Optimal regularity for variational solutions of free transmission problems



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# OPTIMAL REGULARITY FOR VARIATIONAL SOLUTIONS OF FREE TRANSMISSION PROBLEMS.

DIEGO MOREIRA* AND HARISH SHRIVASTAVA†

ABSTRACT. In this article we study functionals of the type considered in [43], i.e.

$$J(v) := \int_{B_1} A(x,u)|\nabla u|^2 + f(x,u)u + Q(x)\lambda(u)\,dx$$

here $A(x,u) = A_+(x)\chi_{\{u>0\}} + A_-(x)\chi_{\{u<0\}}$, $f(x,u) = f_+(x)\chi_{\{u>0\}} + f_-(x)\chi_{\{u<0\}}$ and $\lambda(x,u) = \lambda_+(x)\chi_{\{u>0\}} + \lambda_-(x)\chi_{\{u\leq 0\}}$. We prove the optimal $C^{0,1^-}$ regularity of minimizers of the functional indicated above (with precise Hölder estimates) when the coefficients $A_\pm$ are continuous functions and $\mu \leq A_\pm \leq \frac{1}{\mu}$ for some $0 < \mu < 1$, with $f \in L^N(B_1)$ and $Q$ bounded. We do this by presenting a new compactness argument and approximation theory similar to the one developed by L. Caffarelli in [10] to treat the regularity theory for solutions to fully nonlinear PDEs. Moreover, we introduce the $\mathcal{T}_{a,b}$ operator that allows one to transfer minimizers from the transmission problems to the Alt-Caffarelli-Friedman type functionals, in small scales, allowing this way the study of the regularity theory of minimizers of Bernoulli type free transmission problems.

**Keywords:** variational calculus, transmission problems, free boundary, optimal regularity.
**2010 Mathematics Subjects Classification:** 35B65, 35Q35, 35R35, 35J20.

CONTENTS



## 1. INTRODUCTION

As nicely pointed out in [12], M. Picone introduced transmission problems in theory of elasticity in 1954 ([39]) and the theory was further developed by Lions [34], Stampaccia [44] and Campanato [13]. The non-divergence case was treated by Schechter in 1960 ([41]). For further details, we refer to [12], from where we learn about the history of those developments.

Mathematical models involving composite materials have gained significant attention in recent years as we can see, for instance, from the works of [12], [19], [23], [15], [40], [47] (see also a slightly older and interesting article [14]). It is a challenging task to model the electromagnetic or heat conduction properties of certain materials, particularly when they are close to threshold points which lead to abrupt changes in the nature of the material. The same issue arises when one tries to model properties of composite mixtures. Applications and results that deal with such class of models can be found in [7], [1] and others. Interesting discussions related to those models in the applied sciences are also present in [6], [45], [5].

In a very rough way, the models discussed above can be represented by the PDE

$$\operatorname{div} A(x,u,\nabla u) = f(x,u) \tag{1.1}$$

where $A(x,u,\nabla u)$ and $f(x,u)$ have jump discontinuities with respect to the variable $u$. These jump discontinuities model the change in the diffusion coefficients as soon as $u$ crosses a threshold value or else, the phase of the material changes. Few examples of phenomenas that can be modelled through transmission problems are mixture of different chemicals, conductivity (electric and thermal) of composite materials or a material operating close to thresholds like triple point, melting point or breakdown potential.

Transmission problems can also be posed in variational formulations where configurations of least energy are of interest. This means that one is led to study minimizers of functionals of the following type

$$\int_{\Omega} H(x,u,\nabla u)\,dx \tag{1.2}$$

where $H(x,u,\nabla u)$ has a jump discontinuity with respect to its variables. Some results related to the above mentioned variational formulation (1.2) can be found in [43], [4]. Moreover, the form (1.1) is also studied in [29], [28].

In this work, we study the optimal regularity result for variational solutions to transmission problems of Bernoulli type. In fact, in our context here, we have a free transmission problem in the sense that there is no a priori knowledge on position, geometry or regularity of the separating interface. More precisely, we focus on (local) minimizers of functionals of the form

$$J_{A,f,Q}(v;B_1) = \int_{B_1} A(x,v)|\nabla v|^2 + f(x,v)v + Q(x)\lambda(v)\,dx \tag{1.3}$$

where for $v \in H^1(B_1)$ we set

$$\begin{aligned} A(x,v) &:= A_+(x)\chi_{\{v>0\}} + A_-(x)\chi_{\{v<0\}},\\ f(x,v) &:= f_+(x)\chi_{\{v>0\}} + f_-(x)\chi_{\{v<0\}},\\ \lambda(v) &:= \lambda_+\chi_{\{v>0\}} + \lambda_-\chi_{\{v\le 0\}}. \end{aligned}$$

In a way, our transmission problem is a inhomogeneous version of the classical two phase free boundary problem studied by H. Alt, L. Caffarelli and A. Friedman in 1984 (c.f. [3]) where now the leading gradient term on the functional has jump discontinuities across the free boundary and also a inhomogeneity term (which is not present in [3]). In our context here, the following assumptions are enforced throughout the paper

- **(G1)** (**Continuity**) $A_\pm$ are continuous in $B_1$, the open unit ball in $\mathbb{R}^N$.
- **(G2)** $f_\pm \in L^N(B_1)$.
- **(G3)** (**Ellipticity**) There exists $\mu \in (0,1)$ such that
$$\text{for every } x \in B_1 \text{ we have } \mu \le A_\pm(x) \le \tfrac{1}{\mu}.$$
- **(G4)** $\Lambda := \lambda_+ - \lambda_- > 0$.
- **(G5)** $0 \le Q(x) \le q_2 < \infty$ for every $x \in B_1$. Without loss of generality, we assume $q_2 \ge 1$.

Here, unlike [3] (also [29] for the non-variational setting), local minimizers are not locally Lipschitz continuous. In fact, we show here that they are $C^{0,1^-}_{loc}$ (with precise estimates) and this is the optimal regularity. As a matter of fact, this optimal regularity is already manifested at the PDE level as it is proven in [27] (see Remark 2.5 below).

Our strategy in this paper can be divided in two macro parts. In the first one, we obtain the optimal growth rate for local minimizers in balls centered on the zero level set. In the second part, we prove the optimal regularity of minimizers by using the Euler-Lagrange equations that are satisfied in both phases. We then match those regularities by using Harnack type arguments to obtain the final optimal regularity (see for instance, Proposition 7.4).

In order to implement the second part pointed out above, we also need analogous estimates for weak solutions of PDEs of the form (4.1) (c.f. Theorem 2.4). We were unable to find such results in the literature showing how precisely the constants depend on the data of the problem as well as results that hold in any dimension $N \ge 2$. This is done in Section 4 of the paper.

For the optimal growth rate of minimizers along the zero level set, we implement an approximation theory similar to the one developed by L. Caffarelli in the seminal paper [10] to treat regularity theory for solutions to fully non-linear PDEs. There are essentially two main steps. The first one is to prove the proximity of local minimizers to regular profiles by compactness argument (c.f Theorem 6.3 and Proposition 6.4). In our case, the regular profiles inherit regularity from (local) minimizers of functionals of Alt-Caffarelli-Friedman type (as in [3]) via $\mathcal{T}_{a,b}$ operator discussed in the sequel. The second one is to reduce the problem to a so called "small regime configuration" (c.f. Proposition 7.2) via the scaling invariance of the problem under the appropriate regularity assumptions of the data.

We observe that compactness here appears to be subtler than the analogous situation for viscosity solutions in the theory PDEs, as stability properties of minimizers (by which we mean minimality property being kept under limits) are in general more delicate to assure. Moreover, in the context of nonlinear variational Bernoulli free boundary type problems, local Lipschitz regularity (unavailable in our case here) appears to be an effective auxiliary instrument to prove stability (at least in the level of blow-up of minimizers) as one can see in the works of D. Danielli and A. Petrosyan (Lemma 5.3 in [16]), S. Martinez and N. Wolanski (Lemma 7.2 in [36]) and H. Alt, L. Caffarelli, A. Friedman themselves (Lemma 3.3 in [3]).

Here, we follow a different route inspired on the ideas due to H. Alt and L. Caffarelli present in [2, Lemma 5.4] (for blow-ups of minimizers) that account for careful perturbations of minimizers to create suitable admissible competitors for energy comparison. Furthermore, these ideas (in [2]) also account for smart cancelations driven by the weak convergence, showing somehow a dependence on the quadratic structure of the functional.

In our case, although we do not deal with blow-ups (of minimizers), we still need a careful construction in the spirit of the proof of [2, Lemma 5.4] in order to produce suitable competitors for minimizers. The construction is more involved since our problem is a nonlinear one (it is a transmission problem instead of a pure free boundary one of Bernoulli type). Moreover, the competitors need to be constructed in such a way that the $L^1_{loc}$ convergence of positivity sets holds, recovering this way some kind of upper semi-continuity of the phase transition part of the functional, namely, $\int Q(x)\lambda(v)\,dx$ [1]. We close the compactness argument using the Widman's hole filling technique (Proposition 6.4).

Finally, we highlight that to handle the transmission problem *per se*, we introduce a new tool, namely, the $\mathcal{T}_{a,b}$ operator. The idea is that this operator "bridges" minimizers of transmission functionals of the form (2.3) with minimizers of functionals of Alt-Caffarelli-Friedman type in [3] (c.f Lemma 3.10). This permits a perturbation theory to be carried out (like in [10]) by passing from one functional to the other in small scales, recovering this way the regularity theory for the original minimizers. We choose to present the results pertaining to the $\mathcal{T}_{a,b}$ operator in Section 3 for general value of $p \geq 1$, i.e, transmission functionals involving $p-$energy, as it can be of independent interest and of use in other transmission problems. At this point, we do not know how to implement the compactness argument for the case where $p \in [1, \infty)$ and $p \neq 2$ since the cancellation effect yielded by the weak convergence (see for instance, the proof of **(EIV)** in Theorem 6.3) seems to be delicate to reproduce in the case $p \neq 2$.

Our results extend the ones in [4] and [43] in the particular case where matrix coefficients $A_\pm$ are variable multiples of the identity matrix. Indeed, in those papers the authors prove that minimizers become more and more regular as the "jump" distance between the phase coefficients $A_+$ and $A_-$ tends to zero. Here, as pointed out before, we show that any minimizer of (1.3) is locally $C^{0,1^-}$ regular with precise corresponding Hölder estimates, independent of the "jump" distance between $A_+$ and $A_-$ across the phases. This paper is essentially self contained and it has detailed proofs.

**Outline of the paper:** In Section 2, we list the main definitions and results. In Section 3 we introduce the $\mathcal{T}_{a,b}$ operator which converts a functional of the form (2.3) into a functional of Alt-Caffrelli-Friedman type. We show that the $\mathcal{T}_{a,b}$ operator preserves the regularity of minimizers of the original functional with constant coefficients (c.f. Proposition 3.13). Then in Section 4, we prove the interior $C^{0,1^-}$ regularity estimates for weak solutions to the PDE of the form (4.1) (c.f. Theorem 2.4). In Section 6, we make use of ideas from [2] and show that under the small regime, minimizers of (1.3) are close to minimizers of functional of the form (2.3), whose regularity estimates are well understood thanks to Section 3. Finally, in Section 7 and 8 we prove the main result via rescaling arguments, namely Theorem 2.3.

## 2. Main definitions and results

In this paper, we study the regularity of minimizers of the energy functional (1.3), under the assumptions **(G1)-(G5)**. For the coming sections, we simplify the notation as needed by dropping some subscripts whenever no ambiguity is present. For instance, $J_{A,f,Q}(\cdot, B_1)$ can be denoted only by $J$ when definitions of $A, f, Q$ and $B_1$ are clear from the context. We will note $F$ as

$$F := |f_+| + |f_-|. \tag{2.1}$$

[1]This is reflected, for instance, in equations (6.20) and (6.33) in the proof of Theorem 6.3.

Let $D \subset \mathbb{R}^N$ be a bounded, Lipschitz and open set. Then for all $\phi \in W^{1,p}(D)$

$$\begin{aligned} W^{1,p}_{\phi}(D) &:= \Big\{ v \in W^{1,p}(D) \,:\, v - \phi \in W^{1,p}_0(D) \Big\} \\ &= \Big\{ v \in W^{1,p}(D) \,:\, \mathrm{Tr}_{\partial D}(v) = \mathrm{Tr}_{\partial D}(\phi) \Big\}. \end{aligned} \tag{2.2}$$

When $p = 2$, $W^{1,2}(D)$ is denoted by $H^1(D)$ and analogously we define $H^1_{\phi}(D)$. We say that a function $v \in W^{1,p}(D)$ takes the boundary value $\phi$ on $\partial D$ whenever $v \in W^{1,p}_{\phi}(D)$. In order to study the traces, we use the notation $L^p(\partial D)$ for $L^p(\partial D, d\mathcal{H}^{N-1})$.

Existence of a minimizer for $J_{A,f,Q}(\cdot, B_1)$ in (1.3) follows from [43, Theorem 2.1]. For the analysis that we carry out in later sections, we define the following notation (2.3) for $x_0 \in D \subset B_1$

$$J_{x_0}(v; D) := \int_D A_+(x_0)|\nabla v^+|^2 + A_-(x_0)|\nabla v^-|^2 + Q(x_0)\lambda(v)\, dx. \tag{2.3}$$

**Definition 2.1.** *Given a functional $J(\cdot;\Omega) : W^{1,p}(\Omega) \to \mathbb{R}$, a function $u \in W^{1,p}(\Omega)$ is an absolute minimizer of $J(\cdot;\Omega)$ if for every $v \in W^{1,p}(\Omega)$ such that $u - v \in W^{1,p}_0(\Omega)$ we have*

$$J(u;\Omega) \le J(v;\Omega).$$

*A function $u \in W^{1,p}(\Omega)$ is a local minimizer of $J(\cdot;\Omega)$ if for every open Lipschitz subset $D \subset\subset \Omega$ and for every $v \in W^{1,p}(D)$ such that $\mathrm{Tr}_{\partial D}(v) = u$ we have*

$$J(u;D) \le J(v;D).$$

In fact, absolute minimizers of functionals of the form (1.3) and (2.3) are also local minimizers (c.f. Lemma 3.11). In order to better state our results, we define the modulus of continuity of continuous function $A : \Omega \to \mathbb{R}$ as follows

**Definition 2.2.** *A modulus of continuity is a function $\omega : [0,\infty) \to \mathbb{R}$ such that $\omega$ is non-negative, non-decreasing, and $\lim_{t\to 0} \omega(t) = 0$. In particular if $A : \Omega \to \mathbb{R}$ is continuous and $D \subset\subset \Omega$. We define $\omega_{A,D}$ as*

$$\omega_{A,D}(t) := \begin{cases} \sup\Big( |A(x) - A(y)| \,:\, x,y \in D \text{ such that } |x-y| \le t \Big) & t \le \mathrm{diam}(D) \\ \omega_{A,D}(\mathrm{diam}(D)) & t > \mathrm{diam}(D). \end{cases}$$

*Then $\omega_{A,D}$ is a modulus of continuity (we call $\omega_{A,D}$ as the modulus of continuity of $A$ in $D$).*

We denote $C(\Omega)^{N\times N}$ as the set of $N \times N$ matrices with entries as continuous functions on $\Omega$.

**Main results:** We now state the main results in this paper.

**Theorem 2.3** (Optimal regularity for absolute minimizers)**.** *Let $u \in H^1(B_1)$ be a bounded absolute minimizer of $J_{A,f,Q}(\cdot, B_1)$ given in (1.3) and satisfying **(G1)-(G5)**. Then $u \in C^{0,1^-}_{loc}(B_1)$ with estimates. More precisely, for any $0 < \alpha < 1$ and $B_r \subset\subset B_1$, we have that $u \in C^{0,\alpha}(B_r)$ with the following estimate*

$$\|u\|_{C^{\alpha}(B_r)} \le \frac{C(N, \alpha, \mu, q_2, \lambda_+, \omega_{A_\pm, B_{r^*}})}{(1-r)^{\alpha}} \left(1 + \|u\|_{L^\infty(B_1)} + \|f\|_{L^N(B_1)}\right). \tag{2.4}$$

*Here $r^* = \frac{1+r}{2}$. The function $\omega_{A_\pm, B_{r^*}}$ is the maximum of modulus of continuity of $A_+$ and $A_-$ in $B_{r^*}$.*

In the process of proving the main result, we obtain optimal PDE regularity result which seems to us to be unavailable in the literature in that form (where there is a precise constant dependence and the result holding in every dimension $N \ge 2$)

**Theorem 2.4** (Optimal regularity of bounded weak solutions)**.** *Let $u \in H^1(B_1)$ be a bounded weak solution to*

$$\mathrm{div}\,(A(x)\nabla u) = f \;\text{ in } B_1$$

*where $A \in C(B_1)^{N\times N}$ and there exists $0 < \mu < 1$ such that $\mu|\xi|^2 \le \langle A(x)\xi, \xi\rangle \le \frac{1}{\mu}|\xi|^2$ a.e. in $B_1$ for all $\xi \in \mathbb{R}^N$, $f \in L^N(B_1)$. Then $u \in L^\infty_{loc}(B_1)$ and for any $0 < \alpha < 1$ and $B_r \subset\subset B_1$, we have that $u \in C^{0,\alpha}(B_r)$ with the following estimate*

$$\|u\|_{C^{\alpha}(B_r)} \le \frac{C(N, \alpha, \mu, \omega_{A, B_{r^*}})}{(1-r)^{\alpha}} \left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^N(B_1)}\right). \tag{2.5}$$

*Here $r^* = \frac{1+r}{2}$. The function $\omega_{A, B_{r^*}}$ is the modulus of continuity of $A$ in $B_{r^*}$.*

*Remark* 2.5. The regularity provided in Theorem 2.4 and Theorem 2.3 are sharp even in the case where right hand side $f$ is equal to zero. For this, we refer to [27]. For recent developments in the subject, we refer to [38].

## 3. Preliminary tool: The $\mathcal{T}_{a,b}$ operator

In this section $D$ is a bounded, open and Lipschitz subset on $\mathbb{R}^N$. We start by considering the following functionals

$$\mathcal{J}(v;D) := \int_D \Big(a^p|\nabla v^+|^p + b^p|\nabla v^-|^p + \lambda(v)\Big)\,dx \text{ and } \mathcal{F}(v;D) := \int_D \Big(|\nabla v|^p + \lambda(v)\Big)\,dx,$$

where $p \in (1,\infty)$ and $a, b > 0$. We make the observation that when $u$ is a minimizer of the energy functional $\mathcal{J}(\cdot;;D)$, then, this very function carry the regularity properties available for a minimizer of the classical functional $\mathcal{F}(\cdot;D)$. The purpose of this section is to develop the ingredients that prove the above remark. For that, we introduce the $\mathcal{T}_{a,b}$-operator.

**Definition 3.1.** *Let $a, b > 0$, $p \in [1,\infty)$ and $D$ be an open Lipschitz set, we define $\mathcal{T}_{a,b} : W^{1,p}(D) \to W^{1,p}(D)$ as follows*

$$\mathcal{T}_{a,b}(v) := av^+ - bv^-.$$

*Remark* 3.2 (Definition of $\mathcal{T}_{a,b}$-operator on $\partial D$). In the coming sections, we deal with the action of the $\mathcal{T}_{a,b}$-operator on functions in $W^{1,p}(D)$ and also keep track of their traces. For this reason, in order to maintain compatibility of notations, we define the $\mathcal{T}_{a,b}$-operator also on the boundary level acting on $L^p(\partial D)$ to be $\mathcal{T}^{\partial D}_{a,b} : L^p(\partial D) \to L^p(\partial D)$ given by

$$\mathcal{T}^{\partial D}_{a,b}(\varphi) := a\varphi^+ - b\varphi^-.$$

*Remark* 3.3. In the sequel, we use the following simple fact from the basic theory of topological spaces. Consider $(X,\tau)$ to be a Hausdorff topological space and let $v_k$ be a sequence in $X$ and $v \in X$. Suppose that for every subsequence $v_{k_j}$, we can extract yet another subsequence $v_{k_{j_l}}$ so that $v_{k_{j_l}} \to v$ in $(X,\tau)$ as $l \to \infty$. Then $v_k \to v$ in $(X,\tau)$ as $k \to \infty$.

*Remark* 3.4. We recall the following decomposition for any $a \in \mathbb{R}$. There exists unique numbers $p_a$ and $n_a$ such that $a = p_a - n_a$ and $p_a, n_a \geq 0$ and $p_a \cdot n_a = 0$. Clearly, $p_a = \max(a,0) =: a^+$ and $n_a = -\min(a,0) =: a^-$.

