{1/2} \\ &\leq c(\varepsilon) |A_k|^{2(1-1/2^*_Q)} + \varepsilon \int_{\Omega} |\nabla_{\mathbb{G}} [(w_\lambda - k)_+]|^2, \end{aligned}$$

where $c(\varepsilon) > 0$ is a suitable constant depending on ε , and

$$A_k = \{w_\lambda \geq k\}.$$

From this, by choosing $\varepsilon > 0$ sufficiently small, we get

$$\int_{\Omega} |\nabla_{\mathbb{G}} [(w_\lambda - k)_+]|^2 \leq c |A_k|^{2(1-1/2^*_Q)}. \quad (2.41)$$

With estimate (2.41) at hand, we can easily complete the proof of this step (and of the whole theorem). Indeed, owing to the cited (2.41), for every $1 < k < h$ we have

$$\begin{aligned} (h-k)^2 |A_h|^{2/2_\varrho^*} \left(\int_{A_h} (w_\lambda - k)_+^{2_\varrho^*} \right)^{2/2_\varrho^*} &\leq \left(\int_{A_k} (w_\lambda - k)_+^{2_\varrho^*} \right)^{2/2_\varrho^*} \\ &\quad \text{(here we use once again (2.12))} \\ &\leq c \int_\Omega |\nabla_{\mathbb{G}}[(w_\lambda - k)_+]|^2 \leq c |A_k|^{2(1-1/2_\varrho^*)}, \end{aligned}$$

for some constant $c > 0$ independent of k, h ; as a consequence, we get

$$|A_h| \leq \frac{c}{(h-k)^{2_\varrho^*}} |A_k|^{2_\varrho^*-1} \quad \text{for all } 1 < k < h.$$

Since, obviously, $q = 2_\varrho^* - 1 > 1$, we can apply [40, Lemma B.1]: this gives

$$|A_d| = |\{w_\lambda \geq d\}| = 0$$

for some $d > 0$, and thus $w_\lambda \in L^\infty(\Omega)$, as desired.

STEP V). In this last step, we prove that w_λ is the *unique* weak solution of problem (2.36). To this end, let us suppose that $z_\lambda \in S_0^1(\Omega)$ is another weak solution of the same problem; we then choose a smooth function $\theta \in C^\infty(\mathbb{R})$ satisfying the following properties

- $\theta(t) = 0$ for $t \leq 0$ and $\theta(t) = 1$ for $t \geq 1$;
- θ is non-decreasing on $\mathbb{R}$;

and we define (for every $\varepsilon > 0$)

$$\theta_\varepsilon(t) := \theta\left(\frac{t}{\varepsilon}\right).$$

Now, since $\theta_\varepsilon \in S_0^1(\Omega)$, and since w_λ, z_λ solve (2.36), we get

$$\begin{aligned} (*) \quad &\int_\Omega \langle \nabla_{\mathbb{G}} w_\lambda, \nabla_{\mathbb{G}}(w_\lambda - z_\lambda) \rangle_{\mathbb{G}^1} \theta'_\varepsilon(w_\lambda - z_\lambda) - \lambda \int_\Omega \frac{\theta_\varepsilon(w_\lambda - z_\lambda)}{w_\lambda^\gamma} = 0; \\ (*) \quad &\int_\Omega \langle \nabla_{\mathbb{G}} z_\lambda, \nabla_{\mathbb{G}}(w_\lambda - z_\lambda) \rangle_{\mathbb{G}^1} \theta'_\varepsilon(w_\lambda - z_\lambda) - \lambda \int_\Omega \frac{\theta_\varepsilon(w_\lambda - z_\lambda)}{z_\lambda^\gamma} = 0. \end{aligned}$$

As a consequence, by subtracting the above identities, we obtain

$$\begin{aligned} 0 &\geq - \int_\Omega |\nabla(z_\lambda - w_\lambda)|_{\mathbb{G}}^2 \theta'_\varepsilon(w_\lambda - z_\lambda) \\ &= \lambda \int_\Omega \left(\frac{1}{z_\lambda^\gamma} - \frac{1}{w_\lambda^\gamma} \right) \theta_\varepsilon(w_\lambda - z_\lambda). \end{aligned} \tag{2.42}$$

From this, by letting $\varepsilon \rightarrow 0^+$ we derive that

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\Omega} \left(\frac{1}{z_{\lambda}^{\gamma}} - \frac{1}{w_{\lambda}^{\gamma}} \right) \theta_{\varepsilon}(w_{\lambda} - z_{\lambda}) = \int_{\{w_{\lambda} > z_{\lambda}\}} \left(\frac{1}{z_{\lambda}^{\gamma}} - \frac{1}{w_{\lambda}^{\gamma}} \right) \leq 0,$$

from which we derive that $|\{w_{\lambda} > z_{\lambda}\}| = 0$, that is, $w_{\lambda} \leq z_{\lambda}$ a.e. on Ω . Finally, by interchanging the role of z_{λ} and w_{λ} , we conclude that

$$w_{\lambda} \equiv z_{\lambda} \text{ a.e. in } \Omega,$$

and this proves the uniqueness of w_{λ} . $\square$

3 Existence of the First Solution

To begin with, we define

$$\Lambda := \sup\{\lambda > 0 : (P_{\lambda}) \text{ admits a weak solution}\}. \quad (3.1)$$

We then turn to prove in this first part of the section the following facts:

- a) Λ is well-defined and $\Lambda < +\infty$;
- b) problem (P_{λ}) admits a weak solution for every $0 < \lambda \leq \Lambda$.

We begin by proving assertion a).

Lemma 3.1 *Let Λ be as in (3.1). Then $\Lambda \in (0, +\infty)$.*

Proof We consider the functional

$$I_{\lambda}(u) := \frac{1}{2} \int_{\Omega} |\nabla_{\mathbb{G}} u|^2 - \frac{\lambda}{1-\gamma} \int_{\Omega} |u|^{1-\gamma} - \frac{1}{2_{\mathbb{Q}}^*} \int_{\Omega} |u|^{2_{\mathbb{Q}}^*}, \quad u \in S_0^1(\Omega). \quad (3.2)$$

First of all, by combining Hölder's and Sobolev's inequalities, we have

$$\begin{aligned} \text{a) } \int_{\Omega} |u|^{2_{\mathbb{Q}}^*} &\leq C \|\nabla_{\mathbb{G}} u\|_{L^2(\Omega)}^{2_{\mathbb{Q}}^*}; \\ \text{b) } \int_{\Omega} |u|^{1-\gamma} &\leq C \|u\|_{L^{2_{\mathbb{Q}}^*}(\Omega)}^{(1-\gamma)/2_{\mathbb{Q}}^*} \leq C \|\nabla_{\mathbb{G}} u\|_{L^2(\Omega)}^{1-\gamma}; \end{aligned} \quad (3.3)$$

as a consequence, denoting by

$$B_r := \left\{ u \in S_0^1(\Omega) : \|u\|_{S_0^1(\Omega)} \leq r \right\},$$

the above estimates (3.3-a) imply the existence of $r_0 > 0$ and $\delta_0 > 0$ such that

$$\begin{cases} \frac{1}{2} \|\nabla_{\mathbb{G}} u\|_{L^2(\Omega)}^2 - \frac{1}{2^*} \|u\|_{L^{2^*}(\Omega)}^{2^*} \geq 2\delta_0 \text{ for all } u \in \partial B_{r_0}, \\ \frac{1}{2} \|\nabla_{\mathbb{G}} u\|_{L^2(\Omega)}^2 - \frac{1}{2^*} \|u\|_{L^{2^*}(\Omega)}^{2^*} \geq 0 \text{ for all } u \in B_{r_0}, \end{cases} \quad (3.4)$$

hence, again by (3.3-b) we conclude that there exists $\lambda_\star > 0$ such that

$$I_\lambda|_{\partial B_{r_0}} \geq 2\delta_0 - \frac{\lambda C}{1-\gamma} r_0^{1-\gamma} \geq \delta_0, \quad \text{for every } \lambda \in (0, \lambda_\star]. \quad (3.5)$$

We now define $c_\star := \inf_{B_{r_0}} I_{\lambda_\star}$, and we notice that $c_\star < 0$. Indeed, for every $v \neq 0$, it holds that

$$I_{\lambda_\star}(tv) = t^2 \|\nabla_{\mathbb{G}} v\|_{L^2(\Omega)}^2 - \frac{\lambda_\star}{1-\gamma} t^{1-\gamma} \int_{\Omega} |v|^{1-\gamma} - \frac{t^{2^*}}{2^*} \int_{\Omega} |v|^{2^*},$$

which becomes negative for $t > 0$ small enough because $0 < 1-\gamma < 1$. The argument is now pretty standard. We first consider a minimizing sequence $\{u_j\}_j \subset B_{r_0}$ related to $c_\star$ and we know that there exists $u_\star$ such that, up to subsequences,

- $u_j \rightarrow u_\star$ as $j \rightarrow +\infty$ weakly in $S_0^1(\Omega)$;
- $u_j \rightarrow u_\star$ as $j \rightarrow +\infty$ strongly in $L^r(\Omega)$ for every $r \in [2, 2^*)$;
- $u_j \rightarrow u_\star$ as $j \rightarrow +\infty$ pointwise a.e. in Ω .

Moreover, since $I_\lambda(|u|) = I_\lambda(u)$ for every $\lambda > 0$, we may also assume that $u_j \geq 0$. Combining (3.5) with $c_\star < 0$, we realize that there exists a positive, and independent of j , constant $\varepsilon_0 > 0$ such that

$$\|u_j\|_{S_0^1(\Omega)} \leq r_0 - \varepsilon_0. \quad (3.6)$$

Now, by combining the algebraic inequality $(a+b)^p \leq a^p + b^p$ (holding true for all $a, b \geq 0$ and $0 < p < 1$) with the Hölder inequality, as $j \rightarrow +\infty$ we have

$$\begin{aligned} \int_{\Omega} u_j^{1-\gamma} &\leq \int_{\Omega} u_\star^{1-\gamma} + \int_{\Omega} |u_j - u_\star|^{1-\gamma} \\ &\leq \int_{\Omega} u_\star^{1-\gamma} + C \|u_j - u_\star\|_{L^2(\Omega)}^{1-\gamma} = \int_{\Omega} u_\star^{1-\gamma} + o(1), \end{aligned}$$

and similarly

$$\int_{\Omega} u_\star^{1-\gamma} \leq \int_{\Omega} u_j^{1-\gamma} + \int_{\Omega} |u_j - u_\star|^{1-\gamma} = \int_{\Omega} u_j^{1-\gamma} + o(1),$$

which in turn implies

$$\int_{\Omega} u_j^{1-\gamma} = \int_{\Omega} u_\star^{1-\gamma} + o(1), \quad \text{as } j \rightarrow +\infty. \quad (3.7)$$

By Brezis-Lieb lemma, see [12], it is now well known that

$$\|u_j\|_{L^{2^*}(\Omega)}^{2^*} = \|u_\star\|_{L^{2^*}(\Omega)}^{2^*} + \|u_j - u_\star\|_{L^{2^*}(\Omega)}^{2^*} + o(1), \quad \text{as } j \rightarrow +\infty, \quad (3.8)$$

and

$$\|\nabla_{\mathbb{G}} u_j\|_{L^2(\Omega)}^2 = \|\nabla_{\mathbb{G}} u_\star\|_{L^2(\Omega)}^2 + \|\nabla_{\mathbb{G}}(u_j - u_\star)\|_{L^2(\Omega)}^2 + o(1), \quad \text{as } j \rightarrow +\infty. \quad (3.9)$$

By combining (3.9) and (3.6), it follows that $u_\star \in B_{r_0}$ and that $u_j - u_\star \in B_{r_0}$ for big enough j , and this allows to use the second line of (3.4) on $u_j - u_\star$. Using now (3.7)–(3.9), as $j \rightarrow +\infty$, we find

$$\begin{aligned} c_\star &= I_{\lambda_\star}(u_j) + o(1) \\ &= I_{\lambda_\star}(u_\star) + \frac{1}{2} \|\nabla_{\mathbb{G}}(u_j - u_\star)\|_{L^2(\Omega)}^2 - \frac{1}{2^*} \|u_j - u_\star\|_{L^{2^*}(\Omega)}^{2^*} + o(1) \\ &\geq I_{\lambda_\star}(u_\star) + o(1) \geq c_\star + o(1), \end{aligned}$$

which proves that $u_\star \geq 0$, $u_\star \not\equiv 0$ is a local minimizer of $I_{\lambda_\star}$ in the $S_0^1(\Omega)$ -topology. From this, by arguing exactly as in the proof of Theorem 2.7 (the unique difference being the presence of the critical term, see also [34, Lemma 2.1]), we show that $u_\star$ is actually a weak solution of (P_λ) , and hence we get that

$$\Lambda \geq \lambda_\star > 0.$$

Let us now prove that $\Lambda < +\infty$. Following [34], we consider the *first Dirichlet eigenvalue* μ_1 of the operator $-\Delta_{\mathbb{G}}$ in Ω , namely

$$\mu_1 = \min \left\{ \|\nabla_{\mathbb{G}} u\|_{L^2(\Omega)}^2 : u \in S_0^1(\Omega) \text{ and } \|u\|_{L^2(\Omega)}^2 = 1 \right\},$$

and we let $e_1 \in S_0^1(\Omega)$ be the principal eigenfunction associated with μ_1 , i.e.,

- a) $\|e_1\|_{L^2(\Omega)} = 1$ and $e_1 > 0$ a.e. in Ω ;
- b) $\|\nabla_{\mathbb{G}} e_1\|_{\mathbb{G}}^2 = \mu_1$.

The proof of the existence of such a function e_1 follows by rather standard arguments.

Now, assuming that there exists a weak solution $u \in S_0^1(\Omega)$ of problem (P_λ) (for some $\lambda > 0$), and using this e_1 as a test function in identity (2.33), we get

$$\mu_1 \int_{\Omega} u e_1 = \int_{\Omega} \langle \nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}} e_1 \rangle_{\mathbb{G}} = \int_{\Omega} (\lambda u^{-\gamma} + u^{2^*-1}) e_1.$$

Setting $\overline{\Lambda}$ a constant such that

$$\overline{\Lambda} t^{-\gamma} + t^{2^*-1} > \mu_1 t, \quad \text{for every } t > 0,$$

we find that $\lambda < \overline{\Lambda}$ and then $\Lambda < \overline{\Lambda} < +\infty$, as desired. $\square$

Now we have established Lemma 3.1, we then turn to prove assertion b), namely the existence of at least one weak solution of problem (P_λ) for every $0 < \lambda \leq \Lambda$.

To begin with, we prove the following simple yet important technical lemma.

Lemma 3.2 *Let $w, u \in S_0^1(\Omega)$ be a weak subsolution [resp. weak supersolution] and a weak solution of problem (P_λ) , respectively. We assume that*

- a) $w \leq u$ [resp. $w \geq u$] a.e. in Ω ;*
- b) for every open set $\mathcal{O} \Subset \Omega$ there exists $C = C(\mathcal{O}, w) > 0$ such that*

$$w \geq C \text{ a.e. in } \mathcal{O}.$$

Then, either $w \equiv u$ or $w < u$ [resp. $w > u$] a.e. in Ω .

Proof We limit ourselves to consider only the case when w is a weak subsolution of problem (P_λ) , since the case when w is a weak supersolution is analogous.

