




**Quantum backreaction in an analog black hole**G. Ciliberto <sup>1,2,3</sup> R. Balbinot,<sup>4</sup> A. Fabbri <sup>5</sup> and N. Pavloff <sup>2,6</sup><sup>1</sup>*Laboratoire d'Informatique de Paris 6, CNRS, Sorbonne Université, 4 place Jussieu, 75005 Paris, France*<sup>2</sup>*Université Paris-Saclay, CNRS, LPTMS, 91405 Orsay, France*<sup>3</sup>*Physikalisches Institut, Albert-Ludwigs-Universität Freiburg, Hermann-Herder-Straße 3, 79104 Freiburg, Germany*<sup>4</sup>*Dipartimento di Fisica e Astronomia dell'Università di Bologna and INFN sezione di Bologna, Via Irnerio 46, 40126 Bologna, Italy*<sup>5</sup>*Departamento de Física Teórica and IFIC, Universidad de Valencia-CSIC, Calle Dr. Moliner 50, 46100 Burjassot, Spain*<sup>6</sup>*Institut Universitaire de France*

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We extend the Gross-Pitaevskii equation to incorporate the effect of quantum fluctuations onto the flow of a weakly interacting Bose-Einstein condensate. Applying this framework to an analog black hole in a quasi-one-dimensional, transonic flow, we investigate how acoustic Hawking radiation backreacts on the background condensate. Our results point to the emergence of stationary density and velocity undulations in the supersonic region (analogous to the black-hole interior) and enable to evaluate the change in upstream and downstream Mach numbers caused by Hawking radiation. These findings provide new insight into the interplay between quantum fluctuations and analog gravity in Bose-Einstein condensates.

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As first noted in Ref. [1], a one-dimensional weakly interacting Bose gas provides an excellent platform for realizing an analogy originally proposed by Unruh. According to this analogy, the interface between the supersonic and the subsonic regions of a transonic flow acts as a “sonic horizon” mimicking the event horizon of a gravitational black hole [2]. Within the Bose-Einstein condensate (BEC) framework, this line of research culminated in the experimental observation of analog Hawking radiation [3], manifested as nonlocal correlations in density fluctuations emitted from the acoustic horizon [4].

From an experimental standpoint, the appeal of BEC platforms lies in their ultralow temperatures, their paradigmatic quantum nature, and the high degree of experimental control achievable in such systems. On the theoretical side, the Unruh analogy arises naturally: following Bogoliubov's approach [5], it is customary to separate a classical field, representing the background flow that serves as an effective metric, from quantum fluctuations. However, the Bogoliubov approach is approximate, and it is natural to seek improvements by incorporating higher-order interactions among the classical and/or quantum components of the total field. This topic has been extensively studied in BEC physics, beginning with the pioneering work of Beliaev [6,7].

In the context of analog gravity, this line of research is referred to as quantum backreaction, drawing the analogy with the study of how Hawking radiation affects the black-hole metric [8–12]. In General Relativity, this problem is notoriously difficult and fundamentally constrained by the absence of a quantum theory of gravitation (for a recent review on the backreaction problem, see Ref. [13] and references therein). In contrast, the prospects are far more promising in BEC physics, where one effectively has a “theory of everything” since a single quantum field simultaneously governs the dynamics of both the effective metric and its quantum fluctuations. As a result, a number of theoretical studies addressed this

issue (actually in a more general framework than the mere analogous Hawking radiation), both numerically [14–17] and using perturbative methods [18–23]. The latter closely parallel the semiclassical approach used in General Relativity: the equation governing the classical field (analogous to Einstein's equations) is modified by a contribution from the expectation value of the quantum fluctuations, encompassed in the equivalent of a stress-energy tensor.

However, in one-dimensional BECs, such approaches face technical challenges physically rooted in the Hohenberg-Mermin-Wagner theorem: at this dimensionality, the standard Gross-Pitaevskii description is particularly simple, but quantum phase fluctuations diverge at long wavelengths. A way around issues of this type has been proposed by Popov [24–26], who developed an amplitude-phase formalism to treat the long-wavelength degrees of freedom of a Bose field operator. This approach has since proven highly effective in BEC physics [27–36]. In the analysis below, we integrate it with the perturbative framework developed in Refs. [37–42]. This set of studies incorporates corrections beyond the leading Gross-Pitaevskii order, capturing both the normal and anomalous averages [defined in Eq. (7)]. Such contributions are crucial for a consistent treatment of both quantum and finite-temperature fluctuations.

