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The mLPDA approach: The case of study of Josephson critical currents in ultracold atomic superfluid sytems

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Summary. — In this work, we provide a brief review of the mLPDA approach and its application to investigate the Josephson critical currents in ultracold atomic systems, where a potential barrier acts as a weak-link. The strength of this method lies in its ability to treat spatial inhomogeneities and pairing fluctuations on the same footing thus allowing for a reliable description of the entire BCS-BEC crossover at all temperatures in the superfluid phase.

1. – Introduction

The persistent interest in inhomogeneous superconducting or superfluid systems, such as SNS junctions [1], twisted junctions realized with heterostructures [2], moving barriers [3, 4], or optical lattices [5] in superfluid atomic systems, demands theoretical approaches that can efficiently investigate and predict the diverse features of those systems at a reasonable computational cost. In this context, the Bogoliubov-de Gennes (BdG) equations [6] are among the most widely used approaches. However, their application is often impractical due to the requirement for a huge memory space and long computational time. Therefore, when possible, it is more convenient to solve simpler equations, like the Ginzburg-Landau [7] or the Gross-Pitaevskii equation [8], whose validity however is limited to the BCS limit at temperatures close to the critical temperature T_c and to the BEC limit at $T = 0$, respectively.

For these reasons, various approximations of the BdG equations have been developed over the years, such as the Eilenberger [9] and Usadel [10] equations, which, though handier than the BdG equations, are strictly valid only in the BCS limit of the crossover. More

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recently, a new approximation, called Local Phsase Density Approximation (LPDA) [11], was proposed. This method relies on the application of a double coarse-graining procedure to the BdG equations and, instead of solving for a complete set of orthonormal eigenfunctions, requires the solution of a single, though highly non-linear, second-order differential equation for the order parameter $\Delta(\mathbf{r})$. The LPDA approach can be applied across the whole BCS-BEC crossover at finite temperature, provided that the size of any source of inhomogeneity is larger than the pair coherence length [12].

Nevertheless, the LPDA approach can at most recover the same physical content as the BdG equations, meaning, its results are expected to be not fully reliable at finite temperature not only in the BEC limit but also at unitarity. For this reason, we have recently developed the modified-LPDA (mLPDA) approach [13], which includes pairing fluctuations directly on top of the LPDA equation in the form of the non-selfconsistent t-matrix approximation, thereby retaining its numerical advantages.

This article aims at providing a brief review of the mLPDA approach, highlighting its purpose and potential. We will also present its results when applied to investigate the Josephson effect in ultracold atomic systems [14]. The structure of the article is as follows. Section **2** introduces the mLPDA approach. Section **3** reports on the comparison between the experimental data for the Josephson critical current and the outcomes of the mLPDA simulation when investigating the experimental systems [3, 4]. Section **4** gives our conclusions.

2. – The mLPDA approach

The LPDA equation for the order parameter $\Delta(\mathbf{r})$ is given by

$$-\frac{m}{4\pi a_F}\Delta(\mathbf{r}) = \mathcal{I}_0(\mathbf{r})\Delta(\mathbf{r}) + \mathcal{I}_1(\mathbf{r})\frac{\nabla^2}{4m}\Delta(\mathbf{r}) - \mathcal{I}_1(\mathbf{r})i\frac{\mathbf{A}(\mathbf{r})}{m}\cdot\nabla\Delta(\mathbf{r}), \tag{1}$$

where m is the fermion mass, a_F the scattering length of the two-fermion problem, $\mathbf{A}$ is the effective vector potential, and the coefficients $\mathcal{I}_0$ and $\mathcal{I}_1$ depend on the local value of the order parameter $\Delta(\mathbf{r})$, chemical potential μ, and any external potential, if present (see ref. [11] for the complete expressions).

When investigating the Josephson effect [15], we identify $\mathbf{A}(\mathbf{r}) = -\mathbf{q}$, being $\mathbf{q}$ the wavevector associated with the superfluid flow, and solve eq. (1) along with the physical constraint of the current conservation, which in the LPDA approach takes the form [11]

$$\mathbf{j}(\mathbf{r}) = \frac{1}{m}\left(\mathbf{q} + \nabla\phi(\mathbf{r})\right) n(\mathbf{r}) + 2k_BT\sum_n e^{i\omega_n\eta}\int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\frac{\mathbf{k}}{m}\mathcal{G}_{11}^{\mathrm{mf}}\left(k,\omega_n;\mathbf{q}\mid\mathbf{r}\right) = j_{\mathrm{bulk}}, \tag{2}$$

where $\mathbf{j}(\mathbf{r})$ and j_{bulk} are the local and superfluid current density, respectively and ϕ is the contribution to the phase of the order parameter due to the presence of the weak-link. Here, k_B is the Boltzmann constant, $\omega_n = (2n+1)k_BT$ (n integer) a fermionic Matsubara frequency, η a positive infinitesimal, $\mathcal{G}_{11}^{\mathrm{mf}}$ the finite temperature Green function evaluated at the mean-field level (see ref. [11] for further details), and $n(\mathbf{r})$ the local density, which in this approach is written as follows:

$$n(\mathbf{r}) = 2k_BT\sum_n e^{i\omega_n\eta}\int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\mathcal{G}_{11}^{\mathrm{mf}}\left(k,\omega_n;\mathbf{q}\mid\mathbf{r}\right). \tag{3}$$