To begin with, we arbitrarily fix a bounded open set $\mathcal{O} \Subset \Omega$ and we observe that, since w is a weak subsolution of problem (P_λ) and since u is a weak solution of the same problem, we have the following computations:

$$\begin{aligned} -\Delta_{\mathbb{G}}(u - w) &\geq \lambda(u^{-\gamma} - w^{-\gamma}) + (u^{2^*_{\mathbb{G}}-1} - w^{2^*_{\mathbb{G}}-1}) \\ &\quad (\text{since } w \leq u, \text{ see assumption a)}) \\ &\geq \lambda(u^{-\gamma} - w^{-\gamma}) \\ &\quad (\text{by the Mean Value Theorem, for some } \theta \in (0, 1)) \\ &= -\gamma\lambda(\theta u + (1 - \theta)w)^{-\gamma-1}(u - w) \\ &\geq -\gamma\lambda w^{-\gamma-1}(u - w) \\ &\quad (\text{by assumption b)}) \\ &\geq -\gamma\lambda C^{-\gamma-1}(u - w), \end{aligned}$$

in the weak sense on $\mathcal{O}$ (here, $C > 0$ is a constant depending on $\mathcal{O}$ and on w).

As a consequence of this fact, and since $w \leq u$ a.e. in Ω , we are then entitled to apply the Strong Maximum Principle in Proposition 2.4 to the function $v = u - w$ (with $c \equiv \gamma\lambda C^{-\gamma-1}$), obtaining that

$$\text{either } v \equiv 0 \text{ or } v > 0 \text{ a.e. in } \mathcal{O}.$$

Due to the arbitrariness of $\mathcal{O} \Subset \Omega$, this completes the proof. $\square$

Remark 3.3 We explicitly observe that, if $w \in S_0^1(\Omega)$ is a weak supersolution of problem (P_λ) , it follows from Remark 2.6-3) that assumption b) in Lemma 3.2 is always satisfied. Hence, if $u \in S_0^1(\Omega)$ is a weak solution of (P_λ) , we get

$$(u \leq w \text{ a.e. in } \Omega) \implies (\text{either } u \equiv w \text{ or } u < w \text{ a.e. in } \Omega).$$

We now turn to establish a crucial Perron-type lemma which extends [34, Lemma 2.2] to the case of Carnot groups.

Lemma 3.4 *Let $\underline{u}, \bar{u} \in S_0^1(\Omega)$ be a weak subsolution and a weak supersolution, respectively, of problem (P_λ) . We assume that*

- a) $\underline{u}(g) \leq \bar{u}(g)$ for a.e. $g \in \Omega$;
- b) for every open set $\mathcal{O} \Subset \Omega$ there exists $C = C(\mathcal{O}, \underline{u}) > 0$ such that

$$\underline{u} \geq C \text{ a.e. in } \mathcal{O}.$$

Then, there exists a weak solution $u \in S_0^1(\Omega)$ of (P_λ) such that

$$\underline{u}(g) \leq u(g) \leq \bar{u}(g) \text{ for a.e. } g \in \Omega.$$

Proof We adapt to our setting the proof of [34, Lemma 2.2]. We consider the set

$$M := \left\{ u \in S_0^1(\Omega) : \underline{u} \leq u \leq \bar{u} \text{ a.e. in } \Omega \right\},$$

which is closed and convex.

STEP 1: we claim that there exists a relative minimizer u_λ of I_λ on M . It is enough to show that I_λ is w.l.s.c. on M . To this aim, let $u_j \in M$ be weakly convergent to u in $S_0^1(\Omega)$. Without loss of generality, possibly passing to a subsequence, we may assume that $u_j \rightarrow u$ pointwise a.e. in Ω , so that $u \in M$. Thanks to the continuous embedding of $S_0^1(\Omega)$, we have that

$$\int_{\Omega} \bar{u}^{2^*_Q} < +\infty \quad \int_{\Omega} \bar{u}^{1-\gamma} < +\infty,$$

where in the latter we used first Hölder inequality. Hence, by dominated convergence, we have that, as $j \rightarrow +\infty$,

$$\|u_j\|_{L^{2^*_Q}(\Omega)}^{2^*_Q} \rightarrow \|u\|_{L^{2^*_Q}(\Omega)}^{2^*_Q} \quad \text{and} \quad \int_{\Omega} |u_j|^{1-\gamma} \rightarrow \int_{\Omega} |u|^{1-\gamma}.$$

Therefore,

$$\liminf_{j \rightarrow +\infty} I_\lambda(u_j) \geq I_\lambda(u),$$

as desired.

STEP 2: we prove that u_λ is a weak solution of (P_λ) . Just in this paragraph we will use more compact notation u instead of u_λ .

We take $\varphi \in S_0^1(\Omega)$ and $\varepsilon > 0$. We define the function

$$v_\varepsilon := u + \varepsilon\varphi - \varphi^\varepsilon + \varphi_\varepsilon \in M,$$

where

$$\varphi^\varepsilon := (u + \varepsilon\varphi - \bar{u})_+ \quad \varphi_\varepsilon := (u + \varepsilon\varphi - \underline{u})_-.$$

Since $u + t(v_\varepsilon - u) \in M$ for $t \in (0, 1)$, we have that

$$\begin{aligned} 0 &\leq \lim_{t \rightarrow 0^+} \frac{I_\lambda(u + t(v_\varepsilon - u)) - I_\lambda(u)}{t} \\ &= \int_{\Omega} \langle \nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}}(v_\varepsilon - u) \rangle_{\mathfrak{g}_1} - \lambda \int_{\Omega} \frac{v_\varepsilon - u}{u^\gamma} - \int_{\Omega} u^{2^*_Q-1} (v_\varepsilon - u). \end{aligned} \quad (3.10)$$

We omit the details concerning the second integral, we refer to [34] for the details.

Set now

$$E^\varepsilon := \int_{\Omega} \langle \nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}} \varphi^\varepsilon \rangle_{\mathfrak{g}_1} - \lambda \int_{\Omega} \frac{\varphi^\varepsilon}{u^\gamma} - \int_{\Omega} u^{2^*_Q-1} \varphi^\varepsilon,$$

and

$$E_\varepsilon := \int_{\Omega} \langle \nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}} \varphi_\varepsilon \rangle_{\mathfrak{g}_1} - \lambda \int_{\Omega} \frac{\varphi_\varepsilon}{u^\gamma} - \int_{\Omega} u^{2^*_Q-1} \varphi_\varepsilon.$$

With this notation at hand, we can write (3.10) as

$$\int_{\Omega} \langle \nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}} \varphi \rangle_{\mathfrak{g}_1} - \lambda \int_{\Omega} \frac{\varphi}{u^\gamma} - \int_{\Omega} u^{2^*_Q-1} \varphi \geq \frac{E^\varepsilon - E_\varepsilon}{\varepsilon}.$$

We want to show that

$$\frac{E^\varepsilon}{\varepsilon} \geq o(1) \quad \text{and} \quad \frac{E_\varepsilon}{\varepsilon} \leq o(1), \quad \text{as } \varepsilon \rightarrow 0^+.$$

We will show only the first one, being the second very similar. Firstly, we define the sets

$$\begin{aligned} \Omega^\varepsilon &:= \{g \in \Omega : u(g) + \varepsilon \varphi(g) \geq \bar{u}(g) > u(g)\}, \\ \mathcal{C}\Omega^\varepsilon &:= \{g \in \Omega : u(g) + \varepsilon \varphi(g) < \bar{u}(g)\}. \end{aligned}$$

and notice that, being u and $\bar{u}$ measurable functions,

$$|\Omega^\varepsilon| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0^+.$$

Indeed, $\bigcap_{\varepsilon>0} \Omega^\varepsilon = \emptyset$, and as $|\Omega| < +\infty$, this implies

$$\lim_{\varepsilon \rightarrow 0^+} |\Omega^\varepsilon| = \left| \bigcap_{\varepsilon>0} \Omega^\varepsilon \right| = 0.$$

Therefore, following [34], we have that

$$\begin{aligned} \frac{E^\varepsilon}{\varepsilon} &= \frac{1}{\varepsilon} \left(\int_{\Omega} \langle \nabla_{\mathbb{G}}(u - \bar{u}), \nabla_{\mathbb{G}}\varphi^\varepsilon \rangle_{\mathfrak{g}_1} + \int_{\Omega} \langle \nabla_{\mathbb{G}}\bar{u}, \nabla_{\mathbb{G}}\varphi^\varepsilon \rangle_{\mathfrak{g}_1} - \int_{\Omega} (\lambda u^{-\gamma} + u^{2^*_{\mathbb{G}}-1})\varphi^\varepsilon \right) \\ &\geq \frac{1}{\varepsilon} \int_{\Omega} \langle \nabla_{\mathbb{G}}(u - \bar{u}), \nabla_{\mathbb{G}}\varphi^\varepsilon \rangle_{\mathfrak{g}_1} + \frac{1}{\varepsilon} \int_{\Omega^\varepsilon} (\lambda \bar{u}^{-\gamma} + \bar{u}^{2^*_{\mathbb{G}}-1} - \lambda u^{-\gamma} - u^{2^*_{\mathbb{G}}-1})\varphi^\varepsilon \\ &\geq \frac{1}{\varepsilon} \int_{\Omega} \langle \nabla_{\mathbb{G}}(u - \bar{u}), \nabla_{\mathbb{G}}\varphi^\varepsilon \rangle_{\mathfrak{g}_1} - \frac{\lambda}{\varepsilon} \int_{\Omega^\varepsilon} |\bar{u}^{-\gamma} - u^{-\gamma}| |\varphi|. \end{aligned} \quad (3.11)$$

Now, owing to (2.15) and since $\varphi^\varepsilon = 0$ on $\mathcal{C}\Omega^\varepsilon$, we have that

$$\begin{aligned} \frac{1}{\varepsilon} \int_{\Omega} \langle \nabla_{\mathbb{G}}(u - \bar{u}), \nabla_{\mathbb{G}}\varphi^\varepsilon \rangle_{\mathfrak{g}_1} &= \frac{1}{\varepsilon} \int_{\Omega^\varepsilon} \langle \nabla_{\mathbb{G}}(u - \bar{u}), \nabla_{\mathbb{G}}\varphi^\varepsilon \rangle_{\mathfrak{g}_1} \\ &= \frac{1}{\varepsilon} \int_{\Omega^\varepsilon} |\nabla(u - \bar{u})|_{\mathbb{G}}^2 + \int_{\Omega^\varepsilon} \langle \nabla_{\mathbb{G}}(u - \bar{u}), \nabla_{\mathbb{G}}\varphi \rangle_{\mathfrak{g}_1} \\ &\geq \int_{\Omega^\varepsilon} \langle \nabla_{\mathbb{G}}(u - \bar{u}), \nabla_{\mathbb{G}}\varphi \rangle_{\mathfrak{g}_1} = o(1), \quad \text{as } \varepsilon \rightarrow 0^+. \end{aligned} \quad (3.12)$$

Combining (3.11) and (3.12), we finally get

$$\frac{E^\varepsilon}{\varepsilon} \geq o(1), \quad \text{as } \varepsilon \rightarrow 0^+. \quad (3.13)$$

Similarly

$$\frac{E_\varepsilon}{\varepsilon} \leq o(1), \quad \text{as } \varepsilon \rightarrow 0^+. \quad (3.14)$$

Combining (3.13) and (3.14), we get that

$$\int_{\Omega} \langle \nabla_{\mathbb{G}}u, \nabla_{\mathbb{G}}\varphi \rangle_{\mathfrak{g}_1} - \int_{\Omega} (\lambda u^{-\gamma} - u^{2^*_{\mathbb{G}}-1})\varphi \geq o(1) \quad \text{as } \varepsilon \rightarrow 0^+.$$

Taking now $-\varphi$, and passing to the limit as $\varepsilon \rightarrow 0^+$, one closes the proof. $\square$

We are now ready to prove the existence of a weak solution of (P_λ) . It is an adaptation to our setting of [34, Lemma 2.3].

Lemma 3.5 *Problem (P_λ) admits (at least) one weak solution $u_\lambda \in S_0^1(\Omega)$ for every $\lambda \in (0, \Lambda]$.*

Proof The idea of the proof is rather standard: we want to construct both a weak subsolution and a weak supersolution and then apply Lemma 3.4.

Let us start with the weak subsolution. By Theorem 2.7, we know that for every $\lambda \in (0, \Lambda)$ (and actually for all $\lambda > 0$) there exists a unique solution w_λ of (2.36), which is the Euler-Lagrange equation naturally associated with the functional J_λ defined in (2.37). The function w_λ is a weak subsolution of (P_λ) .

Let us now look for a weak supersolution. By the very definition of Λ , we know that there necessarily exists $\lambda' \in (\lambda, \Lambda)$ such that $(P_{\lambda'})$ admits a weak solution $u_{\lambda'}$, and this can be easily taken as a weak supersolution of (P_λ) .

We now claim that

$$w_\lambda(g) \leq u_{\lambda'}(g), \quad \text{for a.e. } g \in \Omega. \quad (3.15)$$

To this aim, we proceed essentially as in the proof of Theorem 2.7, STEP V). First of all, let us consider a smooth non-decreasing function $\theta : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\theta(t) = 1 \text{ for } t \geq 1 \quad \text{and} \quad \theta(t) = 0 \text{ for } t \leq 0,$$

and it is linked in a smooth way for $t \in (0, 1)$. We further define the function

$$\theta_\varepsilon(t) := \theta\left(\frac{t}{\varepsilon}\right), \quad \varepsilon > 0, \quad t \in \mathbb{R}.$$

Due to its definition, we are entitled to use the function $\theta_\varepsilon(w_\lambda - u_{\lambda'})$ as a test function in both $(P_{\lambda'})$ (solved by $u_{\lambda'}$) and (2.36) (solved by w_λ). Thus, we have

$$\int_{\Omega} \langle \nabla_{\mathbb{G}} w_\lambda, \nabla_{\mathbb{G}}(w_\lambda - u_{\lambda'}) \rangle_{\mathfrak{g}_1} \theta'_\varepsilon(w_\lambda - u_{\lambda'}) - \lambda \int_{\Omega} \frac{\theta_\varepsilon(w_\lambda - u_{\lambda'})}{w_\lambda^\gamma} = 0, \quad (3.16)$$

and

$$\begin{aligned} & \int_{\Omega} \langle \nabla_{\mathbb{G}} u_{\lambda'}, \nabla_{\mathbb{G}}(w_\lambda - u_{\lambda'}) \rangle_{\mathfrak{g}_1} \theta'_\varepsilon(w_\lambda - u_{\lambda'}) \\ & - \lambda' \int_{\Omega} \frac{\theta_\varepsilon(w_\lambda - u_{\lambda'})}{u_{\lambda'}^\gamma} - \int_{\Omega} u_{\lambda'}^{2^*_Q-1} \theta_\varepsilon(w_\lambda - u_{\lambda'}) = 0. \end{aligned} \quad (3.17)$$

Subtracting (3.16) from (3.17) we get

$$\begin{aligned} 0 & \geq - \int_{\Omega} |\nabla(u_{\lambda'} - w_\lambda)|_{\mathbb{G}}^2 \theta'_\varepsilon(w_\lambda - u_{\lambda'}) \\ & = \int_{\Omega} \left(\frac{\lambda'}{u_{\lambda'}^\gamma} - \frac{\lambda}{w_\lambda^\gamma} + u_{\lambda'}^{2^*_Q-1} \right) \theta_\varepsilon(w_\lambda - u_{\lambda'}) \\ & \geq \lambda \int_{\Omega} \left(\frac{1}{u_{\lambda'}^\gamma} - \frac{1}{w_\lambda^\gamma} \right) \theta_\varepsilon(w_\lambda - u_{\lambda'}). \end{aligned} \quad (3.18)$$

Now, letting $\varepsilon \rightarrow 0^+$ we find that

$$\int_{\{w_\lambda > u_{\lambda'}\}} \left(\frac{1}{u_{\lambda'}^\gamma} - \frac{1}{w_\lambda^\gamma} \right) \leq 0,$$

and this implies that

$$|\{g \in \Omega : w_\lambda(g) > u_{\lambda'}(g)\}| = 0,$$

as claimed in (3.15).

