Supplemental Material

Here we provide additional details on the results stated in the main text.

CONTENTS

Appendix A: Preliminaries

In this SM, we employ the Choi-Jamiolkowski isomorphism [1] to write a state $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$ in a 2-replica space as

$$|\rho(t)\otimes\rho(t)\rangle = \mathbb{1}\otimes\rho(t)\otimes\mathbb{1}\otimes\rho(t)|\mathcal{I}\rangle\,, \tag{SA.1}$$

where $|\mathcal{I}\rangle = |I^+\rangle^{\otimes N} \in \mathcal{H}^{\otimes 4} \simeq (\mathbb{C}^q)^{\otimes N}$ and $|I^+\rangle = \sum_{a,b=0}^{q-1}|aabb\rangle$ is the maximally entangled state in the 2-replica space. We will also make use of the following states:

$$|I^-\rangle = \sum_{a,b=0}^{q-1}|abba\rangle\,, \quad |I^x\rangle = \sum_{a,b=0}^{q-1}|abab\rangle\,, \quad |I^0\rangle = \sum_{a=0}^{q-1}|aaaa\rangle\,. \tag{SA.2}$$

To obtain the continuous-time evolution of the state (SA.1), we follow a simplified version of the model in [59]. Namely, at every discrete time step t, we apply a two-body gate $U_{j,k}$, $1\le j<k\le N$, with a certain probability p, so that at time $t+\Delta t$ the state is:

$$|\rho(t+\Delta t)\otimes\rho(t+\Delta t)\rangle = \mathbb{1}\otimes U(t+\Delta t)\rho(t)U^\dagger(t+\Delta t)\otimes\mathbb{1}\otimes U(t+\Delta t)\rho(t)U^\dagger(t+\Delta t)\,|I^+\rangle\,, \tag{SA.3}$$

where $U(t+\Delta t) = U_{j,k}U(t)$ with probability $p=\lambda N\Delta t$, $\lambda>0$, and $U(t+\Delta t)=U(t)$ with probability $1-p$. Then we average over the sites j,k and over the ensemble $\mathcal{E}$ from which the operators $U_{j,k}$ are drawn. After taking the limit $\Delta t\to 0$, this yields a Lindblad equation for the state $\mathbb{E}[|\rho(t)\otimes\rho(t)\rangle]$:

$$\frac{d}{dt}\mathbb{E}\left[|\rho(t)\otimes\rho(t)\rangle\right] = -\mathcal{L}\mathbb{E}\left[|\rho(t)\otimes\rho(t)\rangle\right], \tag{SA.4}$$

with

$$\mathcal{L} = \frac{2\lambda}{N-1}\sum_{1\le j<k\le N}(1-\mathcal{U}_{j,k}), \quad \mathcal{U}_{j,k} = \mathbb{E}\left[U_{j,k}^*\otimes U_{j,k}\otimes U_{j,k}^*\otimes U_{j,k}\right]. \tag{SA.5}$$

The ensemble-averaged purity $\mathrm{Tr}_A\mathbb{E}[\rho_A^2(t)]$ of the state $|\rho(t)\otimes\rho(t)\rangle$ in a bipartition of the total system, $\mathcal{S}=A\cup\bar{A}$. is obtained by introducing a swap operator X_A which exchanges the two copies of the subsystem A. In the Choi-Jamiolkowki representation, the swap operator reads:

$$|X_A\rangle = \bigotimes_{i\in A} |I^-\rangle_i \bigotimes_{j\in\bar{A}} |I^+\rangle_j \,, \tag{SA.6}$$

and the averaged purity is given by the overlap

$$\mathrm{Tr}_A\mathbb{E}[\rho_A^2(t)] = \mathrm{Tr}\{X_A\mathbb{E}[\rho(t)\otimes\rho(t)]\} = \langle X_A|\mathbb{E}[\rho(t)\otimes\rho(t)]\rangle \,. \tag{SA.7}$$

We will consider two ensembles: random permutation circuits, $\mathcal{E}_{\mathrm{PC}}$, and circuits made of random permutation and random phase gates, $\mathcal{E}_{\mathrm{PPC}}$. When the matrices $U_{i,j}$ are drawn from these two ensembles, the averaged two-body gates $\mathcal{U}_{i,j}$ are operators acting on (tensor products of) replica Hilbert spaces of small local dimension d_{eff}. Namely, in the case of $\mathcal{E}_{\mathrm{PPC}}$ the space $\mathcal{H}_{\mathrm{eff}}$ is spanned by tensor products of the states $|I^+\rangle_i\,, |I^-\rangle_i\,, |I^0\rangle_i$, while for the ensemble $\mathcal{E}_{\mathrm{PC}}$, when the local dimension is $q=2$ and the initial state is a random product state, $\mathcal{H}_{\mathrm{eff}}$ is spanned by tensor products of $|I^+\rangle_i\,, |I^-\rangle_i\,, |I^x\rangle_i\,, |I^0\rangle_i$.

1. Local dimensional reduction for 2-qubit permutation gates

In this section, we show that, for random-permutation circuits, the averaged 2-replica dynamics from random-product states takes place in an effective Hilbert space of dimension 4.

We consider random product states

$$|\psi\rangle = \bigotimes_j |\phi_j\rangle \,, \tag{SA.8}$$

where $|\phi_j\rangle = v_j|0\rangle$ and v_j are independent Haar random single-qubit unitaries. Next, we denote by $\tilde{\mathcal{P}}_N$ the set of all N-qubit Pauli strings with an overall phase $\phi\in[\pm1,\pm i]$, and by $\mathcal{P}_N$ the set of all Pauli strings without overall phases. The invariance of the Haar measure under left multiplication implies

$$(P^*\otimes P)^{\otimes 2}\mathbb{E}_{\{\phi_j\}}[\otimes_j(|\phi_j\rangle^*|\phi_j\rangle)^{\otimes 2}] = \mathbb{E}_{\{\phi_j\}}[\otimes_j(|\phi_j\rangle^*|\phi_j\rangle)^{\otimes 2}] \,, \tag{SA.9}$$

where $\mathbb{E}_{\{\phi_j\}}[\cdot]$ denotes averaging over the initial random product states. Since all 2-qubit permutation gates U are in the Clifford group, we have

$$U^\dagger P U = \alpha Q \,, \tag{SA.10}$$

where $Q\in\mathcal{P}_N$, $|\alpha|=1$, and so

$$(U^\dagger)^* P^* U^* = \alpha^* Q^* \,. \tag{SA.11}$$

These equations imply

$$(P^*\otimes P)^{\otimes 2}(U^*\otimes U\otimes U^*\otimes U) = (U^*\otimes U\otimes U^*\otimes U^*)(Q^*\otimes Q)^{\otimes 2} \,, \tag{SA.12}$$

and so finally

$$\frac{1}{4^N}\sum_{P\in\mathcal{P}_N}(P^*\otimes P)^{\otimes 2}(U^*\otimes U\otimes U^*\otimes U) = (U^*\otimes U\otimes U^*\otimes U)\frac{1}{4^N}\sum_{P\in\mathcal{P}_N}(P^*\otimes P)^{\otimes 2} \,. \tag{SA.13}$$

On the other hand,

$$\frac{1}{4}\sum_{\alpha=0}^{3}(\sigma_j^\alpha)^*\otimes\sigma_j^\alpha\otimes(\sigma_j^\alpha)^*\otimes\sigma_j^\alpha = \Pi_j \,, \tag{SA.14}$$

where Π_j is a projector onto a 4 dimensional subspace of $\mathcal{H}_j^{\otimes 4}$, so that $\mathbb{E}_{\{\phi_j\}}[\otimes_j(|\phi_j\rangle^*|\phi_j\rangle)^{\otimes 2}] = \mathbb{E}_{\{\phi_j\}}[\otimes_j\Pi_j(|\phi_j\rangle^*|\phi_j\rangle)^{\otimes 2}]$. Using now Eq. (SA.13) we have

$$\begin{aligned}\mathbb{E}_{\{\phi_j\}}[(1-\mathcal{L})^k\otimes_j(|\phi_j\rangle^*|\phi_j\rangle)^{\otimes 2}] &= \left[(1-\mathcal{L})^k\frac{1}{4^N}\sum_{P\in\mathcal{P}_N}(P^*\otimes P)^{\otimes 2}\otimes_j(|\phi_j\rangle^*|\phi_j\rangle)^{\otimes 2}\right]\\ &= \left[(1-\mathcal{L})^{k-1}\left(\frac{1}{4^N}\sum_{P\in\mathcal{P}_N}(P^*\otimes P)^{\otimes 2}\right)(1-\mathcal{L})\left(\frac{1}{4^N}\sum_{P\in\mathcal{P}_N}(P^*\otimes P)^{\otimes 2}\right)\otimes_j(|\phi_j\rangle^*|\phi_j\rangle)^{\otimes 2}\right]\\ &= \mathbb{E}_{\{\phi_j\}}[(1-\tilde{\mathcal{L}})^k\otimes_j\Pi_j(|\phi_j\rangle^*|\phi_j\rangle)^{\otimes 2}],\end{aligned} \tag{SA.15}$$

where $\tilde{\mathcal{L}} = (\otimes_j\Pi_j)\mathcal{L}(\otimes_j\Pi_j)$. Putting all together, we obtain the following solution to Eq. (10), after averaging over the random initial states:

$$|\rho(t)\otimes\rho(t)\rangle = e^{-\tilde{\mathcal{L}}t}\mathbb{E}_{\phi_j}[\otimes_j\Pi_j(|\phi_j\rangle^*|\phi_j\rangle)^{\otimes 2}\,. \tag{SA.16}$$

That is, the dynamics takes place in the site-permutation symmetric subspace of $\tilde{\mathcal{H}} = \otimes_j\tilde{\mathcal{H}}_j$ with $\dim(\tilde{\mathcal{H}}_j) = 4$.

Appendix B: Bounds on the entanglement Page curves

In this section, we provide further details on the Page curve bound for the case of the von Neumann entanglement entropy

$$S_1(t)\le\min\left[|A|, S_1^{\mathrm{PE}}(|\psi\rangle), |A|\sqrt{1-z^2}+O(1)\right]\,. \tag{SB.1}$$

We derive in particular the last term, as the first two terms in the right-hand side follow immediately from the case of Rényi entropies derived in the main text.

First, setting $\delta = \sqrt{1-z^2}$, where $z = |\langle\psi_0|a_N\rangle|$, we see that δ is equal to the trace distance $\Delta(|a_N\rangle\langle a_N|, |\psi_0\rangle\langle\psi_0|)$ between $|a_N\rangle\langle a_N|$ and $|\psi_0\rangle\langle\psi_0|$. Therefore, using the contractivity of the trace distance with respect to the partial trace, we have

$$\delta_A \equiv \Delta(\mathrm{Tr}_{\bar{A}}[|a_N\rangle\langle a_N|], \rho_A = \mathrm{Tr}_{\bar{A}}[|\psi_0\rangle\langle\psi_0|)\le\delta\,. \tag{SB.2}$$

Now, $\mathrm{Tr}_{\bar{A}}[|a_N\rangle\langle a_N| = (|+\rangle\langle+|)^{\otimes|A|}$ is a pure state. Thus, using the Fannes inequality [71], we get

$$S_1(\rho_A)\le|A|\delta+O(1)\,, \tag{SB.3}$$

where we used that the binary entropy is $H(\delta_A, 1-\delta_A)$ is in $[0,1]$ and $\delta_A\le\delta$. Therefore, assuming $|A|\le L/2$, we obtain

$$S_1(\rho_A)\le|A|\sqrt{1-z^2}+O(1)\,. \tag{SB.4}$$

Appendix C: Derivation of the system of differential equations

Here, we show that the averaged purity can be obtained as the solution to a system of differential equations. We will start with the circuit of random permutation and random-phase gates, as it is technically simpler.