The paper is organized as follows. In Sec. I, we introduce the model and the symmetry-breaking approach, which involves decomposing the quantum field operator into a classical order parameter and a (small) quantum correction. In Sec. II, we present an alternative, though not identical, separation that proves better suited to our objectives. Section III details our perturbative expansion. The leading order corresponds to the standard Gross-Pitaevskii equation (Sec. III A). The next order, the Bogoliubov level, is presented in Sec. III B, and is used in Sec. III C to compute the source terms that appear in the final step of our

expansion. This last step, presented in Sec. III D, leads to a modified equation for the classical order parameter that incorporates the effects of quantum fluctuations. The equations obtained in Sec. III D are general and apply to time- and space-dependent configurations in arbitrary dimensions. In Sec. IV we focus on stationary states. We first examine a uniform system in Sec. IV A to benchmark our approach, and then turn in Sec. IV B to the specific case of a one-dimensional analog black hole, the primary system for which our theoretical framework has been devised. Our conclusions and prospects for future work are presented in Sec. V. Relevant results from earlier studies, as well as technical aspects, are summarized in the Appendixes. Appendixes A and B present the acoustic black-hole configurations we consider and the associated quantum (outgoing or ingoing) modes. These modes are used in Appendix C as a basis for expanding the quantum fluctuation field, which makes it possible to explicitly compute source terms relevant to our backreaction equations. Appendix D details some steps of the computations presented in the main text. Our method is applied in Sec. IV B for a specific type of back-hole configuration (the so-called waterfall), and Appendix E complements this discussion by presenting results for different analog settings.

### I. SELF-CONSISTENT APPROACH

We consider bosons of mass  $m$  interacting through a pointlike potential in a  $d$ -dimensional space ( $d = 1, 2, \text{ or } 3$ ). The Bose quantum field operator  $\hat{\Psi}(\mathbf{x}, t)$  obeys the Heisenberg equation

$$i\hbar\partial_t\hat{\Psi} = -\frac{\hbar^2}{2m}\nabla^2\hat{\Psi} + [U - \mu + g\hat{\Psi}^\dagger\hat{\Psi}]\hat{\Psi}. \quad (1)$$

In this equation  $\mu$  is the chemical potential,  $g (>0)$  is the intensity of the pointlike interparticle repulsive potential and  $U(\mathbf{x})$  an external potential.

The Bogoliubov approach amounts to separate in a Bose-condensed system a classical and a quantum contribution. This can be achieved by writing [43,44]

$$\hat{\Psi}(\mathbf{x}, t) = \Phi(\mathbf{x}, t) + \hat{\psi}(\mathbf{x}, t), \quad (2)$$

where

$$\Phi = \langle\hat{\Psi}\rangle \quad \text{and thus} \quad \langle\hat{\psi}\rangle = 0. \quad (3)$$

$\Phi(\mathbf{x}, t)$  is known as the order parameter. In Eq. (3) the average is taken with respect to some statistical operator which needs not correspond to thermodynamical equilibrium (and could not in the case of an analog black hole). The fact that  $\langle\hat{\Psi}\rangle \neq 0$  implies that this is not a number-conserving state, but rather a coherent superposition of states with different numbers of particles.

The field  $\hat{\psi}$  describes quantum fluctuations (in all the text we note quantum operators acting in Fock space with hats). It obeys, as does  $\hat{\Psi}$ , the standard Bose commutation rules

$$\begin{aligned} [\hat{\psi}(\mathbf{x}, t), \hat{\psi}^\dagger(\mathbf{y}, t)] &= \delta(\mathbf{x} - \mathbf{y}), \\ [\hat{\psi}(\mathbf{x}, t), \hat{\psi}(\mathbf{y}, t)] &= 0, \end{aligned} \quad (4)$$

where  $[\dots, \dots]$  denotes the commutator.