With (3.15) at hand, we are ready to complete the proof of the lemma: in fact, setting $\bar{u} = u_{\lambda'}$ and $\underline{u} = w_{\lambda}$, by (3.15) and Theorem 2.7 we know that

- i) $\underline{u}$ is weak subsolution and $\bar{u}$ is a weak supersolution of problem (P_{λ}) ;
- ii) $\underline{u}$ and $\bar{u}$ satisfy assumptions a)-b) in Lemma 3.4.

We can then apply Lemma 3.4, thereby proving that problem (P_{λ}) admits a weak solution u_{λ} for every $\lambda \in (0, \Lambda)$, further satisfying

$$I_{\lambda}(u_{\lambda}) = \min\{I_{\lambda}(u) : u \in S_0^1(\Omega) \text{ and } w_{\lambda} \leq u \leq u_{\lambda'}\} \leq I_{\lambda}(w_{\lambda}).$$

In particular, by Theorem 2.7 we have

$$I_{\lambda}(u_{\lambda}) \leq I_{\lambda}(w_{\lambda}) \leq J_{\lambda}(w_{\lambda}) < 0. \quad (3.19)$$

We now turn to consider the ‘limit case’ $\lambda = \Lambda$. The proof in this case is analogous to that in [34, Lemma 2.3], but we present it here for the sake of completeness.

To begin with, we choose an increasing sequence $\{\lambda_k\}_k \subseteq (0, \Lambda)$ which converges to Λ as $k \rightarrow +\infty$; accordingly, we let

$$u_k = u_{\lambda_k} \in S_0^1(\Omega)$$

be the weak solution of problem (P_{λ_k}) constructed above (via the Perron method). On account of (3.19), for every $k \geq 1$ we have

$$I_{\lambda_k}(u_k) = \frac{1}{2} \int_{\Omega} |\nabla_{\mathbb{G}} u_k|^2 - \frac{\lambda_k}{1-\gamma} \int_{\Omega} |u_k|^{1-\gamma} - \frac{1}{2_Q^*} \int_{\Omega} |u_k|^{2_Q^*} < 0. \quad (3.20)$$

Moreover, by using $\varphi = u_k$ in (2.33) (recall that u_k solves (P_{λ_k})), we get

$$\int_{\Omega} |\nabla_{\mathbb{G}} u_k|^2 - \lambda_k \int_{\Omega} u_k^{1-\gamma} - \int_{\Omega} u_k^{2_Q^*} = 0. \quad (3.21)$$

By combining (3.20)-(3.21), it is then easy to recognize that the sequence $\{u_k\}_k$ is bounded in $S_0^1(\Omega)$; as a consequence, we can find a function

$$u_{\Lambda} \in S_0^1(\Omega)$$

such that (up to a subsequence and as $k \rightarrow +\infty$)

- a) $u_k \rightarrow u_{\Lambda}$ weakly in $S_0^1(\Omega)$ and strongly in $L^p(\Omega)$ for $1 \leq p < 2_Q^*$;
- b) $u_k \rightarrow u_{\Lambda}$ a.e. in Ω .

We now observe that, since $\lambda_k \geq \lambda_1$ for every $k \geq 1$ (recall that the sequence $\{\lambda_k\}_k$ is increasing), by arguing as above we see that $u_{\lambda_k} \geq w_{\lambda_1}$, and thus

$$u_{\Lambda} > 0 \quad \text{a.e. in } \Omega.$$

Moreover, since u_k solves problem (P_{λ_k}) , we have

$$\int_{\Omega} \langle \nabla_{\mathbb{G}} u_k, \nabla_{\mathbb{G}} \varphi \rangle_{\mathfrak{g}_1} - \lambda_k \int_{\Omega} u_k^{-\gamma} \varphi - \int_{\Omega} u_k^{2_{\mathbb{Q}}^* - 1} \varphi = 0 \quad \text{for every } \varphi \in S_0^1(\Omega).$$

As a consequence, by letting $k \rightarrow +\infty$ in the above identity with the aid of the Lebesgue Dominated Convergence theorem (see, e.g., the proof of [34, Lemma 2.2] and take into account Remark 2.1) we conclude that u_{Λ} satisfies

$$\int_{\Omega} \langle \nabla_{\mathbb{G}} u_{\Lambda}, \nabla_{\mathbb{G}} \varphi \rangle_{\mathfrak{g}_1} - \Lambda \int_{\Omega} u_{\Lambda}^{-\gamma} \varphi - \int_{\Omega} u_{\Lambda}^{2_{\mathbb{Q}}^* - 1} \varphi = 0 \quad \text{for every } \varphi \in S_0^1(\Omega),$$

and this proves that u_{Λ} is a weak solution of problem (P_{Λ}) .

We explicitly point out that the convergence of the $\int_{\Omega} \langle \nabla_{\mathbb{G}} u_k, \nabla_{\mathbb{G}} \varphi \rangle_{\mathfrak{g}_1}$ to $\int_{\Omega} \langle \nabla_{\mathbb{G}} u_{\Lambda}, \nabla_{\mathbb{G}} \varphi \rangle_{\mathfrak{g}_1}$ follows from the weak convergence of u_k to u_{Λ} , since

$$v \mapsto \int_{\Omega} \langle \nabla_{\mathbb{G}} v, \nabla_{\mathbb{G}} \varphi \rangle_{\mathfrak{g}_1}$$

is a linear and continuous functional on $S_0^1(\Omega)$. This closes the proof. $\square$

Lemma 3.6 *Let $\underline{u}, \bar{u}, u_{\lambda} \in S_0^1(\Omega)$ be, respectively, the weak subsolution, the weak supersolution and the weak solution of problem (P_{λ}) obtained in Lemma 3.5, and assume that $0 < \lambda < \Lambda$. Then, u_{λ} is a local minimizer of I_{λ} in (3.2).*

Proof By contradiction, suppose that u_{λ} is not a local minimizer for I_{λ} . Then, we can construct a sequence $\{u_j\}_j \subseteq S_0^1(\Omega)$ satisfying the following properties:

- i) $u_j \rightarrow u_{\lambda}$ in $S_0^1(\Omega)$ as $j \rightarrow +\infty$;
- ii) $I_{\lambda}(u_j) < I_{\lambda}(u_{\lambda})$ for every $j \in \mathbb{N}$.

We explicitly observe that, by possibly replacing u_j with $z_j = |u_j|$, we may assume that $u_j \geq 0$ a.e. in Ω for every $j \geq 1$. In fact, since $u_j \rightarrow u_{\lambda}$ in $S_0^1(\Omega)$ and since $u_{\lambda} > 0$ almost everywhere in Ω , it is easy to recognize that

$$|u_j| \rightarrow |u_{\lambda}| = u_{\lambda} \text{ in } S_0^1(\Omega) \text{ as } j \rightarrow +\infty,$$

and this shows that property i) is still satisfied by $\{z_j\}_j$. Moreover, we have

$$I_{\lambda}(|u_j|) = I_{\lambda}(u_j) < I_{\lambda}(u_{\lambda}) \quad \text{for every } j \geq 1,$$

and this shows that also property ii) is still satisfied by the sequence $\{z_j\}_j$. Hence, from now on we tacitly understand that $\{u_j\}_j$ is a sequence of *non-negative functions* satisfying properties i)-ii) above. Accordingly, we set

$$v_j := \max\{\underline{u}, \min\{\bar{u}, u_j\}\} \in S_0^1(\Omega)$$

and we define

$$(*) \quad \bar{w}_j = (u_j - \bar{u})_+ \in S_0^1(\Omega)_+(\Omega) \text{ and } \bar{S}_j = \text{supp}(\bar{w}_j) = \{u_j \geq \bar{u}\};$$

$$(*) \quad \underline{w}_j = (u_j - \underline{u})_- \in S_0^1(\Omega)_+(\Omega) \text{ and } \underline{S}_j = \text{supp}(\underline{w}_j) = \{u_j \leq \underline{u}\}.$$

We explicitly observe that, by definition, the following identities hold:

$$\begin{aligned} \text{a) } & v_j \in M = \{u \in S_0^1(\Omega) : \underline{u} \leq u \leq \bar{u}\}; \\ \text{b) } & v_j \equiv \bar{u} \text{ on } \bar{S}_j, v_j \equiv \underline{u} \text{ on } \underline{S}_j \text{ and } v_j \equiv u_j \text{ on } \{\underline{u} < u_j < \bar{u}\}; \\ \text{c) } & u_j = \bar{u} + \bar{w}_j \text{ on } \bar{S}_j \text{ and } u_j = \underline{u} - \underline{w}_j \text{ on } \underline{S}_j. \end{aligned} \quad (3.22)$$

Following [34], we now claim that

$$\lim_{j \rightarrow +\infty} |\bar{S}_j| = \lim_{j \rightarrow +\infty} |\underline{S}_j| = 0. \quad (3.23)$$

Indeed, let $\sigma > 0$ be arbitrary and let $\delta > 0$ be such that $|\Omega \setminus \Omega_\delta| < \frac{\sigma}{2}$, where we have set $\Omega_\delta = \{g \in \Omega : d_{\mathbb{G}}(g, \partial\Omega) > \delta\} \Subset \Omega$. Since, by construction,

$$\underline{u} = w_\lambda \in S_0^1(\Omega)$$

is the unique solution of problem (2.36), we know from Theorem 2.7 that

$$u_\lambda \geq \underline{u} \geq C > 0 \text{ a.e. in } \Omega_\delta, \quad (3.24)$$

where $C = C(\delta, \underline{u}) > 0$ is a suitable constant (recall that $\underline{u} \leq u_\lambda \leq \bar{u}$).

On the other hand, since u_λ is a weak solution of problem (P_λ) , and since $\bar{u} = u_{\lambda'}$ for some $\lambda < \lambda' < \Lambda$ (see the proof of Lemma 3.5), by (3.24) we have

$$\begin{aligned} -\Delta_{\mathbb{G}}(\bar{u} - u_\lambda) &= \lambda' \bar{u}^{-\gamma} - \lambda u_\lambda^{-\gamma} + (\bar{u}^{2^*_Q-1} - u_\lambda^{2^*_Q-1}) \\ &\quad (\text{since, by construction, } \underline{u} \leq u_\lambda \leq \bar{u} \text{ and } \lambda < \lambda') \\ &\geq \lambda(\bar{u}^{-\gamma} - u_\lambda^{-\gamma}) \\ &\quad (\text{by the Mean Value Theorem, for some } \theta \in (0, 1)) \\ &= -\gamma\lambda(\theta\bar{u} + (1-\theta)u_\lambda)^{-\gamma-1}(\bar{u} - u_\lambda) \\ &\geq -\gamma\lambda\underline{u}^{-\gamma-1}(\bar{u} - u_\lambda) \\ &\quad (\text{here we use (3.24)}) \\ &\geq -\gamma\lambda C^{-\gamma-1}(\bar{u} - u_\lambda), \end{aligned}$$

in the weak sense on Ω_δ ; as a consequence, we see that $v := \bar{u} - u_\lambda \in S_0^1(\Omega)$ is a *weak supersolution* (in the sense of Definition 2.5) of equation (2.25), with

$$c(x) = \gamma\lambda C(\delta, \bar{u})^{-\gamma-1} > 0.$$

Since $v > 0$ a.e. on every ball $B \Subset \Omega$ (as $\bar{u} = u_{\lambda'}$ and $\lambda \neq \lambda'$), we can apply again Corollary 2.3, ensuring the existence of $C_1 = C_1(\delta, u_\lambda, \bar{u}) > 0$ such that

$$v = \bar{u} - u_\lambda \geq C_1 > 0 \text{ a.e. in } \Omega_\delta. \quad (3.25)$$

With (3.25) at hand, we can finally complete the proof of (3.23): in fact, recalling that $u_j \rightarrow u_\lambda$ in $S_0^1(\Omega) \hookrightarrow L^2(\Omega)$ as $j \rightarrow +\infty$, from (3.25) we obtain

$$\begin{aligned} |\bar{S}_j| &\leq |\Omega \setminus \Omega_\delta| + |\Omega_\delta \cap \bar{S}_j| < \frac{\sigma}{2} + \frac{1}{C_1^2} \int_{\Omega_\delta \cap \bar{S}_j} (\bar{u} - u_\lambda)^2 \\ &\quad (\text{since } 0 \leq \bar{u} - u_\lambda \leq u_j - u_\lambda \text{ a.e. in } \bar{S}_j) \\ &< \frac{\sigma}{2} + \frac{1}{C_1^2} \|u_j - u_\lambda\|_{L^2(\Omega)}^2 < \sigma, \end{aligned}$$

provided that j is large enough, and this proves that $|\bar{S}_j| \rightarrow 0$ as $j \rightarrow +\infty$. In a very similar fashion, one can prove that $|\underline{S}_j| \rightarrow 0$ as $j \rightarrow +\infty$.

Now we have established (3.23), we can proceed toward the end of the demonstration of the lemma. To begin with, using identities b)-c) in (3.22) we write

$$\begin{aligned} I_\lambda(u_j) &= I_\lambda(v_j) + \frac{1}{2} \left(\int_\Omega |\nabla_{\mathbb{G}} u_j|_{\mathbb{G}}^2 - \int_\Omega |\nabla_{\mathbb{G}} v_j|_{\mathbb{G}}^2 \right) \\ &\quad - \frac{\lambda}{1-\gamma} \int_\Omega (|u_j|^{1-\gamma} - |v_j|^{1-\gamma}) - \frac{1}{2_\star^*} \int_\Omega (|u_j|^{2_\star^*} - |v_j|^{2_\star^*}) \\ &= I_\lambda(v_j) + \frac{1}{2} \int_{\bar{S}_j \cup \underline{S}_j} (|\nabla_{\mathbb{G}} u_j|_{\mathbb{G}}^2 - |\nabla_{\mathbb{G}} v_j|_{\mathbb{G}}^2) \\ &\quad - \frac{\lambda}{1-\gamma} \int_{\bar{S}_j \cup \underline{S}_j} (|u_j|^{1-\gamma} - |v_j|^{1-\gamma}) - \frac{1}{2_\star^*} \int_{\bar{S}_j \cup \underline{S}_j} (|u_j|^{2_\star^*} - |v_j|^{2_\star^*}) \\ &= I_\lambda(v_j) + \mathcal{R}_j^{(1)} + \mathcal{R}_j^{(2)} = (\star), \end{aligned}$$

where we have introduced the shorthand notation

$$\begin{aligned} (*) \mathcal{R}_j^{(1)} &= \frac{1}{2} \int_{\bar{S}_j} (|\nabla_{\mathbb{G}} (\bar{u} + \bar{w}_j)|_{\mathbb{G}}^2 - |\nabla_{\mathbb{G}} \bar{u}|_{\mathbb{G}}^2) \\ &\quad - \int_{\bar{S}_j} \left\{ \frac{\lambda}{1-\gamma} (|\bar{u} + \bar{w}_j|^{1-\gamma} - |\bar{u}|^{1-\gamma}) + \frac{1}{2_\star^*} (|\bar{u} + \bar{w}_j|^{2_\star^*} - |\bar{u}|^{2_\star^*}) \right\}; \\ (*) \mathcal{R}_j^{(2)} &= \frac{1}{2} \int_{\underline{S}_j} (|\nabla_{\mathbb{G}} (\underline{u} - \underline{w}_j)|_{\mathbb{G}}^2 - |\nabla_{\mathbb{G}} \underline{u}|_{\mathbb{G}}^2) \\ &\quad - \int_{\underline{S}_j} \left\{ \frac{\lambda}{1-\gamma} (|\underline{u} - \underline{w}_j|^{1-\gamma} - |\underline{u}|^{1-\gamma}) + \frac{1}{2_\star^*} (|\underline{u} - \underline{w}_j|^{2_\star^*} - |\underline{u}|^{2_\star^*}) \right\}, \end{aligned}$$

and then we obtain

$$I_\lambda(u_j) = I_\lambda(v_j) + A_j + B_j,$$

where we have used the notation

$$\begin{aligned} (*)A_j &= \frac{1}{2} \int_{\Omega} |\nabla_{\mathbb{G}} \bar{w}_j|_{\mathbb{G}}^2 + \int_{\Omega} \langle \nabla_{\mathbb{G}} \bar{u}, \nabla_{\mathbb{G}} \bar{w}_j \rangle_{\mathfrak{g}_1} \\ &\quad - \int_{\bar{S}_j} \left\{ \frac{\lambda}{1-\gamma} (|\bar{u} + \bar{w}_j|^{1-\gamma} - |\bar{u}|^{1-\gamma}) + \frac{1}{2_Q^*} (|\bar{u} + \bar{w}_j|^{2_Q^*} - |\bar{u}|^{2_Q^*}) \right\}; \\ (*)B_j &= \frac{1}{2} \int_{\Omega} |\nabla_{\mathbb{G}} \underline{w}_j|_{\mathbb{G}}^2 + \int_{\Omega} \langle \nabla_{\mathbb{G}} \underline{u}, \nabla_{\mathbb{G}} \underline{w}_j \rangle_{\mathfrak{g}_1} \\ &\quad - \int_{\underline{S}_j} \left\{ \frac{\lambda}{1-\gamma} (|\underline{u} - \underline{w}_j|^{1-\gamma} - |\underline{u}|^{1-\gamma}) + \frac{1}{2_Q^*} (|\underline{u} - \underline{w}_j|^{2_Q^*} - |\underline{u}|^{2_Q^*}) \right\}. \end{aligned}$$