1. Random Permutation Circuit with Random Phases (RPPC)

In this section, we derive a system of differential equations that describe the time evolution of the averaged purity $\mathcal{P}_A(t)$:

$$\mathcal{P}_A(t) := \mathrm{Tr}_A\mathbb{E}[\rho_A^2(t)] = \langle X_A|\mathbb{E}\left[\rho(t)\otimes\rho(t)\right]\rangle\,, \tag{SC.1}$$

for a 2-local random circuit consisting of random permutations with the addition of random phases. By choosing a site-permutation invariant initial state, the purity at time $t=0$ depends only on $n=|A|$, $\mathcal{P}_A(0) = \mathcal{P}_n(0)$. Because the Lindbladian (SA.5) is site-permutation invariant, the evolved state will also retain this property.

The presence of additional random phases reduces the dimension of the invariant subspace of $\mathcal{U}_{i,k}$, which is now spanned by the states

$$|I^+I^+\rangle_{j,k}\ ,\quad |I^-I^-\rangle_{j,k}\ ,\quad |I^0I^0\rangle_{j,k}\ , \tag{SC.2}$$

where

$$|I^\alpha I^\alpha\rangle_{j,k} = |I^\alpha\rangle_j\,|I^\alpha\rangle_k\ ,\qquad \alpha=+,-,0\,. \tag{SC.3}$$

Because of site-permutation invariance, in order to obtain a closed system of ODEs, we only need to introduce the $\binom{N+2}{2}$ states:

$$|G_{n_-,n_+,n_0}\rangle := \frac{1}{N!}\sum_{\pi\in\mathcal{S}_N}\pi\,|I^-\rangle^{\otimes n_-}\otimes|I^+\rangle^{\otimes n_+}\otimes|I^0\rangle^{\otimes n_0}\ , \tag{SC.4}$$

where we employ the same symbol π to denote a permutation and its unitary implementation on a state. We then define the functions

$$G_{n_-,n_+,n_0}(t) = \langle G_{n_-,n_+,n_0}|\rho(t)\otimes\rho(t)\rangle\,, \tag{SC.5}$$

where n_-, n_+, n_0 are non-negative integers such that $n_-+n_++n_0=N$. The purity is recovered by setting $n_0=0$:

$$\mathcal{P}_n(t) = \langle G_{n,N-n,0}|\mathbb{E}\left[\rho(t)\otimes\rho(t)\right]\rangle\ = \langle X_n|\mathbb{E}\left[\rho(t)\otimes\rho(t)\right]\rangle\,, \tag{SC.6}$$

where $|X_n\rangle=(N!)^{-1}\sum_{\pi\in\mathcal{S}_N}\pi\,|I^-\rangle^{\otimes n}|I^+\rangle^{\otimes(N-n)}$ is the symmetrized swap state. The system of differential equations is:

$$\begin{aligned}\frac{dG_{n_-,n_+,n_0}(t)}{dt} =&\langle G_{n_-,n_+,n_0}(t)|(-\mathcal{L})\mathbb{E}\left[\rho(t)\otimes\rho(t)\right]\rangle\\ =&\frac{\lambda[(n_-)^2+(n_+)^2+(n_0)^2-N^2]}{N-1}G_{n_-,n_+,n_0}(t)\\ &+\frac{2\lambda n_-n_+}{(N-1)(q+1)}\left[G_{n_--1,n_++1,n_0}(t)+G_{n_-+1,n_+-1,n_0}(t)+(q-1)G_{n_--1,n_+-1,n_0+2}(t)\right]\\ &+\frac{2\lambda n_0n_-}{(N-1)(q+1)}(G_{n_-+1,n_+,n_0-1}(t)+q\,G_{n_--1,n_+,n_0+1}(t))\\ &+\frac{2\lambda n_0n_+}{(N-1)(q+1)}(G_{n_-,n_++1,n_0-1}(t)+q\,G_{n_-,n_+-1,n_0+1}(t)).\end{aligned} \tag{SC.7}$$

The system closes by adopting the convention that $G_{n_-,n_+,n_0}(t)=0$ whenever any of the three indices is -1 or $N+1$. In particular, the equation for the purity is:

$$\frac{d\mathcal{P}_n(t)}{dt} = -\frac{2\lambda n(N-n)}{N-1}\mathcal{P}_n(t)+\frac{2\lambda n(N-n)}{(N-1)(q+1)}\left(\mathcal{P}_{n-1}(t)+\mathcal{P}_{n+1}(t)\right)+\frac{2\lambda n(N-n)(q-1)}{(N-1)(q+1)}G_{n-1,N-n-1,2}(t). \tag{SC.8}$$

To prove (SC.7) we rewrite Eq. (SC.4) as

$$|G_{n_-,n_+,n_0}\rangle = \frac{1}{N!}\sum_{\pi\in\mathcal{S}_N}\bigotimes_{i\in A_\pi^-}|I^-\rangle_i\bigotimes_{i\in A_\pi^+}|I^+\rangle_i\bigotimes_{i\in A_\pi^0}|I^0\rangle_i\ , \tag{SC.9}$$

where $A_\pi^-=\{\pi(1),\ldots,\pi(n_-)\}$, $A_\pi^+=\{\pi(n_-+1),\ldots,\pi(n_-+n_+)\}$, $A_\pi^0=\{\pi(n_-+n_++1),\ldots,\pi(N)\}$, and we note that, for a fixed permutation π, there are exactly n_-n_+ ways one could remove one index from the set A_π^- and one from the set A_π^+, and move them to A_π^0. This gives the following representation of $|G_{n_--1,n_+-1,n_0+2}\rangle$ in terms of the r.h.s. of Eq. (SC.9),

$$|G_{n_--1,n_+-1,n_0+2}\rangle = \frac{1}{N!}\sum_{\pi\in\mathcal{S}_N}\frac{1}{n_-n_+}\sum_{\substack{j\in A_\pi^-\\ k\in A_\pi^+}}\bigotimes_{\substack{i\in A_\pi^-\\ i\neq j}}|I^-\rangle_i\bigotimes_{\substack{i\in A_\pi^+\\ i\neq k}}|I^+\rangle_i\bigotimes_{i\in A_\pi^0\cup\{j,k\}}|I^0\rangle_i\ , \tag{SC.10}$$

and the rest of the states appearing in Eq. (SC.7) can be expressed in an analogous form. Thus, from:

$$\begin{aligned}
\mathcal{U}_{j,k}\left|I^-I^-\right\rangle_{j,k} &= \left|I^-I^-\right\rangle_{j,k}\ , \\
\mathcal{U}_{j,k}\left|I^+I^+\right\rangle_{j,k} &= \left|I^+I^+\right\rangle_{j,k}\ , \\
\mathcal{U}_{j,k}\left|I^0I^0\right\rangle_{j,k} &= \left|I^0I^0\right\rangle_{j,k}\ , \\
\mathcal{U}_{j,k}\left|I^-I^0\right\rangle_{j,k} &= \frac{1}{q+1}(\left|I^-I^-\right\rangle_{j,k} + q\ \left|I^0I^0\right\rangle_{j,k})\,, \\
\mathcal{U}_{j,k}\left|I^+I^0\right\rangle_{j,k} &= \frac{1}{q+1}(\left|I^+I^+\right\rangle_{j,k} + q\ \left|I^0I^0\right\rangle_{j,k})\,, \\
\mathcal{U}_{j,k}\left|I^+I^-\right\rangle_{j,k} &= \frac{1}{q+1}[\left|I^-I^-\right\rangle_{j,k} + \left|I^+I^+\right\rangle_{j,k} + (q-1)\ \left|I^0I^0\right\rangle_{j,k}]\,,
\end{aligned} \tag{SC.11}$$

we obtain:

$$\begin{aligned}
&\sum_{1\le j<k\le N} \langle G_{n_-,n_+,n_0}(t)|\mathcal{U}_{j,k} \\
&= \frac{1}{N!}\sum_{\pi\in\mathcal{S}_N}\left(\sum_{\substack{j,k\in A_\pi^-\\ j<k}} + \sum_{\substack{j,k\in A_\pi^+\\ j<k}} + \sum_{\substack{j,k\in A_\pi^0\\ j<k}} + \sum_{\substack{j\in A_\pi^-\\ k\in A_\pi^+}} + \sum_{\substack{j\in A_\pi^-\\ k\in A_\pi^0}} + \sum_{\substack{j\in A_\pi^+\\ k\in A_\pi^0}}\right)\bigotimes_{i\in A_\pi^-}\langle I^-|_i \bigotimes_{i\in A_\pi^+}\langle I^+|_i \bigotimes_{i\in A_\pi^0}\langle I^0|_i\,\mathcal{U}_{j,k} \\
&= \left(\frac{n_-(n_--1)}{2} + \frac{n_+(n_+-1)}{2} + \frac{n_0(n_0-1)}{2}\right)\langle G_{n_-,n_+,n_0}(t)| \\
&+ \frac{n_-n_+}{q+1}\left(\langle G_{n_--1,n_++1,n_0}(t)| + \langle G_{n_-+1,n_+-1,n_0}(t)| + (q-1)\,\langle G_{n_--1,n_+-1,n_0+2}(t)|\right) \\
&+ \frac{n_-n_0}{q+1}\left(\langle G_{n_-+1,n_+,n_0-1}(t)| + q\ \langle G_{n_--1,n_+,n_0+1}|\right) + \frac{n_+n_0}{q+1}\left(\langle G_{n_-,n_++1,n_0-1}(t)| + q\ \langle G_{n_-,n_+-1,n_0+1}|\right).
\end{aligned} \tag{SC.12}$$

Equation (SC.7) then follows straightforwardly from the definition (SA.5).

Let us now derive the initial conditions for the homogeneous product state:

$$|\Psi_0\rangle = \left(\sum_{a=0}^{q-1}\lambda_a\,|a\rangle\right)^{\otimes N}, \qquad \sum_{a=0}^{q-1}|\lambda_a|^2 = 1, \tag{SC.13}$$

which corresponds to:

$$|\rho_0\otimes\rho_0\rangle = \bigotimes_{j=1}^{N}\left[\sum_{a,b,c,d=0}^{q-1}\lambda_a^*\lambda_b\lambda_c^*\lambda_d\,|abcd\rangle_j\right]. \tag{SC.14}$$

The computation is straightforward:

$$\begin{aligned}
&G_{n_-,n_+,n_0}(0) = \langle G_{n_-,n_+,n_0}|\rho_0\otimes\rho_0\rangle \\
=&\frac{1}{N!}\sum_{\pi\in S_N}\left(\langle I^-|\sum_{a,b,c,d}\lambda_a^*\lambda_b\lambda_c^*\lambda_d\,|abcd\rangle\right)^{n_-}\left(\langle I^+|\sum_{a,b,c,d}\lambda_a^*\lambda_b\lambda_c^*\lambda_d\,|abcd\rangle\right)^{n_+}\left(\langle I^0|\sum_{a,b,c,d}\lambda_a^*\lambda_b\lambda_c^*\lambda_d\,|abcd\rangle\right)^{n_0} \\
=&\frac{1}{N!}\sum_{\pi\in\mathcal{S}_N}\left(\sum_{a=0}^{q-1}|\lambda_a|^4\right)^{n_0} = \left(\sum_{a=0}^{q-1}|\lambda_a|^4\right)^{n_0},
\end{aligned} \tag{SC.15}$$

a result consistent with $\mathcal{P}_n(0) = 1$ for this state. In particular, for a qubit state:

$$|\Psi_0\rangle = (\cos\theta\,|0\rangle + \sin\theta\,|1\rangle)^{\otimes N}\,, \tag{SC.16}$$

$$G_{n_-,n_+,n_0}(0) = \left(1 - \frac{1}{2}\sin^2(2\theta)\right)^{n_0}, \tag{SC.17}$$

so that $G_{n_-,n_+,n_0}(0) = 1$ for $\theta = 0, \pi/2$ (corresponding to *classical states* $|\Psi_0\rangle = |a\rangle^{\otimes N}$) and the minimum of the initial value is achieved when $\theta = \pi/4$, for which $G_{n_-,n_+,n_0}(0) = 2^{-n_0}$. In the q-dimensional case, one has:

$$G_{n_-,n_+,n_0}(0) = q^{-n_0} \quad \text{for} \quad |\Psi_0\rangle = \left(\sum_{a=0}^{q-1} \frac{e^{i\theta_a}}{\sqrt{q}} |a\rangle\right)^{\otimes N} . \tag{SC.18}$$

2. Entangled Initial States and Anticoncentration

In this section, we compute the initial conditions for a 2-local random permutation circuit with random phases, that is, the initial conditions for the system (SC.7), when the initial state is a site-permutation invariant entangled state. In particular, we focus on the following:

1. The GHZ state:

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{q}} \sum_{a=0}^{q-1} |a\rangle^{\otimes N} ; \tag{SC.19}$$

2. The Dicke states ($q = 2$)

$$|D\rangle = \sum_{k=0}^{N} c_k |D_k\rangle , \quad |D_k\rangle = \binom{N}{k}^{-1/2} \sum_{\pi} \pi |0\rangle^{\otimes(N-k)} \otimes |1\rangle^{\otimes k} , \quad \sum_{k=0}^{N} |c_k|^2 = 1 , \tag{SC.20}$$

where the sum is over the distinct permutations of the qubits. Notice that by considering $|0\rangle$, $|1\rangle$ as two spin states of a spin-1/2 chain, the state $|W\rangle \equiv |D_1\rangle$ is a one-magnon state with zero momentum. Its purity in a bipartiton, as well as that of $|D_2\rangle$, was computed in [**?**].