Equation (2) yields

$$\begin{aligned} \hat{\Psi}^\dagger\hat{\Psi} &= |\Phi|^2 + \Phi\hat{\psi}^\dagger + \Phi^*\hat{\psi} + \hat{\psi}^\dagger\hat{\psi}, \\ \hat{\Psi}\hat{\Psi} &= \Phi^2 + 2\Phi\hat{\psi} + \hat{\psi}\hat{\psi}, \\ \hat{\Psi}^\dagger\hat{\Psi}\hat{\Psi} &= |\Phi|^2\Phi + 2|\Phi|^2\hat{\psi} + \Phi^2\hat{\psi}^\dagger \\ &\quad + 2\Phi\hat{\psi}^\dagger\hat{\psi} + \Phi^*\hat{\psi}\hat{\psi} + \hat{\psi}^\dagger\hat{\psi}\hat{\psi}. \end{aligned} \quad (5)$$

The expectation values of expressions (5) read as

$$\begin{aligned} \langle\hat{\Psi}^\dagger\hat{\Psi}\rangle &= |\Phi|^2 + \bar{n}, \quad \langle\hat{\Psi}\hat{\Psi}\rangle = \Phi^2 + \bar{m}, \\ \langle\hat{\Psi}^\dagger\hat{\Psi}\hat{\Psi}\rangle &= |\Phi|^2\Phi + 2\Phi\bar{n} + \Phi^*\bar{m} + \langle\hat{\psi}^\dagger\hat{\psi}\hat{\psi}\rangle, \end{aligned} \quad (6)$$

where

$$\bar{n}(\mathbf{x}, t) = \langle\hat{\psi}^\dagger\hat{\psi}\rangle \quad \text{and} \quad \bar{m}(\mathbf{x}, t) = \langle\hat{\psi}\hat{\psi}\rangle \quad (7)$$

are known as the normal and anomalous averages, respectively. From these expressions, the average of Eq. (1) reads as [41]

$$\begin{aligned} i\hbar\partial_t\Phi &= \left[ -\frac{\hbar^2}{2m}\nabla^2 + U - \mu + g|\Phi|^2 + 2g\bar{n} \right] \Phi \\ &\quad + g\bar{m}\Phi^* + g\langle\hat{\psi}^\dagger\hat{\psi}\hat{\psi}\rangle. \end{aligned} \quad (8)$$

The difference between Eqs. (8) and (1) reads as [41]

$$\begin{aligned} i\hbar\partial_t\hat{\psi} &= \left[ -\frac{\hbar^2}{2m}\nabla^2 + U - \mu + 2g|\Phi|^2 \right] \hat{\psi} + g\Phi^2\hat{\psi}^\dagger \\ &\quad + 2g\Phi(\hat{\psi}^\dagger\hat{\psi} - \bar{n}) + g\Phi^*(\hat{\psi}\hat{\psi} - \bar{m}) \\ &\quad + g\langle\hat{\psi}^\dagger\hat{\psi}\hat{\psi}\rangle - \langle\hat{\psi}^\dagger\hat{\psi}\hat{\psi}\rangle. \end{aligned} \quad (9)$$

At this point Eqs. (8) and (9) are exact. They form a self-consistent system describing the reciprocal effects of the classical and quantum fields  $\Phi$  and  $\hat{\psi}$ , respectively. We will solve them perturbatively, assuming that the effects of the quantum fluctuations on the classical background are small.

### II. AMPLITUDE-PHASE FORMALISM

As an alternative to the decomposition (2) we use an amplitude-phase formalism which consists in writing the quantum field as

$$\hat{\Psi} = \exp\{i(\Theta + \hat{\theta})\}\sqrt{\rho(1 + \hat{\eta})}, \quad (10)$$

where  $\Theta(\mathbf{x}, t)$  and  $\rho(\mathbf{x}, t)$  are the classical phase and density fields, respectively. The Hermitian field operators  $\hat{\theta}(\mathbf{x}, t)$  and  $\hat{\eta}(\mathbf{x}, t)$  describe quantum fluctuations of the phase and of the relative density, respectively, with  $\langle\hat{\theta}\rangle = 0 = \langle\hat{\eta}\rangle$ .

