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A Linear State Game of Advertising *à la* Vidale-Wolfe

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Abstract

We revisit the tradition of advertising models stemming from Vidale and Wolfe (1957), to illustrate the possibility of building up a game delivering a (degenerate) feedback equilibrium under open-loop rules. To this aim, we reformulate the state equation of the generic firm in such a way that its own advertising effort and the rivals' reaction to it enter the state dynamics additively. This approach amounts to envisaging situations where advertising has an essentially predatory/defensive nature, as it is not designed to modify the natural growth rate of a firm's sales or market share. This modelling strategy gives the game a state-linear structure, which also delivers an Arrovian result concerning the relationship between the aggregate advertising effort and industry structure.

JEL Codes: C73, L13, M37

Keywords: advertising; oligopoly; differential games; strong time consistency

1 Introduction

Given its inherently dynamic nature, advertising stands out as one of the most debated topics within the realm of optimal control and differential game theory since seminal works of Friedman (1958), Clemhout *et al.* (1971), and Leitmann and Schmitendorf (1978), among others. Notably, advertising efforts are typically categorized into three main types: informative, persuasive, and complementary, as outlined by Stigler and Becker (1977) and Becker and Murphy (1993). The latter category suggests that advertising plays a role in defining the overall features of a product, thus complementing it. In the applications of dynamic techniques, the focus has often been directed towards the specific state variable affected by advertising efforts or on examining the impact of advertising throughout the product life cycle.

Dynamic models addressing demand (or output) expansion are often associated with the concept of persuasive advertising. These models typically involve firms investing to increase choke prices, which become the relevant state variable, as in Cellini and Lambertini (2003,a,b) and Cellini *et al.* (2008). Alternatively, other models consider output (or sales) levels as the relevant state variables reacting to firms' advertising efforts, as exemplified by Vidale and Wolfe (1957) and its many follows ups.

Another scenario is that in which advertising aimed at enhancing goodwill, as in Nerlove and Arrow (1962), Gould (1970), Fershtman (1984) and many others, where the individual firm's profit is augmented by a state variable inflating revenues or gross profits. Additionally, in market growth or product diffusion models, such as those pioneered by Bass (1969), the relevant state variable is a firm's cumulative volume of sales. Notably, models belonging to this latter class are often characterized by a finite time horizon, as further product diffusion is inevitably constrained by the availability of competing goods from existing rivals or new entrants.

The wide literature concerning the dynamic analysis of advertising in competitive markets features several examples of games generating open-loop solutions which are Markov-perfect. This is the case, for instance, in Leitmann and Schmitendorf (1978), Feichtinger (1983) and other contributions sharing the property of state redundancy although not being state-linear, including some formulations of games with advertising for goodwill, as in Lambertini and Zaccour (2015).[1]

In this paper, we revisit the tradition of advertising models originating from Vidale and Wolfe (1957) to show that it is possible to build a game that achieves a (degenerate) feedback equilibrium under open-loop rules. In line with Vidale and Wolfe (1957), previous formalizations of the cross-effects of firms' advertising efforts and sales across the set of state variables were characterized by the lack of additive separability between states and controls, as in the competitive version of the model appearing in Deal (1979). Consequently, a fully analytical characterization of feedback equilibria through the solution of the relevant set of Hamilton-Jacobi-Bellman equations remains out of reach as one cannot formulate a plausible guess about the shape of the relevant value function. This limitation also yields an open-loop solution which is weakly time consistent.

We reformulate the state equation of the generic firm in such a way that its own advertising effort and the rivals' best replies to it enter the state

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[1] It must also be stressed that there exists a class of advertising games based upon Lanchester (1956) and Case (1979) whose formulation is neither state-linear nor linear-quadratic, which nonetheless can be analytically solved in feedback strategies using linear value functions. See Sethi and Thompson (1981), Sethi (1983) and Sorger (1989), *inter alia*.

dynamics additively. We preserve the role of the carrying capacity while obtaining an open-loop solution which is indeed a degenerate feedback one. This approach enables us to characterize situations where advertising has an essentially dual role, being both predatory and defensive at the same time, as it is not designed to directly modify the natural growth rate of a firm's sales or market share. This modeling strategy gives the game a state-linear structure, which also yields an Arrovian result concerning the relationship between the aggregate advertising effort and industry structure.