This decomposition also applies to functions. For $w \in L^p(D)$, $\mathcal{T}_{a,b}(v) = [\mathcal{T}_{a,b}(v)]^+ - [\mathcal{T}_{a,b}(v)]^-$ a.e. in $D$ and from the above mentioned uniqueness of decomosition

$$\begin{aligned} [\mathcal{T}_{a,b}(v)]^+ &= av^+ = (av)^+ \text{ a.e. in } D \\ [\mathcal{T}_{a,b}(v)]^- &= bv^- = (bv)^- \text{ a.e. in } D. \end{aligned} \tag{3.1}$$

Similarly, (3.1) hold for every $w \in L^p(\partial D)$ $\mathcal{H}^{N-1}$ a.e. on $\partial D$. Moreover,

$$\chi_{\{v>0\}} = \chi_{\{\mathcal{T}_{a,b}(v)>0\}} \text{ and } \chi_{\{v\leq 0\}} = \chi_{\{\mathcal{T}_{a,b}(v)\leq 0\}} \text{ a.e. in } D.$$

In particular, we have

$$\lambda(v) = \lambda(\mathcal{T}_{a,b}(v)) \text{ a.e. in } D. \tag{3.2}$$

We also note the following observation

*Remark* 3.5. For every $w \in W^{1,p}(D)$, we observe that

$$\begin{aligned} |w^+ \pm w^-|^p &= |w^+ \pm w^-|^p\chi_{\{w>0\}} + |w^+ \pm w^-|^p\chi_{\{w\leq 0\}} \\ &= \big|(w^+ \pm w^-)\chi_{\{w>0\}}\big|^p + \big|(w^+ \pm w^-)\chi_{\{w\leq 0\}}\big|^p \\ &= |w^+|^p + |w^-|^p. \end{aligned} \tag{3.3}$$

Also,

$$\begin{aligned} |\nabla w^+ \pm \nabla w^-|^p &= |\nabla w^+ \pm \nabla w^-|^p\chi_{\{w>0\}} + |\nabla w^+ \pm \nabla w^-|^p\chi_{\{w\leq 0\}} \\ &= \big|(\nabla w^+ \pm \nabla w^-)\chi_{\{w>0\}}\big|^p + \big|(\nabla w^+ \pm \nabla w^-)\chi_{\{w\leq 0\}}\big|^p \\ &= |\nabla w^+|^p + |\nabla w^-|^p. \end{aligned} \tag{3.4}$$

The above two computations imply that

$$\|w\|^p_{L^p(D)} = \|w^+ \pm w^-\|^p_{L^p(D)} = \|w^+\|_{L^p(D)} + \|w^-\|^p_{L^p(D)} \tag{3.5}$$

$$\|\nabla w\|^p_{L^p(D)} = \|\nabla w^+ \pm \nabla w^-\|^p_{L^p(D)} = \|\nabla w^+\|_{L^p(D)} + \|\nabla w^-\|^p_{L^p(D)} \tag{3.6}$$

The above remarks lead to the following important corollary

**Corollary 3.6.** *Let $p \in (1, \infty)$ then, the $\mathcal{T}_{a,b}$ operator splits as follows*

($a$) *For every $w \in L^p(D)$*

$$|\mathcal{T}_{a,b} w|^p = a^p |w^+|^p + b^p |w^-|^p \ \text{a.e. in } D \tag{3.7}$$

*in particular*

$$\|\mathcal{T}_{a,b}(w)\|^p_{L^p(D)} = a^p \|w^+\|^p_{L^p(D)} + b^p \|w^-\|^p_{L^p(D)}.$$

($b$) (3.7) *also holds for $\varphi \in L^p(\partial D)$ replacing the Lebesgue measure by $H^{N-1}_{\lfloor \partial D}$.*

($c$) *For every $w \in W^{1,p}(D)$*

$$|\nabla \mathcal{T}_{a,b}(w)|^p = a^p |\nabla w^+|^p + b^p |\nabla w^-|^p \ \text{a.e. in } D \tag{3.8}$$

*in particular*

$$\|\nabla \mathcal{T}_{a,b}(w)\|_{L^p(D)} = a^p \|\nabla w^+\|^p_{L^p(D)} + b^p \|\nabla w^-\|^p_{L^p(D)}.$$

($d$) *$w \in W^{1,p}(\Omega)$ if and only if $\mathcal{T}_{a,b}(w) \in W^{1,p}(\Omega)$.*

*Proof.* The claims (a), (b) and (c) follow readily by applying (3.3), (3.4), (3.5) and (3.6) for $\mathcal{T}_{a,b}(w)$ and then using (3.1). (d) readily follows from (a) and (c). □

We show that the $\mathcal{T}_{a,b}$ operator is sequentially continuous in weak and strong $W^{1,p}(D)$ topologies and commutes with the trace operator. Moreover, it preserves the pointwise convergence of sequence of functions. The proofs of these facts are inspired by the ideas from [35] and [30].

**Proposition 3.7.** *The following properties hold*

***(1)*** *Assume $v_k \rightharpoonup v$ (weakly) in $W^{1,p}(D)$. Then*

***(1a)*** *$v_k^\pm \rightharpoonup v^\pm$ (weakly) in $W^{1,p}(D)$.*

***(1b)*** *$\mathcal{T}_{a,b}(v_k) \rightharpoonup \mathcal{T}_{a,b}(v)$ (weakly) in $W^{1,p}(D)$.*

***(1c)*** *The maps $v \mapsto |v|$ and $\mathcal{T}_{a,b}$ operator are sequentially weakly continuous in $W^{1,p}(D)$.*

***(2)*** *Assume $v_k \to v$ (strongly) in $W^{1,p}(D)$. Then*

***(2a)*** *$v_k^\pm \to v^\pm$ (strongly) in $W^{1,p}(D)$.*

***(2b)*** *$\mathcal{T}_{a,b}(v_k) \to \mathcal{T}_{a,b}(v)$ (strongly) in $W^{1,p}(D)$.*

***(2c)*** *The maps $v \mapsto |v|$ and $\mathcal{T}_{a,b}$ operator are continuous in $W^{1,p}(D)$.*

***(3)*** *Assume $v_k \to v$ in $L^p(D)$. Then*

***(3a)*** *$v_k^\pm \to v^\pm$ in $L^p(D)$.*

***(3b)*** *$\mathcal{T}_{a,b}(v_k) \to \mathcal{T}_{a,b}(v)$ in $L^p(D)$.*

***(4)*** *Assume $v \in W^{1,p}(D)$. Then*

***(4a)*** $\mathrm{Tr}(v^\pm) = (\mathrm{Tr}(v))^\pm$ *$\mathcal{H}^{N-1}$ a.e. on $\partial D$.*

***(4b)*** $\mathrm{Tr}(\mathcal{T}_{a,b}(v)) = \mathcal{T}^{\partial D}_{a,b}(\mathrm{Tr}(v))$ *$\mathcal{H}^{N-1}$ a.e. on $\partial D$.*

***(5)*** ***(1a)***, ***(1b)***, ***(2a)*** *and* ***(2b)*** *also hold when $W^{1,p}(D)$ is replaced by $W^{1,p}_0(D)$.*

***(6)*** *Assume $v_k \to v$ pointwise almost everywhere in $D$. Then $\mathcal{T}_{a,b}(v_k) \to \mathcal{T}_{a,b}(v)$ pointwise almost everywhere in $D$. Moreover, if $v \in C(D)$, then $|v|, \mathcal{T}_{a,b}(v) \in C(D)$.*

*Remark* 3.8. Although in **(1c)** we showed the sequential weakly continuity of $\mathcal{T}_{a,b}$-operator in $W^{1,p}(D)$, we in fact, can prove that $\mathcal{T}_{a,b}$-operator is weakly continuous. For that we need to repeat the same arguments replacing the sequences by nets.

*Proof.* In order to prove **(1)** we observe that from the assumptions that $v_k \in W^{1,p}(D)$ is a bounded sequence in $W^{1,p}(D)$. Moreover,

$$\int_D |v_k^+|^p \, dx \le \int_D |v_k|^p \, dx$$
$$\int_D |\nabla v_k^+|^p \, dx = \int_D |\nabla v_k|^p \chi_{\{v_k>0\}} \, dx \le \int_D |\nabla v_k|^p \, dx.$$

Hence, $v_k^+$ is also a bounded sequence in $W^{1,p}(D)$. Therefore there exists $w \in W^{1,p}(D)$ such that up to a subsequence which we still denote as $v_k^+$

$$v_k^+ \rightharpoonup w \text{ weakly in } W^{1,p}(D) \text{ and}$$
$$v_k^+ \to w \text{ in } L^p(D).$$

From compact embedding of $W^{1,p}(D)$ in $L^p(D)$, we know that $v_k \to v$ in $L^p(D)$. Furthermore, we observe that

$$\int_D |v_k^\pm - v^\pm|^p \, dx \le \int_D |v_k - v|^p \, dx \to 0 \text{ as } k \to \infty. \tag{3.9}$$

From (3.9), we conclude $v_k^+ \to v^+$ in $L^p(D)$. Therefore, from Remark 3.3 we have $v_k^+ \to v^+ = w$ in $L^p(D)$. So,

$$v_k^+ \rightharpoonup v^+ \text{ weakly in } W^{1,p}(D).$$

Similarly, we can prove that $v_k^- \rightharpoonup v^-$ weakly in $W^{1,p}(D)$. This proves **(1a)**, **(1b)** and **(1c)** follow readily. In order to prove **(2)**, from (3.6) and [24, Lemma 7.6, Lemma 7.7] we observe that for every $w \in W^{1,p}(D)$,

$$\begin{aligned} \int_D \left| \nabla \left( w^\pm \mp \frac{w}{2} \right) \right|^p dx &= \int_D \left| \nabla \left( \frac{w^+}{2} + \frac{w^-}{2} \right) \right|^p dx \\ &= \left( \frac{1}{2} \right)^p \int_D |\nabla w|^p \, dx \end{aligned} \tag{3.10}$$

Since $v_k \to v$ in $W^{1,p}(D)$, we have

$$\begin{aligned} \lim_{k\to\infty} \int_D \left| \nabla \left( v_k^\pm \mp \frac{v_k}{2} \right) \right|^p dx &= \lim_{k\to\infty} \left( \frac{1}{2} \right)^p \int_D |\nabla v_k|^p \, dx \\ &= \left( \frac{1}{2} \right)^p \int_D |\nabla v|^p \, dx \\ &= \int_D \left| \nabla \left( v^\pm \mp \frac{v}{2} \right) \right|^p dx. \end{aligned} \tag{3.11}$$

From **(1a)** we know that

$$\left( v_k^\pm \mp \frac{v_k}{2} \right) \rightharpoonup \left( v^\pm \mp \frac{v}{2} \right) \text{ weakly in } W^{1,p}(D).$$

From weak convergence and covergence of norms (3.11) by [18, Chapter 6, Proposition 9.1, pp. 259] (Radon's theorem), we obtain

$$\left( v_k^\pm \mp \frac{v_k}{2} \right) \to \left( v^\pm \mp \frac{v}{2} \right) \text{ strongly in } W^{1,p}(D).$$

Therefore, $v_k^\pm \to v^\pm$ in $W^{1,p}(D)$. This proves **(2a)**, **(2b)** and **(2c)** follow readily. Moreover, the argument in (3.9) also proves **(3a)** and also **(3b)**.

Now, let us discuss the proof of **(4a)** and **(4b)**. Let $v \in W^{1,p}(D)$. From [22, Theorem 4.3] there exists a sequence $v_k \in C(\overline{D}) \cap W^{1,p}(D)$ such that $v_k \to v$ in $W^{1,p}(D)$. Then from [22, Theorem 4.6], we have

$$\mathrm{Tr}(v_k^\pm) = v_k^\pm|_{\partial D} = \mathrm{Tr}(v_k)^\pm \text{ on } \partial D. \tag{3.12}$$

From **(2a)**, we know that $v_k^\pm \to v^\pm$ in $W^{1,p}(D)$ and then

$$\mathrm{Tr}(v_k^\pm) \to \mathrm{Tr}(v^\pm) \text{ in } L^p(\partial D). \tag{3.13}$$

Also, by continuity of trace operator $\mathrm{Tr}(v_k) \to \mathrm{Tr}(v)$ in $L^p(\partial D)$. Now by (3.12) and from the argument (3.9) with Lebesgue measure replaced by $d\mathcal{H}^{N-1}$ restricted to $\partial D$ we obtain

$$\mathrm{Tr}(v_k)^\pm \to \mathrm{Tr}(v)^\pm \text{ in } L^p(\partial D). \tag{3.14}$$

Combining the equations (3.12), (3.13) and (3.14) we conclude

$$\mathrm{Tr}(v)^{\pm} \overset{(3.14)}{=} \lim_{k\to\infty} \mathrm{Tr}(v_k)^{\pm} \overset{(3.12)}{=} \lim_{k\to\infty} \mathrm{Tr}(v_k^{\pm}) \overset{(3.13)}{=} \mathrm{Tr}(v^{\pm}) \quad \mathcal{H}^{N-1} \text{ a.e. on } \partial D.$$

Commutation of the Trace operator Tr and $\mathcal{T}_{a,b}$-operator (i.e. **(4b)**) follows directly from **(4a)**.

In order to see **(5)** we note that since $v_k \in W_0^{1,p}(D)$, $\mathrm{Tr}(v_k) = 0$ by [21, Section 5.5, Theorem 2]. From **(4a)** $\mathrm{Tr}(v_k^{\pm}) = 0$ and thus $v_k^{\pm} \in W_0^{1,p}(D)$ by [21, Section 5.5, Theorem 2]. Now, **(5)** follows from the fact that $W_0^{1,p}(D)$ is closed and weakly closed in $W^{1,p}(D)$ [9, Theorem 3.7].

Finally we have **(6)** by observing that

$$v_k \to v \text{ a.e. in } D \implies v_k^{\pm} \to v^{\pm} \text{ a.e. in } D \implies \mathcal{T}_{a,b}(v_k) \to \mathcal{T}_{a,b}(v) \text{ a.e. in } D.$$

□

**Proposition 3.9.** *Regarding the operator $\mathcal{T}_{a,b} : W^{1,p}(D) \to W^{1,p}(D)$. We have*

*(1)* $\mathcal{T}_{a,b} : W^{1,p}(D) \to W^{1,p}(D)$ *is a homeomorphism and*

$$\Big[\mathcal{T}_{a,b}\Big]^{-1} = \mathcal{T}_{\frac{1}{a},\frac{1}{b}}.$$

*(2)* $\mathcal{T}^{\partial D}_{a,b} : L^p(\partial D) \to L^p(\partial D)$ *is a homeomorphism and*

$$\Big[\mathcal{T}^{\partial D}_{a,b}\Big]^{-1} = \mathcal{T}^{\partial D}_{\frac{1}{a},\frac{1}{b}}.$$

*(3)* $\mathcal{T}_{a,b}$ *and* $\mathcal{T}^{-1}_{a,b}$ *are bounded maps in* $W^{1,p}(D)$ *and* $L^p(D)$ *with the following estimates,*

$$\min(a,b)\cdot\|w\|_{L^p(D)} \le \|\mathcal{T}_{a,b}(w)\|_{L^p(D)} \le \max(a,b)\cdot\|w\|_{L^p(D)}$$

$$\min(a,b)\cdot\|\nabla w\|_{L^p(D)} \le \|\nabla\mathcal{T}_{a,b}(w)\|_{L^p(D)} \le \max(a,b)\cdot\|\nabla w\|_{L^p(D)} \quad \forall w \in W^{1,p}(D).$$

*Furthermore,* $\mathcal{T}^{\partial D}_{a,b}$ *and* $(\mathcal{T}^{\partial D}_{a,b})^{-1}$ *are bounded maps in* $L^p(\partial D)$ *with the following estimates,*

$$\min(a,b)\cdot\|w\|_{L^p(\partial D)} \le \|\mathcal{T}^{\partial D}_{a,b}(w)\|_{L^p(\partial D)} \le \max(a,b)\cdot\|w\|_{L^p(\partial D)} \quad \forall w \in L^p(\partial D).$$

*(4)* *Let $\phi \in W^{1,p}(D)$. Then, the image of $W^{1,p}_{\phi}(D)$ under the operator $\mathcal{T}_{a,b}$ is $W^{1,p}_{\mathcal{T}_{a,b}(\phi)}(D)$. Moreover, the map*

$$\mathcal{T}_{a,b} : W^{1,p}_{\phi}(D) \to W^{1,p}_{\mathcal{T}_{a,b}(\phi)}(D)$$

*is a homeomorphism.*

*Proof.* In order to prove **(1)**, it is enough to show that

$$\mathcal{T}_{\frac{1}{a},\frac{1}{b}}(\mathcal{T}_{a,b}) = \mathcal{T}_{a,b}\left(\mathcal{T}_{\frac{1}{a},\frac{1}{b}}\right) = Id \text{ in } W^{1,p}(D). \tag{3.15}$$

Let $v \in W^{1,p}(D)$, then for any $\alpha, \beta > 0$

$$\begin{aligned}\mathcal{T}_{\frac{1}{\alpha},\frac{1}{\beta}}\Big(\mathcal{T}_{\alpha,\beta}(v)\Big) &= \frac{1}{\alpha}\Big[\mathcal{T}_{\alpha,\beta}(v)\Big]^{+} - \frac{1}{\beta}\Big[\mathcal{T}_{\alpha,\beta}(v)\Big]^{-} \\ &= \frac{1}{\alpha}(\alpha v^{+}) - \frac{1}{\beta}(\beta v^{-}) = v.\end{aligned} \tag{3.16}$$

The last identity follows from (3.1). The claim (3.15) readily follows by putting the pairs $(\alpha,\beta) = (a,b)$ and then $(\alpha,\beta) = (\frac{1}{a},\frac{1}{b})$ in (3.16).

Since the operators $\mathcal{T}_{a,b}$ and $\Big[\mathcal{T}_{a,b}\Big]^{-1} = \mathcal{T}_{\frac{1}{a},\frac{1}{b}}$ are continuous in $W^{1,p}(D)$ (Proposition 3.7-**(2)**) then $\mathcal{T}_{a,b}$-operator is a homeomorphism. The proof of **(2)** follows similarly. The proof of **(3)** follows from Corollary 3.6.

In order to prove **(4)**, we take $\phi \in W^{1,p}(D)$. We only need to show that

$$\mathcal{T}_{a,b}\left(W^{1,p}_{\phi}(D)\right) = W^{1,p}_{\mathcal{T}_{a,b}(\phi)}(D). \tag{3.17}$$

Let $\alpha, \beta > 0$ and $w \in W^{1,p}_{\phi}(D)$. Clearly, $\mathcal{T}_{a,b}(w) \in W^{1,p}(D)$. Now, we compute the trace of $\mathcal{T}_{a,b}(w)$. By Proposition 3.7-**(4b)**, we have

$$\mathrm{Tr}(\mathcal{T}_{\alpha,\beta}(w)) = \mathcal{T}^{\partial D}_{\alpha,\beta}(\mathrm{Tr}(w)) = \mathcal{T}^{\partial D}_{\alpha,\beta}(\mathrm{Tr}(\phi)) = \mathrm{Tr}(\mathcal{T}_{\alpha,\beta}(\phi)).$$

From this, we conclude that

$$\mathcal{T}_{\alpha,\beta}\left(W^{1,p}_{\phi}(D)\right) \subset W^{1,p}_{\mathcal{T}_{\alpha,\beta}(\phi)}(D). \tag{3.18}$$

Therefore by taking the pair $(\alpha,\beta)=(a,b)$ we arrive to

$$\mathcal{T}_{a,b}\left(W^{1,p}_{\phi}(D)\right) \subset W^{1,p}_{\mathcal{T}_{a,b}(\phi)}(D) \tag{3.19}$$

From Proposition 3.9-**(1)** there exists a unique $\psi \in W^{1,p}(D)$ such that

$$\mathcal{T}_{\frac{1}{a},\frac{1}{b}}(\psi)=\phi. \tag{3.20}$$

Then plugging $(\alpha,\beta)=(\frac{1}{a},\frac{1}{b})$ and replacing $\phi$ by $\psi$ in (3.18)

$$\mathcal{T}_{\frac{1}{a},\frac{1}{b}}\left(W^{1,p}_{\psi}(D)\right) \subset W^{1,p}_{\mathcal{T}_{\frac{1}{a},\frac{1}{b}}(\psi)}(D).$$

applying the $\mathcal{T}_{a,b}$-operator on both sides of set inclusion above, using (3.20) we arrive at

$$W^{1,p}_{\psi}(D)=\mathcal{T}_{a,b}\left(\mathcal{T}_{\frac{1}{a},\frac{1}{b}}\left(W^{1,p}_{\psi}(D)\right)\right) \subset \mathcal{T}_{a,b}\left(W^{1,p}_{\mathcal{T}_{\frac{1}{a},\frac{1}{b}}(\psi)}(D)\right)=\mathcal{T}_{a,b}\left(W^{1,p}_{\phi}(D)\right). \tag{3.21}$$

Recalling definition of $\psi$ in (3.20) and from (3.21) we obtain

$$W^{1,p}_{\mathcal{T}_{a,b}(\phi)}(D) \subset \mathcal{T}_{a,b}\left(W^{1,p}_{\phi}(D)\right). \tag{3.22}$$

Finally, combining (3.19) and (3.22) we obtain (3.17). □

**Lemma 3.10.** *Consider the two functionals defined in $W^{1,p}(D)$, $p\in(1,\infty)$*

$$\mathcal{F}(v;D):=\int_D |\nabla v|^p+\lambda(v)\,dx,\quad \mathcal{J}(v;D):=\int_D a^p|\nabla v^+|^p+b^p|\nabla v^-|^p+\lambda(v)\,dx.$$

*Then $u_0\in W^{1,p}(D)$ is an absolute minimizer of $\mathcal{J}(\cdot;D)$ if and only if $\mathcal{T}_{a,b}(u_0)$ is an absolute minimizer of $\mathcal{F}(\cdot;D)$.*

*Proof.* Minimality of $\mathcal{T}_{a,b}(u_0)$ for $\mathcal{F}(\cdot;D)$ can be written as

$$\int_D |\nabla\mathcal{T}_{a,b}(u_0)|^p+\lambda(\mathcal{T}_{a,b}(u_0))\,dx \le \int_D |\nabla v|^p+\lambda(v)\,dx$$

for every $v\in W^{1,p}(D)$ such that $v-\mathcal{T}_{a,b}(u_0)\in W^{1,p}_0(D)$ or $v\in W^{1,p}_{\mathcal{T}_{a,b}(u_0)}(D)$. From Proposition 3.9-**(4)** and Proposition 3.7-**(4)**, if $w$ runs over $W^{1,p}_{u_0}(D)$ if and only if $\mathcal{T}_{a,b}(w)$ runs over $W^{1,p}_{\mathcal{T}_{a,b}(u_0)}(D)$. Replacing $v$ by $\mathcal{T}_{a,b}(w)$ in the inequality above, we obtain

$$\int_D |\nabla\mathcal{T}_{a,b}(u_0)|^p+\lambda(\mathcal{T}_{a,b}(u_0))\,dx \le \int_D |\nabla\mathcal{T}_{a,b}(w)|^p+\lambda(\mathcal{T}_{a,b}(w))\,dx$$

for every $w\in W^{1,p}_{u_0}(D)$. Now, using (3.8) and recalling the equation (3.2) $\lambda(\mathcal{T}_{a,b}(u_0))=\lambda(u_0)$ and $\lambda(\mathcal{T}_{a,b}(w))=\lambda(w)$, the proof of Proposition 3.10 is finished. □

We call the functionals of the type (1.3), (2.3) or $\mathcal{F}$ and $\mathcal{J}$ defined in the beginning of this section as "integral functionals". More precisely, for any $j:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ the funtional $J$ for the following form

$$J(v;\Omega):=\int_\Omega j(x,v,\nabla v)\,dx$$

is called an integral functional. In order to proceed with the proof of the next lemma, we require the integral functional $J(\cdot;D)$ and $J(\cdot;\Omega\setminus D)$ to be finite for every compact set $D\subset\subset\Omega$ for every $v\in W^{1,p}(\Omega)$.

**Lemma 3.11.** *Assume $\Omega$ be an open, bounded and Lipschitz subset in $\mathbb{R}^N$ and $u\in W^{1,p}(\Omega)$ is an absolute minimizer of an integral functional $J(\cdot;\Omega)$ such that $J(v;D)<\infty$ for every $v\in W^{1,p}(\Omega)$ and $D\subset\subset\Omega$. Then, $u$ is a local minimizer of $J(\cdot;\Omega)$.*

*Remark* 3.12. Lemma 3.11 applies to the functionals of the form (1.3), (2.3). Indeed, assume $\Omega$ be an open and bounded subset of $\mathbb{R}^N$. We can easily see that for any $D\subset\subset\Omega$, $x_0\in D$ and $v\in H^1(D)$

$$\max\Big(|J_{A,f,Q}(v;D)|,|J_{x_0}(v;D)|,|J_{A,f,Q}(v;\Omega\setminus D)|,|J_{x_0}(v;\Omega\setminus D)|\Big)<\infty.$$

For more precise details, we refer to the computations in (6.30) and (6.31).