Now, since we have already recognized that $v_j \in M$ and since, by construction, we know that $I_\lambda(u_\lambda) = \inf_M I_\lambda$ (see the proof of Lemma 3.5), we get

$$I_\lambda(u_j) \geq I_\lambda(u_\lambda) + A_j + B_j. \quad (3.26)$$

On the other hand, since $\bar{u} = u_{\lambda'}$ is a *weak supersolution* of (P_λ) , we have

$$\begin{aligned} A_j &= \frac{1}{2} \int_{\Omega} |\nabla_{\mathbb{G}} \bar{w}_j|_{\mathbb{G}}^2 + \int_{\Omega} \langle \nabla_{\mathbb{G}} \bar{u}, \nabla_{\mathbb{G}} \bar{w}_j \rangle_{\mathfrak{g}_1} \\ &\quad - \int_{\bar{S}_j} \left\{ \frac{\lambda}{1-\gamma} (|\bar{u} + \bar{w}_j|^{1-\gamma} - |\bar{u}|^{1-\gamma}) + \frac{1}{2_Q^*} (|\bar{u} + \bar{w}_j|^{2_Q^*} - |\bar{u}|^{2_Q^*}) \right\} \\ &\quad (\text{by the Mean Value Theorem, for some } \theta \in (0, 1)) \\ &\geq \frac{1}{2} \int_{\Omega} |\nabla_{\mathbb{G}} \bar{w}_j|_{\mathbb{G}}^2 + \int_{\bar{S}_j} (\lambda \bar{u}^{-\gamma} + \bar{u}^{2_Q^*-1}) \bar{w}_j \, dx \\ &\quad - \int_{\bar{S}_j} \{ \lambda (\bar{u} + \theta \bar{w}_j)^{-\gamma} \bar{w}_j + (\bar{u} + \theta \bar{w}_j)^{2_Q^*-1} \bar{w}_j \} dx \\ &\geq \frac{1}{2} \int_{\Omega} |\nabla_{\mathbb{G}} \bar{w}_j|_{\mathbb{G}}^2 - \int_{\bar{S}_j} ((\bar{u} + \theta \bar{w}_j)^{2_Q^*-1} - \bar{u}^{2_Q^*-1}) \bar{w}_j \, dx \\ &\quad (\text{again by the Mean Value Theorem}) \\ &\geq \frac{1}{2} \int_{\Omega} |\nabla_{\mathbb{G}} \bar{w}_j|_{\mathbb{G}}^2 - C \int_{\bar{S}_j} (\bar{u}^{2_Q^*-2} + \bar{w}_j^{2_Q^*-2}) \bar{w}_j^2, \end{aligned}$$

where $C > 0$ is a suitable constant only depending on the dimension Q . From this, by exploiting Hölder's and Sobolev's inequalities, we obtain

$$\begin{aligned} A_j &= \frac{1}{2} \int_{\Omega} |\nabla_{\mathbb{G}} \bar{w}_j|_{\mathbb{G}}^2 - C \int_{\bar{S}_j} (\bar{u}^{2^*_{\mathbb{G}}-2} + \bar{w}_j^{2^*_{\mathbb{G}}-2}) \bar{w}_j^2 \\ &\geq \frac{1}{2} \int_{\Omega} |\nabla_{\mathbb{G}} \bar{w}_j|_{\mathbb{G}}^2 \left\{ 1 - \hat{C} \left(\int_{\bar{S}_j} \bar{u}^{2^*_{\mathbb{G}}} dx \right)^{\frac{2^*_{\mathbb{G}}-2}{2^*_{\mathbb{G}}}} - \hat{C} \left(\int_{\Omega} |\nabla_{\mathbb{G}} \bar{w}_j|_{\mathbb{G}}^2 \right)^{(2^*_{\mathbb{G}}-2)/2} \right\}, \end{aligned} \quad (3.27)$$

where $\hat{C} > 0$ is another constant depending on Q .

With (3.27) at hand, we are finally ready to complete the proof. Indeed, taking into account the above (3.23), we have

$$\lim_{j \rightarrow +\infty} \left(\int_{\bar{S}_j} \bar{u}^{2^*_{\mathbb{G}}} dx \right)^{\frac{2^*_{\mathbb{G}}-2}{2^*_{\mathbb{G}}}} = 0;$$

moreover, since $u_j \rightarrow u_{\lambda}$ in $S_0^1(\Omega)$ as $j \rightarrow +\infty$, one also gets

$$\begin{aligned} 0 &\leq \int_{\Omega} |\nabla_{\mathbb{G}} \bar{w}_j|_{\mathbb{G}}^2 = \int_{\bar{S}_j} |\nabla_{\mathbb{G}} (u_j - \bar{u})|^2 \\ &\leq 2 \|u_j - u_{\lambda}\|_{S_0^1(\Omega)}^2 + 2 \int_{\bar{S}_j} |\nabla_{\mathbb{G}} (u_{\lambda} - \bar{u})|^2 \rightarrow 0 \quad \text{as } j \rightarrow +\infty. \end{aligned}$$

Gathering these facts, we then infer the existence of some $j_0 \geq 1$ such that

$$\begin{aligned} A_j &\geq \frac{1}{2} \int_{\Omega} |\nabla_{\mathbb{G}} \bar{w}_j|_{\mathbb{G}}^2 \left\{ 1 - \hat{C} \left(\int_{\bar{S}_j} \bar{u}^{2^*_{\mathbb{G}}} dx \right)^{\frac{2^*_{\mathbb{G}}-2}{2^*_{\mathbb{G}}}} - \hat{C} \left(\int_{\Omega} |\nabla_{\mathbb{G}} \bar{w}_j|_{\mathbb{G}}^2 \right)^{(2^*_{\mathbb{G}}-2)/2} \right\} \\ &\geq 0 \quad \forall j \geq j_0. \end{aligned}$$

By arguing in a very similar way, one can prove that $B_j \geq 0$ for every $j \geq j_0$ (by possibly enlarging j_0 if needed); as a consequence, from (3.26) we get

$$I_{\lambda}(u_j) \geq I_{\lambda}(u_{\lambda}) + A_j + B_j \geq I_{\lambda}(u_{\lambda}),$$

but this is contradiction with property ii) of the sequence $\{u_j\}_j$. $\square$

4 Existence of the Second Solution

In this section we are going to find a second solution to (P_{λ}) . For this purpose, we will apply the Ekeland's variational principle similarly to what done in [34]. For any

$\lambda < \Lambda$, the set where we develop the method is

$$H_\lambda := \left\{ u \in S_0^1(\Omega) : u \geq u_\lambda \text{ a.e. in } \Omega \right\} \quad (4.1)$$

where u_λ is the (first, weak) solution to (P_λ) determined in Lemma 3.5.

We know by Lemma 3.6 that $r_0 > 0$ exists such that $r_0 < \|u_\lambda\|_{S_0^1(\Omega)}$ and $I_\lambda(u_\lambda) \leq I_\lambda(u)$ for any $u \in S_0^1(\Omega)$ such that $\|u - u_\lambda\|_{S_0^1(\Omega)} \leq r_0$. Therefore, we are in one of the following two cases

(1) for every $r \in (0, r_0)$,

$$\inf_{\|u - u_\lambda\|_{S_0^1(\Omega)} = r} I_\lambda(u) = I_\lambda(u_\lambda)$$

(2) there exists $r_1 \in (0, r_0)$ such that

$$\inf_{\|u - u_\lambda\|_{S_0^1(\Omega)} = r_1} I_\lambda(u) > I_\lambda(u_\lambda).$$

We treat the two cases separately.

4.1 First Case

We are going to prove that for any $r \in (0, r_0)$ there exists a solution v_λ of (P_λ) such that $\|v_\lambda - u_\lambda\|_{S_0^1(\Omega)} = r$, therefore in particular $v_\lambda \neq u_\lambda$.

By hypothesis, we can find a sequence $\{u_k\}_k \subset H_\lambda$ satisfying the following properties:

- $\|u_k - u_\lambda\|_{S_0^1(\Omega)} = r$ for every $k \geq 1$;
- $I_\lambda(u_k) \rightarrow I_\lambda(u_\lambda) =: c_\lambda$ as $k \rightarrow +\infty$.

We then choose $\bar{r} > 0$ so small that $r - \bar{r} \leq 0$ and $r + \bar{r} < r_0$ and, accordingly, we consider the subset of H_λ defined as follows:

$$X_\lambda = \{u \in H_\lambda : r - \bar{r} \leq \|u - u_\lambda\|_{S_0^1(\Omega)} \leq r + \bar{r}\}.$$

By construction $u_k \in X_\lambda$ for every $k \geq 1$. Since it is *closed*, this set X_λ is a *complete metric space* when endowed with the distance induced by $\|\cdot\|_{S_0^1(\Omega)}$; moreover, since I_λ is a *real-valued and continuous functional* on X_λ , and since

$$\inf_{X_\lambda} I_\lambda = I_\lambda(u_\lambda)$$

we are entitled to apply Ekeland's Variational Principle [19] to the functional I_λ on X_λ , providing us with a sequence $\{v_k\}_k \subset X_\lambda$ such that

$$\begin{aligned} \text{i) } & I_\lambda(v_k) \leq I_\lambda(u_k) \leq I_\lambda(u_\lambda) + 1/k^2, \\ \text{ii) } & \|v_k - u_k\|_{S_0^1(\Omega)} \leq 1/k, \\ \text{iii) } & I_\lambda(v_k) \leq I_\lambda(u) + 1/k \|v_k - u\|_{S_0^1(\Omega)} \quad \text{for every } u \in X. \end{aligned} \quad (4.2)$$

We now observe that, since $\{v_k\}_k \subset X_\lambda$ and since the set X_λ is *bounded* in $S_0^1(\Omega)$, there exists $v_\lambda \in S_0^1(\Omega)$ such that (as $k \rightarrow +\infty$ and up to a sub-sequence)

$$\begin{aligned} \text{i) } & v_k \rightarrow v_\lambda \text{ weakly in } S_0^1(\Omega); \\ \text{ii) } & v_k \rightarrow v_\lambda \text{ strongly in } L^p(\Omega) \text{ for every } 1 \leq p < 2_Q^*; \\ \text{iii) } & v_k \rightarrow v_\lambda \text{ pointwise a.e. in } \Omega. \end{aligned} \quad (4.3)$$

where we have also used the *compact embedding* $S_0^1(\Omega) \hookrightarrow L^2(\Omega)$ (see Section 2.2).

Lemma 4.1 *The function v_λ is a weak solution of (P_λ) .*

Proof We fix $w \in H_\lambda$ and choose $\varepsilon_0 = \varepsilon_0(w, \lambda)$ sufficiently small that $v_k + \varepsilon(w - v_k) \in X_\lambda$ for every $0 < \varepsilon < \varepsilon_0$. We point out that such an ε_0 exists (for k sufficiently big) because by the properties above $\|u_k - u_\lambda\|_{S_0^1(\Omega)} = r$ and

$$\begin{aligned} r - \frac{1}{k} & \leq \|u_k - u_\lambda\|_{S_0^1(\Omega)} - \|v_k - u_k\|_{S_0^1(\Omega)} \leq \|v_k - u_\lambda\|_{S_0^1(\Omega)} \\ & \leq \|u_k - u_\lambda\|_{S_0^1(\Omega)} + \|v_k - u_k\|_{S_0^1(\Omega)} \leq r + \frac{1}{k}. \end{aligned}$$

By setting $u = v_k + \varepsilon(w - v_k)$ in (4.2) we get

$$\frac{I_\lambda(v_k + \varepsilon(w - v_k)) - I_\lambda(v_k)}{\varepsilon} \geq -\frac{1}{k} \|w - v_k\|_{S_0^1(\Omega)}.$$

Taking the limit for $\varepsilon \rightarrow 0^+$, we get

$$\begin{aligned} -\frac{1}{k} \|w - v_k\|_{S_0^1(\Omega)} & \leq \int_{\Omega} \langle \nabla_{\mathbb{G}} v_k, \nabla_{\mathbb{G}}(w - v_k) \rangle_{\mathfrak{g}_1} - \int_{\Omega} v_k^{2_Q^*-1} (w - v_k) \\ & \quad - \lambda \lim_{\varepsilon \rightarrow 0^+} \int_{\Omega} (v_k + \theta \varepsilon(w - v_k))^{-\gamma} (w - v_k) \end{aligned}$$

with $0 < \theta < 1$. Observe that $v_k + \varepsilon(w - v_k) \geq u_\lambda$ a.e. in Ω , and $w - v_k \in S_0^1(\Omega)$, therefore

$$\int_{\Omega} |(v_k + \theta \varepsilon(w - v_k))^{-\gamma} (w - v_k)| \leq \int_{\Omega} u_\lambda^{-\gamma} |w - v_k| < \infty.$$

By the dominated convergence theorem, this implies

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\Omega} (v_k + \theta \varepsilon(w - v_k))^{-\gamma} (w - v_k) = \int_{\Omega} v_k^{-\gamma} (w - v_k),$$

and therefore we get the inequality

$$\begin{aligned} & -\frac{1}{k} \|w - v_k\|_{S_0^1(\Omega)} \\ & \leq \int_{\Omega} \langle \nabla_{\mathbb{G}} v_k, \nabla_{\mathbb{G}}(w - v_k) \rangle_{\mathfrak{g}_1} - \int_{\Omega} v_k^{2_{\mathbb{Q}}^*-1} (w - v_k) - \lambda \int_{\Omega} v_k^{-\gamma} (w - v_k) \quad \forall w \in H_{\lambda}. \end{aligned} \quad (4.4)$$

For any $\varphi \in S_0^1(\Omega)$ and any $\varepsilon > 0$ we introduce

- $\varphi_{k,\varepsilon} := v_k + \varepsilon\varphi - u_{\lambda}$;
- $\varphi_{\varepsilon} := v_{\lambda} + \varepsilon\varphi - u_{\lambda}$.

