

Restrictions of Pfaffian systems for Feynman integrals

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ABSTRACT: This work studies limits of Pfaffian systems, a class of first-order PDEs appearing in the Feynman integral calculus. Such limits appear naturally in the context of scattering amplitudes when there is a separation of scale in a given set of kinematic variables. We model these limits, which are often singular, via *restrictions* of $\mathcal{D}$ -modules. We thereby develop two different restriction algorithms: one based on gauge transformations, and another relying on the Macaulay matrix. These algorithms output Pfaffian systems containing fewer variables and of smaller rank. We show that it is also possible to retain logarithmic corrections in the limiting variable. The algorithms are showcased in many examples involving Feynman integrals and hypergeometric functions coming from GKZ systems. This work serves as a continuation of [1].

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1 Introduction

The evaluation of Feynman integrals in perturbative quantum field theory is a central, albeit challenging, problem. A widely used approach is to obtain a set of differential equations (DEQs) whose solution yields the Feynman integrals in question. In this setting, one considers a vector I of unknown functions, dubbed the *master integrals*, which depend on a collection of kinematic variables $z = (z_1, \dots, z_m)$, such as the masses and momenta involved

in a scattering experiment. Using the shorthand notation $\partial_i = \frac{\partial}{\partial z_i}$, the DEQs obeyed by the master integrals take the form

$$\partial_i I = P_i \cdot I \quad \text{for } i = 1, \dots, m, \quad (1.1)$$

where each P_i is an $r \times r$ matrix with entries being rational in z . This is called the *Pfaffian system* for I . Using (1.1), the unknown vector I can be analytically expressed in terms of iterated integrals [2, 3], or computed by numerical integration [4–6] for fixed values of z .

In practical calculations pertaining to high-energy physics, there is often a scale separation between the z variables. Namely, one variable, say z_1 , is *small* compared to the rest (or large, in which case $1/z_1$ is small). A severely incomplete list of examples includes the large top mass expansion [7], the small transverse momentum expansion [8], the multi-Regge limit [9], the soft limit in gravity [10], various kinematic regions in effective field theories such as the soft-collinear effective theory [11], and the computation of boundary conditions using expansion by regions [12] in the threshold [13] or massless [14] limits. There are also interesting limits such as threshold expansions [15, 16] and limits of the x -parameter in the Simplified Differential Equations approach [17, 18] which involve special kinematic configurations rather than scale-separated variables. We can model these situations as a (typically singular!) limit $z_1 \rightarrow 0$ on a Pfaffian system. Starting from (1.1), the aim of this article is then to derive a *simpler* Pfaffian system which holds in the limit (see also [19, chapter 12] and [20] for similar approaches). By simpler, we mean that the new Pfaffian system has a smaller rank than r , and it depends on one variable less, because z_1 decouples from the system.

Our algorithms are rooted in the theory of *restrictions* (in the sense that z_1 is being “restricted” to 0), which is historically tied to the theory of $\mathcal{D}$ -modules. As a supplement to studying DEQs from the viewpoint of analysis, $\mathcal{D}$ -modules leverage the algebraic structure of differential operators. When a particular $\mathcal{D}$ -module is *holonomic*, meaning that it is associated to a system of DEQs with a finite number of solutions, one can take advantage of the remarkable work by M. Kashiwara [21] and subsequent series of works (see [22] and references therein), which generalized the theory of regular connections by P. Deligne [23]. This is now established as the theory of regular holonomic systems [22]. In the context of holonomic $\mathcal{D}$ -modules, we can construct bases of differential operators, obtain relations between basis elements, count the number of solutions to DEQs, and more. This naturally links $\mathcal{D}$ -modules to Pfaffian systems for Feynman integrals. Indeed, in this work, restrictions are studied from two complementary points of view: one at the level of Pfaffian systems, and the other at the level of $\mathcal{D}$ -modules.

We take our inspiration from a particularly well-studied holonomic $\mathcal{D}$ -module: the GKZ hypergeometric system [24] — though, as we show, the algorithms presented here also apply beyond this case. In the GKZ framework [25–43], one generalizes parametric representations of a Feynman integral to include extra variables, such that now $z = (z_1, \dots, z_N)$ where $N \geq m$, with the inequality generally being strict. On one hand, this means that there are more variables to manage. On the other hand, additional structure arises which can be facilitated in the computation of Pfaffian systems. In particular, one can immediately write a collection of (higher-order) differential operators which annihilate the generalized

Feynman integral. These operators generate the GKZ $\mathcal{D}$ -module. Using the fast algorithm of [44], it is possible to construct a basis of operators (the so-called standard monomials) for the GKZ $\mathcal{D}$ -module, which translates into a basis of integrals for the Pfaffian system. Encoding algebraic relations between the generators of the GKZ system into a special matrix, called the *Macaulay matrix*, one can readily obtain the Pfaffian matrices associated to this basis too. This strategy for computing Pfaffian systems was carried out in the manuscript [1], and we regard this work as continuation of the latter.

After having made use of this GKZ technology, one seeks to dispense with the auxiliary z variables to match with the proper Feynman integral. In other words, one asks for a *restriction* to the case $N = m$, such that the number of variables agree. In this work, we present two strategies with different benefits that solve the restriction problem.

Pfaffian-level restriction. In this strategy, we assume that a Pfaffian system (1.1) is given. For instance, the system could have been derived via integration-by-parts (IBP) following Laporta’s algorithm [45], intersection theory [46–50], creative telescoping [51], or by the Macaulay matrix method.

Suppose that we want to restrict $z_1 = 0$, and that the Pfaffian matrices P_i are singular there. The first step in our protocol is to bring the Pfaffian system into *normal form*, meaning (roughly) that the Pfaffian matrix P_1 has a simple pole at $z_1 = 0$, and the remaining matrices $P_2, \dots$ are finite there. Normal form is achieved by Moser reduction, an efficient algorithm involving a series of gauge transformations that we review and give references for in appendix B.

Next we must choose between two cases: (i) do we seek solutions for the integrals I that are holomorphic at $z_1 = 0$, or (ii) do we seek solutions of the form $I^{(n)}(z_2, \dots) \log^n(z_1)$ that behave logarithmically at the origin of z_1 ? In case (i), one can immediately write a gauge transformation — see (3.15) — which yields the restricted Pfaffian system in the variables $z_2, \dots$. In case (ii), one repeats this gauge transformation procedure several times, once for each power of $\log(z_1)$.

This Pfaffian-level restriction protocol is efficient and applicable to any Pfaffian system. While we have only tested it for restrictions to hyperplanes, namely restrictions of the form $z_1 = f(z_2, \dots)$ where f is a linear function of the remaining variables, we expect that it will be applicable to restrictions onto general hypersurfaces too.

While the whole strategy ultimately only requires gauge transformations applied to Pfaffian systems, we derive and motivate it by studying $\mathcal{D}$ -modules. The derivation is partly conjectural, but we verify it in many examples.

$\mathcal{D}$ -module-level restriction. In this approach, we assume that a holonomic $\mathcal{D}$ -module is given. One example is the GKZ $\mathcal{D}$ -module.¹

Traditional $\mathcal{D}$ -module restriction algorithms stem from the breakthrough work by T. Oaku [54]. They rely on Gröbner bases in non-commutative rings, which can become

¹Alternative $\mathcal{D}$ -modules related to Feynman integrals can also be envisaged, such as the annihilating operators coming from conformal symmetry [52] and the Yangian bootstrap, see [53] and references therein.

computationally expensive when many variables are involved. In this work, instead, we extend the Macaulay matrix method from [1] to incorporate restriction of variables.

Our approach contains two steps. In the first step, we guess a basis of differential operators for the restricted $\mathcal{D}$ -module. This is done systematically via Algorithm 4.7. In the second step, we construct a Macaulay matrix for this basis. This matrix is polynomial in the z variables, and so we can immediately fix desired values for the z 's without running into singularities. The Macaulay matrix induces a linear system of equations, the solution to which is the restricted Pfaffian system (such a linear system can be swiftly solved by rational reconstruction over finite fields [55, 56]). The restriction method based on the Macaulay matrix is summarized in Algorithm 4.11.

In comparison to the Pfaffian-level restriction protocol, the $\mathcal{D}$ -module level restriction does not require a Pfaffian system as input. This is an appealing feature, because the original, un-restricted Pfaffian system can often be computationally expensive to obtain. Furthermore, while the Pfaffian-level restriction method is partly conjectural, the $\mathcal{D}$ -module-level restriction protocol is proven in this text to yield the correct restriction module.

The $\mathcal{D}$ -module-level protocol works for restriction to general hypersurfaces, in addition to hyperplanes. As far as we are aware, this is the first instance of such an algorithm. We illustrate this feature in section 5.6 by computing the restriction of the Appell $F_4(x, y)$ function onto its singular locus $xy((x - y)^2 - 2(x + y) + 1) = 0$.

The computer algebra software RISA/ASIR was used extensively in our $\mathcal{D}$ -module computations, especially the newly developed package `mt_mm`. RISA/ASIR can be built from source by cloning the `git` repository <https://github.com/openxm-org/OpenXM>. Source code for `mt_mm` can be found in the directory `OpenXM/src/asir-contrib/packages/src/mt_mm`.

Symmetry relations. In addition to integration-by-parts relations, some Feynman integrals are also linearly related to one another via *symmetry relations*. These symmetries stem from reparametrizations of the Feynman *integrand* resulting in the same function after integration. Although this case is not modeled as a singular limit on a Pfaffian system, we nevertheless observe that symmetry relations can be treated using the Pfaffian-level restriction protocol. Our test case is the equal-mass limit of the Pfaffian system for the three-mass two-loop sunrise integral — see section 5.4.

Outline. The remaining part of this text is structured as follows.

In section 2, we collect preliminary material on $\mathcal{D}$ -modules associated to Euler integrals (a class of integrals which includes Feynman integrals). This section also defines the notions of Pfaffian systems and restrictions. To further familiarize oneself with the language of $\mathcal{D}$ -modules, we strongly recommend reading the review in appendix A.

Section 3 introduces the central concept of a *logarithmic integrable connection*, a particular $\mathcal{D}$ -module which represents the Pfaffian system we seek to restrict. The story presented in this section leads us to the three equations (3.15), (3.18) and (3.23) which together outline the Pfaffian-level restriction protocol. Some steps in the construction are technical, so in section 3.4 we summarize how to apply the method to Feynman integrals in practical terms. In particular, we there write the holomorphic restriction Algorithm 3.8 and the logarithmic restriction Algorithm 3.9.

Moving on to section 4, we begin our quest to formulate restriction algorithms at the level of $\mathcal{D}$ -modules. We begin by recalling the important notions of weight vectors and b -functions in section 4.1, as they pertain to restriction algorithms based on Gröbner bases. This subsection culminates in Algorithm 4.7, which systematically searches for a differential operator basis for the restricted $\mathcal{D}$ -module. Having laid out this groundwork, we are then ready in section 4 to formulate the $\mathcal{D}$ -module-level restriction algorithm based on the Macaulay matrix — see Algorithm 4.11. Here we heavily rely on earlier results from the manuscript [1].

Section 5, making up the largest body of this work, contains a plethora of examples studying restrictions on Feynman integrals and hypergeometric functions. The goal of each example is always to derive a Pfaffian system which holds in some limit.

We give conclusions and speculate on future developments in section 6.

A long appendix follows the main text. We have already mentioned appendix A and appendix B above. Appendix C describes improvements of algorithms mentioned in the main text. Finally, appendix D is a collection of proofs which require technical details on connections and $\mathcal{D}$ -modules. We hope that this layout makes it easier to read the paper.

2 Preliminaries

In this section, we recall the framework of twisted cohomology groups for Feynman integrals and fix basic notation used throughout the paper. We use terminology on $\mathcal{D}$ -modules, such as *regular holonomic*, which is explained in the standard textbooks [22] and [57]. See [1, appendix B] for a short introduction to holonomic $\mathcal{D}$ -modules (for a pedagogical presentation which is tailored to Feynman integrals, see also the recent article [52]). Appendix A in the present manuscript gives an introduction to tensor products of modules, as they are used extensively throughout this text.

2.1 Twisted cohomology groups as $\mathcal{D}$ -modules

We study Euler integrals of the form

$$f_{\Gamma}(z) = \int_{\Gamma} g(z; x)^{\beta_0} x_1^{-\beta_1} \cdots x_n^{-\beta_n} \frac{dx}{x}, \quad \frac{dx}{x} := \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}, \quad (2.1)$$

which are integrated along a *twisted cycle* Γ , namely an integration contour without boundary along which the branch of the integrand is specified [58, chapter 3]. The exponents $\beta = (\beta_0, \dots, \beta_n) \in \mathbb{C}^{n+1}$ are complex parameters. $g(z; x)$ is a Laurent polynomial in x with monomial coefficients z_i :

$$g(z; x) = \sum_{i=1}^N z_i x^{\alpha_i}. \quad (2.2)$$

This is written in multivariate exponent notation, i.e. given an integer vector $\alpha_i \in \mathbb{Z}^n$ we set

$$x^{\alpha_i} := x_1^{\alpha_{i,1}} \cdots x_n^{\alpha_{i,n}}, \quad (2.3)$$

where $\alpha_{i,j}$ stands for the j -th component of the vector α_i . The monomial exponents α_i induce an $(n+1) \times N$ matrix

$$A = (a_1 \dots a_N) \quad (2.4)$$

with columns defined by $a_i := (1, \alpha_i)$, under the assumption that $\text{Span}\{a_1, \dots, a_N\} = \mathbb{Z}^{n+1}$. If the variables z_i are all independent, the Euler integral (2.1) is a solution to a PDE system called the GKZ hypergeometric system [24]. Generalized versions of Feynman integrals are also solutions to GKZ systems [25, 26]. Indeed, the Lee-Pomeransky representation [59] takes the form (2.1), but in this case the z_i are subject to linear constraints determined by the topology of a Feynman diagram — some of them may be equal to unity, or equal to each other. Therefore, we allow z to vary on an affine subspace $Y = \mathbb{A}^m \subset \mathbb{A}^N$ of dimension $m \leq N$. The coordinates $z = (z_1, \dots, z_m)$ denote an affine parametrization of Y . Physically, Y is the space of kinematic variables, e.g. the masses and momenta associated to a given scattering process.

Let $\mathbb{G}_m$ (resp. $\mathbb{A}$) be the complex torus (resp. the complex affine line)² equipped with the Zariski topology. We define

$$X := \{(z, x) \in Y \times (\mathbb{G}_m)^n \mid g(z; x) \neq 0\}. \quad (2.5)$$

$\pi : X \rightarrow Y$ is then the natural projection from the space of kinematic and integration variables (z, x) to the space of kinematic variables z .

Let $\mathcal{D}_Y$ represent the ring of linear differential operators with polynomial coefficients in z . It is generated by $z_1, \dots, z_m$ and $\partial_1, \dots, \partial_m$, over $\mathbb{C}$, with generators subject to the commutation relations $[z_i, z_j] = [\partial_i, \partial_j] = 0$, $[\partial_i, z_j] = \delta_{ij}$. Setting $\mathcal{O}(X) := \mathbb{C}[z_1, \dots, z_m, x_1^{\pm 1}, \dots, x_n^{\pm 1}, \frac{1}{g}]$, we define an action of $\mathcal{D}_Y$ on $h = h(z; x) \in \mathcal{O}(X)$ by

$$\frac{\partial}{\partial z_i} \bullet h := \frac{\partial h}{\partial z_i} + \beta_0 \left(\frac{1}{g(z; x)} \frac{\partial g(z; x)}{\partial z_i} \right) h. \quad (2.6)$$

The collection of (relative) k -forms on X is an $\mathcal{O}(X)$ -module

$$\Omega_{X/Y}^k := \bigoplus_{K \subset \{1, \dots, n\}, |K|=k} \mathcal{O}(X) dx^K. \quad (2.7)$$

This space is acted on by the covariant derivative in integration variables only:

$$\nabla_x := d_x + \beta_0 \frac{d_x g(z; x)}{g(z; x)} \wedge - \sum_{i=1}^n \beta_i \frac{dx_i}{x_i} \wedge. \quad (2.8)$$

The n -th *relative de Rham cohomology group* is then defined as

$$M_A(\beta; Y) := \Omega_{X/Y}^n / \text{Im}(\nabla_x : \Omega_{X/Y}^{n-1} \longrightarrow \Omega_{X/Y}^n). \quad (2.9)$$

The action of $\mathcal{D}_Y$ on $\mathcal{O}(X)$ defined by (2.6) induces an action on the quotient $M_A(\beta; Y)$, thereby endowing $M_A(\beta; Y)$ with the structure of a $\mathcal{D}_Y$ -module. If $Y = \mathbb{A}^N$ then $M_A(\beta; Y)$ equals the GKZ system [60], but we do not assume this unless stated explicitly.

²The torus $\mathbb{G}_m$ (resp. the complex affine line $\mathbb{A}$) is equal to $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ (resp. $\mathbb{C}$) as a set.

Define the *holonomic rank* of a (left) $\mathcal{D}_Y$ -module to be the number of holomorphic solutions³ at a generic point in Y . While this definition is *analytic*, we also give an *algebraic* definition in the next subsection.

We have the following result on $M_A(\beta; Y)$:

Theorem 2.1. *$M_A(\beta; Y)$ is regular holonomic. If the parameter β is generic, its holonomic rank is given by the absolute value of the Euler characteristic of a generic fiber, $|\chi(\pi^{-1}(z))|$. The latter equals the number of roots of the likelihood equation [61]*

$$\frac{\beta_0}{g(z; x)} \frac{\partial g(z; x)}{\partial x_i} - \frac{\beta_i}{x_i} = 0, \quad i = 1, \dots, n, \quad (2.10)$$

when β be a generic complex vector.

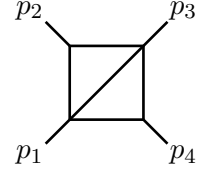
See the proof in appendix D.1.

Example 2.2. Let

$$I_{\nu_1, \dots, \nu_5} = \int \frac{d^d k_1 d^d k_2}{[k_1^2]^{\nu_1} [k_2^2]^{\nu_2} [(p_1 - k_1)^2]^{\nu_3} [(p_{12} - k_{12})^2]^{\nu_4} [(-p_{123} + k_{12})^2]^{\nu_5}} \quad (2.11)$$

denote the massless two-loop N-box integral family in $d = 4 - 2\epsilon$ dimensions with kinematics

$$p_i^2 = 0, \quad s = 2p_1 \cdot p_2, \quad t = 2p_2 \cdot p_3. \quad (2.12)$$



We used the notation $p_I = \sum_{i \in I} p_i$ in the propagators.

Up to a prefactor, the Lee-Pomeransky representation of $I_{\nu_1, \dots, \nu_5}$ can be brought into the form

$$\int_{(0, \infty)^5} g(z; x)^{\epsilon-2} x_1^{\nu_1+\epsilon\delta} \dots x_5^{\nu_5+\epsilon\delta} \frac{dx}{x} \quad (2.13)$$

with

$$\begin{aligned} g(z; x) = & x_1 x_2 + x_1 x_4 + x_1 x_5 + x_2 x_3 + x_2 x_4 \\ & + x_2 x_5 + x_3 x_4 + x_3 x_5 + x_1 x_2 x_4 + z x_2 x_3 x_5 \end{aligned} \quad (2.14)$$

and $z = t/s$. Thus, the space of kinematic variables is given by

$$Y = \{(z_1, \dots, z_{10}) \in \mathbb{A}^{10} \mid z_1 = \dots = z_9 = 1\}. \quad (2.15)$$

The two parameters $\epsilon, \delta \ll 1$ are analytic regulators in the sense of [62, 63].

The holonomic rank of $M_A(\beta; Y)$ is 3, as seen counting the number of solutions to (2.10), but the holonomic rank of the GKZ system $M_A(\beta; \mathbb{A}^{10})$ is 9. ■

³The definition of solutions for a $\mathcal{D}$ -module is given in appendix A.6.

2.2 Zero-dimensional $\mathcal{R}$ -modules and Pfaffian systems

Let $\mathcal{R}_Y$ denote the ring of differential operators with coefficients in the field of *rational* functions on Y (this generalizes $\mathcal{D}_Y$, which only contains polynomial coefficient functions). The field of rational functions on Y is denoted by $\mathcal{K}_Y$. When there is no fear of confusion, we write $\mathcal{R}$ (resp. $\mathcal{K}$) instead of $\mathcal{R}_Y$ (resp. $\mathcal{K}_Y$). An $\mathcal{R}$ -module naturally arises from a $\mathcal{D}_Y$ -module $\mathcal{M}$ by taking a tensor product $\mathcal{R} \otimes_{\mathcal{D}_Y} \mathcal{M}$. Note that given two $\mathcal{D}_Y$ -modules $\mathcal{M}_1$ and $\mathcal{M}_2$, they may fail to be isomorphic as $\mathcal{D}_Y$ -modules even if $\mathcal{R} \otimes_{\mathcal{D}_Y} \mathcal{M}_1$ and $\mathcal{R} \otimes_{\mathcal{D}_Y} \mathcal{M}_2$ are isomorphic as $\mathcal{R}$ -modules. However, as we will see, it is enough to deal with the $\mathcal{R}$ -module structure to derive a Pfaffian system.

A zero-dimensional $\mathcal{R}$ -module $\mathcal{M}$ is, by definition, a left $\mathcal{R}$ -module whose dimension over $\mathcal{K}$ is finite. The following proposition holds [64, Theorem 6.9.1]:

Proposition 2.3. *If $\mathcal{M}$ is a holonomic $\mathcal{D}_Y$ -module, then $\mathcal{R} \otimes_{\mathcal{D}_Y} \mathcal{M}$ is a zero-dimensional $\mathcal{R}$ -module.*

When a $\mathcal{D}_Y$ -module $\mathcal{M}$ is holonomic, its *holonomic rank* is *algebraically* defined to be $\dim_{\mathcal{K}} \mathcal{R} \otimes_{\mathcal{D}_Y} \mathcal{M}$. Theorem 2.12 will show agreement between the analytic and algebraic definitions of the holonomic rank.

Given a basis $\{e_1, \dots, e_r\}$ of a zero-dimensional $\mathcal{R}$ -module $\mathcal{M}$, there exist *Pfaffian matrices* $P_i(z) = (p_{ij}^k)_{j,k=1}^r \in \mathcal{K}^{r \times r}$, $i = 1, \dots, m$, such that the identities

$$\partial_i e_j = \sum_{k=1}^r p_{ij}^k e_k \quad (2.16)$$

hold true. It follows that these matrices are subject to the integrability condition:

$$\partial_i P_j(z) - \partial_j P_i(z) = P_i(z) \cdot P_j(z) - P_j(z) \cdot P_i(z). \quad (2.17)$$

Conversely, due to (2.16), a set of matrices $\{P_1(z), \dots, P_m(z)\}$ satisfying the integrability condition (2.17) uniquely determines a left $\mathcal{R}$ -module on $\mathcal{M} = \oplus_{i=1}^r \mathcal{K} e_i$.

For an r -dimensional column vector of *functions* $I = I(z)$, the system of PDEs

$$\partial_i I(z) = P_i(z) \cdot I(z) \quad (2.18)$$

is called a *Pfaffian system*. In the context of Feynman integrals, I would correspond to a basis of master integrals.

Example 2.4. Let us consider the case of an ODE to give intuition on how an *algebraic* basis e , consisting of differential operators with rational function coefficients, is related to an *analytic* basis of functions I . For the moment, we set $m = 1$. Take a univariate differential operator $D = \sum_{i=0}^{r-1} a_i(z) \partial^i$, where $\partial := \frac{\partial}{\partial z}$ and $a_i \in \mathcal{K}$. D then induces an $\mathcal{R}$ -module $\mathcal{M}_D := \mathcal{R}/\mathcal{R}D$. For an unknown function $f = f(z)$, the ODE

$$Df = 0 \quad (2.19)$$

is equivalent to the Pfaffian system

$$\partial I = P \cdot I, \quad P = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ -a_0 & -a_1 & \cdots & \cdots & -a_{r-1} \end{bmatrix}, \quad I = [f, \partial f, \dots, \partial^{r-1} f]^T. \quad (2.20)$$

The corresponding algebraic basis is $e = [1, \partial, \dots, \partial^{r-1}] \subset \mathcal{M}_D$.

More generally, consider any $m > 0$ and a left ideal $\mathcal{J} \subset \mathcal{R}$ generated by differential operators $D_1, \dots, D_\mu \in \mathcal{R}$. Assume that the quotient $\mathcal{R}/\mathcal{J}$ is zero-dimensional. Then we can take a finite subset of monomials of partial derivatives $\{e_i = \partial^{\alpha_i}\}_{i=1}^r$ such that its equivalence classes in $\mathcal{R}/\mathcal{J}$ form a basis. For an unknown function $f(z)$, the system of PDEs

$$D_i f = 0 \quad i = 1, \dots, \mu \quad (2.21)$$

is equivalent to the Pfaffian system (2.18) with $I = [e_1 f, \dots, e_\mu f]^T$. Let us mention that any zero-dimensional $\mathcal{R}$ -module is isomorphic to a quotient $\mathcal{R}/\mathcal{J}$ by a left ideal $\mathcal{J} \subset \mathcal{R}$. This is proved along similar lines of [65, Theorem 1].⁴ ■

The following example gives a free basis for the GKZ system. It is essential for linking an operator basis to a basis of differential forms, which in turn defines integrands for a basis of Euler integrals.

Example 2.5. Let us consider the GKZ $\mathcal{D}$ -module $M_A(\beta; \mathbb{A}^N)$, where we recall that N equals to the number of monomials in (2.2). After choosing a term order in the ring $\mathcal{R}$, a set of standard monomials $\mathbf{Std}$ is well-defined [64, chapter 6]. Each element $e_i = \partial^{\alpha_i}$ of $\mathbf{Std}$ corresponds to a cohomology class represented by a monomial differential form

$$\beta_0 (\beta_0 - 1) \dots (\beta_0 - |\alpha| + 1) \frac{x^{A\alpha}}{g^{|\alpha|}} \frac{dx}{x}, \quad (2.22)$$

where we set $A\alpha := \sum_{i=1}^N \alpha_i \cdot a_i$. These cohomology classes provide a free basis of $M_A(\beta; \mathbb{A}^N)$ on a non-empty open subset of $\mathbb{A}^N$. A Pfaffian system for $\{e_i\}$ can e.g. be obtained via a Gröbner basis computation [64, Example 6.2.1]. ■

We note that Pfaffian matrices are basis-dependent. If we choose a different basis $\{\tilde{e}_1, \dots, \tilde{e}_r\}$ specified by an invertible matrix $G = (g_i^j)_{i,j=1}^r$ such that $e_i = \sum_{j=1}^r g_i^j \tilde{e}_j$, then the new Pfaffian matrix is determined by a *gauge transformation*:

$$G[P_i] := G^{-1}(P_i G - \partial_i G). \quad (2.23)$$

⁴The statement in this reference concerns $\mathcal{D}$ -modules, but one can write the corresponding proof for $\mathcal{R}$ -modules simply by replacing the length of a module by its dimension over $\mathcal{K}$.

2.3 Secondary-like equation

This section contains theorems which will be used in section 5.4 to extract certain symmetry relations among Feynman integrals.

For a left $\mathcal{R}$ -module $\mathcal{M}$, we write $\text{End}_{\mathcal{R}}(\mathcal{M})$ for the set of $\mathcal{R}$ -morphisms $\varphi : \mathcal{M} \rightarrow \mathcal{M}$. We recall that a left $\mathcal{R}$ -module $\mathcal{M}$ is said to be *simple* if any of its $\mathcal{R}$ -submodules equal $\mathcal{M}$ or 0. A proof of the following theorem is given in appendix D.2.

Theorem 2.6. *If an $\mathcal{R}$ -module $\mathcal{M}$ is simple, then $\text{End}_{\mathcal{R}}(\mathcal{M})$ is a one-dimensional $\mathbb{C}$ -vector space.*

In section 5.4, we use the contraposition of this theorem: if $\dim_{\mathbb{C}} \text{End}_{\mathcal{R}}(\mathcal{M}) > 1$, then the $\mathcal{R}$ -module $\mathcal{M}$ has a submodule $\mathcal{M}' \subset \mathcal{M}$ which is neither $\mathcal{M}$ nor 0. It is also important to note that the converse to Theorem 2.6 is true for semi-simple $\mathcal{R}$ -modules. An $\mathcal{R}$ -module is said to be semi-simple if for any $\mathcal{R}$ -submodule $\mathcal{N}$ of $\mathcal{M}$, there exists an $\mathcal{R}$ -submodule $\mathcal{N}'$ such that $\mathcal{M} = \mathcal{N} \oplus \mathcal{N}'$.

Theorem 2.7. *When the exponent β is generic and real, the $\mathcal{R}$ -module $\mathcal{R} \otimes_{\mathcal{D}_Y} M_A(\beta; Y)$ is a semi-simple $\mathcal{R}$ -module.*

Theorem 2.8. *Suppose that an $\mathcal{R}$ -module $\mathcal{M}$ is semi-simple. Then $\mathcal{M}$ is simple if and only if $\text{End}_{\mathcal{R}}(\mathcal{M})$ is a one-dimensional $\mathbb{C}$ -vector space.*

See the proofs of these theorems in appendices D.3 and D.4 respectively.

In practice, it is useful to describe the set $\text{End}_{\mathcal{R}}(\mathcal{M})$ in terms of a differential equation. Fixing a basis $\{e_1, \dots, e_r\}$ of $\mathcal{M}$, any $\mathcal{K}$ -linear map $\varphi : \mathcal{M} \rightarrow \mathcal{M}$ is represented by an $r \times r$ matrix Φ with entries in $\mathcal{K}$. Starting from (2.16) and using that $\varphi(\partial_i e_j) = \partial_i \varphi(e_j)$, a short calculation shows that φ is $\mathcal{R}$ -linear if and only if the following differential equations hold:

$$\partial_i \Phi = \Phi \cdot P_i^T - P_i^T \cdot \Phi, \quad i = 1, \dots, m, \quad (2.24)$$

which we call the *secondary-like equation* (in analogy with the secondary equation from [66]).

2.4 Restriction of a $\mathcal{D}$ -module

Consider an affine subspace $Y' \subset Y$, and let $\mathcal{O}_{Y'}$ denote the coordinate ring of Y' . More precisely, $\mathcal{O}_{Y'}$ is given by the quotient $\mathcal{O}_Y / \mathcal{I}_{Y'}$, where $\mathcal{I}_{Y'}$ is the ideal of $\mathcal{O}_Y$ which vanishes on Y' . When $Y' = \{z_1 = \dots = z_{m'} = 0\}$, $\mathcal{I}_{Y'}$ is generated by $z_1, \dots, z_{m'}$, in which case $\mathcal{O}_{Y'} \simeq \mathbb{C}[z_1, \dots, z_m] / \mathcal{I}_{Y'} = \mathbb{C}[z_{m'+1}, \dots, z_m]$. As a note on notation, primed symbols, such as Y' , will henceforth be associated to restrictions.

For a $\mathcal{D}_Y$ -module $\mathcal{M}$, we set

$$\iota^* \mathcal{M} := \mathcal{O}_{Y'} \otimes_{\mathcal{O}_Y} \mathcal{M}. \quad (2.25)$$

We write $\iota : Y' \hookrightarrow Y$ for the natural inclusion. By equation (A.2) of the appendix, $\iota^* \mathcal{M}$ can also be written as

$$\iota^* \mathcal{M} = \frac{\mathcal{M}}{z_1 \mathcal{M} + \dots + z_{m'} \mathcal{M}}. \quad (2.26)$$

The left action of $\mathcal{D}_Y$ on $\mathcal{M}$ naturally induces that of $\mathcal{D}_{Y'}$ on $\iota^* \mathcal{M}$. We thus have

Definition 2.9. The restriction of a $\mathcal{D}_Y$ -module $\mathcal{M}$ to Y' is defined by the $\mathcal{D}_{Y'}$ -module $\iota^*\mathcal{M}$.

Recall that the $\mathcal{D}_Y$ -module $M_A(\beta; Y)$ from (2.9) characterizes the vector space structure of Feynman integrals. The following proposition thus links restrictions to our study of Feynman integrals.

Proposition 2.10. Let Y' be an affine subspace of Y . Then there is an isomorphism $\iota^*M_A(\beta; Y) \simeq M_A(\beta; Y')$ of $\mathcal{D}_{Y'}$ -modules.

Proof. It follows immediately from the definition of $M_A(\beta; Y)$ and the base change formula of [57, Theorem 8.4] $\square$

Definition 2.11. The $\mathcal{R}_{Y'}$ -restriction of $\mathcal{M}$ to $z_1 = \dots = z_{m'} = 0$ is defined by $\mathcal{R}_{Y'} \otimes_{\mathcal{D}_{Y'}} \iota^*\mathcal{M}$. If no confusion arises regarding the specification of $\mathcal{R}_{Y'}$, we simply call $\mathcal{R}_{Y'} \otimes_{\mathcal{D}_{Y'}} \iota^*\mathcal{M}$ the rational restriction of $\mathcal{M}$ to $z_1 = \dots = z_{m'} = 0$.