The perturbative expansions based on decompositions (2) and (10) are not equivalent. To make the difference explicit, we carry out a series expansion of expression (10):

$$\hat{\Psi} = \varphi \left( 1 + i\hat{\theta} + \frac{1}{2}\hat{\eta} + \frac{i}{2}\hat{\theta}\hat{\eta} - \frac{1}{2}\hat{\theta}^2 - \frac{1}{8}\hat{\eta}^2 + \dots \right), \quad (11)$$

where

$$\varphi(\mathbf{x}, t) = \sqrt{\rho(\mathbf{x}, t)} \exp\{i\Theta(\mathbf{x}, t)\}. \quad (12)$$

In all the following it will suffice to keep in this expansion only the terms which have been explicitly written in (11). This

amounts to including fluctuations up to quadratic order in  $\hat{\eta}$  and  $\hat{\theta}$  and to write

$$\hat{\Psi} = \varphi(1 + \hat{a} + \hat{A}), \quad (13)$$

where

$$\begin{aligned} \hat{a}(\mathbf{x}, t) &= i\hat{\theta} + \frac{1}{2}\hat{\eta}, \\ \hat{A}(\mathbf{x}, t) &= \frac{i}{2}\hat{\theta}\hat{\eta} - \frac{1}{2}\hat{\theta}^2 - \frac{1}{8}\hat{\eta}^2 \\ &= -\frac{1}{2}\hat{a}^\dagger\hat{a} - \frac{1}{4}\hat{a}^{\dagger 2} + \frac{1}{4}\hat{a}^2. \end{aligned} \quad (14)$$

The truncation of expansion (11) is justified by assuming that the quantum contributions  $\hat{\psi}$ ,  $\hat{\theta}$ , and  $\hat{\eta}$  are “small.” As will be shown in Sec. III A, this assumption permits the neglect of all terms of third order and higher in these fields. While the assumption of small quantum fluctuations holds in three dimensions, it fails in lower dimensions, where terms involving products of these nominally small quantities may actually diverge. We will address this issue in due course; for now, we naively proceed with the expansion.

In terms of the quantities  $\varphi$ ,  $\hat{a}$ , and  $\hat{A}$ , the different terms appearing in (8) read as

$$\Phi = \varphi(1 + \langle \hat{A} \rangle), \quad (15)$$

$$\tilde{n} = |\varphi|^2 \langle \hat{a}^\dagger \hat{a} \rangle, \quad \tilde{m} = \varphi^2 \langle \hat{a} \hat{a} \rangle, \quad (16)$$

and

$$\begin{aligned} |\Phi|^2 \Phi &= |\varphi|^2 \varphi (1 + 2\langle \hat{A} \rangle + \langle \hat{A}^\dagger \rangle), \\ \tilde{n} \Phi &= |\varphi|^2 \varphi \langle \hat{a}^\dagger \hat{a} \rangle, \quad \tilde{m} \Phi^* = |\varphi|^2 \varphi \langle \hat{a} \hat{a} \rangle, \end{aligned} \quad (17)$$

where the quantum fields contribution has been kept only up to second order.

Equation (15) shows that the classical fields  $\Phi(\mathbf{x}, t)$  and  $\varphi(\mathbf{x}, t)$  are not identical, indicating that the separation between quantum and classical contributions in Eq. (2) does not exactly match the one in Eq. (10). The interest in the amplitude-phase formalism lies in the fact that, whereas in low dimension  $\Phi$  is an ill-defined quantity which obeys a singular equation, we will see in Sec. III D that the quantity  $\varphi$  is well behaved. Accordingly, Eq. (8) which governs the dynamics of the classical field is written as

$$\begin{aligned} i\hbar\partial_t(\varphi + \varphi\langle \hat{A} \rangle) &= \left[ -\frac{\hbar^2}{2m}\nabla^2 + U - \mu \right] (\varphi + \varphi\langle \hat{A} \rangle) \\ &+ g|\varphi|^2\varphi(1 + 2\langle \hat{A} \rangle + \langle \hat{A}^\dagger \rangle) \\ &+ g|\varphi|^2\varphi(2\langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a} \hat{a} \rangle), \end{aligned} \quad (18)$$

where the term  $\langle \hat{\psi}^\dagger \hat{\psi} \hat{\psi} \rangle$  has been dropped in accordance with our accounting for the contribution of quantum fields only up to second order. A technical remark is in order here: It is possible to include this term, to keep all the terms in expansion (11) without stopping at second order, and likewise to keep all the contributions to  $\hat{A}$  [without restricting oneself to the terms included in (14)] and then to start the perturbative treatment only in the next section. This choice of presentation would be indeed more logical, but it leads to unnecessarily awkward

expressions which would be immediately expanded to second order in the next section.