*Proof of Lemma 3.11.* In order to prove that $u$ is a local minimizer of $J(\cdot;\Omega)$, we take $D \subset\subset \Omega$ an open Lipschitz subset. Let $v \in W^{1,p}(D)$ such that $\mathrm{Tr}_{\partial D}(v) = u$. We now set $w$ as

$$w = \begin{cases} v & \text{in } D \\ u & \text{in } \Omega \setminus D. \end{cases}$$

Since $D$ is a Lipschitz domain and $\mathrm{Tr}_{\partial D}(v) = u$, from [17, Theorem 3.44] $w \in W^{1,p}(\Omega)$. (Although [17, Theorem 3.44] is stated for $C^1$ domains, the same proof applies to Lipschitz domains by using the trace theorem [22, Theorem 4.6]). Moreover, by the definition of $w$ and local properties of traces [22, Theorem 5.7], $\mathrm{Tr}_{\partial\Omega}(w) = u$. Thus from the assumption we have

$$J(u;\Omega) \leq J(w;\Omega).$$

Now,

$$J(u;D) + J(u;\Omega \setminus D) \leq J(w;D) + J(w;\Omega \setminus D).$$

From the definition of $w$, $w = u$ in $\Omega \setminus D$ and from Remark 3.12 we cancel the second terms on both sides. We obtain

$$J(u;D) \leq J(v;D).$$

This proves that $u$ is a local minimizer of $J(\cdot;\Omega)$. □

**Proposition 3.13.** *For $p \in (1,\infty)$, assume $u_0 \in W^{1,p}(D) \cap L^\infty(D)$ is an absolute minimizer of $\mathcal{J}(\cdot;D)$ where*

$$\mathcal{J}(v;D) := \int_D a^p|\nabla v^+|^p + b^p|\nabla v^-|^p + \lambda(v)\,dx.$$

*Then, $u_0 \in C^{0,1^-}_{loc}(D)$ with estimates.*

*More precisely, for every $0 < \alpha < 1$ and $D' \subset\subset D$ we have the following estimates*

$$\|u_0\|_{C^{0,\alpha}(D')} \leq \frac{1}{\min(a,b)}[\mathcal{T}_{a,b}(u_0)]_{C^{0,\alpha}(D)} \leq \frac{1}{\min(a,b)}C(N,\alpha,\lambda_+,\mathrm{dist}(D',\partial D),\|u_0\|_{L^\infty(D)}). \tag{3.23}$$

*Proof.* From Lemma 3.10 we know that $\mathcal{T}_{a,b}(u_0)$ is an absolute minimizer of the $\mathcal{F}(\cdot;D)$ defined in Lemma 3.10. This means

$$\int_D |\nabla(\mathcal{T}_{a,b}(u_0))|^2\,dx + \lambda(\mathcal{T}_{a,b}(u_0))\,dx \leq \int_D |\nabla v|^2\,dx + \lambda(v)\,dx \quad \forall\, v \in W^{1,p}_{\mathcal{T}_{a,b}(u_0)}(D).$$

From Lemma 3.11 $\mathcal{T}_{a,b}(u_0)$ is a local minimizer of $\mathcal{F}(\cdot;D)$ as in Lemma 3.10. Hence, from [8, Theorem 5.1] we know that $\mathcal{T}_{a,b}(u_0) \in C^{0,1^-}_{loc}(D)$ with the following estimates.

$$[\mathcal{T}_{a,b}(u_0)]_{C^\alpha(D')} \leq C(N,\mu,\lambda_+,\alpha,\mathrm{dist}(D',\partial D),\|u_0\|_{L^\infty(D)})$$

(One phase version of [8, Theorem 5.1] can be found in [36, Theorem 4.1]). This implies that $u_0$ also belongs to $C^{0,1^-}_{loc}(D)$ with estimates

$$[u_0]_{C^\alpha(D')} \leq \frac{1}{\min(a,b)}C(N,\lambda_+,\alpha,\mathrm{dist}(D',\partial D),\|u_0\|_{L^\infty(D)}). \tag{3.24}$$

Indeed, let $x, y \in D' \subset\subset D$. Then for any given $0 < \alpha < 1$, we have

$$|\mathcal{T}_{a,b}(u_0)(x) - \mathcal{T}_{a,b}(u_0)(y)| \leq [\mathcal{T}_{a,b}(u_0)]_{C^{0,\alpha}(D)}|x-y|^\alpha. \tag{3.25}$$

From Proposition 3.7-**(6)**, we know that $\mathcal{T}_{a,b}$-operator preserves continuity. We rewrite the above inequality in different regions of the domain $D'$. It is easy to see that

- $\bar{x} \in \{u_0 \geq 0\}$, $\bar{y} \in \{u_0 \geq 0\} \implies |u_0(\bar{x}) - u_0(\bar{y})| \leq \frac{[\mathcal{T}_{a,b}(u_0)]_{C^{0,\alpha}(D')}}{a}|\bar{x}-\bar{y}|^\alpha$.
- $\bar{x} \in \{u_0 \leq 0\}$, $\bar{y} \in \{u_0 \leq 0\} \implies |u_0(\bar{x}) - u_0(\bar{y})| \leq \frac{[\mathcal{T}_{a,b}(u_0)]_{C^{0,\alpha}(D')}}{b}|\bar{x}-\bar{y}|^\alpha$.
- $\bar{x} \in \{u_0 \geq 0\}$, $\bar{y} \in \{u_0 \leq 0\} \implies |u_0(\bar{x}) - u_0(\bar{y})| \leq \frac{[\mathcal{T}_{a,b}(u_0)]_{C^{0,\alpha}(D')}}{\min(a,b)}|\bar{x}-\bar{y}|^\alpha$.

To check the last inequality we observe that if $\bar{x} \in \{u_0 > 0\}$, $\bar{y} \in \{u_0 \leq 0\}$ then

$$\begin{aligned}\min(a,b)\big|u_0(\bar{x}) - u_0(\bar{y})\big| &\leq \big(au_0(\bar{x}) - bu_0(\bar{y})\big) \ \text{(since the term inside is non-negative and } u_0(\bar{y}) \leq 0) \\ &= |au_0^+(x) - b(-u_0^-(x))| \\ &= \big|\mathcal{T}_{a,b}u_0(\bar{x}) - \mathcal{T}_{a,b}u(\bar{y})\big| \leq [\mathcal{T}_{a,b}(u_0)]_{C^{0,\alpha}(D')}|\bar{x}-\bar{y}|^\alpha.\end{aligned}$$

Then we combine the above inequality with (3.25). □

We present a basic lemma useful in upcoming proofs. The weak convergence in Lebesgue spaces preserve pointwise almost everywhere inequality.

**Lemma 3.14.** *Let $U \subset \mathbb{R}^N$ be a Lebesgue measurable set of finite measure and $f_k$ and $g_k$ be two sequences of functions in $L^p(U)$ with $p \in [1,\infty)$ such that*

$$f_k \rightharpoonup f \text{ and } g_k \rightharpoonup g \text{ weakly in } L^p(U) \text{ and } f_k \le g_k \text{ a.e. in } U \text{ for all } k \in \mathbb{N}.$$

*Then*

$$f \le g \text{ a.e. in } U.$$

*Proof.* For any $E \subset U$ Lebesgue measurable, $\chi_E \in L^{p'}(U)$. Then by the weak convergence

$$\int_D f\chi_E\,dx = \lim_{k\to\infty}\int_D f_k\chi_E\,dx \le \lim_{k\to\infty}\int_D g_k\chi_E\,dx = \int_D g\chi_E\,dx.$$

Therefore, $\int_E f\,dx \le \int_E g\,dx$ for every Lebesgue measurable set $E \subset U$. Hence $f \le g$ a.e in $U$. □

## 4. Regularity of solutions of PDEs with continuous coefficients

In this section we set $A(x) \in C^{N\times N}(D)$ and $f \in L^N(B_1)$. We are concerned about the regularity of (weak) solutions to the following PDE in $B_1$

$$\operatorname{div}(A(x)\nabla u) = f. \tag{4.1}$$

Our setup condition regarding the above equation in this paper are

- **(Ellipticity)** There exists $0 < \mu < 1$ such that for almost every $x \in B_1$ and for all $\xi \in \mathbb{R}^N$ we have
$$\frac{1}{\mu}|\xi|^2 \le \langle A(x)\xi,\xi\rangle \le \mu|\xi|^2.$$
- **(Continuity)** $A_{i,j} \in C(B_1)$ for every $1 \le i,j \le N$.
- $f \in L^N(B_1)$.

We remind the reader on the definition of weak solution to (4.1).

**Definition 4.1.** *$u \in H^1(B_1)$ is a weak solution to (4.1) in $B_1$ if for all $\varphi \in H_0^1(B_1)$ we have*

$$\int_{B_1}\Big(A(x)\nabla u\cdot\nabla\varphi\Big)\,dx = -\int_{B_1} f(x)\varphi(x)\,dx \tag{4.2}$$

*Remark* 4.2. Let $h \in H^1_{loc}(B_1)\cap L^\infty(B_{1/2})$ such that $h$ weak solution to the following PDE in $B_{1/2}$

$$\operatorname{div}(A(0)\nabla h) = 0. \tag{4.3}$$

Then $h$ satisfies the following classical interior gradient estimate

$$\|\nabla h\|_{L^\infty(B_{1/4})} \le C(N,\mu)\|h\|_{L^\infty(B_{1/2})}. \tag{4.4}$$

*Remark* 4.3. Let the function $u \in H^1_{loc}(B_\Theta(x_0))$ be a weak solution to the PDE (4.1) in $B_\Theta(x_0)$, we set $w$ as follows

$$w(y) := \Phi u(\Theta y + x_0) + \Psi, \quad y \in B_1.$$

It is easy to check by scaling and change of variables, $w \in H^1_{loc}(B_1)$ and $w$ is a weak solution to the following PDE

$$\operatorname{div}(\bar{A}(x)\nabla w) = \bar{f} \text{ in } B_1$$

where $\bar{A}$ and $\bar{f}$ are defined as follows for $x \in B_1$

$$\bar{A}(x) := A(x_0 + \Theta x)$$
$$\bar{f}(x) := \Phi\Theta^2 f(x_0 + \Theta x).$$

**Proposition 4.4.** *Assume $u \in H^1(B_{1/2})\cap L^\infty(B_1)$ be a weak solution to (4.1) in $B_{1/2}$ with $\|u\|_{L^\infty(B_{1/2})} \le 1$, $\|\nabla u\|_{L^2(B_{1/2})} \le M$ and $f \in L^N(B_{1/2})$. Then for every $\varepsilon > 0$ there exists $\delta(\varepsilon, M, N, \mu) > 0$ such that if*

$$\max\Big(\|A(x) - A(0)\|_{L^\infty(B_1)}, \|f\|_{L^N(B_{1/2})}\Big) \le \delta$$

*then*

$$\|u - h\|_{L^\infty(B_{1/4})} \le \varepsilon,$$

*for some $h \in H^1(B_{1/2})$ such that $\|h\|_{L^\infty(B_{1/2})} \le 1$ and $h$ is a weak solution to $\operatorname{div}(A(0)\nabla h) = 0$ in $B_{1/2}$.*

*Proof.* Let us suppose for the sake of contradiction that the statement of Proposition 4.4 is not true. This implies that there exists $\varepsilon_0 > 0$ and sequences $A_k, f_k$ such that $A_k \in C(B_{1/2})$ and $f_k \in L^N(B_{1/2})$ satisfying $\|A_k - A(0)\|_{L^\infty(B_{1/2})} \leq \frac{1}{k}$, $\|f_k\|_{L^N(B_{1/2})} \leq \frac{1}{k}$. So that the corresponding weak solutions $u_k$ of the following PDE

$$\operatorname{div}(A_k(x)\nabla u_k) = f_k(x) \text{ in } B_{1/2} \tag{4.5}$$

are such that $\|\nabla u_k\|_{L^2(B_{1/2})} \leq M$ and $\|u_k\|_{L^\infty(B_{1/2})} \leq 1$ and for any $h$ satisfying the following PDE

$$\operatorname{div}(A(0)\nabla h) = 0 \tag{4.6}$$

we have

$$\|u_k - h\|_{L^\infty(B_{1/4})} > \varepsilon_0. \tag{4.7}$$

Now, we see that $u_k$ are uniformly equicontinuous since they satisfy a uniform interior Hölder estimates. Indeed, from [26, Corollary 4.18] we know that there exists $\beta_0 := \beta_0(\mu, N) \in (0,1)$ and $C_0 := C_0(\mu, N) > 0$ so that for every $k \in \mathbb{N}$, $u_k$ satisfy the following uniform estimate

$$|u_k(x) - u_k(y)| \leq C_0|x-y|^{\beta_0}\Big\{\|u_k\|_{L^2(B_{1/2})} + \|f_k\|_{L^N(B_{1/2})}\Big\}, \quad \forall x, y \in B_{1/4}. \tag{4.8}$$

Since $\|u_k\|_{L^2(B_{1/2})} \leq C(N)\cdot\|u_k\|_{L^\infty(B_{1/2})} \leq C(N)$ and $\|f_k\|_{L^N(B_{1/2})} \leq \frac{1}{k} \leq 1$, we can write

$$\|u_k\|_{C^{\beta_0}(B_{1/4})} \leq C_1(N, \mu).$$

By Arzela Ascoli theorem, there exists $u_0 \in C^{0,\beta_0}(B_{1/4})$ such that

$$u_k \to u_0 \text{ in } L^\infty(B_{1/4}) \text{ up to a subsequence.} \tag{4.9}$$

Since $\|\nabla u_k\|_{L^2(B_{1/2})} \leq M$ and $\|u_k\|_{L^2(B_{1/2})} \leq C(N)\|u_k\|_{L^\infty(B_{1/2})} \leq C(N)$ for every $k$, we conclude that $u_k$ is a bounded sequence in $H^1(B_{1/2})$. That is to say

$$\|u_k\|_{H^1(B_{1/2})} \leq C(M, N).$$

Therefore, $u_k$ converges weakly to $u_0 \in H^1(B_{1/2})$ (up to a subsequence)

$$u_k \rightharpoonup u_0 \text{ weakly in } H^1(B_{1/2}).$$

We claim that $u_0$ is a weak solution to $\operatorname{div}(A(0)\nabla u_0) = 0$, this gives us a contradiction and hence proves the Proposition 4.4. Indeed, we know that $\|\nabla u_0\|_{L^2(B_{1/2})} \leq M$ from lower semicontinuity of $H^1$ norm and $\|u_0\|_{L^\infty(B_{1/2})} \leq 1$ from Lemma 3.14. Once $u_k$ are weak solutions of (4.5), we have for every $\Phi \in H^1_0(B_{1/2})$

$$\int_{B_{1/2}} \big(A_k(x)\nabla u_k \cdot \nabla\Phi\big)\, dx = -\int_{B_{1/2}} f_k \Phi\, dx \tag{4.10}$$

Therefore

$$\int_{B_{1/2}} \Big(\big(A_k(x) - A(0)\big)\nabla u_k \cdot \nabla\Phi\Big)\, dx + \int_{B_{1/2}} \big(A(0)\nabla u_k \cdot \nabla\Phi\big)\, dx = -\int_{B_{1/2}} f_k\Phi\, dx \tag{4.11}$$

Since $A_{k,\pm} \to A_\pm(0)$ uniformly in $B_{1/2}$ we obtain

$$\begin{aligned}\int_{B_{1/2}} \Big(\big(A_k(x) - A(0)\big)\nabla u_k \cdot \nabla\Phi\Big)\, dx &\leq \|A_k - A(0)\|_{L^\infty(B_{1/2})} \int_{B_{1/2}} \big|\nabla u_k \cdot \nabla\Phi\big|\, dx \\ &\leq \frac{1}{k}\, \|\nabla u_k\|_{L^2(B_{1/2})}\|\nabla\Phi\|_{L^2(B_{1/2})} \to 0.\end{aligned}$$

Moreover if $N > 2$ then $2^{*\prime} = \frac{2N}{N+2} < N$. So by Sobolev embedding we arrive at

$$\begin{aligned}\int_{B_{1/2}} |f_k(x)\Phi(x)|\, dx &\leq \|f_k\|_{L^{2^{*\prime}}(B_{1/2})}\|\Phi\|_{L^{2^*}(B_{1/2})} \\ &\leq C(N)\|f_k\|_{L^N(B_{1/2})}\|\Phi\|_{L^{2^*}(B_{1/2})} \\ &\leq \frac{C(N)}{k}\|\Phi\|_{H^1(B_{1/2})} \to 0.\end{aligned}$$

In the case where if $N = 2$ we have $H^1(B_{1/2}) \hookrightarrow L^N(B_{1/2})$, therefore

$$\begin{aligned}\int_{B_{1/2}} |f_k(x)\Phi(x)|\, dx &\leq \|f_k\|_{L^N(B_{1/2})}\|\Phi\|_{L^{N'}(B_{1/2})} \\ &\leq \frac{C(N)}{k}\|\Phi\|_{H^1(B_{1/2})} \to 0.\end{aligned}$$

Therefore, the first and third terms in (4.11) tend to zero as $k\to\infty$. We obtain that for every $\Phi\in H^1_0(B_{1/2})$

$$\lim_{k\to\infty}\int_{B_{1/2}}\big(A(0)\nabla u_k\cdot\nabla\Phi\big)\,dx=\int_{B_{1/2}}\big(A(0)\nabla u_0\cdot\nabla\Phi\big)\,dx=0. \tag{4.12}$$

This proves $u_0\in H^1(B_{1/2})$ is a weak solution to $\operatorname{div}(A(0)\nabla u_0)=0$ in $B_{1/2}$. □

We employ Widman's hole filling technique in the proceeding proposition. The following classical lemma is useful

**Lemma 4.5.** (*c.f.* [25, Lemma 6.1]) *Let $Z(t)$ be a bounded non negative function defined in $[\rho,r]$. And assume that we have for $\rho\le s<t\le r$*

$$Z(s)\le\theta Z(t)+\frac{A}{|s-t|^2}+C$$

*for some $0\le\theta<1$ and $A,C\ge0$. Then we have*

$$Z(\rho)\le C(\theta)\Big[\frac{A}{|\rho-r|^2}+C\Big].$$

**Proposition 4.6.** *Let $u\in H^1(B_1)\cap L^\infty(B_1)$ be a weak solution of (4.1) such that $\|u\|_{L^\infty(B_1)}\le1$. Then for every $\varepsilon>0$ there exists $0<\delta(\varepsilon,N,\mu)<\frac12$ such that if*

$$\max\Big(\|A-A(0)\|_{L^\infty(B_1)},\|f\|_{L^N(B_1)}\Big)\le\delta$$

*then*

$$\|u-h\|_{L^\infty(B_{1/4})}\le\varepsilon$$

*for some $h\in H^1(B_{1/2})$ such that $\|h\|_{L^\infty(B_{1/2})}\le1$ and $h$ is a weak solution to $\operatorname{div}(A(0)\nabla h)=0$ in $B_{1/2}$.*

*Proof.* We start off by claiming that $\|\nabla u\|_{L^2(B_{1/2})}\le M(N,\mu)$.

Indeed, let $s,t>0$ be such that $1/2\le s<t\le1$ and $\eta\in C^\infty_0(B_1)$ such that $0\le\eta\le1$ and

$$\eta(x)=\begin{cases}1 & x\in B_s\\ 0 & x\in B_1\setminus B_t.\end{cases}$$

we can assume that

$$|\nabla\eta|\le\frac{C(N)}{|s-t|}\quad\text{in } B_1. \tag{4.13}$$

Now, consider $\varphi=\eta u\in H^1_0(B_1)$ a test function and from the definition of weak solutions in (4.2) we have

$$\int_{B_1}\Big(A(x)\nabla u\cdot\nabla(\eta u)\Big)\,dx=-\int_{B_1}f(x)(\eta u)\,dx.$$

We expand the integral on the LHS, after rearranging the terms and from ellipticity of $A$ we obtain

$$\mu\int_{B_s}|\nabla u|^2\,dx\le\int_{B_t}\eta\big(A(x)\nabla u\cdot\nabla u\big)\,dx=-\int_{B_t}f(x)(\eta u)\,dx-\int_{B_t}u\big(A(x)\nabla u\cdot\nabla\eta\big)\,dx. \tag{4.14}$$

We observe that since $\nabla\eta=0$ in $B_s$

$$\begin{aligned}\int_{B_t}\big|u\big(A(x)\nabla u\cdot\nabla\eta\big)\big|\,dx&=\int_{B_t\setminus B_s}\big|u\big(A(x)\nabla u\cdot\nabla\eta\big)\big|\,dx\\&\le\frac1\mu\|\nabla u\|_{L^2(B_t\setminus B_s)}\|\nabla\eta\|_{L^2(B_1)}.\end{aligned} \tag{4.15}$$

Moreover, since $\|f\|_{L^N(B_1)}\le\frac12$, $\|\eta\|_{L^\infty(B_1)}\le1$ and $\|u\|_{L^\infty(B_1)}\le1$

$$\int_{B_1}\Big|f(x)(\eta u)\,dx\Big|\le C(N)\|f\|_{L^N(B_1)}\le C(N). \tag{4.16}$$

using the (4.15), (4.16) on the third and fourth term of (4.14), for some $0<\tau<1$, we arrive to

$$\begin{aligned}\mu\int_{B_s}|\nabla u|^2\,dx&\le\frac1\mu\|\nabla u\|_{L^2(B_t\setminus B_s)}\|\nabla\eta\|_{L^2(B_1)}+\Big|\int_{B_t}f(x)(\eta u)\,dx\Big|\\&\le\frac{C(\mu)}{2\tau^2}\|\nabla u\|^2_{L^2(B_t\setminus B_s)}+2\tau^2\|\nabla\eta\|^2_{L^2(2)}+C(N).\end{aligned}$$

By taking $\tau = \frac{1}{2}$ and using (4.14) we obtain

$$\int_{B_s} |\nabla u|^2 \, dx \leq C_1(\mu) \int_{B_t \setminus B_s} |\nabla u|^2 \, dx + \frac{C_2(N)}{|s-t|^2} + C(N). \tag{4.17}$$

Now, we use the Widman's hole filling procedure. Adding the term $C_1(\mu) \int_{B_s} |\nabla u|^2 \, dx$ on both sides of (4.17) and we arrive at

$$\int_{B_s} |\nabla u|^2 \, dx \leq \frac{C_1}{C_1+1} \int_{B_t} |\nabla u|^2 \, dx + \frac{C_2 \cdot (C_1+1)^{-1}}{|s-t|^2} + \frac{C(N)}{C_1+1} \tag{4.18}$$