When the initial state is not a product state, a direct use of the Choi-Jamiolkowski isomorphism is generally not the most efficient way to proceed. This is because the expression of the two-replica state $|\rho \otimes \rho\rangle$ may be quite cumbersome. Instead, we will derive an alternative expression for the functions G_{n_-,n_+,n_0} which is useful in the case of permutation-invariant entangled states. Let us define a tripartite Hilbert space:

$$\mathcal{H} = \mathcal{H}_- \otimes \mathcal{H}_+ \otimes \mathcal{H}_0 := (\mathcal{H}_{\text{loc}})^{\otimes n_-} \otimes (\mathcal{H}_{\text{loc}})^{\otimes n_+} \otimes (\mathcal{H}_{\text{loc}})^{\otimes n_0} , \quad \mathcal{H}_{\text{loc}} \simeq \mathbb{C}^q, \quad n_- + n_+ + n_0 = N, \tag{SC.21}$$

corresponding to three non overlapping regions A_-, A_+, A_0, of sizes $|A_-| = n_-$, $|A_+| = n_+$, $|A_0| = n_0$. We then consider the functions $G_{n_-,n_+,n_0}(t)$ defined in (SC.5), and let $\rho(t)$ be a permutation-invariant state. By using the explicit expression (SA.1) for the 2-replica state $|\rho(t) \otimes \rho(t)\rangle$, after some algebra we arrive at:

$$G_{n_-,n_+,n_0}(t) = \sum_{\{x_i\}_{i\in A_0}} \text{Tr}_{\mathcal{H}_-}(\langle\{x_i\}| \,\text{Tr}_{\mathcal{H}_+}\rho(t)\, |\{x_i\}\rangle)^2, \tag{SC.22}$$

where the states $|\{x_i\}\rangle$, for $x_i = 0, \dots, q-1$ and $i \in A_0$, form the computational basis of $\mathcal{H}_0$. For $n_- = n$, $n_+ = N-n$, $n_0 = 0$, the right-hand side of equation (SC.22) yields the averaged purity $\mathcal{P}_n(t)$, as expected. On the other hand, for $\rho(t) = |\psi(t)\rangle \langle\psi(t)|$:

$$G_{0,0,N}(t) = \sum_{\{x_i\}_{i=1,\dots,N}} (\langle\{x_i\}| \,\rho(t)\, |\{x_i\}\rangle)^2 = \sum_{\{x_i\}} |\langle\{x_i\}|\psi(t)\rangle|^4 = I_2(|\psi(t)\rangle), \tag{SC.23}$$

where $I_2(|\psi\rangle)$ is the second inverse participation ratio of the state $|\psi\rangle$ (cf. Eq. (3)). For generic values of n_-, n_+, n_0 an intermediate situation arises, whereby partial traces are taken over the regions A_-, A_+ while the off-diagonal matrix elements in the region A_0 are washed away. We observe that this is consistent with the result (SC.18) for maximally delocalized product states, for which it is known that $I_2 = q^{-N}$.

By using equation (SC.22), the computation of the functions G_{n_-,n_+,n_0} for $|\text{GHZ}\rangle$ is straightforward. Indeed, in this case $\rho = |\text{GHZ}\rangle\langle\text{GHZ}|$ and

$$\begin{gathered}\text{Tr}_{\mathcal{H}_+} \rho = \frac{1}{q} \sum_{a,b=0}^{q-1} \text{Tr}_{\mathcal{H}_+} (|a\rangle \langle b|)^{\otimes N} = \frac{1}{q} \sum_{a=0}^{q-1} (|a\rangle \langle b|)^{\otimes(N-n_+)}, \\ \langle\{x_i\}| \,\text{Tr}_{\mathcal{H}_+} \rho\, |\{x_i\}\rangle = \frac{1}{q} \sum_{a=0}^{q-1} (|a\rangle \langle a|)^{\otimes n_-} \prod_{i\in A_0} \delta_{x_i,a} .\end{gathered} \tag{SC.24}$$

Hence:

$$G_{n_-,n_+,n_0}(|\mathrm{GHZ}\rangle) = \frac{1}{q^2}\sum_{a=0}^{q-1}\sum_{\{x_i\}_{i\in A_0}}\prod_{i\in A_0}\delta_{x_i,a} = \frac{1}{q}, \tag{SC.25}$$

for all triples (n_-, n_+, n_0).

Let us now consider the Dicke state (SC.20). The computation of $I_2(|D\rangle)$ is straightforward, and yields:

$$I_2(|D\rangle) = \sum_{m=1}^{N} |c_m|^4 \binom{N}{m}^{-1}. \tag{SC.26}$$

This implies in particular that we can express the second PE of $|D\rangle$ as:

$$S_2^{\mathrm{PE}}(|D\rangle) = -\log I_2(|D\rangle) = \Delta_2 \log(2^N), \quad \Delta_2 = -\frac{1}{N}\log\left(\sum_{m=1}^{N}|c_m|^4\binom{N}{m}^{-1}\right). \tag{SC.27}$$

The quantity Δ_2 is a fractal dimension $0 \leq \Delta_2 \leq 1$, which signals the amount of delocalization (or anticoncentration) of the state: the latter is maximally localized when $\Delta_2 = 0$ and maximally delocalized when $\Delta_2 = 1$.

Computing the initial conditions $G_{n_-,n_+,n_0}(0)$ for the state $|D\rangle$ when (n_-, n_+, n_0) is a generic triplet is a difficult task. We limit ourselves to the case of $|D\rangle = c_1|D_1\rangle + c_2|D_2\rangle$, $|c_1|^2 + |c_2|^2 = 1$, that is a linear superposition of a one- and a two-magnon state. First, we observe that by defining for any region $A \in \{A_-, A_+, A_0\}$:

$$|D_k\rangle_A = \binom{|A|}{k}^{-1/2}\sum_{\pi}\pi\,|0\rangle^{\otimes(|A|-k)}\otimes|1\rangle^{\otimes k} \in \mathcal{H}_{\mathrm{loc}}^{\otimes|A|}, \tag{SC.28}$$

then in the Hilbert space (SC.21) we can write, for $k = 0, \ldots, N$:

$$|D_k\rangle = \binom{N}{k}^{-\frac{1}{2}}\sum_{k_-=0}^{k}\sum_{k_+=0}^{k-k_-}\binom{n_-}{k_-}^{\frac{1}{2}}\binom{n_+}{k_+}^{\frac{1}{2}}\binom{n_0}{k_0}^{\frac{1}{2}}|k_-,k_+,k_0\rangle, \quad |k_-,k_+,k_0\rangle := |D_{k_-}\rangle_{A_-}|D_{k_+}\rangle_{A_+}|D_{k_0}\rangle_{A_0}, \tag{SC.29}$$

where $n_0 = N - n_- - n_+$, and $k_0 = k - k_- - k_+$ for every k_-, k_+. We also adopted the convention that $\binom{N}{k} = 0$ if $k < 0$ or $k > N$. Thus, defininig the ratios:

$$r_- := \frac{n_-}{N}, \quad r_+ := \frac{n_+}{N}, \quad r_0 := \frac{n_0}{N}, \tag{SC.30}$$

we have:

$$|D_1\rangle = r_-^{\frac{1}{2}}|1,0,0\rangle + r_+^{\frac{1}{2}}|0,1,0\rangle + r_0^{\frac{1}{2}}|0,0,1\rangle, \tag{SC.31}$$

$$\begin{aligned}|D_2\rangle = {}& \left[r_-\left(\frac{n_- - 1}{N-1}\right)\right]^{\frac{1}{2}}|2,0,0\rangle + \left[r_+\left(\frac{n_+ - 1}{N-1}\right)\right]^{\frac{1}{2}}|0,2,0\rangle + \left[r_0\left(\frac{n_0 - 1}{N-1}\right)\right]^{\frac{1}{2}}|0,0,2\rangle \\ & + \left(\frac{2r_-n_+}{N-1}\right)^{\frac{1}{2}}|1,1,0\rangle + \left(\frac{2r_-n_0}{N-1}\right)^{\frac{1}{2}}|1,0,1\rangle + \left(\frac{2r_+n_0}{N-1}\right)^{\frac{1}{2}}|0,1,1\rangle.\end{aligned} \tag{SC.32}$$

We note that the expression (SC.29) can be generalized to compute multipartite entanglement, and it was first employed for the description of kink states in the ferromagnetic Ising chain in [?]. The reduced density matrix of $|D\rangle$ in $\mathcal{H}_- \otimes \mathcal{H}_0$ is obtained as

$$\begin{aligned}\mathrm{Tr}_{\mathcal{H}_+}|D\rangle\langle D| = {}& |c_1|^2(r_-|1,0\rangle\langle 1,0| + r_+|0,0\rangle\langle 0,0| + r_0|0,1\rangle\langle 0,1|) \\ & + |c_2|^2\left[r_-\left(\frac{n_- - 1}{N-1}\right)|2,0\rangle\langle 2,0| + r_-\left(\frac{n_+ - 1}{N-1}\right)|0,0\rangle\langle 0,0| + r_0\left(\frac{n_0 - 1}{N-1}\right)|0,2\rangle\langle 0,2|\right. \\ & \left. + \frac{2r_-n_+}{N-1}|1,0\rangle\langle 1,0| + \frac{2r_-n_+}{N-1}|1,0\rangle\langle 1,0| + \frac{2r_-n_0}{N-1}|1,1\rangle\langle 1,1|\right] \\ & + c_1c_2^*\left[r_-\left(\frac{n_- - 1}{N-1}\right)^{\frac{1}{2}}|1,0\rangle\langle 2,0| + r_+^{\frac{1}{2}}\left(\frac{2r_-n_+}{N-1}\right)|0,0\rangle\langle 1,0| + r_0^{\frac{1}{2}}\left(\frac{2r_-n_0}{N-1}\right)|0,1\rangle\langle 1,0|\right] + \mathrm{h.c.} \\ & + \text{off-diagonal in } \mathcal{H}_0\,.\end{aligned} \tag{SC.33}$$

In the above expression, we only list those terms with non-zero diagonal matrix elements in $\mathcal{H}_0$. In order to evaluate the diagonal matrix elements in $\mathcal{H}_0$ we make use of the following decomposition of the Hilbert space,

$$\mathcal{H}_0 = \oplus_{k=0}^{n_0} \mathcal{H}_0^{(k)}, \qquad \mathcal{H}_0^{(k)} = \text{Span}\left\{ |\{x_i\}\rangle_{i\in A_0} \middle| x_i \in \{0,1\}, \sum_{i\in A_0} x_i = k \right\}. \tag{SC.34}$$

The space $\mathcal{H}_0^{(k)}$ is the sector of $\mathcal{H}_0$ in which the number of local states $|1\rangle$ in the region A_0 is fixed to k. As expected, $\sum_{k=0}^{n_0} \dim\mathcal{H}_0^{(k)} = \sum_{k=0}^{n_0} \binom{n_0}{k} = 2^{n_0}$. By using this above decomposition, for $|\{x_i\}\rangle, |D_{k_0}\rangle \in \mathcal{H}_0$:

$$\langle\{x_i\}|D_{k_0}\rangle = \binom{n_0}{k_0}^{-\frac{1}{2}} \delta_{\sum_i x_i - k_0, 0}\,. \tag{SC.35}$$