### III. PERTURBATIVE EXPANSION

In this section we present the different steps of the perturbative expansion of Eqs. (18) and (9) leading to the backreaction equations (54) and (55).

#### A. Classical fields and Gross-Pitaevskii equation

At lowest order all the contributions of the quantum fields are discarded from Eq. (18), or equivalently from Eq. (8). The corresponding approximation of the classical field  $\varphi$  is the Gross-Pitaevskii order parameter  $\varphi_{\text{GP}}(\mathbf{x}, t)$  solution of

$$i\hbar\partial_t\varphi_{\text{GP}} = \left[ -\frac{\hbar^2}{2m}\nabla^2 + U - \mu + g|\varphi_{\text{GP}}|^2 \right] \varphi_{\text{GP}}. \quad (19)$$

At this order, the classical fields,  $\varphi$  defined in Eq. (12) and  $\Phi$  defined in Eq. (2), are both equal to the solution  $\varphi_{\text{GP}}$  of the Gross-Pitaevskii equation (19). Their modifications induced by the quantum fluctuations appear at higher order. To take them into account we write

$$\varphi = \varphi_{\text{GP}} + \delta\varphi \quad \text{and} \quad \Phi = \varphi_{\text{GP}} + \delta\Phi. \quad (20)$$

Comparing Eqs. (18) and (19) makes it clear that the first effects of quantum fluctuations on  $\delta\varphi(\mathbf{x}, t)$  and  $\delta\Phi(\mathbf{x}, t)$  are of second order in  $\hat{a}$  and  $\hat{a}^\dagger$ , which legitimates our discarding of contributions involving terms of third order in these fields in Sec. II, since they arise at the next order in perturbation. We may thus replace Eq. (15) by

$$\delta\Phi = \delta\varphi + \varphi_{\text{GP}}\langle \hat{A} \rangle. \quad (21)$$

The correction to  $\varphi_{\text{GP}}$  is accordingly computed from an expansion of Eq. (18) in which are kept only the terms linear in  $\delta\varphi$  (or  $\delta\Phi$ ) and quadratic in the quantum fields  $\hat{a}$  and  $\hat{a}^\dagger$  (or equivalently  $\hat{\eta}$  and  $\hat{\theta}$ ):

$$(i\hbar\partial_t - L)\delta\Phi - g\varphi_{\text{GP}}^2\delta\Phi^* = g|\varphi_{\text{GP}}|^2\varphi_{\text{GP}}(2\langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a} \hat{a} \rangle), \quad (22)$$

where

$$L = -\frac{\hbar^2}{2m}\nabla^2 + U - \mu + 2g|\varphi_{\text{GP}}|^2. \quad (23)$$

Equation (22) describes the modification of the classical fields beyond the Gross-Pitaevskii approximation. It involves terms of second order in the quantum fields, the evaluation of which necessitates the solution of the first-order Bogoliubov equation (see Sec. III B below). Some of these terms are ill defined in low dimension. This issue will be addressed in Sec. III D.

#### B. First order: Bogoliubov equation for the quantum fields

According to our perturbative scheme, in Eqs. (21) and (22) the quantum averages should be evaluated from the solution of (9) where the classical fields are the solutions of the Gross-Pitaevskii equation (19) and only linear terms in the quantum fields are included. This leads to the celebrated Bogoliubov equation

$$(i\hbar\partial_t - L)\hat{\psi} - g\varphi_{\text{GP}}^2\hat{\psi}^\dagger = 0. \quad (24)$$

It is convenient for our purpose to express this equation in terms of the amplitude and phase fields  $\hat{\eta}$  and  $\hat{\theta}$  defined in Eq. (10). To this end we write the solution  $\varphi_{\text{GP}}$  of the Gross-Pitaevskii equation (19) as

$$\varphi_{\text{GP}}(\mathbf{x}, t) = \sqrt{\rho_{\text{GP}}(\mathbf{x}, t)} \exp\{i\Theta_{\text{GP}}(\mathbf{x}, t)\}, \quad (25)$$

so that the Gross-Pitaevskii density and velocity read as

$$\rho_{\text{GP}}(\mathbf{x}, t) = |\varphi_{\text{GP}}|^2 \quad \text{and} \quad \mathbf{V}_{\text{GP}}(\mathbf{x}, t) = \frac{\hbar}{m} \nabla \Theta_{\text{GP}}. \quad (26)$$