There remain to add a few relevant elements about a crucial aspect of the model and the nature of its equilibrium outcome. The results attained through the solution of this advertising game have some interesting implications concerning the potential, too often overlooked, of the open-loop equilibrium concept and the very fact that open-loop information be assumed, the common objection being that doing so leads to the replication of static equilibria. Indeed, this is not the case in general, a fortiori in games sharing the property we are referring here. More explicitly, the ensuing analysis hinges upon the possibility of specifying the setup (in this case, belonging to a well established tradition which is relevant for industrial economics and business and management alike) so as to capture a plausible and relevant real-world scenario and enjoying the property of state-linearity, yielding strong time consistency under open-loop rules. This makes room for a properly dynamic characterisation of firms' behaviour as well as the aggregate performance of a whole industry, through feedback rules emerging from the Hamiltonian formulation of the game itself, thereby validating the adoption of the open-loop approach.

The remainder of the chapter is structured as follows. Section 2 contains the motivation and layout of the game. The Markov-perfect open-loop solution and the analysis of its stability properties is in Section 3. A few concluding remarks are in Section 4.

2 Setup

The differential game describes a sales-response model *à la* Vidale and Wolfe (1957), which has in common with Bass (1969) and the ensuing literature the presence of state equations mimicking a logistic growth curve, although we look at the case of non-durables.[2] In particular, we propose a modified version of Deal's (1979), in terms of the formalization of the cross-effects of firms' advertising efforts and sales across the set of the state variables.

Consider a population of firms $\mathcal{N} = \{1, 2, 3, ...n\}$ endowed with the same technology, summarized by a constant marginal production cost $c > 0$. Firms sell a homogeneous good over continuous time $t \in [0, \infty)$. The individual volume of instantaneous sales is $x_i(t) \geq 0$, $i = 1, 2, ...n$, and the unit margin is $P = p - c > 0$. Here, we posit that either the sector is perfectly competitive (so that firms permanently face an infinitely elastic demand function at p) or price is regulated (and time-invariant, as is the case of marginal cost). At every instant, each firm may boost its sales volume through an advertising effort $k_i(t)$, which entails an instantaneous cost $\Gamma_i(t) = bk_i^2(t)$, where b is a positive constant. therefore, the individual firm's instantaneous profit function is $\pi_i(t) = (p - c)x_i(t) - \Gamma_i(t) = Px_i(t) - bk_i^2(t)$.

Before defining the state dynamics, it is useful to briefly discuss how it has been specified in the extant literature, and what implications this has engendered. In Deal (1979), originally formulated as a duopoly model, the

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[2] For overviews of the related literature, see Dockner *et al.* (2000), Erickson (2003), Jørgensen and Zaccour (2004) and Lambertini (2018).

individual state equation is

$$\dot{x}_i(t) = \zeta k_i(t)\left[1 - \frac{x_i(t) + X_{-i}(t)}{X_{\max}}\right] - \delta x_i(t), \tag{1}$$

which can be extended to include the negative effect exerted by the $n-1$ rivals, $K_{-i}(t) = \sum_{i \neq j} k_j(t)$:

$$\dot{x}_i(t) = \zeta\left[k_i(t) - \beta K_{-i}(t)\right]\left[1 - \frac{x_i(t) + X_{-i}(t)}{X_{\max}}\right] - \delta x_i(t), \tag{2}$$

where $X_{\max} \geq \sum_{i=1}^{n} x_i(t)$ is the maximum volume of output consumers may absorb from this industry,[3] and $\{\beta, \delta, \zeta\}$ are positive constants. In particular, ζ is the 'natural' growth rate of firm i's share. In (1) and (2), states appear at the first degree, and therefore the instantaneous growth rate is not implying a logistic growth, just because the element that would imply it is replaced by a function of advertising controls, either $k_i(t)$ or $k_i(t) - \beta K_{-i}(t)$.