Therefore, it is a matter of algebra to obtain from (SC.22):

$$\begin{aligned} G_{n_-,n_+,n_0}(c_1|D_1\rangle + c_2|D_2\rangle) =& |c_1|^4\left(r_+^2 + r_-^2 + \frac{r_0^2}{N}\right) + |c_2|^4\left[r_+^2\left(\frac{n_+-1}{N-1}\right)^2 + r_-^2\left(\frac{n_--1}{N-1}\right)^2 + \left(\frac{2r_-n_+}{N-1}\right)^2\right.\\ &\left. + \frac{1}{n_0}\left(\frac{2r_+n_0}{N-1}\right)^2 + \frac{1}{n_0}\left(\frac{2r_-n_0}{N-1}\right)^2 + \frac{2}{n_0(n_0-1)}\left(r_0\left(\frac{n_0-1}{N-1}\right)\right)^2\right]\\ &+2|c_1|^2|c_2|^2\left[r_+^2\left(\frac{n_+-1}{N-1}\right) + r_-^2\left(\frac{n_--1}{N-1}\right) + \frac{2r_-^2n_+}{N-1} + \frac{2r_+^2n_-}{N-1} + \frac{2r_+r_0}{N-1} + \frac{2r_-r_0}{N-1}\right]. \end{aligned} \tag{SC.36}$$

3. Random Permutation Circuit with Initial Random Product States

We now derive the system of differential equations for the averaged purity in a 2-local Random Permutation Circuit with an initial random product state, that is, a state in which local (on-site) scrambling is applied before the action of the 2-local gates. We focus on qubits ($q = 2$). Since permutation gates acting on two-qubit states are Clifford, the considerations of Section A 1 apply, and the evolution takes place in an effective 4-dimensional Hilbert space.

Differential equations

By inspection, one can show that the effective local Hilbert space is spanned the following states in the four-replica space

$$\begin{aligned} |J^0\rangle &= \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)\,, \qquad & |J^+\rangle &= \frac{1}{\sqrt{2}}(|0011\rangle + |1100\rangle)\,,\\ |J^x\rangle &= \frac{1}{\sqrt{2}}(|0101\rangle + |1010\rangle) & |J^-\rangle &= \frac{1}{\sqrt{2}}(|1001\rangle + |0110\rangle)\,. \end{aligned} \tag{SC.37}$$

Indeed, defining

$$\begin{aligned} |v_0\rangle_{i,j} &= |J^0\rangle_i\,|J^0\rangle_j\\ |v_1\rangle_{i,j} &= \frac{1}{\sqrt{3}}(|J^0\rangle_i\,|J^+\rangle_j + |J^+\rangle_i\,|J^0\rangle_j + |J^+\rangle_i\,|J^+\rangle_j)\,,\\ |v_2\rangle_{i,j} &= \frac{1}{\sqrt{3}}(|J^0\rangle_i\,|J^x\rangle_j + |J^x\rangle_i\,|J^0\rangle_j + |J^x\rangle_i\,|J^x\rangle_j)\,,\\ |v_3\rangle_{i,j} &= \frac{1}{\sqrt{3}}(|J^0\rangle_i\,|J^-\rangle_j + |J^-\rangle_i\,|J^0\rangle_j + |J^-\rangle_i\,|J^-\rangle_j)\,,\\ |v_4\rangle_{i,j} &= \frac{1}{\sqrt{6}}(|J^+\rangle_i\,|J^x\rangle_j + |J^x\rangle_i\,|J^+\rangle_j + |J^+\rangle_i\,|J^-\rangle_j + |J^-\rangle_i\,|J^+\rangle_j + |J^x\rangle_i\,|J^-\rangle_j + |J^-\rangle_i\,|J^x\rangle_j)\,, \end{aligned} \tag{SC.38}$$

one can show

$$\tilde{\mathcal{U}}_{i,j} = \Pi_i\Pi_j\,\mathcal{U}_{i,j}\,\Pi_i\Pi_j = \sum_{k=0}^{4} |v_k\rangle_{i,j}\,\langle v_k|_{i,j}\,, \tag{SC.39}$$

with the projectors defined in (SA.14). Then, the Lindbladian $\tilde{\mathcal{L}}$ is simply obtained from (SA.5) by replacing the averaged gate $\mathcal{U}_{i,j}$ with $\tilde{\mathcal{U}}_{i,j}$.

The time evolution of the averaged purity in a 2-local random permutation circuit with a random initial state is obtained by solving a system of differential equations for the following $\binom{N+3}{3}$ functions

$$G_{p,q,r,s}(t) := \langle G_{p,q,r,s}|\mathbb{E}[\rho(t)\otimes\rho(t)]\rangle\,, \qquad p,q,r,s\in\mathbb{N}_0, \quad p+q+r+s=N, \tag{SC.40}$$

where:

$$|G_{p,q,r,s}\rangle := \frac{1}{N!}\sum_{\pi\in\mathcal{S}_N}\pi\,|I^-\rangle^{\otimes p}\otimes|I^+\rangle^{\otimes q}\otimes|I^0\rangle^{\otimes r}\otimes|I^x\rangle^{\otimes s}\,, \tag{SC.41}$$

and we introduced the state $|I^x\rangle = \sum_{a,b=0}^{1}|abab\rangle$. The $|I^\alpha\rangle$ and $|J^\beta\rangle$ states are related via:

$$|J^0\rangle = \frac{|I^0\rangle}{\sqrt{2}}, \quad |J^+\rangle = \frac{|I^+\rangle-|I^0\rangle}{\sqrt{2}}, \quad |J^x\rangle = \frac{|I^x\rangle-|I^0\rangle}{\sqrt{2}}, \quad |J^-\rangle = \frac{|I^-\rangle-|I^0\rangle}{\sqrt{2}}. \tag{SC.42}$$

A closed system of ODEs can be obtained by imposing the condition that $G_{p,q,r,s}(t)=0$ whenever any of the indices is either 0 or $N+1$. In order to derive the equations, we first compute the action of $\tilde{\mathcal{U}}_{i,j}$ on the replica states $|I^\alpha I^\beta\rangle_{i,j}$, observing that $\tilde{\mathcal{U}}_{i,j}\,|I^\alpha I^\beta\rangle_{i,j} = \tilde{\mathcal{U}}_{i,j}\,|I^\beta I^\alpha\rangle_{i,j}$:

$$\begin{aligned}
\tilde{\mathcal{U}}_{i,j}\,|I^-I^-\rangle_{i,j} &= |I^-I^-\rangle_{i,j}\,,\\
\tilde{\mathcal{U}}_{i,j}\,|I^+I^+\rangle_{i,j} &= |I^+I^+\rangle_{i,j}\,,\\
\tilde{\mathcal{U}}_{i,j}\,|I^xI^x\rangle_{i,j} &= |I^xI^x\rangle_{i,j}\,,\\
\tilde{\mathcal{U}}_{i,j}\,|I^0I^0\rangle_{i,j} &= |I^0I^0\rangle_{i,j}\,,\\
\tilde{\mathcal{U}}_{i,j}\,|I^-I^0\rangle_{i,j} &= |I^0I^0\rangle_{i,j} + \frac{1}{3}(|I^-I^-\rangle_{i,j} - |I^0I^0\rangle_{i,j})\,,\\
\tilde{\mathcal{U}}_{i,j}\,|I^+I^0\rangle_{i,j} &= |I^0I^0\rangle_{i,j} + \frac{1}{3}(|I^+I^+\rangle_{i,j} - |I^0I^0\rangle_{i,j})\,,\\
\tilde{\mathcal{U}}_{i,j}\,|I^xI^0\rangle_{i,j} &= |I^0I^0\rangle_{i,j} + \frac{1}{3}(|I^xI^x\rangle_{i,j} - |I^0I^0\rangle_{i,j})\,,\\
\tilde{\mathcal{U}}_{i,j}\,|I^-I^+\rangle_{i,j} &= \frac{1}{3}(|I^-I^-\rangle_{i,j} + |I^+I^+\rangle_{i,j} + |I^0I^0\rangle_{i,j}) + \frac{2}{\sqrt{6}}\,|v_4\rangle_{i,j}\,,\\
\tilde{\mathcal{U}}_{i,j}\,|I^-I^x\rangle_{i,j} &= \frac{1}{3}(|I^-I^-\rangle_{i,j} + |I^xI^x\rangle_{i,j} + |I^0I^0\rangle_{i,j}) + \frac{2}{\sqrt{6}}\,|v_4\rangle_{i,j}\,,\\
\tilde{\mathcal{U}}_{i,j}\,|I^+I^x\rangle_{i,j} &= \frac{1}{3}(|I^+I^+\rangle_{i,j} + |I^xI^x\rangle_{i,j} + |I^0I^0\rangle_{i,j}) + \frac{2}{\sqrt{6}}\,|v_4\rangle_{i,j}\,.
\end{aligned} \tag{SC.43}$$

Then, proceeding as in Section C 1, we rewrite $|G_{p,q,r,s}\rangle$ as

$$|G_{p,q,r,s}\rangle = \frac{1}{N!}\sum_{\pi\in\mathcal{S}_N}\bigotimes_{i\in A_\pi^p}|I^-\rangle_i\bigotimes_{i\in A_\pi^q}|I^+\rangle_i\bigotimes_{i\in A_\pi^r}|I^0\rangle_i\bigotimes_{i\in A_\pi^s}|I^x\rangle_i\,, \tag{SC.44}$$

where

$$\begin{aligned}
A_\pi^p &= \{\pi(1),\dots,\pi(p)\}, & A_\pi^q &= \{\pi(p+1),\dots,\pi(p+q)\},\\
A_\pi^r &= \{\pi(p+q+1),\dots,\pi(p+q+r)\}, & A_\pi^s &= \{\pi(p+q+r+1),\dots,\pi(N)\}.
\end{aligned} \tag{SC.45}$$

This follows from appropriately splitting the sum over indices in:

$$\begin{aligned}
&\sum_{1\le i<j\le N}\langle G_{p,q,r,s}(t)|\tilde{\mathcal{U}}_{i,j}\,|\mathbb{E}[\rho(t)\otimes\rho(t)]\rangle\\
&=\frac{1}{N!}\sum_{\pi\in\mathcal{S}_N}\left(\sum_{\substack{i,j\in A_\pi^p\\ i<j}}+\sum_{\substack{i,j\in A_\pi^q\\ i<j}}+\sum_{\substack{i,j\in A_\pi^r\\ i<j}}+\sum_{\substack{i,j\in A_\pi^s\\ i<j}}+\sum_{\substack{i\in A_\pi^p\\ j\in A_\pi^q}}+\sum_{\substack{i\in A_\pi^p\\ j\in A_\pi^r}}+\sum_{\substack{i\in A_\pi^p\\ j\in A_\pi^s}}+\sum_{\substack{i\in A_\pi^q\\ j\in A_\pi^r}}+\sum_{\substack{i\in A_\pi^q\\ j\in A_\pi^s}}+\sum_{\substack{i\in A_\pi^r\\ j\in A_\pi^s}}\right)\\
&\bigotimes_{k\in A_\pi^p}\langle I^-|_k\bigotimes_{k\in A_\pi^q}\langle I^+|_k\bigotimes_{k\in A_\pi^r}\langle I^0|_k\bigotimes_{k\in A_\pi^s}\langle I^x|_k\,\tilde{\mathcal{U}}_{i,j}\,|\mathbb{E}[\rho(t)\otimes\rho(t)]\rangle\rangle.
\end{aligned} \tag{SC.46}$$