Pfaffian matrices for rational restrictions at the level of $\mathcal{D}$ -modules can be obtained algorithmically as follows. First, compute a basis e of the rational restriction $\mathcal{R}_{Y'} \otimes_{\mathcal{D}_{Y'}} \iota^*\mathcal{M}$ as a vector space over the rational function field $\mathcal{K}_{Y'}$ on Y' . Second, express $\partial_i e$ as a linear combination of the basis e . The coefficients of e in this linear combination gives the Pfaffian matrix. For more details, see Example 4.6, Theorem 4.10, Theorem 4.14 and Algorithm 4.11.

Next we relate restrictions to solution spaces of PDEs. Elements of $\mathcal{D}_Y$ act on the holomorphic functions of any open set $U \subset Y$ by $\partial_i \bullet f = \frac{\partial f}{\partial z_i}$. We denote by $\mathcal{O}^{an}(U)$,⁵ the set of the holomorphic functions on U . $\mathcal{O}^{an}(U)$ is equipped with a left $\mathcal{D}_Y$ -module structure by this action. Let $\mathcal{I}$ be a left ideal of $\mathcal{D}_Y$. It is well-known that when $\mathcal{M} = \mathcal{D}_Y/\mathcal{I}$ is holonomic, i.e. when $\mathcal{I}$ is a holonomic ideal, then the space of holomorphic solutions on U

$$\mathrm{Hom}_{\mathcal{D}_Y}(\mathcal{M}, \mathcal{O}^{an}(U)) = \{f \in \mathcal{O}^{an}(U) \mid \ell \bullet f = 0 \text{ for all } \ell \in \mathcal{I}\} \quad (2.27)$$

is a finite-dimensional vector space over $\mathbb{C}$ (see appendix A). Let c be a point in Y . We denote by $B(c, R)$ the open ball with the center c and radius R . There exists a number $R_0 > 0$ such that $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{D}_Y}(\mathcal{M}, \mathcal{O}^{an}(B(c, R)))$ is a constant for any $R \leq R_0$. We call it the space of the holomorphic solutions of $\mathcal{M}$ at $z = c$. The following result is fundamental.

Theorem 2.12. See the expositions [64, chapter 6], [67], [68, section 5.2].

1. The left $\mathcal{D}_{Y'}$ -module $\iota^*\mathcal{M}$ is a holonomic $\mathcal{D}_{Y'}$ -module.
2. Let $\mathcal{M}$ be regular holonomic and let $c \in Y'$ be a point outside of the singular locus of $\iota^*\mathcal{M}$. Then the holonomic rank of $\iota^*\mathcal{M}$ agrees with the dimension of the space of holomorphic solutions $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{D}_Y}(\mathcal{M}, \mathcal{O}^{an}(B(c, R_0)))$ when $R_0 > 0$ is sufficiently small. In particular, if $\mathcal{M}$ takes the form $\mathcal{M} = \mathcal{D}_Y/\mathcal{I}$ for some left ideal $\mathcal{I} \subset \mathcal{D}_Y$, we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{D}_Y}(\mathcal{M}, \mathcal{O}^{an}(B(c, R_0))) = \dim_{\mathbb{C}} \frac{\mathcal{D}_Y}{\mathcal{I} + (z_1 - c_1)\mathcal{D}_Y + \dots + (z_m - c_m)\mathcal{D}_Y} \quad (2.28)$$

by setting $m' = m$. In other words, we set $Y' = \{c\}$.

⁵The superscript *an* stands for analytic; see the discussion in Remark A.10 for clarification.

3. If $c \in Y$ does not belong to the singular locus of $\mathcal{M}$, $\dim_{\mathbb{C}} \text{Hom}_{\mathcal{D}_Y}(\mathcal{M}, \mathcal{O}^{an}(B(c, R_0)))$ is equal to the holonomic rank of $\mathcal{M}$.

Let $\mathcal{D}_{Y'}^k := \overbrace{\mathcal{D}_{Y'} \times \cdots \times \mathcal{D}_{Y'}}^k$ be a free $\mathcal{D}_{Y'}$ -module. It is well-known that the restriction $\iota^* \mathcal{M}$ can be expressed as $\iota^* \mathcal{M} = \mathcal{D}_{Y'}^k / \mathcal{J}$ for some k , where $\mathcal{J}$ is a left $\mathcal{D}_{Y'}$ -submodule of $\mathcal{D}_{Y'}^k$ [64, 67]. The restriction can also be expressed in terms of a Gröbner basis as in (4.7). The holonomic rank of $\iota^* \mathcal{M}$ can be determined by computing a Gröbner basis of $\mathcal{R}_{Y'} \mathcal{J}$ in a free left $\mathcal{R}_{Y'}$ -module $\mathcal{R}_{Y'}^k$.⁶

For a given holonomic $\mathcal{D}_{Y'}$ -module $\mathcal{D}_{Y'}^k / \mathcal{J}$, there exists a left ideal $\mathcal{I}$ of $\mathcal{D}_{Y'}$ such that $\mathcal{D}_{Y'} / \mathcal{I}$ is isomorphic to $\mathcal{D}_{Y'}^k / \mathcal{J}$ as a $\mathcal{D}_{Y'}$ -module. An algorithmic construction of $\mathcal{I}$, relying on the so-called *cyclic vector*, is given in [65]. Our Remark A.9 gives an easy, albeit inexact, derivation. The cyclic vector also allows for the construction of an ideal associated to a Pfaffian system.

3 Restriction of a Pfaffian system

In this section, we present our algorithm for the restriction of a generic Pfaffian system. In section 3.1, we revise key points in the theory of integrable connections. Then in section 3.2 and section 3.3, we show the restriction procedure for a logarithmic connection and the corresponding solution of the Pfaffian system. Finally, in section 3.4, we apply the developed theory to the Pfaffian systems coming from Feynman integrals.

3.1 Integrable connection

The arguments we present in this section are Zariski local, meaning that we always work on a Zariski open subset Y_0 of the total space Y . Let $\mathcal{O}_{Y_0}$ denote the ring of regular functions on Y_0 . When Y_0 is a complement of a hypersurface $\{f = 0\}$ defined by $f \in \mathcal{O}_Y \setminus \{0\}$, $\mathcal{O}_{Y_0}$ is given by $\mathbb{C}[z_1, \dots, z_m, \frac{1}{f}]$. We set $\mathcal{D}_{Y_0} := \mathcal{O}_{Y_0} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y$. The ring structure of $\mathcal{D}_Y$ induces that of $\mathcal{D}_{Y_0}$. An *integrable connection* $\mathcal{M}$ on Y_0 is a free $\mathcal{D}_{Y_0}$ -module of finite rank as an $\mathcal{O}_{Y_0}$ -module.⁷ More concretely, $\mathcal{M}$ is isomorphic to $\mathcal{O}_{Y_0}^r$ as an $\mathcal{O}_{Y_0}$ -module for some integer r . The action of ∂_i (for $i = 1, \dots, m$) is determined by an $r \times r$ matrix $P_i(z)$ with entries in $\mathcal{O}_{Y_0}$ according to the formula (2.16), where $\{e_1, \dots, e_r\}$ is any $\mathcal{O}_{Y_0}$ -free basis. The integrability condition (2.17) follows straightforwardly for the matrices $P_i(z)$ obtained in this way. Conversely, any set of matrices $\{P_i\}_{i=1}^r$ with entries in $\mathcal{O}_{Y_0}$ satisfying the integrability condition (2.17) gives rise to a connection

$$\nabla_i = \partial_i - P_i \tag{3.1}$$

⁶For introductions to Gröbner bases of a submodule in a free module, see [64, section 3.5], [69, chapter 3], and appendix A. The first two references discuss modules over a polynomial ring, but it is easy to generalize their exposition to left modules over $\mathcal{D}_{Y'}$ or $\mathcal{R}_{Y'}$.

⁷In the standard literature, an integrable connection is usually defined as a $\mathcal{D}_Y$ -module which is locally free as an $\mathcal{O}_Y$ -module. However, a locally free $\mathcal{O}_Y$ -module is a free module when it is restricted to an open subset. As we are only interested in the structure of $\mathcal{R}$ -modules in this paper, the global structure of a locally free module does not play any role. Hence, our definition of an integrable connection does not involve locally free modules.

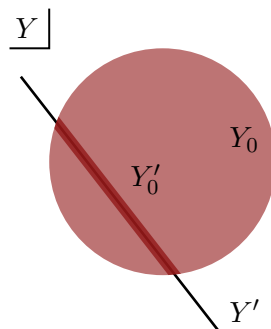


Figure 1. The different spaces Y, Y', Y_0 and Y'_0 wherein the z -variables live. The coordinates $\{z_1, \dots, z_m\}$ and $\{z_{m'+1}, \dots, z_m\}$ parameterize Y and Y' respectively. Y_0 is an open subset in Y containing Y' . The intersection $Y'_0 = Y_0 \cap Y'$ is shown as a dark red line.

by the formula (2.16). In this representation, the integrability condition translates into the commutator relation $[\nabla_i, \nabla_j] = 0$.

Any holonomic $\mathcal{D}_Y$ -module is an integrable connection on a non-empty open subset Y_0 . For a $\mathcal{D}_Y$ -module $\mathcal{M}$, we define a left $\mathcal{D}_{Y_0}$ -module $\mathcal{M}|_{Y_0}$ by $\mathcal{M}|_{Y_0} := \mathcal{O}_{Y_0} \otimes_{\mathcal{O}_Y} \mathcal{M}$. As the following proposition shows, studying the $\mathcal{R}$ -module structure of a $\mathcal{D}_Y$ -module is the same as restricting it onto an open subset wherein it is an integrable connection.

Proposition 3.1. *Let $\mathcal{M}_1, \mathcal{M}_2$ be a pair of $\mathcal{D}_Y$ -modules. $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic as $\mathcal{R}$ -modules if and only if there is a non-empty open subset $Y_0 \subset Y$ such that $\mathcal{M}_1|_{Y_0}$ and $\mathcal{M}_2|_{Y_0}$ are both integrable connections and isomorphic as $\mathcal{D}_{Y_0}$ -modules.*

Next, we recall the definition of a logarithmic connection. To simplify the exposition, let Y' be an affine hyperplane of Y . By an affine linear change of coordinates, we may assume that $Y' = \{z_1 = 0\}$. We write $\mathcal{D}_Y(\log Y')$ for the subring of $\mathcal{D}_Y$ generated as

$$\mathcal{D}_Y(\log Y') := \mathbb{C}\langle z_1, \dots, z_m, z_1 \partial_1, \partial_2, \dots, \partial_m \rangle. \quad (3.2)$$

The use of the symbol $\log Y'$ is justified because (3.4) below has a logarithmic singularity along Y' . We further introduce $Y'_0 := Y_0 \cap Y'$ and assume $Y'_0 \neq \emptyset$. The relations between the sets Y, Y', Y_0 , and Y'_0 are shown in figure 1.

Next we define

$$\mathcal{D}_{Y_0}(\log Y') := \mathcal{O}_{Y_0} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y(\log Y'). \quad (3.3)$$

The ring structure of $\mathcal{D}_Y(\log Y')$ induces that of $\mathcal{D}_{Y_0}(\log Y')$.

Definition 3.2. A *logarithmic connection* $\mathcal{M}$ on Y_0 along Y' is a $\mathcal{D}_{Y_0}(\log Y')$ -module that is free as an $\mathcal{O}_{Y_0}$ -module. More concretely, $\mathcal{M}$ is isomorphic to $\mathcal{O}_{Y_0}^r$ as an $\mathcal{O}_{Y_0}$ -module for some integer r , and the action of ∂_i is given by an $r \times r$ matrix $P_i(z)$ with entries in $\mathcal{O}_{Y_0}$ by the formulas

$$z_1 \partial_1 e_j = \sum_{k=1}^r p_{1j}^k(z) e_k \quad (3.4)$$

$$\partial_i e_j = \sum_{k=1}^r p_{ij}^k(z) e_k, \quad i = 2, \dots, m \quad (3.5)$$

where $\{e_1, \dots, e_r\}$ is some $\mathcal{O}_{Y_0}$ -free basis.

Setting

$$z' = (z_2, \dots, z_m), \quad (3.6)$$

the residue matrix of $\mathcal{M}$ along Y' is given by $P_1(0, z') = (p_{1j}^k(0, z'))_{j,k=1}^r$. The residue matrix with respect to a different free basis is obtained by a conjugation of $P_1(0, z')$. Therefore, matrix multiplication by $P_1(0, z')$ from the left is a well-defined linear operator on $\mathcal{O}_{Y'_0} \otimes_{\mathcal{O}_{Y_0}} \mathcal{M} \simeq \mathcal{O}_{Y'_0}^r$, as it does not depend on a particular choice of the free basis. Given a logarithmic connection $\mathcal{M}$ on Y_0 along Y' we associate to it a $\mathcal{D}_Y$ -module $\mathcal{D}_Y \otimes_{\mathcal{D}_Y(\log Y')} \mathcal{M}$, which is again denoted by the symbol $\mathcal{M}$ when there is no fear of confusion. As an $\mathcal{R}$ -module, $\mathcal{M}$ corresponds to a Pfaffian system (2.18) with

$$P_1 \text{ replaced by } \frac{1}{z_1} P_1(z), \quad (3.7)$$

where the factor of $1/z_1$ comes from (3.4).

An integrable connection $\mathcal{M}$ on $Y_0 \setminus Y'$ is said to be *regular* along Y' if there exists a logarithmic connection $\widetilde{\mathcal{M}}$ on Y_0 along Y' whose restriction onto $Y_0 \setminus Y'$ is $\mathcal{M}$. This terminology is compatible with the regular holonomicity of a $\mathcal{D}_Y$ -module in the following sense: if $\mathcal{M}$ is a regular holonomic $\mathcal{D}_Y$ -module and $\mathcal{M}|_{Y_0 \setminus Y'}$ is an integrable connection, then $\mathcal{M}$ is regular. Although the notion of regularity is well-defined, $\widetilde{\mathcal{M}}$ is not uniquely determined by $\mathcal{M}$. Let us finally mention that the regularity of $\mathcal{M}$ is equivalent to the corresponding Pfaffian matrices taking the form (3.7) after a gauge transformation.

3.2 Restriction of a logarithmic connection in normal form

A matrix is called *resonant*⁸ when it has two distinct eigenvalues with an integral difference. When $\mathcal{M}$ is a regular connection along Y' defined on $Y_0 \setminus Y'$, there exists a logarithmic connection $\widetilde{\mathcal{M}}$ along Y' on Y_0 such that

1. The restriction of $\widetilde{\mathcal{M}}$ onto $Y_0 \setminus Y'$ is $\mathcal{M}$,
2. The residue matrix $P_1(0, z')$ is non-resonant and the only integral eigenvalue, if it exists, is 0.

We say that such an $\widetilde{\mathcal{M}}$ is in *normal form*. If $Y' = \{z_1 = 0\}$, the normal form Pfaffian matrices associated to $\widetilde{\mathcal{M}}$ will expand as

$$\frac{P_1}{z_1} = \sum_{n=-1}^{\infty} P_{1,n}(z') z_1^n \quad (3.8)$$

$$P_i = \sum_{n=0}^{\infty} P_{i,n}(z') z_1^n, \quad i = 2, \dots, m. \quad (3.9)$$

In other words, P_1 has a logarithmic singularity on Y' , and the other Pfaffian matrices are finite there (see appendix B for more details and proofs).

⁸This is different from resonance in GKZ systems.

The existence of a normal form of a logarithmic connection is well-known [23, Proposition 5.4]. Working at the level of Pfaffian matrices, normal form can be found via the standard Moser reduction algorithms of [70, 71], which we review in appendix B.

The next proposition follows from Definition 3.2:

Proposition 3.3. *Let $\mathcal{M}$ be a logarithmic connection along Y' on Y_0 . Let $\{e_1, \dots, e_r\}$ be a free basis of $\mathcal{M}$ and let $P_i(z)$ be the matrices determined by (3.4) and (3.5). Let $\mathcal{N}$ be the left $\mathcal{D}_{Y_0}$ -submodule of $\mathcal{D}_{Y_0}^r$ generated by the row vectors of $\mathbb{1} z_1 \partial_1 - P_1(z)$ and $\mathbb{1} \partial_i - P_i(z)$ for $i = 2, \dots, m$, with $\mathbb{1}$ being the identity matrix. Then the left $\mathcal{D}_{Y_0}$ -module $\mathcal{M}$ is isomorphic to $\mathcal{D}_{Y_0}^r / \mathcal{N}$.*

Let $\mathcal{M}$ be a logarithmic connection on Y_0 along Y' , and suppose non-resonance, so that none of the eigenvalues of the residue matrix $P_1(0, z')$ is a positive integer. The restriction $\iota^* \mathcal{M}$ as an integrable connection can be calculated as follows.⁹ We fix a free basis $\{e_1, \dots, e_r\}$ of $\mathcal{M}$ and identify it with $\mathcal{O}_{Y_0}^r$ as an $\mathcal{O}_{Y_0}$ -module. We set

$$\mathcal{M}' := \mathcal{O}_{Y_0'}^r / \mathcal{N}', \quad (3.10)$$

where $\mathcal{N}'$ denotes an $\mathcal{O}_{Y_0'}$ -submodule of $\mathcal{O}_{Y_0'}^r$ spanned by the row vectors of $P_1(0, z')$. From the integrability condition (2.17), we have

$$\partial_i P_1(0, z') = [P_i(0, z'), P_1(0, z')] , \quad \text{for } i = 2, \dots, m . \quad (3.11)$$

It follows easily that the action of $\mathcal{D}_{Y_0'}$ on $\mathcal{O}_{Y_0'}^r$ given by

$$\partial_i e_j = \sum_{k=1}^r p_{ij}^k(0, z') e_k \quad (3.12)$$

induces an action on the quotient $\mathcal{M}'$.

Proposition 3.4. *The restriction $\iota^* \mathcal{M}$ is isomorphic to the integrable connection $\mathcal{M}'$ as a $\mathcal{D}_{Y_0'}$ -module.*

See the proof in appendix D.5.

As the rank of the residue matrix $P_1(0, z')$ is constant on Y_0' , $\mathcal{M}'$ is free as an $\mathcal{O}_{Y_0'}$ -module (with Y_0' being replaced by a smaller Zariski open subset if necessary), so we may denote by $Q_i(z')$ the Pfaffian matrices associated to this integrable connection. In what follows, we give a procedure for finding the Pfaffian matrices $Q_i(z')$ associated to the restriction module $\mathcal{M}'$.

Let r' be the holonomic rank of $\mathcal{M}'$. For an element m of $\mathcal{O}_{Y_0'}^r$, we write $[m]$ for the equivalence class in $\mathcal{M}'$. We suppose the existence of a special, free basis $\{f_1, \dots, f_r\}$ for $\mathcal{O}_{Y_0'}^r$ with associated Pfaffian matrices $\tilde{Q}_2(z'), \dots, \tilde{Q}_m(z')$. This basis is assumed to have the following special property: $\{f_{r'+1}, \dots, f_r\}$ is a basis of $\mathcal{N}'$. The Pfaffian matrix $Q_i(z')$ of $\mathcal{M}'$ with respect to the basis $\{[f_1], \dots, [f_{r'}]\}$ is then given by the first $r' \times r'$ block of the matrix $\tilde{Q}_i(z')$.

⁹Recall that ι is the natural inclusion introduced in section 2.4.

In general, our given basis e_i with associated Pfaffian matrices $P_2(0, z'), \dots, P_m(0, z')$ will not automatically yield this block matrix structure. In other words, we know e_i and $P_i(0, z')$ but we do not know the special basis f_i and $\tilde{Q}_i(z')$. Let us thus proceed to construct the gauge transformation mapping e_i to the special basis f_i .

Given our free basis $\{e_1, \dots, e_r\}$ of $\mathcal{O}_{Y_0}^r$, we can always find an $r' \times r$ matrix $B = (b_{ij})$ with entries in $\mathcal{O}_{Y_0}$ such that $\{[f_1], \dots, [f_{r'}]\}$ with $f_i = \sum_{j=1}^r b_{ij} e_j$ forms a free basis of $\mathcal{M}'$. Then, by the definition of $\mathcal{M}'$, there are exactly $r - r'$ linearly independent rows of the residue matrix $P_1(0, z')$. Let R be the matrix containing these independent rows. Writing $R = (r_{ij})_{i=r'+1, \dots, r}^{j=1, \dots, r}$, then $f_i = \sum_{j=1}^r r_{ij} e_j$ forms a free basis of $\mathcal{N}'$. Aligning the rows of B and R produces an invertible $r \times r$ matrix M :

$$M = \begin{bmatrix} B \\ R \end{bmatrix} = \underbrace{\begin{bmatrix} \text{red block } r' \times r' \\ \text{purple block } (r-r') \times r' \\ \text{white block } (r-r') \times (r-r') \end{bmatrix}}_r \quad (3.13)$$

In vector notation, we now have $f = M \cdot e$.

Borrowing notation from (2.23), we thus obtain the Pfaffian matrix $\tilde{Q}_i(z')$ associated to $\{f_1, \dots, f_r\}$ from the known matrices $P_i(0, z')$ via a gauge transformation by M^{-1} :

$$\tilde{Q}_i(z') = (M^{-1})[P_i(0, z')] = (M \cdot P_i(0, z') + \partial_i M) \cdot M^{-1}, \quad i = 2, \dots, m. \quad (3.14)$$

The sought-after Pfaffian matrix $Q_i(z')$ for the restriction ideal $\mathcal{M}'$, written in the basis B , is finally obtained from the first $r' \times r'$ block of $\tilde{Q}_i(z')$:

$$(\partial_i M + M \cdot P_i(0, z')) \cdot M^{-1} = \begin{bmatrix} Q_i(z') & \star \\ \emptyset & \star \end{bmatrix}. \quad (3.15)$$

This formula for obtaining $Q_i(z')$ will be essential for calculating restrictions of Pfaffian systems in section 5. Note that the lower-left block is zero because $\{f_{r'+1}, \dots, f_r\}$ is a free basis of $\mathcal{N}'$ (see also appendix C.2.1 for a matrix-based derivation of this fact). This $\emptyset$ -block can be used as an intermediate check on practical computations.

Let us conclude this section by stating an assumption used throughout the rest of this paper. First, recall that on a non-empty Zariski open subset Y_0 of Y , the $\mathcal{D}_{Y_0}$ -module $M(\beta; Y)|_{Y_0}$ is a regular integrable connection.

Assumption: let β be generic. The restriction of a normal form of $M_A(\beta; Y)|_{Y_0}$ to Y' is isomorphic to $M_A(\beta; Y')$ as an $\mathcal{R}_{Y'}$ -module.

By Proposition 3.4 and appendix B, the restriction of a normal form of $M_A(\beta; Y)|_{Y_0}$ to Y' can be computed by means of linear algebra. By this assumption, we can then compute the restriction of $\mathcal{D}$ -modules $M_A(\beta; Y') \simeq \iota^* M_A(\beta; Y)$ from the data of $\mathcal{R} \otimes_{\mathcal{D}_Y} M_A(\beta; Y)$.

Remark 3.5. Although we are presently unable to prove this assumption, we have verified in all our examples that the corank of the residue matrix is always identical to the expected holonomic rank shown in Theorem 2.1. Moreover, we compute the Q_i matrices in various examples and observe that they agree with the Pfaffian matrices derived independently using IBP software.

3.3 Restriction of solutions to a Pfaffian system

We use the same notation as the previous subsection. In this subsection, we investigate the boundary value problem of Pfaffian systems in normal form. This can be seen as a singular boundary value problem; see [72, section 2], [73] and references therein.

Holomorphic restriction Let $\mathcal{M}$ be a logarithmic connection on Y_0 along Y' and let $\{e_1, \dots, e_r\}$ be its free basis. Take $\{\phi_1, \dots, \phi_r\}$ to be the dual basis to $\{e_1, \dots, e_r\}$, i.e. each ϕ_i is an $\mathcal{O}_{Y_0}$ -linear map from $\mathcal{M}$ to $\mathcal{O}_{Y_0}$ satisfying $\phi_i(e_j) = \delta_{ij}$. By the definition of the action of ∂_i on $\mathcal{M}$, we can write the space of holomorphic solutions of $\mathcal{M}$ on an open set U (recall the definition (2.27)) as

$$\mathrm{Hom}_{\mathcal{D}_{Y_0}}(\mathcal{M}, \mathcal{O}^{an}(U)) = \left\{ \sum_{i=1}^r I_i(z) \phi_i \mid [I_1(z), \dots, I_r(z)]^T \text{ satisfies (2.18) with (3.7)} \right\}. \quad (3.16)$$

The vector of functions $I(z) = [I_1(z), \dots, I_r(z)]^T$ is a solution¹⁰ to the Pfaffian system (2.18) and it will be associated to Feynman master integrals later on.

Set $U' := Y' \cap U$. We derive a realization of $\mathrm{Hom}_{\mathcal{D}_{Y_0}}(\mathcal{M}', \mathcal{O}^{an}(U'))$ similar to (3.16). Because $\mathcal{M}$ is logarithmic, we can expand $P_1(z)$ as

$$P_1(z) = P_{1,-1}(z')z_1^{-1} + P_{1,0}(z') + P_{1,1}(z')z_1 + \dots \quad (3.17)$$

The column vector $I(z') := I(0, z') = [I_1(0, z'), \dots, I_r(0, z')]^T$ then satisfies the following system of PDEs:

$$\begin{aligned} \partial_i I(z') &= P_{i,0}(z') \cdot I(z'), \quad i = 2, \dots, m \\ P_{1,-1}(z') \cdot I(z') &= 0. \end{aligned} \quad (3.18)$$

Thus, we have an identification

$$\mathrm{Hom}_{\mathcal{D}_{Y_0'}}(\mathcal{M}', \mathcal{O}^{an}(U')) = \left\{ \sum_{i=1}^r I_i(0, z') \phi_i \mid I(z') \text{ is subject to (3.18)} \right\}. \quad (3.19)$$

We call (3.18) the *holomorphic restriction* of the Pfaffian system $\partial_i I(z) = P_i \cdot I(z)$, as it pertains to solution vectors $I(z)$ which are holomorphic at $z_1 = 0$. Note that the first condition involving the residue matrix $P_{1,-1}(z')$ imposes constraints on the solution vector

¹⁰Indeed, equation (3.16) implies that for any $m \in \mathcal{M}$ we have the homomorphism property $\partial_i(\sum_{k=1}^r I_k(z) \phi_k(m)) = \sum_{k=1}^r I_k(z) \phi_k(\partial_i m)$. So if we choose $m = e_j$, then $\partial_i(\sum_{k=1}^r I_k(z) \phi_k(e_j)) = \sum_{k=1}^r I_k(z) \phi_k(\partial_i e_j)$, resulting in $\partial_i I_j(z) = \sum_{k=1}^r I_k(z) p_{ij}^k(0, z')$, after we use $\phi_k(e_j) = \delta_{kj}$ and (3.12). Thus we see that the functions I_j indeed satisfy the Pfaffian system (2.18).

$I(z')$. These extra relations lead to a drop in rank in comparison to the unrestricted Pfaffian system.

Solutions to Pfaffian systems (3.7) and solutions of (3.18) are related to each other in the following sense:

Theorem 3.6. *Suppose that the Pfaffian system (3.7) is non-resonant. Then there is a one-to-one correspondence between the solutions $I(z)$ of (3.7) being holomorphic at $(z_1, z') = (0, 0)$ and the holomorphic solutions $I(z')$ of (3.18). The correspondence is given by the boundary value map $I(z) \mapsto I(0, z')$.*

See the proof in appendix D.6.

Logarithmic restriction So far, we have only focused on limits of Pfaffian systems for which the solution I is holomorphic. However, in physics applications, it is interesting to study solutions that develop logarithmic singularities in the restriction limit $z_1 \rightarrow 0$. Let us hence define $\tilde{\mathcal{O}}^{(N)}$ to be the vector space of functions of the form

$$f_0(z) + f_1(z) \log z_1 + \dots + f_N(z) \log^N z_1, \quad (3.20)$$

where the $f_i(z)$ denote convergent power series at the origin. Then we have an identification

$$\text{Hom}_{\mathcal{D}_{Y_0}}(\mathcal{M}, \tilde{\mathcal{O}}^{(N)}) = \left\{ \sum_{i=1}^r I_i(z) \phi_i \mid I_i(z) \in \tilde{\mathcal{O}}^{(N)}, [I_1, \dots, I_r]^T \text{ satisfies (2.18) with (3.7)} \right\}. \quad (3.21)$$

We fix a free basis $\{e_1, \dots, e_r\}$ of $\mathcal{M}$ to identify it with $\mathcal{O}_{Y_0}^r$ as an $\mathcal{O}_{Y_0}$ -module. For any non-negative integer N , we set $\mathcal{M}'_N := \mathcal{O}_{Y_0}^r / \text{Im } P_1(0, z')^{N+1}$. As discussed in the construction of $\mathcal{M}'$ in section 3.2, it follows that the action of $\mathcal{D}_{Y'_0}$ on $\mathcal{O}_{Y_0}^r$ given by (3.12) induces that on the quotient $\mathcal{M}'$.

Consider now a vector expression of the form

$$I^{(0)}(z) + I^{(1)}(z) \log z_1 + \dots + I^{(N)}(z) \log^N z_1. \quad (3.22)$$

We present the following generalization of (3.18), which is a system of PDEs for the vector functions $I^{(i)}(0, z') = I^{(i)}(z')$:

$$\begin{aligned} \partial_i I^{(n)}(z') &= P_{i,0}(z') \cdot I^{(n)}(z'), & i = 2, \dots, m, \quad n = 0, \dots, N \\ P_{1,-1}(z') \cdot I^{(n)}(z') &= (n+1)I^{(n+1)}(z'), & n = 0, \dots, N \end{aligned} \quad (3.23)$$

with $I^{(N+1)}(z') = 0$. We call (3.23) the *logarithmic restriction* of (3.7) at order N . We have an identification

$$\text{Hom}_{\mathcal{D}_{Y'_0}}(\mathcal{M}'_N, \tilde{\mathcal{O}}^{(N)}) = \left\{ \sum_{i=1}^r I^{(i)}(z') \phi_i \mid I^{(i)}(z') \in \mathcal{O}^{an}, I^{(i)} \text{ is subject to (3.23)} \right\}. \quad (3.24)$$

Theorem 3.7. *Suppose that the Pfaffian system (3.7) is non-resonant. Then there is a one-to-one correspondence between the solutions $I(z)$ to (3.7) of the form $I^{(0)}(z) + I^{(1)}(z) \log z_1 + \dots + I^{(N)}(z) \log^N z_1$ and the holomorphic solutions $I^{(n)}(z')$ of (3.23). More explicitly, the correspondence is given by the boundary value map $I(z) \mapsto I^{(0)}(0, z')$.*

See the proof in appendix D.7.

3.4 Relation to Feynman integrals

Let us summarize how the preceding sections relate to the study of Feynman integrals.

Fix a single Feynman integral. Its associated de Rham cohomology group $M_A(\beta; Y)$ (see (2.9)) carries the structure of a holonomic $\mathcal{D}_Y$ -module, so the previous results on $\mathcal{D}$ -modules and Pfaffian systems also apply to $M_A(\beta; Y)$. In particular, there exist an r -dimensional vector of master integrals $I = I(z)$ and Pfaffian matrices $P_i(z)$ such that $\partial_i I = P_i(z) \cdot I(z)$ holds for $i = 1, \dots, m$. Assume that this Pfaffian system has a singularity at $z_1 = 0$. We recall the notation $z' = (0, z_2, \dots, z_m)$.

According to the discussion of section 3.2, there exists a normal form of $M_A(\beta; Y)$. In practice, this means that we can use Moser reduction to find a set of new Pfaffian matrices $\tilde{P}_i(z)$ such (i) $\tilde{P}_1(z)$ has a simple pole at $z_1 = 0$, (ii) $\tilde{P}_2(z_1), \dots, \tilde{P}_m(z)$ are holomorphic along $\{z_1 = 0\}$, and (iii) the spectrum of the residue matrix $\tilde{P}_{1,-1}(z')$ is non-resonant (i.e., no difference of eigenvalues is integral) with zero being the unique integral eigenvalue. By abuse of notation, we keep using the notation I for the solution vector in this new basis.

We are interested in a system of *simpler* PDEs which holds in the limit $z_1 \rightarrow 0$. This problem is split into two cases: (i) the solution vector I is finite at $z_1 = 0$, or (ii) the solution vector is log-singular at $z_1 = 0$.