When including pairing fluctuations on top of the LPDA equation, we have adopted the procedure outlined in ref. [16], where the equation for the order parameter was retained at the mean-field level and pairing fluctuations were added only in the density equation. Similarly, in our approach, we include pairing fluctuations in a minimal albeit essential way, that is, we retain the LPDA eq. (1) for the order parameter and in place of $\mathcal{G}_{11}^{\rm mf}$ in eqs. (2), (3) we use $\mathcal{G}_{11}^{\rm pf}$ which includes pairing fluctuations. In order to determine the expression for the "normal" single-particle Green's function $\mathcal{G}_{11}^{\rm pf}$ (see eq. (27) of ref. [13]), we have generalized the expression of the t-matrix propagator in the presence of a stationary flow of momentum $\mathbf{q}$ and evaluated $\mathcal{G}_{11}^{\rm pf}$ in the spirit of a local density approximation, where the external potential enters $\mathcal{G}_{11}^{\rm pf}$ via the substitution $\mu(\mathbf{r}) = \mu - V(\mathbf{r})$ [13].

3. – Numerical results

We have applied the mLPDA approach to investigate the experimental systems realized in refs. [3,4], where the Josephson critical current was measured at low temperature for various couplings across the BCS-BEC crossover and for temperatures ranging from 0 to T_c at unitarity, respectively. In both experiments, N_w ^{6}Li atoms were initially confined by a harmonic potential and then by two hard-walls placed at spatial coordinates $\pm x_w$. At the center of the atomic cloud a repulsive barrier was raised adiabatically and then moved along the x-axis with different velocities to determine the critical current.

In our simulations we neglected the trapping frequency along the x-axis and considered the actual barrier shape generated by a laser beam, namely $V(\mathbf{r}) = V_0(z)\exp(-2x^2/w(z)^2)$, with $V_0(z) = V_0/\sqrt{1+(z/z_R)^2}$ and $w(z) = w\sqrt{1+(z/z_R)^2}$, where $z_R = \pi w^2/\lambda$ is the Rayleigh range of the laser (λ being its wavelength).

In fig. 1 we show the comparison between the experimental and numerical data for the Josephson critical currents. Panels (a), (b), and (c) report the Josephson critical current *vs.* the dimentionless coupling parameter $(k_F^t a_F)^{-1}$ (k_F^t being the Fermi wavevector associated with the harmonic trap Fermi energy E_F^t) for different heights of the potential barrier indicated in the upper right of each panel. The shaded green area corresponds to the experimental uncertainty on the physical quantities involved in the simulations. Panel (d) shows the temperature dependence of the Josephson critical current at unitarity. In both cases, there is good agreement between the experimental and theoretical results. Upon examining panels (a), (b), and (c), it is clear that such agreement would not have been possible without the inclusion of pairing fluctuations.

4. – Conclusions

In this work, we have provided a brief overview of the recently developed mLPDA approach. This method, while requiring a reasonable computational effort, allows for the investigation of inhomogeneous superfluid systems beyond the mean-field approximation. The mLPDA approach has been applied to investigate the experimental systems realized in refs. [3,4], obtaining a good quantitative agreement with the experimental data for the Josephson critical current, both across the BCS-BEC crossover at low temperature [3] and at unitarity for temperatures ranging from 0 to T_c [4].

In general, the mLPDA approach is a powerful tool, more sophisticated than the simple BdG method. It can be applied whenever one aims to treat inhomogeneity and pairing fluctuations simultaneously. These features are particularly valuable in light of the growing number of condensed-matter devices developed for quantum technology and sensing, which could benefit from such detailed investigations.

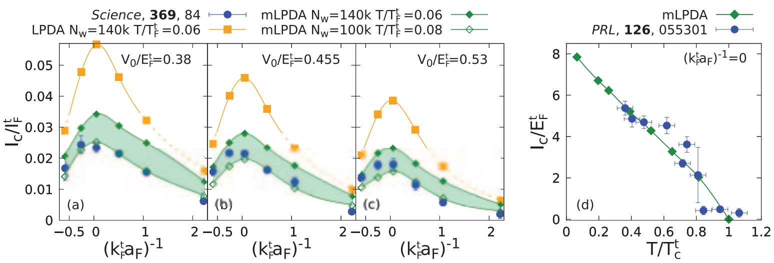

Fig. 1. – Panels (a), (b), and (c) report the comparison between the theoretical results (orange squares and green diamonds represent the LPDA and mLPDA outcomes, respectively) and the experimental data (blue dots) for the Josephson critical current *vs.* coupling at low temperature [3]. In the upper right of each panel the barrier height is reported in units of the harmonic trap Fermi energy. The associated wavevector k_F^t allows for the definition of the normalization current $I_F^t = k_F^t \int dy\, dz\, n(x_w, y, z)$. Panel (d) shows the comparison between the theoretical results (green diamonds) and the experimental data (blue dots) for the Josephson critical curret at unitarity as a function of temperature [4] (Reproduced from figs. 3 and 4 of ref. [14]).

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