By construction $w := v_k + \varepsilon\varphi + (\varphi_{k,\varepsilon})_- \in H_{\lambda}$, then, by (4.4),

$$\begin{aligned} -\frac{1}{k} \|\varepsilon\varphi + (\varphi_{k,\varepsilon})_-\|_{S_0^1(\Omega)} & \leq \int_{\Omega} \langle \nabla_{\mathbb{G}} v_k, \nabla_{\mathbb{G}}(\varepsilon\varphi + (\varphi_{k,\varepsilon})_-) \rangle_{\mathfrak{g}_1} - \int_{\Omega} v_k^{2_{\mathbb{Q}}^*-1} (\varepsilon\varphi + (\varphi_{k,\varepsilon})_-) \\ & \quad - \lambda \int_{\Omega} v_k^{-\gamma} (\varepsilon\varphi + (\varphi_{k,\varepsilon})_-) \end{aligned} \quad (4.5)$$

We aim to pass to the limit for $k \rightarrow +\infty$ and $\varepsilon \rightarrow 0^+$. We first observe that, because of (4.3-iii),

$$(\varphi_{\varepsilon,k})_- \rightarrow (\varphi_{\varepsilon})_- \text{ pointwise a.e. in } \Omega, \text{ for } k \rightarrow +\infty.$$

Since $v_k \geq u_{\lambda} > 0$ a.e. in Ω for any k , we get the following,

- $v_k^{2_{\mathbb{Q}}^*-1} (\varphi_{k,\varepsilon})_- = v_k^{2_{\mathbb{Q}}^*-1} (u_{\lambda} - \varepsilon\varphi - v_k) \cdot \mathbf{1}_{\{u_{\lambda} - \varepsilon\varphi - v_k \geq 0\}} \leq (u_{\lambda} + \varepsilon|\varphi|)^{2_{\mathbb{Q}}^*}$
- $v_k^{-\gamma} (\varphi_{k,\varepsilon})_- \leq 2\varepsilon |\varphi| u_{\lambda}^{-\gamma}$.

By the results above and Remark 2.6-(2), we can use the Dominated Convergence Theorem, obtaining

$$\begin{aligned} \lim_{k \rightarrow +\infty} & \left(\int_{\Omega} v_k^{2_{\mathbb{Q}}^*-1} (\varepsilon\varphi + (\varphi_{k,\varepsilon})_-) + \lambda \int_{\Omega} v_k^{-\gamma} (\varepsilon\varphi + (\varphi_{k,\varepsilon})_-) \right) \\ & = \int_{\Omega} v_{\lambda}^{2_{\mathbb{Q}}^*-1} (\varepsilon\varphi + (\varphi_{\varepsilon})_-) + \lambda \int_{\Omega} v_{\lambda}^{-\gamma} (\varepsilon\varphi + (\varphi_{\varepsilon})_-) \end{aligned} \quad (4.6)$$

Focusing on the remaining term on the RHS of (4.5), with computations similar to the ones carried out in [3, Lemma 3.4], we get

$$\int_{\Omega} \langle \nabla_{\mathbb{G}} v_k, \nabla_{\mathbb{G}}(\varphi_{k,\varepsilon})_- \rangle_{\mathfrak{g}_1} \leq \int_{\Omega} \langle \nabla_{\mathbb{G}} v_{\lambda}, \nabla_{\mathbb{G}}(\varphi_{\varepsilon})_- \rangle_{\mathfrak{g}_1} + o(1) \text{ as } k \rightarrow +\infty$$

and as a consequence of the weak convergence $v_k \rightarrow v_{\lambda}$ in $S_0^1(\Omega)$, we obtain

$$\int_{\Omega} \langle \nabla_{\mathbb{G}} v_k, \nabla_{\mathbb{G}}(\varepsilon\varphi + (\varphi_{k,\varepsilon})_-) \rangle_{\mathfrak{g}_1} \leq \int_{\Omega} \langle \nabla_{\mathbb{G}} v_{\lambda}, \nabla_{\mathbb{G}}(\varepsilon\varphi + (\varphi_{\varepsilon})_-) \rangle_{\mathfrak{g}_1} + o(1) \text{ as } k \rightarrow +\infty. \quad (4.7)$$

Gathering (4.6) and (4.7), and taking into account that $\|(\varphi_{k,\varepsilon})_-\|_{S_0^1(\Omega)}$ is *uniformly bounded* with respect to k (as the same is true of v_k), we can finally pass to the limit as $k \rightarrow +\infty$ in (4.5), obtaining

$$\int_{\Omega} \langle \nabla_{\mathbb{G}} v_{\lambda}, \nabla_{\mathbb{G}} (\varepsilon \varphi + (\varphi_{\varepsilon})_-) \rangle_{\mathfrak{g}_1} \geq \int_{\Omega} v_{\lambda}^{2_{\mathbb{Q}}^*-1} (\varepsilon \varphi + (\varphi_{\varepsilon})_-) + \lambda \int_{\Omega} v_{\lambda}^{-\gamma} (\varepsilon \varphi + (\varphi_{\varepsilon})_-). \quad (4.8)$$

Exploiting the computations carried out in [34, Lemma 2.6], we get

$$\begin{aligned} & \int_{\Omega} \langle \nabla_{\mathbb{G}} v_{\lambda}, \nabla_{\mathbb{G}} \varphi \rangle_{\mathfrak{g}_1} - \lambda \int_{\Omega} v_{\lambda}^{-\gamma} \varphi - \int_{\Omega} v_{\lambda}^{2_{\mathbb{Q}}^*-1} \varphi \\ & \geq -\frac{1}{\varepsilon} \left(\int_{\Omega} \langle \nabla_{\mathbb{G}} v_{\lambda}, \nabla_{\mathbb{G}} (\varphi_{\varepsilon})_- \rangle_{\mathfrak{g}_1} - \lambda \int_{\Omega} v_{\lambda}^{-\gamma} (\varphi_{\varepsilon})_- - \int_{\Omega} v_{\lambda}^{2_{\mathbb{Q}}^*-1} (\varphi_{\varepsilon})_- \right) \\ & \quad (\text{since } u_{\lambda} \text{ is a solution of } (\mathbf{P}_{\lambda})) \\ & = -\frac{1}{\varepsilon} \left(\int_{\Omega} \langle \nabla_{\mathbb{G}} (v_{\lambda} - u_{\lambda}), \nabla_{\mathbb{G}} (\varphi_{\varepsilon})_- \rangle_{\mathfrak{g}_1} - \lambda \int_{\Omega} (v_{\lambda}^{-\gamma} - u_{\lambda}^{-\gamma}) (\varphi_{\varepsilon})_- \right. \\ & \quad \left. - \int_{\Omega} (v_{\lambda}^{2_{\mathbb{Q}}^*-1} - u_{\lambda}^{2_{\mathbb{Q}}^*-1}) (\varphi_{\varepsilon})_- \right) \\ & \quad (\text{since } v_{\lambda} \geq u_{\lambda} \text{ a.e. and } v_{\lambda} = \lim_{k \rightarrow +\infty} v_k) \\ & \geq -\frac{1}{\varepsilon} \left(\int_{\{u_{\lambda} > v_{\lambda} + \varepsilon \varphi\}} \langle \nabla_{\mathbb{G}} (v_{\lambda} - u_{\lambda}), \nabla_{\mathbb{G}} (v_{\lambda} - u_{\lambda} + \varepsilon \varphi) \rangle_{\mathfrak{g}_1} \right. \\ & \quad \left. - \lambda \int_{\{u_{\lambda} > v_{\lambda} + \varepsilon \varphi\}} (v_{\lambda}^{-\gamma} - u_{\lambda}^{-\gamma}) (v_{\lambda} - u_{\lambda} + \varepsilon \varphi) \right) \\ & \geq o(1) \quad \text{as } \varepsilon \rightarrow 0^+, \end{aligned}$$

where we used (2.15) and, as before, that

$$\bigcap_{\varepsilon > 0} \{u_{\lambda} > v_{\lambda} + \varepsilon \varphi\} = \emptyset.$$

This implies that $|\{u_{\lambda} > v_{\lambda} + \varepsilon \varphi\}| \rightarrow 0$ and therefore, by letting $\varepsilon \rightarrow 0^+$ in the inequality above, we conclude that

$$\int_{\Omega} \langle \nabla_{\mathbb{G}} v_{\lambda}, \nabla_{\mathbb{G}} \varphi \rangle_{\mathfrak{g}_1} - \lambda \int_{\Omega} v_{\lambda}^{-\gamma} \varphi - \int_{\Omega} v_{\lambda}^{2_{\mathbb{Q}}^*-1} \varphi \geq 0, \quad (4.9)$$

and as $\varphi \in S_0^1(\Omega)$ is arbitrary, this allows to conclude that v_{λ} is a weak solution of $(\mathbf{P}_{\lambda})$ as claimed. $\square$

It remains to show that v_{λ} and u_{λ} are different solutions. In order to do that we use the following lemma.

Lemma 4.2 *If v_{λ} is the solution introduced above, then $\|v_{\lambda} - u_{\lambda}\|_{S_0^1(\Omega)} = r$.*

Proof We want to prove that

$$v_k \rightarrow v_\lambda \text{ strongly in } S_0^1(\Omega) \text{ as } k \rightarrow +\infty. \quad (4.10)$$

Indeed, if this is the case, owing to the fact that $\|u_k - u_\lambda\|_{S_0^1(\Omega)} = r$ for any k , we have

$$r - \|v_k - u_k\|_{S_0^1(\Omega)} \leq \|v_k - u_\lambda\|_{S_0^1(\Omega)} \leq r + \|v_k - u_k\|_{S_0^1(\Omega)},$$

which with the strong convergence and the fact that $\|v_k - u_k\|_{S_0^1(\Omega)} \leq \frac{1}{k}$ implies

$$\|v_\lambda - u_\lambda\|_{S_0^1(\Omega)} = r.$$

We turn to prove (4.10). As $v_k \rightarrow v_\lambda$ weakly in $S_0^1(\Omega)$ as $k \rightarrow +\infty$, we proceed as in the proof of Lemma 3.1 obtaining the following analogs of (3.7)–(3.9)

$$\int_{\Omega} v_k^{1-\gamma} = \int_{\Omega} v_\lambda^{1-\gamma} + o(1) \quad (4.11)$$

$$\|v_k\|_{L^{2^*}(\Omega)}^{2^*} = \|v_\lambda\|_{L^{2^*}(\Omega)}^{2^*} + \|v_k - v_\lambda\|_{L^{2^*}(\Omega)}^{2^*} + o(1) \quad (4.12)$$

$$\|v_k\|_{S_0^1(\Omega)}^2 = \|v_\lambda\|_{S_0^1(\Omega)}^2 + \|v_k - v_\lambda\|_{S_0^1(\Omega)}^2 + o(1) \quad (4.13)$$

Also, because of (4.3)–(ii) we get

$$\lim_{k \rightarrow +\infty} \int_{\Omega} |v_k - v_\lambda|^{1-\gamma} = 0. \quad (4.14)$$

Therefore, choosing $w = v_\lambda$ in (4.4), we obtain

$$\begin{aligned} \int_{\Omega} |\nabla_{\mathbb{G}}(v_k - v_\lambda)|^2 + \lambda \int_{\Omega} v_k^{-\gamma} v_\lambda &\leq \lambda \int_{\Omega} v_k^{1-\gamma} + \int_{\Omega} v_k^{2^*-1} (v_k - v_\lambda) + o(1) \\ &\leq \lambda \int_{\Omega} v_\lambda^{1-\gamma} + \|v_k - v_\lambda\|_{L^{2^*}(\Omega)}^{2^*} + o(1). \end{aligned}$$

Observing that $0 \leq v_k^{-\gamma} v_\lambda \leq v_\lambda^{-\gamma} v_\lambda \in L^1(\Omega)$ we can use again the Dominated Convergence Theorem,

$$\int_{\Omega} v_k^{-\gamma} v_\lambda \rightarrow \int_{\Omega} v_\lambda^{1-\gamma} \text{ as } k \rightarrow +\infty. \quad (4.15)$$

To proceed further, we choose $w = 2v_k \in H_\lambda$, yielding

$$\|v_k\|_{S_0^1(\Omega)}^2 - \|v_k\|_{L^{2^*}(\Omega)}^{2^*} - \lambda \int_{\Omega} v_k^{1-\gamma} \geq -\frac{1}{k} \|v_k\|_{S_0^1(\Omega)}^2 = o(1),$$

combining the former equality with the following consequence of v_λ being a solution of (P_λ) ,

$$\|v_\lambda\|_{S_0^1(\Omega)}^2 - \|v_\lambda\|_{L^{2^*_Q}(\Omega)}^{2^*_Q} - \lambda \int_{\Omega} v_k^{1-\gamma} \geq o(1) \text{ as } k \rightarrow +\infty$$

we get

$$\|v_k - v_\lambda\|_{S_0^1(\Omega)}^2 \geq \|v_k - v_\lambda\|_{L^{2^*_Q}(\Omega)}^{2^*_Q} + o(1) \text{ as } k \rightarrow +\infty. \quad (4.16)$$

Assuming without loss of generality that $I_\lambda(u_\lambda) \leq I_\lambda(v_\lambda)$, from (4.2) and (4.11)-(4.13) we obtain

$$\begin{aligned} I_\lambda(v_k - v_\lambda) &= I_\lambda(v_k) - I_\lambda(v_\lambda) + o(1) \\ &\leq I_\lambda(u_\lambda) - I_\lambda(v_\lambda) + \frac{1}{k^2} + o(1) \\ &= o(1) \text{ as } k \rightarrow +\infty \end{aligned}$$

which, together with (4.14), gives

$$\frac{1}{2} \|v_k - v_\lambda\|_{S_0^1(\Omega)}^2 - \frac{1}{2^*_Q} \|v_k - v_\lambda\|_{L^{2^*_Q}(\Omega)}^{2^*_Q} = I_\lambda(v_k - v_\lambda) + \frac{\lambda}{1-\gamma} \int_{\Omega} |v_k - v_\lambda|^{1-\gamma} \leq o(1). \quad (4.17)$$

From (4.16) and (4.17) we finally conclude

$$\lim_{k \rightarrow +\infty} \|v_k - v_\lambda\|_{L^{2^*_Q}(\Omega)}^{2^*_Q} = \lim_{k \rightarrow +\infty} \|v_k - v_\lambda\|_{S_0^1(\Omega)}^2 = 0,$$

proving (4.10). $\square$

We showed that for any $r \in (0, r_0)$ such that $\inf I_\lambda(u) = I_\lambda(u_\lambda)$ (where the infimum is taken on the set of $u \in S_0^1(\Omega)$ such that $\|u - u_\lambda\|_{S_0^1(\Omega)} = r$) there exists a solution v_λ to (P_λ) such that $v_\lambda \neq u_\lambda$.

4.2 Second Case

We are going to prove that if there exists $r_1 \in (0, r_0)$ such that

$$\inf_{\|u - u_\lambda\|_{S_0^1(\Omega)} = r_1} I_\lambda(u) > I_\lambda(u_\lambda) \quad (4.18)$$

for some $r_1 \in (0, r_0)$, then there exists a (second) solution v_λ of the problem (P_λ) such that $0 < u_\lambda < v_\lambda$.

Consider the space of continuous curves $C([0, 1], H_\lambda)$ endowed with the max distance

$$d(\eta, \eta') = \max_{t \in [0, 1]} \|\eta(t) - \eta'(t)\|_{S_0^1(\Omega)}. \quad (4.19)$$

We recall that the space H_λ was introduced in (4.1). We define the following complete metric space,

$$\Gamma_\lambda := \left\{ \eta \in C([0, 1], H_\lambda) : \begin{array}{l} \eta(0) = u_\lambda, \\ \|\eta(1) - u_\lambda\|_{S_0^1(\Omega)} > r_1, \\ I_\lambda(\eta(1)) < I_\lambda(u_\lambda) \end{array} \right\}. \quad (4.20)$$

In order to apply Ekeland's variational principle, we need to show that $\Gamma_\lambda \neq \emptyset$ and to estimate the minimax level

$$\ell_0 := \inf_{\eta \in \Gamma_\lambda} \max_{t \in [0, 1]} I_\lambda(\eta(t)).$$

We consider the functions U_ε introduced at (2.21). For any $a \in \mathbb{G}$, we also consider the functions

$$U_{\varepsilon,a}(g) := U_\varepsilon(a^{-1} \diamond g) = \varphi(a^{-1} \diamond g) T_\varepsilon(a^{-1} \diamond g),$$

where $\{T_\varepsilon\}$ is the one-parameter family of functions defined in (2.20) starting from a fixed minimizer T of the Sobolev Inequality (2.12).