Now, from Lemma 4.5 with $Z(t) := \int_{B_t} |\nabla u|^2 \, dx$ we have

$$\int_{B_{1/2}} |\nabla u|^2 \, dx \leq C_3(N, \mu). \tag{4.19}$$

Once $u$ satisfies (4.1) in $B_1$ we can apply Proposition 4.4. This way, for a given $\varepsilon > 0$, we choose $\delta(\varepsilon, M, \mu, N) > 0$ in Proposition 4.4 which corresponds to $M = C_3(N,\mu)^{1/2}$. Therefore obtain $\delta := \delta(\varepsilon, \mu, N)$ satisfying the required properties. □

**Proposition 4.7** (Key lemma for PDE)**.** *Let $u \in H^1(B_1) \cap L^\infty(B_1)$ be a weak solution to the PDE (4.1) in $B_1$ with $\|u\|_{L^\infty(B_1)} \leq 1$. Then, for every $0 < \alpha < 1$, there exists $\delta(N, \mu, \alpha) > 0$ and $r_0(N, \mu, \alpha) < 1/4$ such that if*

$$\max\left( \|A(x) - A(0)\|_{L^\infty(B_1)}, \|f\|_{L^N(B_1)} \right) \leq \delta$$

*Then*

$$\sup_{B_{r_0}} |u - u(0)| \leq r_0^\alpha.$$

*Proof.* Let $\varepsilon > 0$ that will be suitably chosen later. By Proposition 4.6 there exists $\delta(\varepsilon, N, \mu) > 0$ such that

$$\|u - h\|_{L^\infty(B_{1/4})} < \varepsilon, \tag{4.20}$$

Where $h$ is a weak solution to $\operatorname{div}(A(0)\nabla h) = 0$ in $B_{1/2}$ and $\|h\|_{L^\infty(B_{1/2})} \leq 1$. By classical elliptic regularity theory, $h$ satisfies the interior gradient estimates (4.4). Hence

$$\|\nabla h\|_{L^\infty(B_{1/4})} \leq C(N, \mu). \tag{4.21}$$

Fix $\beta = (1+\alpha)/2 < 1$, we claim the following

$$\sup_{B_r} |h - h(0)| \leq C(N,\mu) r^\beta \quad \forall r < 1/4. \tag{4.22}$$

Indeed, $h - h(0)$ is a weak solution to the same PDE $\operatorname{div}(A(0)\nabla h) = 0$ and $\|h - h(0)\|_{L^\infty(B_{1/2})} \leq 2$ from Proposition 4.6. By mean value theorem and internal gradient estimates (4.4), if $x \in B_r$ and $r < 1/4$ we have

$$\begin{aligned} |h(x) - h(0)| &\leq \left|\nabla(h(\theta x) - h(0))\right| \cdot |x|, \ (0 < \theta < 1) \\ &\leq 2C(N,\mu)\|h\|_{L^\infty(B_{1/2})} r \leq C(N,\mu) r^\beta \end{aligned}$$

Combining equations (4.20) and (4.22) we get for $r < 1/4$

$$\begin{aligned} \sup_{B_r} |u(x) - u(0)| &\leq \sup_{B_r} \Big( |u(x) - h(x)| + |h(x) - h(0)| + |h(0) - u(0)| \Big) \\ &\leq 2\varepsilon + C(N,\mu) r^\beta. \end{aligned} \tag{4.23}$$

We select $r_0(N, \mu, \alpha) < 1/4$ such that

$$C(N,\mu) r_0^\beta \leq \frac{r_0^\alpha}{3}$$

that is

$$r_0 = \left( \frac{1}{3C(N,\mu)} \right)^{1/\beta - \alpha} \leq \left( \frac{1}{3C(N,\mu)} \right)^{2/1-\alpha}.$$

Now, we choose $\varepsilon(N, \mu, \alpha)$ in such a way that

$$\varepsilon < \frac{r_0^\alpha}{3}.$$

We see that the choice of $\delta$ depends on $\varepsilon$ and since $\varepsilon$ depends on $N$, $\mu$ and $\alpha$ therefore $\delta$ is actually chosen depending on $N$, $\mu$ and $\alpha$. From (4.23) we get

$$\sup_{B_{r_0}} |u - u(0)| \leq r_0^\alpha.$$

□

**Proposition 4.8.** *Suppose $u \in H^1(B_1) \cap L^\infty(B_1)$ is a weak solution to the the PDE (4.1) in $B_1$ with $\|u\|_{L^\infty(B_1)} \le 1$. Then, for every $0 < \alpha < 1$, there exists $\delta(N, \mu, \alpha) > 0$ and $C(N, \mu, \alpha) > 0$ such that if*

$$\max\left(\|A(x) - A(0)\|_{L^\infty(B_1)}, \|f\|_{L^N(B_1)}\right) \le \delta$$

*Then*

$$\sup_{B_r} |u - u(0)| \le C(N, \mu, \alpha) r^\alpha \ \forall r < r_0,$$

*where $r_0$ is as in Proposition 4.7. Precisely speaking, we have $C(N, \mu, \alpha) = r_0^{-\alpha}$.*

*Proof.* We argue by scaling and claim that for all $k \in \mathbb{N}$

$$\sup_{B_{r_0^k}} |u - u(0)| \le r_0^{k\alpha}. \tag{4.24}$$

It follows readily from Proposition 4.7 that (4.24) holds for $k = 1$. Let us suppose it also holds up to $k_0 \in \mathbb{N}$. We prove that (4.24) holds also for $k = k_0 + 1$. In order to do that, we define the following rescaled function

$$\tilde{u}(y) = \frac{1}{r_0^{k_0\alpha}} \left( u(r_0^{k_0} y) - u(0) \right), \ \ y \in B_1. \tag{4.25}$$

From Remark 4.3, we see that $\tilde{u}$ satisfies the following PDE

$$\operatorname{div}(\tilde{A}(y)\nabla\tilde{u}) = \tilde{f} \ \text{ in } B_1 \tag{4.26}$$

where $\tilde{A}$ and $\tilde{f}$ are given by

$$\begin{aligned} \tilde{A}(y) &:= A(r_0^{k_0} y) \\ \tilde{f}(y) &:= r_0^{k_0(2-\alpha)} f(r_0^{k_0} y). \end{aligned}$$

Now, we verify that the PDE (4.26) and its solution $\tilde{u}$ satisfy the assumptions of Proposition 4.7. Indeed, from (4.24) for $k = k_0$ and (4.25), we have

$$\sup_{B_1} |\tilde{u}| = r_0^{-k_0\alpha} \sup_{B_{r_0^{k_0}}} |u - u(0)| \le 1.$$

Moreover, for $\delta > 0$ as in Proposition 4.7 we see that

$$\sup_{B_1} |\tilde{A} - \tilde{A}(0)| = \sup_{B_{r_0^{k_0}}} |A - A(0)| \le \delta.$$

Additionally

$$\|\tilde{f}\|_{L^N(B_1)} = r_0^{k_0(1-\alpha)} \|f\|_{L^N(B_{r_0^{k_0}})} \le \delta.$$

So we can apply the Proposition 4.7 for $\tilde{A}$, $\tilde{f}$ and $\tilde{u}$. Hence we obtain

$$\sup_{B_{r_0}} |\tilde{u}| \le r_0^\alpha.$$

Rescaling back to $u$ we arrive at

$$\sup_{B_{r_0^{k_0+1}}} |u - u(0)| \le r_0^{(k_0+1)\alpha}.$$

So the claim (4.24) is proven for all $k \in \mathbb{N}$. To finish the proof of Proposition 4.8, let $r \in (0, r_0)$ be given then we can find $k \in \mathbb{N}$ such that $r_0^{k+1} \le r < r_0^k$. From (4.24), we see that

$$\sup_{B_r} |u - u(0)| \le \sup_{B_{r_0^k}} |u - u(0)| \le r_0^{k\alpha} = r_0^{(k+1)\alpha} \frac{1}{r_0^\alpha} \le r_0^{-\alpha} r^\alpha. \tag{4.27}$$

Finally we can take $C(N, \mu, \alpha) = \frac{1}{r_0^\alpha}$. Then Proposition 4.8 is proven. □

In the sequel, we remove the small regime condition on the previous results. As a matter of fact we prove the main result of this section.

**Proposition 4.9.** *Suppose $u \in H^1(B_1) \cap L^\infty(B_1)$ be a weak solution to (4.1) in $B_1$. Then for any $0<\alpha<1$, we have $u \in C^\alpha(B_{1/2})$ with the following estimates*

$$[u]_{C^\alpha(B_{1/2})} \leq C(N,\alpha,\mu,\omega_{A,\overline{B_{3/4}}}) \left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^N(B_1)}\right). \tag{4.28}$$

*Here $\omega_{A,\overline{B_{3/4}}}$ is uniform the modulus of continuity of $A$ in $\overline{B_{3/4}}$ (c.f. Definition 2.2). In particular, we have*

$$\|u\|_{C^\alpha(B_{1/2})} \leq C(N,\alpha,\mu,\omega_{A,\overline{B_{3/4}}}) \left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^N(B_1)}\right). \tag{4.29}$$

*Proof.* Since $A \in C^{N\times N}(B_1)$, then the coefficient matrix $A$ is uniformly continuous in $\overline{B_{3/4}}$. Hence, there exists $s_0 := s_0(\omega_{A,\overline{B_{3/4}}}, N, \mu, \alpha)$ such that $0 < s_0 < 1/4$ and for every $x_0 \in B_{1/2}$ we have

$$\sup_{B_{s_0}(x_0)} |A - A(x_0)| \leq \delta(N,\mu,\alpha) \tag{4.30}$$

for $\delta(N,\mu,\alpha) > 0$ as in Porposition 4.8. We fix now $x_0 \in B_{1/2}$ and define the following rescaled function

$$w(y) := \frac{u(x_0 + s_0 y)}{\left(\|u\|_{L^\infty(B_1)} + \frac{s_0}{\delta}\|f\|_{L^N(B_1)}\right)}, \quad y \in B_1. \tag{4.31}$$

By Remark 4.3, we can easily verify that the function $w \in W^{1,p}(B_1)$ satisfies the following PDE

$$\operatorname{div}(\bar{A}(y)\nabla v) = \bar{f} \text{ in } B_1. \tag{4.32}$$

Where $\bar{A}$ and $\bar{f}$ are defined as follows

$$\bar{A}(y) := A(x_0 + s_0 y)$$
$$\bar{f}(y) := \frac{s_0^2 \cdot f(x_0 + s_0 y)}{\left(\|u\|_{L^\infty(B_1)} + \frac{s_0}{\delta}\|f\|_{L^N(B_1)}\right)}, \; y \in B_1.$$

From (4.31), we can see that

$$\|w\|_{L^\infty(B_1)} \leq 1. \tag{4.33}$$

We rewrite (4.30) as follows

$$\sup_{B_{s_0}(x_0)} |A - A(x_0)| = \|\bar{A} - \bar{A}(0)\|_{L^\infty(B_1)} \leq \delta(N,\mu,\alpha). \tag{4.34}$$

Moreover, from the definition of $\bar{f}$, we see that

$$\|\bar{f}\|_{L^N(B_1)} = \frac{s_0 \cdot \|f\|_{L^N(B_{s_0}(x_0))}}{\left(\|u\|_{L^\infty(B_1)} + \frac{s_0}{\delta}\|f\|_{L^N(B_1))}\right)} \leq \delta(N,\mu,\alpha). \tag{4.35}$$

From (4.33), (4.32), (4.34) and (4.35) we see that $w, \bar{A}$ and $\bar{f}$ satisfy the assumptions of Proposition 4.8, and therefore from Proposition 4.8 we have

$$\sup_{B_r} |w - w(0)| \leq \frac{1}{r_0^\alpha} r^\alpha, \quad \forall r \leq r_0(N,\mu,\alpha)$$

Rescaling back to $u$, we obtain for every $x_0 \in B_{1/2}$

$$\sup_{z \in B_{s_0 r}(x_0)} \left| \frac{u - u(x_0)}{\left(\|u\|_{L^\infty(B_1)} + \frac{s_0}{\delta}\|f\|_{L^N(B_1)}\right)} \right| \leq C r^\alpha, \quad r \leq r_0.$$

Therefore

$$\sup_{B_r(x_0)} |u - u(x_0)| \leq \frac{1}{r_0^\alpha s_0^\alpha} \left(\|u\|_{L^\infty(B_1)} + \frac{s_0}{\delta}\|f\|_{L^N(B_1)}\right) r^\alpha, \quad \forall r \leq s_0 r_0. \tag{4.36}$$

Let $x, y \in B_{1/2}$. If $|x-y| \leq s_0 r_0$, then we replace $x$ by $x_0$ and take $r = |x-y|$ in the above equation and we have

$$|u(x) - u(y)| \leq \frac{1}{r_0^\alpha s_0^\alpha} \left(\|u\|_{L^\infty(B_1)} + \frac{s_0}{\delta}\|f\|_{L^N(B_1)}\right) |x-y|^\alpha.$$

Otherwise, if $|x-y| > s_0 r_0$ then $\frac{|u(x)-u(y)|}{|x-y|^\alpha} \leq \frac{2\|u\|_{L^\infty(B_1)}}{s_0^\alpha r_0^\alpha}$. In other words,

$$|u(x) - u(y)| \leq \frac{2\|u\|_{L^\infty(B_1)}}{s_0^\alpha r_0^\alpha} |x-y|^\alpha.$$

Finally combining the two estimates above we have for all $x, y \in B_{1/2}$

$$\begin{aligned} |u(x)-u(y)| &\leq \frac{C}{r_0^\alpha s_0^\alpha}\left(\|u\|_{L^\infty(B_1)}+\frac{s_0}{\delta}\|f\|_{L^N(B_1)}\right)|x-y|^\alpha \\ &\leq \frac{C(1+\frac{1}{\delta})}{r_0^\alpha s_0^\alpha}\left(\|u\|_{L^\infty(B_1)}+\|f\|_{L^N(B_1)}\right)|x-y|^\alpha. \end{aligned} \tag{4.37}$$

This proves Proposition 4.9. □

*Remark* 4.10. The dependence of constant at right hand side of (4.28) and (4.29) on the modulus of continuity $\omega_{A,\overline{B_{3/4}}}$ can be removed as long as we are under the smallness condition on the oscillations of the coefficient $A$. That is if we have

$$\|A-A(0)\|_{L^\infty(B_1)} \leq \delta \tag{4.38}$$

for $\delta(N,\mu,\alpha)$ as in Lemma 4.8. Then under this condition we have

$$\|u\|_{C^\alpha(B_{1/2})} \leq C(N,\mu,\alpha)\left(\|u\|_{L^\infty(B_1)}+\|f\|_{L^N(B_1)}\right). \tag{4.39}$$

To see this, we observe that in (4.37), the dependence on $\omega_{A,\overline{B_{3/4}}}$ is coming from $s_0$. This positive number $s_0$ is chosen so small that $\|A-A(0)\|_{L^\infty(B_{s_0(x_0)})} \leq \delta$. This step is not required if we already assume (4.38).

Proposition 4.9 is scaled version of the regularity estimates for weak solution to the PDE (4.1). From the classical rescaling argument, we can also prove the following result for any general domain $\Omega$.

**Corollary 4.11.** *Let $u \in H^1(B_\rho(x_0)) \cap L^\infty(B_\rho(x_0))$ be a weak solution to*

$$\operatorname{div}(A(x)\nabla u) = f \quad \text{in } B_\rho(x_0)$$

*where $A \in C(B_\rho(x_0))^{N\times N}$ and there exists $0<\mu<1$ such that $\mu|\xi|^2 \leq \langle A(x)\xi,\xi\rangle \leq \frac{1}{\mu}|\xi|^2$ a.e. in $B_\rho(x_0)$ and for all $\xi \in \mathbb{R}^N$. $f \in L^N(B_\rho(x_0))$. Then for any $0<\alpha<1$, $u \in C^{0,\alpha}(B_{\frac{\rho}{2}}(x_0))$ and if $0<\rho<1$ we have*

$$\|u\|_{C^{0,\alpha}(B_{\frac{\rho}{2}}(x_0))} \leq \frac{C(N,\mu,\alpha,\omega_{A,\overline{B_{\frac{3\rho}{4}}(x_0)}})}{\rho^\alpha}\left(\|u\|_{L^\infty(B_\rho(x_0))}+\rho\|f\|_{L^N(B_\rho(x_0))}\right). \tag{4.40}$$

*Here $\omega_{A,\overline{B_{\frac{3\rho}{4}}(x_0)}}$ is the uniform modulus of continuity of the coefficient $A$ in $\overline{B_{\frac{3\rho}{4}}(x_0)}$. Furthermore, the dependence on $\omega_{A,\overline{B_{\frac{3\rho}{4}}(x_0)}}$ can be dropped under the smallness assumption*

$$\|A-A(x_0)\|_{L^\infty(B_\rho(x_0))} \leq \delta \tag{4.41}$$

*for ($\delta$ as in Lemma 4.8 ). Then if $0<\rho<1$*

$$\|u\|_{C^{0,\alpha}(B_{\frac{\rho}{2}}(x_0))} \leq \frac{C(N,\mu,\alpha)}{\rho^\alpha}\left(\|u\|_{L^\infty(B_\rho(x_0))}+\rho\|f\|_{L^N(B_\rho(x_0))}\right). \tag{4.42}$$

*Proof.* The proof of Corollary 4.11 follows readily by scaling. In fact, we just consider the function defined as $v(y) := \frac{u(x_0+\rho y)}{\rho}$ and apply Proposition 4.9 together with Remark 4.10. Indeed, from Remark 4.3, $v \in H^1(B_1)\cap L^\infty(B_1)$ is a weak solution to the PDE $\operatorname{div}(\bar{A}(y)\nabla v) = \bar{f}$ in $B_1$. Where $\bar{A}$ and $\bar{f}$ are given by

$$\bar{A}(y) = A(x_0+\rho y), \quad \bar{f}(y) = \rho f(x_0+\rho y).$$

We can obtain the estimates from Proposition 4.9 on $v$ and by translating information back to $u$. □

We prove the main result of this section

## 5. Proof of Theorem 2.4

*Proof of Theorem 2.4.* Let $D \subset\subset B_1$. We observe that $D \subset\subset B_d \subset\subset B_1$ where $d = 1-\frac{\operatorname{dist}(D,\partial B_1)}{2}$. Since $A \in C(B_1)^{N\times N}$, then the matrix $A$ is uniformly continuous in $\overline{B_d}$. Thus, there exists a modulus of continuity $\omega_{A,\overline{B_d}}$ (c.f. Definition 2.2) given by

$$\omega_{A,\overline{B_d}}(t) := \begin{cases} \left(\sup_{\substack{|x-y|<t \\ x,y\in\overline{B_d}}} |A(x)-A(y)|\right) & t \leq 2d \\ \omega_{A,\overline{B_d}}(2d) & t > 2d. \end{cases}$$

We set $t_0$ as

$$t_0 := t_0(\omega_{A,\overline{B_d}}, \delta) = \sup\left\{t \,\middle|\, \omega_{A,\overline{B_d}}(t) \le \delta\right\}$$

as well as $s_0 := \min\left(t_0, \frac{\operatorname{dist}(D,\partial B_1)}{4}\right)$. Since $\omega_{A,\overline{B_d}}$ is a non-decreasing function we have $\omega_{A,\overline{B_d}}(s_0) \le \delta$. Now since

$$D \subset \bigcup_{x\in D} B_{s_0}(x) \subset \overline{B_d}$$

we have

$$\sup_{B_{s_0}(x)} |A - A(x)| \le \omega_{A,\overline{B_d}}(s_0) \le \delta, \ \forall x \in D. \tag{5.1}$$

We note that $u$ is also an absolute minimizer of $J(\cdot; B_{s_0}(x))$, by (5.1) and (4.42) in Corollary 4.11 we have for all $y \in B_{s_0/2}(x) \cap D$

$$|u(x) - u(y)| \le \frac{C(N,\alpha,\mu)}{s_0^\alpha}\left(\|u\|_{L^\infty(B_1)} + s_0\|f\|_{L^N(B_1)}\right)|x-y|^\alpha. \tag{5.2}$$

Now, if $x, y \in D$ are such that $|x-y| \ge s_0/2$, then

$$|u(x) - u(y)| \le 2^{1+\alpha}\frac{\|u\|_{L^\infty(B_1)}}{s_0^\alpha}|x-y|^\alpha. \tag{5.3}$$

By combining (5.2) and (5.3), we arrive to $\left(\text{since } s_0 \le \frac{\operatorname{dist}(D,\partial B_1)}{4} \le \frac{\operatorname{diam}(B_1)}{4} = \frac{1}{2} < 1\right)$

$$[u]_{C^\alpha(D)} \le \frac{C(N,\alpha,\mu)}{s_0^\alpha}\left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^N(B_1)}\right). \tag{5.4}$$

We observe from the definition of $s_0$ that

$$[u]_{C^\alpha(D)} \le \begin{cases} \frac{C(N,\alpha,\mu)}{t_0^\alpha}\left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^N(B_1)}\right) & \text{if } t_0 \le \frac{\operatorname{dist}(D,\partial B_1)}{4} \\ \frac{4^\alpha\cdot C(N,\alpha,\mu)}{\operatorname{dist}(D,\partial B_1)^\alpha}\left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^N(B_1)}\right) & \text{if } t_0 \ge \frac{\operatorname{dist}(D,\partial B_1)}{4}. \end{cases} \tag{5.5}$$

In order to control the first term in the equation above by a universal multiple of $\operatorname{dist}(D,\partial B_1)^{-\alpha}$, we observe that once $t_0 > 0$ depends only on the modulus of continuity $\omega_{A,\overline{B_d}}$ and $\delta$, there exists a $n_0 := n_0(\omega_{A,\overline{B_d}}, \delta) = n_0(N,\mu,\alpha,\omega_{A,\overline{B_d}}) \in \mathbb{N}$ depending only on $t_0$ such that $\frac{2}{n_0} \le t_0$. Hence

$$\frac{\operatorname{dist}(D,\partial B_1)}{n_0} \le \frac{\operatorname{diam}(B_1)}{n_0} = \frac{2}{n_0} \le t_0 \implies \frac{1}{t_0^\alpha} \le \frac{n_0^\alpha}{\operatorname{dist}(D,\partial B_1)^\alpha} = \frac{C(N,\mu,\alpha,\omega_{A,\overline{B_d}})}{\operatorname{dist}(D,\partial B_1)^\alpha}. \tag{5.6}$$

Now, (5.5) becomes

$$[u]_{C^\alpha(D)} \le \frac{C(N,\alpha,\mu,\omega_{A_\pm,\overline{B_d}})}{\operatorname{dist}(D,\partial B_1)^\alpha}\left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^N(B_1)}\right). \tag{5.7}$$

To finish we observe that Theorem 2.4 follows from (5.7) by taking $D$ as $B_r$ for any $r < 1$, once $\operatorname{dist}(D,\partial B_1)^\alpha = \operatorname{dist}(B_r,\partial B_1)^\alpha = (1-r)^\alpha$ and $d = \frac{1+r}{2}$. □

## 6. Approximation lemma via compactness

Our strategy is to make use of regularity theory of limiting functional and prove regularity of minimizers. We were inspired from the method used in [2, Lemma 5.4]. First we show that if the functions $A_+$ and $A_-$ are very close to different constants, the regularity of a minimizer is close to $C^{0,1^-}$.