From the definitions (2) and (13) we have

$$\hat{\psi} = \hat{\Psi} - \langle \hat{\Psi} \rangle = \varphi \hat{a} + \varphi(\hat{A} - \langle \hat{A} \rangle). \quad (27)$$

At the order of accuracy at which Eq. (24) is solved, one may replace (27) by

$$\hat{\psi} \simeq \varphi_{\text{GP}} \hat{a} = \varphi_{\text{GP}} \left( \frac{1}{2} \hat{\eta} + i \hat{\theta} \right). \quad (28)$$

Then, since  $\varphi_{\text{GP}}$  is a solution of (19),  $(i\hbar\partial_t - L)\varphi_{\text{GP}} = -g|\varphi_{\text{GP}}|^2\varphi_{\text{GP}}$  and the first term in Eq. (24) can be cast under the form

$$(i\hbar\partial_t - L)\hat{\psi} = \varphi_{\text{GP}}(-g|\varphi_{\text{GP}}|^2 + i\hbar\partial_t - \mathcal{L})\hat{a}, \quad (29)$$

where

$$\mathcal{L} = -\frac{\hbar^2}{m} \left( \frac{\nabla \varphi_{\text{GP}}}{\varphi_{\text{GP}}} \right) \cdot \nabla - \frac{\hbar^2}{2m} \nabla^2. \quad (30)$$

It follows from Eq. (25) that

$$\frac{\nabla \varphi_{\text{GP}}}{\varphi_{\text{GP}}} = \frac{1}{2} \frac{\nabla \rho_{\text{GP}}}{\rho_{\text{GP}}} + i \nabla \Theta_{\text{GP}}, \quad (31)$$

which makes it possible to recast the operator  $\mathcal{L}$  under the form

$$\mathcal{L} = -\frac{\hbar^2}{2m\rho_{\text{GP}}} \nabla \cdot \rho_{\text{GP}} \nabla - i\hbar \mathbf{V}_{\text{GP}} \cdot \nabla, \quad (32)$$

and Eq. (24) as

$$(i\hbar\partial_t - \hat{\mathcal{L}})\hat{a} = g|\varphi_{\text{GP}}|^2(\hat{a} + \hat{a}^\dagger) = g\rho_{\text{GP}}\hat{\eta}. \quad (33)$$

Adding and subtracting Eq. (33) with its Hermitian conjugate yields

$$\hbar(\partial_t + \mathbf{V}_{\text{GP}} \cdot \nabla)\hat{\theta} - \frac{\hbar^2}{4m\rho_{\text{GP}}} \nabla \cdot (\rho_{\text{GP}} \nabla \hat{\eta}) + g\rho_{\text{GP}}\hat{\eta} = 0, \quad (34)$$

and

$$(\partial_t + \mathbf{V}_{\text{GP}} \cdot \nabla)\hat{\eta} + \frac{\hbar}{m\rho_{\text{GP}}} \nabla \cdot (\rho_{\text{GP}} \nabla \hat{\theta}) = 0. \quad (35)$$

Equations (34) and (35) are the amplitude-phase versions of (24). They are typically solved in cases where  $\varphi_{\text{GP}}$  is time independent. We here consider the more general situation where the density  $\rho_{\text{GP}}$  and velocity  $\mathbf{V}_{\text{GP}}$  may depend not only on space but also on time.

It will appear convenient in future computations to introduce the following operators [which act on  $(\mathbf{x}, t)$ -dependent scalar quantities]:

$$\mathcal{T} = \partial_t + \mathbf{V}_{\text{GP}} \cdot \nabla, \quad (36a)$$

$$\mathcal{X} = \frac{\hbar}{2m\rho_{\text{GP}}} \nabla \cdot \rho_{\text{GP}} \nabla. \quad (36b)$$

In terms of operators  $\mathcal{T}$  and  $\mathcal{X}$  one may write

$$i\hbar\partial_t - \mathcal{L} = \hbar(i\mathcal{T} + \mathcal{X}), \quad (37)$$

and the Bogoliubov equations (34) and (35) read as

$$\frac{1}{2} \mathcal{X} \hat{\eta} - \mathcal{T} \hat{\theta} = \frac{g\rho_{\text{GP}}}{\hbar} \hat{\eta}, \quad (38a)$$

$$\frac{1}{2} \mathcal{T} \hat{\eta} + \mathcal{X} \hat{\theta} = 0. \quad (38b)$$

### C. Averages of quantum fields

In this section we give explicit expressions the quadratic averages of the quantum fields which appear in Eq. (22).