Case (i): Holomorphic restriction The relevant set of PDEs was given in (3.18). However, this is written in a redundant fashion because we have not resolved the constraint¹¹

$$\tilde{P}_{1,-1} \cdot I(0, z') = 0, \quad (3.25)$$

where the limit $z_1 \rightarrow 0$ on $I(z_1, z')$ is taken at the *integrand* level. There are $\text{Rank}[\tilde{P}_{1,-1}]$ many relations above, so we would like a restricted PDE system which is manifestly of dimension $r' = r - \text{Rank}[\tilde{P}_{1,-1}]$. Assume we know an r' -dimensional basis $J(z')$ for the restricted system (in section 5.2 and appendix C.2.2, we show how to construct such a basis from the original, r -dimensional one). This basis fixes the matrix B in (3.13) according to the relation $J = B \cdot I(0, z')$. Moreover, the matrix R in (3.13) equals $\text{RowReduce}[\tilde{P}_{1,-1}]$, where the operation RowReduce includes the deletion of zero-rows. So we now know the matrix M from that same formula. Equation (3.15) finally yields the $r' \times r'$ -dimensional Pfaffian matrices $Q_i(z')$ satisfying

$$\partial_i J = Q_i \cdot J, \quad i = 2, \dots, m. \quad (3.26)$$

This restricted Pfaffian system is simpler than the original one in the sense that it has a smaller rank, and it depends on one less variable. If needed, once J has been determined, it is possible to recover I by inverting the relation

$$M \cdot I(0, z') = \begin{bmatrix} B \cdot I(0, z') \\ R \cdot I(0, z') \end{bmatrix} = \begin{bmatrix} J \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad (3.27)$$

where the block of $r - r'$ zeros below J on the r.h.s. arises due to (3.25).

¹¹This constraint can be thought of as a collection of IBP relations which hold in the limit.

Let us summarize these steps in an algorithm.

Algorithm 3.8. (Pfaffian-level holomorphic restriction to $z_1 = 0$)

Input: The r -dimensional Pfaffian system $\partial_i I(z) = P_i(z) \cdot I(z)$, $i = 1, \dots, m$.

Output: The solution vector $I(0, z')$.

- 1: Gauge transform $P_i \rightarrow \tilde{P}_i := G[P_i]$ to *normal form* via Moser reduction.
- 2: Find the residue matrix $\tilde{P}_{1,-1}$ and its row-reduced form $R := \text{RowReduce}[\tilde{P}_{1,-1}]$. Set $r' = r - \text{Rank}[R]$.
- 3: Choose an r' -dimensional basis J . This yields the $r' \times r$ -dimensional matrix B via $J = B \cdot I(0, z')$. If there is no obvious choice for J , then B can be set to $B = \text{NullSpace}[R]$.
- 4: Construct the invertible block matrix $M = \begin{bmatrix} B \\ R \end{bmatrix}$.
- 5: For each $i = 2, \dots, m$, compute $(\partial_i M + M \cdot P_i(0, z')) \cdot M^{-1}$ and save the upper-left $r' \times r'$ block matrices $Q_i(z')$.
- 6: Solve the Pfaffian system $\partial_i J(z') = Q_i(z') \cdot J(z')$ for J .
- 7: **return** $I(0, z') = M^{-1} \cdot [J | 0 \dots 0]^T$.

We visualize this algorithm in figure 2 by showing the building blocks of the restricted Pfaffian system and its solution introduced in section 3.2. The **red** surface depicts every possible solution $I(0, z_2)$ which satisfies the constraint $\tilde{P}_{1,-1}(z_2) \cdot I(0, z_2) = 0$, i.e. the nullspace of $\tilde{P}_{1,-1}$. For a fixed value of z_2 , this nullspace spans a 1-dimensional vector space, shown as **red** lines. As z_2 varies, these lines trace out the **red** surface. This surface thus portrays a basis for the restricted Pfaffian system, as given by the row-span of the **B** matrix in (3.13). For fixed z_2 , the **purple** lines represent the 1-dimensional span of the row vectors in $\tilde{P}_{1,-1}$. As z_2 varies, we obtain the **purple** surface. This surface hence depicts the **R** matrix in (3.13). Orthogonality of the **purple** surface w.r.t. the **red** surface ensures invertibility of the M matrix in (3.13). The **yellow** curve illustrates a *particular* solution vector $I(0, z_2) := [I_1(0, z_2), I_2(0, z_2)]^T$ given some initial condition $I(0, 0)$. Note that this solution is constrained to lie on the **red** surface.

Case (ii): Logarithmic restriction By the general theory of PDEs [19], we expect the solution vector $I(z)$ to take the following form in the limit $z_1 \rightarrow 0$:

$$I^{(\log)}(z) = \sum_{\lambda \in \text{Spec}[\tilde{P}_{1,-1}]} z_1^\lambda \sum_{n=0}^{N_\lambda} I^{(\lambda, n)}(z') \log^n(z_1). \quad (3.28)$$

The outer sum runs over the unique eigenvalues λ of the residue matrix $\tilde{P}_{1,-1}$, and for each λ there is a collection of $N_\lambda + 1$ unknown vectors $I^{(\lambda, 0)}, \dots, I^{(\lambda, N_\lambda)}$. The integer N_λ depends on the structure of the Jordan blocks in the Jordan decomposition of $\tilde{P}_{1,-1}$.

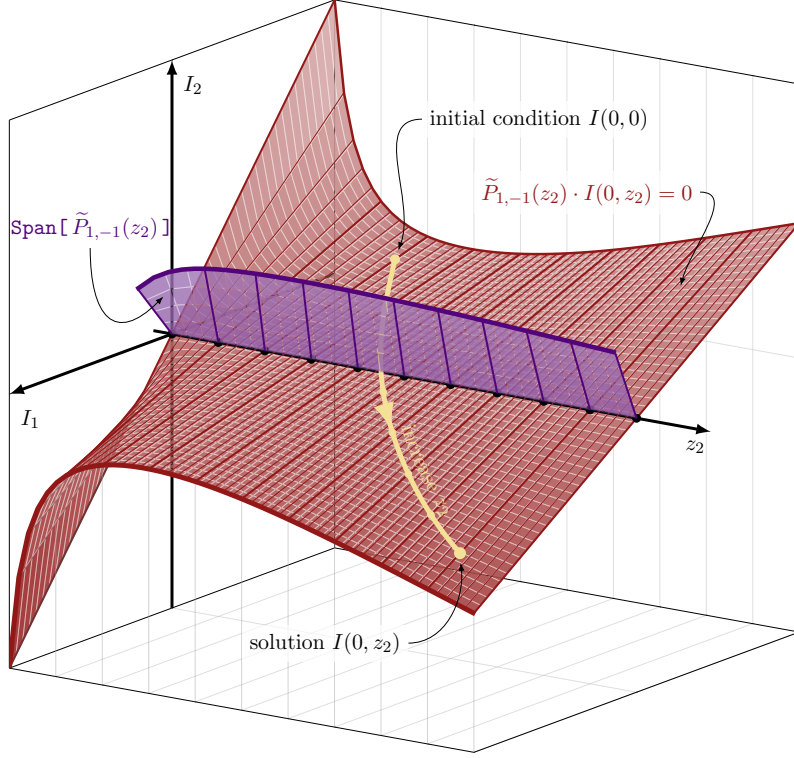


Figure 2. Algorithm 3.8 exemplified as the restriction $z_1 \rightarrow 0$ of a rank $r = 2$ Pfaffian system in two variables z_1 and z_2 .

For fixed λ , each $I^{(\lambda,n)}$ is subject to the system of PDEs (3.23). This system includes a list of constraints

$$(\tilde{P}_{1,-1} - \lambda) \cdot I^{(\lambda,n)} = (n+1)I^{(\lambda,n+1)}, \quad n = 1, \dots, N_\lambda, \quad (3.29)$$

which leads to a drop in rank. Equation (3.23) was specifically written for the eigenvalue $\lambda = 0$, but in the general case there is a shift by $-\lambda$ as on the l.h.s. We observe that the constraints above imply

$$I^{(\lambda,n)} = (n+1)! (\tilde{P}_{1,-1} - \lambda)^{n+1} \cdot I^{(\lambda,0)}, \quad n = 1, \dots, N_\lambda, \quad (3.30)$$

for which reason we only need to compute $I^{(\lambda,0)}$. Setting $n = N_\lambda$ above, we have

$$(\tilde{P}_{1,-1} - \lambda)^{N_\lambda+1} \cdot I^{(\lambda,0)} = 0, \quad (3.31)$$

giving $\text{Rank}[\tilde{P}_{1,-1}^{N_\lambda+1}]$ many constraints among the entries of the solution vector $I^{(\lambda,0)}$.

For each λ , we calculate $I^{(\lambda,0)}$ by repeating the procedure from case (i). The first step is to find an r'_λ -dimensional basis $J^{(\lambda)}$, where $r'_\lambda = r - \text{Rank}[(\tilde{P}_{1,-1} - \lambda)^{N_\lambda+1}]$, so as to fix the matrix $B^{(\lambda)}$ in

$$M^{(\lambda)} = \begin{bmatrix} B^{(\lambda)} \\ R^{(\lambda)} \end{bmatrix}. \quad (3.32)$$

It is important that $B^{(\lambda)}$ be chosen such that $M^{(\lambda)}$ is invertible. The $R^{(\lambda)}$ block encodes the constraints from the residue matrix in (3.31):

$$R^{(\lambda)} = \text{RowReduce}[(\tilde{P}_{1,-1} - \lambda)^{N_\lambda+1}]. \quad (3.33)$$

Because of (3.31), we thus get a block of zeros in

$$M^{(\lambda)} \cdot I^{(\lambda,0)} = \begin{bmatrix} \frac{B^{(\lambda)} \cdot I^{(\lambda,0)}}{R^{(\lambda)} \cdot I^{(\lambda,0)}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (3.34)$$

This can be inverted to recover $I^{(\lambda,0)}$ once we have computed $J^{(\lambda)}$. Inserting $M^{(\lambda)}$ into (3.15), we obtain a collection of $r'_\lambda \times r'_\lambda$ -dimensional Pfaffian matrices $Q_i^{(\lambda)}(z')$ such that

$$\partial_i J^{(\lambda)} = Q_i^{(\lambda)} \cdot J^{(\lambda)}, \quad i = 2, \dots, m. \quad (3.35)$$

The equations (3.30), (3.34) and (3.35) together determine the unknown vectors $I^{(\lambda,n)}$ in (3.28), apart from fixing the boundary constants stemming from solutions of the Pfaffian system (3.35). These constants can e.g. be fixed by comparing with the original integral $I(z)$ at special numerical points.

Summarizing all the steps above, we have

Algorithm 3.9. (Pfaffian-level logarithmic restriction to $z_1 = 0$)

Input: The r -dimensional Pfaffian system $\partial_i I(z) = P_i(z) \cdot I(z)$, $i = 1, \dots, m$.

Output: The solution vector $I^{(\log)}(z)$.

- 1: Gauge transform $P_i \rightarrow \tilde{P}_i := G[P_i]$ to *normal form* via Moser reduction.
- 2: Find the residue matrix $\tilde{P}_{1,-1}$.
- 3: **for** each unique $\lambda \in \text{Spec}[\tilde{P}_{1,-1}]$ **do**
- 4: Compute $R^{(\lambda)} = \text{RowReduce}[(\tilde{P}_{1,-1} - \lambda)^{N_\lambda+1}]$ and set $r'_\lambda = r - \text{Rank}[R^{(\lambda)}]$.
- 5: Choose an r'_λ -dimensional basis $J^{(\lambda)}$. This fixes an $r'_\lambda \times r$ -dimensional matrix $B^{(\lambda)}$ via $J^{(\lambda)} = B^{(\lambda)} \cdot I^{(\lambda,0)}$. One choice is $B^{(\lambda)} = \text{NullSpace}[R^{(\lambda)}]$.
- 6: Construct the invertible block matrix $M^{(\lambda)} = \begin{bmatrix} B^{(\lambda)} \\ R^{(\lambda)} \end{bmatrix}$.
- 7: For each $i = 2, \dots, m$, compute $(\partial_i M^{(\lambda)} + M^{(\lambda)} \cdot P_i(0, z')) \cdot (M^{(\lambda)})^{-1}$ and save the upper-left $r'_\lambda \times r'_\lambda$ block matrices $Q_i^{(\lambda)}(z')$.
- 8: Solve the Pfaffian system $\partial_i J^{(\lambda)}(z') = Q_i^{(\lambda)}(z') \cdot J^{(\lambda)}(z')$ for $J^{(\lambda)}$.
- 9: Compute $I^{(\lambda,0)}(z') = (M^{(\lambda)})^{-1} \cdot [J^{(\lambda)} | 0 \dots 0]^T$
- 10: Compute $I^{(\lambda,n)}(z') = (n+1)! (\tilde{P}_{1,-1} - \lambda)^{n+1} \cdot I^{(\lambda,0)}$ for each $n = 1, \dots, N_\lambda$.
- 11: **end for**
- 12: **return** The sum (3.28).

This is a novel approach for computing logarithmic corrections to scattering events. A pedagogical example is given in section 5.2, where we compute logarithmic corrections to a one-loop box diagram contributing to Bhabha scattering. In that example we imagine that z_1 denotes a small mass, such as that of the electron. While the goal of the example is to compute $I^{(\lambda, \bullet)}(z')$, the actual, full, solution $I(z)$ would read

$$I(z) = \sum_{\lambda \in \text{Spec}[\tilde{P}_{1,-1}]} z_1^\lambda \sum_{n=0}^{N_\lambda} I^{(\lambda,n)}(z_1, z') \log^n(z_1), \quad (3.36)$$

where $I^{(\lambda,n)}(z_1, z')$ is the analytic function satisfying $I^{(\lambda,n)}(0, z') = I^{(\lambda,n)}(z')$. Using the recursion relations (D.10), (D.21) from the proofs of Theorem 3.6 and Theorem 3.7, presented in appendix D, it is possible to reconstruct $I^{(\lambda,n)}(z_1, z')$ from $I^{(\lambda,n)}(z')$ as a power series in z_1 . In other words, one would thereby obtain terms of the form $z_1^a \log^b(z_1)$ for $a, b \in \mathbb{Z}_{>0}$. Explicit calculations of this nature are left for future work.

4 Restrictions as $\mathcal{D}$ -modules by the Macaulay matrix method

In this section, we will discuss rational restrictions from the $\mathcal{D}$ -module point of view. The first subsection gives an algorithm for finding bases of rational restrictions. The second subsection uses these bases to construct rational restrictions via the Macaulay matrix method.

4.1 Restrictions as $\mathcal{D}$ -modules

Recall the notion of rational restrictions from Definition 2.11. There, Y is the m -dimensional complex plane and Y' is $(m - m')$ -dimensional complex plane in Y defined by $z_1 = \dots = z_{m'} = 0$. The following proposition will be used to prove the correctness of Algorithm C.2 later. A proof is given in appendix D.8.

Proposition 4.1. *Let $\mathcal{I}$ be a regular holonomic ideal of $\mathcal{D}_Y$. We have the isomorphism of left $\mathcal{R}_{Y'}$ -modules*

$$\mathcal{R}_{Y'} \otimes_{\mathcal{D}_{Y'}} \mathcal{D}_Y / (\mathcal{I} + \sum_{j=1}^{m'} z_j \mathcal{D}_Y) \simeq \mathcal{R}_{Y'} \otimes_{\mathcal{D}_{Y'}} \mathcal{D}_Y / ((\mathcal{R}_Y \mathcal{I}) \cap \mathcal{D}_Y + \sum_{j=1}^{m'} z_j \mathcal{D}_Y). \quad (4.1)$$

The ideal $(\mathcal{R}_Y \mathcal{I}) \cap \mathcal{D}_Y$ is known as the Weyl closure of $\mathcal{I}$ [74, Definition 5.1].

Let $\mathcal{I}$ be a left holonomic ideal in $\mathcal{D}_Y$. The restriction of the left holonomic $\mathcal{D}_Y$ -module $\mathcal{M} = \mathcal{D}_Y / \mathcal{I}$ to Y' is defined by $\iota^* \mathcal{M} = \mathcal{D}_Y / (\mathcal{I} + z_1 \mathcal{D}_Y + \dots + z_{m'} \mathcal{D}_Y)$. We denote the restriction $\iota^* \mathcal{M}$ by $\mathcal{N}_{Y'}$. When $m = m'$, we denote the restriction by $\mathcal{N}_0$.

Let $w \in \mathbb{Z}^m$ be a vector and let w_i denote its i -th component. Define the $(-w, w)$ -order for $cz^\alpha \partial^\beta$, $c \in \mathbb{C}$, by

$$\text{ord}_{(-w, w)}(cz^\alpha \partial^\beta) = -w \cdot \alpha + w \cdot \beta, \quad (4.2)$$

where $z^\alpha = \prod_{i=1}^m z_i^{\alpha_i}$, $\partial^\beta = \prod_{i=1}^m \partial_i^{\beta_i}$ and $\cdot$ is the standard inner product. The order is defined for $\ell \in \mathcal{D}_Y$ as the maximum of all monomial $(-w, w)$ -orders appearing in ℓ . For

$\ell, \ell' \in \mathcal{D}_Y$, we have $\text{ord}_{(-w,w)}(\ell\ell') = \text{ord}_{(-w,w)}(\ell) + \text{ord}_{(-w,w)}(\ell')$. Putting $\theta_i = z_i \partial_i$, we note that $\text{ord}_{(-w,w)}(\theta_i) = 0$. Since w defines the order among monomials, it is called the *weight vector*.

Now, let us briefly review an algorithm for constructing the restriction $\mathcal{N}_{Y'}$ of $\mathcal{M}$ by following the exposition [68, section 5.2]. Let $w \in \mathbb{Z}^m$ be a weight vector such that

$$w_{m'+1} = \dots = w_m = 0, \quad w_i > 0, \quad 1 \leq i \leq m'. \quad (4.3)$$

Let $b(u)$ be the *b-function* (also called the indicial polynomial) of the left ideal $\mathcal{I}$ in $\mathcal{D}_Y$ with respect to w . More precisely, $b(\sum_{i=1}^{m'} w_i z_i \partial_i)$ is a monic generator of the principal ideal

$$\mathbb{C} \left[\sum_{i=1}^{m'} w_i z_i \partial_i \right] \cap \text{in}_{(-w,w)} \mathcal{I}, \quad (4.4)$$

where $\text{in}_{(-w,w)}(f)$ is the sum of the highest order terms in $f \in \mathcal{D}_Y$ such that the weight for z_i is $-w_i$ and the weight for ∂_i is w_i [68, section 1.1].¹²

Fix a weight vector w satisfying (4.3). Let $\mathcal{F}_k$ be the set of all elements in $\mathcal{D}_Y$ such that the $(-w, w)$ -order is less than or equal to k . In other words, we have

$$\mathcal{F}_k = \left\{ \sum c_{\alpha\beta} z^\alpha \partial^\beta \mid -w \cdot \alpha + w \cdot \beta \leq k \right\}, \quad (4.5)$$

which is a free left $\mathcal{D}_{Y'}$ -module.

Theorem 4.2. [54], [68, section 5.2]

1. If $b(u) = 0$ does not have a non-negative integral root, then the restriction $\mathcal{N}_{Y'} = 0$.
2. Let u_0 be the maximal non-negative integral root of $b(u) = 0$ and let $\{g_1, \dots, g_t\}$ be a $(-w, w)$ -Gröbner basis of $\mathcal{I}$ (see [68, section 1.1]). Then the restriction $\mathcal{N}_{Y'}$ is expressed as

$$\mathcal{N}_{Y'} \simeq \frac{\mathcal{F}_{u_0}}{\sum_{\{i \in \{1, \dots, t\} \mid \text{ord}_{(-w,w)} g_i \leq u_0\}} \mathcal{F}_{u_0 - \text{ord}_{(-w,w)}(g_i)} g_i + \sum_{j=1}^{m'} z_j \mathcal{F}_{u_0 + w_j}} \quad (4.6)$$

as a left $\mathcal{D}_{Y'}$ -module.

The right hand side of (4.6) can also be expressed as

$$\frac{\sum_{w \cdot \alpha \leq u_0} \mathcal{D}_{Y'} \prod_{j=1}^{m'} \partial_j^{\alpha_j}}{\sum_{\{i \in \{1, \dots, t\} \mid \text{ord}_{(-w,w)} g_i \leq u_0\}} \mathcal{D}_{Y'} \sum_{w \cdot \beta \leq u_0 - \text{ord}_{(-w,w)}(g_i)} : \prod_{j=1}^{m'} \partial_j^{\beta_j} g_i : |_{z_1=\dots=z_{m'}=0}}}. \quad (4.7)$$

The notation $: \ell : |_{z_1=\dots=z_{m'}=0}$ means that we first order each monomial in ℓ such that ∂_i 's come to the right and z_j 's come to the left (via $[\partial_i, z_j] = \delta_{ij}$), whereafter we set $z_1 = \dots = z_{m'} = 0$. For example, we have

$$: \partial_1 z_1 : |_{z_1=0} = (z_1 \partial_1 + 1) |_{z_1=0} = 1.$$

¹²We note that several computer algebra systems have commands for computing *b-functions* — see in particular [68, p. 194] for a definition of the *b-function* which is computer algebra friendly and [64] for details on computer algebra systems which are useful in this context.

Note that u_0 may be set equal to a non-negative integer larger than or equal to the maximal non-negative integral root of $b(u) = 0$. This procedure will be utilized in the following sections.

Remark 4.3. Let us regard $\mathcal{R}_{Y'} \otimes_{\mathcal{D}_{Y'}} \mathcal{N}_{Y'}$ as a vector space over $\mathbb{C}(z_{m'+1}, \dots, z_m)$. Considering a Gröbner basis in a free $\mathcal{R}_{Y'}$ -module for the denominator of the expression (4.7), we can take a basis of the vector space consisting only of monomials in ∂_i 's.

We proceed by constructing an algorithm for finding the Pfaffian system associated to the restriction $\mathcal{N}_{Y'}$ via Gröbner bases for submodules. This algorithm will lend itself to the Macaulay matrix restriction method presented in section 4.2 by providing a standard basis.

In order to simplify our presentation, we first illustrate our algorithms for $m = 2$ and $m' = 1$. The generalization to more than 2 variables is not difficult, and will be discussed later. In the rest of this section, we set $x = z_1, y = z_2$ and $\mathcal{D}' = \mathcal{D}_m, \mathcal{D} = \mathcal{D}_{m-m'}$.

We want to restrict a given left ideal $\mathcal{I} = \mathcal{D}' \cdot \{f_1, \dots, f_\mu\}$ in $\mathcal{D}' = \mathbb{C}\langle x, y, \partial_x, \partial_y \rangle$ to $x = 0$. The restriction module is defined as

$$\mathcal{D}'/(\mathcal{I} + x\mathcal{D}').$$

As stated in Theorem 2.12, when $\mathcal{D}'/\mathcal{I}$ is a holonomic $\mathcal{D}'$ -module, the restriction module is a holonomic $\mathcal{D}$ -module with $\mathcal{D} = \mathbb{C}\langle y, \partial_y \rangle$.

Let $\mathcal{J}$ be a left submodule of $\mathcal{D}^{k+1}$. The holonomic rank of the left $\mathcal{D}$ -module $\mathcal{D}^{k+1}/\mathcal{J}$ is, by definition, $\dim_{\mathbb{C}(y)} \mathcal{R}^{k+1}/(\mathcal{R}\mathcal{J})$, where $\mathcal{R} = \mathbb{C}(y)\langle \partial_y \rangle$.

Let $\ell \in \mathcal{D}'$ and take k to be a sufficiently large number. We regard the operator

$$\ell = \sum_{j=0}^k c_j(x, y, \partial_y) \partial_x^j$$

as an element of $\mathcal{D}[x]^{k+1}$ by

$$v_k(\ell) := (c_k, c_{k-1}, \dots, c_1, c_0) \in \mathcal{D}[x]^{k+1}. \quad (4.8)$$

We regard $\mathcal{D}[x]^{k+1}$ (resp. $\mathcal{D}^{k+1}$) as a subset of $\mathcal{D}[x]^{k+2}$ (resp. $\mathcal{D}^{k+2}$) by the inclusion

$$\mathcal{D}[x]^{k+1} \ni (c_k, c_{k-1}, \dots, c_1, c_0) \mapsto (0, c_k, c_{k-1}, \dots, c_1, c_0) \in \mathcal{D}[x]^{k+2}.$$

Let u_0 be the maximal non-negative integral root of the b -function of $\mathcal{I}$ for the weight $w = (1, 0)$. If there is no such u_0 , the restriction is 0. Let u_1 be the maximal $(-w, w)$ -order of the elements in a $(-w, w)$ -Gröbner basis for $\mathcal{I}$ (see [68, section 1.1] for details on Gröbner bases of Weyl algebras).

Algorithm 4.4. (Rational restriction to $x = 0$)

Input: Generators $\{f_1, \dots, f_\mu\}$ of a holonomic ideal $\mathcal{I}$ in $\mathcal{D}'$. Holonomic rank r of the restriction of $\mathcal{D}'/\mathcal{I}$ to $x = 0$. A positive integer γ such that $\gamma \geq \max(u_0 + 1, u_1 + 1)$.

Output: Generators of a left submodule $\mathcal{J} \cap \mathcal{D}^\gamma$ of $\mathcal{D}^\gamma$ such that $\mathcal{R}^\gamma/\mathcal{R}(\mathcal{J} \cap \mathcal{D}^\gamma) \simeq \mathcal{R} \otimes_{\mathcal{D}} \mathcal{D}'/(\mathcal{I} + x\mathcal{D}')$, that is the rational restriction.

```

1:  $w = (1, 0)$ 
2:  $k = \gamma - 1$ 
3: repeat
4:    $\mathcal{J} = \mathcal{D} \cdot \left\{ v_k \left( : \partial_x^j f_i : \Big|_{x=0} \right) \mid \text{ord}_{(-w, w)}(\partial_x^j f_i) \leq k, 1 \leq i \leq N, j \in \mathbb{N}_0 \right\}$ 
5:    $k := k + 1.$ 
6: until  $\text{rank}(\mathcal{D}^\gamma / \mathcal{J} \cap \mathcal{D}^\gamma) = r$ 
7: return  $\mathcal{J} \cap \mathcal{D}^\gamma$ 
    
```

We can obtain generators of $\mathcal{J} \cap \mathcal{D}^\gamma$ by the POT (position over term) order in $\mathcal{D}^{k+1}$ (see appendix A for Gröbner bases in a free module and the POT order). Once we get a set of generators of $\mathcal{J} \cap \mathcal{D}^\gamma$, the Pfaffian system of the restriction $\mathcal{D}'/(\mathcal{I} + x\mathcal{D}')$ can be obtained from a Gröbner basis of $\mathcal{R}(\mathcal{J} \cap \mathcal{D}^\gamma)$. An improvement of this algorithm is given in appendix C.1.

Theorem 4.5. *Algorithm 4.4 stops at some number k and gives the correct answer.*

The proof of this theorem is technical; see appendix D.9.

Let us illustrate Algorithm 4.4 with a small input.

Example 4.6. Let $\mathcal{I}$ be the left ideal generated by

$$\begin{aligned} f_1 &= 2\theta_x\theta_y - 1 = 2xy\partial_x\partial_y - 1, \\ f_2 &= 2\theta_y^2 + \theta_x - \theta_y - 1 = 2(y^2\partial_y^2 + y\partial_y) + x\partial_x - y\partial_y - 1. \end{aligned}$$

The solution space for $\mathcal{I}$ is spanned by $\{xy^{1/2}, x^{1/\sqrt{2}}y^{1/\sqrt{2}}, x^{-1/\sqrt{2}}y^{-1/\sqrt{2}}\}$. We will construct the restriction to $x = 0$. The listed solutions imply that the expected rank of the restriction is 1, standing for the solution $xy^{1/2}$, which is the only solution holomorphic at $x = 0$. The b -function for the weight $w = (1, 0)$ is $(u - 1)(u^2 - 1/2)$, implying that $u_0 = 1$. The maximum $(-w, w)$ -order of the $(-w, w)$ -Gröbner basis of $\mathcal{I}$ is 0, hence $u_1 = 0$. Thus we put $\gamma = 2$. We start with $k = 1$ in Algorithm 4.4. The submodule $\mathcal{J}$ is generated by

$$\begin{aligned} (0, -1) &= v_1(: f_1 : |_{x=0}), \\ (2y\partial_y - 1, 0) &= v_1(: \partial_x f_1 : |_{x=0}) = v_1(2y\partial_x\partial_y + 2xy\partial_x^2\partial_y - \partial_x |_{x=0}), \\ (0, 2y^2\partial_y^2 + y\partial_y - 1) &= v_1(: f_2 : |_{x=0}), \\ (2y^2\partial_y^2 + y\partial_y, 0) &= v_1(: \partial_x f_2 : |_{x=0}) \end{aligned}$$

The Gröbner basis of $\mathcal{J}$ with the POT order is

$$\{(0, -1), (2y\partial_y - 1, 0)\},$$

whose holonomic rank is 1. The set of standard monomials of the Gröbner basis in $\mathcal{D}^2$ is $S = \{s_1 := (1, 0) \in \mathcal{D}^2\}$. Then, we have the 1×1 Pfaffian matrix $\left(\frac{1}{2y}\right)$ for S as

$$\partial_y s_1 = (\partial_y, 0) \equiv \left(\frac{1}{2y}, 0\right) = \frac{1}{2y} s_1.$$

■

Let us now generalize to the case of several variables $z = (z_1, \dots, z_m)$. Fix an integer m and a weight vector w satisfying (4.3). We put

$$c(k) = \#E_k, \quad E_k = \{\alpha \mid \alpha \cdot w \leq k, \alpha_1, \dots, \alpha_{m'} \in \mathbb{N}_0, \alpha_{m'+1} = \dots = \alpha_m = 0\}. \quad (4.9)$$

We regard $\{\partial_1^{\alpha_1} \dots \partial_{m'}^{\alpha_{m'}} \mid \alpha \in E_k\}$ as the standard basis of $\mathcal{D}_{Y'}[z_1, \dots, z_{m'}]^{c(k)}$. We define the map v_k from $\mathcal{D}_Y$ to $\mathcal{D}_{Y'}[z_1, \dots, z_{m'}]^{c(k)}$ by

$$v_k(\ell) = \sum_{\alpha \in E_k} \ell_\alpha \partial_1^{\alpha_1} \dots \partial_{m'}^{\alpha_{m'}}, \quad (4.10)$$

where we decompose $\mathcal{D} \ni \ell$ as $\ell = \sum_{\alpha \in E_k} \ell_\alpha \partial_1^{\alpha_1} \dots \partial_{m'}^{\alpha_{m'}}$, $\ell_\alpha \in \mathcal{D}_{Y'}[z_1, \dots, z_{m'}]$. Algorithm 4.4 can be generalized using v_k . The map v_k takes values in a $c(k)$ -dimensional vector space. We start the procedure from $k \geq \max(u_1, u_0)$, where u_0 is the maximal non-negative integral root of the b -function with respect to w , and u_1 is the maximum of $\text{ord}_{(-w, w)}$ for the $(-w, w)$ -Gröbner basis of $\mathcal{I}$. Note that $c(k) = k + 1$ when $m = 2$, $m' = 1$.

The case $m' = m$ will be used to find a rational restriction by the Macaulay matrix method, so the following algorithm assumes this equality.

Algorithm 4.7. (Basis for the restriction to $z_1 = \dots = z_m = 0$)

Input: Generators $\{f_1, \dots, f_\mu\}$ of a holonomic ideal $\mathcal{I}$ in $\mathcal{D}_Y$. The vector space dimension r of holomorphic solutions at $z = (z_1, \dots, z_m) = 0$. A positive integer γ such that $\gamma \geq \max(u_0, u_1)$.

Output: A $\mathbb{C}$ -basis of the restriction $\mathcal{N}_0$ to the point $z = (z_1, \dots, z_m) = 0$.

- 1: $w = (1, \dots, 1)$
- 2: $k = \gamma$
- 3: **repeat**
- 4: $\mathcal{J} = \mathbb{C} \cdot \left\{ v_k(\partial^\alpha f_i) \mid \text{ord}_{(-w, w)} \left(\prod_{j=1}^m \partial_j^{\alpha_j} f_i \right) \leq k, \alpha \in \mathbb{N}_0^m \right\} \subseteq \mathbb{C}^{c(k)}$
- 5: $k := k + 1$
- 6: **until** $\dim(\mathbb{C}^{c(\gamma)} / \mathcal{J} \cap \mathbb{C}^{c(\gamma)}) = r$
- 7: **return** A vector space basis of $\mathbb{C}^{c(\gamma)} / \mathcal{J} \cap \mathbb{C}^{c(\gamma)}$.

Here we identify the vector space $\mathbb{C}^{c(k)}$ with the vector space spanned by $\partial^\alpha, \alpha \in E_k$. For example, when $m = 2$, $k = 2$, and ∂^α is sorted as $(\partial_1^2, \partial_1 \partial_2, \partial_2^2, \partial_1, \partial_2, 1)$, then $(0, 0, 0, 0, 0, 1)$ stands for 1, $(0, 0, 0, 0, 1, 0)$ for ∂_2 , and $(1, 0, 0, 0, 0, 0)$ for ∂_1^2 . Note that this algorithm only utilizes numerical linear algebra, so it is enough to compute a basis of the linear space $\mathbb{C}^{c(\gamma)} / \mathcal{J} \cap \mathbb{C}^{c(\gamma)}$. The numerical matrix representing $\mathcal{J}$ is large in general, so in our implementations we perform the steps 6 and 7 over a finite field instead of $\mathbb{C}$, which greatly speeds up the computation. Note also that we do not know γ beforehand without computing a Gröbner basis, for which reasons we also choose γ probabilistically in our implementation.