*Remark* 6.1. Although elementary, we make a systematic use of limsup and liminf properties (especially $*$ and $**$) in our proof of compactness. For the reader's convenience, we record them here. For any bounded sequences of real numbers $a_k$ and $b_k$, the following properties hold (for the proofs check for instance [20, Theorem 3.127])

**($\mathcal{I}$1)** $\limsup_{k\to\infty}(-a_k) = -\liminf_{k\to\infty}(a_k)$ and $\liminf_{k\to\infty}(-a_k) = -\limsup_{k\to\infty}(a_k)$.

**($\mathcal{I}$2)** Also we have

$$\begin{aligned}\liminf_{k\to\infty} a_k + \liminf_{k\to\infty} b_k &\overset{*}{\leq} \liminf_{k\to\infty}(a_k + b_k)\\ &\leq \limsup_{k\to\infty} a_k + \liminf_{k\to\infty} b_k\\ &\leq \limsup_{k\to\infty}(a_k + b_k)\\ &\overset{**}{\leq} \limsup_{k\to\infty} a_k + \limsup_{k\to\infty} b_k.\end{aligned}$$

*Remark* 6.2. In the forthcoming proofs, we consider the functional $J(\cdot; B_1) := J_{A,f,Q}(\cdot; B_1)$, where $B_1 := B_1(0)$ and

$$J(\cdot; B_1) := J_{A,f,Q}(v; B_1) = \int_{B_1} A(x,u)|\nabla u|^2 + f(x,u)u + Q(x)\lambda(v)\,dx.$$

Up to rescaling, $B_1$ represents a small ball contained inside a general domain $\Omega$. We already know that local minimizers of $J(\cdot;\Omega)$ are locally bounded in $\Omega$ c.f. [25, Theorem 7.3], [43, Theorem 2.3]. Therefore it is reasonable to assume that for a minimizer $u \in H^1(B_1)$ of $J_{A,f,Q}(u; B_1) < \infty$ and $\|u\|_{L^\infty(B_1)} \leq 1$. These assumptions do not compromise any generality.

We now approximate minimizers by regular ones under the small regime scenario.

**Theorem 6.3.** *Let $u \in H^1(B_{1/2})$ be an absolute minimizer of $J(\cdot, B_{1/2})$ such that $J(\cdot; B_{1/2})$ satisfies structural conditions* ***(G1)-(G5)****. Assume also that $\|\nabla u\|_{L^2(B_{1/2})} \leq M$ and $\|u\|_{L^\infty(B_{1/2})} \leq 1$. Then for any $\varepsilon > 0$ there exists $\delta(\varepsilon, M, \mu, q_2, \lambda_+, N) > 0$ such that if*

$$\max\Big(\|A_\pm - A_\pm(0)\|_{L^\infty(B_{1/2})},\ \|f_\pm\|_{L^N(B_{1/2})},\ \|Q - Q(0)\|_{L^\infty(B_{1/2})}\Big) \leq \delta$$

*then*

$$\|u - h\|_{L^\infty(B_{1/4})} \leq \varepsilon$$

*where $h$ is an absolute minimizer of*

$$J_0(v; B_{1/2}) := \int_{B_{1/2}} \Big(A(0,v)|\nabla v|^2 + Q(0)\lambda(v)\Big)\,dx.$$

*Moreover, $\|h\|_{L^\infty(B_{1/2})} \leq 1$.*

*Proof.* Let us suppose for the sake of contradiction that the statement of Theorem 6.3 is not true. This implies that there exists an $\varepsilon_0 > 0$ and sequences $A_k, f_k, Q_k$ such that $A_{\pm,k} \in C(B_{1/2})$, $Q_k \in L^\infty(B_{1/2})$ and $f_k \in L^N(B_{1/2})$ with $\|A_{\pm,k} - A_\pm(0)\|_{L^\infty(B_{1/2})} < \frac{1}{k}$, $\|f_{\pm,k}\|_{L^N(B_{1/2})} < \frac{1}{k}$ and $\|Q_k - Q(0)\|_{L^\infty(B_{1/2})} < \frac{1}{k}$ as well as corresponding absolute minimizers $u_k \in H^1(B_{1/2})$ of the functional $J_k(\cdot; B_{1/2})$ (defined in (6.2) below) such that $\|\nabla u_k\|_{L^p(B_{1/2})} \leq M$ and $\|u_k\|_{L^\infty(B_{1/2})} \leq 1$ also satisfy

$$\|u_k - h\|_{L^\infty(B_{1/4})} > \varepsilon_0 \tag{6.1}$$

for all $h \in H^1(B_{1/2}) \cap L^\infty(B_{1/2})$ with $\|h\|_{L^\infty(B_{1/2})}$ that are absolute minimizers of the functional of the type

$$J_0(v; B_{1/2}) := \int_{B_{1/2}} \Big(A(0,v)|\nabla v|^2 + Q(0)\lambda(v)\Big)\,dx.$$

The functional $J_k(\cdot; B_{1/2})$ is given by

$$J_k(v; B_{1/2}) := \int_{B_{1/2}} \Big(A_k(x,v)|\nabla v|^2 + f_k(x,v)v + Q_k(x)\lambda(v)\Big)\,dx. \tag{6.2}$$

We recall that $\lambda(s) = \lambda_+\chi_{\{s>0\}} + \lambda_-\chi_{\{s\leq 0\}}$ and

$$A_k(x,v) = A_{+,k}(x)\chi_{\{v>0\}} + A_{-,k}(x)\chi_{\{v\leq 0\}},\ f_k(x,v) = f_{+,k}(x)\chi_{\{v>0\}} + f_{-,k}(x)\chi_{\{v\leq 0\}}. \tag{6.3}$$

From **(G1)-(G5)**, it follows that the functional $J_k(\cdot; B_{1/2})$ and its integrand satisfy the structural condition in [25, equation 7.2] uniformly in $k$. Therefore, from [25, Theorem 7.6] it follows the existence of $\beta_0 := \beta_0(\mu, q_2, \lambda_+, N) \in (0,1)$ and $C_0 := C_0(\mu, q_2, \lambda_+, N) > 0$ such that for every $k \in \mathbb{N}$ we have

$$\|u_k\|_{C^{\beta_0}(\overline{B_{1/4}})} \leq C_0.$$

By Arzela Ascoli theorem, there exists $u_0 \in C^{0,\beta_0}(B_{1/4})$ such that

$$u_k \to u_0 \text{ in } L^\infty(B_{1/4}) \text{ up to a subsequence.} \tag{6.4}$$

Since $\|\nabla u_k\|_{L^2(B_{1/2})} \le M$ and $\|u_k\|_{L^2(B_{1/2})} \le C(N)\cdot \|u_k\|_{L^\infty(B_{1/2})} \le C(N)$ for every $k$, $u_k$ is a bounded sequence in $H^1(B_{1/2})$. That is

$$\|u_k\|_{H^1(B_{1/2})} \le C(M,N). \tag{6.5}$$

Therefore, $u_k$ converges weakly to $u_0$ (up to a subsequence)

$$u_k \rightharpoonup u_0 \;\text{ weakly in } H^1(B_{1/2}).$$

From (6.5) and weak lower semicontinuity of $H^1(B_{1/2})$ norm, this implies

$$\|u_0\|_{H^1(B_1)} \le C(M,N). \tag{6.6}$$

Also $\|u_0\|_{L^\infty(B_{1/2})} \le 1$ from Lemma 3.14. In order to simplify the steps to come, we introduce $\mathcal{G}_k(\cdot;B_{1/2})$ as follows

$$\mathcal{G}_k(v;B_{1/2}) := \int_{B_{1/2}} \Big( (A_k(x,v) - A(0,v))|\nabla v|^2 + f_k(x,v)v + (Q_k(x) - Q(0))\lambda(v) \Big)\, dx.$$

We observe that for any function $v \in H^1(B_{1/2})$, we have the following splitting

$$J_k(v;B_{1/2}) = J_0(v;B_{1/2}) + \mathcal{G}_k(v;B_{1/2}). \tag{6.7}$$

Now we make a claim

$$\textbf{Claim: } u_0 \text{ is an absolute minimizer of } J_0(\cdot;B_{1/2}). \tag{$\boldsymbol{\mathcal{M}}$}$$

Proving the claim ($\boldsymbol{\mathcal{M}}$) will lead us to a contradiction to (6.1) and thus to the proof of Theorem 6.3.

In order to prove the claim ($\boldsymbol{\mathcal{M}}$), let us take a function $v \in H^1(B_{1/2})$ be such that $v - u_0 \in H^1_0(B_{1/2})$. To ease the notation, we relabel the constants $A_\pm(0)$ as follows

$$A_+(0) =: a^2$$
$$A_-(0) =: b^2.$$

We observe that $\mu \le a^2, b^2 \le \frac{1}{\mu}$ (c.f. **(G3)**). For $r \in (0, \frac{1}{2})$ we set $\eta_r \in C^\infty_c(\mathbb{R}^N)$ so that

$$\eta_r = 1 \;\text{ in } B_{\frac{1}{2}-r}, \qquad \operatorname{supp}(\eta_r) \subset \overline{B_{1/2}}, \qquad |\nabla \eta| \le \tfrac{C(N)}{r}. \tag{6.8}$$

Now, we consider $v_{k,r} \in H^1(B_{1/2})$ such that

$$\mathcal{T}_{a,b}(v_{k,r}) := \mathcal{T}_{a,b}(v) + (1-\eta_r)(\mathcal{T}_{a,b}(u_k) - \mathcal{T}_{a,b}(u_0)). \tag{6.9}$$

The function $v_{k,r} \in H^1(B_{1/2})$ above is well defined due to the fact that the $\mathcal{T}_{a,b}$-operator is a bijective map in $H^1(B_{1/2})$ (c.f. Lemma 3.9-**(1)**).

Let us verify that $v_{k,r}$ is an admissible competitor for the minimality of $u_k$ for the functional $J_k(\cdot;B_{1/2})$. Indeed, we observe that since $\eta_r \in C^\infty_c(B_{1/2}) \subset H^1_0(B_{1/2})$ then

$$\operatorname{Tr}\left(\eta_r \left(\mathcal{T}_{a,b}(u_k) - \mathcal{T}_{a,b}(u_0)\right)\right) = 0 \text{ on } \partial B_{1/2}.$$

Now, we have

$$\begin{aligned}
\operatorname{Tr}\left(\mathcal{T}_{a,b}(v_{k,r})\right) &= \operatorname{Tr}(\mathcal{T}_{a,b}(v)) + \operatorname{Tr}\left((1-\eta_r)(\mathcal{T}_{a,b}(u_k) - \mathcal{T}_{a,b}(u_0))\right) \\
&= \operatorname{Tr}\left(\mathcal{T}_{a,b}(v)\right) + \operatorname{Tr}\left(\mathcal{T}_{a,b}(u_k) - \mathcal{T}_{a,b}(u_0)\right) - \operatorname{Tr}\left(\eta_r \left(\mathcal{T}_{a,b}(u_k) - \mathcal{T}_{a,b}(u_0)\right)\right) \\
&= \operatorname{Tr}\left(\mathcal{T}_{a,b}(v)\right) + \operatorname{Tr}\left(\mathcal{T}_{a,b}(u_k) - \mathcal{T}_{a,b}(u_0)\right) = \operatorname{Tr}\left(\mathcal{T}_{a,b}(v)\right) + \operatorname{Tr}\left(\mathcal{T}_{a,b}(u_k)\right) - \operatorname{Tr}\left(\mathcal{T}_{a,b}(u_0)\right).
\end{aligned}$$

Using Proposition 3.7-**(4b)**, and the fact that $\operatorname{Tr}(u_0) = \operatorname{Tr}(v)$ on $\partial B_{1/2}$ we arrive at

$$\mathcal{T}^{\partial B_{1/2}}_{a,b}\big(\operatorname{Tr}(v_{k,r})\big) = \mathcal{T}^{\partial B_{1/2}}_{a,b}\big(\operatorname{Tr}(v)\big) + \mathcal{T}^{\partial B_{1/2}}_{a,b}\big(\operatorname{Tr}(u_k)\big) - \mathcal{T}^{\partial B_{1/2}}_{a,b}\big(\operatorname{Tr}(u_0)\big) = \mathcal{T}^{\partial B_{1/2}}_{a,b}\big(\operatorname{Tr}(u_k)\big).$$

Since $\mathcal{T}^{\partial B_{1/2}}_{a,b}$ is a bijective map (c.f. Proposition 3.9-**(2)**), we conclude

$$\operatorname{Tr}(v_{k,r}) = \operatorname{Tr}(u_k) \text{ on } \partial B_{1/2}. \tag{6.10}$$

Therefore, by minimality of $u_k$ for $J_k(\cdot;B_{1/2})$

$$J_k(u_k;B_{1/2}) \le J_k(v_{k,r};B_{1/2}). \tag{6.11}$$

From the splitting in (6.7), we can write

$$J_0(u_k;B_{1/2}) + \mathcal{G}_k(u_k;B_{1/2}) \le J_0(v_{k,r};B_{1/2}) + \mathcal{G}_k(v_{k,r};B_{1/2}). \tag{6.12}$$

In order to further ease the notation, we relabel $\mathcal{T}_{a,b}(v_{k,r})$, $\mathcal{T}_{a,b}(v)$, $\mathcal{T}_{a,b}(u_0)$, $\mathcal{T}_{a,b}(u_k)$ as follows

$$\mathcal{T}_{a,b}(v_{k,r}) =: V_{k,r}, \;\; \mathcal{T}_{a,b}(v) =: V, \;\; \mathcal{T}_{a,b}(u_0) =: U_0, \;\; \mathcal{T}_{a,b}(u_k) =: U_k. \tag{6.13}$$

We can rewrite the definition of $v_{k,r}$ in (6.9) as

$$V_{k,r} = V + (1-\eta_r)(U_k - U_0). \tag{6.14}$$

Next, we show that $\|v_{k,r}\|_{H^1(B_{1/2})}$ and $\|V_{k,r}\|_{H^1(B_{1/2})}$ are bounded sequences independent of $k$. First we observe that from Proposition 3.9-**(3)**, (6.5) and (6.6) we have

$$\begin{aligned}
\|U_k\|_{H^1(B_{1/2})} &\le \max(a,b)\cdot\|u_k\|_{H^1(B_{1/2})} \le \frac{1}{\sqrt{\mu}}C(N,M)\\
\|U_0\|_{H^1(B_{1/2})} &\le \max(a,b)\cdot\|u_0\|_{H^1(B_{1/2})} \le \frac{1}{\sqrt{\mu}}C(N,M)\\
\|V\|_{H^1(B_{1/2})} &\le \max(a,b)\cdot\|v\|_{H^1(B_{1/2})} \le \frac{1}{\sqrt{\mu}}\|v\|_{H^1(B_{1/2})}.
\end{aligned} \tag{6.15}$$

In order to bound $\int_{B_{1/2}}|V_{k,r}|^2\,dx$ we observe that from (6.15) we have

$$\begin{aligned}
\int_{B_{1/2}}|V_{k,r}|^2\,dx &= \int_{B_{1/2}}\left|V+(1-\eta_r)(U_k-U_0)\right|^2dx\\
&\le 2\int_{B_{1/2}}\left(|V|^2+|U_k|^2+|U_0|^2\right)dx\\
&\le \frac{1}{\mu}\Big(\|v\|^2_{H^1(B_{1/2})}+C(N,M)\Big) = C(N,M,\mu)\Big(\|v\|^2_{H^1(B_{1/2})}+1\Big).
\end{aligned}$$

Now, we estimate $\int_{B_{1/2}}|\nabla V_{k,r}|^2\,dx$. Again using (6.15) and (6.8) we have

$$\begin{aligned}
\int_{B_{1/2}}|\nabla V_{k,r}|^2\,dx &= \int_{B_{1/2}}\left|\nabla\big(V+(1-\eta_r)(U_k-U_0)\big)\right|^2dx\\
&\le 2\left[\int_{B_{1/2}}|\nabla V|^2\,dx + \frac{C(N)}{r^2}\int_{B_{1/2}}|U_0-U_k|^2\,dx + \int_{B_{1/2}}|\nabla(U_0-U_k)|^2\,dx\right]\\
&\le 4\left[\int_{B_{1/2}}\Big(|\nabla V|^2+|\nabla U_0|^2+|\nabla U_k|^2\Big)dx + \frac{C(N)}{r^2}\int_{B_{1/2}}\Big(|U_0|^2+|U_k|^2\Big)dx\right]\\
&\le 4\left[\frac{1}{\mu}\|v\|^2_{H^1(B_{1/2})}+\frac{1}{\mu}C(N,M)+\frac{1}{r^2\mu}C(N,M)\right]\\
&\le \frac{\bar{C}(N,M)}{r^2\mu}\Big(\|v\|^2_{H^1(B_{1/2})}+1\Big) = C_1(N,M,\mu,r)\Big(\|v\|^2_{H^1(B_{1/2})}+1\Big).
\end{aligned} \tag{6.16}$$

Thus, $r>0$ we have a bound on $\|V_{k,r}\|_{H^1(B_{1/2})}$ independent of $k$. Again from Proposition 3.9-**(3)** we have

$$\|v_{k,r}\|^2_{H^1(B_{1/2})} \le \frac{1}{\big(\min(a,b)\big)^2}\|V_{k,r}\|^2_{H^1(B_{1/2})} \le c_1(N,M,\mu,r)\Big(\|v\|^2_{H^1(B_{1/2})}+1\Big). \tag{6.17}$$

Since $u_k \rightharpoonup u_0$ weakly in $H^1(B_{1/2})$, Lemma 3.7-**(1b)** gives $U_k \rightharpoonup U_0$ weakly in $H^1(B_{1/2})$. From the properties of weak convergence in $H^1(B_{1/2})$ and Sobolev embeddings, we can verify that the following convergences hold up to a subsequence

- **(C1)** $\nabla u_k \rightharpoonup \nabla u_0$ and $\nabla U_k \rightharpoonup \nabla U_0$ weakly in $H^1_0(B_{1/2})$ (c.f. Proposition 3.7-**(1b)**).
- **(C2)** $u_k \to u_0$ and $U_k \to U_0$ strongly in $L^2(B_{1/2})$ (c.f. Proposition 3.7-**(3b)**).
- **(C3)** $u_k \to u_0$ and $U_k \to U_0$ pointwise almost everywhere in $B_{1/2}$ (c.f. Proposition 3.7-**(6)**).

In order to proceed with the proof, we need some auxiliary estimates which are listed below

- **(EI)** For every $w\in H^1(B_{1/2})$ we have

$$\mathcal{G}_k(w;B_{1/2}) \le \frac{C(N,\lambda_+)}{k}\Big(\|w\|^2_{H^1(B_{1/2})}+1\Big). \tag{6.18}$$

- **(EII)** For $U_0 = \mathcal{T}_{a,b}(u_0)$ we have

$$\int_{B_{1/2}} Q(0)\lambda(U_0)\,dx \le \liminf_{k\to\infty}\int_{B_{1/2}} Q(0)\lambda(U_k)\,dx. \tag{6.19}$$

**(EIII)** For $V_{k,r} := \mathcal{T}_{a,b}(v_{k,r})$,

$$\limsup_{r\to 0}\left(\limsup_{k\to\infty}\int_{B_{1/2}} Q(0)\lambda(V_{r,k})\,dx\right) \leq \int_{B_{1/2}} Q(0)\lambda(V)\,dx. \tag{6.20}$$

**(EIV)** For $U_k := \mathcal{T}_{a,b}(u_k)$ and $V := \mathcal{T}_{a,b}(v)$ we have for every $k \in \mathbb{N}$

$$\liminf_{k\to\infty}\int_{B_{1/2}} \left(|\nabla U_k|^2 - |\nabla V_{k,r}|^2\right) dx \geq \int_{B_{1/2}} \left(|\nabla U_0|^2 - |\nabla V|^2\right) dx. \tag{6.21}$$

Before delving into the proofs of **(EI)**, **(EII)**, **(EIII)** and **(EIV)**, we assume for the moment that all of them hold and prove minimality of $u_0$ for $J_0(\cdot; B_{1/2})$. We come back to their proofs after that.