Lemma 4.3 *There exists $\varepsilon_0 > 0$, $a \in \Omega$ and $R_0 \geq 1$ such that*

$$I_\lambda(u_\lambda + RU_{\varepsilon,a}) < I_\lambda(u_\lambda) \quad \forall \varepsilon \in (0, \varepsilon_0), \quad \forall R \geq R_0 \quad (4.21)$$

$$I_\lambda(u_\lambda + tR_0U_{\varepsilon,a}) < I_\lambda(u_\lambda) + \frac{1}{Q} S_{\mathbb{G}}^{Q/2} \quad \forall t \in [0, 1], \quad \forall \varepsilon \in (0, \varepsilon_0) \quad (4.22)$$

where $S_{\mathbb{G}}$ is the best Sobolev constant introduced at (2.16).

Proof We start by breaking down the term $I_\lambda(u + tRU_{\varepsilon,a})$. From now on we will use the notation $p := 2_\lambda^* - 1 = \frac{Q+2}{Q-2}$ for the sake of a simpler writing. We have

$$\begin{aligned} I_\lambda(u + tRU_{\varepsilon,a}) &= \frac{1}{2} \int |\nabla_{\mathbb{G}} u|^2 - \frac{1}{p+1} \int u^{p+1} - \frac{\lambda}{1-\gamma} \int u^{1-\gamma} \\ &\quad + tR \left(\int \langle \nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}} U_{\varepsilon,a} \rangle_{\mathfrak{g}_1} - \int u^p U_{\varepsilon,a} - \lambda \int u^{-\gamma} U_{\varepsilon,a} \right) \quad (C) \\ &\quad - \frac{1}{p+1} \left(\int (u + tRU_{\varepsilon,a})^{p+1} - \int u^{p+1} \right) + tR \int u^p U_{\varepsilon,a} \quad (A) \end{aligned}$$

$$+ \frac{t^2 R^2}{2} \int |\nabla_{\mathbb{G}} U_{\varepsilon,a}|^2 \quad (B)$$

$$- \frac{\lambda}{1-\gamma} \left(\int (u + tRU_{\varepsilon,a})^{1-\gamma} - \int u^{1-\gamma} \right) + \lambda tR \int u^{-\gamma} U_{\varepsilon,a} \quad (D)$$

where all the integrals are over $\Omega \subset \mathbb{G}$. We treat the conclusion of the calculation and after that we pass to the breakdown of each part of the above sum. In particular, we are going to prove that, as $\varepsilon \rightarrow 0^+$, we have

$$\begin{aligned} \text{(C)} &= 0 \\ \text{(A)} &= -\frac{A t^{p+1} R^{p+1}}{p+1} - t^p R^p K \varepsilon^{\frac{Q-2}{2}} + o\left(\varepsilon^{\frac{Q-2}{2}}\right) \\ \text{(B)} &= \frac{B t^2 R^2}{2} + o\left(\varepsilon^{\frac{Q-2}{2}}\right) \\ \text{(D)} &= o\left(\varepsilon^{\frac{Q-2}{2}}\right) \end{aligned}$$

for an opportune choice of positive constants A, B, K . Again in the spirit of a simpler notation, we define $s := tR$ and $S := \left(\frac{B}{A}\right)^{\frac{1}{p-1}}$.

We follow the approach of [64] and introduce

$$f_\varepsilon(s) := \frac{B s^2}{2} - \frac{A s^{p+1}}{p+1} - s^p K \varepsilon^n.$$

We denote by $s_\varepsilon > 0$ the point where f_ε achieves its max. Observe that

$$f'_\varepsilon(s) = Bs - As^p - ps^{p-1}K\varepsilon^n \quad (4.23)$$

and also

$$BS - AS^p = 0. \quad (4.24)$$

The last equations (4.23) and (4.24) imply

$$S > s_\varepsilon > 0 \text{ and } s_\varepsilon \rightarrow S \text{ as } \varepsilon \rightarrow 0^+.$$

Let's write

$$s_\varepsilon = S(1 - t_\varepsilon).$$

We can develop for t_ε ,

$$t_\varepsilon(p-1) \left(\frac{B^p}{A}\right)^{\frac{1}{p-1}} = p \frac{B}{A} K \varepsilon^n + o(\varepsilon^n), \quad \text{as } \varepsilon \rightarrow 0^+.$$

and we obtain, as $\varepsilon \rightarrow 0^+$,

$$\begin{aligned}
 I_\lambda(u + tRU_{\varepsilon,a}) &< I_\lambda(u) + \frac{Bs_\varepsilon^2}{2} - \frac{As_\varepsilon^{p+1}}{p+1} - s_\varepsilon^p K \varepsilon^n + o(\varepsilon^n) \\
 &= I_\lambda(u) + \frac{BS^2}{2} - \frac{AS^{p+1}}{p+1} - BS t_\varepsilon + AS^p t_\varepsilon - S^p K \varepsilon^n + o(\varepsilon^n) \\
 &\quad \text{(and therefore by Eq. (4.24))} \\
 &= I_\lambda(u) + \frac{BS^2}{2} - \frac{AS^{p+1}}{p+1} - S^p K \varepsilon^n + o(\varepsilon^n) \\
 &= I_\lambda(u) + \left(\frac{1}{2} - \frac{1}{p+1} \right) \frac{B^{\frac{p+1}{p-1}}}{A^{\frac{2}{p-1}}} - S^p K \varepsilon^n + o(\varepsilon^n)
 \end{aligned}$$

which allows to conclude proving (4.22), because $\frac{1}{2} - \frac{1}{p+1} = \frac{1}{Q}$ and moreover, we will prove below the following equalities

$$A = \|T\|_{L^2_\mathcal{Q}(\mathbb{G})}^2, \quad B = \|\nabla_\mathbb{G} T\|_{L^2(\mathbb{G})}^2,$$

therefore, by definition

$$S_\mathbb{G} = \frac{B}{A^{\frac{2}{p+1}}}.$$

Breaking down part (C)

This part comes directly from the Definition 2.5 of a weak solution, by using $U_{\varepsilon,a}$ as test function. Therefore, this term is null.

Breaking down part (B)

We set the constant B to be

$$B := \|\nabla_\mathbb{G} T\|_{L^2(\mathbb{G})}^2 = \|\nabla_\mathbb{G} T_{\varepsilon,a}\|_{L^2(\mathbb{G})}^2 = S_\mathbb{G}^{Q/2},$$

then, as already stated in (2.23), we know that

$$\|\nabla_\mathbb{G} U_{\varepsilon,a}\|_{L^2(\mathbb{G})}^2 = B + O(\varepsilon^{Q-2}).$$

Therefore, if we set φ and a as to have $(a \diamond \text{supp}(\varphi)) \Subset \Omega$, then we can summarize part (B) by

$$\frac{Bt^2 R^2}{2} + t^2 R^2 O(\varepsilon^{Q-2}). \quad (4.25)$$

Breaking down part (A)

This part requires a further breaking down,

$$-\frac{1}{p+1} \left(\int_{\Omega} (u + t R U_{\varepsilon,a})^{p+1} - \int_{\Omega} u^{p+1} \right) + t R \int_{\Omega} u^p U_{\varepsilon,a}$$

$$= -\frac{t^{p+1} R^{p+1}}{p+1} \int_{\Omega} U_{\varepsilon,a}^{p+1} \quad (\text{A1})$$

$$-t^p R^p \int_{\Omega} u U_{\varepsilon,a}^p \quad (\text{A2})$$

$$+ X_{\varepsilon} + Y_{\varepsilon} \quad (\text{A3})$$

In particular, we will prove that for almost every $a \in \Omega$, the following are true,

$$(\text{A1}) = -\frac{A t^{p+1} R^{p+1}}{p+1} + O(\varepsilon^Q)$$

for a suitable positive constant A ,

$$(\text{A2}) = -t^p R^p K \varepsilon^{\frac{Q-2}{2}} + o(\varepsilon^{\frac{Q-2}{2}})$$

for a suitable positive constant K ,

$$(\text{A3}) = t^{\beta} R^{\beta} o(\varepsilon^{\frac{Q-2}{2}})$$

for some β such that $1 < \beta < \frac{Q}{Q-2}$.

We will use widely [14, Lemma 4], that we state here precisely

Lemma 4.4 *For any $x, y \in \mathbb{R}$ and $q \geq 1$ we have*

$$\left| |x+y|^q - |x|^q - |y|^q - qxy(|x|^{q-2} + |y|^{q-2}) \right| = X + Y$$

where the numbers $X, Y \in \mathbb{R}$ satisfy the following conditions:

(1) if $q \leq 3$, then $X = 0$ when $|x| < |y|$ and $Y = 0$ when $|x| > |y|$, moreover there exists a constant $C = C(q)$ such that

$$|X| \leq C |x| |y|^{q-1} \text{ if } |x| \geq |y|$$

$$|Y| \leq C |x|^{q-1} |y| \text{ if } |x| \leq |y|.$$

(2) if $q > 3$, there exists a constant $C = C(q)$ such that

$$|X + Y| \leq C \left(|x|^{q-2} y^2 + x^2 |y|^{q-2} \right)$$

We apply this lemma after imposing

$$\begin{aligned}q &= 2_Q^* = p + 1 \\x &= u \\y &= tR U_{\varepsilon,a}.\end{aligned}$$

- *Part (A1)*: we write

$$U_{\varepsilon,a}^{p+1} = T_{\varepsilon,a}^{p+1} + T_{\varepsilon,a}^{p+1} (\varphi_a^{p+1} \mathbf{1}_\Omega - 1). \quad (4.26)$$

where $T_{\varepsilon,a}(g) := T_\varepsilon(a^{-1} \diamond g)$ and $\varphi_a(g) := \varphi(a^{-1} \diamond g)$. By definition there exists a neighborhood V of $a \in \Omega$ such that $\varphi_a|_V \equiv 1$. Moreover, as a consequence of (2.18), we have that

$$T_{\varepsilon,a} = \varepsilon^{\frac{Q-2}{2}} \tilde{T}_{\varepsilon,a}$$

with $\tilde{T}_{\varepsilon,a}$ uniformly bounded on $\Omega \setminus V$ for $\varepsilon \rightarrow 0^+$. As $\varphi_a|_V \equiv 1$, the integral of the second term in (4.26) is an $O\left(\varepsilon^{\frac{Q-2}{2}(p+1)}\right) = O(\varepsilon^Q)$.

For the first term, we set the constant

$$A := \|T_{\varepsilon,a}\|_{L^{p+1}(\mathbb{G})}^{p+1} = \|T\|_{L^{2_Q^*}(\mathbb{G})}^{2_Q^*} = S_{\mathbb{G}}^{Q/2}.$$

Therefore, combining these terms, the integral of $U_{\varepsilon,a}^{p+1}$ equals

$$A + O(\varepsilon^Q) \quad \text{as } \varepsilon \rightarrow 0^+,$$

as we wanted to prove.

- *Part (A2)*: to treat this part we consider some known results about convolutions on homogeneous Lie groups, that we use over the Carnot group $\mathbb{G}$.

Consider two measurable functions $f, h: \mathbb{G} \rightarrow \mathbb{R}$, then their convolution is defined by

$$(f * h)(x) := \int_{\mathbb{G}} f(y)h(y^{-1} \diamond x) = \int_{\mathbb{G}} f(x \diamond y^{-1})h(y).$$

Consider $\psi: \mathbb{G} \rightarrow \mathbb{R}$ measurable such that $\int_{\mathbb{G}} \psi$ is finite. From this we get

$$\psi_t := \frac{1}{t^Q} \psi \circ \delta_{\frac{1}{t}}$$

where Q is the homogeneous dimension of $\mathbb{G}$ and δ_ε the associated family of dilations. Then, by [23, Proposition 1.20], we know that if $f \in L^q(\mathbb{G})$, with $1 \leq q \leq +\infty$, then

$$\left\| f * \psi_t - f \int_{\mathbb{G}} \psi \right\|_{L^q(\mathbb{G})} \rightarrow 0, \quad \text{as } t \rightarrow 0^+.$$

In particular this implies that for a.e. $g \in \mathbb{G}$, $f * \psi_t(g) \rightarrow f(g) \int_{\mathbb{G}} \psi$ as $t \rightarrow 0^+$.

We consider

$$u U_{\varepsilon,a}^p = u \varphi_a^p T_{\varepsilon,a}^p = u T_{\varepsilon,a}^p + u(\varphi_a^p - 1) T_{\varepsilon,a}^p,$$

and we integrate separately the two terms. In order to integrate the first one, we set

$$\psi(g) := T^p(g^{-1}) \quad \text{for all } g \in \mathbb{G},$$

therefore, $\psi_{\varepsilon}(g) = \varepsilon^{\frac{Q-2}{2}} T_{\varepsilon}^p(g^{-1})$, and

$$\int_{\mathbb{G}} u T_{\varepsilon,a}^p = \varepsilon^{\frac{Q-2}{2}} u * \psi_{\varepsilon}(a) \quad \text{for all } a \in \Omega.$$

We point out that we are considering u as a function defined on the whole group $\mathbb{G}$ with $u|_{\mathbb{G} \setminus \Omega} \equiv 0$. As we said above, for a.e. a the convolution converges to $u(a) \int_{\mathbb{G}} \psi$. Observe that in this case $\int_{\mathbb{G}} \psi = \int_{\mathbb{G}} T^p$ because the inversion has Jacobian identically equal to 1 in the case of Carnot groups. Therefore for a.e. $a \in \mathbb{G}$,

$$u * \psi_{\varepsilon}(a) \rightarrow u(a) \int_{\mathbb{G}} T^p =: K \quad \text{as } \varepsilon \rightarrow 0^+.$$

Regarding the second term, we observe as before that $\varphi_a|_V \equiv 1$ in an opportune neighborhood V of a , and $T_{\varepsilon,a} = \varepsilon^{\frac{Q-2}{2}} \tilde{T}_{\varepsilon,a}$ with $\tilde{T}_{\varepsilon,a}$ uniformly bounded on $\Omega \setminus V$ as $\varepsilon \rightarrow 0^+$. Therefore, for an opportune positive constant M ,

$$\int_{\Omega} u(\varphi_a^p - 1) T_{\varepsilon,a}^p \leq M \varepsilon^{\frac{Q-2}{2}p} \int_{\Omega} u = O\left(\varepsilon^{\frac{Q+2}{2}}\right).$$

We may conclude that

$$\int_{\Omega} u U_{\varepsilon,a}^p = K \varepsilon^{\frac{Q-2}{2}} + o\left(\varepsilon^{\frac{Q-2}{2}}\right)$$

for a.e. $a \in \Omega$, and for an opportune positive constant K (depending on a).

- *Part (A3)*: we follow again the idea of [14]. As suggested by Lemma 4.4, we have to consider two sub-cases. First when $q = p + 1 \leq 3$, which implies $Q \geq 6$, and secondly the case $q > 3$, that gives $Q = 4$ or $Q = 5$. We observe indeed that there exist no non-trivial Carnot groups with $Q < 4$ (trivial here means with step 1 and therefore a Euclidean space structure); moreover, we have $Q = 4$ only in the case when $\mathbb{G} = \mathbb{H}^1$ is the first Heisenberg group.