After some algebra, the above yields the following system of equations:

$$\begin{aligned}
\frac{dG_{p,q,r,s}(t)}{dt} =& \langle G_{p,q,r,s}(t)|(-\tilde{\mathcal{L}})\mathbb{E}\left[\rho(t)\otimes\rho(t)\right]\rangle \\
=& -\lambda N G_{p,q,r,s}(t) + \frac{\lambda[p(p-1)+q(q-1)+r(r-1)+s(s-1)}{N-1} G_{p,q,r,s}(t) \\
&+ \frac{2\lambda ps}{3(N-1)}\left[G_{p+1,q,r,s-1}(t) + 2G_{p-1,q,r,s+1}(t)\right] \\
&+ \frac{2\lambda qs}{3(N-1)}\left[G_{p,q+1,r,s-1}(t) + 2G_{p,q-1,r,s+1}(t)\right] \\
&+ \frac{2\lambda rs}{3(N-1)}\left[G_{p,q,r+1,s-1}(t) + 2G_{p,q,r-1,s+1}(t)\right] \\
&+ \frac{2\lambda pq}{3(N-1)}\left[G_{p+1,q-1,r,s}(t) + G_{p-1,q+1,r,s}(t) + G_{p-1,q,r+1,s}(t) + G_{p,q-1,r+1,s}(t) + G_{p,q,r,s}(t)\right. \\
&\quad \left. +4G_{p-1,q-1,r,s+2}(t) - 2G_{p-1,q,r,s+1}(t) - 2G_{p,q-1,r,s+1}(t) - 2G_{p-1,q-1,r+1,s+1}(t)\right] \\
&+ \frac{2\lambda pr}{3(N-1)}\left[G_{p+1,q,r-1,s}(t) + G_{p-1,q,r+1,s}(t) + G_{p-1,q+1,r,s}(t) + G_{p,q+1,r-1,s}(t) + G_{p,q,r,s}(t)\right. \\
&\quad \left. +4G_{p-1,q,r-1,s+2}(t) - 2G_{p,q,r-1,s+1}(t) - 2G_{p-1,q+1,r-1,s+1}(t) - 2G_{p-1,q,r,s+1}(t)\right] \\
&+ \frac{2\lambda qr}{3(N-1)}\left[G_{p,q+1,r-1,s}(t) + G_{p,q-1,r+1,s}(t) + G_{p+1,q-1,r,s}(t) + G_{p+1,q,r-1,s}(t) + G_{p,q,r,s}(t)\right. \\
&\quad \left. +4G_{p,q-1,r-1,s+2}(t) - 2G_{p+1,q-1,r-1,s+1}(t) - 2G_{p,q,r-1,s+1}(t) - 2G_{p,q-1,r,s+1}(t)\right].
\end{aligned} \tag{SC.47}$$

We consider a random initial state,

$$|\psi_0\rangle = \bigotimes_{j=1}^{N}\left(U_j\,|0\rangle_j\right), \qquad U_j \in U(2), \tag{SC.48}$$

so that

$$\begin{aligned}
|\mathbb{E}[\rho_0\otimes\rho_0]\rangle &= \bigotimes_{j=1}^{N}\left[\mathbb{E}(U_j^*\otimes U_j\otimes U_j^*\otimes U_j)\,|0000\rangle_j\right] \\
&= \bigotimes_{j=1}^{N}\left[\frac{1}{d^2-1}\left(|I^+\rangle_{j\,j}\langle I^+| + |I^-\rangle_{j\,j}\langle I^-|\right) - \frac{1}{d(d^2-1)}\left(|I^+\rangle_{j\,j}\langle I^-| + |I^-\rangle_{j\,j}\langle I^+|\right)\right]|0000\rangle_j \\
&= \bigotimes_{j=1}^{N}\left[\frac{|I^+\rangle_j + |I^-\rangle_j}{d(d+1)}\right].
\end{aligned} \tag{SC.49}$$

The normalisation of the state gives

$$\langle I^+|\mathbb{E}[\rho_0\otimes\rho_0]\rangle = \prod_{j=1}^{N}\left[\frac{{}_j\langle I^+|I^+\rangle_j + {}_j\langle I^+|I^-\rangle_j}{d(d+1)}\right] = 1, \tag{SC.50}$$

and therefore the initial conditions for the systems immediately follow from the definition of $|G_{p,q,r,s}\rangle$:

$$\langle G_{p,q,r,s}|\mathbb{E}[\rho_0\otimes\rho_0]\rangle = \left(\frac{2d}{d(d+1)}\right)^{r+s} = \left(\frac{2}{3}\right)^{r+s}. \tag{SC.51}$$

Steady-state solution

Let us now derive the steady-state solution to the system (SC.47), that is, the solution to:

$$\frac{dG_{p,q,r,s}(t)}{dt} = 0. \tag{SC.52}$$

By inspecting the right-hand side of (SC.47), one observes that the system admits the five linearly independent solutions:

$$\{2^p,\, 2^q,\, 2^r,\, 2^{-r},\, 1\}. \tag{SC.53}$$

The above functions span the space of steady-state solutions, since when the system (SC.47) is written in matrix form, $d\underline{G}(t)/dt = A \cdot \underline{G}$, the matrix A has a five-dimensional kernel, therefore any steady-state solution can be expressed as a linear combination of the above stationary solutions,

$$G^{(\infty)}_{p,q,r,s} = \alpha 2^p + \beta 2^q + \gamma 2^r + \delta 2^{-s} + \epsilon, \tag{SC.54}$$

where $\{\alpha, \beta, \gamma, \delta, \epsilon\}$ are (so far) free parameters. To determine them, we recall that there must also exist 5 integrals of motion (spanning the kernel of the *transpose* of the matrix A), which can be understood as linear combinations of functions $G_{p,q,r,s}(t)$ that stay constant for any initial state,

$$\mathcal{F}^{(j)}(t) = \sum_{\substack{p,q,r,s\\ p+q+r+s=N}} \alpha^{(j)}_{p,q,r,s} G_{p,q,r,s}(t), \qquad \frac{d\mathcal{F}^{(j)}(t)}{dt} = 0, \text{ for } j = 1, \dots, 5. \tag{SC.55}$$

In particular, these are given as

$$\mathcal{F}^{(1)}(t) = G_{N,0,0,0}(t), \qquad \mathcal{F}^{(2)}(t) = G_{0,N,0,0}(t), \qquad \mathcal{F}^{(3)}(t) = G_{0,0,N,0}(t), \qquad \mathcal{F}^{(4)}(t) = G_{0,0,0,N}(t),$$
$$\mathcal{F}^{(5)}(t) = \sum_{\substack{p,q,r,s\\ p+q+r+s=N}} (-2)^s \frac{(p+q+r+s)!}{p!q!r!s!} G_{p,q,r,s}(t). \tag{SC.56}$$

The first four of the above are obviously integrals of motion, whereas it is not immediate a priori that $d\mathcal{F}^{(5)}(t)/dt = 0$; however, this can be directly checked from (SC.47).

Using now the fact that the value of Eq. (SC.56) is the same in the initial and stationary state, and combining it with the initial condition (SC.51), we can determine the coefficients $\{\alpha, \beta, \gamma, \delta, \epsilon\}$ in Eq. (SC.54) to finally yield

$$G^{(\infty)}_{p,q,r,s} = \frac{[1 - 2\left(\frac{2}{3}\right)^N + 3^{-N}][2^{-(N-p)} + 2^{-(N-q)}] + [\left(\frac{2}{3}\right)^N - 2^{-(N-1)} + 3^{-N}][1 - 2^{-(N-r)} + 2^{-s}]}{[1 - 2^{-(N-1)}](1 - 2^{-N})}. \tag{SC.57}$$

The expression of the steady-state averaged purity is therefore given as

$$\mathcal{P}^{(\infty)}_n = G^{(\infty)}_{n,N-n,0,0} = \frac{[1 - 2\left(\frac{2}{3}\right)^N + 3^{-N}][2^{-(N-n)} + 2^{-n}] + [\left(\frac{2}{3}\right)^N - 2^{-(N-1)} + 3^{-N}][2 - 2^{-N}]}{[1 - 2^{-(N-1)}][1 - 2^{-N}]}. \tag{SC.58}$$

Random Permutation Circuit with Arbitrary Initial States

One can repeat an analogous analysis also in the case of generic initial states, but now the number of invariant states increases, and apart from $\{|I^0\rangle, |I^x\rangle, |I^+\rangle, |I^-\rangle\}$ we need to introduce 11 additional states $\{|I^j\rangle\}$ labelled with $j = 1, \dots, 11$ [53],

$$|I^1\rangle = \sum_{a,b=0}^{q-1} |baaa\rangle, \qquad |I^2\rangle = \sum_{a,b=0}^{q-1} |abaa\rangle, \qquad |I^3\rangle = \sum_{a,b=0}^{q-1} |aaba\rangle, \qquad |I^4\rangle = \sum_{a,b=0}^{q-1} |aaab\rangle,$$
$$|I^5\rangle = \sum_{a,b,c=0}^{q-1} |bcaa\rangle, \qquad |I^6\rangle = \sum_{a,b,c=0}^{q-1} |baca\rangle, \qquad |I^7\rangle = \sum_{a,b,c=0}^{q-1} |baac\rangle,$$
$$|I^8\rangle = \sum_{a,b,c=0}^{q-1} |abca\rangle, \qquad |I^9\rangle = \sum_{a,b,c=0}^{q-1} |abac\rangle, \qquad |I^{10}\rangle = \sum_{a,b,c=0}^{q-1} |aabc\rangle,$$
$$|I^{11}\rangle = \sum_{a,b,c,d=0}^{q-1} |abcd\rangle. \tag{SC.59}$$

Note that for conciseness we will interchangeably use $|I^{12}\rangle = |I^0\rangle$, $|I^{13}\rangle = |I^x\rangle$, $|I^{14}\rangle = |I^-\rangle$, $|I^{15}\rangle = |I^+\rangle$. Using these one-site states we introduce a N-site basis of symmetric states

$$|G_{\underline{n}}\rangle := \frac{1}{N!}\sum_{\pi\in S_N}\pi\bigotimes_{j=1}^{15}|I^j\rangle^{\otimes n_j}\,, \qquad \underline{n}=(n_1,n_2,\ldots,n_{15}), \quad n_1+n_2+\cdots+n_{15}=15, \quad n_j\geq 0, \tag{SC.60}$$

and the overlap functions

$$G_{\underline{n}}(t) := \langle G_{\underline{n}}|\mathbb{E}[\rho(t)\otimes\rho(t)]\rangle\,. \tag{SC.61}$$

They satisfy a system of equations similar to (SC.47),

$$\frac{dG_{\underline{n}}(t)}{dt} = \frac{2\lambda}{L-1}\sum_{j=1}^{15}\sum_{k=j+1}^{15} n_j n_k \sum_{i=1}^{15}\alpha^i_{j,k}\left(G_{\underline{n}^i_{j,k}} - G_{\underline{n}}\right), \tag{SC.62}$$

where we introduced $\underline{n}^i_{j,k}$ to mean

$$\underline{n}^i_{j,k} = \begin{cases} (n_1,n_2,\ldots,n_j-1,\ldots,n_k-1,\ldots,n_i+2,\ldots,n_{15}), & i\neq j,k,\\ (n_1,n_2,\ldots,n_j+1,\ldots,n_k-1,\ldots,n_{15}), & i=j,\\ (n_1,n_2,\ldots,n_j-1,\ldots,n_k+1,\ldots,n_{15}), & i=k,\end{cases} \tag{SC.63}$$

and we use the convention $G_{\underline{n}}=0$ if $n'_j<0$ or $n'_j>N$ for some j'. The coefficients $\alpha^i_{j,k}$ are given as [53],

$$\alpha^i_{j,k} = \sum_{l=1}^{15} W_{i,l}\,\langle I^l|I^j\rangle\,\langle I^l|I^k\rangle\,, \qquad [W^{-1}]_{i,j} = \langle I^i|I^j\rangle^2\,. \tag{SC.64}$$

Even though compact, Eq. (SC.62) is not easy to study numerically due to a very high number of relevant states, however we can make some progress to study the stationary state. Let us focus on the $q=2$ case, as for $q\geq 3$ the stationary states in case of two-site coupling coincides with the stationary state under global dynamics [83–85].