A few additional words are in order, concerning the intensity of the negative effect of the rivals's advertising campaigns. In line with the parallel literature on R&D for process innovation with technological spillovers, stemming from d'Aspremont and Jacquemin (1988), it seems appropriate to assume $\beta \in [0, 1/(n-1)]$, in such a way that the instantaneous impact of $K_{-i}(t)$ is at most as large as that of firm i's own effort. And indeed, after solving the game, we will see that there exists a solid reason - not directly related to firms' interplay in the R&D space - to adopt

Assumption A $\beta \in [0, 1/(n-1)]$.

Moreover, we also introduce

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[3] Borrowing from the jargon of the parallel literature on biological natural resources, $X_{\max}$ is the industry's *carring capacity* (see, among many others, Clark, 1990).

Assumption B $X_{\max} \in \left(0, \overline{X}_{\max}\right)$, with

$$\overline{X}_{\max} \equiv \frac{nP\left[1-\beta\left(n-1\right)\right]-2bn\zeta\left(\delta+\rho\right)+\sqrt{\Omega}}{4b\left(\delta+\rho\right)^{2}} \in \mathbb{R}^{+},$$

$$\begin{aligned}\Omega \equiv\ & n[8bP\left(1+\beta\right)\left(1-\beta\left(n-1\right)\right)\left(n-1\right)\zeta\left(\delta+\rho\right)+ \\ & n(P\left(1-\beta\left(n-1\right)\right)+2b\zeta\left(\delta+\rho\right)^{2}].\end{aligned}$$

The role of Assumption B is to ensure $\sum_{i=1}^{n} x_i\left(t\right)/X_{\max} < 1$ throughout the game as well as at the steady state equilibrium, thereby excluding the arising of a corner solution with demand saturation at all times.

Now we may focus on the difference between (1) and (2). While in the former the presence of rivals is signalled by their sales levels only, in the latter it also takes the form of the countervailing effect associated to advertising efforts (which may have comparative nature, for example). However, both specifications of the state equation(s) have a fundamental implication as far as the solvability of the game under feedback rules is concerned. This is due to the lack of additive separability between states and controls in both (1) and (2), which are clearly nonlinear and therefore do not allow for either an intuitive conjecture of the value function or, consequently, for a fully analytical characterization of feedback equilibria through the solution of the relevant set of Hamilton-Jacobi-Bellman equations. And, of course, the specification of state equations as in either (1) or (2) makes the open-loop solution weakly time consistent.

This has triggered several efforts aimed at delivering models with largely although not entirely equivalent economic interpretations, but producing strongly time consistent equilibria, possibly in the form of degenerate feedback strategies designed under open-loop information (see, e.g., Leitmann and Schmitendorf, 1978; Feichtinger, 1983; Dragone *et al.*, 2010; and Jørgensen *et al.*, 2010), by formulating state equations in such a way that the game becomes state-redundant, at least in its open-loop form.

Yet, another avenue - which, to the best of our knowledge, has been overlooked thus far - is open for interesting extensions. This consists in constructing additively separable state equations in which the role of carrying capacity is preserved and, nonetheless, the open-loop solution is indeed a degenerate feedback one. To this purpose, we specify the state dynamics as follows:

$$\dot{x}_i(t) = \zeta \left[1 - \frac{x_i(t) + X_{-i}(t)}{X_{\max}}\right] + k_i(t) - \beta K_{-i}(t) - \delta x_i(t). \qquad (3)$$

Here, the whole net advertising effort adds up to the 'natural' growth rate of the individual firm's sales, in the same way as the harvest rate of firms in a renewable resource extraction game (Lambertini and Leitmann, 2019). Since the control used in this game is not a harvest rate, the interpretation of (3) is the following. The $n-1$ rivals are aware that their advertising efforts may exert a business stealing effect by shifting downwards firm i sales, all else equal (specifically, for any given δ and ζ), and therefore firm i reacts to diminish the impact of the negative spillover associated with $K_{-i}(t)$. Naturally, firm i also knows that, by doing so, it is impacting each of the rivals' sales in an analogous way. In a sense, this formulation has some features in common with the concept of *conformance quality* dating back to Garvin (1988), which appears in an additively separable way in analogous extensions of the Vidale-Wolfe model (see, e.g., El Ouardighi *et al.*, 2013).