(1) CASE I): $Q \geq 6$. We have

$$\begin{aligned} |X_{\varepsilon}| &\leq C t^p R^p \int_{\{u \geq t R U_{\varepsilon,a}\}} u U_{\varepsilon,a}^p \\ |Y_{\varepsilon}| &\leq C t R \int_{\{u < t R U_{\varepsilon,a}\}} u^p U_{\varepsilon,a} \end{aligned}$$

- Consider X_ε . Let α, β real numbers such that $\alpha + \beta = p$ and $0 < \beta < \frac{Q}{Q-2}$. As a consequence $u t^p R^p U_{\varepsilon,a}^\beta \leq u^{1+\alpha} t^\beta R^\beta U_{\varepsilon,a}^\beta$ on the set $\{u \geq t R U_{\varepsilon,a}\}$. Moreover, by (2.18) we have the estimate

$$U_{\varepsilon,a}(g) \leq \frac{\varepsilon^{\frac{Q-2}{2}}}{|a^{-1} \diamond g|_{\mathbb{G}}^{Q-2}} \quad \text{for all } g \in \mathbb{G}$$

and

$$|X_\varepsilon| \leq t^\beta R^\beta \int_{\Omega} u^{1+\alpha} U_{\varepsilon,a}^\beta \leq t^\beta R^\beta \int_{\Omega} u^{1+\alpha} \frac{\varepsilon^\beta \frac{Q-2}{2}}{|a^{-1} \diamond g|_{\mathbb{G}}^{\beta(Q-2)}}.$$

By definition $1 + \alpha < p + 1 = 2_Q^*$, this implies that $u^{1+\alpha}$ is in $L^1(\Omega)$ because $u \in S_0^1(\Omega) \subseteq L^{\frac{2}{Q}}(\Omega)$. Moreover, $\frac{1}{|g|_{\mathbb{G}}^{\beta(Q-2)}}$ is an L^1 function as well, because $\beta(Q-2) < Q$, the homogeneous dimension of $\mathbb{G}$. We can conclude by observing that for almost every $a \in \mathbb{G}$,

$$\int_{\mathbb{G}} u^{1+\alpha} \frac{1}{|a^{-1} \diamond g|_{\mathbb{G}}^{Q-2}} < \infty.$$

Indeed, this is the convolution $(u^{1+\alpha} * f)(a)$ where $f(g) := \frac{1}{|g^{-1}|_{\mathbb{G}}^{Q-2}}$, and by [23, Proposition 1.19] it has finite value for a.e. $a \in \mathbb{G}$. Therefore, we conclude that $|X_\varepsilon| \leq t^\beta R^\beta O\left(\varepsilon^\beta \frac{Q-2}{2}\right)$ for any $\beta < \frac{Q}{Q-2}$, that gives

$$|X_\varepsilon| = t^\beta R^\beta o(\varepsilon^{\frac{Q-2}{2}}) \quad \text{for } \beta : 1 < \beta < \frac{Q}{Q-2} \text{ and a.e. } a \in \mathbb{G}.$$

- Consider Y_ε . Analogously to what we did for X_ε , we take $\alpha, \beta > 0$ such that $\alpha + \beta = p = 2_Q^* - 1$ and $\beta < \frac{2}{Q-2}$. Therefore, $u^p U_{\varepsilon,a} \leq u^\alpha t^{1+\beta} R^{1+\beta} U_{\varepsilon,a}^{1+\beta}$ on the set $\{u < t R U_{\varepsilon,a}\}$. Moreover, from the definition of Y_ε we get

$$|Y_\varepsilon| \leq t^{1+\beta} R^{1+\beta} \int_{\Omega} u^\alpha \frac{\varepsilon^{\frac{Q-2}{2}(1+\beta)}}{|a^{-1} \diamond g|_{\mathbb{G}}^{(Q-2)(1+\beta)}}.$$

Observe that $\alpha < p$, therefore $u^\alpha \in L^1(\Omega)$. Moreover, by definition we have the inequality $1 < 1 + \beta < \frac{Q}{Q-2}$, therefore $\frac{1}{|g|_{\mathbb{G}}^{(Q-2)(1+\beta)}}$ is a function $L^1(\Omega)$. As before we have the convergence of the integral,

$$\int_{\mathbb{G}} u^\alpha \frac{1}{|a^{-1} \diamond g|_{\mathbb{G}}^{(Q-2)(1+\beta)}}$$

for almost every $a \in \mathbb{G}$, therefore $|Y_\varepsilon| \leq t^{1+\beta} R^{1+\beta} O\left(\varepsilon^{\frac{Q-2}{2} \cdot (1+\beta)}\right)$ with the constraint $1 < 1 + \beta < \frac{Q}{Q-2}$, and as a consequence

$$|Y_\varepsilon| = t^{1+\beta} R^{1+\beta} o\left(\varepsilon^{\frac{Q-2}{2}}\right) \text{ for } \beta : 0 < \beta < \frac{2}{Q-2}, \text{ and for a.e. } a \in \mathbb{G}.$$

This concludes the case $Q \geq 5$.

(2) CASE II): $Q \in \{4, 5\}$. We continue to follow Lemma 4.4. As $p + 1 = 4 > 3$ we have

$$|X_\varepsilon + Y_\varepsilon| \leq C t^2 R^2 \int u^2 U_{\varepsilon,a}^2.$$

By the estimate (2.18), we know that

$$T_{\varepsilon,a}(g) \leq \min \left\{ \frac{1}{\varepsilon}, \frac{\varepsilon}{|a^{-1} \diamond g|_{\mathbb{G}}^2} \right\}.$$

Therefore, if we consider $\alpha, \beta > 0$ such that $\alpha + \beta = 2$, we have

$$U_{\varepsilon,a}^2 \leq \frac{1}{\varepsilon^\alpha} \frac{\varepsilon^\beta}{|a^{-1} \diamond g|_{\mathbb{G}}^{2\beta}}$$

and as a consequence

$$|X_\varepsilon + Y_\varepsilon| \leq C t^2 R^2 \varepsilon^{2-2\alpha} \int_{\mathbb{G}} \frac{u^2}{|a^{-1} \diamond g|_{\mathbb{G}}^{2\beta}}.$$

The integral is again the convolution of two functions in $L^1(\mathbb{G})$, therefore it is finite for almost every $a \in \mathbb{G}$.

We conclude that $|X_\varepsilon + Y_\varepsilon| = t^2 R^2 O(\varepsilon^{2-2\alpha})$ for any α such that $0 < \alpha < 2$ and almost every $a \in \mathbb{G}$. Therefore, by setting $\alpha < \frac{1}{2}$ we get

$$|X_\varepsilon + Y_\varepsilon| = t^2 R^2 o(\varepsilon) \quad \text{as } \varepsilon \rightarrow 0^+.$$

Part (D): we finally need to estimate

$$-\frac{\lambda}{1-\gamma} \left(\int_{\Omega} (u + t R U_{\varepsilon,a})^{1-\gamma} - \int_{\Omega} u^{1-\gamma} \right) + \lambda t R \int_{\Omega} u^{-\gamma} U_{\varepsilon,a}.$$

We consider a constant $0 < \tau < \frac{1}{4}$. We divide the domain in two parts

$$\begin{aligned} \Omega_\varepsilon^{\text{in}} &:= \{g \in \Omega : |a^{-1} \diamond g|_{\mathbb{G}} \leq \varepsilon^\tau\} \\ \Omega_\varepsilon^{\text{out}} &:= \{g \in \Omega : |a^{-1} \diamond g|_{\mathbb{G}} > \varepsilon^\tau\}. \end{aligned}$$

We start by focusing on the inner domain. Observe that $u + tRU_{\varepsilon,a} > u > 0$, therefore

$$\begin{aligned}
 & -\frac{\lambda}{1-\gamma} \left(\int_{\Omega_{\varepsilon}^{\text{in}}} (u + tRU_{\varepsilon,a})^{1-\gamma} - \int_{\Omega_{\varepsilon}^{\text{in}}} u^{1-\gamma} \right) + \lambda tR \int_{\Omega_{\varepsilon}^{\text{in}}} u^{-\gamma} U_{\varepsilon,a} \\
 & \leq \lambda tR \int_{\Omega_{\varepsilon}^{\text{in}}} u^{-\gamma} U_{\varepsilon,a} \\
 & \quad \text{(by Lemma 3.4)} \\
 & \leq CtR \int_{\Omega_{\varepsilon}^{\text{in}}} U_{\varepsilon,a} \\
 & \quad \text{(by estimate (2.18))} \\
 & \leq CtR \int_{\Omega_{\varepsilon}^{\text{in}}} \frac{\varepsilon^{\frac{Q-2}{2}}}{|a^{-1} \diamond g|_{\mathbb{G}}^{Q-2}} \\
 & \quad \text{(by the co-area formula)} \\
 & = CtR \varepsilon^{\frac{Q-2}{2}} \int_0^{\varepsilon^{\tau}} r dr = o\left(\varepsilon^{\frac{Q-2}{2}}\right).
 \end{aligned} \tag{D-in}$$

Indeed, after observing that for ε sufficiently small $\Omega_{\varepsilon}^{\text{in}} \Subset \Omega$, we can apply point b) of Lemma 3.4 and obtain that u is bounded from below by a positive constant in $\Omega_{\varepsilon}^{\text{in}}$. Moreover, as u is in $L^1(\Omega)$ we can use the co-area formula in the last step.

We treat now the outer domain. Consider the Taylor expansion of $(u + \theta tRU_{\varepsilon,a})^{1-\gamma}$ with respect to θ and evaluate it at $\theta = 1$ obtaining

$$\frac{1}{1-\gamma} (u + tRU_{\varepsilon,a})^{1-\gamma} = \frac{u^{1-\gamma}}{1-\gamma} + u^{-\gamma} tRU_{\varepsilon,a} - \gamma(u + \theta_{\varepsilon} tRU_{\varepsilon,a})^{-1-\gamma} t^2 R^2 U_{\varepsilon,a}^2$$

with $0 < \theta_{\varepsilon} < 1$ a real number.

Therefore, if $\text{supp}(\varphi_a) \Subset \Omega$, we have

$$\begin{aligned}
 & -\frac{\lambda}{1-\gamma} \left(\int_{\Omega_{\varepsilon}^{\text{out}}} (u + tRU_{\varepsilon,a})^{1-\gamma} - \int_{\Omega_{\varepsilon}^{\text{out}}} u^{1-\gamma} \right) + \lambda tR \int_{\Omega_{\varepsilon}^{\text{out}}} u^{-\gamma} U_{\varepsilon,a} \\
 & = t^2 R^2 \int_{\Omega_{\varepsilon}^{\text{out}}} \frac{\gamma \cdot U_{\varepsilon,a}^2}{(u + \theta_{\varepsilon} tRU_{\varepsilon,a})^{1+\gamma}} \\
 & \quad \text{(by estimate (2.18) and Lemma 3.4)} \\
 & \leq Ct^2 R^2 \int_{\Omega_{\varepsilon}^{\text{out}}} \frac{\varepsilon^{Q-2}}{\varepsilon^{2\tau(Q-2)}} \\
 & \leq Ct^2 R^2 |\Omega| \varepsilon^{(Q-2)(1-2\tau)} \\
 & = o\left(\varepsilon^{\frac{Q-2}{2}}\right).
 \end{aligned} \tag{D-out}$$

Here we used the definition of $\Omega_\varepsilon^{\text{out}}$ and the fact that by Lemma 3.4, $u + \theta_\varepsilon t R U_{\varepsilon,a}$ is bounded below by a positive constant on any open set $\mathcal{O} \Subset \Omega$. Therefore, we choose the cut-off function φ as to have $\text{supp}(\varphi_a) \Subset \Omega$. This concludes our proof of Lemma 4.3. $\square$

4.2.1 Existence of the Second Solution

By Lemma 4.3, we see that

$$\tilde{\eta}(t) := u_\lambda + t R_0 U_{\varepsilon,a} \in \Gamma_\lambda \quad \text{for all } \varepsilon \in (0, \varepsilon_0) \quad (4.27)$$

(by enlarging R_0 if needed), and thus $\Gamma_\lambda \neq \emptyset$, as claimed. Now we observe that, since it is non-empty, this set Γ_λ is a complete metric space endowed with the distance d introduced in (4.19).

Now, we follow the idea of [3, Lemma 3.5]. Given a continuous and locally Lipschitz functional such as $I_\lambda: H_\lambda \rightarrow \mathbb{R}$ we consider its generalized directional derivative defined in the Appendix at (A.1).

Observe that

$$I_\lambda^0(x, y) = \int_\Omega \langle \nabla_{\mathbb{G}} x, \nabla_{\mathbb{G}} y \rangle_{\mathfrak{g}_1} - \lambda \int_\Omega x^{-\gamma} y - \int_\Omega x^{2_\mathbb{Q}^*-1} y.$$

We then apply Lemma A.2. The Φ functional becomes

$$\Phi(\eta) := \max_{t \in [0,1]} I_\lambda(\eta(t)).$$

and we denote the minimax level as

$$\ell_0 := \inf_{\eta \in \Gamma_\lambda} \Phi(\eta).$$

From the application of the Ekeland's Variational Principle and the lemma, we get a sequence $\{\eta_k\}_k \in \Gamma_\lambda$ verifying

- $\Phi(\eta_k) \leq \ell_0 + \frac{1}{k}$
- $\Phi(\eta_k) \leq \Phi(\eta) + \frac{1}{k} d(\eta_k, \eta)$

and we find another sequence $\{t_k\}_k \subset [0, 1]$ such that

- $\eta_k(t_k) \in H_\lambda$
- $I_\lambda(v_k) \rightarrow \ell_0$ as $k \rightarrow +\infty$
- there exists $C > 0$ such that for every $w \in H_\lambda$, the following is verified,

$$\begin{aligned} & \int_\Omega \langle \nabla_{\mathbb{G}} v_k, \nabla_{\mathbb{G}}(w - v_k) \rangle_{\mathfrak{g}_1} - \lambda \int_\Omega v_k^{-\gamma} (w - v_k) - \int_\Omega v_k^{2_\mathbb{Q}^*-1} (w - v_k) \\ & \geq -\frac{C}{k} (1 + \|w\|_{S_0^1(\Omega)}). \end{aligned} \quad (4.28)$$

In particular, if we choose $w = 2v_k$ in (A.4), we get

$$\|v_k\|_{S_0^1(\Omega)}^2 - \|v_k\|_{L^{2_Q^*}(\Omega)}^{2_Q^*} - \lambda \int v_k^{1-\gamma} \geq -\frac{1}{k} \max(1, \|v_k\|_{S_0^1(\Omega)}).$$

Summing up this to the fact that $I_\lambda(v_k) \rightarrow \ell_0$ as $k \rightarrow +\infty$, we obtain

$$\ell_0 + o(1) \geq \left(\frac{1}{2} - \frac{1}{2_Q^*}\right) \|v_k\|_{S_0^1(\Omega)}^2 - \lambda \left(\frac{1}{1-\gamma} - \frac{1}{2_Q^*}\right) \int v_k^{1-\gamma}. \quad (4.29)$$

Since $\frac{1}{2} - \frac{1}{2_Q^*} > 0$, then if v_k is unbounded in $S_0^1(\Omega)$ we would have that (up to a subsequence) $\|v_k\|_{S_0^1(\Omega)} \rightarrow +\infty$ and this contradicts (4.29). Therefore v_k is bounded. From (A.4) we get

$$\|v_k\|_{S_0^1(\Omega)}^2 - \lambda \int_\Omega v_k^{1-\gamma} - \int_\Omega v_k^{2_Q^*} \geq -\frac{C}{k} (1 + 2\|v_k\|_{S_0^1(\Omega)}). \quad (4.30)$$