To characterise the stationary state of Eq. (SC.62) we need to find all the integrals of motion, that is linear combinations of $G_{\underline{n}}$ which don't change for *any* initial state,

$$\mathcal{F} = \sum_{\underline{n}} d_{\underline{n}} G_{\underline{n}}, \qquad \frac{d\mathcal{F}(t)}{dt} = 0. \tag{SC.65}$$

We can very quickly find 15 of them by observing that the r.h.s. of (SC.62) has no terms with $j=k$, which gives

$$\mathcal{F}^{(k)}(t) = G_{\underline{n}^{(k)}}(t), \qquad \underline{n}^{(k)} = (\underbrace{0,0,\ldots,0}_{k-1},N,\underbrace{0,0,\ldots,0}_{15-k}), \qquad k=1,\ldots 15. \tag{SC.66}$$

This is a consequence of a stronger property of the time-evolution, namely that applying any two-site gate on a state $\langle I^k|^{\otimes N}$ from the right leaves the state invariant,

$$\langle I^k|^{\otimes N}\,\mathcal{U}_{i,j} = \langle I^k|^{\otimes N}\,. \tag{SC.67}$$

Similarly, we observe that the following holds for all $\underline{n}$ and j, k

$$\left(-2\,\langle I^0| + \langle I^x| + \langle I^+| + \langle I^-|\right)^{\otimes N}(\mathcal{U}_{j,k}-1)\,|G_{\underline{n}}\rangle = 0, \tag{SC.68}$$

which gives the 16-th conservation law,

$$\mathcal{F}^{(16)}(t) = \sum_{\underline{n}}\left(\prod_{j=1}^{11}\delta_{n_j,0}\right)\frac{(n_0+n_x+n_++n_-)!}{n_0!n_x!n_+!n_-!}(-2)^{n_0}G_{\underline{n}}(t). \tag{SC.69}$$

Note that $\mathcal{F}^{(16)}$ is analogous to $\mathcal{F}^{(5)}$ from (SC.56). While unable to prove that these are all the conservation laws for any system size N, analysis of the r.h.s. of Eq. (SC.62) for small N suggests the kernel of the map is 16-dimensional.

Similarly we find 16 independent stationary states, $\underline{G}^\infty = \{G^\infty_{\underline{n}}\}_{\underline{n}}$, given by

$$
\begin{aligned}
G^{\infty,1}_{\underline{n}} &= 2^{n_1+n_5+n_6+n_7+n_{11}}, & G^{\infty,2}_{\underline{n}} &= 2^{n_2+n_5+n_8+n_9+n_{11}},\\
G^{\infty,3}_{\underline{n}} &= 2^{n_3+n_6+n_8+n_{10}+n_{11}}, & G^{\infty,4}_{\underline{n}} &= 2^{n_4+n_7+n_9+n_{10}+n_{11}},\\
G^{\infty,5}_{\underline{n}} &= 2^{-n_3-n_4+n_5+n_{11}-n_{12}-n_{13}-n_{15}}, & G^{\infty,6}_{\underline{n}} &= 2^{-n_2-n_4+n_6+n_{11}-n_{12}-n_{14}-n_{15}},\\
G^{\infty,7}_{\underline{n}} &= 2^{-n_2-n_3+n_7+n_{11}-n_{12}-n_{13}-n_{14}}, & G^{\infty,8}_{\underline{n}} &= 2^{-n_1-n_4+n_8+n_{11}-n_{12}-n_{13}-n_{14}},\\
G^{\infty,9}_{\underline{n}} &= 2^{-n_1-n_3+n_9+n_{11}-n_{12}-n_{14}-n_{15}}, & G^{\infty,10}_{\underline{n}} &= 2^{-n_1-n_2+n_{10}+n_{11}-n_{12}-n_{13}-n_{15}},\\
G^{\infty,11}_{\underline{n}} &= 2^{n_5+n_6+n_7+n_8+n_9+n_{10}+2n_{11}-n_{12}}, & G^{\infty,12}_{\underline{n}} &= 1,\\
G^{\infty,13}_{\underline{n}} &= 2^{n_6+n_9+n_{11}+n_{13}}, & G^{\infty,14}_{\underline{n}} &= 2^{n_5+n_10+n_{11}+n_{14}},\\
G^{\infty,15}_{\underline{n}} &= 2^{n_7+n_8+n_{11}+n_{15}}, & G^{\infty,16}_{\underline{n}} &= 2^{-n_1-n_2-n_3-n_4+n_{11}-n_{12}}.
\end{aligned}
\tag{SC.70}
$$

Now we finally have everything needed to get a prediction for the stationary state under 2-qubit dynamics,

$$
G^\infty_{\underline{n}} = \sum_{j=1}^{16} c_j G^{\infty,j}_{\underline{n}},
\tag{SC.71}
$$

with the coefficients c_j determined by requiring $\mathcal{F}^{(j),\infty} = \mathcal{F}^{(j)}(0)$.

In the case of a random product initial state, the expression can be checked to reduce to the previous form. Assuming now that the initial state is not random, the stationary density depends on 4 real initial-state dependent real parameters, and three phases,

$$
\begin{aligned}
z(|\psi\rangle)e^{i\theta_1(|\psi\rangle)} &= \frac{1}{\sqrt{d}}\sum_s \langle s|\psi\rangle\,, \qquad y(|\psi\rangle)e^{i\theta_2(|\psi\rangle)} = \sum_s \langle s|\psi\rangle^2\,,\\
w(|\psi\rangle)e^{i\theta_3(|\psi\rangle)} &= \sum_s |\langle s|\psi\rangle|^2 \langle s|\psi\rangle\,, \qquad 0\le z(|\psi\rangle), y(|\psi\rangle), w(|\psi\rangle)\le 1,\\
Q(|\psi\rangle) &= \langle \mathcal{F}^{(16)}| \left(|\psi\rangle^* \otimes |\psi\rangle \otimes |\psi\rangle^* \otimes |\psi\rangle\right), \qquad Q\ge 0,
\end{aligned}
\tag{SC.72}
$$

and the stationary purity for an arbitrary non-random initial state is given by

$$
\begin{aligned}
P_x = \frac{1}{(1-\frac{2}{d})(1-\frac{1}{d})(1-\frac{6}{d}+\frac{2}{d^2})}\Big\{ &\left(d^{-\frac{n}{N}} + d^{-\frac{N-n}{N}}\right)\left(1-\frac{4}{d}\right)\left(1-\frac{2}{d}-\frac{2}{d^2}\right) - \frac{4}{d}\left(1-\frac{11}{2d}+\frac{8}{d^2}+\frac{1}{d^3}\right)\\
&+\left(1-d^{-\frac{n}{N}}\right)\left(1-d^{-\frac{N-n}{n}}\right)\Big[\frac{Q}{d}\left(1-\frac{4}{d}\right)\left(1-\frac{2}{d}\right) I_2 \left(1-\frac{6}{d}+\frac{14}{d^2}\right) - \frac{y^2}{d}\left(1-\frac{4}{d}+\frac{6}{d^2}\right)\\
&+\frac{4z^2}{d}\left(1-\frac{3z^2}{2}+\frac{2}{d}\right) + \frac{2yz^2}{d}\left(1+\frac{2}{d}\right)\cos(2\vartheta_1-\vartheta_2) - \frac{12}{d^{\frac{3}{2}}} wz\cos(\vartheta_1-\vartheta_3)\Big]\Big\},
\end{aligned}
\tag{SC.73}
$$

where we have used the short-hand notation $d = 2^N$. In the limit of large sizes the leading order contribution becomes for $x = n/N$

$$
P_x = d^{-x} + d^{-(1-x)} + (1 - d^{-x} - d^{-(1-x)})\left(I_2 + \frac{Q}{d}\right) + \mathcal{O}d^{-1}.
\tag{SC.74}
$$

For appropriate values of Q, this does not necessarily coincide with the global-permutation result (cf. (SD.20)). For example, for homogeneous non-random product initial states one gets

$$
\frac{Q}{d} - z^4 \ge 0, \qquad \frac{Q}{d} - z^4 = \mathcal{O}(1),
\tag{SC.75}
$$

and therefore we do not necessarily saturate the bound (6) even in the limit of large system sizes.

Appendix D: Global permutation ensembles

In this section, we compute the average Rényi-2 entropy in the ensemble $\mathcal{E}_{\rm GPP}$ (global phase transformations followed by global permutations) and — for a locally scrambled state — in the ensemble $\mathcal{E}_{\rm GP}$ (global permutations).

Namely, we compute the average Rényi-2 entropy of $U^{\rm RPP}|\psi_0\rangle$ and $U^{\rm RP}|\psi_0\rangle$, where $U^{\rm RPP}$, $U^{\rm RP}$ are $d\times d$ matrices drawn from the two ensembles. We then show that for the ensemble $\mathcal{E}_{\rm GPP}$ the result coincides with that obtained in the ensemble of infinite-depth circuits $\mathcal{E}_{\rm PPC}(D\to\infty)$ at any system size, while for the ensemble $\mathcal{E}_{\rm GP}$ the average Rényi-2 entropy coincides with that in $\mathcal{E}_{\rm PC}(D\to\infty)$ only as $N\to\infty$.

1. Page curves

To begin with, we consider a tripartite Hilbert space $\mathcal{H}=\mathcal{H}_-\otimes\mathcal{H}_+\otimes\mathcal{H}_0$ as in (SC.21), associated to three regions A_-, A_+, A_0 with $|A_-|=n_-$, $|A_+|=n_+$, $|A_0|=n_0$. The dimension of the Hilbert space is:

$$d=d_-\,d_+\,d_0=q^{n_-}q^{n_+}q^{n_0}=q^N. \tag{SD.1}$$

For $\alpha=-,+,0$, we let $|I^\alpha\rangle=\otimes_{j\in A_\alpha}|I^\alpha\rangle_j$ and we consider the overlaps $\langle I^\alpha|\rho_0\otimes\rho_0\rangle$ for a pure state $\rho_0=|\psi_0\rangle\langle\psi_0|$. These can be obtained from the expression (SC.22),

$$\begin{aligned}
G_{N,0,0}(|\psi_0\rangle)&=\langle I^-|\rho_0\otimes\rho_0\rangle=\mathrm{Tr}_{\mathcal{H}}(\rho_0)^2=1,\\
G_{0,N,0}(|\psi_0\rangle)&=\langle I^+|\rho_0\otimes\rho_0\rangle=(\mathrm{Tr}_{\mathcal{H}}\,\rho_0)^2=1,\\
G_{0,0,N}(|\psi_0\rangle)&=\langle I^0|\rho_0\otimes\rho_0\rangle=\sum_{\{x_i\}}|\langle\{x_i\}|\psi_0\rangle|^4=I_2(|\psi_0\rangle).
\end{aligned} \tag{SD.2}$$

The above expressions are valid for a generic initial state $|\psi_0\rangle$. From now on, we specialize to the case of permutation-invariant states, namely:

1. Product states of the form (SC.13), for which $I_2(|\psi_0\rangle)=\left(\sum_{a=0}^{q-1}|\lambda_a|^4\right)^N$;
2. The Dicke states $|D\rangle$, eq. (SC.20), for which $I_2(|D\rangle)=\sum_{m=1}^N|c_m|^4\binom{N}{m}^{-1}$;
3. The GHZ state $|\mathrm{GHZ}\rangle$, for which $I_2(|\mathrm{GHZ}\rangle)=q^{-1}$.

The states above are chosen as they provide three different scaling laws for the Inverse Participation Ratio, with a localization which is exponential, polynomial, and constant in the system size, respectively.

Let us consider the ensemble $\mathcal{E}_{\rm GPP}$. The average purity of the state $U^{\rm RPP}|\psi_0\rangle$ is obtained by computing the quantities:

$$\mathcal{G}_{n_-,n_+,n_0}(|\psi_0\rangle):=\langle G_{n_-,n_+,n_0}|\mathcal{U}^{\rm RPP}|\rho_0\otimes\rho_0\rangle\,, \tag{SD.3}$$

where

$$\begin{aligned}
\mathcal{U}^{\rm RPP}&:=\mathbb{E}\left[(U^{\rm RPP})^*\otimes U^{\rm RPP}\otimes(U^{\rm RPP})^*\otimes U^{\rm RPP}\right]\\
&=\frac{1}{d(d-1)}\left[|I^-\rangle\langle I^-|+|I^+\rangle\langle I^+|+(d+1)\,|I^0\rangle\langle I^0|-|I^-\rangle\langle I^0|-|I^0\rangle\langle I^-|-|I^+\rangle\langle I^0|-|I^0\rangle\langle I^+|\right],
\end{aligned} \tag{SD.4}$$

is the ensemble average.