Accordingly, the Hamiltonian function of firm i is:

$$\mathcal{H}_i\left(x_i(t), X_{-i}\left(t\right), k_i(t), K_{-i}\left(t\right), \lambda_{ij}\left(t\right)\right) = e^{-\rho t}\left\{Px_i\left(t\right) - bk_i^2\left(t\right) + \right.$$

$$\lambda_{ii}\left(t\right)\left[\zeta\left(1 - \frac{x_i\left(t\right) + X_{-i}\left(t\right)}{X_{\max}}\right) + k_i\left(t\right) - \beta K_{-i}\left(t\right) - \delta x_i\left(t\right)\right] +$$

$$\left.\textstyle\sum_{j\neq i}\lambda_{ij}\left(t\right)\left[\zeta\left(1 - \frac{x_j\left(t\right) + X_{-j}\left(t\right)}{X_{\max}}\right) + k_j\left(t\right) - \beta K_{-j}\left(t\right) - \delta x_j\left(t\right)\right]\right\}, \tag{4}$$

where $\lambda_{ij}\left(t\right)$ is the relevant capitalized costate variable, for all i and j. We are now ready to illustrate the solution of the game under open-loop information, and its key properties.

3 Solving the game

The individual firm's first order condition (FOC) w.r.t. $k_i(t)$ is

$$\lambda_{ii}\left(t\right) - 2bk_i\left(t\right) - \beta\sum_{j\neq i}\lambda_{ij}\left(t\right) = 0, \tag{5}$$

which is accompanied by the following set of costate equations:

$$-\frac{\partial\mathcal{H}_i(\cdot)}{\partial x_i(t)} = \dot{\lambda}_{ii}(t) - \rho\lambda_{ii}\left(t\right) \Rightarrow$$

$$\dot{\lambda}_{ii}(t) = \frac{\left[\lambda_i(t)\left(\delta+\rho\right) - P\right]X_{\max} + \zeta\left[\lambda_{ii}\left(t\right) + \sum_{j\neq i}\lambda_{ij}\left(t\right)\right]}{X_{\max}} \tag{6}$$

$$-\frac{\partial\mathcal{H}_i(\cdot)}{\partial x_j(t)} = \dot{\lambda}_{ij}(t) - \rho\lambda_{ij}\left(t\right) \Rightarrow$$

$$\dot{\lambda}_{ij}(t) = \frac{\lambda_{ij}(t)\left(\delta+\rho\right)X_{\max} + \zeta\left[\lambda_{ii}\left(t\right) + \sum_{j\neq i}\lambda_{ij}\left(t\right)\right]}{X_{\max}}, \tag{7}$$

while the set of transversality conditions is summarized by

$$\lim_{t\to\infty} e^{-\rho t}\lambda_{ij}\left(t\right)x_{j}\left(t\right)=0\,\forall\, j=1,2,3,...n. \tag{8}$$

We are now in a position to assess some essential properties of the open-loop solution by looking at the system (5-7). To begin with, (5) does not explicitly feature any state variable, and this is true also for all costate equations (6-7). Intuitively, this is due to the fact that the present game is linear in states, thanks to its additively separable reformulation. These elements prove the following:

Lemma 1 *Since the game has a state-linear structure, its open-loop solution is a degenerate feedback one.*