Example 4.8. We denote z_1, z_2 by x, y respectively. The function $(y^2 - x^3)^{1/2}$ spans the holomorphic solution space of the operators $\{2y\partial_x + 3x^2\partial_y, 2x\partial_x + 3y\partial_y - 3\}$ around $(x, y) = (1, 2)$. Let us compute the restriction to this point. We make the change of variables

$x := x + 1$ and $y := y + 2$ and apply Algorithm 4.7 with $w = (1, 1)$ and $m = 2$. We have $b(u) = u$, so $u_0 = 0$. The $(-w, w)$ -initial terms of the $(-w, w)$ -Gröbner basis are

$$\{\partial_x + \dots, \partial_y + \dots, x\partial_y^2 + \dots, x^2\partial_x^3 + \dots\}.$$

The $(-w, w)$ -orders all equal 1. Hence, we can take $u_1 = 1$. We have $E_0 = \{(0, 0)\}$ and $E_1 = \{(1, 0), (0, 1), (0, 0)\}$. Therefore, we can set $\gamma = 1$ and $c(\gamma) = 2$. The input operators are

$$f_1 = 2(y + 2)\partial_x + 3(x + 1)^2\partial_y \quad (4.11)$$

$$f_2 = 2(x + 1)\partial_x + 3(y + 2)\partial_y - 3. \quad (4.12)$$

Then the generators of $\mathcal{J}$ for $k = 1$ are

$$v_k(f_1|_{x=y=0}) = (4, 3, 0), \quad v_k(f_2|_{x=y=0}) = (2, 6, -3), \quad (4.13)$$

with the vector v_k being indexed by $(\partial_x, \partial_y, 1)$. The row echelon form of $\begin{bmatrix} 4 & 3 & \cdot \\ 2 & 6 & -3 \end{bmatrix}$ is $\begin{bmatrix} -2 & \cdot & -1 \\ \cdot & -3 & 2 \end{bmatrix}$, meaning that the rank of $\mathbb{C}^3/\mathcal{J} \cap \mathbb{C}^3$ is 1. Note that $\gamma = 1$ does not work even when we increase k to larger values. $\blacksquare$

4.2 Macaulay matrix method for restrictions

In this subsection, we consider the case when $Y' \subset Y$ is an irreducible, closed and smooth subvariety. Building on the work of [1], the aim of this section is to give a Macaulay matrix method for rational restrictions, as per Definition 2.11, when Y' is either $z_1 = \dots = z_{m'} = 0$ or Y' is an irreducible hypersurface. The rational restriction gives a description of holomorphic solutions on Y' .

Let $\mathcal{M} = \mathcal{D}_Y/\mathcal{I}$ be a regular holonomic $\mathcal{D}_Y$ -module. The space Y can be decomposed into algebraic varieties Y_α such that $Y = \sqcup Y_\alpha$ and the space of holomorphic solutions of $\mathcal{M}$ on Y_α is a locally constant sheaf. In other words, all holomorphic solutions on Y_α are analytic continuations of holomorphic solutions at a point in Y_α along paths in Y_α . In particular, the number of holomorphic solutions at a point $c \in Y_\alpha$ only depends on Y_α and not c . This decomposition is called a *stratification*. Stratifications were proved to exist in a broader context in the work of M. Kashiwara [21]. For instance, a stratification of the system in the Example 4.6 is

$$(\mathbb{C}^2 \setminus V(xy)) \sqcup (V(x) \setminus (0, 0)) \sqcup (V(y) \setminus (0, 0)) \sqcup (0, 0). \quad (4.14)$$

Here, $V(f)$ is the variety defined by $f = 0$. The number of holomorphic solutions on these stratifications are respectively 4, 1, 0 and 0.

Consider the restriction $\mathcal{N} := \iota^*\mathcal{M}$ and let $W \subset Y'$ be the maximum-dimensional stratum of Y' . W is known to be a Zariski open set, and it does not intersect with the singular locus of $\mathcal{N}$. We suppose that the origin 0 belongs to W .

Lemma 4.9. *Let $S = \{s_i\}$ be a finite set consisting of monomials in ∂ . Assume that a set consisting of equivalence classes represented by elements of S is a $\mathbb{C}$ -basis for a $\mathbb{C}$ -vector space $\mathcal{D}_Y/(\mathcal{I} + \sum_{i=1}^m z_i \mathcal{D}_Y)$. Then the set of equivalence classes in $\mathcal{N}|_W$ represented by elements of the form $1 \otimes s_i \in \mathcal{N}|_W$, for some $s_i \in S$, gives a free basis of $\mathcal{N}|_W$ as an $\mathcal{O}_{Y'}$ -module.*

It follows from this lemma that S gives a basis for the rational restriction of $\mathcal{M}$ to Y' . The lemma is well-known folklore in the theory of $\mathcal{D}$ -modules and we give a constructive and elementary proof in the appendices D.10 and D.12.

Let $\{f_1, \dots, f_\mu\}$ be a set of generators for a regular holonomic ideal $\mathcal{I}$ in $\mathcal{D}_Y$. We make a parallel change of variables $(z_1, \dots, z_m) = (z'_1, \dots, z'_m) + (c_1, \dots, c_m)$ by choosing a numerical vector $(c_1, \dots, c_m)$ in $W \subset Y'$ such that we can apply Lemma 4.9. We dispose of the old coordinates z_i and denote the new coordinates z'_i by z_i . A set of monomials S as in Lemma 4.9 can be obtained by Algorithm 4.7. Since we do not know the stratum W , the chosen variable shift is probabilistic. If we unluckily choose a point outside of W , the method fails. However, the relative measure of $Y' \setminus W$ is 0, so the algorithm succeeds with probability 1. The following theorem easily follows from Lemma 4.9.

Theorem 4.10. *Let S be as in Lemma 4.9, and regard it as a column vector. There exists an $r \times \mu$ matrix Q_i with entries in $\mathcal{R}_{Y'}[\partial_1, \dots, \partial_{m'}]$ and an $r \times r$ matrix P_i with entries in $\mathbb{C}(z') := \mathbb{C}(z_{m'+1}, \dots, z_m)$ such that*

$$\partial_i S = P_i \cdot S + : Q_i \cdot [f_1, \dots, f_\mu]^T : \Big|_{z_1=\dots=z_{m'}=0} \quad (4.15)$$

holds in $\mathcal{R}_{Y'}[\partial_1, \dots, \partial_{m'}]$.

The expression (4.15) leads to the following Macaulay matrix method for finding a Pfaffian system associated to the restriction to $z_1 = \dots = z_{m'} = 0$.

Algorithm 4.11. (Rationally restricted Pfaffian system via the Macaulay matrix)

Input: The basis S . Generators $\{f_1, \dots, f_\mu\}$ of a regular holonomic ideal $\mathcal{I}$ in $\mathcal{D}_m$.

Output: Pfaffian matrices of the restriction $\mathcal{R}_{Y'} \otimes_{\mathcal{D}_{Y'}} \mathcal{D}_Y/(\mathcal{I} + z_1 \mathcal{D}_Y + \dots + z_{m'} \mathcal{D}_Y)$.

- 1: **RStd** = S .
- 2: Call **find_macaulay_and_pfaffian**($\{f_1, \dots, f_\mu\}$, **RStd**, m').
- 3: **return** The Pfaffian matrices P_i , $i = m' + 1, \dots, m$.

Here **find_macaulay_and_pfaffian**($\{f_1, \dots, f_n\}$, **RStd**, m') is Algorithm 1 of [1] with the following modification in steps 2 and 6:

$$M_D := \left[M_{\text{Ext}} \mid M_{\text{RStd}} \right] \Big|_{z_1=\dots=z_{m'}=0}. \quad (4.16)$$

The Macaulay matrix M_D is constructed from the coefficients of certain normally ordered differential operators. These operators come from acting on the generators of $\mathcal{I}$ with a set of monomials $\text{Mons} := \{\prod_{d_1+\dots+d_m \leq D, d_i \in \mathbb{Z}_0} \partial_1^{d_1} \dots \partial_m^{d_m}\}$, where the integer D is called the degree of the Macaulay matrix. The columns of M_D are thus indexed by **Mons**. Setting $\text{Ext} := \text{Mons} \setminus \text{RStd}$, we obtain matrix blocks M_{RStd} and M_{Ext} that are indexed by the

monomials in RStd and Ext respectively. Note that RStd can be a subset of a standard basis Std of $\mathcal{R}_m/\mathcal{R}_m\mathcal{I}$.

With these modifications in comparison to [1], let us briefly describe the Macaulay matrix method for finding Pfaffian systems. First, define the matrices C_{Ext} and C_{RStd} , with entries consisting of 1's and 0's, by the expression

$$\partial_i \text{RStd} =: C_{\text{Ext}} \cdot \text{Ext} + C_{\text{RStd}} \cdot \text{RStd}. \quad (4.17)$$

Here RStd and Ext are regarded as column vectors. The Macaulay matrix method then instructs us to solve the linear equation

$$C_{\text{Ext}} - C \cdot M_{\text{Ext}}|_{z_1=\dots=z_{m'}=0} = 0 \quad (4.18)$$

for an unknown matrix C . The Pfaffian matrix in direction ∂_i is then given by

$$P_i(z') = C_{\text{RStd}} - C \cdot M_{\text{RStd}}|_{z_1=\dots=z_{m'}=0}. \quad (4.19)$$

See [1, section 4] for additional details.

Remark 4.12. Suppose that we know the holonomic rank r of the restriction. One strategy to find S is to use the probabilistic method of Algorithm 4.7, as explained after Lemma 4.9. A second strategy is to try all possible standard monomials until the Macaulay matrix method succeeds.

Example 4.13. We consider a left ideal generated by

$$\partial_x(\theta_x + c_1 - 1) - (\theta_x + \theta_y + a)(\theta_x + b_1) \quad (4.20)$$

$$\partial_y(\theta_y + c_2 - 1) - (\theta_x + \theta_y + a)(\theta_y + b_2), \quad (4.21)$$

where $\theta_x = x\partial_x$ and $\theta_y = y\partial_y$. This system annihilates the Appell F_2 function (see [75] for a definition). Let us try to find the restriction to $x = 0$ by applying Algorithm 4.7 with $\gamma = 2, k = 4$ and Algorithm 4.11 with $\gamma = 1$ and $r = 2$. We have implemented the algorithms in RISA/ASIR in the package `mt_mm.rr` [76]. The output RStd (a standard basis for the rational restriction) is $[1, \partial_y]$ and the Pfaffian matrix P_2 is constructed by the Macaulay matrix method as follows.

```
import("mt_mm.rr")$
Ideal = [(-x^2+x)*dx^2+(-y*x)*dx*dy+((-a-b1-1)*x+c1)*dx-b1*y*dy-b1*a,
        (-y^2+y)*dy^2+(-x*y)*dy*dx+((-a-b2-1)*y+c2)*dy-b2*x*dx-b2*a]$
Xvars = [x,y]$
//Rule gives a probabilistic determination of RStd (Std for the restriction)
Rule=[ [y,y+1/3], [a,1/2], [b1,1/3], [b2,1/5], [c1,1/7], [c2,1/11]]$
Ideal_p = base_replace(Ideal,Rule);
RStd=mt_mm.restriction_to_pt_(Ideal_p,Gamma=2,KK=4,[x,y] | p=10^8);
RStd=reverse(map(dp_ptod,RStd[0],[dx,dy]));
Id = map(dp_ptod,Ideal,poly_dvar(Xvars))$
MData = mt_mm.find_macaulay(Id,RStd,Xvars | restriction_var=[x]);
//For larger problems, use FiniteFlow instead of find_pfaffian
P2 = mt_mm.find_pfaffian(MData,Xvars,2 | use_orig=1);
```

The output P2 is

$$P_2 = \begin{bmatrix} 0 & 1 \\ \frac{-b_2 a}{y(y-1)} & \frac{-(a+b_2+1)y+c_2}{y(y-1)} \end{bmatrix}, \quad (4.22)$$

which appears in the Pfaffian system for the restriction as $\partial_y S = P_2 \cdot S$, $S = [1, \partial_y]^T$.

Programs for larger problems are posted at [77]. ■

The method presented above can also be applied to restrictions to irreducible hyper-surfaces. Suppose that Y' is the vanishing locus of an irreducible polynomial $L \in \mathbb{C}[z] = \mathbb{C}[z_1, \dots, z_m]$ and that Y' is non-singular. We then have the following theorem, which is proved in appendix D.11:

Theorem 4.14. *Let S be a set as in Lemma 4.9 and regard it as a column vector. There exists an $r \times \mu$ matrix Q_i with entries in $\mathcal{D}_Y$, an $r \times r$ matrix P_i with entries in $\mathbb{C}[z]$ and a polynomial $q_i \in \mathbb{C}[z]$ such that*

$$q_i \partial_i S = P_i \cdot S + : Q_i \cdot [f_1, \dots, f_\mu]^T : \mod L \quad (4.23)$$

holds in $\mathcal{D}_Y$. Here $\mathcal{D}_Y \ni \sum c_\alpha(z) \partial^\alpha = 0 \mod L$ means that each $c_\alpha(z)$ is divisible by L .

We call P_i/q_i the Pfaffian matrix on the codimension-1 stratum W .

Let us now give a procedure for obtaining P_i and q_i . Our column vector S will be denoted by **RStd**. In this context, **Mons** consists of the monomials in ∂ appearing in $:\partial^\alpha f_j:$ which construct the Macaulay matrix. Working modulo L , the equation (4.18) is now modified to

$$C_{\text{Ext}} - C \cdot M_{\text{Ext}} \equiv 0 \mod L, \quad (4.24)$$

where the unknown matrix C is solved for over the fraction field of the quotient ring $\mathcal{O}_{Y'} = \mathbb{C}[z]/LC[z]$. In other words, we find a C with rational function entries and an $|\mathbf{RStd}| \times |\mathbf{Ext}|$ -dimensional matrix H with *polynomial entries* such that

$$q_i(C_{\text{Ext}} - C \cdot M_{\text{Ext}}) = HL \quad (4.25)$$

holds, where q_i is a polynomial which is relatively prime to L that cancels the denominators of the elements of C . Note that the entries of M_{Ext} are polynomial in z . Then (4.25) can be solved by a syzygy computation in the polynomial ring $\mathbb{C}[z]$. For example, when $C_{\text{Ext}} = [1, 0]$, $M_{\text{Ext}} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$, $H = [H_1, H_2]$ and $C = [c_1/q_i, c_2/q_i]$, we may solve the syzygy equation

$$q_i \begin{bmatrix} 1 \\ 0 \end{bmatrix} - c_1 \begin{bmatrix} m_{11} \\ m_{12} \end{bmatrix} - c_2 \begin{bmatrix} m_{21} \\ m_{22} \end{bmatrix} - H_1 \begin{bmatrix} L \\ 0 \end{bmatrix} - H_2 \begin{bmatrix} 0 \\ L \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (4.26)$$

for the unknowns H_1 , H_2 , c_1 , c_2 and q_i . This can be solved via a Gröbner basis computation [69]. Finally, the matrix

$$\frac{\tilde{P}_i}{q_i} = C_{\text{RStd}} - C \cdot M_{\text{RStd}} \quad (4.27)$$

gives a Pfaffian matrix on the codimension-1 stratum W . Note that the entries of $\tilde{P}_i/q_i$ belong to the fraction field of the ring $\mathcal{O}_{Y'}$.

In section 5.6, we use this procedure to find the restriction to the hypersurface singularity $(x-y)^2 - 2(x+y) + 1 = 0$ of the Appell F_4 hypergeometric system (see [75] for a definition) and Horn's hypergeometric function H_i , $i = 1, \dots, 7$. As far as we are aware, the results on H_i were not known before.

5 Examples

We have explained general algorithms for restriction of Pfaffian systems in section 3, and restrictions for regular holonomic $\mathcal{D}$ -modules in section 4. Now we apply these methods to Pfaffian systems associated to Feynman integrals and GKZ-hypergeometric systems. Data and code for these examples can be found in [77].

When no confusion arises, 0-entries in matrices and vectors are denoted by $\cdot$ in this section.

5.1 From GKZ to N-box

In this first example, we compute the restriction of a GKZ system at the level of Pfaffian matrices, i.e. by the holomorphic restriction Algorithm 3.8.

5.1.1 Setup

Let $I_{\nu_1, \dots, \nu_5}$ denote the 2-loop massless N-box integral family presented in Example 2.2. We study this family as a restriction of a GKZ system:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & 1 & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 \end{bmatrix} \xrightarrow[i \neq 10]{z_i=1} \begin{array}{c} p_2 \quad p_3 \\ \diagdown \quad \diagup \\ \square \\ \diagup \quad \diagdown \\ p_1 \quad p_4 \end{array} \quad (5.1)$$

While the Lee-Pomeransky polynomial for the GKZ system has generic monomial coefficients $z_1, \dots, z_{10}$, to match the proper N-box Feynman integral family we must take the limits

$$z_1, \dots, z_9 \rightarrow 1, \quad z_{10} \rightarrow \frac{t}{s}. \quad (5.2)$$

As noted in Example 2.2, this restriction drops the rank from 9 to 3. The goal of this example is to obtain the 3×3 Pfaffian matrix $Q_{10}(z_{10})$ by the holomorphic restriction procedure.

Before employing the restrictions at the level of Pfaffians, let us first use the homogeneity property described in [1, appendix A] to rescale 6 ($= 5 + 1$, where 5 is the number of integration variables) variables to 1:

$$z_1, z_2, z_3, z_4, z_5, z_9 \rightarrow 1. \quad (5.3)$$

In the notation of [1, appendix A], this corresponds to choosing a *simplex* $\sigma = [1, 2, 3, 4, 5, 9]$. This rescaling is the maximal restriction which does not change the rank of the GKZ system.

By the method of [44], we obtain 9 standard monomials for the rescaled GKZ system:

$$\mathbf{Std} = \left[\partial_8^2, \partial_6 \partial_{10}, \partial_7 \partial_{10}, \partial_{10}^2, \partial_6, \partial_7, \partial_8, \partial_{10}, 1 \right]^T. \quad (5.4)$$

The Macaulay matrix method of [1] yields a Pfaffian system

$$\partial_i \mathbf{Std} = P_i \cdot \mathbf{Std}, \quad i = 6, 7, 8, 10. \quad (5.5)$$

We want to apply the restriction $z_6, z_7, z_8 \rightarrow 1$ onto this system.

The 3-dimensional integral basis I for the restricted system is taken to be

$$\begin{bmatrix} I_{02202} \\ I_{22020} \\ I_{11111} \end{bmatrix} = G_0 \cdot I, \quad (5.6)$$

where the mass dimension is factored out via G_0 , a diagonal matrix containing powers of s . By [1, Proposition A.1], we can translate this integral basis into a $\mathcal{D}_Y$ -module basis, finding

$$\begin{aligned} I_{02202} &\rightarrow \frac{c_1}{\epsilon(\epsilon-1)} \partial_{10}^2 \\ I_{22020} &\rightarrow \frac{c_2}{\epsilon(\epsilon-1)} \left[\epsilon(2+5\delta)(1+2\epsilon+5\epsilon\delta) + 2z_{10}(1+2\epsilon+5\epsilon\delta) \partial_{10} + z_{10}^2 \partial_{10}^2 \right] \\ I_{11111} &\rightarrow \frac{c_3}{\epsilon(\epsilon-1)} \left[(1+\epsilon+2\delta\epsilon) \partial_{10} + z_{10} \partial_{10}^2 - z_7 \partial_7 \partial_{10} + z_6 \partial_6 \partial_{10} \right], \end{aligned} \quad (5.7)$$

where we recall that ϵ and δ are analytic regulators stemming from the integral representation (2.13). The prefactors

$$c_1 = \frac{s^{-4-\epsilon-9\delta\epsilon}\Gamma(2-\epsilon)}{\Gamma(\delta\epsilon)^6\Gamma(-3\epsilon-9\delta\epsilon)\Gamma(2+\delta\epsilon)^3}, \quad c_2 = \frac{s^{-4-\epsilon-9\delta\epsilon}\Gamma(2-\epsilon)}{\Gamma(\delta\epsilon)^6\Gamma(-3\epsilon-9\delta\epsilon)\Gamma(2+\delta\epsilon)^3}, \quad c_3 = \frac{\epsilon^2(1+2\epsilon)s^{-4-\epsilon-9\delta\epsilon}\Gamma(2-\epsilon)}{\Gamma(\delta\epsilon)^4\Gamma(1-3\epsilon-9\delta\epsilon)\Gamma(1+\delta\epsilon)^5} \quad (5.8)$$

come from the Lee-Pomeransky representation together with factors of s from G_0 .

5.1.2 Normal form

We choose to do the restriction in the order¹³ z_6, z_7, z_8 . For the application of the restriction method, we must be careful in checking that the Pfaffian system is in normal form (recall section 3.2). We observe that the system is in normal form w.r.t. the variables z_6 and z_7 . However, the system is not logarithmic w.r.t. z_8 :

$$P_8|_{z_6, z_7 \rightarrow 1} = (z_8 - 1)^{-2} P_{8,-2}|_{z_6, z_7 \rightarrow 1} + (z_8 - 1)^{-1} P_{8,-1}|_{z_6, z_7 \rightarrow 1} + \dots \quad (5.9)$$

This 2nd order pole can be cured by Moser reduction, in this case a gauge transformation by

$$G = \text{Diag}[(z_8 - 1)^{-1}, 1, \dots, 1], \quad G[P_i] = G^{-1}(P_i \cdot G - \partial_i G). \quad (5.10)$$

¹³We have checked that the final result does not depend on the order of the individual limits.

We check that the transformed residue matrices $G[P_i]_{-1}$, for $i = 6, 7, 8$, all have non-resonant spectra. Hence, the Pfaffian system is now in normal form w.r.t. all the variables we seek to restrict. To simplify the notation in the following, let us

replace $P_i \rightarrow G_1[P_i]$ in this example

to denote the gauge transformed Pfaffian system.

5.1.3 Restriction

The restriction procedure implores us to build the matrix

$$M = \begin{bmatrix} B \\ R \end{bmatrix} \quad (5.11)$$

coming from (3.13). The block matrix B translates between the unrestricted, 9-dimensional basis and the restricted, 3-dimensional basis. This requires us to expand the basis (5.7) in terms of the **Std** from (5.4),¹⁴ leading to the 3×9 matrix

$$B = \frac{1}{\epsilon(\epsilon-1)} \begin{bmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & z_{10}^2 & \cdot & \cdot & 2z_{10}(1+2\epsilon+5\delta\epsilon) & \epsilon(2+5\delta)(1+2\epsilon+5\delta\epsilon) & \cdot \\ \cdot & z_6 & -z_7 & z_{10} & \cdot & \cdot & 1+\epsilon+2\delta\epsilon & \cdot & \cdot \end{bmatrix}, \quad (5.12)$$

with columns labeled by **Std**, and rows labeled by I . The block matrix R is built from the independent rows of the residue matrices:

$$R = \text{RowReduce} \left[\begin{bmatrix} P_{6,-1} \\ P_{7,-1} \\ P_{8,-1} \end{bmatrix} \right]. \quad (5.13)$$

Here we have stacked the matrices on top of each other, row reduced, and finally deleted zero-rows.

Next we insert M into (3.15):

$$\left(\partial_{10} M + M \cdot P_{10,0} \right) \cdot M^{-1} = \begin{bmatrix} Q_{10} & \star \\ 0_{6 \times 3} & \star \end{bmatrix}, \quad (5.14)$$

where $P_{10,0}$ is defined by the limit $P_{10}|_{z_6, z_7, z_8 \rightarrow 0}$, which is a well-defined because we are in normal form. Sending the analytic regular $\delta \rightarrow 0$ in Q_{10} , we finally obtain:

$$Q_{10} = \begin{bmatrix} -\frac{2(1+\epsilon)}{z_{10}} & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ -\frac{z_{10}\epsilon}{6(1+z_{10})} & \frac{\epsilon}{6z_{10}(1+z_{10})} & -\frac{z_{10}+2\epsilon}{z_{10}(1+z_{10})} \end{bmatrix}. \quad (5.15)$$

This is the 3-dimensional Pfaffian matrix for the restricted system. This result agrees with an independent calculation performed with the IBP software LITERED[78, 79].

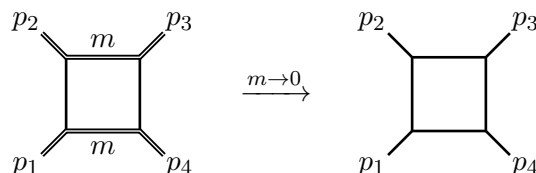
¹⁴It can happen that monomials in ∂_i appear in the restricted basis which are not part of **Std**. This situation is treated in [1, section 3.2].

5.2 Logarithmic corrections to the one-loop Bhabha box integral

Let us now turn to non-GKZ example to illustrate Algorithm 3.9.

5.2.1 Setup

We would like to study logarithmic corrections in the small-mass expansion of the following topology:



The corresponding family of Feynman integrals in $d = 4 - 2\epsilon$ dimensions is

$$I_{\nu_1 \nu_2 \nu_3 \nu_4}(m) = \int \frac{d^d k / i\pi^{d/2}}{[k^2 - m^2]^{\nu_1} [(k + p_1)^2]^{\nu_2} [(k + p_{12})^2 - m^2]^{\nu_3} [(k + p_{123})^2]^{\nu_4}}, \quad (5.16)$$

with $p_K := \sum_{k \in K} p_k$ for $K \subset \{1, 2, 3, 4\}$. The kinematic variables are given by

$$p_1^2 = p_2^2 = p_3^2 = p_4^2 = m^2, \quad s = p_{12}^2, \quad t = p_{23}^2. \quad (5.17)$$

This family of integrals contributes to Bhabha scattering at 1-loop order.

After IBP reduction, we find 5 master integrals, rescaled by a matrix G_0 to become unitless:

$$\begin{bmatrix} I_{0010}(m) \\ I_{0101}(m) \\ I_{1010}(m) \\ I_{0111}(m) \\ I_{1111}(m) \end{bmatrix} =: G_0 \cdot I, \quad G_0 = (-s)^{-\epsilon} \text{Diag} [(-s), 1, 1, (-s)^{-1}, (-s)^{-2}]. \quad (5.18)$$

The master integrals I obey a Pfaffian system

$$\partial_i I = P_i \cdot I, \quad z_1 = \frac{m^2}{-s}, \quad z_2 = \frac{-t}{-s} \quad (5.19)$$

in terms of two 5×5 Pfaffian matrices $P_i = P_i(z_1, z_2)$.

Our goal is to find an approximation to the master integrals I which holds in the limit $m \rightarrow 0$, or equivalently

$$z_1 \rightarrow 0. \quad (5.20)$$

The approximation is found by the restriction procedure laid out in section 3.4. In particular, we will find and solve *simpler* Pfaffian systems in comparison to (5.19), in the sense that they will be of smaller rank and contain one variable less.

5.2.2 Normal form

In the basis I , we find that P_1 and P_2 respectively have 2nd and 1st order poles at $z_1 = 0$. Moreover, the spectrum of the residue matrix of P_1 is resonant. We therefore need to bring the Pfaffian system to normal form before we can apply the restriction algorithm. We find that a single gauge transformation

$$I \rightarrow G_1 \cdot I, \quad G_1[P_i] := G_1^{-1}(P_i \cdot G_1 - \partial_i G_1) \quad (5.21)$$

by a diagonal matrix

$$G_1 = \text{Diag}[z_1, 1, 1, 1, 1] \quad (5.22)$$

does the trick. To simplify the notation, we

$$\text{replace } P_i \rightarrow G_1[P_i] \text{ in this example.} \quad (5.23)$$

Now the following expansions in z_1 hold true:

$$P_1 = \frac{1}{z_1} P_{1,-1}(z_2) + O(z_1^0) \quad (5.24)$$

$$P_2 = z_1^0 P_{2,0}(z_2) + O(z_1), \quad (5.25)$$

with $P_{1,-1}$ having unique eigenvalues $\{0, -\epsilon\}$, which is a non-resonant spectrum.

Jordan decomposition of the residue matrix $P_{1,-1}$ provides information on how to subdivide the approximated solution to I . More precisely, given

$$\text{JordanDecomposition}[P_{1,-1}] = \begin{bmatrix} 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & -\epsilon & 1 \\ \cdot & \cdot & \cdot & \cdot & -\epsilon \end{bmatrix}, \quad (5.26)$$

we observe two blocks: a 3×3 block associated to the eigenvalue $\lambda_1 = 0$ and a 2×2 block associated to the eigenvalue $\lambda_2 = -\epsilon$. As in (3.28), we then expect the approximated solution to take the form

$$I^{(\log)}(z_1, z_2) \stackrel{z_1 \rightarrow 0}{\equiv} z_1^{\lambda_1} I^{(\lambda_1)}(z_2) + z_1^{\lambda_2} \left[I^{(\lambda_2,0)}(z_2) + I^{(\lambda_2,1)}(z_2) \log(z_1) \right]. \quad (5.27)$$

The vector $I^{(\lambda_1)}$ will be found from the solution to a rank-3 Pfaffian system, and the vectors $I^{(\lambda_2,0)}, I^{(\lambda_2,1)}$ from the solution to a rank-2 Pfaffian system. The logarithm in z_1 appears due to the "1" in the superdiagonal of the λ_2 -eigenvalue block.

5.2.3 Restriction w.r.t. eigenvalue $\lambda_1 = 0$

Here we describe how to find the 5-dimensional vector $I^{(\lambda_1)}$ by solving a 3×3 subsystem in the variable z_2 only. Why is this subsystem of rank 3? Recalling (3.18), this question is

answered by the following constraints which hold in the limit $z_1 \rightarrow 0$:

$$\text{RowReduce}[P_{2,-1}] \cdot I^{(\lambda_1)} := R^{(\lambda_1)} \cdot I^{(\lambda_1)} \quad (5.28)$$

$$= \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \frac{z_2 \epsilon}{1-2\epsilon} & \cdot \end{bmatrix} \cdot I^{(\lambda_1)} \quad (5.29)$$

$$= 0. \quad (5.30)$$

We may regard $I^{(\lambda_1)}$ as the piece of the full solution vector I which is holomorphic in the massless limit, wherefore we expect every entry of $I^{(\lambda_1)}$ to be proportional to $I_i(m=0)$, for some $i = 1, \dots, 5$, with the massless limit being taken at the *integrand* level.¹⁵ The first row of (5.29) thus states that the massless tadpole $I_{0010}(0)$ vanishes, and the 2nd row yields an IBP relation between the massless t -channel bubble $I_{0101}(0)$ and the massless triangle $I_{0111}(0)$. These two constraints drop the rank from 5 to 3. Summing up, $I^{(\lambda_1)}$ is of the form

$$I^{(\lambda_1)} = G_0 \cdot \begin{bmatrix} \cdot \\ I_{0101}(0) \\ I_{1010}(0) \\ \frac{2\epsilon-1}{z_2 \epsilon} I_{0101}(0) \\ I_{1111}(0) \end{bmatrix}. \quad (5.31)$$

Next we find a 3-dimensional basis for the vector space associated to the eigenvalue $\lambda_1 = 0$. One possible basis is

$$\text{NullSpace}[P_{1,-1}] = \begin{bmatrix} \cdot & \frac{z_2 \epsilon}{2\epsilon-1} & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}, \quad (5.32)$$

where the rows of this matrix, viewed as vectors, span the nullspace of $P_{1,-1}$. The 1st row suggests a linear combination between $I_{0101}(0)$ and $I_{0111}(0)$ as a basis element. We could use this as it is, but by virtue of (5.29) it will be simpler to rewrite $I_{0111}(0)$ in terms of $I_{0101}(0)$ up to a non-zero prefactor, which we are free to omit in our choice of basis. The 2nd and 3rd rows of (5.32) respectively instruct us to pick the s -channel bubble $I_{1010}(0)$ and the box $I_{1111}(0)$ as the last two basis elements. Summing up, we have the following basis for the restricted Pfaffian system associated to the eigenvalue λ_1 :

$$J^{(\lambda_1)} := (-s)^\epsilon \begin{bmatrix} I_{0101}(0) \\ I_{1010}(0) \\ (-s)^2 I_{1111}(0) \end{bmatrix}, \quad (5.33)$$

with prefactors of s stemming from G_0 in (5.18). The 3×5 matrix relating the bases $J^{(\lambda_1)}$ and $I^{(\lambda_1)}$ is thus

$$B^{(\lambda_1)} = \begin{bmatrix} \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}, \quad J^{(\lambda_1)} = B^{(\lambda_1)} \cdot I^{(\lambda_1)}. \quad (5.34)$$

¹⁵This corresponds to the so-called “hard region” in the expansion by regions method [16].