From the expression (SC.9), we obtain

$$\begin{aligned}
&\mathcal{G}_{n_-,n_+,n_0}(|\psi_0\rangle)=\frac{1}{N!}\sum_{\pi\in S_N}\bigotimes_{j\in A_\pi^-}\langle I^-|_j\bigotimes_{j\in A_\pi^+}\langle I^+|_j\bigotimes_{j\in A_\pi^0}\langle I^0|_j\\
&\frac{1}{d(d-1)}\left[|I^-\rangle\langle I^-|+|I^+\rangle\langle I^+|+(d+1)\,|I^0\rangle\langle I^0|-|I^-\rangle\langle I^0|-|I^0\rangle\langle I^-|-|I^+\rangle\langle I^0|-|I^0\rangle\langle I^+|\right]|\rho_0\otimes\rho_0\rangle\\
=&\frac{1}{d-1}\left[(d_-+d_+)(1-I_2(|\psi_0\rangle))+(d+1)I_2(|\psi_0\rangle)-2\right].
\end{aligned} \tag{SD.5}$$

Notice that $\mathcal{G}_{n,N-n_0-n,n_0}=\mathcal{G}_{N-n_0-n,n,n_0}$, $\forall n_0=0,\dots,N-n$, which entails the known symmetry of the purity $(n_0=0)$ $\mathcal{P}_n=\mathcal{P}_{N-n}$.

We note that the expression (SD.5) is surprisingly general: for *every* permutation-invariant initial state $|\psi_0\rangle$ the function $\mathcal{G}_{n_-,n_+,n_0}(|\psi_0\rangle)$ depends only on the (sub)system sizes N, n, on the Hilbert-space dimension d, and on

$I_2(|\psi_0\rangle)$, the latter being the only state-dependent quantity. Letting $x = \frac{n_-}{N}$, $1-x = \frac{n_+}{N}$, at large N the average purity at leading order in N can be read out from the above expression:

$$\mathcal{P}_x^{\mathrm{GPP}} \simeq q^{-xN} + q^{(x-1)N} + I_2(|\psi\rangle)[1 - q^{-xN} - q^{(x-1)N}], \tag{SD.6}$$

which implies

$$-\frac{1}{N}\log \mathcal{P}_x^{\mathrm{GPP}} = \log(q)\min\left\{x, 1-x, -\frac{\log I_2(|\psi\rangle)}{N\log q}\right\} + O\left(\frac{1}{N}\right). \tag{SD.7}$$

From the above expression we can define a condition of *high localization*: if $\exists\, x < 1/2$ such that the inequality

$$-\frac{\log I_2(|\psi\rangle)}{N\log q} < x\,, \tag{SD.8}$$

is verified, then maximal scrambling is prevented in the state $|\psi\rangle$, meaning that the Page curve does not reproduce that of the Haar random ensemble at large system size. This condition is always satisfied for the product state with non-maximal tilting ($\theta \neq \pi/4$ for $q = 2$), for the GHZ state and for the Dicke states.

2. Proof of equivalence at late times

Here, we show that the functions $\mathcal{G}_{n_-,n_+,n_0}[|\psi_0\rangle]$ obtained in (SD.5) solve the steady-state form of the system (SC.7), which is obtained from the latter by removing the time derivative term. We do so by first observing that d and $I_2(|\psi_0\rangle)$ is invariant under the changes of n_-, n_+, n_0 that preserve their sum. Therefore we have

$$\begin{aligned}
&\mathcal{G}_{n_--1,n_++1,n_0} + \mathcal{G}_{n_-+1,n_+-1,n_0} + (q-1)\mathcal{G}_{n_--1,n_+-1,n_0+2}\\
=&\frac{q^{-1}\,d_-(1-\mathrm{I}_2) + q\,d_+(1-\mathrm{I}_2) + [(d+1)\mathrm{I}_2 - 2]}{d-1} + \frac{q\,d_-(1-\mathrm{I}_2) + q^{-1}\,d_+(1-\mathrm{I}_2) + [(d+1)\mathrm{I}_2 - 2]}{d-1}\\
+&(q-1)\frac{q^{-1}\,d_-(1-\mathrm{I}_2) + q^{-1}\,d_+(1-\mathrm{I}_2) + [(d+1)\mathrm{I}_2 - 2]}{d-1}\\
=&(q+1)\mathcal{G}_{n_-,n_+,n_0}\,,
\end{aligned} \tag{SD.9}$$

and analogously

$$\mathcal{G}_{n_-+1,n_+,n_0-1} + q\,\mathcal{G}_{n_--1,n_+,n_0+1} = \mathcal{G}_{n_-,n_++1,n_0-1} + q\,\mathcal{G}_{n_-,n_+-1,n_0+1} = (q+1)\mathcal{G}_{n_-,n_+,n_0}.$$

A simple inspection of equation (SC.7) then reveals that

$$G_{n_-,n_+,n_0}(t\to\infty) = \mathcal{G}_{n_-,n_+,n_0}(|\psi_0\rangle) \tag{SD.10}$$

is a steady state solution for every choice of the permutation-invariant initial state $|\psi_0\rangle$.

3. Page curves for the $\mathcal{E}_{GP}$

Finally, let us consider the averaged purity in the ensemble $\mathcal{E}_{GP}$, i.e. the averaged purity after acting with an all-to-all permutation gate on an initial state. As before, we use d to denote the full Hilbert-space dimension $d = q^N$, and we are interested in a bipartition of the system into subsystems with n and $N-n$ sites. Similarly to before, the central object is the averaged 4-replica gate $\mathcal{U}^{\mathrm{RP}}$, defined as

$$\mathcal{U}^{\mathrm{RP}} := \mathbb{E}\left[(U^{\mathrm{RP}})^* \otimes U^{\mathrm{RP}} \otimes (U^{\mathrm{RP}})^* \otimes U^{\mathrm{RP}}\right]. \tag{SD.11}$$

The matrix $\mathcal{U}^{\mathrm{RP}}$ is again a projector to the set of invariant states, but the number of invariant states is now 15. Apart from $|I^\alpha\rangle$ for $\alpha \in \{0,+,-,x\}$, we also have 11 additional states $|I^j\rangle$, $j = 1,\ldots,11$, as introduced in Eq. (SC.59). Using now the fact that $\mathcal{U}^{\mathrm{RP}}$ is a projector to the space spanned by $|I^\alpha\rangle^{\otimes N}$ for $\alpha \in \{1,2,\ldots,11,0,x,+,-\}$, we can express the averaged purity as

$$\mathcal{P}_n^{\mathrm{GP}} = \sum_{\alpha,\beta\in\{1,2,\ldots,11,0,x,+,-\}} W_{\alpha,\beta}(N)\,\langle I^+|I^\alpha\rangle^n\,\langle I^-|I^\alpha\rangle^{N-n}\left(\langle I^\beta|^{\otimes N}\right)|\rho_0\otimes\rho_0\rangle\,, \tag{SD.12}$$

where $W(N)$ is the inverse of the matrix of overlaps

$$\left[W^{-1}(N)\right]_{\alpha,\beta} = \langle I^{\alpha}|I^{\beta}\rangle^{N}. \tag{SD.13}$$

All the overlaps between states $|I^{\alpha}\rangle$ and the initial state are given in terms of the amplitudes and phases given in Eq. (SC.72), as well as the inverse participation ratio $I_2(|\psi\rangle)$ (cf. (3)). Explicitly, the overlaps are given as

$$\begin{aligned}
&\langle I^0|^{\otimes N}|\rho_0\otimes\rho_0\rangle = I_2(|\psi\rangle), \qquad \langle I^x|^{\otimes N}|\rho_0\otimes\rho_0\rangle = y(|\psi\rangle)^2, \qquad \langle I^+|^{\otimes N}|\rho_0\otimes\rho_0\rangle = \langle I^-|^{\otimes N}|\rho_0\otimes\rho_0\rangle = 1,\\
&\langle I^1|^{\otimes N}|\rho_0\otimes\rho_0\rangle = \langle I^2|^{\otimes N}|\rho_0\otimes\rho_0\rangle^* = \langle I^3|^{\otimes N}|\rho_0\otimes\rho_0\rangle = \langle I^4|^{\otimes N}|\rho_0\otimes\rho_0\rangle^* = \sqrt{d}z(|\psi\rangle)w(|\psi\rangle)e^{i[\theta_3(|\psi\rangle)-\theta_1(|\psi\rangle)]},\\
&\langle I^5|^{\otimes N}|\rho_0\otimes\rho_0\rangle = \langle I^7|^{\otimes N}|\rho_0\otimes\rho_0\rangle = \langle I^8|^{\otimes N}|\rho_0\otimes\rho_0\rangle = \langle I^{10}|^{\otimes N}|\rho_0\otimes\rho_0\rangle = dz(|\psi\rangle)^2,\\
&\langle I^6|^{\otimes N}|\rho_0\otimes\rho_0\rangle = \langle I^9|^{\otimes N}|\rho_0\otimes\rho_0\rangle^* = dz(|\psi\rangle)^2y(|\psi\rangle)e^{i[\vartheta_2(|\psi\rangle)-2\vartheta_1(|\psi\rangle)]},\\
&\langle I^{11}|^{\otimes N}|\rho_0\otimes\rho_0\rangle = d^2z(|\psi\rangle)^4.
\end{aligned} \tag{SD.14}$$

Inserting this into the equation for averaged purity gives the following general expression,

$$\begin{aligned}
\mathcal{P}_n = {} & \frac{\left(1-\frac{4}{d}\right)\left(q^{-(N-n)}+q^{-n}\right)-\frac{2}{d}\left(1-\frac{5}{d}\right)}{\left(1-\frac{3}{d}\right)\left(1-\frac{2}{d}\right)} + \frac{\left(1-q^{-(N-n)}\right)\left(1-q^{-n}\right)}{\left(1-\frac{1}{d}\right)\left(1-\frac{2}{d}\right)\left(1-\frac{3}{d}\right)}\Bigg[\left(1-\frac{5}{d}\right)I_2(|\psi\rangle)+\frac{1}{d^2}y(|\psi\rangle)\\
&+8d^{-\frac{3}{2}}w(|\psi\rangle)z(|\psi\rangle)\cos\left[\vartheta_1(|\psi\rangle)-\vartheta_3(|\psi\rangle)\right]-z(|\psi\rangle)^2\frac{2}{d}\left(2+y(|\psi\rangle)\cos\left[2\vartheta_1(|\psi\rangle)-\vartheta_2(|\psi\rangle)\right]\right)+z(|\psi\rangle)^4\Bigg].
\end{aligned} \tag{SD.15}$$

Note that this expression is completely general and holds for any initial state. For convenience let us also explicitly consider an average over Haar-random initial product states, which is obtained from the general form (SD.12) upon replacing the initial-state overlap by an averaged initial state overlap,

$$\left(\langle I^{\beta}|^{\otimes N}\right)|\rho_0\otimes\rho_0\rangle \mapsto \left(\langle I^{\beta}|^{\otimes N}\right)\mathbb{E}[|\rho_0\otimes\rho_0\rangle], \tag{SD.16}$$

and these can be straightforwardly obtained by noting

$$\mathbb{E}[|\rho_0\otimes\rho_0\rangle] = \bigotimes_{j=1}^{N}\frac{1}{q(q+1)}\left(|I^+\rangle+|I^-\rangle\right). \tag{SD.17}$$

Explicitly evaluating the overlaps and plugging it into Eq. (SD.12) gives (after some straightforward manipulations) the general form

$$\mathbb{E}[\mathcal{P}_n] = \frac{(q^{-n}+q^{-(N-n)})(1-\frac{5}{d}+\frac{8}{d^2})+\left((1-q^{-n})(1-q^{-(N-n)})\mathbb{E}[I_2]-\frac{2}{d}\right)(1-\frac{4}{d}+\frac{7}{d^2})}{(1-\frac{1}{d})(1-\frac{2}{d})(1-\frac{3}{d})}, \tag{SD.18}$$

where we have used $\mathbb{E}[I_2]$ to denote the value of I_2 averaged over the random product initial state,

$$\mathbb{E}[I_2] = \left(\frac{2}{q+1}\right)^{N}. \tag{SD.19}$$

Note that specialising to $q=2$ this gives the expression in Eq. (15).