There remains to analytically characterize it. To this aim, we may (i) drop the time argument for the sake of brevity, and (ii) impose symmetry upon all variables not pertaining to firm i and solve (5) to find the expression of the optimal $\lambda_{ii}\left(t\right)$ at a generic instant,

$$\lambda_{ii}=2bk_{i}+\beta\left(n-1\right)\lambda_{ij}, \tag{9}$$

and then solve the same equation again w.r.t. k_i, to get

$$k_{i}=\frac{\lambda_{ii}-\beta\left(n-1\right)\lambda_{ij}}{2b}. \tag{10}$$

The above expression can be differentiated w.r.t. time to deliver the following control equation:

$$\dot{k}_{i}=\frac{\dot{\lambda}_{ii}-\beta\left(n-1\right)\dot{\lambda}_{ij}}{2b}, \tag{11}$$

which, using (6-9), simplifies as follows:

$$\dot{k}_{i}=\frac{\left(2bk_{i}-P\right)X_{\max}-\xi\left[1-\beta\left(n-1\right)\right]\left[2bk_{i}+\left(1+\beta\right)\left(n-1\right)\right]\lambda_{ij}}{2bX_{\max}}. \tag{12}$$

Then, we may solve the system (7-12) to find the expressions of the optimal k_i and λ_{ij} at any point in time. This is done by posing equal to zero the related integration constants, thus obtaining

$$\begin{aligned} k_i^* &= \frac{P\left[(\delta+\rho) X_{\max}+\zeta(1+\beta)(n-1)\right]}{2b(\delta+\rho)\left[(\delta+\rho) X_{\max}+n\zeta\right]} \\ \lambda_{ij}^* &= -\frac{P\zeta}{(\delta+\rho)\left[(\delta+\rho) X_{\max}+n\zeta\right]}, \end{aligned} \tag{13}$$

with $k_i^* > 0 > \lambda_{ij}^*$ everywhere. The state linearity of the game also implies that k_i^* and λ_{ij}^* also solve $\dot{k}_i = 0$ and $\dot{\lambda}_{ij} = 0$, and the same holds for λ_{ii}^* (originating from (9)) and (6).

At this point, we may impose symmetry upon all variables, thereby dropping indexes. It is evident that the optimal advertising effort is linearly increasing in β, as intuition would suggest *a priori*:

$$\frac{\partial k^*}{\partial\beta} = \frac{P\beta(n-1)}{2b(\delta+\rho)\left[(\delta+\rho) X_{\max}+n\zeta\right]} > 0. \tag{14}$$

Less obvious is the interpretation of the following partial derivatives:

$$\begin{aligned} \frac{\partial k^*}{\partial\zeta} &= -\frac{P\left[1-\beta(n-1)\right] X_{\max}}{2b\left[(\delta+\rho) X_{\max}+n\zeta\right]^2} < 0 \\ \frac{\partial^2 k^*}{\partial\zeta^2} &= \frac{nP\left[1-\beta(n-1)\right] X_{\max}}{b\left[(\delta+\rho) X_{\max}+n\zeta\right]^3} > 0, \end{aligned} \tag{15}$$

which have opposite signs. This tells that the equilibrium individual effort is decreasing and convex in ζ for all $\beta \in [0, 1/(n-1))$, becoming insensitive to the natural growth rate in correspondence of the upper bound of the parameter scaling the negative advertising spillover.

Remark 2 *The individually optimal advertising effort increases in β while being monotonically decreasing in ζ.*

The above Remark prompts for the analysis of the marginal rate of substitution between β and ζ, by looking at the total differential of k^* in the space (β, ζ), whereby k^* is constant provided that

$$\frac{\partial k^*}{\partial \beta} d\beta + \frac{\partial k^*}{\partial \zeta} d\zeta = 0. \tag{16}$$

Its solution,

$$\frac{\partial k^* / \partial \beta}{\partial k^* / \partial \zeta} = \frac{(\delta + \rho)\left[1 + \beta (n-1)\right] X_{\max}}{(n-1)\, \zeta \left[(\delta + \rho) X_{\max} + n\zeta\right]} \tag{17}$$

is positive everywhere. This yields

Corollary 3 *Parameters β and ζ are complements along any isoquant along which k^* is constant.*