Using (3.13), we proceed to build the invertible matrix

$$M^{(\lambda_1)} = \begin{bmatrix} B^{(\lambda_1)} \\ R^{(\lambda_1)} \end{bmatrix}, \quad (5.35)$$

which has the property

$$\begin{bmatrix} J^{(\lambda_1)} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = M^{(\lambda_1)} \cdot I^{(\lambda_1)}. \quad (5.36)$$

The 3×3 Pfaffian matrix $Q_2^{(\lambda_1)}$ associated to the vector $J^{(\lambda_1)}$ is now found via (3.15):

$$\left(\partial_2 M^{(\lambda_1)} + M^{(\lambda_1)} \cdot P_{2,0} \right) \cdot (M^{(\lambda_1)})^{-1} = \begin{bmatrix} Q_2^{(\lambda_1)} & \star \\ 0_{2 \times 3} & \star \end{bmatrix}, \quad (5.37)$$

with $P_{2,0}$ defined in (5.25). Explicitly, the Pfaffian system restricted to $z_1 = 0$ w.r.t. the eigenvalue $\lambda_1 = 0$ is

$$\partial_2 J^{(\lambda_1)} = Q_2^{(\lambda_1)} \cdot J^{(\lambda_1)}, \quad Q_2^{(\lambda_1)} = \begin{bmatrix} -\frac{\epsilon}{z_2} & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \frac{2(1-2\epsilon)}{z_2^2(z_2+1)} & \frac{2(2\epsilon-1)}{z_2(z_2+1)} & \frac{-z_2-\epsilon-1}{z_2(z_2+1)} \end{bmatrix}. \quad (5.38)$$

This agrees with the system one would find by performing IBP reduction directly on the massless integrals $J^{(\lambda_1)}$, as can be checked with LITERED. It is standard to solve such a system by passing to a canonical basis [80] (see [81] for this specific example).

5.2.4 Restriction w.r.t. eigenvalue $\lambda_2 = -\epsilon$

Next we determine the 5-dimensional vectors $I^{(\lambda_2,0)}, I^{(\lambda_2,1)}$ by solving a rank-2 Pfaffian system. The rank is 2 because of (3.31), which imposes the following constraint on the solution vector $I^{(\lambda_2,0)}$:

$$\text{RowReduce} \left[(P_{1,-1} - \lambda_2)^2 \right] \cdot I^{(\lambda_2,0)} := R^{(\lambda_2)} \cdot I^{(\lambda_2,0)} \quad (5.39)$$

$$= \begin{bmatrix} 1 & \cdot & \cdot & \frac{z_2\epsilon}{1-\epsilon} & \frac{z_2\epsilon}{2(1-\epsilon)} \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \end{bmatrix} \cdot I^{(\lambda_2,0)} \quad (5.40)$$

$$= 0. \quad (5.41)$$

Since there are 3 relations imposed on the entries of $I^{(\lambda_2,0)}$, the rank drops from 5 to 2. Let us further note that, according to (3.30), we obtain $I^{(\lambda_2,1)}$ for free once we know $I^{(\lambda_2,0)}$:

$$(P_{1,-1} - \lambda_2) \cdot I^{(\lambda_2,0)} = I^{(\lambda_2,1)}. \quad (5.42)$$

Let

$$J^{(\lambda_2)} = \begin{bmatrix} J^{(\lambda_2)} \\ J_1^{(\lambda_2)} \\ J_2^{(\lambda_2)} \end{bmatrix} \quad (5.43)$$

denote the basis of the Pfaffian system associated to the eigenvalue λ_2 . Unlike the zero eigenvalue case of section 5.2.3, here we no longer have a natural choice for the basis matrix $B^{(\lambda_2)}$ that relates $J^{(\lambda_2)}$ with $I^{(\lambda_2)}$. Luckily, when building the transformation matrix $M^{(\lambda_2)}$ from (3.13), we only need to ensure that our choice of $B^{(\lambda_2)}$ together with the $R^{(\lambda_2)}$ of (5.41) produce a full rank square matrix, so let us take¹⁶

$$B^{(\lambda_2)} := \begin{bmatrix} \cdot & \cdot & \cdot & \frac{z_2 \epsilon}{\epsilon-1} & \frac{z_2 \epsilon}{2(\epsilon-1)} \\ \cdot & \cdot & \cdot & 1 & \cdot \end{bmatrix}. \quad (5.44)$$

Using (3.13) to write

$$M^{(\lambda_2)} = \begin{bmatrix} B^{(\lambda_2)} \\ R^{(\lambda_2)} \end{bmatrix} \Rightarrow I^{(\lambda_2,0)} = (M^{(\lambda_2)})^{-1} \cdot \begin{bmatrix} J^{(\lambda_2)} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} J_1^{(\lambda_2)} \\ \cdot \\ \cdot \\ J_2^{(\lambda_2)} \\ \frac{2(\epsilon-1)}{z_2 \epsilon} J_2^{(\lambda_2)} - 2J_2^{(\lambda_2)} \end{bmatrix}, \quad (5.45)$$

we then insert $M^{(\lambda_2)}$ into (3.15) to obtain the 2×2 Pfaffian matrix associated to $J^{(\lambda_2)}$:

$$\left(\partial_2 M^{(\lambda_2)} + M^{(\lambda_2)} \cdot P_{2,0} \right) \cdot (M^{(\lambda_2)})^{-1} = \begin{bmatrix} Q_2^{(\lambda_2)} & \star \\ \mathbb{0}_{3 \times 2} & \star \end{bmatrix}. \quad (5.46)$$

Explicitly, the Pfaffian system restricted to $z_1 = 0$ w.r.t. the eigenvalue λ_2 is

$$\partial_2 J^{(\lambda_2)} = Q_2^{(\lambda_2)} \cdot J^{(\lambda_2)}, \quad Q_2^{(\lambda_2)} = \begin{bmatrix} \cdot & \cdot \\ \frac{\epsilon-1}{z_2^2} & \frac{-1}{z_2} \end{bmatrix}. \quad (5.47)$$

The general solution is

$$J_1^{(\lambda_2)} = c_1(\epsilon), \quad J_2^{(\lambda_2)} = \frac{c_1(\epsilon)(1-\epsilon) \log(z_2)}{z_2} + \frac{c_2(\epsilon)}{z_2}. \quad (5.48)$$

The constants $c_1(\epsilon), c_2(\epsilon)$ will be fixed by matching with the full solution vector I .

5.2.5 Full result

Let us combine the solution vectors from the previous two sections. Given $J^{(\lambda_1)}$ and $J^{(\lambda_2)}$, the vectors $I^{(\lambda, \bullet)}$ from (5.27) are determined by

$$I^{(\lambda_1)} = (M^{(\lambda_1)})^{-1} \cdot \begin{bmatrix} J^{(\lambda_1)} \\ \cdot \\ \cdot \end{bmatrix} \quad (5.49)$$

$$I^{(\lambda_2,0)} = (M^{(\lambda_2)})^{-1} \cdot \begin{bmatrix} J^{(\lambda_2)} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \quad (5.50)$$

$$I^{(\lambda_2,1)} = (P_{1,-1} - \lambda_2) \cdot I^{(\lambda_2,0)}. \quad (5.51)$$

¹⁶See also the procedure described in appendix C.2.2 that gives an alternative choice of the basis matrix.

Reinstating prefactors and gauge transformations, the expression (5.27) becomes

$$\begin{bmatrix} I_{0010}(m) \\ I_{0101}(m) \\ I_{1010}(m) \\ I_{0111}(m) \\ I_{1111}(m) \end{bmatrix} \xrightarrow{m \rightarrow 0} z_1^{\lambda_1} G_0 \cdot G_1 \cdot \begin{bmatrix} 0 \\ I_{0101}(0) \\ I_{1010}(0) \\ \frac{1-2\epsilon}{t\epsilon} I_{0101}(0) \\ I_{1111}(0) \end{bmatrix} + z_1^{\lambda_2} G_0 \cdot G_1 \cdot \begin{bmatrix} \frac{c_1}{\epsilon-1} \\ 0 \\ 0 \\ \frac{c_1 \log(z_2/z_1) + c_2}{z_2} \\ \frac{-2}{\epsilon} \frac{c_1 [\epsilon \log(z_2/z_1) - 1] + \epsilon c_2}{z_2} \end{bmatrix}, \quad (5.52)$$

where $c_1 \rightarrow \frac{c_1}{\epsilon-1}$ has been rescaled to simplify the expression. c_1 can be found by matching against the exact result for the tadpole:

$$\begin{aligned} \text{l.h.s.: } I_{0010}(m) &= -\Gamma(\epsilon-1)(m^2)^{1-\epsilon} \\ \text{r.h.s.: } (-s)^{1-\epsilon} \cdot z_1^{1-\epsilon} \cdot \frac{c_1}{\epsilon-1} &= (-s)^{1-\epsilon} \cdot \left(\frac{m^2}{-s}\right)^{1-\epsilon} \cdot \frac{c_1}{\epsilon-1} \\ \Rightarrow c_1 &= (1-\epsilon)\Gamma(\epsilon-1). \end{aligned} \quad (5.53)$$

The constant c_2 can be found order by order in ϵ by matching with the l.h.s. at a particular phase space point. Expanding¹⁷ $c_2 = \sum_{i=-2}^2 c_{2,i} \epsilon^i$ and matching with $I_{1111}(m)$ at $(s, t, m^2) = (-1, -9, 10^{-3})$ using AMFlow [4], we obtain

$$(c_{2,-2}, c_{2,-1}, c_{2,0}, c_{2,1}, c_{2,2}) = (1, -0.588997, -7.12155, -4.69031, -20.4068). \quad (5.54)$$

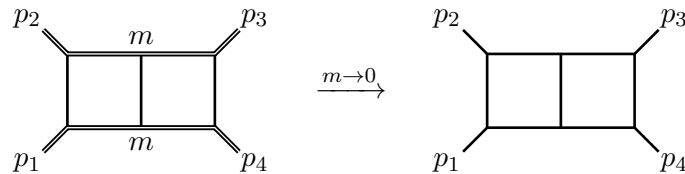
After inserting the solutions for c_1, c_2 into the l.h.s. of (5.53), we find good agreement at other phase space points. For instance, at $(s, t, m^2) = (-1, -8, 10^{-3})$ we find that, up to $O(\epsilon^3)$, the l.h.s. agrees with $I_{0111}(m)$ within 0.01% accuracy, and with $I_{1111}(m)$ within 0.001% accuracy.

5.3 From massive Bhabha to the massless double-box

Here we apply the Pfaffian-level restriction protocol to a more complicated two-loop diagram to illustrate that the method is computationally cheap.

5.3.1 Setup

We study the massless limit of a planar 2-loop integral family contributing to Bhabha scattering in $d = 4 - 2\epsilon$ dimensions:



¹⁷Note that leading ϵ -poles for $I_{1111}(m)$ and $I_{1111}(0)$ are ϵ^{-2} and ϵ^{-1} respectively. We therefore start the Laurent series for c_2 at $O(\epsilon^{-2})$ to cancel the ϵ^{-2} poles on the r.h.s.

5.3.2 Normal form

The massive Pfaffian system (5.62) for the basis (5.59) is not in normal form. The first issue is that the Pfaffians P_i have higher order singularities in the restriction limit:

$$P_1 = z_1^{-3} P_{1,-3} + z_1^{-2} P_{1,-2} + z_1^{-1} P_{1,-1} + \dots \quad (5.63)$$

$$P_2 = z_1^{-2} P_{2,-2} + z_1^{-1} P_{2,-1} + P_{1,0} + \dots \quad (5.64)$$

To cure it, we perform Moser reduction via two gauge transformations by

$$G_1 = \text{Diag}[\underbrace{z_1, \dots, z_1}_{8 \text{ times}}, 1, \dots, 1], \quad G_2 = \text{Diag}[\underbrace{z_1, \dots, z_1}_{12 \text{ times}}, 1, \dots, 1] \quad (5.65)$$

whereby P_1 becomes logarithmically singular at $z_1 = 0$,

$$G_{21}[P_1] = z_1^{-1} G_{21}[P_1]_{-1} + \dots \quad (5.66)$$

Here we introduced the short-hand notation

$$G_{i\dots j} := G_i \cdot \dots \cdot G_j. \quad (5.67)$$

The transformations (5.66) still leave $G_{21}[P_2]$ with an unwanted singularity at $z_1 = 0$:

$$G_{21}[P_2] = z_1^{-1} G_{21}[P_2]_{-1} + G_{21}[P_2]_0 + \dots \quad (5.68)$$

This issue arises because the residue matrix $G_{21}[P_1]_{-1}$ is resonant, meaning that its spectrum contains eigenvalues with integral difference:

$$\begin{aligned} \text{Spec}[G_{21}[P_2]_{-1}] &= \{-2, -2, -2, -1, -1, 0, 0, 0\} \\ &\cup \{-2 - \epsilon, -2 - \epsilon, -1 - \epsilon, -1 - \epsilon, -\epsilon, -\epsilon, -\epsilon\} \\ &\cup \{-3 - 2\epsilon, -2 - 2\epsilon, -2 - 2\epsilon, -1 - 2\epsilon, -1 - 2\epsilon, -1 - 2\epsilon, -2\epsilon\} \\ &\cup \{-2 - 4\epsilon\}. \end{aligned} \quad (5.69)$$

These resonances are cured via an additional gauge transformation by a matrix G_3 , resulting in the non-resonant spectrum that reads

$$\text{Spec}[G_{321}[P_1]_{-1}] = \{0, \dots, -\epsilon, \dots, -2\epsilon, \dots, -2 - 4\epsilon, \dots\}, \quad (5.70)$$

where we only wrote unique eigenvalues. We review the algorithmic procedure to construct such a G_3 in appendix B.¹⁸

With this final composition of gauge transformations applied to our Pfaffian system, we now have that

$$G_{321}[P_2] = G_{321}[P_2]_0 + \dots, \quad (5.71)$$

which means that the system is in normal form.

¹⁸In our basic computer implementation, we perform sequential Jordan decompositions of the residue matrix, which can be computed in a matter of seconds using MATHEMATICA on a laptop.

5.3.3 Restriction

Given our choice of restricted basis, the matrix R from (3.13) is simply

$$R = \text{RowReduce} [G_{321}[P_1]_{-1}] = \begin{bmatrix} 0_{15 \times 8} & \mathbb{1}_{15 \times 15} \end{bmatrix}. \quad (5.72)$$

Had we chosen a restricted basis which was not a subset of $I(m=0)$ (cf. (5.59)), say a Laporta basis where all non-zero propagator powers equal to ± 1 , then R would generally depend on z_2 and ϵ . Given the matrix B from (5.61), we are now ready to build the restriction transformation matrix (3.13):

$$M = \begin{bmatrix} B \cdot G_{321} \\ R \end{bmatrix}. \quad (5.73)$$

The upper matrix block encodes information about the restricted basis, and the lower block contains linear relations (IBPs) which hold in the massless limit.

We finally compute the gauge transformation (3.15)

$$\left(\partial_2 M + M \cdot G_{321}[P_2]_0 \right) \cdot M^{-1} = \begin{bmatrix} Q_2 & \star \\ 0_{15 \times 8} & \star \end{bmatrix}, \quad (5.74)$$

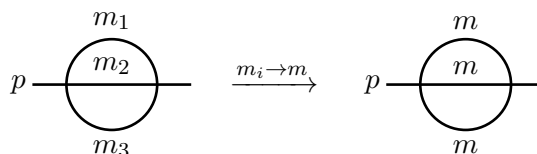
and obtain the 8-dimensional restricted Pfaffian matrix Q_2 associated to the massless double-box topology. In other words, Q_2 satisfies $\partial_2 J = Q_2 \cdot J$, with J given in (5.60). We have verified the form of Q_2 by an independent calculation using the IBP software KIRA [84].

5.4 Unequal to equal-mass sunrise

This example studies symmetry relations of Feynman integrals from the point of view of Pfaffian-level restriction.

5.4.1 Setup

We examine the equal-mass limit of the 2-loop 3-mass sunrise integral family:



The family of momentum space Feynman integrals is

$$I_{\nu_1 \nu_2 \nu_3}(m_1, m_2, m_3) = \int \frac{d^d k_1 d^d k_2}{[k_1^2 - m_1^2]^{\nu_1} [k_2^2 - m_2^2]^{\nu_2} [(k_1 + k_2 + p)^2 - m_3^2]^{\nu_3}}. \quad (5.75)$$

We choose the following 3 dimensionless kinematic variables:

$$z_1 = \frac{m_1^2}{s}, \quad z_2 = \frac{m_2^2}{s}, \quad z_3 = \frac{m_3^2}{s}, \quad \text{where } s = p^2. \quad (5.76)$$

We are interested in the equal-mass limit, that is

$$m_i \rightarrow m, \quad z_i \rightarrow z := \frac{m^2}{s}, \quad \text{for } i = 1, 2, 3. \quad (5.77)$$

The details of the two Pfaffian systems before and after restriction are as follows:

System	Matrices	Rank	Variables
Unequal mass	$\{P_1, P_2, P_3\}$	7	$\{z_1, z_2, z_3\}$
Equal mass	$\{Q_z\}$	3	$\{z\}$

(5.78)

For the unequal-mass integral family we choose the following basis of master integrals:

$$G_0 \cdot I(m_1, m_2, m_3) = [I_{011}, I_{101}, I_{110}, I_{111}, I_{211}, I_{121}, I_{112}]^T, \quad (5.79)$$

where, as before, G_0 is a diagonal matrix containing factors of s which render the integrals unitless. In the equal-mass case, our target basis reads

$$J(m) = [I_{011}(m, m, m), I_{111}(m, m, m), I_{211}(m, m, m)]^T, \quad (5.80)$$

so we introduce the following binary matrix

$$B = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{bmatrix} \quad (5.81)$$

relating the bases by $J(m) = B \cdot I(m, m, m)$.

5.4.2 Restriction

The unequal-mass Pfaffian system turns out to have the following denominators:

$$\begin{aligned} \text{Denominator}[P_i] = \{z_1, z_2, z_3, \\ 1 - 4z_1 + 6z_1^2 - 4z_1^3 + z_1^4 - 4z_2 + 4z_1z_2 + 4z_1^2z_2 - 4z_1^3z_2 + \\ 6z_2^2 + 4z_1z_2^2 + 6z_1^2z_2^2 - 4z_2^3 - 4z_1z_2^3 + z_2^4 - 4z_3 + 4z_1z_3 + 4z_1^2z_3 - \\ 4z_1^3z_3 + 4z_2z_3 - 40z_1z_2z_3 + 4z_1^2z_2z_3 + 4z_2^2z_3 + 4z_1z_2^2z_3 - 4z_2^3z_3 + \\ 6z_3^2 + 4z_1z_3^2 + 6z_1^2z_3^2 + 4z_2z_3^2 + 4z_1z_2z_3^2 + 6z_2^2z_3^2 - 4z_3^3 - 4z_1z_3^3 - 4z_2z_3^3 + z_3^4\}. \end{aligned} \quad (5.82)$$

The last polynomial reduces to $(z-1)^3(9z-1)$ under the identification (5.77), so none of the denominators vanish in the equal-mass limit! This example is hence not a restriction to a singular locus. In particular, we have no residue matrix to utilize, so we need to slightly modify the restriction procedure to derive the equal-mass rank-3 Pfaffian system.

To this end, we begin by observing that some of the MIs (5.79) become identical in the equal-mass limit, e.g. $I_{011}(m, m, m) = I_{101}(m, m, m)$.¹⁹ Let us collect the four such

¹⁹In the momentum space representation (5.75), this can be shown by shifts $k_i \rightarrow k_i + p$ in the loop momenta. These shifts have no effect on the integrated result, because the integration contour is over all of Minkowski space.

symmetry relations arising in the equal-mass limit into the matrix

$$R = \begin{bmatrix} 1 & -1 & . & . & . & . & . \\ 1 & . & -1 & . & . & . & . \\ . & . & . & . & 1 & -1 & . \\ . & . & . & . & 1 & . & -1 \end{bmatrix}. \quad (5.83)$$

In analogy with the residue matrix relations from (3.18), we have

$$R \cdot I(m, m, m) = 0. \quad (5.84)$$

Now the holomorphic restriction protocol can proceed as before. Namely, taking the equal-mass limit²⁰ of the initial Pfaffian system, we compute

$$(\partial_z M + M \cdot P_{z,0}) \cdot M^{-1} = \begin{bmatrix} Q_z & \star \\ 0_{4 \times 3} & \star \end{bmatrix}, \quad \text{with} \quad M = \begin{bmatrix} B \\ R \end{bmatrix} \quad (5.85)$$

and B given in (5.81). This gives

$$Q_z = \begin{bmatrix} \frac{-2\epsilon}{z} & . & . \\ . & . & 3 \\ \frac{2\epsilon^2}{(z-1)z^2(9z-1)} - \frac{(1+2\epsilon)(1+3\epsilon)(3z-1)}{(z-1)z(9z-1)} & - \frac{1+\epsilon-20z-30\epsilon z+27z^2+45\epsilon z^2}{(z-1)z(9z-1)} \end{bmatrix}, \quad (5.86)$$

in agreement with an independent LITERED calculation.

5.4.3 Deriving symmetry relations

In this previous calculation, we merely *guessed* the symmetry relation matrix (5.83). In the following we show how it can be derived.

Let us recall the secondary-like equation (2.24), which in the present context reads

$$\partial_z \Phi + P_{z,0}^T \cdot \Phi - \Phi \cdot P_{z,0}^T = 0, \quad (5.87)$$

for some unknown matrix Φ . The INTEGRABLECONNECTIONS [85] library in MAPLE can find rational solutions to such PDEs via the function `RationalSolutions` (see also the function `EigenRing`). We find five non-trivial solutions

$$\{\Phi_1, \dots, \Phi_5\} = \left\{ \begin{bmatrix} . & -1 & 1 & . & . & . & . \\ . & 1 & -1 & . & . & . & . \\ . & . & . & . & 1 & -1 & . \\ . & . & . & . & . & 1 & -1 \end{bmatrix}, \begin{bmatrix} . & 1 & . & . & . & . & . \\ 1 & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . \\ . & . & . & 1 & . & . & . \\ . & . & . & . & 1 & . & . \\ . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & 1 \end{bmatrix}, \begin{bmatrix} . & 1 & . & . & . & . & . \\ . & 1 & . & . & . & . & . \\ 1 & -1 & 1 & . & . & . & . \\ . & . & . & 1 & . & . & . \\ . & . & . & . & 1 & . & . \\ . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & 1 \end{bmatrix}, \begin{bmatrix} . & 1 & . & . & . & . & . \\ . & 1 & . & . & . & . & . \\ . & . & 1 & . & . & . & . \\ . & . & . & 1 & . & . & . \\ . & . & . & . & 1 & . & . \\ . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . & . \\ . & . & . & 1 & -1 & . & . \\ . & . & . & . & . & 1 & -1 \end{bmatrix} \right\}. \quad (5.88)$$

²⁰In practice, we take $P_{z,0} = (P_1 + P_2 + P_3)|_{z_i \rightarrow z}$, which is obtained from the unequal mass system (5.78) by shifting the variables $\{z_1, z_2, z_3\} \mapsto \{z, z + x_1, z + x_2\}$ and then taking the smooth limit of $x_1, x_2 \rightarrow 0$.

We are interested in eigenspaces of these matrices, so let us first record their eigenvalues:

$$\begin{aligned}
\text{Spec}[\Phi_1] &= \{-1, -1, 0, 0, 0, 0\}, \\
\text{Spec}[\Phi_2] &= \{-1, -1, 1, 1, 1, 1\}, \\
\text{Spec}[\Phi_3] &= \{0, 0, 1, 1, 1, 1\}, \\
\text{Spec}[\Phi_4] &= \{e^{2/3\pi\sqrt{-1}}, e^{2/3\pi\sqrt{-1}}, e^{4/3\pi\sqrt{-1}}, e^{4/3\pi\sqrt{-1}}, 1, 1, 1\}, \\
\text{Spec}[\Phi_5] &= \{0, 0, 0, 0, 0, 1, 1\}.
\end{aligned} \tag{5.89}$$

Next we collect the eigenvectors of Φ_i corresponding to an eigenvalue λ into the rows of a matrix $V_i^{(\lambda)}$, meaning that it satisfies

$$[\Phi_i - \lambda \mathbb{1}] \cdot \left(V_i^{(\lambda)}\right)^T = 0, \quad i = 1, \dots, 5. \tag{5.90}$$

Now comes our main observation: the rows of the symmetry relations matrix (5.83) span the same linear space as the rows of (some of the) $V_i^{(\lambda)}$ matrices! We have

$$\text{RowReduce}[R] = \text{RowReduce}\left[\begin{array}{c} V_1^{(-1)} \\ V_2^{(-1)} \\ V_3^{(0)} \\ V_5^{(1)} \end{array}\right], \tag{5.91}$$

where, as before, the **RowReduce** operation includes deletion of zero-rows. The correspondence (5.91) is the reason why our choice of the symmetry relations matrix (5.83) actually gave the correct restricted Pfaffian (5.86): the rows of this R span an invariant subspace $\mathcal{N}$ of the $\mathcal{R} = \mathbb{C}(z)\langle\partial_z\rangle$ -module $\mathcal{M}$ associated to the Pfaffian matrix $P_{z,0}$ (see the discussion around Theorem 2.6). Equation (5.86) is thus the Pfaffian matrix for the quotient $\mathcal{R}$ -module $\mathcal{M}/\mathcal{N}$ with respect to a basis B , and can be determined by the restriction method as shown above.

5.5 Restriction of GKZ systems by the Macaulay matrix method

Next we present examples of Feynman integrals arising as restrictions of GKZ systems. We employ the Macaulay matrix based algorithms from section 4. For simplicity, in all examples we omit Γ -prefactors coming from the Lee-Pomeransky representation. To obtain Pfaffian matrices for the proper Feynman integrals, one would need to include these prefactors and then send $\delta \rightarrow 0$.

Given n integration variables in the Euler integral (2.1), for every example below we fix the GKZ β -parameters to be the $(n+1)$ -dimensional vector $\beta = [\epsilon, -\epsilon\delta, \dots, -\epsilon\delta]^T$. This follows the convention of [1].

5.5.1 One-loop triangle

We first consider the one-loop triangle diagram with one internal mass m . After rescaling the integration variables by m^2 , the Lee-Pomeransky polynomial associated to this diagram

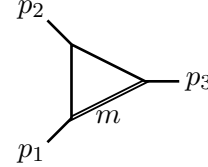
takes the form

$$g(x; z) = z_1 x_1 + z_2 x_2 + z_3 x_3 + z_4 x_1^2 + z_5 x_1 x_2 + z_6 x_1 x_3 \quad (5.92)$$

$$z_1 = \dots = z_5 = 1, \quad z_6 = \frac{(p_1 + p_2)^2}{m^2} \quad (5.93)$$

where p_1, p_2 denote the incoming momenta of two massless particles.

We are interested in obtaining the Pfaffian system associated to the proper Feynman integral as a restriction of the GKZ system:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \cdot & \cdot & 2 & 1 & 1 \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 \end{bmatrix} \xrightarrow[i \neq 6]{z_i=1} \text{Diagram} \quad (5.94)$$


With all variables $z_1, \dots, z_6$ being generic, the GKZ system has rank 3. However, an Euler characteristic computation (see Theorem 2.1) gives $\chi = 2$ for the case $z_1 = \dots = z_5 = 1$ and generic z_6 . We thus seek to drop the rank from 3 to 2 via a $\mathcal{D}_Y$ -module restriction.

Following [1, appendix A], we are free to immediately rescale $z_1 = \dots = z_4 = 1$ without changing the rank of GKZ system. The remaining variables are thus z_5 and z_6 , and we seek the restriction $z_5 = 1$. We first find a standard basis when $z_5 = 1$ by applying Algorithm 4.7 with the randomly chosen value $z_6 = 1/17$ and the algorithm parameters $\gamma = 2, k = 3$. The output is a basis consisting of 2 elements:

$$\text{RStd} = [\partial_6, 1]^T, \quad (5.95)$$

which agrees with the rank being 2. We can obtain a Pfaffian matrix in the variable z_6 for this restricted basis by Algorithm 4.11. The Macaulay matrix succeeds at degree $D = 1$, and we get

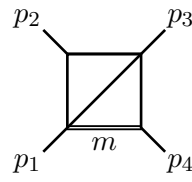
$$P_6 = \begin{bmatrix} \frac{((4\epsilon\delta + \epsilon + 1)z_6 - 4\epsilon\delta - 2\epsilon - 1)}{z_6(1 - z_6)} & \frac{\epsilon^2\delta(3\delta + 1)}{z_6(1 - z_6)} \\ 1 & 0 \end{bmatrix}.$$

5.5.2 Two-loop N-box

The two-loop N-box with one internal mass has the following Lee-Pomeransky polynomial:

$$\begin{aligned} g(z; x) = & z_1 x_1 x_2 + z_2 x_1 x_4 + z_3 x_1 x_5 + z_4 x_2 x_3 + z_5 x_2 x_4 + z_6 x_2 x_5 + \\ & z_7 x_3 x_4 + z_8 x_3 x_5 + z_9 x_1 x_2 x_4 + z_{10} x_1 x_4^2 + z_{11} x_1 x_4 x_5 + z_{12} x_2 x_3 x_4 + \\ & z_{13} x_2 x_3 x_5 + z_{14} x_2 x_4^2 + z_{15} x_2 x_4 x_5 + z_{16} x_3 x_4^2 + z_{17} x_3 x_4 x_5. \end{aligned} \quad (5.96)$$

We study it as a restriction of a GKZ system:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & \cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & 2 & 1 & 1 & \cdot & 2 & 1 & 2 \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot \end{bmatrix} \xrightarrow[i \neq 9, 13]{z_i \rightarrow 1} \text{Diagram} \quad (5.97)$$


The 17-variable GKZ system has rank 33 for generic ϵ, δ . To match with the proper Feynman integral, we seek the restriction

$$z_i = 1, i \neq 9, 13 \quad (5.98)$$

with $z_9 = 1 + \frac{s}{m^2}$ and $z_{13} = \frac{t}{m^2}$ being the remaining kinematic variables. By computing the Euler characteristic with restricted variables, we expect a rank drop from 33 to 7.

To begin with, we choose the simplex $\sigma = [1, 2, 3, 4, 5, 10]$. The remaining variables to restrict are thus $z_i = 1$ for $i \in \{6, 7, 8, 11, 12, 14, 15, 16, 17\}$, for which we apply Algorithm 4.7. Taking parameters $\gamma = 3, k = 4$ and randomly chosen values $z_9 = 1/17, z_{13} = 1/19$, we are able to guess the basis

$$\mathbf{RStd} = [\partial_8, \partial_{15}, \partial_{16}, \partial_{17}, \partial_9, \partial_{13}, 1]^T \quad (5.99)$$

by running the algorithm²¹ over the finite field of characteristic $p = 100000007$. Note that the basis is 7-dimensional, as expected.

Next we use Algorithm 4.11 to obtain a Macaulay matrix, which succeeds²² at degree $D = 1$. Since the Macaulay matrix is valid, it means that the probabilistic basis from Algorithm 4.7 is indeed a basis. Using FINTEFLOW [55], in a matter of seconds we obtain a solution to equation (4.18) for $i = 9, 13$ via rational reconstruction over finite fields. The result is a set of 7×7 Pfaffian matrices P_9 and P_{13} . We have checked that they satisfy integrability.

5.5.3 Two-loop diagonal-box

Here we consider the massless two-loop diagonal-box diagram, whose Lee-Pomeransky polynomial takes the form

$$\begin{aligned} g(z; x) = & z_1 x_1 x_4 + z_2 x_1 x_5 + z_3 x_1 x_6 + z_4 x_2 x_4 + z_5 x_2 x_5 + z_6 x_2 x_6 + \\ & z_7 x_3 x_4 + z_8 x_3 x_5 + z_9 x_3 x_6 + z_{10} x_4 x_5 + z_{11} x_4 x_6 + z_{12} x_4 x_7 + \\ & z_{13} x_5 x_7 + z_{14} x_6 x_7 + z_{15} x_1 x_3 x_4 + z_{16} x_1 x_3 x_5 + z_{17} x_1 x_3 x_6 + \\ & z_{18} x_1 x_4 x_5 + z_{19} x_2 x_4 x_6 + z_{20} x_2 x_4 x_7 + z_{21} x_2 x_5 x_7 + z_{22} x_2 x_6 x_7. \end{aligned} \quad (5.100)$$

We consider the following restriction of the GKZ system:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & \cdot & \cdot & 1 & 1 & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \end{bmatrix} \xrightarrow[z_i=1, z_j=t/s]{i \leq 18, 19 \leq j \leq 22} \begin{array}{c} p_2 \quad p_3 \\ \diagdown \quad \diagup \\ \square \\ \diagup \quad \diagdown \\ p_1 \quad p_4 \end{array} \quad (5.101)$$

²¹It takes 7.32s on an m-PC(Intel(R) Core(TM) i7-10700K CPU @ 3.80GHz, 8G memory).

²²It takes 3.28s on the m-PC.