In the limit of large N with $n/N=x$ fixed, the two expressions reduce to

$$\mathcal{P}_x = q^{-xN}+q^{-(1-x)N}+\left(1-q^{-xN}-q^{-(1-x)N}\right)\left[I_2(|\Psi\rangle)+z(|\psi\rangle)^4\right]+\mathcal{O}(q^{-N}), \tag{SD.20}$$

and

$$\mathbb{E}[\mathcal{P}_x] = q^{-xN}+q^{-(1-x)N}+\left(1-q^{-xN}-q^{-(1-x)N}\right)\mathbb{E}[I_2]+\mathcal{O}(q^{-N}). \tag{SD.21}$$

Appendix E: Bosonic-space formulation

An alternative way of solving the dynamics of the circuit is to map the problem to a bosonic Hilbert space by taking advantage of the (site) permutation-invariance of the dynamics and the initial state. Following Ref. [59], we define site permutation-invariant states of the form

$$|n_-, n_+, n_x, n_0\rangle := \frac{1}{\sqrt{N!\, n_-!\, n_+!\, n_x!\, n_0!}} \sum_{\pi \in \mathcal{S}_N} \pi \, |J^-\rangle^{\otimes n_-} \otimes |J^+\rangle^{\otimes n_+} \otimes |J^x\rangle^{\otimes n_x} \otimes |J^0\rangle^{\otimes n_0} \tag{SE.1}$$

for every $n_-, n_+, n_x, n_0 \geq 0$ such that they sum to N. In the above formula, $\mathcal{S}_N$ is the permutation group over N elements and π is the unitary operator corresponding to an element of $\mathcal{S}_N$. The states (SE.1) are (site) permutation-invariant and form an orthonormal basis by construction. As a result, they can be rewritten in terms of bosonic creation and annihilation operators acting on a vacuum $|\Omega\rangle$

$$|n_-, n_+, n_x, n_0\rangle := \frac{1}{\sqrt{n_-!\, n_+!\, n_x!\, n_0!}} \left(a_-^\dagger\right)^{n_-} \left(a_+^\dagger\right)^{n_+} \left(a_x^\dagger\right)^{n_x} \left(a_0^\dagger\right)^{n_0} |\Omega\rangle \;, \tag{SE.2}$$

which satisfy the usual canonical commutation relations

$$[a_\alpha, a_\beta^\dagger] = \delta_{\alpha,\beta} \,, \qquad [a_\alpha, a_\beta] = [a_\alpha^\dagger, a_\beta^\dagger] = 0 \,. \tag{SE.3}$$

The Lindbladian (SC.39) can be expressed in terms of these operators by using [59]

$$\sum_{j=1}^{N} (|x\rangle\langle y|)_j = a_x^\dagger a_y \tag{SE.4}$$

and

$$\sum_{j<k} \sum_{x,y,z,t} \left[\Gamma^{x,y} \, |xy\rangle_{j,k}\right] \left[\Lambda^{z,t}{}_{j,k}\langle zt|\right] = \frac{1}{2} \sum_{x,y,z,t} \Gamma^{x,y} \Lambda^{z,t} \left[a_x^\dagger a_y^\dagger a_z a_t\right] \tag{SE.5}$$

for every pair of symmetric matrices Γ, Λ. Here, x, y, z, t run over the different "species" of bosons, $-, +, x$ and 0. Now the operators appearing in the above formula can be simply matched to those of Eq. (SC.39), allowing us to express the averaged gate in terms of the global bosonic operators:

$$\begin{aligned}
&\sum_{1\leq i<j\leq N} \tilde{\mathcal{U}}_{i,j} = \sum_{1\leq i<j\leq N} \Pi_i \Pi_j \, \mathcal{U}_{i,j} \Pi_i \Pi_j = \\
&\frac{1}{2} n_0(n_0-1) + \frac{1}{6}\Big[2n_-(n_- - 1) + 2n_+(n_+ - 1) + 2n_x(n_x - 1) + 4n_0 n_- + 4n_0 n_+ + 4n_0 n_x \\
&+ 2a_-^\dagger n_- a_0 + 2a_0^\dagger n_- a_- + 2a_+^\dagger n_+ a_0 + 2a_0^\dagger n_+ a_+ + 2a_x^\dagger n_x a_0 + 2a_0^\dagger n_x a_x\Big] \\
&+ \frac{1}{3}\Big[n_- n_+ + n_- n_x + n_+ n_x + a_+^\dagger n_- a_x + a_+^\dagger n_+ a_x + a_x^\dagger n_- a_x + a_-^\dagger n_x a_+ + a_x^\dagger n_+ a_- + a_+^\dagger n_x a_-\Big] \,,
\end{aligned} \tag{SE.6}$$

where we have introduced the "particle number" operators

$$n_- = a_-^\dagger a_-, \qquad n_+ = a_+^\dagger a_+, \qquad n_x = a_x^\dagger a_x, \qquad n_0 = a_0^\dagger a_0 \,. \tag{SE.7}$$

The operator (SE.6) can be represented by a finite (sparse) matrix in the basis (SE.1), allowing for simple, numericaly exact simulation of the dynamics starting from typical initial states given by (SC.49). These states are easily represented in the bosonic basis and take the form

$$|\mathbb{E}[\rho_0 \otimes \rho_0]\rangle = \frac{1}{6^N} \sum_{n_+, n_-, n_0} 2^{N/2+n_0} \sqrt{\frac{N!}{n_-! n_+! n_0!}} \, |n_-, n_+, 0, n_0\rangle \;, \tag{SE.8}$$

where the sum is over non-negative integers n_-, n_+, n_0 such that their sum is N. Similar computation yields the expression for the symmetrized boundary (swap) state

$$\begin{aligned} |X_n\rangle :=& \frac{1}{N!}\sum_{\pi\in\mathcal{S}_N} \pi\, |I^-\rangle^{\otimes n} \otimes |I^+\rangle^{\otimes N-n} \\ =& \sum_{n_+=0}^{N-n}\sum_{n_-=0}^{n} 2^{N/2}\frac{n!(N-n)!}{(n-n_-)!\,(N-n-n_+)!}\sqrt{\frac{(N-n_--n_+)!}{N!n_-!n_+!}}\,|n_-,n_+,0,N-n_+-n_-\rangle\ . \end{aligned} \tag{SE.9}$$

One can easily check that $\langle X_n|\rho_0\otimes\rho_0\rangle = 1$.

Appendix F: Numerical results

We now provide some details about the numerical implementation of the dynamics as well as some additional results.

Solving the dynamics of the Rényi entropies for finite system sizes amounts to solving the differential equations (14) together with appropriate initial conditons corresponding to typical initial states (SC.51). Eq. (14) corresponds to a set of coupled linear ordinary differential equations. As such, the system can be straightforwardly solved via well-known methods. Importantly, however, when the system size N grows above a certain value (in our calculations, this was $N \gtrsim 32$) numerical instabilities appear and the results (unphysically) diverge.

A way around this is given by the bosonic formulation of the problem detailed in App. E. Mappig the dynamics to the bosonic Hilbert space allows for exact (numerical) vector representation of the system states and the Lindbladian encoding the dynamics. As the Lindbladian has only positive eigenvalues, unphysical divergences are not a problem and it is possible to solve much larger system sizes. In our calculations, we managed to go up to $N = 168$. In this approach, the most computationally demanding part is the exponentialization of the Lindbladian matrix. One can, however, take advantage of the sparse nature of $\mathcal{L}$ and solve the differential equation corresponding to (SA.16) via simple numerical methods, such as Euler or Runge-Kutta type methods. For random permutation circuits starting from typical initial states, all of our results are obtained via forward Euler with 12 digit precision, as apparent from comparison with the 4th order Runge-Kutta solutions.

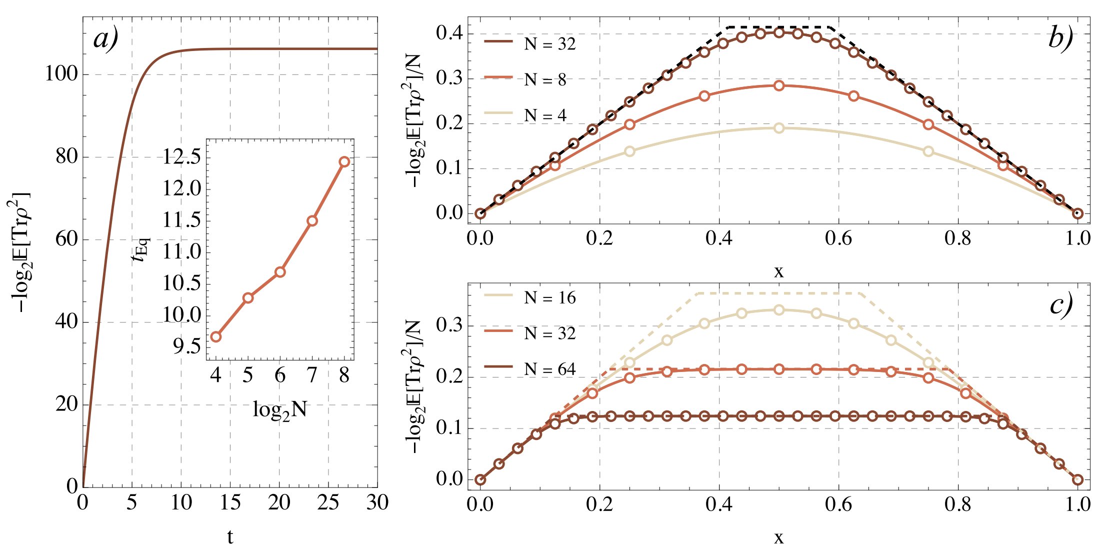

FIG. 1. Rényi-2 entropy dynamics in random permutation circuits with additional random phases. *a)* The time evolution of the averaged Rényi-2 entropy from the initial product state $|\psi\rangle = |\phi(\theta)\rangle^{\otimes N}$, with $|\phi(\theta)\rangle = \cos\theta|0\rangle + \sin(\theta)|1\rangle$ and $\theta = \pi/8$. The system size is set to $N = 256$. *Inset:* the equilibration time as a function of the logarithm of the system size N. *b)* Comparison of the Page curves obtained by averaging over the random circuit (open markers) and the random global (solid lines) ensemble starting from the product state corresponding to $\theta = \pi/8$. The dashed lines show the (tight) upper bounds. *c)* Same as *b)*, starting from the initial state $|D\rangle = (|D_1\rangle + |D_2\rangle)/\sqrt{2}$. It is clear that the global and circuit results coincide for finite N, as opposed to the case with no random phases.

For different initial states, such as product states $|\psi\rangle = |\phi(\theta)\rangle^{\otimes N}$ with $|\phi(\theta)\rangle = \cos\theta|0\rangle + \sin(\theta)|1\rangle$ and fixed θ, the corresponding differential equations are generally very complicated and is restricted to using numerical exact diagonalisation (ED) techniques, which are limited to small system sizes $N \lesssim 10$. In particular, we simulate the random dynamics by sampling different random realizations of the circuits; as such, the finite sample size of the gates result in some numerical error. All numerical methods detailed in the paper were crosschecked by ED for the available system sizes, together with the analytical global ensemble results.

We end this section by presenting an example of data generated by studying the dynamics of random-permutation circuits incorporating additional random phases, as discussed in the main text. The random phases simplify the problem, and one can derive the differential equations (SC.7), independently of the initial state of the system. As a result, the Rényi entropy dynamics of the RPP circuit can be studied for both product and entangled initial states, see Fig. 1. Moreover, integrating the differential equations did not give rise to numerical instabilities for the system sizes studied, and all results presented here are obtained via solving (SC.7) (up to $N = 256$). From our plots, we see the results anticipated in the main text. Notably, we see from Fig. 1 that the Page curves corresponding to the late-time circuit ensemble and the global ensemble coincide for all finite-N, and that they are both bounded by the PEs.