We are also interested in evaluating the impact of firms' number on individual and aggregate advertising efforts. To evaluate this aspect, we may define $K^* = nk^*$ and look at

$$\begin{aligned} \frac{\partial k^*}{\partial n} &= \frac{P\zeta \left[\beta (\delta + \rho) X_{\max} + \zeta (1 + \beta)\right]}{2b (\delta + \rho) \left[(\delta + \rho) X_{\max} + n\zeta\right]^2} > 0 \\ \frac{\partial^2 k^*}{\partial n^2} &= -\frac{n\zeta^2 \left[\beta (\delta + \rho) X_{\max} + \zeta (1 + \beta)\right]}{b (\delta + \rho) \left[(\delta + \rho) X_{\max} + n\zeta\right]^3} < 0 \end{aligned} \tag{18}$$

and

$$\begin{aligned} \frac{\partial K^*}{\partial n} &= k^* + n \cdot \frac{\partial k^*}{\partial n} > 0 \\ \frac{\partial^2 K^*}{\partial n^2} &= \frac{P\zeta X_{\max} \left[\beta (\delta + \rho) X_{\max} + \zeta (1 + \beta)\right]}{b \left[(\delta + \rho) X_{\max} + n\zeta\right]^3} > 0, \end{aligned} \tag{19}$$

which can be summarized in

Remark 4 *All else equal, any increase in the number of firms induces an increase in individual and aggregate advertising efforts.*

This result has a neatly Arrovian flavour (Arrow, 1962), as increasing market fragmentation increases aggregate investment. Here, we are dealing with advertising campaigns, while in the diachronic debate between Arrow (1962) and Schumpeter (1942) the subject was technological innovation, with Schumpeter claiming that the industry structure endowed with the highest innovation incentives should be pure monopoly, and Arrow advocating exactly the opposite.[4] Of course, this finding shall not be taken literally, as efforts and therefore costs shooting up to infinity are inadmissible as they would drive profits below zero well before that.

Before addressing this issue, we must characterize the steady state solution, by inserting k^* into (3), which under symmetry becomes

$$\dot{x} = \zeta \left(1 - \frac{nx}{X_{\max}}\right) + k^* \left[1 - \beta \left(n - 1\right)\right] - \delta x, \tag{20}$$

and impose stationarity, to obtain the following expression, measuring steady state individual sales:

$$x^* = \frac{\left[\zeta + k^* \left(1 - \beta \left(n - 1\right)\right)\right] X_{\max}}{\left(\delta + \rho\right) X_{\max} + n\zeta}, \tag{21}$$

which is always positive under Assumption A. Moreover, it can be easily checked that Assumption B ensures that $nx^*/X_{\max} < 1$.

It is now time to deal with the phase diagram and the associated stability properties, in the state-control space. The equation of the steady state locus $\dot{x} = 0$ is

$$k_{ss} = \frac{x_{ss} \left(n\zeta + \delta X_{\max}\right) - \zeta X_{\max}}{\left(1 - \beta \left(n - 1\right)\right) X_{\max}}, \tag{22}$$

which appears in Figure 1, together with the flat line identifying the optimal advertising control k^* at any point in time and for any admissible state.