First, we recall that for any $w \in H^1(B_{1/2})$ we have (c.f. (3.2) and (3.8))

$$\begin{aligned} J_0(w; B_{1/2}) &= \int_{B_{1/2}} \left(a^2|\nabla w^+|^2 + b^2|\nabla w^-|^2 + Q(0)\lambda(w)\right) dx \\ &= \int_{B_{1/2}} \left(|\nabla \mathcal{T}_{a,b}(w)|^2 + Q(0)\lambda(\mathcal{T}_{a,b}(w))\right) dx. \end{aligned} \tag{6.22}$$

From Lemma 3.10, we observe that to prove that $u_0$ is an absolute minimizer of $J_0(\cdot; B_{1/2})$, it is equivalent to show that $U_0 = \mathcal{T}_{a,b}(u_0)$ is an absolute minimizer of $\mathcal{F}_0(\cdot; B_{1/2})$ which is given by

$$\mathcal{F}_0(W; B_{1/2}) := \int_{B_{1/2}} \left(|\nabla W|^2 + Q(0)\lambda(W)\right) dx.$$

So, our task is reduced to prove that $U_0$ is an absolute minimizer of $\mathcal{F}_0(\cdot; B_{1/2})$. We make the following claim

**Claim:** $U_0$ is an absolute minimizer of $\mathcal{F}_0(\cdot; B_{1/2})$. $\qquad (\boldsymbol{\mathcal{M}'})$

From (6.11), (6.12) and (6.22), we have

$$\int_{B_{1/2}} \left(|\nabla U_k|^2 + Q(0)\lambda(U_k)\right) dx + \mathcal{G}_k(u_k) \leq \int_{B_{1/2}} \left(|\nabla V_{k,r}|^2 + Q(0)\lambda(V_{k,r})\right) dx + \mathcal{G}_k(v_{k,r}).$$

We rearrange the terms in the inequality above and take $\liminf_{k\to\infty}$ on the both sides to arrive to

$$\liminf_{k\to\infty}\left[\int_{B_{1/2}} \left(|\nabla U_k|^2 - |\nabla V_{k,r}|^2\right) dx + \int_{B_{1/2}} Q(0)\Big(\lambda(U_k) - \lambda(V_{k,r})\Big)\, dx\right] \leq \liminf_{k\to\infty}\left(\mathcal{G}_k(v_{k,r}) - \mathcal{G}_k(u_k)\right). \tag{6.23}$$

From (6.5), (6.17) and **(EI)** we have

$$\lim_{k\to\infty}\mathcal{G}_k(v_{k,r}; B_{1/2}) = \lim_{k\to\infty}\mathcal{G}_k(u_k; B_{1/2}) = 0. \tag{6.24}$$

Thus, from (6.23) we conclude

$$\liminf_{k\to\infty}\left[\int_{B_{1/2}} \left(|\nabla U_k|^2 - |\nabla V_{k,r}|^2\right) dx + \int_{B_{1/2}} Q(0)\Big(\lambda(U_k) - \lambda(V_{k,r})\Big)\, dx\right] \leq 0. \tag{6.25}$$

Moreover, from (6.15) and (6.16) the first integral in the estimates above is bounded independent of $k$, i.e. for every $k \in \mathbb{N}$

$$\left|\int_{B_{1/2}} \left(|\nabla U_k|^2 - |\nabla V_{k,r}|^2\right) dx\right| \leq C(N, M, \mu, r)\Big(\|v\|^2_{H^1(B_{1/2})} + 1\Big).$$

Furthermore,

$$\left|\int_{B_{1/2}} Q(0)(\lambda(U_k) - \lambda(V_{k,r}))\, dx\right| \leq 2C(N)q_2\lambda_+.$$

Therefore, we are entitled to use the $(\boldsymbol{\mathcal{I}2})^*$ in (6.25) to obtain

$$\liminf_{k\to\infty}\int_{B_{1/2}} \left(|\nabla U_k|^2 - |\nabla V_{k,r}|^2\right) dx + \liminf_{k\to\infty}\int_{B_{1/2}} Q(0)\Big(\lambda(U_k) - \lambda(V_{k,r})\Big)\, dx \leq 0.$$

Now rearranging the terms and from **$(\mathcal{I}1)^*$** and **$(\mathcal{I}1)^{**}$** together with **(EIV)** we have

$$\begin{aligned}\int_{B_{1/2}}(|\nabla U_0|^2-|\nabla V|^2)\,dx &\le \liminf_{k\to\infty}\int_{B_{1/2}}\Big(|\nabla U_k|^2-|\nabla V_{k,r}|^2\Big)\,dx\\ &\le \limsup_{k\to\infty}\int_{B_{1/2}}Q(0)\Big(\lambda(V_{k,r})-\lambda(U_k)\Big)\,dx\\ &\le \limsup_{k\to\infty}\int_{B_{1/2}}Q(0)\lambda(V_{k,r})\,dx+\limsup_{k\to\infty}\int_{B_{1/2}}-Q(0)\lambda(U_k)\,dx\\ &= \limsup_{k\to\infty}\int_{B_{1/2}}Q(0)(\lambda(V_{k,r})\,dx-\liminf_{k\to\infty}\int_{B_{1/2}}Q(0)\lambda(U_k)\,dx.\end{aligned}\tag{6.26}$$

Taking $\limsup_{r\to 0}$ on both sides of the inequality above and using **(EII)** and **(EIII)**, we arrive at

$$\begin{aligned}\int_{B_{1/2}}(|\nabla U_0|^2-|\nabla V|^2)\,dx &\le \limsup_{r\to0}\left(\limsup_{k\to\infty}\int_{B_{1/2}}Q(0)\lambda(V_{k,r})\,dx-\liminf_{k\to\infty}\int_{B_{1/2}}Q(0)\lambda(U_k)\,dx\right)\\ &\le \limsup_{r\to0}\left(\limsup_{k\to\infty}\int_{B_{1/2}}Q(0)\lambda(V_{k,r})\,dx\right)-\liminf_{k\to\infty}\int_{B_{1/2}}Q(0)\lambda(U_k)\,dx\\ &\le \int_{B_{1/2}}Q(0)\Big(\lambda(V)-\lambda(U_0)\Big)\,dx.\end{aligned}\tag{6.27}$$

Hence, we find

$$\int_{B_{1/2}}(|\nabla U_0|^2-|\nabla V|^2)\,dx\le\int_{B_{1/2}}Q(0)(\lambda(V)-\lambda(U_0))\,dx.$$

This proves $U_0$ is an absolute minimizer of $\mathcal{F}_0(\cdot;B_{1/2})$, i.e. Claim **$\mathcal{M}'$** is proven. In order to finish the proof of Theorem 6.3, Now, let us prove **(EI)-(EIV)**. To prove **(EI)**, we observe that since $\|A_\pm-A_\pm(0)\|_{L^\infty(B_{1/2})}\le\frac{1}{k}$, then

$$\begin{aligned}\int_{B_{1/2}}\Big((A_\pm(0)-A_{\pm,k})|\nabla v|^2\Big)\,dx &\le \|A_\pm(0)-A_{\pm,k}\|_{L^\infty(B_{1/2})}\int_{B_{1/2}}|\nabla v|^2\,dx\\ &\le \frac{\|\nabla v\|^2_{L^2(B_{1/2})}}{k}\le\frac{\|v\|^2_{H^1(B_{1/2})}}{k}.\end{aligned}\tag{6.28}$$

Moreover, since $\|Q_k-Q(0)\|_{L^\infty(B_{1/2})}\le\frac{1}{k}$, we have

$$\int_{B_{1/2}}(Q_k-Q(0))\lambda(v)\,dx\le C(N)\lambda_+\|Q_k-Q(0)\|_{L^\infty(B_{1/2})}\le\frac{C(N)\lambda_+}{k}.\tag{6.29}$$

Let $F_k:=|f_{+,k}|+|f_{-,k}|$. In the case $N>2$, we have $2^{*\prime}=\frac{2N}{N+2}<N$. So, by Sobolev embedding, Young's inequality and (6.3) we find

$$\begin{aligned}\int_{B_{1/2}}|f_k(x,v_k)v|\,dx &\le \int_{B_{1/2}}\Big(|f_{+,k}|+|f_{-,k}|\Big)|v|\,dx=\int_{B_{1/2}}|F_k||v|\,dx\\ &\le \|F_k\|_{L^{2^{*\prime}}(B_{1/2})}\|v\|_{L^{2^*}(B_{1/2})}\\ &\le \frac{C(N)}{k}\|v\|_{H^1(B_{1/2})}\le\frac{C(N)}{2k}\Big(\|v\|^2_{H^1(B_{1/2})}+1\Big).\end{aligned}\tag{6.30}$$

If $N=2$, we know that $H^1(B_{1/2})$ embeds in $L^{N'}(B_{1/2})$ and therefore

$$\begin{aligned}\int_{B_{1/2}}|f_k(x,v_k)v|\,dx &\le \int_{B_{1/2}}(|f_{+,k}|+|f_{-,k}|)|v|\,dx=\int_{B_{1/2}}|F_k||v|\,dx\\ &\le \|F_k\|_{L^N(B_{1/2})}\|v\|_{L^{N'}(B_{1/2})}\\ &\le \frac{C(N)}{k}\|v\|_{H^1(B_{1/2})}\le\frac{C(N)}{2k}(\|v\|^2_{H^1(B_{1/2})}+1).\end{aligned}\tag{6.31}$$

From (6.28), (6.29) and (6.30) or (6.31) the claim **(EI)** is proven. To prove **(EII)** we observe that

$$\lambda(U_0)=\lambda_+\chi_{\{U_0>0\}}+\lambda_-\chi_{\{U_0\le0\}}\le\liminf_{k\to\infty}\Big(\lambda_+\chi_{\{U_k>0\}}+\lambda_-\chi_{\{U_k\le0\}}\Big)=\liminf_{k\to\infty}\lambda(U_k)\text{ a.e. in }B_{1/2}\tag{6.32}$$

Indeed, since $\lambda_- < \lambda_+$, we have $\lambda_- \leq \liminf_{k\to\infty}(\lambda_+\chi_{\{U_k>0\}} + \lambda_-\chi_{\{U_k\leq 0\}})$. Hence (6.32) holds in the set $\{U_0 = 0\}$. From (C3) we have $U_k \to U_0$ pointwise almost everywhere in $B_{1/2}$. Suppose $x_0 \in \{U_0 > 0\}$ is such that $U_k(x_0) \to U_0(x_0)$. Then, for $k$ sufficiently large we have $x_0 \in \{U_k > 0\}$. Hence (6.32) holds almost everywhere in $\{U_0 > 0\}$. Similarly we can show the same for the set $\{U_0 < 0\}$. This proves the claim (6.32). Since $Q(0)\lambda(U_k) \leq q_2\lambda_+$ for all $k$. By (6.32) and since $Q(0) \geq 0$ we have by Fatou's lemma

$$\int_{B_{1/2}} Q(0)\lambda(U_0)\,dx \leq \int_{B_{1/2}} Q(0)\liminf_{k\to\infty}\lambda(U_k)\,dx \leq \liminf_{k\to\infty}\int_{B_{1/2}} Q(0)\lambda(U_k)\,dx.$$

This proves **(EII)**. Now we prove **(EIII)**. The integral in the LHS of **(EIII)** can be written as

$$\int_{B_{1/2}} Q(0)\lambda(V_{r,k})\,dx = Q(0)\Big(\lambda_+|\{V_{k,r} > 0\} \cap B_{1/2}| + \lambda_-|\{V_{k,r} \leq 0\} \cap B_{1/2}|\Big).$$

In order to prove **(EIII)**, we first observe

$$\begin{aligned} &\lim_{r\to 0}\left(\limsup_{k\to\infty}|\{V_{k,r} > 0\} \cap B_{1/2}|\right) = |\{V > 0\} \cap B_{1/2}| \text{ and} \\ &\lim_{r\to 0}\left(\limsup_{k\to\infty}|\{V_{k,r} \leq 0\} \cap B_{1/2}|\right) = |\{V \leq 0\} \cap B_{1/2}|. \end{aligned} \tag{6.33}$$

Indeed, let us set $\mathcal{R}_r := B_{1/2} \setminus B_{\frac{1}{2}-r}$. Now, since $\eta_r = 1$ in $B_{1/2} \setminus \mathcal{R}_r = B_{\frac{1}{2}-r}$ (c.f. (6.8)), recalling the definition of $V_{k,r}$ in (6.14) we observe that

$$V_{k,r} = V \ \text{ in } B_{1/2} \setminus \mathcal{R}_r.$$

With this in mind we have

$$\begin{aligned} |\{V_{r,k} > 0\} \cap B_{1/2}| &= |(\{V_{r,k} > 0\} \cap B_{1/2}) \setminus \mathcal{R}_r| + |\{V_{r,k} > 0\} \cap \mathcal{R}_r| \\ &= |(\{V > 0\} \cap B_{1/2}) \setminus \mathcal{R}_r| + |\{V_{r,k} > 0\} \cap \mathcal{R}_r| \\ &= |\{V > 0\} \cap B_{1/2}| - |\mathcal{R}_r \cap \{V > 0\}| + |\{V_{r,k} > 0\} \cap \mathcal{R}_r|. \end{aligned} \tag{6.34}$$

From above computations we have

$$|\{V > 0\} \cap B_{1/2}| - |\mathcal{R}_r \cap \{V > 0\}| \leq |\{V_{r,k} > 0\} \cap B_{1/2}| \leq |\{V > 0\} \cap B_{1/2}| + |\{V_{r,k} > 0\} \cap \mathcal{R}_r|.$$

and therefore

$$|\{V > 0\} \cap B_{1/2}| - |\mathcal{R}_r| \leq |\{V_{r,k} > 0\} \cap B_{1/2}| \leq |\{V > 0\} \cap B_{1/2}| + |\mathcal{R}_r|.$$

Now, taking $\limsup_{k\to\infty}$ and $\limsup_{r\to 0}$ in the inequality above we find

$$\begin{aligned} |\{V > 0\} \cap B_{1/2}| &= \lim_{r\to 0}\Big(|\{V > 0\} \cap B_{1/2}| - |\mathcal{R}_r|\Big) \\ &\leq \limsup_{r\to 0}\Big(\limsup_{k\to\infty}|\{V_{k,r} > 0\} \cap B_{1/2}|\Big) \\ &\leq \limsup_{r\to 0}\Big(|\{V > 0\} \cap B_{1/2}| + |\mathcal{R}_r|\Big) = |\{V > 0\} \cap B_{1/2}|. \end{aligned}$$

We can proceed similarly by replacing $\{V_{k,r} > 0\}$ by $\{V_{k,r} \leq 0\}$ in (6.34) to arrive to

$$\limsup_{r\to 0}\Big(\limsup_{k\to\infty}|\{V_{k,r} \leq 0\} \cap B_{1/2}|\Big) = |\{V \leq 0\} \cap B_{1/2}|.$$

This proves (6.33). Since the sequences $|\{V_{k,r} > 0\}|$ and $|\{V_{k,r} \leq 0\}|$ are bounded in $k$ and $r$, we can use **($\mathcal{I}$2)**** twice (first in $k$ then in $r$) and (6.33) to obtain

$$\begin{aligned} &\limsup_{r\to 0}\left(\limsup_{k\to\infty}\int_{B_{1/2}} Q(0)\lambda(V_{r,k})\,dx\right) \\ &= Q(0)\limsup_{r\to 0}\left(\limsup_{k\to\infty}\Big(\lambda_+|\{V_{k,r} > 0\} \cap B_{1/2}| + \lambda_-|\{V_{k,r} \leq 0\} \cap B_{1/2}|\Big)\right) \\ &\leq Q(0)\limsup_{r\to 0}\left(\limsup_{k\to\infty}\lambda_+|\{V_{k,r} > 0\} \cap B_{1/2}|\right) + Q(0)\limsup_{r\to 0}\left(\limsup_{k\to\infty}\lambda_-|\{V_{k,r} \leq 0\} \cap B_{1/2}|\right) \\ &= Q(0)\big(\lambda_+|\{V > 0\} \cap B_{1/2}| + \lambda_-|\{V \leq 0\} \cap B_{1/2}|\big) \\ &= \int_{B_{1/2}} Q(0)\lambda(v)\,dx. \end{aligned}$$

This shows **(EIII)**. Finally, we prove **(EIV)**. We start by observing that

$$\int_{B_{1/2}} \left(|\nabla U_k|^2 - |\nabla V_{k,r}|^2\right) dx = \int_{B_{1/2}} \left(\nabla(U_k - V_{k,r}) \cdot \nabla(U_k + V_{k,r})\right) dx.$$

Now we plug in the definition of $V_{k,r}$ (c.f. (6.14)) into the previous equation and arrive to

$$\begin{aligned} \nabla(U_k - V_{k,r}) &= \nabla(U_k - V - (1-\eta_r)(U_k - U_0)) \\ &= \nabla(U_0 - V + \eta_r(U_k - U_0)) \end{aligned} \tag{6.35}$$

and

$$\begin{aligned} \nabla(U_k + V_{k,r}) &= \nabla(U_k + V + (1-\eta_r)(U_k - U_0)) \\ &= \nabla(U_0 + V + (2-\eta_r)(U_k - U_0)). \end{aligned} \tag{6.36}$$

From (6.35) and (6.36) we write

$$\int_{B_{1/2}} \nabla(U_k - V_{k,r}) \cdot \nabla(U_k + V_{k,r})\, dx = \int_{B_{1/2}} \nabla(U_0 - V) \cdot \nabla(U_0 + V)\, dx + \mathbf{T^1_k} + \mathbf{T^2_k} + \mathbf{T^3_k}. \tag{6.37}$$

where

$$\begin{aligned} \mathbf{T^1_k} &:= \int_{B_{1/2}} \nabla(U_0 - V) \cdot \nabla((2-\eta_r)(U_k - U_0))\, dx \\ \mathbf{T^2_k} &:= \int_{B_{1/2}} \nabla(\eta_r(U_k - U_0)) \cdot \nabla(U_0 + V)\, dx \\ \mathbf{T^3_k} &:= \int_{B_{1/2}} \nabla(\eta_r(U_k - U_0)) \cdot \nabla((2-\eta_r)(U_k - U_0))\, dx. \end{aligned} \tag{6.38}$$

Let us study each term separately. First, we see that

$$\begin{aligned} \mathbf{T^1_k} = \int_{B_{1/2}} \nabla(U_0 - V) \cdot \nabla((2-\eta_r)(U_k - U_0))\, dx &= \int_{B_{1/2}} (2-\eta_r)\nabla(U_0 - V) \cdot \nabla(U_k - U_0)\, dx \\ &\quad - \int_{B_{1/2}} \left((U_k - U_0)\nabla(U_0 - V) \cdot \nabla\eta_r\right) dx. \end{aligned}$$

In the identity above, the first term on the right hand side tends to zero as $k \to \infty$ since from (C1), $\nabla(U_k - U_0)$ goes to zero weakly in $L^2(B_{1/2})$. Also, the second term on the right hand side tends to zero as $k \to \infty$ because from (C2), $U_k$ converges to $U_0$ strongly in $L^2(B_1)$. Therefore

$$\lim_{k\to\infty} \mathbf{T^1_k} = \lim_{k\to\infty} \int_{B_{1/2}} \nabla(U_0 - V) \cdot \nabla((2-\eta_r)(U_k - U_0))\, dx = 0. \tag{6.39}$$

Now, let us look at the following term

$$\begin{aligned} \mathbf{T^2_k} = \int_{B_{1/2}} \nabla(\eta_r(U_k - U_0)) \cdot \nabla(U_0 + V)\, dx &= \int_{B_{1/2}} \eta_r \nabla(U_0 + V) \cdot \nabla(U_k - U_0)\, dx \\ &\quad + \int_{B_{1/2}} (U_k - U_0)\nabla\eta_r \cdot \nabla(U_0 + V)\, dx. \end{aligned}$$

Thus, by similar arguments as those in (6.39), we obtain that

$$\lim_{k\to\infty} \mathbf{T^2_k} = \lim_{k\to\infty} \int_{B_{1/2}} \nabla(\eta_r(U_k - U_0)) \cdot \nabla(U_0 + V)\, dx = 0. \tag{6.40}$$

Now, we estimate $\mathbf{T^3_k}$. We write $\mathbf{T^3_k}$ as follows

$$\mathbf{T^3_k} = \mathbf{S^3_k} + \mathbf{T^3_k}^*. \tag{6.41}$$

where

$$\mathbf{S^3_k} := \int_{B_{1/2}} \eta_r(2-\eta_r)|\nabla(U_k - U_0)|^2\, dx$$

and

$$\mathbf{T^3_k}^* := \int_{B_{1/2}} \Big((2-\eta_r)(U_k - U_0)\nabla(U_k - U_0) \cdot \nabla\eta_r - \eta_r(U_k - U_0)\nabla(U_k - U_0) \cdot \nabla\eta_r - |U_k - U_0|^2|\nabla\eta_r|^2\Big)\, dx.$$

From (C2) and (C3) we have

$$\lim_{k\to\infty} \mathbf{T^3_k}^* = 0. \tag{6.42}$$

Since $0 \le \eta_r \le 1$ we have for every $k \in \mathbb{N}$

$$\mathbf{S^3_k} = \int_{B_{1/2}} \Big( \eta_r(2-\eta_r)|\nabla(U_k - U_0)|^2 \Big)\, dx \ge 0. \tag{6.43}$$

Therefore from (6.43) $\mathbf{T^3_k} \ge \mathbf{T^3_k}^*$ for all $k$. From (6.41) and (6.42) we obtain

$$\liminf_{k\to\infty} \mathbf{T^3_k} \ge \liminf_{k\to\infty} \mathbf{T^3_k}^* = 0. \tag{6.44}$$

Now, by using (6.39), (6.40), (6.44) and obtain

$$\begin{aligned}\liminf_{k\to\infty} \int_{B_{1/2}} \left(|\nabla U_k|^2 - |\nabla V_{k,r}|^2\right) dx &= \liminf_{k\to\infty} \Big( \int_{B_{1/2}} \left(|\nabla U_0|^2 - |\nabla V|^2\right) dx + \mathbf{T^1_k} + \mathbf{T^2_k} + \mathbf{T^3_k} \Big) \\ &\ge \int_{B_{1/2}} (U_0 - V)\cdot \nabla(U_0 + V)\, dx = \int_{B_{1/2}} \left(|\nabla U_0|^2 - |\nabla V|^2\right) dx.\end{aligned}$$

This shows **(EIV)** and this finally proves $u_0$ is an absolute minimizer of $J_0(\cdot; B_{1/2})$. This is a contradiction to the assumption (6.1). Thus, we finish the proof of Theorem 6.3. □

Now, we prove that the value of $\delta > 0$ in the Lemma 6.3 is dependent only on $L^\infty$ bounds of absolute minimizer $u$ and other universal constants related to the problem. For this we use the Widman's hole filling technique by following ideas from [25, Chapter 7] (see also [46]).

**Proposition 6.4.** *Let $u \in H^1(B_1) \cap L^\infty(B_1)$ be an absolute minimizer of $J(\cdot; B_1)$ such that $\|u\|_{L^\infty(B_1)} \le 1$. Then for every $\varepsilon > 0$ there exists $0 < \delta(\varepsilon, \mu, q_2, \lambda_+, N) < 1$ such that if*

$$\max \Big( \|A_\pm - A_\pm(0)\|_{L^\infty(B_1)},\ \|f_\pm\|_{L^N(B_1)},\ \|Q - Q(0)\|_{L^\infty(B_1)} \Big) \le \delta$$

*then*

$$\|u - h\|_{L^\infty(B_{1/4})} \le \varepsilon.$$

*Here $h \in H^1(B_{1/2}) \cap L^\infty(B_{1/2})$ is such that $\|h\|_{L^\infty(B_{1/2})} \le 1$ and $h$ is also an absolute minimizer of*

$$J_0(v; B_{1/2}) := \int_{B_{1/2}} \Big( A(0,v)|\nabla v|^2 + Q(0)\lambda(v) \Big)\, dx.$$

*Proof.* We observe that $|f(x,s)| \le F := (|f_+| + |f_-|)$ and $|Q(x)\lambda(s)| \le q_2\lambda_+$. Hence, we can easily verify that the integrand of the functional $J(\cdot; B_1)$ satisfies the following for almost every $x \in B_1$ and for all $\xi \in \mathbb{R}^N$, $s \in \mathbb{R}$.

$$\mu|\xi|^2 - F|s| - q_2\lambda_+ \le A(x,s)|\xi|^2 + f(x,s)s + Q(x)\lambda(s) \le \frac{1}{\mu}|\xi|^2 + F|s| + q_2\lambda_+ \tag{6.45}$$

where $F := |f_+| + |f_-|$. Let $r, s > 0$ be such that $1/2 \le r < s \le \frac{3}{4}$. Since, $u$ is an absolute minimizer of $J(\cdot; B_1)$, so it is of $J(\cdot; B_s)$ (c.f. Lemma 3.11). This means that for every $v \in H^1(B_s)$ such that $v - u \in H^1_0(B_s)$, we have $J(u; B_s) \le J(v; B_s)$. In other words

$$\int_{B_s} \Big( A(x,u)|\nabla u|^2 + f(x,u)u + Q(x)\lambda(u) \Big)\, dx \le \int_{B_s} \Big( A(x,v)|\nabla v|^2 + f(x,v)v + Q(x)\lambda(v) \Big)\, dx. \tag{6.46}$$

Therefore, from (6.45) we can write

$$\int_{B_s} \left( \mu|\nabla u|^2 - F|u| - q_2\lambda_+ \right)\, dx \le \int_{B_s} \left( \frac{1}{\mu}|\nabla v|^2 + F|v| + q_2\lambda_+ \right)\, dx \tag{6.47}$$

for every $v \in H^1(B_s)$ such that $v - u \in H^1_0(B_s)$. Let $\eta \in C_0^\infty(B_1)$ be such that

$$0 \le \eta \le 1, \qquad \eta(x) = \begin{cases} 1 & x \in B_r \\ 0 & x \in B_1 \setminus B_s. \end{cases} \qquad \Big(\text{recall } \tfrac{1}{2} \le r < s \le \tfrac{3}{4}\ \Big)$$

We can assume that $|\nabla \eta| \le \frac{C(N)}{|s-r|}$ in $B_1$. Now, we consider the test function $v = u(1-\eta)$. Since, $v - u = -\eta u \in H^1_0(B_s)$, the function $v \in H^1(B_s)$ is an admissible function for the inequality (6.47).