Now, we can proceed as in the proofs of Lemmas 4.1 and 4.2, to prove that v_k weakly converges (up to a subsequence) to a weak solution v_λ of (P_λ) and moreover

$$\|v_k - v_\lambda\|_{S_0^1(\Omega)}^2 - \|v_k - v_\lambda\|_{L^{2_Q^*}(\Omega)}^{2_Q^*} = o(1) \text{ as } k \rightarrow +\infty. \quad (4.31)$$

To complete this case, it remains to show that $v_\lambda \neq u_\lambda$. Observe that for any $\eta \in \Gamma_\lambda$,

$$\|\eta(0) - u_\lambda\|_{S_0^1(\Omega)} = 0 \text{ and } \|\eta(1) - u_\lambda\|_{S_0^1(\Omega)} > r_1,$$

therefore there exists $t_\eta \in [0, 1]$ such that $\|\eta(t_\eta) - u_\lambda\|_{S_0^1(\Omega)} = r_1$. Since we are assuming (4.18), we have

$$\ell_0 = \inf_{\Gamma_\lambda} \Phi(\eta) \geq \inf_{\Gamma_\lambda} I_\lambda(\eta(t_\eta)) \geq \inf_{\|u - u_\lambda\|_{S_0^1(\Omega)} = r_1} I_\lambda(u) > I_\lambda(u_\lambda)$$

where the last infimum is taken among the $u \in H_\lambda$ such that $\|u - u_\lambda\|_{S_0^1(\Omega)} = r_1$. At the same time, if we consider $\tilde{\eta}$ defined in (4.27), then by (4.22) we get,

$$\ell_0 \leq \Phi(\tilde{\eta}) = \max_{t \in [0, 1]} I_\lambda(\tilde{\eta}(t)) < I_\lambda(u_\lambda) + \frac{1}{Q} S_{\mathbb{G}}^{Q/2}.$$

Summing up,

$$I_\lambda(u_\lambda) < \ell_0 < I_\lambda(u_\lambda) + \frac{1}{Q} S_{\mathbb{G}}^{Q/2}. \quad (4.32)$$

Observe that, since $v_k \rightarrow v_\lambda$ weakly in $S_0^1(\Omega)$, then the equalities (4.11)-(4.13) hold also in this context. Therefore, considering also (4.32) and the fact that $I_\lambda(v_k) \rightarrow \ell_0$,

we get (for k sufficiently large)

$$\begin{aligned}
 & \frac{1}{2} \|v_k - v_\lambda\|_{S_0^1(\Omega)}^2 - \frac{1}{2_Q^*} \|v_k - v_\lambda\|_{L^{2_Q^*}(\Omega)}^{2_Q^*} \\
 &= \frac{1}{2} (\|v_k\|_{S_0^1(\Omega)}^2 - \|v_\lambda\|_{S_0^1(\Omega)}^2) - \frac{1}{2_Q^*} (\|v_k\|_{L^{2_Q^*}(\Omega)}^{2_Q^*}) + o(1) \\
 &= I_\lambda(v_k) - I_\lambda(u_\lambda) + o(1) \\
 &= \ell_0 - I_\lambda(u_\lambda) + o(1) \\
 &< \frac{1}{Q} S_{\mathbb{G}}^{Q/2} - \delta_0
 \end{aligned} \tag{4.33}$$

for some $\delta_0 > 0$ such that the last term is positive. From (4.31), (4.32) and (4.33), by reasoning as in [64, Proposition 3.1], we get that $v_k \rightarrow v_\lambda$ strongly in $S_0^1(\Omega)$. This, together with (4.30) and (4.29), gives

$$I_\lambda(u_\lambda) < \ell_0 = \lim_{k \rightarrow +\infty} I_\lambda(v_k) = I_\lambda(v_\lambda),$$

thus implying that $u_\lambda \neq v_\lambda$.

Gathering all the results established so far, we can finally provide the

Proof of Theorem 1.1 Let Λ be as in (3.1), that is,

$$\Lambda := \sup\{\lambda > 0 : (P_\lambda) \text{ admits a weak solution}\}.$$

Taking into account Lemma 3.1, we know that $\Lambda \in (0, +\infty)$; moreover, by combining Lemma 3.5 with the computations in Sections 4.1-4.2, we know that

- i) there exist *at least two distinct weak solutions* u_λ, v_λ of (P_λ) for $\lambda \in (0, \Lambda)$;
- ii) there exists *at least one weak solution* u_λ of (P_λ) .

Hence, assertions a)-b) in the statement of the theorem are established.

Finally, by the very definition of Λ we derive that (P_λ) *does not admit* weak solutions when $\lambda > \Lambda$; this establishes also assertion c), and the proof is complete. $\square$

Appendix A. Generalized directional derivative

In this Appendix we introduce the notion of *generalized directional derivative* and we prove a related technical lemma; both this notion and this result have been exploited in the proof of the existence of a second solution of (P_λ) (see, precisely, Section 4.2.1).

Consider the convex cone $H_\lambda \subseteq S_0^1(\Omega)$ defined at (4.1), and consider any continuous and locally Lipschitz functional $I : H_\lambda \rightarrow \mathbb{R}$. Moreover, to simplify the notation, let

$$\|u\| = \|u\|_{S_0^1(\Omega)} \quad \text{for every } u \in H_\lambda.$$

The *generalized directional derivative* at $x, v \in H_\lambda$ is defined as

$$I^0(x, v) := \limsup_{\|h\| \rightarrow 0, \rho \rightarrow 0^+} \frac{I(x + h + \rho v) - I(x + h)}{\rho}, \quad (\text{A.1})$$

Lemma A.1 *For fixed x , the map $I \mapsto I^0(x, v)$ is subadditive with respect to v .*

Proof We use the notation $I_{x,h,v}^*(\rho) := I(x + h + \rho v) - I(x + h)$. Therefore,

$$I_{x,h,v+w}^*(\rho) = I_{x,h+\rho v,w}^*(\rho) + I_{x,h,v}^*(\rho).$$

Then, the lemma follows by observing that

$$\limsup_{\|h\| \rightarrow 0, \rho \rightarrow 0^+} I_{x,h+\rho v,w}^*(\rho) = \limsup_{\|h\| \rightarrow 0, \rho \rightarrow 0^+} I_{x,h,w}^*(\rho)$$

because any h' sufficiently small can be represented as $h + \rho v$ for h and ρ sufficiently small. $\square$

We now suppose that I respects property (4.18) and we apply the Ekeland's principle to the continuous functional

$$\Phi(\eta) := \max_{t \in [0,1]} I(\eta(t)),$$

setting $\ell := \inf_{\eta \in \Gamma_\lambda} \Phi(\eta)$. As a direct consequence, there exists a sequence $\{\eta_k\}_k \subset \Gamma_\lambda$ verifying

$$\Phi(\eta_k) \leq \ell + \frac{1}{k} \quad (\text{A.2})$$

$$\Phi(\eta_k) \leq \Phi(\eta) + \frac{1}{k} d(\eta_k, \eta) \quad \forall \eta \in \Gamma_\lambda \quad (\text{A.3})$$

Lemma A.2 *For every k consider $\Lambda_k := \{t \in (0, 1) : I(\eta_k(t)) = \max_{s \in [0,1]} I(\eta_k(s))\}$, then for every k there exists $t_k \in \Lambda_k$ such that, $v_k := \eta_k(t_k)$, we have*

$$I^0\left(v_k; \frac{w - v_k}{\max(1, \|w - v_k\|)}\right) \geq -\frac{1}{k} \quad \forall w \in H_\lambda. \quad (\text{A.4})$$

Proof Observe that Λ_k is always a compact subset of $(0, 1)$. We start by considering the case when Λ_k is a single point t_k .

We argue by contradiction supposing that there exists $\omega \in H_\lambda$ such that

$$I^0\left(\eta_k(t_k); \frac{\omega - \eta_k(t_k)}{\max(1, \|\omega - \eta_k(t_k)\|)}\right) < -\frac{1}{k}. \quad (\text{A.5})$$

Consider a continuous cut-off function $g: [0, 1] \rightarrow [0, 1]$ such that $g \equiv 1$ in a neighborhood of t_k and $g(0) = g(1) = 0$. We also define $m_k(t) := \max(1, \|\omega - \eta_k(t)\|)$ and we introduce

$$\eta_{k,\varepsilon}(t) := \eta_k(t) + \frac{g(t)\varepsilon}{m_k(t)}(\omega - \eta_k(t)).$$

By construction $\eta_{k,\varepsilon}(t) \in \Gamma_\lambda$ for every $t \in [0, 1]$. Therefore by (A.3) we obtain

$$\max_{t \in [0, 1]} I(\eta_k(t)) \leq \max_{t \in [0, 1]} I(\eta_{k,\varepsilon}(t)) + \frac{\varepsilon}{k} \max_{t \in [0, 1]} \frac{g(t)\|\omega - \eta_k(t)\|}{m_k(t)}. \quad (\text{A.6})$$

If $t_{k,\varepsilon}$ is a point where $I(\eta_{k,\varepsilon})$ reaches its max, then there exists $\varepsilon_n \rightarrow 0$ sequence such that $t_{k,\varepsilon_n} \rightarrow t_k$. Therefore $g(t_{k,\varepsilon_n}) = 1$ for n sufficiently big. We denote by $v_{k,n} := \eta_k(t_{k,\varepsilon_n})$ and $m_{k,n} := \max(1, \|\omega - v_{k,n}\|)$ (observe that this is a scalar value), then

$$I(v_{k,n}) \leq I(v_k) \leq I(\eta_{k,\varepsilon_n}(t_{k,\varepsilon_n})) + \frac{\varepsilon_n}{k} = I\left(v_{k,n} + \frac{\varepsilon_n}{m_{k,n}}(\omega - v_{k,n})\right) + \frac{\varepsilon_n}{k}. \quad (\text{A.7})$$

where the first inequality comes from the fact that $v_k = \eta_k(t_k)$ is the max for the I value for any $\eta_k(t)$, and the second inequality comes from (A.6).

We set $m_k^* := \max(1, \|\omega - v_k\|)$, then by the continuity of η_k , we get $v_{k,n} \rightarrow v_k$ in $S_0^1(\Omega)$ and $m_{k,n} \rightarrow m_k^*$ as $n \rightarrow \infty$. Therefore, we can re-write (A.7) obtaining

$$\frac{1}{\varepsilon_n} \left(I\left(v_k + h_n^* + \varepsilon_n \frac{\omega - v_k}{m_k^*}\right) - I(v_k + h_n) \right) \geq -\frac{1}{k}$$

with

$$\begin{aligned} h_n &= v_{k,n} - v_k \\ h_n^* &= v_{k,n} - v_k + \frac{\varepsilon_n}{m_{k,n}}(\omega - v_{k,n}) - \frac{\varepsilon_n}{m_k^*}(\omega - v_k). \end{aligned}$$

By what we proved, we derive that $h_n^* - h_n = o(\varepsilon_n)$ as $n \rightarrow \infty$ and this in turn implies

$$I^0\left(v_k, \frac{\omega - v_k}{m_k^*}\right) \geq -\frac{1}{k},$$

thus contradicting the assumption (A.5).

If Λ_k has more than one element, we have to correct the above proof. For any $t_k \in \Lambda_k$, consider the $\omega \in H_\lambda$ that satisfies (A.5) for t_k .

Claim. For any $t_k \in \Lambda_k$ there exists a neighborhood J of t_k such that

$$I^0\left(\eta_k(s), \frac{\omega - \eta_k(s)}{\max(1, \|\omega - \eta_k(s)\|)}\right) < -\frac{1}{k} \quad \forall s \in J. \quad (\text{A.8})$$

Let's suppose that the claim above is correct and conclude. By compactness of Λ_k , there exist $t_k^{(1)}, \dots, t_k^{(r)}$ elements of Λ_k and their associated neighborhoods $J_1, \dots, J_r$, such that (A.5) is verified on J_i by some $\omega_i \in H_\lambda$ and $\Lambda_k \subset \cup_i J_i \subset [0, 1]$. For an opportune partition of the unit $g_1, \dots, g_r$ associated to the J_i s, we can update ω to a continuous function $\omega: J := \cup_i J_i \rightarrow H_\lambda$,

$$\omega(s) := \sum_{i=1}^r g_i(s) \omega_i.$$

As a consequence of the subadditivity proved in Lemma A.1, we have

$$I^0(\eta_k(s), \omega(s) - \eta_k(s)) \leq -\frac{1}{k} \sum_{i=1}^r g_i(s) \max(1, \|\omega_i - \eta_k(s)\|).$$

Moreover,

$$\sum_{i=1}^r g_i(s) \|\omega_i - \eta_k(s)\| \geq \left\| \sum_{i=1}^r g_i(s) (\omega_i - \eta_k(s)) \right\| = \|\omega(s) - \eta_k(s)\|,$$

and therefore we get

$$I^0\left(\eta_k(s), \frac{\omega(s) - \eta_k(s)}{\max(1, \|\omega(s) - \eta_k(s)\|)}\right) < -\frac{1}{k} \quad \forall s \in J. \quad (\text{A.9})$$

From here the proof can continue as before if we replace every appearance of ω with $\omega(t)$ (with the opportune value of t) and if we take the cut-off function g to be 1 on Λ_k and 0 outside J .

In this case $t_{k, \varepsilon_n} \rightarrow t_k$ for some $t_k \in \Lambda_k$, $v_k = \eta_k(t_k)$ and therefore we derive (A.6) with $\omega(t_{k, \varepsilon_n})$ in place of ω . We also have to update the definition of h_n^* using $\omega(t_{k, \varepsilon_n})$ when necessary.

It remains to prove the claim above. Consider the function

$$\mathbf{w}(s) := \frac{\omega - \eta_k(s)}{\max(1, \|\omega - \eta_k(s)\|)}$$

which by definition is continuous. By (A.5), there exists $\tilde{\varepsilon}, \tilde{\delta}, \tilde{\rho}$ all positive such that

$$\sup_{\|h\| < \tilde{\delta}, \rho < \tilde{\rho}} \frac{I(\eta_k(t_k) + h + \rho \mathbf{w}(t_k)) - I(\eta_k(t_k) + h)}{\rho} < -\frac{1}{k} - \tilde{\varepsilon}. \quad (\text{A.10})$$

We give a first definition of J by imposing $\|\eta_k(s) - \eta_k(t_k)\| < \frac{\tilde{\delta}}{2}$. Then, for any $h \in H_\lambda$ with $\|h\| < \frac{\tilde{\delta}}{2}$ we have

$$\eta_k(s) + h = \eta_k(t_k) + h', \quad \text{with } \|h'\| < \tilde{\delta}.$$

Therefore we have,

$$\begin{aligned}
 I^0(\eta_k(s), \mathbf{w}(s)) &\leq \sup_{\|h\| < \frac{\delta}{2}, \rho < \tilde{\rho}} \frac{I(\eta_k(s) + h + \rho \mathbf{w}(s)) - I(\eta_k + h)}{\rho} \\
 &\leq \sup_{\|h'\| < \tilde{\delta}, \rho < \tilde{\rho}} \frac{I(\eta_k(t_k) + h' + \rho \mathbf{w}(s)) - I(\eta_k(t_k) + h')}{\rho} \\
 &= \sup_{\|h'\| < \tilde{\delta}, \rho < \tilde{\rho}} \left(\frac{I(\eta_k(t_k) + h' + \rho \mathbf{w}(s)) - I(\eta_k(t_k) + h' + \rho \mathbf{w}(t_k))}{\rho} \right. \\
 &\quad \left. + \frac{I(\eta_k(t_k) + h' + \rho \mathbf{w}(t_k)) - I(\eta_k(t_k) + h')}{\rho} \right)
 \end{aligned}$$

Observe that the first term (in the last sum) can be made arbitrarily small (up to stretching J) by using that I is locally Lipschitz, while the second term respects (A.10). Therefore the claim is proved and this concludes our proof. $\square$

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