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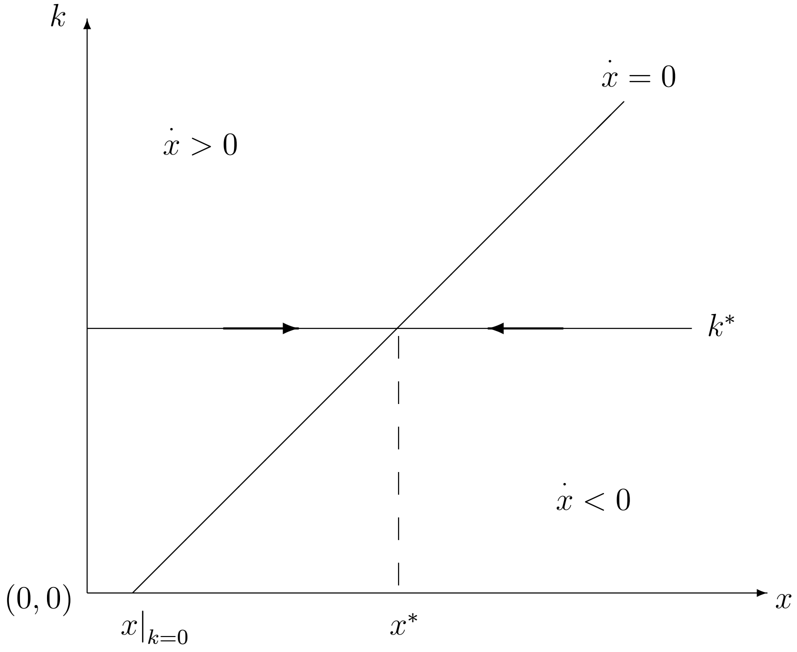
[4] For exhaustive overviews of the ensuing literature, still very livey, see Tirole (1988), Reinganum (1989), Martin (1993), Gilbert (2006) and Aghion *et al.* (2015), among others.

Figure 1 The phase diagram



The steady state locus departs from $x|_{k=0} = \zeta X / (n\zeta + \delta X_{\max})$, i.e., the level dictated by the intrinsic properties of sales dynamics, neither boosted nor diminished by any advertising campaigns.

The sign of $\dot{x}$, which is explicitly indicated and summarized by the arrows along the flat optimal control, reveals that the steady state is stable. Indeed, the inspection of the Jacobian matrix and its determinant reveals that the open-loop equilibrium is a saddle point. The Jacobian matrix is

$$J = \begin{bmatrix} \dfrac{\partial \dot{x}}{\partial x} = -\dfrac{n\zeta + \delta X_{\max}}{\zeta X_{\max}} & \dfrac{\partial \dot{x}}{\partial k} = 1 - \beta (n-1) \\ \\ \dfrac{\partial \dot{k}}{\partial x} = 0 & \dfrac{\partial \dot{k}}{\partial k} = \delta + \rho + \dfrac{[1 - \beta (n-1)] \xi}{X_{\max}} \end{bmatrix}. \tag{23}$$

Since the optimal control is independent of the state at all times, then obviously $\partial \dot{k}/\partial x = 0$, which in turn implies that the determinant of the Jacobian matrix collapses to the product along the main diagonal:

$$\Delta(J) = \frac{\partial \dot{x}}{\partial x} \cdot \frac{\partial \dot{k}}{\partial k} = -\frac{n\zeta + \delta X_{\max}}{\zeta X_{\max}} \left[\delta + \rho + \frac{(1 - \beta(n-1))\,\xi}{X_{\max}} \right]. \tag{24}$$

The expression on the r.h.s. of (24) is negative in view of Assumption A. This suffices to claim

Proposition 5 *The steady state point* (x^*, k^*) *is a saddle point equilibrium.*

4 Concluding remarks

As already stressed in the foregoing discussion, there have been frequent and relevant intersections between the search for strongly time consistent open-loop equilibria and the formulation of the multiple strands of the literature discussing differential games of advertising.

The frequent presence of an explicit non-separability between controls and states in dynamic games of advertising has implied, more often than not, the impossibility of characterizing feedback equilibria, confining attention to open-loop or closed-loop memoryless ones. Therefore, any such games paving the way to a (possibly degenerate) feedback solution under open-loop information has been intensively sought after.

With this in mind, we have proposed a plausible reformulation of the sales expansion model *à la* Vidale and Wolfe (1957) appearing in Deal (1979), transforming the original multiplicative interplay between controls and states into an additive one, so as to make the resulting optimal control delivered by the open-loop solution strongly time consistent without overturn the essential economic features of the model. In addition to preserving the saddle point stability of the resulting equilibrium, this approach has also allowed us to identify a well defined Arrovian nature of the optimal industry investment in advertising campaigns under an admittedly simple and yet robust feedback rule.

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