Since $\|F\|_{L^N(B_s)} \leq 2\delta < 2$ and $\|\big(|u| + |u(1-\eta)|\big)\|_{L^{N'}(B_s)} \leq 2C(N)s^{\frac{N}{N'}} = 2C(N)s^{N-1}$. By plugging $v = u(1-\eta)$ in to (6.47) and rearranging the terms accordingly we obtain,

$$\begin{aligned}\int_{B_s} \mu|\nabla u|^2\,dx &\leq \int_{B_s} \left(\frac{1}{\mu}|\nabla(u(1-\eta))|^2 + F|u(1-\eta)| + q_2\lambda_+\right)dx + \int_{B_s}(F|u| + q_2\lambda_+)\,dx\\ &\leq \int_{B_s}\frac{1}{\mu}\Big|(1-\eta)\nabla u - u\nabla\eta\Big|^2 dx + \|F\|_{L^N(B_s)}\|\big(|u| + |u(1-\eta)|\big)\|_{L^{N'}(B_s)} + C(N)q_2\lambda_+ s^N\\ &\leq \frac{2}{\mu}\int_{B_s}\Big((1-\eta)^2|\nabla u|^2 + |\nabla\eta|^2|u|^2\Big)dx + 4C(N)s^{N-1} + C(N,q_2,\lambda_+)s^N\\ &\leq C(\mu)\int_{B_s}\Big((1-\eta)^2|\nabla u|^2 + |\nabla\eta|^2|u|^2\Big)dx + C(N,q_2,\lambda_+)s^{N-1}.\end{aligned}$$

Since $\|u\|_{L^\infty(B_1)} \leq 1$, $s \in (0,1)$, $|\nabla\eta| \leq \frac{C(N)}{|s-r|}$ in $B_s$ and $\eta = 1$ in $B_r$ we have

$$\begin{aligned}\int_{B_s}|\nabla u|^2\,dx &\leq \bar{C}(\mu)\int_{B_s}\Big((1-\eta)^2|\nabla u|^2 + |\nabla\eta|^2u^2\Big)dx + C(N,\mu,q_2,\lambda_+)\\ &\leq \bar{C}(\mu)\left[\int_{B_s\setminus B_r}|\nabla u|^2\,dx + \frac{C(N)}{|s-r|^2}\int_{B_s}|u|^2\,dx\right] + C(N,\mu,q_2,\lambda_+)\\ &\leq \bar{C}(\mu)\left[\int_{B_s\setminus B_r}|\nabla u|^2\,dx + \frac{C(N)}{|s-r|^2}s^N\right] + C(N,\mu,q_2,\lambda_+).\end{aligned}$$

Since $r < s$ and $s \in (0,1)$ we can write

$$\int_{B_r}|\nabla u|^2\,dx \leq \int_{B_s}|\nabla u|^2\,dx \leq \bar{C}(\mu)\int_{B_s\setminus B_r}|\nabla u|^2\,dx + \frac{C(N,\mu)}{|s-r|^2} + C(N,\mu,q_2,\lambda_+).$$

We now add $\bar{C}(\mu)\int_{B_r}|\nabla u|^2$ on LHS and RHS sides of the inequality above to obtain

$$\int_{B_r}|\nabla u|^2\,dx \leq \frac{\bar{C}(\mu)}{\bar{C}(\mu)+1}\int_{B_s}|\nabla u|^2\,dx + \frac{C'(N,\mu)}{|s-r|^2} + C'(N,\mu,q_2,\lambda_+).$$

Now, from Lemma 4.5 with $Z(t) := \int_{B_t}|\nabla u|^2\,dx$ we conclude that

$$\int_{B_{1/2}}|\nabla u|^2 \leq C_2(N,\mu,q_2,\lambda_+). \tag{6.48}$$

Once we have a universal estimate for the energy in $B_{1/2}$, we can apply Theorem 6.3. Since $u$ is an absolute minimizer of $J(\cdot;B_1)$, so it is of $J(\cdot;B_{1/2})$. Moreover, $\|u\|_{L^\infty(B_{1/2})} \leq \|u\|_{L^\infty(B_1)} \leq 1$. Now, Theorem 6.3 with $M = C_2(\mu,q_2,\lambda_+,N)^{1/2} > 0$ provides a $\delta := \delta(\varepsilon,\mu,q_2,\lambda_+,N) > 0$ for which Proposition 7.2 holds. This finishes the proof. □

## 7. Optimal regularity of minimizers

We recall that in this section $J_{A,f,Q}(\cdot;B_1)$ satisfy the structural conditions **(G1)-(G5)**.

*Remark* 7.1. Let $u \in H^1(B_\Theta(x_0))$ be an absolute minimizer of $J(\cdot;B_\Theta(x_0))$. We define $w$ as follows

$$w(x) := \Phi u(x_0 + \Theta x).$$

We can verify that $w$ is an absolute minimizer the following functional

$$\bar{J}(v) := \int_{B_{r/\Theta}}\big(\bar{A}(x,v)|\nabla v|^2 - \bar{f}(x,v)v + \bar{Q}(x)\lambda(v)\big)\,dx$$

where $\bar{A}$, $\bar{f}$ and $\bar{Q}$ are defined in $B_{r/\Theta}$ as follows

$$\begin{aligned}\bar{A}(x,s) &= A(x_0 + \Theta x, s)\\ \bar{f}(x,s) &= \Phi\Theta^2 f(x_0 + \Theta x)\\ \bar{Q}(x) &= \Phi^2\Theta^2 Q(x_0 + \Theta x).\end{aligned}$$

We now prove the key lemma analogous to Proposition 4.4 in the context of minimizers of $J_{A,f,Q}(\cdot;B_1)$.

**Proposition 7.2** (Key lemma for minimizers)**.** *Let $u \in H^1(B_1) \cap L^\infty(B_1)$ be an absolute minimizer of $J(\cdot; B_1)$ satisfying the structural condition **(G1)-(G5)** with $u(0) = 0$ and $\|u\|_{L^\infty(B_1)} \le 1$. Then for any $0 < \alpha < 1$, there exists $\delta(N, \mu, q_2, \lambda_+, \alpha) > 0$ and $0 < R_0(N, \mu, q_2, \lambda_+, \alpha) < 1/4$ such that if*

$$\max(\|A_\pm - A_\pm(0)\|_{L^\infty(B_1)}, \|f_\pm\|_{L^N(B_1)}, \|Q - Q(0)\|_{L^\infty(B_1)}) \le \delta$$

*then we have*

$$\sup_{B_{R_0}} |u| \le R_0^\alpha. \tag{7.1}$$

*Proof.* Let $\varepsilon > 0$ which will be suitably chosen later. We know that for $\delta(\varepsilon) > 0$ and $h$ as in Lemma 6.3 we have

$$\|u - h\|_{L^\infty(B_{1/2})} < \varepsilon. \tag{7.2}$$

Fix $\beta = \frac{1+\alpha}{2}$ from Proposition 3.13 we have

$$\sup_{B_r} |h - h(0)| \le C(N, \mu, q_2, \lambda_+, \alpha) r^\beta, \quad r < \frac{1}{4}. \tag{7.3}$$

By (7.2) and (7.3) we get for all $r < 1/4$

$$\begin{aligned} \sup_{B_r} |u(x) - u(0)| &\le \sup_{B_r} \Big( |u(x) - h(x)| + |h(x) - h(0)| + |h(0) - u(0)| \Big) \\ &\le 2\varepsilon + C(N, \mu, q_2, \lambda_+, \alpha) r^\beta. \end{aligned} \tag{7.4}$$

We can select $R_0(N, \mu, q_2, \lambda_+, \alpha) < 1/4$ such that

$$C(N, \mu, q_2 \lambda_+, \alpha) R_0^\beta \le \frac{R_0^\alpha}{3}.$$

This means that

$$R_0 \le \left(\frac{1}{3C}\right)^{2/(1-\alpha)}.$$

Now, we can make our choice of $\varepsilon$. Choose $\varepsilon(N, \mu, q_2, \lambda_+, \alpha) > 0$ in such a way that

$$\varepsilon < \frac{R_0^\alpha}{3}.$$

We see that the choice of $\delta$ depends on $\varepsilon$ and since $\varepsilon$ depends on $N, \mu, q_2, \lambda_+$ and $\alpha$ so does $\delta$. Since $u(0) = 0$ and $C(N, \mu, q_2, \lambda_+, \alpha) R_0^\beta$ and $\varepsilon$ are bounded by $R_0^\alpha/3$. We obtain from (7.4)

$$\sup_{B_{R_0}} |u| \le R_0^\alpha.$$

□

We now have the ingredients to show the $C^{0,1^-}$ regularity estimates for an absolute minimizer $u$ around the points in the zero level set.

**Proposition 7.3.** *Let $u \in H^1(B_1) \cap L^\infty(B_1)$ be an absolute minimizer of $J(\cdot; B_1)$ with $u(0) = 0$ and $\|u\|_{L^\infty(B_1)} \le 1$. Then for every $0 < \alpha < 1$, there exists a $\delta(N, \mu, q_2, \lambda_+, \alpha) > 0$ and $C(N, \mu, q_2, \lambda_+, \alpha) > 0$ such that if*

$$\max\Big(\|A_\pm - A_\pm(0)\|_{L^\infty(B_1)}, \|f_\pm\|_{L^N(B_1)}, \|Q - Q(0)\|_{L^\infty(B_1)}\Big) \le \delta$$

*then for $R_0(N, \mu, q_2, \lambda_+, \alpha)$ as in Proposition 7.2 we have*

$$\sup_{B_r} |u(x)| \le C(N, \mu, q_2, \lambda_+, \alpha) \cdot r^\alpha \quad \forall\, r \le R_0. \tag{7.5}$$

*Precisely speaking, we have $C(N, \mu, q_2, \lambda_+, \alpha) = R_0^{-\alpha}$.*

*Proof.* We argue by scaling and claim that for all $k \in \mathbb{N}$

$$\sup_{B_{R_0^k}} |u(x)| \le R_0^{k\alpha}. \tag{7.6}$$

It follows readily from Proposition 7.2 that (7.6) holds for $k = 1$. Now, let us assume it holds up to $k_0 \in \mathbb{N}$. We prove that (7.6) also holds for $k = k_0 + 1$. In order to do that, we set the following scaled function

$$\tilde{u}(y) = \frac{1}{R_0^{k_0\alpha}} u(R_0^{k_0} y).$$

By Remark 7.1 we see that $\tilde{u}$ minimizes the functional $\tilde{J}$ given by

$$\tilde{J}(v) := \int_{B_1} \tilde{A}(y,v)|\nabla v|^2 - \tilde{f}(y,v)v + \tilde{Q}(y)\lambda(v)\,dx$$

where $\tilde{A}$, $\tilde{f}$ and $\tilde{Q}$ are defined as follows for $y \in B_1$

$$\begin{aligned}\tilde{A}(y,s) &:= A(R_0^{k_0}y,s)\\ \tilde{f}(y,s) &:= R_0^{(2k_0(1-\alpha)+k_0\alpha)} f(R_0^{k_0}y,s)\\ \tilde{Q}(y) &:= R_0^{2k_0(1-\alpha)}Q(r_0^{k_0}y).\end{aligned}$$

We observe that $\tilde{J}$ and $\tilde{u}$ satisfy the assumptions of Proposition 7.2. Indeed, by induction hypothesis (from (7.6) for $k = k_0$) we have

$$\sup_{B_1}|\tilde{u}| = R_0^{-k_0\alpha} \sup_{B_{R_0^{k_0}}} |u| \leq 1.$$

Also, for $\delta > 0$ as in Proposition 7.2, we can see that

$$\sup_{B_1}|\tilde{A}_\pm - \tilde{A}_\pm(0)| = \sup_{B_{R_0^{k_0}}} |A_\pm - A_\pm(0)| \leq \delta$$

and

$$\sup_{B_1}|\tilde{Q} - \tilde{Q}(0)| = R_0^{2k_0(1-\alpha)} \sup_{B_{R_0^{k_0}}} |Q - Q(0)| \leq \delta, \quad |\tilde{Q}| \leq q_2.$$

Furthermore,

$$\|\tilde{f}\|_{L^N(B_1)} = R_0^{k_0(1-\alpha)}\|f\|_{L^N(B_{R_0^k})} \leq \delta.$$

Moreover $\tilde{u}(0) = 0$. By applying Proposition 7.2 to the pair $(\tilde{J}, \tilde{u})$, we see from (7.1) that

$$\sup_{B_{R_0}}|\tilde{u}| \leq R_0^\alpha.$$

Rescaling back to $u$, we obtain

$$\sup_{B_{R_0^{k_0+1}}} |u| \leq R_0^{(k_0+1)\alpha}.$$

This finishes the proof by induction and proves the claim (7.6) for all $k \in \mathbb{N}$. To prove Proposition 7.3 let us take $r \in (0, R_0)$ and $k$ such that $R_0^{k+1} \leq r < R_0^k$. From (7.6), we see that

$$\sup_{B_r}|u| \leq \sup_{B_{R_0^k}}|u| \leq R_0^{k\alpha} = R_0^{(k+1)\alpha}\frac{1}{R_0^\alpha} \leq \frac{1}{R_0^\alpha}r^\alpha.$$

Now, we just take $C(N,\mu,q_2,\lambda_+,\alpha) = \frac{1}{R_0^\alpha}$. This finishes the proof of Proposition 7.3. □

Now we claim that only the smallness in oscillations of coefficients $A_\pm$ is sufficient to show the regularity estimates for an absolute minimizer $u$ of $J(\cdot; B_1)$. We prove this result in the following rescaled version of previous lemma.

**Proposition 7.4.** *Suppose $u \in H^1(B_\rho(x_0)) \cap L^\infty(B_\rho(x_0))$ is an absolute minimizer of $J(\cdot; B_\rho(x_0))$ $(\rho < 1)$ and $u(x_0) = 0$. Then for all $0 < \alpha < 1$, there exists $C(N,\mu,q_2,\lambda_+,\alpha) > 0$ such that if*

$$\|A_\pm - A_\pm(x_0)\|_{L^\infty(B_\rho(x_0))} \leq \delta \tag{7.7}$$

*for $\delta(N,\mu,q_2,\lambda_+,\alpha) > 0$, then for $R_0(N,\mu,q_2,\lambda_+,\alpha)$ as in Proposition 7.3 we have*

$$\sup_{B_r(x_0)} |u(x)| \leq \frac{C(N,\mu,q_2,\lambda_+,\alpha)}{\rho^\alpha}\left(\rho + \|u\|_{L^\infty(B_\rho(x_0))} + \rho\|F\|_{L^N(B_\rho(x_0))}\right)r^\alpha \quad \forall r \leq \rho R_0. \tag{7.8}$$

*Remark* 7.5. In fact, $C(N,\mu,q_2,\lambda_+,\alpha)$ in Proposition 7.4 can be taken as

$$C(N,\mu,q_2,q_2\lambda_+,\alpha) := \frac{1}{\sqrt{\delta}}C(N,\mu,q_2\lambda_+,\alpha)(1+\sqrt{q_2})$$

where $\delta$ and $C(N,\mu,q_2,\lambda_+,\alpha)$ is as in Proposition 7.3.

*Proof.* We define the following rescaled function

$$w(y) := \frac{u(x_0+\rho y)}{\left(\sqrt{\frac{2q_2}{\delta}}\cdot\rho + \|u\|_{L^\infty(B_\rho(x_0))} + \frac{\rho}{\delta}\|F\|_{L^N(B_\rho(x_0))}\right)}, \quad y \in B_1$$

where $F = |f_+| + |f_-|$. We can easily verify that

$$\|w\|_{L^\infty(B_1)} \le 1. \tag{7.9}$$

From Remark 7.1 we check that $w$ is an absolute minimizer of the following functional

$$\tilde{J} := \int_{B_1} \left(\bar{A}(y,w)|\nabla w|^2 - \bar{f}(y,w)w + \bar{Q}(y)\lambda(w)\right) dy \tag{7.10}$$

where $\bar{A}_\pm$, $\bar{f}_\pm$ and $\bar{Q}$ are defined as follows

$$\begin{aligned}
\bar{A}_\pm(y) &:= A_\pm(x_0+\rho y),\\
\bar{f}_\pm(y) &:= \frac{\rho^2\cdot f_\pm(x_0+\rho y)}{\left(\sqrt{\frac{2q_2}{\delta}}\cdot\rho + \|u\|_{L^\infty(B_\rho(x_0))} + \frac{\rho}{\delta}\|F\|_{L^N(B_\rho(x_0))}\right)},\\
\bar{Q}(y) &= \frac{\rho^2\cdot Q(x_0+\rho y)}{\left(\sqrt{\frac{2q_2}{\delta}}\cdot\rho + \|u\|_{L^\infty(B_\rho(x_0))} + \frac{\rho}{\delta}\|F\|_{L^N(B_\rho(x_0))}\right)^2}, \; y \in B_1.
\end{aligned}$$

We observe that the pair $(\tilde{J}, w)$ satisfies the assumptions of Proposition 7.3. Indeed, we see from (7.7) that

$$\|\bar{A}_\pm - \bar{A}_\pm(0)\|_{L^\infty(B_1)} = \|A_\pm - A_\pm(x_0)\|_{L^\infty(B_\rho(x_0))} \le \delta. \tag{7.11}$$

Also,

$$\|\bar{f}_\pm\|_{L^N(B_1)} = \frac{\rho\cdot\|f_\pm\|_{L^N(B_\rho(x_0))}}{\left(\sqrt{\frac{2q_2}{\delta}}\cdot\rho + \|u\|_{L^\infty(B_\rho(x_0))} + \frac{\rho}{\delta}\|F\|_{L^N(B_\rho(x_0))}\right)} \le \delta. \tag{7.12}$$

and for the term $\bar{Q}$, since $\|Q - Q(0)\|_{L^\infty(B_\rho(x_0))} \le 2q_2$ we have

$$\|\bar{Q} - \bar{Q}(0)\|_{L^\infty(B_1)} = \frac{\rho^2\cdot\|Q - Q(x_0)\|_{L^\infty(B_\rho(x_0))}}{\left(\sqrt{\frac{2q_2}{\delta}}\cdot\rho + \|u\|_{L^\infty(B_\rho(x_0))} + \frac{\rho}{\delta}\|f\|_{L^N(B_{s_0}(\rho))}\right)^2} \le \delta. \tag{7.13}$$

Also we can easily see that

$$|\bar{Q}| \le \frac{\delta}{2} < 1 \le q_2 \text{ in } B_1. \tag{7.14}$$

Therefore from Proposition 7.3 we have

$$\sup_{x\in B_r} |w(x)| \le C(N,\mu,q_2,\lambda_+,\alpha) r^\alpha \quad \forall\, r \le R_0 \tag{7.15}$$

and on plugging in the definition of $w$ in (7.15) we obtain for all $r \le R_0$

$$\begin{aligned}
\sup_{z\in B_{\rho r}(x_0)} |u(z)| &= \sup_{y\in B_r} |u(x_0+\rho y)|\\
&\le C(N,\mu,q_2,\lambda_+,\alpha)\left(\sqrt{\frac{2q_2}{\delta}}\cdot\rho + \|u\|_{L^\infty(B_\rho(x_0))} + \rho\|f\|_{L^N(B_\rho(x_0))}\right) r^\alpha.
\end{aligned} \tag{7.16}$$

By replacing $\rho r$ by $s$ in (7.16), we obtain $\forall s \le \rho R_0$

$$\begin{aligned}
\sup_{B_s(x_0)} |u(x)| &\le \frac{C(N,\mu,q_2,\lambda_+,\alpha)}{\rho^\alpha}\left(\sqrt{\frac{2q_2}{\delta}}\cdot\rho + \|u\|_{L^\infty(B_\rho(x_0))} + \rho\|f\|_{L^N(B_\rho(x_0))}\right) s^\alpha\\
&\le \frac{C_2(N,\mu,q_2,\lambda_+,\alpha)}{\rho^\alpha}\left(1+\sqrt{q_2}\right)\left(\rho + \|u\|_{L^\infty(B_\rho(x_0))} + \rho\|f\|_{L^N(B_\rho(x_0))}\right) s^\alpha.
\end{aligned}$$

□

*Remark* 7.6. We have obtained a local asymptotic $C^{0,\alpha}$ regularity estimates on absolute minimizers $u$ in the balls centred at the zero level sets. Now suppose $x_0 \in \{u > 0\} \cap B_1$. Since we know that $u$ is a continuous function in $B_1$, therefore the set $\{u > 0\}$ is an open set.

Suppose $B_\rho(x_0) \subset\subset \{u > 0\} \cap B_1$. For any $\varphi \in C_c^\infty(B_\rho(x_0))$ we observe that

$$|t| < \frac{\inf_{B_\rho(x_0)}(u)}{4\|\varphi\|_{L^\infty(B_\rho(x_0))}} \implies u + t\varphi > \frac{3\inf_{B_\rho(x_0)}(u)}{4} > 0. \tag{7.17}$$

Hence $B_\rho(x_0) \subset\subset \{u + t\varphi > 0\}$ provided $t \in \mathbb{R}$ is as in (7.17).