The GKZ system has rank 115 for generic ϵ, δ . Upon restricting to the proper Feynman integral case,

$$z_1 = \dots = z_{18} = 1, \quad z_{19} = \dots = z_{22} = z := \frac{t}{s} \quad (5.102)$$

we expect a rank drop to 7, according to the Euler characteristic.

We follow an analogous strategy as in the previous section. We guess **RStd** by making the change of coordinates $z_i = y_i + 1$ for $i = 1, \dots, 18$, $z_i = y_i + z$ for $i = 19, 20, 21$, $z_{22} = z$ and taking the restriction to $y_i = 0$ ($i = 1, \dots, 21$). For $\sigma = [1, 2, 5, 7, 9, 11, 13, 18]$, $\gamma = 3$, $k = 4$, $p = 100000007$ and the randomly chosen value $z = 1/7$, we find a 7-dimensional basis²³

$$\mathbf{RStd} = [\partial_{21}\partial_z, \partial_z^2, \partial_6, \partial_{17}, \partial_{21}, \partial_z, 1]^T, \quad (5.103)$$

and a valid corresponding Macaulay matrix at degree $D = 1$.²⁴ Note that ∂_i means $\partial/\partial y_i$. With this Macaulay matrix, the 7×7 Pfaffian P_z is solved for in less than 3 minutes on a laptop via **FINITEFLOW**.

5.5.4 Two-loop double-box

Our last Feynman integral example is the massless two-loop double-box. We start from the 26-term Lee-Pomeransky polynomial

$$\begin{aligned} g(z; x) = & z_1 x_1 x_2 + z_2 x_1 x_3 + z_3 x_1 x_6 + z_4 x_1 x_7 + z_5 x_2 x_3 + z_6 x_2 x_4 + z_7 x_2 x_5 + \\ & z_8 x_3 x_4 + z_9 x_3 x_5 + z_{10} x_3 x_6 + z_{11} x_3 x_7 + z_{12} x_4 x_6 + z_{13} x_4 x_7 + z_{14} x_5 x_6 + \\ & z_{15} x_5 x_7 + z_{16} x_1 x_2 x_5 + z_{17} x_1 x_2 x_6 + z_{18} x_1 x_3 x_5 + z_{19} x_1 x_3 x_6 + z_{20} x_1 x_5 x_6 + \\ & z_{21} x_1 x_5 x_7 + z_{22} x_2 x_3 x_5 + z_{23} x_2 x_3 x_6 + z_{24} x_2 x_4 x_6 + z_{25} x_2 x_5 x_6 + z_{26} x_3 x_4 x_7, \end{aligned} \quad (5.104)$$

and compute the restriction

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & 1 & 1 & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 & 1 & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix} \xrightarrow[i \leq 25]{z_i=1} \begin{array}{c} p_2 \quad p_3 \\ \diagdown \quad \diagup \\ \text{---} \text{---} \text{---} \text{---} \\ \diagup \quad \diagdown \\ p_1 \quad p_4 \end{array} \quad (5.105)$$

For generic ϵ and δ , the rank of the GKZ system is a whopping 238. While the GKZ system has generic z_i , the proper massless double-box Feynman integral has monomial coefficients

$$z_i = 1 \text{ for all } i = 1, \dots, 25, \quad z_{26} = -1 - s/t =: z. \quad (5.106)$$

²³It takes 40.56s on an m-PC(Intel(R) Core(TM) i7-10700K CPU @ 3.80GHz, 8G memory).

²⁴It takes 65.76s on the m-PC.

The Euler characteristic suggests a rank 12 system for the restricted variables above. However, this Feynman integral is well-known to satisfy a rank 8 Pfaffian system — we comment on this below.

We fix the simplex $\sigma = [1, 2, 3, 4, 5, 6, 7, 25]$ and then run Algorithm 4.7 with parameters $\gamma = 3$, $k = 5$, $p = 100000007$ and the randomly chosen value $z = 1/7$. The result is a 12-dimensional basis²⁵

$$\text{RStd} = [\partial_z \partial_{15}, \partial_{23} \partial_z, \partial_{24} \partial_z, \partial_z^2, \partial_{13}, \partial_{15}, \partial_{21}, \partial_{22}, \partial_{23}, \partial_{24}, \partial_z, 1]^T, \quad (5.107)$$

whose size agrees with the Euler characteristic.

In step 4 of Algorithm 4.7, we need to compute the echelon form of a matrix of size $132,145 \times 33,649$. Algorithm 4.11 succeeds at finding a Macaulay matrix²⁶ at degree $D = 1$. The size of M_{Ext} is 2926×10775 . Running FINITEFLOW²⁷ to solve the matrix equation (4.18) associated to M_{Ext} , we finally obtain a 12×12 Pfaffian matrix P_z , which can be downloaded from [77].

We suspect that symmetry relations (namely, reparametrizations of the Euler *integrand*) would bridge the gap between $\chi = 12$ and the expected rank 8 for the Pfaffian system associated to this Feynman integral. By using the secondary-like equation (2.24), it should be possible to block-diagonalize our 12-dimensional ODE dictated by P_z to reach the minimal size of 8. Indeed, similar to section 5.4, with the help of INTEGRABLECONNECTIONS [85] library we found a 4-dimensional invariant subspace of symmetry relations that reduce the rank from 12 down to 8, as expected for this family of Feynman integrals.²⁸ It is a future problem for us to give a more efficient implementation combining the secondary-like equation with the methods of [86, 87].

5.6 Restriction of Appell's F_4 and Horn's H_i functions to hypersurfaces

The Appell F_4 function has the following series representation:

$$F_4(a, b, c_1, c_2; x, y) = \sum_{i,j=0}^{\infty} \frac{(a)_{i+j} (b)_{i+j}}{(1)_i (1)_j (c_1)_i (c_2)_j} x^i y^j. \quad (5.108)$$

It appears in several contexts related to Feynman integrals [36, 88]. F_4 is annihilated by the following operators:

$$f_1 = \theta_x(\theta_x + c_1 - 1) - x(\theta_x + \theta_y + a)(\theta_x + \theta_y + b) \quad (5.109)$$

$$f_2 = \theta_y(\theta_y + c_2 - 1) - y(\theta_x + \theta_y + a)(\theta_x + \theta_y + b), \quad (5.110)$$

where the Euler operators are $\theta_x = x\partial_x$ and $\theta_y = y\partial_y$. Let us assume that the parameters a, b, c_1, c_2 take on generic values. The operators f_1 and f_2 generate a holonomic ideal $\mathcal{I}$

²⁵It takes 23,815s CPU time on a t-PC(AMD EPYC 7552 four 48-Core Processors @ 1.5GHz, 1T memory).

²⁶It takes 501s CPU time on the t-PC.

²⁷It took 89,021s CPU-hours on the t-PC. However, FINITEFLOW ran in parallel over 96 threads, so the real time was about 20 min.

²⁸It took about 70 CPU-hours and 4Gb of RAM on the t-PC for INTEGRABLECONNECTIONS to find these rational solutions.

in the $\mathcal{D}$ -module $\mathcal{D}_2$ whose coordinates are $z_1 = x$, $z_2 = y$. The left $\mathcal{D}_2$ -module $\mathcal{D}_2/\mathcal{I}$ is regular holonomic with holonomic rank 4, and the singular locus is

$$xy((x-y)^2 - 2(x+y) + 1) = 0. \quad (5.111)$$

Put $L = (x-y)^2 - 2(x+y) + 1$. The point $(x, y) = (1/4, 1/4)$ is on the singular locus, and we will construct a Pfaffian system restricted to a codimension-1 stratum $W = V(L) \setminus \{(1, 0), (0, 1)\}$ containing that point. Note that $V(L)$ and $xy = 0$ intersect at $(1, 0)$ and $(0, 1)$.

Computing the ODE on $x = y$ by restricting the Pfaffian system, we see that the set of exponents at the point $V(x-y) \cap W$ is

$$\{a, b, c_1, c_2\} \rightarrow \{0, 0, 0, 2(a+b-c_1-c_2)+5\}. \quad (5.112)$$

Moreover, the residue matrix at this point is diagonal, meaning that there is no logarithmic solution. By an analytic coordinate transformation, we can locally transform W and $V(x-y)$ into hyperplanes that intersect transversally. Moreover, it can be expressed as a logarithmic connection with the residue matrix along W being the diagonal matrix with eigenvalues (5.112). It follows from Proposition 3.3 that the holonomic rank r on W is 3 for generic parameter values. Applying the restriction algorithm from Theorem 4.2 with the shift $(x, y) \rightarrow (x + 1/4, y + 1/4)$, we obtain the standard basis $S = \text{RStd} = [1, \partial_x, \partial_y]^T$ (see Remark 4.12).

Let us write a modified version of (4.16) modulo L :

$$M_D := \left[M_{\text{Ext}} \mid M_{\text{RStd}} \right] \bmod L. \quad (5.113)$$

The left block matrix above has columns labeled by $\text{Ext} = [\partial_x^3, \partial_x^2 \partial_y, \partial_x \partial_y^2, \partial_y^3, \partial_x^2, \partial_x \partial_y, \partial_y^2]$. We proceed to solve (4.25) for the unknown matrix C , which will produce the Pfaffian matrices by (4.27). Equation (4.25) requires the following two matrices as input:

$$M_{\text{Ext}} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & -x^2+x & -2yx & -y^2 \\ \cdot & -x^2+x & -2yx & -y^2 & -x^2 & -2yx & -y^2+y \\ \cdot & -x^2 & -2yx & -y^2+y & \cdot & (-a-b-3)x+c_1 & (-a-b-3)y \\ -x^2+x & -2yx & -y^2 & \cdot & (-a-b-3)x+c_1+1 & (-a-b-3)y & (-a-b-3)y+c_2+1 \\ -x^2 & -2yx & -y^2+y & \cdot & (-a-b-3)x & (-a-b-3)y+c_2 & \cdot \end{bmatrix}$$

$$C_{\text{Ext}} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{bmatrix}. \quad (5.114)$$

Although we explained a general method to find C by a syzygy computation in section 4.2, we employ a more efficient method for this case. As (4.25) is written modulo $L = \underline{x^2} + \dots$, we can eliminate all instances of x^p for $p \geq 2$ via division by L . Consequently, it is sufficient to write C as a rational ansatz being linear in x :

$$C = \frac{p_{ij}(a, b, c_1, c_2)x + q_{ij}(a, b, c_1, c_2, y)}{r(a, b, c_1, c_2)x + s(a, b, c_1, c_2, y)}. \quad (5.115)$$

	Hypersurface
H_1	$y^2 - 4xy + 2y + 1 = 0$
H_2	$xy - y - 1 = 0$
H_3	$y^2 - y + x = 0$
H_4	$y^2 - 2y - 4x + 1 = 0$
H_4	$27y^2x - 36xy - y + 16x^2 + 8x + 1 = 0$
H_6	$y^2x - y - 1 = 0$
H_7	$4xy^2 - y^2 - 2y - 1 = 0$

Table 1. The singular hypersurfaces for the Horn’s functions $H_1, \dots, H_7$, whose restriction we successfully computed in this work.

After this preprocessing, we solve the system of linear equations

$$\text{Numerator}[C_{\text{Ext}} - C \cdot M_{\text{Ext}}] = 0 \quad (5.116)$$

for p_{ij}, q_{ij}, r and s , which, by (4.27), yields the Pfaffian matrices P_x and P_y in the fraction field $\mathbb{C}[x, y]$ (the matrices can be downloaded at [77]). As per [89], P_x and P_y induce a $\mathcal{D}$ -module on an affine variety.

Let $[g_1, g_2, g_3]$ denote the linearly independent holomorphic solutions at $(x, y) = (1/4, 1/4) \in W$. The standard monomials on the singular locus W are given by $[s_1, s_2, s_3] = [1, \partial_x, \partial_y]$. We then have a 3×3 matrix $F_{ij} = (s_i \bullet g_j)$. On W , it holds that $(\partial_x \bullet F) \cdot F^{-1} = P_x$ and $(\partial_y \bullet F) \cdot F^{-1} = P_y$. The matrices P_x and P_y are Pfaffian matrices on W . Note that these equalities hold on W but not outside of $V(L)$, because the PDE system (5.109) is irreducible [90].

Let us end our discussion on F_4 with a numerical analysis of the restricted Pfaffian system. To this end, we parameterize $L = 0$ by

$$x = q_x(t), y = q_y(t) \quad \text{with} \quad q_x(t) = t, q_y(t) = t + 1 \pm 2\sqrt{t}. \quad (5.117)$$

Fixing the exponent parameters $(a, b, c_1, c_2) = (-2/3, 1/3, 1/3, 1/3)$, we thus obtain an ODE of rank 3 on $V(L)$:

$$\frac{d}{dt}g(t) = \left[P_x(q_x(t), q_y(t)) \frac{dq_x}{dt} + P_y(q_x(t), q_y(t)) \frac{dq_y}{dt} \right] \cdot g(t). \quad (5.118)$$

A solution for the 3-dimensional vector $g(t)$ with initial condition $g = [1, 0, 0]^T$ is given in a PYTHON notebook at [77].

We note that the restriction of F_4 to its singular locus was also studied in [91] from a different point of view, utilizing a double cover map. However, it is not known that this method can be applied to Horn’s functions $H_1, \dots, H_7$ (see [92] for definitions). With our general algorithm, we are able to obtain rational restrictions of every system for H_i onto irreducible components of their singular loci. The relevant hypersurfaces are given the table 1. The holonomic $\mathcal{D}$ -modules for each H_i are of rank 4, and in all cases the restricted standard basis is $\mathbf{RStd} = [1, \partial_x, \partial_y]^T$. In other words, the rational restrictions of each system is of rank 3. See [77] for programs for these restrictions.

6 Conclusion

In this work we studied the process of taking limits on Feynman integrals and, more generally, Euler integrals. Our philosophy has been to study such integrals from the perspective of the differential equations that they obey; in particular their Pfaffian PDE systems. The structure of Pfaffian systems, in turn, is richly encoded in the algebraic language of holonomic $\mathcal{D}$ -modules. It is thereby possible to formalize the notion of a limit as a *restriction* of a $\mathcal{D}$ -module. In our work, the motivating example of a $\mathcal{D}$ -module has been the GKZ hypergeometric system, but our results and methods apply directly to Feynman integrals as well.

A restriction is generally accompanied by a drop in rank. Regarding the Pfaffian system for some Euler integral, this means that we expect a smaller and simpler Pfaffian system to survive in the limit. The main result of this publication is the development of efficient algorithms for computing these smaller Pfaffian systems. We presented two such restriction algorithms, each one having unique benefits. The first algorithm, working at the level of Pfaffian matrices, only requires sequential applications of certain gauge transformations. This algorithm is able to extract logarithmic corrections in the limiting variable, a feature that we expect to be especially useful for particle physics phenomenology. The second algorithm, working at the level of $\mathcal{D}$ -modules, computes the restricted Pfaffian system by passing through the *Macaulay matrix*. This algorithm benefits from the use of linear algebra over finite fields, making it quite efficient, and it does not require the unrestricted Pfaffian system, which can be hard to obtain, as input. While traditional algorithms have been limited to restriction onto hyperplanes, we have shown that the Macaulay matrix based method is able to restrict onto hypersurfaces.

In future work, it would be interesting to test our methods on state-of-the-art examples relevant for current phenomenological applications. While this paper only attempted to compute logarithmic corrections to the one-loop case of Bhabha scattering, there is in principle no computational barrier preventing the push to higher loops and more points. For the sake of increased accuracy, it is then alluring to also include the mixed power-and-log corrections of the form $z_1^a \log^b(z_1)$, $a, b \in \mathbb{Z}_{>0}$, as discussed at the end of section 3.4. On the mathematical side, we believe that it would be interesting to investigate restriction *ideals* using the Macaulay matrix method. This would also aid in the computation of symbol alphabets at two loops, as they can be obtained from restrictions of GKZ systems [39].

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A Modules and tensors in a nutshell

This exposition is independent of the main text and contains a pedagogical review of tensor products in the context of $\mathcal{D}$ -modules, along with several instructive examples of tensor products also appearing in the text.

Let R be a ring with 1 and $\mathcal{M}$ be an abelian group with the group operation denoted additively by “+”. Consider a map from $R \times \mathcal{M}$ to $\mathcal{M}$, whose image $a m$ for given $a \in R$ and $m \in \mathcal{M}$ is denoted multiplicatively²⁹ and is referred to as the result of the *action* of a on m . Then the abelian group $\mathcal{M}$ forms a *left R -module* when it satisfies the following conditions:

1. For any $a \in R$ and $m_i \in \mathcal{M}$, we have $a(m_1 + m_2) = a m_1 + a m_2$.
2. For any $a_i \in R$ and $m \in \mathcal{M}$, we have $(a_1 a_2) m = a_1(a_2 m)$, and $(a_1 + a_2) m = a_1 m + a_2 m$.³⁰
3. For any $m \in \mathcal{M}$, we have $1 m = m$.

When R is a (commutative) field, a left R -module is nothing but an R -vector space.

A *right R -module* is defined analogously by the following conditions: $(m_1 + m_2) a = m_1 a + m_2 a$, $m(a_1 a_2) = (m a_1) a_2$, $m(a_1 + a_2) = m a_1 + m a_2$, and $m = m 1$.

Example A.1. Consider the ring of univariate differential operators with polynomial coefficients $\mathcal{D} = \mathbb{C}\langle z, \partial_z \rangle$. The quotient set $\mathcal{M} = \mathcal{D}/(z\mathcal{D})$ forms a right $\mathcal{D}$ -module by defining the right action $[m a]$ of some $a \in \mathcal{D}$ on $[m] \in \mathcal{M}$ as the product (i.e. composition) of differential operators in $\mathcal{D}$ and a subsequent reduction to the equivalence class³¹ by $z\mathcal{D}$. ■

Example A.2. Let $\mathcal{D}_n$ be the Weyl algebra:

$$\mathcal{D}_n = \mathbb{C}\langle z_1, \dots, z_n, \partial_1, \dots, \partial_n \rangle,$$

that is, the ring of differential operators in n variables $z_1, \dots, z_n$. The set of length k -vectors $\mathcal{D}_n^k$ is an example of a left $\mathcal{D}_n$ -module: the addition is defined by $(f_1, \dots, f_k) + (g_1, \dots, g_k) = (f_1 + g_1, \dots, f_k + g_k)$ and the action of $h \in \mathcal{D}_n$ on $(f_1, \dots, f_k)$ is given by $(h f_1, \dots, h f_k)$. $\mathcal{D}_n^k$ is called the *free $\mathcal{D}_n$ -module* of rank k . ■

Let us now briefly review the construction of a Gröbner basis in the free module $\mathcal{D}_n^k$ that we have just introduced above. We fix a term order $\prec$ in $\mathcal{D}_n$. For a k -vector $f = (0, \dots, 0, f_i, \dots, f_k)$ with $f_i \neq 0$, the index i is called the *top index* of f . Consider two elements f and g in $\mathcal{D}_n^k$. We say that f is larger than g in the *position over term* (POT) order of $\mathcal{D}_n^k$, when the top index of f is smaller than the top index of g , or when the top

²⁹Explicitly, we consider a map $\psi : R \times \mathcal{M} \rightarrow \mathcal{M}$ and write its image as multiplication: $\psi(a, m) \equiv a m$. We use the same notation for multiplication inside of the ring R .

³⁰In other words, $\psi(a_1 a_2, m) = \psi(a_1, \psi(a_2, m))$ and $\psi(a_1 + a_2, m) = \psi(a_1, m) + \psi(a_2, m)$.

³¹Here and in the following the square brackets $[\bullet]$ denote the equivalence class of some element $\bullet$ under some relation. Sometimes we will omit these brackets for the sake of brevity if it does not lead to confusion.

indices of f and g are both i and the corresponding components $f_i \succ g_i$. The reduction (or the division) algorithm follows from this definition of the total order. Namely, the S -pair $S(f, g)$ of f and g is a $(0, \dots, 0)$ vector when the top indices of f and g are not the same. When the top indices are the same and equal to i , then the S -pair of f and g is defined by $h_1 f - h_2 g$, where $h_j \in \mathcal{D}_n$ are chosen so that $h_1 f_i - h_2 g_i$ is the S -pair of f_i and g_i in $\mathcal{D}_n$.

Example A.3. Consider the free module $\mathcal{D}_2^2$ in two variables z_1 and z_2 . When $f = (\partial_1, z_1)$ and $g = (0, z_1)$, then $S(f, g) = (0, 0)$ as the top indices of f and g are not the same. If $f = (z_1 \partial_1, z_1)$ and $g = (z_2 \partial_1, z_1)$, then the S -pair becomes $S(f, g) = z_2 f - z_1 g = (0, z_1 z_2 - z_1^2)$. For further examples and details we refer the interested reader to [69] and [64, sections 6.6–7].

■

Definition A.4. For a left $\mathcal{D}_n$ -module $\mathcal{M}$ and a right $\mathcal{D}_n$ -module P , let us define the tensor product $P \otimes_{\mathcal{D}_n} \mathcal{M}$ as follows. We denote by $p \otimes m$ the element of the direct product (p, m) , where $p \in P$ and $m \in \mathcal{M}$. Then $P \otimes_{\mathcal{D}_n} \mathcal{M}$ is the set of a finite formal sum of elements of this form modulo the equivalence relation $\sim$ given by:

1. $p_1 \otimes m + p_2 \otimes m \sim (p_1 + p_2) \otimes m$, $m \in \mathcal{M}$, $p_i \in P$ (linearity in the left slot).
2. $p \otimes m_1 + p \otimes m_2 \sim p \otimes (m_1 + m_2)$, $p \in P$, $m_i \in \mathcal{M}$ (linearity in the right slot).
3. $pf \otimes m \sim p \otimes fm$ for $f \in \mathcal{D}_n$, $m \in \mathcal{M}$, and $p \in P$.³²

In other words, if we denote by $F(P \otimes \mathcal{M})$ the set of formal finite sums of $p \otimes m$, then the tensor product is just the factor $P \otimes_{\mathcal{D}_n} \mathcal{M} := F(P \otimes \mathcal{M}) / \sim$.

Remark A.5. When the left entry P is also a left $\mathcal{D}_{n-1}$ -module, the tensor product $P \otimes_{\mathcal{D}_n} \mathcal{M}$ can be turned into a left $\mathcal{D}_{n-1}$ -module by specifying the action of $g \in \mathcal{D}_{n-1}$ on $p \otimes m$ as $g(p \otimes m) = (gp) \otimes m$.

Example A.6. Let $\mathcal{M}$ be a left $\mathcal{D}_n$ -module. In order to illustrate the various concepts introduced so far, here we show a proof of the following well-known isomorphism between two left $\mathcal{D}_{n-1}$ -modules written as quotients:

$$\mathcal{D}_n / (z_n \mathcal{D}_n) \otimes_{\mathcal{D}_n} \mathcal{M} \simeq \mathcal{M} / (z_n \mathcal{M}) \quad (\text{A.1})$$

We start with showing that the right $\mathcal{D}_n$ -module $\mathcal{D}_n / (z_n \mathcal{D}_n)$ (see Example A.1), while being also a left $\mathcal{D}_{n-1}$ -module, is in fact *not* a left $\mathcal{D}_n$ -module. Indeed, suppose that it is a left $\mathcal{D}_n$ -module. We denote by $[\bullet]$ the equivalence class of some element $\bullet$ in $\mathcal{D}_n / (z_n \mathcal{D}_n)$. On one hand, $[z_n] = [0]$ in $\mathcal{D}_n / (z_n \mathcal{D}_n)$, so that the action of the derivative ∂_n should also be³³ $\partial_n[z_n] = [0]$ in $\mathcal{D}_n / (z_n \mathcal{D}_n)$. On the other hand, $\partial_n[z_n] = [z_n \partial_n + 1]$ and $[z_n \partial_n] = [0]$ in $\mathcal{D}_n / (z_n \mathcal{D}_n)$, leading to $\partial_n[z_n] = [1] \neq [0]$ in $\mathcal{D}_n / (z_n \mathcal{D}_n)$, which is a contradiction. The following technical observation will become useful shortly: any non-zero element of $\mathcal{D}_n / (z_n \mathcal{D}_n) \otimes_{\mathcal{D}_n} \mathcal{M}$ can be expressed as $[1] \otimes m$ for some $m \notin z_n \mathcal{M}$. Indeed, in a single

³²Hence the subscript $\mathcal{D}_n$ in $\otimes_{\mathcal{D}_n}$ means that any element $f \in \mathcal{D}_n$ can be carried across the $\otimes$ sign.

³³Note that the condition $a([0] + [0]) = a([0]) = a[0] + a[0]$ implies $a[0] = [0]$ for any $a \in \mathcal{D}_n$.

product any factor $f \in \mathcal{D}_n$ can be moved from the left to the right slot: $[f] \otimes m = [1] \otimes fm$. After performing this operation termwise, any finite sum can be united into a single product: $[1] \otimes m_1 + [1] \otimes m_2 = [1] \otimes (m_1 + m_2)$ according to Definition A.4. For two non-zero elements $[1] \otimes m_1$ and $[1] \otimes m_2$, where $m_i \notin z_n \mathcal{M}$, it can be shown that the condition $[1] \otimes m_1 \neq [1] \otimes m_2$ is equivalent to $m_1 \neq m_2$ in $\mathcal{M}$. We also note that $[0] \otimes m$ and $[f] \otimes 0$ are 0 by definition.

We define a $\mathcal{D}_{n-1}$ -morphism φ from $\mathcal{D}_n/(z_n \mathcal{D}_n) \otimes_{\mathcal{D}_n} \mathcal{M}$ to $\mathcal{M}/(z_n \mathcal{M})$ by $\varphi([f] \otimes m) := [fm]$ and extend this definition on finite sums $\varphi([f_1] \otimes m_1 + [f_2] \otimes m_2) = [f_1 m_1] + [f_2 m_2]$ by linearity. Let us now prove that φ is indeed an isomorphism.

φ is well-defined: we prove that if $t_1 \sim t_2$, $t_i \in \mathcal{D}_n/(z_n \mathcal{D}_n) \otimes_{\mathcal{D}_n} \mathcal{M}$, then $\varphi(t_1) = \varphi(t_2)$. From the definition of φ it follows that if $t \sim 0 \otimes 0$, then $\varphi(t) = [0]$. Any 0 in $\mathcal{D}_n/(z_n \mathcal{D}_n) \otimes_{\mathcal{D}_n} \mathcal{M}$ can be expressed as $[z_n] \otimes m$, $m \in \mathcal{M}$ after a simplification by “+”. Note that φ satisfies $\varphi(t_1 + t_2) = \varphi(t_1) + \varphi(t_2)$ for any $t_i \in \mathcal{D}_n/(z_n \mathcal{D}_n) \otimes_{\mathcal{D}_n} \mathcal{M}$. Since $\varphi([z_n] \otimes m) = [z_n m]$, $z_n m \in z_n \mathcal{M}$, then $[0] \otimes m$ is sent to $[0]$ by φ .

φ is $\mathcal{D}_{n-1}$ -morphism: let $c_i \in \mathcal{D}_{n-1}$, $f_i \in \mathcal{D}_n$, and $m_i \in \mathcal{M}$.

$$\begin{aligned}
 & \varphi(c_1 ([f_1] \otimes m_1) + c_2 ([f_2] \otimes m_2)) \\
 &= \varphi([c_1 f_1] \otimes m_1 + [c_2 f_2] \otimes m_2) && \text{(Remark A.5)} \\
 &= [c_1 f_1 m_1] + [c_2 f_2 m_2] && \text{(definition of } \varphi) \\
 &= c_1 [f_1 m_1] + c_2 [f_2 m_2] && \text{(action of } c_i \text{ on } \mathcal{M}/(z_n \mathcal{M})) \\
 &= c_1 \varphi([f_1] \otimes m_1) + c_2 \varphi([f_2] \otimes m_2) && \text{(definition of } \varphi)
 \end{aligned}$$

φ is surjective: clearly the image $\varphi([1] \otimes \mathcal{M})$ covers the whole $\mathcal{M}/z_n \mathcal{M}$.

φ is injective: due to the linearity of φ , it is enough to prove that if $\varphi(t) = 0$ for some $t \in \mathcal{D}_n/(z_n \mathcal{D}_n) \otimes_{\mathcal{D}_n} \mathcal{M}$, then necessarily $t = 0$. Suppose that $t \neq 0$. Then there exists $m \in \mathcal{M}$ such that $t \sim [1] \otimes m$. Since $\varphi([1] \otimes m) = [m] = 0$, we have $m \in z_n \mathcal{M}$. Writing $m = z_n m'$, $m' \in \mathcal{M}$, we have $[1] \otimes (z_n m') \sim [z_n] \otimes m' \sim 0 \otimes m'$, which means that $t = 0$.

Thus we constructed an explicit isomorphism φ between $\mathcal{D}_n/(z_n \mathcal{D}_n) \otimes_{\mathcal{D}_n} \mathcal{M}$ and $\mathcal{M}/(z_n \mathcal{M})$ and proved the identity (A.1). $\blacksquare$

Example A.7. Another example of tensor product is:

$$\mathcal{N} = \mathbb{C}[z_1, \dots, z_n]/\langle z_n \rangle \otimes_{\mathbb{C}[z_1, \dots, z_n]} \mathcal{M}.$$

For $1 \leq i \leq n-1$ we define the action of ∂_i on the product $[f] \otimes m$ as $\partial_i([f] \otimes m) := [\partial_i f] \otimes m + [f] \otimes (\partial_i m)$, where $[f] \in \mathbb{C}[z_1, \dots, z_n]/\langle z_n \rangle$ and $m \in \mathcal{M}$. Multiplication by $h \in \mathbb{C}[z_1, \dots, z_{n-1}]$ is defined by $h([f] \otimes m) := [hf] \otimes m$ (similar to Remark A.5). Then we can prove that $\mathcal{N}$ is a left $\mathcal{D}_{n-1}$ -module

$$\mathcal{N} \simeq \mathcal{M}/(z_n \mathcal{M}). \quad (\text{A.2})$$

Indeed, the isomorphism φ from $\mathcal{N}$ to $\mathcal{M}/(z_n \mathcal{M})$ is given by $\varphi([f] \otimes m) := [fm]$. Note that for $[f] = [z_n h] \sim [0]$ we have $\varphi([z_n h] \otimes m) = [z_n h m] \sim [0]$, which makes the map φ is well-defined. It is left to the reader to prove that φ is an isomorphism. $\blacksquare$

Example A.8. Our final example of tensor product is $\mathbb{C}[z][1/f] \otimes_{\mathbb{C}[z]} \mathcal{D}_n$, where $\mathbb{C}[z] = \mathbb{C}[z_1, \dots, z_n]$ and f is a polynomial in z . Since the left entry $\mathbb{C}[z][1/f] = \mathbb{C}[z_1, \dots, z_n, 1/f]$ can be viewed as a left $\mathcal{D}_n$ -module, we can regard the tensor product $\mathbb{C}[z][1/f] \otimes_{\mathbb{C}[z]} \mathcal{D}_n$ as a left $\mathcal{D}_n$ -module by $\partial_i(g \otimes \ell) = (\partial_i g) \otimes \ell + g \otimes (\partial_i \ell)$, for $g \in \mathbb{C}[z][1/f]$ and $\ell \in \mathcal{D}_n$. It is an exercise for the reader to describe the action of a general element of $\mathcal{D}_n$ explicitly. Furthermore, we can introduce the ring structure on $\mathbb{C}[z][1/f] \otimes_{\mathbb{C}[z]} \mathcal{D}_n$ by defining the product “ $\cdot$ ” between two elements as $(g_1 \otimes \ell_1) \cdot (g_2 \otimes \ell_2) = g_1 \otimes \ell_1 (g_2 \otimes \ell_2)$, where $g_i \in \mathbb{C}[z][1/f]$ and $\ell_i \in \mathcal{D}_n$. Roughly speaking, $\mathbb{C}[z][1/f] \otimes_{\mathbb{C}[z]} \mathcal{D}_n$ is $\mathbb{C}\langle z_1, \dots, z_n, 1/f, \partial_1, \dots, \partial_n \rangle$. ■

In Proposition 2.3, we construct a zero-dimensional $\mathcal{R}$ -module from a holonomic $\mathcal{D}_Y$ -module by a tensor product. The zero-dimensional one can be expressed as a quotient by a left ideal as the right hand side of (A.3) below. Let us sketch a proof of the isomorphism. Take a left ideal $\mathcal{I}$ in the Weyl algebra $\mathcal{D}_n$, and let $\mathcal{M} := \mathcal{D}_n / \mathcal{I}$ be a left $\mathcal{D}_n$ -module. Let $\mathcal{R}_n = \mathbb{C}(z)\langle \partial_1, \dots, \partial_n \rangle$ be the ring of differential operators with rational function coefficients, where $\mathbb{C}(z) := \mathbb{C}(z_1, \dots, z_n)$ is the field of rational functions. The left $\mathcal{R}_n$ -modules $\mathcal{R}_n \otimes_{\mathcal{D}_n} \mathcal{M}$ and $\mathcal{R}_n / (\mathcal{R}_n \mathcal{I})$ are isomorphic as left $\mathcal{R}_n$ -modules. Here the left action of $h \in \mathcal{R}_n$ on $f \otimes g \in \mathcal{R}_n \otimes_{\mathcal{D}_n} \mathcal{M}$ is defined by $h f \otimes g$. The isomorphism is given by

$$\mathcal{R}_n \otimes_{\mathcal{D}_n} \mathcal{M} \ni f \otimes g \mapsto f g \in \mathcal{R}_n / (\mathcal{R}_n \mathcal{I}), \quad (\text{A.3})$$

and we can easily check that it is an isomorphism of left $\mathcal{R}_n$ -modules by noting that $\mathcal{R}_n$ is an associative algebra. Note that $f_1 f_2 \otimes g = f_1 \otimes f_2 g$ in $\mathcal{R}_n \otimes_{\mathcal{D}_n} \mathcal{M}$ when $f_2 \in \mathcal{D}_n$. We can show analogously that $\mathcal{R}_n \otimes_{\mathcal{D}_n} \mathcal{D}_n^m / \mathcal{J}$ is isomorphic to $\mathcal{R}_n^m / (\mathcal{R}_n \mathcal{J})$ as left $\mathcal{R}_n$ -modules, where $\mathcal{J}$ is some left submodule of $\mathcal{D}_n$.