We know that $u$ is an absolute minimizer of $J(\cdot; B_1)$, so is of $J(\cdot; B_\rho(x_0))$ (c.f. Lemma 3.11). Therefore

$$J(u; B_\rho(x_0)) \le J(u + t\varphi; B_\rho(x_0)).$$

Since $B_\rho(x_0) \subset\subset \{u > 0\}$ and $B_\rho(x_0) \subset\subset \{u + t\varphi > 0\}$ for any $t$ such that $|t| < \frac{\min_{B_\rho(x_0)}(u)}{4\|\varphi\|_{L^\infty(B_\rho(x_0))}}$ we have

$$\int_{B_\rho(x_0)} \Big(A_+(x)|\nabla u|^2 + f_+ u\Big)\,dx \le \int_{B_\rho(x_0)} \Big(A_+(x)|\nabla(u + t\varphi)|^2 + f_+(u + t\varphi)\Big)\,dx.$$

In other words, for any $\varphi \in C_c^\infty(B_\rho(x_0))$

$$\frac{d}{dt}\Big|_{t=0} \int_{B_\rho(x_0)} \Big(A_+(x)|\nabla(u + t\varphi)|^2 + f_+(u + t\varphi)\Big)\,dx = 0.$$

Elaborating the above identity

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\left[\int_{B_{\rho(x_0)}} \Big(A_+(x)|\nabla(u + t\varphi)|^2 + f_+(u + t\varphi)\Big)\,dx\right] &= \int_{B_{\rho(x_0)}} \big(2A_+(x)\nabla u \cdot \nabla\varphi + f_+\varphi\big)\,dx \\ &= 0.\end{aligned}$$

Therefore, $u$ is a weak solution to the following Euler-Lagrange PDE

$$\operatorname{div}(A_+(x)\nabla u) = -\frac{1}{2} f_+(x) \ \text{ in } B_\rho(x_0). \tag{7.18}$$

Furthermore whenever $B_\rho(x_0) \subset\subset \{u < 0\} \cap B_1$, proceeding similarly we obtain

$$\operatorname{div}(A_-(x)\nabla u) = -\frac{1}{2} f_-(x) \ \text{ in } B_\rho(x_0). \tag{7.19}$$

In summary, the following PDEs are true in weak sense

$$\begin{cases} \operatorname{div}(A_+(x)\nabla u) = -\frac{1}{2} f_+ & \text{in } \{u > 0\} \cap B_1 \\ \operatorname{div}(A_-(x)\nabla u) = -\frac{1}{2} f_- & \text{in } \{u < 0\} \cap B_1. \end{cases}$$

**Proposition 7.7.** *Let $u \in H^1(B_1)$ be a bounded absolute minimizer of $J(\cdot; B_1)$. Then for every $0 < \alpha < 1$ and for $\delta(N, \mu, q_2, \lambda_+, \alpha) > 0$ as in Proposition 7.3, we have*

$$\|A_\pm - A_\pm(0)\|_{L^\infty(B_1)} \le \frac{\delta}{2} \implies \|u\|_{C^\alpha(B_{1/2})} \le C\left(1 + \|u\|_{L^\infty(B_1)} + \|F\|_{L^N(B_1)}\right) \tag{7.20}$$

*where $C := C(N, \mu, q_2, \lambda_+, \alpha)$.*

*Proof.* In this proof, $R_0$ is as in Proposition 7.4. For the sake of this proof we introduce the following function in the set $\{u \neq 0\} \cap B_1$

$$d(x) = \begin{cases} \operatorname{dist}(x, \overline{\{u \le 0\}}) & \text{if } u(x) > 0 \\ \operatorname{dist}(x, \overline{\{u \ge 0\}}) & \text{if } u(x) < 0. \end{cases}$$

We start by proving the following auxiliary estimates

**(e-1)** For any $y \in B_{1/2}$ and $x \in \{u = 0\} \cap B_{5/8}$ we have

$$|u(x) - u(y)| = |u(y)| \le C_1(N, \mu, \alpha, q_2, \lambda_+)\Big(1 + \|u\|_{L^\infty(B_1)} + \|F\|_{L^N(B_1)}\Big)|x - y|^\alpha. \tag{7.21}$$

**(e-2)** For any $x \in B_{1/2}$

$$|u(x)| \le C_1(N, \mu, \alpha, q_2, \lambda_+)\Big(1 + \|u\|_{L^\infty(B_1)} + \|F\|_{L^N(B_1)}\Big)d(x)^\alpha. \tag{7.22}$$

**(e-3)** For any $x \in \Big(\{u > 0\} \cup \{u < 0\}\Big) \cap B_{1/2}$ such that $d = d(x) \le \frac{R_0}{8}$

$$\|u\|_{C^{0,\alpha}(B_{d/8}(x))} \le \frac{C_2(N, \mu, \alpha)}{d^\alpha}\Big(u(x) + d\|F\|_{L^N(B_1)}\Big). \tag{7.23}$$

**(e-4)** For any $x \in \Big(\{u>0\} \cup \{u<0\}\Big) \cap B_{1/2}$ such that $d = d(x) \leq \frac{R_0}{8}$

$$\|u\|_{C^{0,\alpha}(B_{d/8}(x))} \leq C_3(N,\mu,q_2,\lambda_+,\alpha)\Big(1+\|u\|_{L^\infty(B_1)}+\|F\|_{L^N(B_1)}\Big). \tag{7.24}$$

Before delving into the proofs of **(e-1)**-**(e-4)**. We start by observing that for any $x \in B_1$ we have

$$\|A_\pm - A_\pm(x)\|_{L^\infty(B_d(x))} \leq \|A_\pm - A_\pm(0)\|_{L^\infty(B_d(x))} + |A_\pm(0) - A_\pm(x)| \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta. \tag{7.25}$$

In order to prove **(e-1)**, we observe that $B_{1/4}(x) \subset\subset B_1$. Once $u$ is an absolute minimizer of $J(\cdot; B_1)$, so it is of $J(\cdot; B_{1/4}(x))$. We now divide the proof of **(e-1)** in two cases

**Case e1.A** : $y \in B_{1/2}$, $x \in B_{5/8}$ and $|x-y| < \frac{R_0}{4}$.

In this case, since (7.25) holds, the choice of $\rho = \frac{1}{4}$, $r = |x-y|$ and $x_0 = x$ is an admissible choice in Proposition 7.4. It readily follows that for some constant $C_0 := C_0(N,\mu,q_2,\lambda_+,\alpha)$

$$|u(x)-u(y)| = |u(y)| \leq C_0(N,\mu,q_2,\lambda_+,\alpha)\Big(1+\|u\|_{L^\infty(B_1)}+\|F\|_{L^N(B_1)}\Big)|x-y|^\alpha.$$

**Case e1.B**: $y \in B_{1/2}$, $x \in B_{5/8}$ and $|x-y| \geq \frac{R_0}{4}$.

$$|u(x)-u(y)| \leq \frac{4^\alpha \cdot 2\|u\|_{L^\infty(B_1)}}{R_0^\alpha}|x-y|^\alpha.$$

Now, from the **Case e1.A** and **Case e1.B**, for $y \in B_{1/2}$ and $x \in \{u=0\} \cap B_{5/8}$ we have

$$|u(x)-u(y)| \leq \max\left(C_1, \frac{2\cdot 4^\alpha}{R_0^\alpha}\right)\Big(1+\|u\|_{L^\infty(B_1)}+\|F\|_{L^N(B_1)}\Big)|x-y|^\alpha.$$

This proves **(e-1)** with $C_1 = \max\left(C_0, \frac{2\cdot 4^\alpha}{R_0^\alpha}\right)$.

In order to prove **(e-2)**, let us take $\bar{x} \in \{u=0\}$ be such that $d = d(x) = |x-\bar{x}|$. We again divide the proof in two cases

**Case e2.A**: Assume $d(x) \leq \frac{R_0}{8}$.

We observe that $\bar{x} \in B_{5/8} \cap \{u=0\}$, indeed,

$$|\bar{x}| \leq |x| + |x-\bar{x}| \leq \frac{1}{2} + \frac{R_0}{8} < \frac{5}{8}.$$

Also, we can easily verify that $B_{1/4}(\bar{x}) \subset\subset B_1$. Hence using **(e-1)** with $\bar{x} \in \{u=0\} \cap B_{5/8}$, we have

$$\begin{aligned}|u(x)-u(\bar{x})| = |u(x)| &\leq C_{00}(N,\mu,q_2,\lambda_+,\alpha)\Big(1+\|u\|_{L^\infty(B_1)}+\|F\|_{L^N(B_1)}\Big)|x-\bar{x}|^\alpha \\ &= C_{00}(N,\mu,q_2,\lambda_+,\alpha)\Big(1+\|u\|_{L^\infty(B_1)}+\|F\|_{L^N(B_1)}\Big)d(x)^\alpha.\end{aligned}$$

**Case e2.B**: Assume that $d > \frac{R_0}{8}$. In this case

$$|u(x)| \leq \frac{8^\alpha \cdot \|u\|_{L^\infty(B_1)}}{R_0^\alpha}d(x)^\alpha.$$

Thus, **Case e2.A** and **Case e2.B** prove **(e-2)** with $C_2 = \max\left(C_{00}, \frac{4^\alpha}{R_0^\alpha}\right)$.

For the proof of **(e-3)**, it is enough to consider only the case where $x \in B_{1/2} \cap \{u>0\}$, since the case where $x \in B_{1/2} \cap \{u<0\}$ can be treated similarly. From (7.18) in Remark 7.6, $u$ is a weak solution of the following PDE

$$\operatorname{div}\Big(A_+(x)\nabla u\Big) = \frac{1}{2}f_+ \ \text{ in } B_{d/4}(x). \tag{7.26}$$

From the non-homogenous Moser-Harnack inequality [42, Theorem 1], we have

$$\begin{aligned}\sup_{B_{d/4}(x)} u &\leq C(N,\mu)\left(\inf_{B_{d/8}(x)} u + d\|f_+\|_{L^N(B_{d/4}(x))}\right) \\ &\leq C(N,\mu)\left(\inf_{B_{d/8}(x)} u + d\|F\|_{L^N(B_{d/4}(x))}\right).\end{aligned} \tag{7.27}$$

Moreover from (7.25) and (4.42) in Corollary 4.11 with $B_\rho(x_0) = B_{d/4}(x)$ and recalling that $u \geq 0$ in $B_{d/4}(x)$ we have

$$\|u\|_{C^{0,\alpha}(B_{d/8}(x))} \leq \frac{C(N,\mu,\alpha)}{d^\alpha}\left(\sup_{B_{d/4}}(u) + d\|F\|_{L^N(B_{d/4}(x))}\right). \tag{7.28}$$

Using (7.27) and (7.28) we arrive at

$$\begin{aligned}\|u\|_{C^{0,\alpha}(B_{d/8}(x))} &\leq \frac{C(N,\mu,\alpha)}{d^\alpha}\left(\inf_{B_{d/8}(x)} u + d\|F\|_{L^N(B_{d/4}(x))}\right)\\ &\leq \frac{C(N,\mu,\alpha)}{d^\alpha}\Big(u(x) + d\|F\|_{L^N(B_{d/4}(x))}\Big).\end{aligned}$$

This concludes the proof of **(e-3)**. In order to prove **(e-4)**, we again treat only the case $x \in \{u > 0\} \cap B_{1/2}$. From **(e-2)**

$$u(x) = |u(x)| \leq C(N,\mu,q_2,\lambda_+,\alpha)\Big(1 + \|u\|_{L^\infty(B_1)} + \|F\|_{L^N(B_1)}\Big)d(x)^\alpha.$$

Plugging the above estimates in **(e-3)** we obtain

$$\begin{aligned}\|u\|_{C^\alpha(B_{d(x)/8}(x))} &\leq C_1C_2\left(1 + \|u\|_{L^\infty(B_1)} + \|F\|_{L^N(B_1)}\right) + C_1 d(x)^{1-\alpha}\|F\|_{L^N(B_1)}\\ &\leq C_3(N,\mu,q_2,\lambda_+,\alpha)\left(1 + \|u\|_{L^\infty(B_1)} + \|F\|_{L^N(B_1)}\right).\end{aligned} \tag{7.29}$$

Now, under the possession of **(e-1)**-**(e-4)**, we now finish the proof of Proposition 7.7.

We again divide the proof in cases.

**Case I**. Let $x, y \in B_{1/2}$ are such that $u(x)\cdot u(y) = 0$.

We can assume without loosing generality that $u(x) = 0$. Then, it follows readily from **(e-1)** that

$$|u(x) - u(y)| = |u(y)| \leq C_1(N,\mu,q_2,\lambda_+,\alpha)\Big(1 + \|u\|_{L^\infty(B_1)} + \|F\|_{L^N(B_1)}\Big)|x-y|^\alpha.$$

**Case II**. Let $x, y \in B_{1/2}$ such that $u(x)\cdot u(y) \neq 0$.

Without loss of generality, we can assume the

$$d(y) \leq d(x).$$

Once more, splitting the proof in cases

**Case II.1**. If $|x-y| < \frac{d(x)}{8}$. We now study the two subcases

**Case II.1.A**. If $d(x) \leq \frac{R_0}{8}$.
In this case, $y \in B_{d(x)/8}(x)$. Then it readily follows from **(e-4)** that

$$|u(x) - u(y)| \leq [u]_{C^\alpha(B_{d(x)/8}(x))}|x-y|^\alpha \leq C_3\Big(1 + \|u\|_{L^\infty(B_1)} + \|F\|_{L^N(B_1)}\Big)|x-y|^\alpha$$

where $C_3 := C_3(N, \mu q_2, \lambda_+, \alpha, )$.

**Case II.1.B**. If $d(x) > \frac{R_0}{8}$.
Since $\operatorname{div}\big(A_+(x)\nabla u\big) = \frac{1}{2}f_+$ in $B_{d/8}(x)$ in weak sense, we apply (4.42) in Corollary 4.11 on $u$ in the ball $B_{d/8}(x)$. This leads to

$$\begin{aligned}|u(x) - u(y)| &\leq [u]_{C^\alpha(B_{d(x)/8}(x))}|x-y|^\alpha\\ &\leq \frac{C(N,\mu,\alpha)}{d^\alpha}\Big(\|u\|_{L^\infty(B_1)} + d\|F\|_{L^N(B_1)}\Big)|x-y|^\alpha\\ &\leq \frac{C_4(N,\mu,\alpha)}{R_0^\alpha}\Big(1 + \|u\|_{L^\infty(B_1)} + \|F\|_{L^N(B_1)}\Big)|x-y|^\alpha.\end{aligned}$$

**Case II.2**. If $|x-y| \geq \frac{d(x)}{8}$.

By **(e-2)** and the assumption $d(x) \geq d(y)$ we obtain

$$\begin{aligned} |u(x)-u(y)| &= |u(x)|+|u(y)| \\ &\leq C_1(N,\mu,q_2,\lambda_+,\alpha)\Big(1+\|u\|_{L^\infty(B_1)}+\|F\|_{L^N(B_1)}\Big)(d(x)^\alpha+d(y)^\alpha) \\ &\leq 2C_1(N,\mu,q_2,\lambda_+,\alpha)\Big(1+\|u\|_{L^\infty(B_1)}+\|F\|_{L^N(B_1)}\Big)d(x)^\alpha \\ &\leq C_5(N,\mu,q_2,\lambda_+,\alpha)\Big(1+\|u\|_{L^\infty(B_1)}+\|F\|_{L^N(B_1)}\Big)|x-y|^\alpha. \end{aligned} \tag{7.30}$$

This proves Proposition 7.7. □

Now we present a scaled version of Proposition 7.7.

**Corollary 7.8.** *Let $u \in H^1(B_\rho(x_0))$ be a bounded absolute minimizer of $J(\cdot;B_\rho(x_0))$ and $\rho < 1$. Then for every $0 < \alpha < 1$ and for $\delta(N,\mu,q_2,\lambda_+,\alpha) > 0$ as in Proposition 7.3, we have*

$$\|A_\pm - A_\pm(x_0)\|_{L^\infty(B_\rho(x_0))} \leq \frac{\delta}{2} \implies \|u\|_{C^\alpha(B_{\frac{\rho}{2}}(x_0))} \leq \frac{C}{\rho^\alpha}\left(\rho+\|u\|_{L^\infty(B_\rho(x_0))}+\rho\|F\|_{L^N(B_\rho(x_0))}\right) \tag{7.31}$$

*where $C := C(N,\mu,q_2,\lambda_+,\alpha)$.*

*Proof.* We reduce the Corollary 7.8 to Proposition 7.7 by using the following rescaling

$$w(y) = \frac{1}{\rho}u(x_0+\rho y).$$

From Remark 7.1, $w$ is an absolute minimizer of $\bar{J}(\cdot;B_1)$ given by

$$\bar{J}(v;B_1) = \int_{B_1}\Big(\bar{A}(x,v)|\nabla v|^2+\bar{f}(x,v)v+\bar{Q}(x)\lambda(v)\Big)\,dx$$

where

$$\begin{aligned} \bar{A}_\pm(y) &= A_\pm(x_0+\rho y) \\ \bar{f}_\pm(y) &= \rho f_\pm(x_0+\rho y) \\ \bar{Q}(y) &= Q(x_0+\rho y). \end{aligned}$$

We observe that $w$ satisfies assumptions of the Proposition 7.7. The final estimates are obtained by rescaling back to $u$. □

And now, we present the proof of the Theorem 2.3.

## 8. Proof of the main result (Theorem 2.3).

*Proof of Theorem 2.3.* Let $D \subset\subset B_1$. We observe that $D \subset\subset B_d \subset\subset B_1$ where $d = 1-\frac{\mathrm{dist}(D,\partial B_1)}{2}$. Since $A_\pm \in C(B_1)$, then $A_\pm$ are uniformly continuous in $\overline{B_d}$. Thus, we define $\omega_{A_\pm,B_d}$ as the following modulus of continuity (c.f. Definition 2.2). We define $\omega_{A_\pm,\overline{B_d}}$ to be

$$\omega_{A_\pm,\overline{B_d}}(t) := \max\left(\sup_{\substack{|x-y|<t\\ x,y\in\overline{B_d}}}|A_+(x)-A_+(y)|\,,\ \sup_{\substack{|x-y|<t\\ x,y\in\overline{B_d}}}|A_-(x)-A_-(y)|\right) \text{ for } t \leq 2d$$

and we define $\omega_{A_\pm,\overline{B_d}}(t) := \omega_{A_\pm,\overline{B_d}}(2d)$ for $t > 2d$. We set $t_0$ as

$$t_0 := t_0(\omega_{A_\pm,\overline{B_d}},\delta) = \sup\left\{t \;\middle|\; \omega_{A_\pm,\overline{B_d}}(t) \leq \delta\right\}$$

as well as

$$s_0 := \min\Big(t_0, \frac{\mathrm{dist}(D,\partial B_1)}{4}\Big).$$

Since $\omega_{A_\pm,\overline{B_d}}$ is a non-decreasing function we have $\omega_{A_\pm,\overline{B_d}}(s_0) \leq \delta$. Now since

$$D \subset \bigcup_{x\in D} B_{s_0}(x) \subset \overline{B_d}$$

we have

$$\sup_{B_{s_0}(x)}|A_\pm - A_\pm(x_0)| \leq \omega_{A_\pm,\overline{B_d}}(s_0) \leq \delta,\ \forall x \in D. \tag{8.1}$$

By Lemma 3.11 $u$ is also an absolute minimizer of $J(\cdot; B_{s_0}(x))$, we have from Corollary 7.8 that for all $y \in B_{s_0/2}(x) \cap D$

$$|u(x) - u(y)| \leq \frac{C(N, \mu, q_2, \lambda_+, \alpha, )}{s_0^\alpha} \left(s_0 + \|u\|_{L^\infty(B_1)} + s_0\|f\|_{L^N(B_1)}\right) |x - y|^\alpha. \tag{8.2}$$

Now, if $x, y \in D$ are such that $|x - y| \geq s_0/2$, then

$$|u(x) - u(y)| \leq 2^{1+\alpha} \frac{\|u\|_{L^\infty(B_1)}}{s_0^\alpha} |x - y|^\alpha. \tag{8.3}$$

By combining (8.2) and (8.3), we arrive to $\left(\text{since } s_0 \leq \frac{\text{dist}(D,\partial B_1)}{4} \leq \frac{\text{diam}(B_1)}{4} = \frac{1}{2} < 1\right)$

$$[u]_{C^\alpha(D)} \leq \frac{C(N, \alpha, \mu, q_2, \lambda_+)}{s_0^\alpha} \left(1 + \|u\|_{L^\infty(B_1)} + \|f\|_{L^N(B_1)}\right). \tag{8.4}$$

We observe from the definition of $s_0$ that

$$[u]_{C^\alpha(D)} \leq \begin{cases} \frac{C(N,\mu,q_2,\lambda_+,\alpha)}{t_0^\alpha} \left(1 + \|u\|_{L^\infty(B_1)} + \|f\|_{L^N(B_1)}\right) & \text{if } t_0 \leq \frac{\text{dist}(D,\partial B_1)}{4} \\ \frac{4^\alpha \cdot C(N,\mu,q_2,\lambda_+,\alpha)}{\text{dist}(D,\partial B_1)^\alpha} \left(1 + \|u\|_{L^\infty(B_1)} + \|f\|_{L^N(B_1)}\right) & \text{if } t_0 \geq \frac{\text{dist}(D,\partial B_1)}{4}. \end{cases}$$

In order to control the first term in the equation above by a universal multiple of $\text{dist}(D, \partial B_1)^{-\alpha}$, we observe that once $t_0 > 0$ depends only on the modulus of continuity $\omega_{A_\pm,\overline{B_d}}$ and $\delta$, there exists a $n_0 := n_0(\omega_{A_\pm,\overline{B_d}}, \delta) = n_0(N, \mu, \alpha, q_2, \lambda_+, \omega_{A_\pm,\overline{B_d}}) \in \mathbb{N}$ such that $\frac{2}{n_0} \leq t_0$. Hence

$$\frac{\text{dist}(D, \partial B_1)}{n_0} \leq \frac{\text{diam}(B_1)}{n_0} = \frac{2}{n_0} \leq t_0 \implies \frac{1}{t_0^\alpha} \leq \frac{n_0^\alpha}{\text{dist}(D, \partial B_1)^\alpha} = \frac{C(N, \mu, q_2, \lambda_+, \alpha, \omega_{A_\pm,\overline{B_d}})}{\text{dist}(D, \partial B_1)^\alpha}.$$

Now, (8.4) becomes

$$[u]_{C^\alpha(D)} \leq \frac{C(N, \mu, q_2, \lambda_+, \alpha, \omega_{A_\pm,\overline{B_d}})}{\text{dist}(D, \partial B_1)^\alpha} \left(1 + \|u\|_{L^\infty(B_1)} + \|f\|_{L^N(B_1)}\right). \tag{8.5}$$

To finish we observe that Theorem 2.3 follows from (8.5) by taking $D$ as $B_r$ for any $r < 1$, once $\text{dist}(D, \partial B_1)^\alpha = \text{dist}(B_r, \partial B_1)^\alpha = (1 - r)^\alpha$ and $d = \frac{1+r}{2}$.

□

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*Departamento de Matemática, Universidade Federal da Ceara(Fortaleza, Brazil)
*Email address*: *dmoreira@mat.ufc.br

†Tata Institute of Fundamental Researcher-Centre of of Applicable Mathematics
*Email address*: †harish21@tifrbng.res.in,