Remark A.9. We referred to the algorithm of constructing a *cyclic vector* in section 2. It is a way to construct an ideal given a Pfaffian system.

In order to apply the cyclic vector to $\mathcal{R}_n^m / (\mathcal{R}_n \mathcal{J})$, we first compute the Weyl closure of $\mathcal{J}$, and then apply the algorithm of [65] to the closure. This procedure is exact, but it needs huge computational resources. We present an easy but inexact method. For simplicity, we illustrate the case of $m = 2$, $n = 1$ ($\partial = \partial_1$, $z = z_1$), and $\mathcal{R}_1 \mathcal{J}$ is generated by

$$(\partial, 0) - (a, b), \quad (0, \partial) - (c, d), \quad a, b, c, d \in \mathbb{C}(z).$$

These generators induce a system of differential equations for two unknown functions u^1 and u^2 :

$$\partial \bullet u^1 - a u^1 - b u^2 = 0, \quad \partial \bullet u^2 - c u^1 - d u^2 = 0. \quad (\text{A.4})$$

Differentiating both equations w.r.t. z , we obtain two more identities:

$$u_{zz}^1 - a_z u^1 - a u_z^1 - b_z u^2 - b u_z^2 = 0, \quad u_{zz}^2 - c_z u^1 - c u_z^1 - d_z u^2 - d u_z^2 = 0 \quad (\text{A.5})$$

where u_z and u_{zz} denote the first and second derivatives w.r.t. z . We regard $u^1, u^2, u_z^1, u_z^2, u_{zz}^1$ and u_{zz}^2 as independent variables in the polynomial ring $\mathbb{C}(z)[u^1, u^2, u_z^1, u_z^2, u_{zz}^1, u_{zz}^2]$ and eliminate u^2, u_z^2, u_{zz}^2 from (A.4) and (A.5). Then we obtain an ODE $u_{zz}^1 + c_1(z)u_z^1 + c_2(z)u^1 = 0$ for the function u^1 . In some cases, the left ideal $\mathcal{I} = \langle \partial^2 + c_1 \partial + c_2 \rangle$ in $\mathcal{R}_1$ gives an isomorphism of $\mathcal{R}_1^2 / (\mathcal{R}_1 \mathcal{J})$ and $\mathcal{R}_1 / \mathcal{I}$.

Remark A.10. As a historical concluding remark, let us rephrase the notion of a *solution* to a system of differential equations in the language of homological algebra, which is a starting point of the theory of $\mathcal{D}$ -modules. Let us denote by $\mathcal{O}^{an}(U)$ the set of holomorphic functions³⁴ on an open set U , and by

$$\mathrm{Hom}_{\mathcal{D}_n}(\mathcal{M}, \mathcal{O}^{an}(U)) \quad (\text{A.6})$$

the set of left $\mathcal{D}_n$ -morphisms from the left $\mathcal{D}_n$ -module $\mathcal{M}$ to $\mathcal{O}^{an}(U)$. The correspondence between a homomorphism $\varphi \in \mathrm{Hom}_{\mathcal{D}_n}(\mathcal{M}, \mathcal{O}^{an}(U))$ and a solution f is given by $f = \varphi([1])$. In fact, take $\mathcal{M} := \mathcal{D}_n/\mathcal{I}$, and let ℓ be an element of the left ideal $\mathcal{I}$, so that $[\ell] \sim [0]$ in $\mathcal{M}$. Since φ is a left $\mathcal{D}_n$ -morphism, we have:

$$0 = \varphi([0]) = \varphi([\ell]) = \varphi(\ell[1]) = \ell \bullet \varphi([1]), \quad (\text{A.7})$$

which means that $\varphi([1]) \in \mathcal{O}^{an}(U)$ is a solution to the differential equation $\ell \bullet f = 0$. Conversely, let f be a solution of $\mathcal{I}$, meaning that for any $\ell \in \mathcal{I}$ it satisfies $\ell \bullet f = 0$. Then for any other element $m \in \mathcal{D}_n$, the mapping $m \mapsto m \bullet f \in \mathcal{O}^{an}(U)$ defines a left $\mathcal{D}_n$ -morphism, i.e. an element of $\mathrm{Hom}_{\mathcal{D}_n}(\mathcal{M}, \mathcal{O}^{an}(U))$. The derived functor of Hom gives cohomological solution spaces in the theory of $\mathcal{D}$ -modules [22].

B Normal form by Moser reduction

In section 3.2, it was emphasised that a Pfaffian system must be in *normal form* before the restriction protocol can proceed. A given set of Pfaffian matrices $P_1, \dots, P_m$ is in normal form w.r.t. z_1 when P_1 satisfies two conditions:

1. *Logarithmic*: P_1 has a simple pole at $z_1 = 0$.
2. *Non-resonant*: the eigenvalues of the residue matrix $P_{1,-1}(z')$ have no integral difference, and 0 is the unique integer eigenvalue.

As explained later in this section, the non-resonance condition additionally implies that $P_2, \dots, P_m$ are finite at $z_1 = 0$. This is important, because in the restriction protocol of section 3.2 we use the matrices $P_2(0, z'), \dots, P_m(0, z')$, where $z' = (z_2, \dots, z_m)$.

Normal form can be obtained by following the Moser reduction scheme [93], which was later made algorithmic by Barkatou [94] (see also [95]). For CAS implementations, see e.g. ISOLDE [96] and Fuchsia [97]. For the reader's convenience, in this section we review Moser reduction, largely following [70].

B.1 Logarithmic P_1

Let $\mathbb{1}_r$ denote the r -dimensional identity matrix. The following theorem is a result of [93, 94].

³⁴The superscript “an” stands for “analytic”. On the other hand, $\mathcal{O}^{alg}(U)$ is the set of rational functions that do not have poles on U . They are called regular functions on U . We omit the superscript “alg”, that stands for “algebraic”, in this paper.

Theorem B.1. *Let $P(z)$ be an $r \times r$ matrix, depending on a single variable z , with entries in the field of rational functions $\mathbb{C}(z)$. Assume that a system of ODEs*

$$\frac{dI}{dz} = P(z) \cdot I \quad (\text{B.1})$$

is regular singular at $z = 0$. After a finite sequence of gauge transforms $P \rightarrow G[P] = G^{-1}(P \cdot G - \partial_z G)$ with G of the form

$$G = \text{a constant matrix, or } G = \begin{bmatrix} z\mathbb{1}_{r'} & 0 \\ 0 & \mathbb{1}_{r-r'} \end{bmatrix}, \text{ for } r' \in \mathbb{N}, \quad (\text{B.2})$$

the ODEs (B.1) can be transformed into a system with only simple poles at $z = 0$. Such a sequence of gauge transforms can be found algorithmically.

Remark B.2. The gauge transformation matrices can be constructed without algebraic extensions. In other words, if the entries of $P(z)$ belong to $K(z)$ for some subfield $K \subset \mathbb{C}$, then G can be taken to have coefficients in K . This is useful when the system (B.1) contains additional parameters independent of z .

The theorem above pertains to ODEs, but Pfaffian systems are PDEs in general. Fortunately, we can still apply the theorem to the PDE case by treating one variable at a time. Consider a Pfaffian system in m variables: $\partial_i I = P_i(z) \cdot I$, for $i = 1, \dots, m$. Assume that the P_i have higher-order poles along $z_1 = 0$. Treat

$$\frac{dI}{dz_1} = P_1(z_1, z') \cdot I \quad (\text{B.3})$$

as an ODE in z_1 with parameters $z' = (z_2, \dots, z_m)$. By Theorem B.1 and Remark B.2, we can find a gauge transformation matrix G such that $G[P_i]$ only has a simple pole along $z_1 = 0$. We may thus assume that the Pfaffian matrices take the form:

$$P_1(z) = \sum_{n=-1}^{\infty} P_{1,n}(z') z_1^n. \quad (\text{B.4})$$

This solves the problem of finding a logarithmic set of Pfaffian matrices.

B.2 Non-resonant $P_{1,-1}$

Next we show how to cure the resonant spectrum of $P_{1,-1}(z')$. The first thing to note is that the eigenvalues of $P_{1,-1}(z')$ are independent of z' [19, Theorem 12.1]. Therefore, if $P_{1,-1}(z')$ is non-resonant at a point, then it is non-resonant everywhere. Fix a generic value $z' = a'$, and set $P := P_{1,-1}(a')$. Suppose that P is in Jordan normal form, namely in block-diagonal form with Jordan cells $J_1, \dots, J_k$. Let r_i denote the size of each J_i . Subdivide the matrix $P_1(z)$ into cells as

$$P_1 = \left[\begin{array}{c|c|c|c} (P_1)_{11} & (P_1)_{12} & \dots & (P_1)_{1k} \\ (P_1)_{21} & (P_1)_{22} & & \vdots \\ \vdots & & \ddots & \\ (P_1)_{k1} & \dots & & (P_1)_{kk} \end{array} \right] \quad (\text{B.5})$$

and construct the gauge transformation matrix

$$G = \begin{bmatrix} z_1 \mathbb{1}_{r_1} & 0 & \dots & 0 \\ 0 & \mathbb{1}_{r_2} & & \vdots \\ \vdots & & \ddots & \\ 0 & \dots & & \mathbb{1}_{r_k} \end{bmatrix}. \quad (\text{B.6})$$

By assumption, P_1 has a simple pole along $z_1 = 0$, and the same will hold true for $G[P_1]$. A short calculation shows that the residue matrix of $G[P_1]$ evaluated at $z' = a'$ is of the form

$$G[P] = \begin{bmatrix} J_1 - \mathbb{1}_{r_1} & (P_{1,0})_{12} & (P_{1,0})_{13} & \dots & (P_{1,0})_{1k} \\ 0 & J_2 & 0 & \dots & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & & & J_k \end{bmatrix}. \quad (\text{B.7})$$

We have thereby shifted the eigenvalue corresponding to the Jordan block J_1 by 1. Repeating this procedure sufficiently many times for each J_i , we will render P non-resonant at $z' = a'$, and therefore non-resonant everywhere.

Let us now show that non-resonance implies finiteness of $P_2, \dots, P_m$. The following lemma is well-known.

Lemma B.3. *Suppose that $A, B \in M_r(\mathbb{C})$ are $r \times r$ matrices with their entries in $\mathbb{C}$. Let $\lambda(A), \lambda(B)$ denote the sets of eigenvalues for A, B . Consider the linear map*

$$T_{A,B} : M_r(\mathbb{C}) \ni X \mapsto A \cdot X - X \cdot B \in M_r(\mathbb{C}). \quad (\text{B.8})$$

Then the set of eigenvalues of $T_{A,B}$ equals the set of eigenvalue differences $\lambda(A) - \lambda(B)$.

It follows from the integrability condition (2.17) and (B.4) that

$$[P_{1,-1}(z'), P_{i,n}(z')] = n P_{i,n}(z'), \quad i = 2, \dots, m, \quad n < 0. \quad (\text{B.9})$$

We can rewrite this in terms of the map $T_{A,B}$ as

$$T_{P_{1,-1}, P_{1,-1}}(P_{i,n}) = n P_{i,n}(z'), \quad (\text{B.10})$$

meaning that $P_{i,n}$ is an eigenvector of $T_{P_{1,-1}, P_{1,-1}}$ with eigenvalue n . On the other hand, Lemma B.3 shows that any eigenvalue of $T_{P_{1,-1}, P_{1,-1}}$ must be of the form $\lambda_a - \lambda_b$, where $\lambda_{a,b}$ are eigenvalues of $P_{1,-1}$. If $P_{1,-1}$ is non-resonant, we cannot have that $\lambda_a - \lambda_b = n$. Consequently, $P_{2,n}(z') = 0$ for $n < 0$.

Summarizing the previous two sections, we have a

Theorem B.4. *The algorithm for transforming a Pfaffian system into normal form w.r.t. z_1 is as follows:*

1. *Apply the Moser reduction scheme from Theorem B.1 over the field $\mathbb{C}(z')$ to ensure that P_1 has a simple pole at $z_1 = 0$.*
2. *Apply gauge transformations of the form (B.7) until $P_{1,-1}$ becomes non-resonant.*

This theorem can be sequentially applied to several variables, in case one seeks to restrict more than one variable.

B.3 Jordan normal form without field extensions

We have already noted that Moser reduction can be performed using only linear algebra without field extensions. Step 1 of Theorem B.4 is therefore efficient to implement. Step 2, on the other hand, apparently requires one to solve polynomial equations, as this is the standard way of computing Jordan normal forms. Here we argue that algebraic extensions can *also* be avoided in step 2, wherefore Theorem B.4 does not require them at all. Fix a subfield $K \subset \mathbb{C}$. The purpose of this subsection is to prove the following lemma.

Lemma B.5. *There exists an invertible matrix $Q \in K^{r \times r}$ such that $Q^{-1}PQ$ is block diagonalized as*

$$Q^{-1}PQ = \begin{bmatrix} B_1 & \mathbb{0} & \dots & \mathbb{0} \\ \mathbb{0} & B_2 & & \vdots \\ \vdots & & \ddots & \\ \mathbb{0} & \dots & & B_k \end{bmatrix} \quad (\text{B.11})$$

where each B_i is non-resonant. We write $\text{Spec}[B_i]$ for the set of eigenvalues of the block B_i . Then, for any pair of indices i, j , either $(\text{Spec}[B_i] + \mathbb{Z}) \cap \text{Spec}[B_j] = \emptyset$ or there exists a unique integer m such that $\text{Spec}[B_i] + m = \text{Spec}[B_j]$.

Once this lemma is proved, we can conclude that the gauge transformations involving Jordan normal forms from step 2 of Theorem B.4 can be algorithmically chosen from $GL_r(K[z_1^\pm])$, which is free of field extensions. Indeed, each Jordan block J_i of P is replaced by the block B_i . Let us begin with a simple lemma.

Lemma B.6. *Let $f(t) \in K[t]$ be an irreducible polynomial. If α is a root of $f(t) = 0$, no integral shift of α is a root.*

Proof. Set $f(t) = a_0 + \dots + a_{N-1}t^{N-1} + t^N$ and fix an integer $m \neq 0$ such that $f(\alpha + m) = 0$. We prove that the existence of m leads to a contradiction. A swift computation shows that

$$0 = f(\alpha + m) = f(\alpha) + (\text{terms of order } \leq N-1 \text{ in } \alpha) := f(\alpha) + g(\alpha) = g(\alpha), \quad (\text{B.12})$$

where in the last step we used that α is a root of f . The coefficient of α^{N-1} in $g(\alpha)$ is Nm , which is clearly non-zero. This means that $g(\alpha) = 0$, i.e. α is also a root of $g(t)$. But this is a contradiction, because $f(t)$ is the minimal polynomial.³⁵ $\square$

Suppose we are given a pair of irreducible polynomials $f_1, f_2 \in K[t]$. If they share a root α modulo $\mathbb{Z}$, that means that $f_1(t+s) = f_2(t)$ has an integral solution as a system of equations in s . This is seen by expanding both sides and collecting powers of t , leading to a polynomial system for s of the form $g_0(s) = \dots = g_{N-1}(s) = 0$. The GCD $g(s)$ is used whether we have an integral solution or not.

³⁵The minimal polynomial of α over K is the polynomial with the smallest degree, whose roots contain α .

Let us recall a well-known fact from linear algebra (see e.g. [98, chapter 12]). Given an $r \times r$ matrix P with entries in K , we can equip K^r with the structure of a $K[t]$ -module by letting t act on K^r via left multiplication by P . Consider an exact sequence

$$K[t]^r \xrightarrow{t\mathbb{1}_r - P} K[t]^r \longrightarrow K^r \longrightarrow 0. \quad (\text{B.13})$$

The last morphism sends $f(t)e_i$ to $f(P) \cdot e_i$, where $f(t) \in K[t]$ and e_i is the standard unit vector. The fundamental theorem of principal ideal domains shows that $t\mathbb{1}_r - P$ is transformed into a canonical form

$$L(t)(t\mathbb{1}_r - P)R(t) = \text{Diag}[1, \dots, 1, d_1(t), \dots, d_p(t)] \quad (\text{B.14})$$

for some invertible polynomial matrices $L(t), R(t) \in GL(r, K[t])$ with $d_1(t) \mid \dots \mid d_p(t)$. According to the Chinese remainder theorem, we thus have an isomorphism (which can be made algorithmically explicit)

$$K^r \xleftarrow{\varphi} \bigoplus_{i=1}^p K[t]/d_i(t) \simeq \bigoplus_{i=1}^p \bigoplus_{j=1}^{q_i} K[t]/d_{ij}(t)^{m_{ij}} \quad (\text{B.15})$$

where $d_i(t) = \prod_{j=1}^{q_i} d_{ij}(t)^{m_{ij}}$ is the irreducible decomposition. The morphism φ sends any element $(f_i(t))_{i=1}^p \in \bigoplus_{i=1}^p K[t]/d_i(t)$ to $\sum_{i=1}^p f_i(t)v_i(t)|_{t \mapsto P}$ where $v_i(t)$ is the $(r - p + i)$ -th column vector of $L(t)^{-1}$ and the subscript $t \mapsto P$ stands for the operation of replacing an expression $t^a v$ ($v \in K^r$) by $P^a v$. Each subspace $K[t]/d_{ij}(t)^{m_{ij}}$ is invariant under multiplication by t , i.e. left multiplication by P . This means that we can find a basis of K^r such that the left multiplication by P is block-diagonalized into a direct sum of multiplication by t on $K[t]/d_{ij}(t)^{m_{ij}}$. It is not difficult to check that the $d_{ij}(t)$ share a common root modulo $\mathbb{Z}$, as explained in the previous paragraph. Finally, any basis of each block $K[t]/d_{ij}(t)^{m_{ij}}$ is sent to a set of column vectors B_{ij} . The matrix Q is obtained by aligning these B_{ij} to form a square matrix.

Remark B.7. In the examples of Pfaffian systems for Feynman integrals discussed in this paper, any eigenvalue of the residue matrix is an integral linear combination of ε and $\varepsilon\delta$. Since the Euler integral (2.1) admits a meromorphic continuation as in [99, Theorem 2.5], we expect this to be true in general.

C Improvements on restriction algorithms

Here we remark on subtle improvements to steps in the restriction algorithms presented in the main body of this text.

C.1 $\mathcal{D}$ -module-level restriction

The b -function played a central role in the $\mathcal{D}$ -module restriction protocol of section 4.1. However, it is well-known that the b -function is computationally expensive to obtain. When we can find a logarithmic connection for $\mathcal{D}'/\mathcal{I}$ via another method (recall the definition of $\mathcal{D}'$ below Remark 4.3), such as the Macaulay matrix method, we can utilize the following proposition to compute the b -function more easily:

Proposition C.1. *Let $S = [s_1 = 1, s_2, \dots, s_r]^T$ be a set of standard monomials for $\mathcal{I}$ such that $\text{ord } s_i = 0$. We denote by $\bar{\mathcal{I}} = \mathcal{R}'\mathcal{I} \cap \mathcal{D}'$ the Weyl closure of $\mathcal{I}$. Assume that the Pfaffian system with respect to S has the form*

$$\begin{aligned}\partial_x S &= \left(\frac{P_{-1}}{x} + P_0 + xP_1 + \dots \right) \cdot S \mod \mathcal{R}'\mathcal{I} \\ \partial_y S &= (Q_0 + xQ_1 + \dots) \cdot S \mod \mathcal{R}'\mathcal{I}\end{aligned}$$

Then the eigenvalues of P_{-1} contain the roots of the b -function of $\bar{\mathcal{I}}$ for the weight $w = (1, 0)$.

Proof. We note that $x\partial_x S = P_{-1} \cdot S$ modulo $F_{-1} + \bar{\mathcal{I}}$. Here F_{-1} is the filtration defined in (4.5). Let $\varphi(s)$ be the characteristic polynomial of P_{-1} . By the Cayley-Hamilton theorem, we have $\varphi(P_{-1}) = 0$. Hence, we have $\varphi(x\partial_x)S = 0$ modulo $F_{-1} + \bar{\mathcal{I}}$. In particular, we have $\varphi(x\partial_x) = 0$ modulo $F_{-1} + \bar{\mathcal{I}}$ because $s_1 = 1$. Therefore, the b -function is a factor of $\varphi(s)$. $\square$

If we have Pfaffian matrices for a logarithmic connection with the singular locus $x = 0$, we can accelerate Algorithm 4.4 with help from the residue matrix P_{-1} as follows. Let S be a column vector consisting of standard monomials of a Gröbner basis of $\mathcal{I}$ in $\mathcal{R}' = \mathbb{C}(x, y)\langle \partial_x, \partial_y \rangle$. Suppose that

$$x\partial_x S - (P_{-1} + P_0x + \dots) \cdot S \equiv 0 \mod \bar{\mathcal{I}} \quad (\text{C.1})$$

where $xP = P_{-1} + P_0x + \dots$ is the Pfaffian matrix in the x direction multiplied by x . Taking the limit $x = 0$ in (C.1), we have $P_{-1} \cdot S \equiv 0 \mod \bar{\mathcal{I}} + x\mathcal{D}'$, which gives a relation among standard monomials modulo $\bar{\mathcal{I}} + x\mathcal{D}'$.

We can obtain higher-order relations by applying ∂_x^j to (C.1) and taking the limit $x = 0$. For example, applying ∂_x in conjunction with the relation $\partial_x S \equiv P \cdot S$,

$$\partial_x S + x\partial_x^2 S \equiv (P_{-1} + O(x)) \cdot S + (P_{-1} + P_0x + O(x^2)) \cdot x^{-1}(P_{-1} + P_0x + O(x^2)) \cdot S. \quad (\text{C.2})$$

When $P_{-1} \cdot S \equiv 0$, the limit $x = 0$ yields

$$\partial_x S \equiv P_{-1} \cdot S + (P_{-1} \cdot P_0 + P_0 \cdot P_{-1}) \cdot S \equiv 0 \mod \bar{\mathcal{I}} + x\mathcal{D}'. \quad (\text{C.3})$$

Let $\mathcal{P}$ be the set of the relations coming from $P_{-1} \cdot S \equiv 0$ and higher-order obtained via the Pfaffian equation (C.1).

Algorithm C.2 (Rational restriction to $x = 0$ using the residue matrix P_{-1}).

Input: Generators $\{f_1, \dots, f_\mu\}$ of a holonomic ideal $\mathcal{I}$ in $\mathcal{D}'$. The relations $\mathcal{P}$. Holonomic rank r of the restriction to $x = 0$ of $\mathcal{I}$. A non-negative integer γ such that $\gamma \geq \max(s_0 + 1, s_1 + 1)$.

Output: Generators of a left submodule $\mathcal{J} \cap \mathcal{D}^\gamma$ of $\mathcal{D}^\gamma$ such that $\mathcal{R}^\gamma / \mathcal{R}(\mathcal{J} \cap \mathcal{D}^\gamma) = \mathcal{R} \otimes_{\mathcal{D}} \mathcal{D}' / (\bar{\mathcal{I}} + x\mathcal{D}')$.

1: $w = (1, 0)$

2: $k = \gamma - 1$

3: **repeat**
 4: $\mathcal{J} = \mathcal{D} \cdot \left\{ v_k \left(: \partial_x^j f_i : \Big|_{x=0} \right) \mid \text{ord}_{(-w,w)}(\partial_x^j f_i) \leq m, 1 \leq i \leq N, j \in \mathbb{N}_0 \right\}$
 $\quad \cup \left\{ v_k(p) \mid p \in \mathcal{P}, \text{ord}_{(-w,w)}(p) \leq k \right\}$
 5: $k := k + 1$
 6: **until** $\text{rank}(\mathcal{D}^\gamma / \mathcal{J} \cap \mathcal{D}^\gamma) = r$
 7: **return** $\mathcal{J}$

Theorem C.3. *Algorithm C.2 stops and gives the correct answer.*

Proof. Rows of the Pfaffian system belong to the Weyl closure $\bar{\mathcal{I}} = (\mathcal{R}'\mathcal{I}) \cap \mathcal{D}'$. Then the elements of $\mathcal{P}$ belong to $\bar{\mathcal{I}} + x\mathcal{D}'$, and this gives an expression for $\mathcal{R} \otimes_{\mathcal{D}} (\mathcal{D}' / (\mathcal{I} + x\mathcal{D}'))$ by Proposition 4.1. Combining this fact with the proof of Theorem 4.5, we reach the desired conclusion. $\square$

Example C.4. To illustrate the difference between Algorithm 4.4 and Algorithm C.2, we consider the Feynman integral for the one-loop triangle with one internal mass (for comparison, see the example from section 5.5.1). The Lee-Pomeransky polynomial takes the form

$$g(z; x) = z_1 x_1 + z_2 x_2 + z_3 x_3 + z_4 x_1^2 + z_5 x_1 x_2 + z_6 x_1 x_3 \quad (\text{C.4})$$

$$z_1 = z_2 = z_3 = z_4 = z_5 = 1 \quad \text{and} \quad z_6 \text{ generic.} \quad (\text{C.5})$$

The associated GKZ system has the data

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \cdot & \cdot & 2 & 1 & 1 \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 \end{bmatrix}, \quad \beta = [b_1, b_2, b_3, b_4]. \quad (\text{C.6})$$

The GKZ case with generic $z_1, \dots, z_6$ is of holonomic rank 3. The Feynman integral, subject to (C.5), has rank 2. We proceed to compute this drop in rank via a $\mathcal{D}$ -module restriction.

To begin, choose the simplex $\sigma = [1, 2, 3, 4]$. The remaining variables are z_5 and z_6 , and we seek the restriction $z_5 = 1$. Shifting $z_5 \rightarrow z_5 + 1$, we can alternatively restrict $z_5 = 0$. The generators of the GKZ system then read (see [1, appendix A] for how to rewrite generators given a simplex σ)

$$\begin{aligned}
 f_1 &= (z_5 - z_6 + 1) \underline{\partial_5 \partial_6} + b_4 \partial_5 - b_3 \partial_6 \\
 f_2 &= [(-z_6 + 1)z_5 - z_6 + 1] \underline{\partial_5 \partial_6} + z_6(1 - z_6) \partial_6^2 + (b_4 z_5 + b_4) \partial_5 \\
 &\quad + [(-b_1 + b_2 + b_3 + 2b_4 - 1)z_6 + 2b_1 - b_2 - 2b_3 - 2b_4 + 1] \partial_6 \\
 &\quad - b_4(b_3 - b_1 + b_2 + b_4) \\
 f_3 &= -z_5(z_5 + 1) \underline{\partial_5^2} - z_6 z_5 \partial_5 \partial_6 + [(-b_1 + b_2 + 2b_3 + b_4 - 1)z_5 + b_1 - b_4] \partial_5 + b_3 z_6 \partial_6 \\
 &\quad - b_3(b_3 - b_1 + b_2 + b_4)
 \end{aligned}$$

where we underlined the leading terms according to the weight $w = (1, 0)$. By the Macaulay matrix method, we find Pfaffian matrices P_5 and P_6 written in the basis $S = [\partial_5, \partial_6, 1]^T$.

These matrices have denominators $z_5(z_5+1)(z_6-z_5-1)$ and $z_6(z_6-1)(z_6-z_5-1)$ respectively. Since the Pfaffian system is logarithmic along $z_5 = 0$, we may apply Algorithm C.2 with $x = z_5$ and $y = z_6$. The residue matrix at $x = 0$ is

$$P_{-1} = \begin{bmatrix} b_1 - b_4 & b_3 y & b_3(b_1 - b_2 - b_3 - b_4) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}. \quad (\text{C.7})$$

In (x, y) variables, the basis reads $S = [\partial_y, \partial_x, 1]^T$. The only non-zero element of $P_{-1} \cdot S$ is thus

$$(b_1 - b_4)\partial_x + b_3 y \partial_y + b_3(b_1 - b_2 - b_3 - b_4). \quad (\text{C.8})$$

Next we execute Algorithm C.2 with $k = 1$, $\gamma = 2$ and the target holonomic rank $r = 2$. The operator (C.8) is included in $\mathcal{J}$. The POT Gröbner basis for $\mathcal{J}$ in $\mathcal{D}^2$ is the set

$$\begin{aligned} & \{[0, y(1-y)\partial_y^2 + ((-b_1 + b_2 + b_3 + 2b_4 - 1)y + 2b_1 - b_2 - b_3 - 2b_4 + 1)\partial_y \\ & \quad + b_4(b_1 - b_2 - b_3 - b_4)]^T, \\ & [b_1 - b_4, b_3 y \partial_y + b_3(b_1 - b_2 - b_3 - b_4)]^T \}. \end{aligned}$$

This is a rank 2 system when $b_1 - b_4 \neq 0$, wherefore the terminating condition for the algorithm is met.

Incidentally, the b -function (the indicial polynomial) along $x = 0$ is $s(s - b_1 + b_4 - 1)$. When $s = 1$, it is equal to $-(b_1 - b_4)$, which appears in the second element of the Gröbner basis. ■

C.2 Pfaffian-level restriction

Here we collect technical remarks on the computational side of the restriction method based on equations (3.13), (3.15) of section 3.2.

C.2.1 Details on the M -matrix

We start with a matrix-based argument that motivates the bottom-left $\mathbb{O}$ -block in the gauge-transformed Pfaffian matrix (3.15). Following the row-decomposition of (3.13), let us focus on the first r' columns of M^{-1} :

$$M^{-1} = \left[\begin{array}{c|c} \text{red block} & \text{purple block} \end{array} \right] = \left[\begin{array}{c|c} \underbrace{\quad}_{r'} & \underbrace{\quad}_{r-r'} \end{array} \right] \Bigg\}^r. \quad (\text{C.9})$$

The column block V satisfies the two identities

$$\mathbb{1}_{r \times r} = M \cdot M^{-1} = \begin{bmatrix} B \\ R \end{bmatrix} \cdot \left[\begin{array}{c|c} \text{red} & \text{purple} \end{array} \right] = \begin{bmatrix} B \cdot V & \star \\ R \cdot V & \star \end{bmatrix} \implies \begin{cases} B \cdot V = \mathbb{1}_{r' \times r'} , \\ R \cdot V = \mathbb{O}_{(r-r') \times r'} . \end{cases} \quad (\text{C.10})$$

C.2.3 Inversion of the M -matrix

To find the restricted Pfaffian Q_i in (3.15) it is enough to only partially invert the M -matrix (3.13). Indeed, using the column decomposition (C.9) to find Q_i we only need to determine the V -block from the system (C.10) in two steps. First find (any basis of) the right nullspace of R

$$\tilde{V} := \text{NullSpace}[R], \quad (\text{C.17})$$

and then rotate it from the right to fix the normalization

$$V := \tilde{V} \cdot (B \cdot \tilde{V})^{-1}. \quad (\text{C.18})$$

This second step will require inversion of a smaller $r' \times r'$ matrix $(B \cdot \tilde{V})$, which is easier than inverting the full $r \times r$ matrix M .

With the column block (C.9), the Pfaffian Q_i can be extracted from the full restriction gauge transformation as follows:

$$(\partial_i B + B \cdot P_i(0, z')) \cdot V = Q_i. \quad (\text{C.19})$$

D Proofs

D.1 Theorem 2.1

Setting $\mathcal{O}(X) := \mathbb{C}[z_1, \dots, z_m, x_1^{\pm 1}, \dots, x_n^{\pm 1}, \frac{1}{g}]$, we define an action of $\mathcal{D}_Y$ on $f = f(z, x) \in \mathcal{O}(X)$ by

$$\frac{\partial}{\partial z_i} \bullet f = \frac{\partial f}{\partial z_i} + \beta_0 \left(\frac{1}{g(z; x)} \frac{\partial g(z; x)}{\partial z_i} \right) f, \quad (\text{D.1})$$

$$\frac{\partial}{\partial x_i} \bullet f = \frac{\partial f}{\partial x_i} + \beta_0 \left(\frac{1}{g(z; x)} \frac{\partial g(z; x)}{\partial x_i} \right) f - \beta_i \frac{f}{x_i}. \quad (\text{D.2})$$

The symbol $\mathcal{O}(X) g^{\beta_0} x_1^{-\beta_1} \dots x_n^{-\beta_n}$ denotes the left $\mathcal{D}_Y$ -module $\mathcal{O}(X)$ endowed with this action. By definition, $M_A(\beta; Y)$ is the 0-th cohomology group of the direct image $\mathcal{D}_Y$ -module $\int_{\pi} \mathcal{O}(X) g^{\beta_0} x_1^{-\beta_1} \dots x_n^{-\beta_n}$. The first claim is an immediate consequence of the preservation of regular holonomicity [22, Theorem 6.1.5].

Let us prove the second claim. Since $M_A(\beta; Y)$ is holonomic, it is an integrable connection on a generic open subset of Y . We fix a generic $z \in Y$. Then the restriction of $M_A(\beta; Y)$ to z is the n -th twisted cohomology group on $\pi^{-1}(z)$ with its twist given by the integrand of (2.1). In view of [40, Theorem 2.6], its dimension is given by $|\chi(\pi^{-1}(z))|$.

D.2 Theorem 2.6

Let $\varphi \in \text{End}_{\mathcal{R}}(\mathcal{M})$. Writing $\bar{\mathcal{K}}$ for the algebraic closure of $\mathcal{K}$, the action of $\mathcal{R}$ on $\mathcal{K}$ extends to that on $\bar{\mathcal{K}}$. We can regard φ as an element of $\text{End}_{\bar{\mathcal{K}} \otimes_{\mathcal{K}} \mathcal{R}}(\bar{\mathcal{M}})$ where we set $\bar{\mathcal{M}} := \bar{\mathcal{K}} \otimes_{\mathcal{K}} \mathcal{M}$. We first prove that any eigenvalue of φ is in $\mathbb{C}$. Let us take an eigenvector $v \in \bar{\mathcal{M}}$ of φ . Since $\bar{\mathcal{M}}$ is finite-dimensional over $\bar{\mathcal{K}}$, there exists a minimal k such that

$\text{Span}_{\overline{\mathcal{K}}}\{v, \partial_1 v, \dots, \partial_1^k v\} = \text{Span}_{\overline{\mathcal{K}}}\{v, \partial_1 v, \dots\}$. It follows that there exist $a_0, a_1, \dots, a_k \in \overline{\mathcal{K}}$ such that $\partial_1^k v = \sum_{j=0}^k a_j \partial_1^j v$. It is straightforward to see that

$$\varphi(\partial_1^k v) = \sum_{j=0}^k \binom{k}{j} (\partial_1^{k-j} \alpha) \partial_1^j v = \sum_{j=0}^{k-1} \binom{k}{j} (\partial_1^{k-j} \alpha) \partial_1^j v + \sum_{j=0}^{k-1} a_j \alpha (\partial_1^j v) \quad (\text{D.3})$$

holds true. On the other hand, the fact that φ commutes with the action of ∂_1 implies the following identity:

$$\varphi(\partial_1^k v) = \sum_{j=0}^{k-1} a_j \varphi(\partial_1^j v) = \sum_{j=0}^{k-1} a_j \sum_{\ell=0}^j \binom{j}{\ell} (\partial_1^{j-\ell} \alpha) (\partial_1^\ell v). \quad (\text{D.4})$$

Comparing the coefficients of $\partial_1^{k-1} v$ in (D.3) and (D.4), we obtain that $\partial_1 \alpha = 0$. Similarly, we can prove that $\partial_i \alpha = 0$ for all $i = 1, \dots, n$. Since such a function α must be a constant function, it must belong to $\mathbb{C}$. Now, suppose that $\dim_{\mathbb{C}} \text{End}_{\mathcal{R}}(\mathcal{M}) \geq 2$ and take a morphism $\varphi \in \text{End}_{\mathcal{R}}(\mathcal{M})$ which is linearly independent from $\text{id}_{\mathcal{M}}$ over $\mathbb{C}$. For any eigenvalue $\alpha \in \mathbb{C}$ of φ , $\alpha \cdot \text{id}_{\mathcal{M}} - \varphi$ has a non-trivial kernel, which is a non-trivial $\mathcal{R}$ -submodule of $\mathcal{M}$. This is a contradiction.

D.3 Theorem 2.7

We set $\mathcal{M} := \mathcal{R} \otimes_{\mathcal{D}_Y} M_A(\beta; Y)$. Let Y_0 be a non-empty Zariski open subset of Y . Any $\mathcal{D}_{Y_0}$ -submodule N of $M_A(\beta; Y)|_{Y_0}$ gives rise to an $\mathcal{R}$ -submodule $\mathcal{M}$. Conversely, any $\mathcal{R}$ -submodule $\mathcal{N}$ of $\mathcal{M}$ is of the form $\mathcal{N} = \mathcal{R} \otimes_{\mathcal{D}_{Y_0}} N$ for some non-empty Zariski open subset Y_0 of Y and a $\mathcal{D}_{Y_0}$ -submodule N of $M_A(\beta; Y)|_{Y_0}$. Shrinking Y_0 appropriately if necessary, we may assume that both $M_A(\beta; Y)|_{Y_0}$ and N are free as $\mathcal{O}_{Y_0}$ -modules, i.e., they are integrable connections. Now, the sheaf $\mathcal{L}$ of flat sections of $M_A(\beta; Y)|_{Y_0}$ admits an invariant Hermitian form. This can be shown by adopting the argument of [100, p. 559–560] since our exponent β is real and generic. Then, the sheaf $\mathcal{L}_N$ of flat sections of N admits an orthogonal complement $\mathcal{L}_{N'}$ which is the sheaf of flat sections of a $\mathcal{D}_{Y_0}$ -submodule N' of $M_A(\beta; Y)|_{Y_0}$ in view of [23, Théorème 5.9]. The direct sum decomposition $M_A(\beta; Y)|_{Y_0} = N \oplus N'$ gives rise to the corresponding decomposition $\mathcal{M} = \mathcal{N} \oplus \mathcal{N}'$, which proves that $\mathcal{M}$ is a semi-simple $\mathcal{R}$ -module.

D.4 Theorem 2.8

It is enough to show that $\dim_{\mathbb{C}} \text{End}_{\mathcal{R}}(\mathcal{M}) > 1$ if $\mathcal{M}$ is not simple. By the assumption that $\mathcal{M}$ is semi-simple, there exist a pair of $\mathcal{R}$ -submodules $\mathcal{N}, \mathcal{N}'$ such that $\dim_{\mathcal{K}} \mathcal{N}, \dim_{\mathcal{K}} \mathcal{N}' \geq 1$ and $\mathcal{M} = \mathcal{N} \oplus \mathcal{N}'$. The identity morphism $\text{id}_{\mathcal{N}} : \mathcal{N} \rightarrow \mathcal{N}$ extends uniquely to an $\mathcal{R}$ -endomorphism $\varphi : \mathcal{M} \rightarrow \mathcal{M}$ by the requirement $\varphi(\mathcal{N}') = 0$. In the same way, the identity morphism $\text{id}_{\mathcal{N}'} : \mathcal{N}' \rightarrow \mathcal{N}'$ extends to an $\mathcal{R}$ -endomorphism $\varphi' : \mathcal{M} \rightarrow \mathcal{M}$. It is obvious from the construction that φ and φ' are linearly independent, which shows the desired inequality $\dim_{\mathbb{C}} \text{End}_{\mathcal{R}}(\mathcal{M}) > 1$.

D.5 Proposition 3.4

In order to simplify the discussion, we only prove the case when $m = 1$, i.e. when Y' is a single point and $Y \simeq \mathbb{A}^1$. The case of $m \geq 2$ can be proved in the same way.

We identify $\mathcal{D}_{Y'}$ with the field of complex numbers $\mathbb{C}$. It is easy to see that the following morphism is well-defined:

$$\varphi : \mathcal{M}' = \mathbb{C}^r / \mathcal{N}' \ni [v] \mapsto [v] \in \mathcal{D}_Y^r / (z_1 \mathcal{D}_Y^r + \mathcal{N}) = \iota^* \mathcal{M}, \quad (\text{D.5})$$

where the submodule $\mathcal{N}$ is defined in Proposition 3.3. Let us take a constant column vector $v \in \mathbb{C}^r$ and consider an element of the form $v \partial_1^n \in \mathcal{D}_Y^r$ for integer $n \geq 1$. A straightforward computation shows the identity $v \partial_1^n \equiv (\mathbb{1}_r n - P_1(0)^T)^{-1} \cdot \sum_{k=1}^n \binom{n}{k} (\partial_1^k P_1^T|_{z_1=0}) \cdot v \partial_1^{n-k}$ in $\iota^* \mathcal{M}$, which proves that φ is surjective. Finally, we prove that φ is injective. Let us assume that $\varphi([v]) = [0]$. Since the constant vector v does not contain ∂_1 , there exist an element $p \in \mathcal{D}_Y^r$ and a column vector $w \in \mathbb{C}^r$ such that $v = z_1 p + (\mathbb{1}_r z_1 \partial_1 - P_1(z_1)^T) \cdot w$. Substituting $z_1 = 0$, we obtain $v = -P_1(0)^T \cdot w$, which proves that $[v] = [0]$.

D.6 Theorem 3.6

We prove the case $m = 2$. The proof can easily be generalized to the case of $m > 2$.

Let $I(z) = \sum_{k=0}^{\infty} I_k(z_2) z_1^k$ be a vector of holomorphic functions and let P_1, P_2 be the $r \times r$ Pfaffian matrices given by (3.7). We expand P_1 as shown in (3.17) and P_2 as

$$P_2(z) = \sum_{k=0}^{\infty} P_{2,k}(z_2) z_1^k. \quad (\text{D.6})$$

Then the integrability condition (2.17) is equivalent to

$$\begin{aligned} \partial_2 P_{1,-1} &= [P_{2,0}, P_{1,-1}], \\ \partial_2 P_{1,k} - (k+1)P_{2,k+1} &= \sum_{\substack{i+j=k \\ i \geq 0, j \geq -1}} [P_{2,i}, P_{1,j}]. \end{aligned} \quad (\text{D.7})$$

The fact that $I(z_1, z_2)$ is a solution of (2.18) is equivalent to $\{I_k(z_2)\}_k$ being subject to the following relations:

$$((k+1) - P_{1,-1}) \cdot I_{k+1} = \sum_{\substack{i+j=k \\ i, j \geq 0}} P_{1,i} \cdot I_j, \quad \text{for } k \geq -1, \quad (\text{D.8})$$

$$\partial_2 I_k = \sum_{\substack{i+j=k \\ i, j \geq 0}} P_{2,i} \cdot I_j, \quad \text{for } k \geq 0. \quad (\text{D.9})$$

If $I(z_1, z_2)$ is a solution of (2.18), it is clear that $I_0(z_2)$ is a solution to the restricted Pfaffian equation (3.18). Conversely, let $I_0(z_2)$ be a solution of the restricted Pfaffian equation (3.18). We can uniquely extend $I_0(z_2)$ to the solution $\sum_{k=0}^{\infty} I_k(z_2) z_1^k$ by requiring that the coefficients $I_k(z_2)$ solve the linear system (D.8), that is:

$$((k+1) - P_{1,-1}(z_2)) \cdot I_{k+1}(z_2) = \sum_{\substack{i+j=k \\ i, j \geq 0}} P_{1,i}(z_2) \cdot I_j(z_2), \quad (\text{D.10})$$

with z_2 kept as a parameter. By assumption, the residue matrix $P_{1,-1}(z_2)$ is non-resonant, so the only integer eigenvalue it has is 0, which means that the matrix on the l.h.s. of (D.10) is invertible and there is a unique $I_{k+1}(z_2)$ that solves it for each $k \geq -1$. So by construction of (D.10), the condition (D.8) is satisfied and it is left to show that the other condition (D.9) is satisfied as well. To this end, consider the identity

$$((k+1) - P_{1,-1}(z_2)) \cdot \partial_2 I_{k+1}(z_2) = ((k+1) - P_{1,-1}(z_2)) \cdot \sum_{\substack{i+j=k+1 \\ i, j \geq 0}} P_{2,i} \cdot I_j(z_2), \quad (\text{D.11})$$

which holds by induction in k from acting with ∂_2 on (D.10) and the integrability condition (D.7). Due to invertibility of the common matrix factor here, this equation implies (D.9).

Finally, we prove the convergence of $I(z_1, z_2) = \sum_{k=0}^{\infty} I_k(z_2) z_1^k$. Let $f(t; z_2) = \text{Det}[t - P_{1,-1}(z_2)]$ be the characteristic polynomial of $P_{1,-1}$. The leading coefficient of $f(t; z_2)$ in t is 1, and thus there exists a constant $C > 0$ such that $f(t; z_2) \geq C t^r$ for any $t > 0$ sufficiently large. To be more precise, z_2 belongs to a small region in which we have a uniform bound on the coefficients of $f(t; z_2)$. Therefore, the matrix norm of the cofactor matrix of $(t - P_{1,-1}(z_2))$ can be estimated by $C' t^{r-1}$ for some constant $C' > 0$ and for any $t > 0$ sufficiently large. Taking into account Clamer's formula for matrix inverse, we obtain

$$\|(t - P_{1,-1}(z_2))^{-1}\| \leq C_1 t^{-1} \quad (\text{D.12})$$

for any $t > 0$ sufficiently large. Here $\|\bullet\|$ stands for a matrix norm. On the other hand, convergence of the expansion (3.17) and Cauchy's inequality implies that there are positive constants C_2 and s such that

$$\|P_{1,k}(z_2)\| \leq C_2 s^k, \quad (\text{D.13})$$

for any $k \geq 0$. We claim that the inequality

$$\|I_k(z_2)\| \leq R^k \|I_0(z_2)\| \quad (\text{D.14})$$

is true with $R = \max(C_1 C_2, s)$. Indeed, if the estimate (D.14) is true up to k , then

$$\|I_{k+1}(z_2)\| \leq \|((k+1) - P_{1,-1}(z_2))^{-1}\| \sum_{i+j=k} \|P_i(z_2)\| \|I_j(z_2)\| \quad (\text{D.15})$$

$$\leq C_1 C_2 \frac{1}{k+1} \sum_{i+j=k} s^i R^j \|I_0(z_2)\| \quad (\text{D.16})$$

$$\leq R^{k+1} \|I_0(z_2)\|. \quad (\text{D.17})$$

This proves the convergence of $I(z_1, z_2)$. Uniqueness of I is clear from (D.8) and the assumption of the theorem.

D.7 Theorem 3.7

We use the same notation as in appendix D.6. Suppose we construct a matrix $G(z_1, z_2) \in GL(r; \mathbb{C}\{\{z_1, z_2\}\})$ such that the gauge transformation $I(z_1, z_2) = G(z_1, z_2) \cdot J(z_1, z_2)$ turns

the system (2.18) into

$$\begin{aligned}\partial_1 J(z_1, z_2) &= \frac{\Gamma}{z_1} J(z_1, z_2), \\ \partial_2 J(z_1, z_2) &= 0,\end{aligned}\tag{D.18}$$

for some constant matrix Γ . This system can be solved by the matrix exponential: $J(z_1, z_2) = z_1^\Gamma$.

The transformation matrix G is also subject to its own Pfaffian system:

$$\begin{aligned}\partial_1 G &= T_{P_1, \Gamma/z_1}(G), \\ \partial_2 G &= P_2 \cdot G.\end{aligned}\tag{D.19}$$

Here $T_{A,B} : M(r; \mathbb{C}) \rightarrow M(r; \mathbb{C})$ is a linear map defined by $T_{A,B}(G) := A \cdot G - G \cdot B$. Note that this is a Pfaffian system of the rank r^2 .

Lemma D.1. *The system (D.19) is integrable.*

Proof. For a matrix $A \in M(r; \mathbb{C})$, let us denote by $T_A : M(r; \mathbb{C}) \rightarrow M(r; \mathbb{C})$ the linear map given by the left multiplication with A , explicitly $T_A(G) := A \cdot G$, so that $T_A(G) = T_{A,0}(G)$. We want to prove the following integrability condition:

$$\partial_1 T_{P_2} - \partial_2 T_{P_1, \Gamma/z_1} = [T_{P_1, \Gamma/z_1}, T_{P_2}].\tag{D.20}$$

To show this, we use identities for the three building blocks appearing above, namely for the two derivatives $\partial_1 T_{P_2} = T_{\partial_1 P_2}$ and $\partial_2 T_{P_1, \Gamma/z_1} = T_{\partial_2 P_1}$, and for the commutator $[T_{P_1, \Gamma/z_1}, T_{P_2}] = T_{[P_1, P_2]}$. Combining them and taking into account the integrability condition (2.18) proves the integrability of the system (D.19). $\square$

The Pfaffian system (D.19) is logarithmic along $\{z_1 = 0\}$. Theorem 3.6 shows that there exists a convergent solution G to (D.19) with $G(0,0) = \mathbb{1}_r$, since $\mathbb{1}_r \in \text{NullSpace}[\text{Res}(T_{P_1, \Gamma/z_1})|_{z_2=0}]$ is a valid boundary condition. Therefore, any solution to the system (2.18) can be written in the form

$$I(z_1, z_2) = G(z_1, z_2) \cdot z_1^\Gamma \cdot y = \sum_{n=0}^{r-1} G(z_1, z_2) \cdot \Gamma^n \cdot y \log^n z_1\tag{D.21}$$

for some $y \in \mathbb{C}^r$, which is fixed by the boundary condition. This proves that the boundary value map $I(z_1, z_2) \mapsto I_0(0, z_2)$ of Theorem 3.7 is injective. Let us prove that the boundary value map is surjective. We observe that

$$P_{1,-1}(z_2) \cdot G(0, z_2) = G(0, z_2) \cdot \Gamma\tag{D.22}$$

is true by (D.19). This identity shows that any holomorphic solution $I(z_2)$ of (3.18) can be written as $G(0, z_2) \cdot y$, where $y \in \mathbb{C}^r$ is an element of the generalized 0-eigenspace of Γ . So we conclude that $I(z_1, z_2) = G(z_1, z_2) \cdot z_1^\Gamma \cdot y$ is the desired solution of (2.18) with the boundary value $G(0, z_2) \cdot y$.

Remark D.2. Finally let us comment on how the Pfaffian system (D.19) can be constructed in practice. If we choose to flatten the G matrix into a vector $\vec{G}$ of length r^2 as

$$\vec{G} := \text{Flatten}[G] \quad (\text{D.23})$$

in MATHEMATICA notation, then the matrix representation of the $T_{A,B}$ map (which we denote by the same symbol) becomes³⁶

$$T_{A,B} = \text{Flatten}[\text{Outer}[\text{Times}, A, \mathbb{1}] - \text{Outer}[\text{Times}, \mathbb{1}, B^T], \{\{1, 3\}, \{2, 4\}\}]. \quad (\text{D.24})$$

For example, let $A = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$ and $B = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$ be some 2×2 matrices, then the 4×4 matrix $T_{A,B}$ looks like the following difference:

$$T_{\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}, \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} - \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}. \quad (\text{D.25})$$

So we conclude, that the rank r^2 Pfaffian system (D.19) can be obtained from the original system (2.18) by simple matrix operations, and, as can be seen from (D.25), these two Pfaffian systems share many common properties, such sparsity and block-triangularity, which is important for practical computations.

D.8 Proposition 4.1

The map $\mathcal{D}_Y/\mathcal{I} \ni \ell \xrightarrow{\varphi} \ell \in \mathcal{D}_Y/((\mathcal{R}_Y\mathcal{I}) \cap \mathcal{D}_Y)$ is surjective. Then the induced map $\varphi : \mathcal{D}_Y/(\mathcal{I} + \sum_{j=1}^{m'} z_j \mathcal{D}_Y) \longrightarrow \mathcal{D}_Y/((\mathcal{R}_Y\mathcal{I}) \cap \mathcal{D}_Y + \sum_{j=1}^{m'} z_j \mathcal{D}_Y)$ is surjective, and the induced map for solution spaces

$$\text{Hom}_{\mathcal{D}_Y}(\mathcal{D}_Y/((\mathcal{R}_Y\mathcal{I}) \cap \mathcal{D}_Y), \mathcal{O}(U)) \xrightarrow{\psi} \text{Hom}_{\mathcal{D}_Y}(\mathcal{D}_Y/\mathcal{I}, \mathcal{O}(U))$$

is injective for a sufficiently small open ball U . When U is a simply connected domain outside of the singular locus of $\mathcal{D}_Y/\mathcal{I}$, holomorphic solutions on U of the two systems $\mathcal{D}_Y/\mathcal{I}$ and $\mathcal{D}_Y/((\mathcal{R}_Y\mathcal{I}) \cap \mathcal{D}_Y)$ agree. Therefore, the injective map ψ is an isomorphism for any sufficiently small open ball U by the identity theorem of complex function theory. Let U be a small open ball with a center on $Y' = \{z \mid z_1 = \dots = z_{m'} = 0\}$ and suppose that the center is outside of the singular locus of the restriction $\mathcal{D}_Y/(\mathcal{I} + \sum_{j=1}^{m'} z_j \mathcal{D}_Y)$. Then we have $\text{Hom}_{\mathcal{D}_Y}(\mathcal{D}_Y/\mathcal{I}, \mathcal{O}(U))|_V = \text{Hom}_{\mathcal{D}_{Y'}}(\mathcal{D}_Y/(\mathcal{I} + \sum_{j=1}^{m'} z_j \mathcal{D}_Y), \mathcal{O}(U \cap V))$. It implies that the holonomic ranks of $\mathcal{D}_Y/(\mathcal{I} + \sum_{j=1}^{m'} z_j \mathcal{D}_Y)$ and $\mathcal{D}_Y/((\mathcal{R}_Y\mathcal{I}) \cap \mathcal{D}_Y + \sum_{j=1}^{m'} z_j \mathcal{D}_Y)$ agree. The left $\mathcal{R}_{Y'}$ morphism induced by φ from $\mathcal{R}_{Y'} \otimes_{\mathcal{D}_{Y'}} \mathcal{D}_Y/(\mathcal{I} + \sum_{j=1}^{m'} z_j \mathcal{D}_Y)$ to $\mathcal{R}_{Y'} \otimes_{\mathcal{D}_{Y'}} \mathcal{D}_Y/((\mathcal{R}_Y\mathcal{I}) \cap \mathcal{D}_Y + \sum_{j=1}^{m'} z_j \mathcal{D}_Y)$ is surjective, and the vector space dimensions of these two agree over the field $\mathbb{C}(z_{m'+1}, \dots, z_m)$. Thus, φ is one-to-one.

³⁶In tensorial notation: $T_{A,B} = A \otimes \mathbb{1} - \mathbb{1} \otimes B^T$.

D.9 Theorem 4.5

Let r be the rank of the restriction. Recall that the $(-1, 0, 1, 0)$ -order of the monomial $\ell = x^{\alpha_1} y^{\alpha_2} \partial_x^{\beta_1} \partial_y^{\beta_2}$ is $-\alpha_1 + \beta_1$. We denote it by $\text{ord}(\ell) = -\alpha_1 + \beta_1$, thereby omitting the suffix $(-w, w)$.

Since the input is holonomic, there exists an operator of the form

$$B = b(\theta_x) + xc(x, y, \theta_x, \partial_y), \quad \theta_x = x\partial_x, \quad b(u) \in \mathbb{C}[s], \quad c \in \mathcal{D}' \quad (\text{D.26})$$

in $\mathcal{I}$ (see, e.g., [68, Theorem 5.1.2]). Note that $\text{ord}(b(\theta_x)) = 0$ and $\text{ord}(xc(x, y, \theta_x, \partial_y)) < 0$. We have

$$\begin{aligned} \partial_x^k B &= b(\theta_x + k) \partial_x^k + \partial_x^k xc(x, y, \theta_x, \partial_y) \\ &= b(k) \partial_x^k + (b(\theta_x + k) - b(k)) \partial_x^k + \partial_x^k xc(x, y, \theta_x, \partial_y). \end{aligned} \quad (\text{D.27})$$

We have $(b(\theta_x + k) - b(k))|_{x=0} = 0$, and $\text{ord}(\partial_x^k xc(y, \theta_x, \partial_y)) < k$. Then the equation (D.27) implies that $b(k) \partial_x^k = 0$ modulo $x\mathcal{D}' + \mathcal{I} + F_{k-1}$ where F_{k-1} is defined in (4.5). This property, which was firstly utilized for the restriction algorithm in [54], is a key in proving the termination of the algorithm (see also [68, Lemma 5.1.8]).

The operators f_i , $i = 1, \dots, N$ are generators of the left ideal $\mathcal{I}$. Suppose that the operator B is expressed as

$$B = \sum_{i=1}^N \sum_{j=0}^{m_i} d_{ij}(x, y, \partial_y) \partial_x^j f_i.$$

Let u_0 be the maximal non-negative integral root of $b(u) = 0$. We denote by $e_0, e_1, e_2, \dots$ the standard basis of $\mathcal{D}^{k+1}$. In other words, e_i stands for ∂_x^i , and we use this identification in this proof. Note that the last element of $v_k(\ell)$ is the component for e_0 , the second to last element of $v_k(\ell)$ is the component for e_1 , and so on. When $k \geq \max(u_0 + 1, \max(m_i + \text{ord}(f_i))) =: k_0$, it follows from (D.27) that the left module generated by

$$v_k(: \partial_x^j f_i : |_{x=0}), \quad \text{ord}(\partial_x^j f_i) \leq k, \quad i = 1, \dots, N, \quad j \in \mathbb{N}_0 \quad (\text{D.28})$$

in $\mathcal{D}^{k+1}$ contains an element of the form

$$b(k') e_{k'} + \sum_{i=0}^{k'-1} d_i(y, \partial_y) e_i, \quad k' > u_0, \quad b(k') \neq 0. \quad (\text{D.29})$$

Let G_k be a Gröbner basis by the POT order of

$$\left\{ v_k(: \partial_x^j f_i : |_{x=0}) \mid 1 \leq i \leq N, \text{ord}(\partial_x^j f_i) \leq k, j \in \mathbb{N}_0 \right\} \quad (\text{D.30})$$

in $\mathcal{D}^{k+1}$. The element (D.29) is contained in G_k when $k \geq k_0$, which means that there is no standard monomial of the form $e_{k'}$, $k' > u_0$ in the Gröbner basis. We note that the submodules $\mathcal{D}G_k \cap \mathcal{D}^\gamma$ represent an increasing set, i.e. we have $\mathcal{D}G_k \cap \mathcal{D}^\gamma \subseteq \mathcal{D}G_{k+1} \cap \mathcal{D}^\gamma$. For a sufficiently large k , the submodule $\mathcal{D}G_k \cap \mathcal{D}^\gamma$ contains the denominator module of (4.7)

because the $v_{k+1}(\partial x^\beta g_i : |_{x=1})$'s of g_i in (4.7) are contained in $\mathcal{D}G_k$. Note that e_i for $u_0 < i \leq \gamma$ do not appear in the standard monomials.

Recall the restriction algorithm of $\mathcal{D}$ -modules (see section 4.1 and [68, section 5.2]). The algorithm constructs the restriction by the $(-w, w)$ -Gröbner basis G . Elements for which the $(-w, w)$ -order is more than u_0 are not used to construct the restriction. Our procedure does not have this trashing procedure, so our module $\mathcal{D}G_k$ might contain these trashed elements. This is why we need to take γ larger than u_1 (see also Example 4.8).

Consider two left $\mathcal{R}$ -submodules $\mathcal{R}J$ and $\mathcal{R}J'$ of $\mathcal{R}^\gamma$ such that $\mathcal{R}J \subset \mathcal{R}J'$. If $\dim_{\mathbb{C}(y)} \mathcal{R}^\gamma/(\mathcal{R}J) = \dim_{\mathbb{C}(y)} \mathcal{R}^\gamma/(\mathcal{R}J')$, then these two $\mathcal{R}$ modules are isomorphic. Thus, we obtain the conclusion. $\square$

As final remark, note that the identity (D.27) is the key to proving the existence of γ .

D.10 Lemma 4.9

Consider the holomorphic solution sheaf $\text{Hom}_{\mathcal{D}_{Y'}}(\mathcal{N}, \mathcal{O}^{an})$ on Y' . The germs are finite-dimensional vector spaces over $\mathbb{C}$; let r be the their maximal dimension. Let W' be the maximal subset of Y' such that the dimension of the vector space is r . Since the solution sheaf is a perverse sheaf [22, section 4.6], the codimension of W' in Y' is 0. Let $h_1, \dots, h_r$ be linearly independent holomorphic solutions on W' . Consider the matrix $H = (s_i \bullet h_j)$ under the assumption $0 \in W'$. Since $S = \{s_i\}$ is the basis of $\mathcal{D}_Y/(\mathcal{I} + \sum_{i=1}^m z_i \mathcal{D}_Y)$, the determinant $|H|$ is not identically equal to 0 [68, Proposition 5.2.14]. Let W be the non-zero locus of $|H|$ in W' . We may suppose that 0 is in W from the beginning. Let $\partial_{Y'}$ be an element of $\mathcal{D}_{Y'}$. Note that the restriction $\mathcal{N}$ is regular holonomic. Then $H^{-1} \partial_{Y'} \bullet H$ is a matrix consisting of regular function on W , and it gives the action of $\partial_{Y'}$ to S . $\square$

D.11 Theorem 4.14

Let $\mathbb{C}(Y')$ be the fraction field of the quotient ring $\mathbb{C}[Y'] = \mathbb{C}[z_1, \dots, z_m]/L$. We note that the following identities hold,

$$\mathcal{D}_Y/(\mathcal{I} + L\mathcal{D}_Y) \simeq \mathcal{D}_Y/L\mathcal{D}_Y \otimes_{\mathcal{D}_Y} \mathcal{D}_Y/\mathcal{I} \simeq \mathbb{C}[Y'] \otimes_{\mathbb{C}[z]} \mathcal{D}_Y/\mathcal{I}, \quad (\text{D.31})$$

and regular functions on Y' can be expressed by elements of $\mathbb{C}[Y']$. This is holonomic $\mathcal{D}_{Y'}$ -module and is isomorphic to $\mathcal{N} = \iota^* \mathcal{D}_Y/\mathcal{I}$. Since the support of $\mathcal{D}_Y/(\mathcal{I} + L\mathcal{D}_Y)$ is $Y' = V(L)$, it can also be regarded as the left $\mathcal{D}_Y$ -module on Y supported on Y' by the Kashiwara equivalence (see, e.g., [22, section 1.6]) and then $\partial_i, i = 1, \dots, m$ act on it. Since S is the basis of the connection $\mathcal{N}(W)$ on the maximal dimensional stratum W , $1 \times s_j$'s can be regarded as the basis of

$$\mathbb{C}(Y') \otimes_{\mathbb{C}[z]} \mathbb{C}[Y'] \otimes_{\mathbb{C}[z]} \mathcal{D}_Y/\mathcal{I} \simeq \mathbb{C}(Y') \otimes_{\mathbb{C}[z]} \mathcal{D}_Y/\mathcal{I}$$

as a vector space over $\mathbb{C}(Y')$. Then there exists a matrix $P_i(z) \in \mathbb{C}(Y')^{r \times r}$ such that

$$\begin{bmatrix} 1 \times \partial_i s_1 \\ \vdots \\ 1 \times \partial_i s_r \end{bmatrix} = P_i(z) \cdot \begin{bmatrix} 1 \times s_1 \\ \vdots \\ 1 \times s_r \end{bmatrix}. \quad (\text{D.32})$$

Thus, we obtain the conclusion. $\square$

D.12 Construction of a stratification by a comprehensive Gröbner basis

Here we explain how the stratum W' , or a somewhat smaller one, can be obtained by the comprehensive Gröbner basis [101]. Comprehensive Gröbner bases and an application to $\mathcal{D}$ -module restrictions are discussed in [102]. A different approach to construct strata is given in [103].

Let $\{f_1, \dots, f_\mu\}$ be a set of generators for a holonomic ideal in $\mathcal{D}_Y$. Taking n indeterminates $c_1, \dots, c_m$, we make a change of coordinates

$$z_i \rightarrow z_i + c_i, \quad i = 1, \dots, m \quad (\text{D.33})$$

in every f_j . We apply the Algorithm 4.7 using the input $f_1(c, z, \partial), \dots, f_\mu(c, z, \partial)$. We make the translation to row echelon form or compute the Gröbner basis with the POT order over the rational function field $\mathbb{C}(c_1, \dots, c_m)$. Let $g(c)$ be the least common multiple of all the denominator polynomials needed to obtain row echelon form. Consider the stratum $V(L) \cap \{z \mid g(z) \neq 0\}$. The stratum is maximal dimensional. Specializing c to a number vector in the stratum for the parametric echelon form, we obtain the echelon form to evaluate the rank. It follows from the assumption on g that the rank is constant on the stratum and maximal. This is the first step in constructing the comprehensive Gröbner basis.

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