The first quantity to be considered is the density-density correlation function  $G^{(2)}$  defined as

$$\begin{aligned} G^{(2)}(\mathbf{x}, \mathbf{y}, t) &= \langle : \hat{\rho}(\mathbf{x}, t) \hat{\rho}(\mathbf{y}, t) : \rangle - \langle \hat{\rho}(\mathbf{x}, t) \rangle \langle \hat{\rho}(\mathbf{y}, t) \rangle \\ &= \langle \hat{\rho}(\mathbf{x}, t) \hat{\rho}(\mathbf{y}, t) \rangle - \delta(\mathbf{x} - \mathbf{y}) \langle \hat{\rho}(\mathbf{y}, t) \rangle \\ &\quad - \langle \hat{\rho}(\mathbf{x}, t) \rangle \langle \hat{\rho}(\mathbf{y}, t) \rangle, \end{aligned} \quad (39)$$

where the operators between colons are normal ordered and  $\hat{\rho}$  is the density operator whose exact expression (valid at all orders) is

$$\hat{\rho} = \hat{\Psi}^\dagger \hat{\Psi} = \rho (1 + \hat{\eta}). \quad (40)$$

At the order of the expansion considered in Sec. III B (which is the relevant one) this yields directly

$$g^{(2)}(\mathbf{x}, t) \equiv \frac{G^{(2)}(\mathbf{x}, \mathbf{x}, t)}{\rho^2(\mathbf{x}, t)} = \langle \hat{\eta}^2(\mathbf{x}, t) \rangle - \frac{\delta(\mathbf{0})}{\rho_{\text{GP}}(\mathbf{x}, t)}. \quad (41)$$

This expression is regular in one dimension (1D): the  $\delta$  peak contribution exactly cancels an ultraviolet divergence in  $\langle \hat{\eta}^2 \rangle$  computed from the solution of the Bogoliubov equation. It is ultraviolet diverging in two and three dimensions (2D and 3D) (see below).

We will also need to evaluate the average of the combination

$$\hat{a}^\dagger \hat{a} + \hat{a} \hat{a} = \frac{1}{2} \hat{\eta}^2 + \frac{i}{2} [\hat{\eta}, \hat{\theta}] + \frac{i}{2} (\hat{\eta} \hat{\theta} + \hat{\theta} \hat{\eta}), \quad (42)$$

where the explicit  $(\mathbf{x}, t)$  dependence of all the terms has been omitted for legibility and the second member has been obtained by use of Eq. (14) and of the relation

$$\hat{\theta} \hat{\eta} = -\frac{1}{2} [\hat{\eta}, \hat{\theta}] + \frac{1}{2} (\hat{\eta} \hat{\theta} + \hat{\theta} \hat{\eta}). \quad (43)$$

Relation (28) and the fact that operators  $\hat{\psi}$  and  $\hat{\psi}^\dagger$  obey the standard Bose commutation relations (4) impose<sup>1</sup>

$$[\hat{\eta}(\mathbf{x}, t), \hat{\theta}(\mathbf{y}, t)] = \frac{i}{\rho_{\text{GP}}(\mathbf{x}, t)} \delta(\mathbf{x} - \mathbf{y}), \quad (44)$$

and from (41) the average of (42) thus reads as

$$\langle \hat{a}^\dagger \hat{a} + \hat{a} \hat{a} \rangle = \frac{1}{2} g^{(2)} + i \text{Re} \langle \hat{\eta} \hat{\theta} \rangle. \quad (45)$$

<sup>1</sup>Note that the commutation relation (44) is not exact. It may be shown that it holds up to third order in the quantum fields, which is more than sufficient for our perturbative scheme.

This relation makes it possible to derive an expression for  $g^{(2)}(\mathbf{x}, t)$  alternative to (41):

$$g^{(2)} = \langle \hat{a}^\dagger \hat{a} + \hat{a} \hat{a} \rangle + \text{c.c.}, \quad (46)$$

where c.c. stands for complex conjugate

We also need to evaluate the average of operator  $\hat{A}$  defined in Eq. (14). Use of relation (43) leads to

$$\begin{aligned} \langle \hat{A} \rangle &= -\frac{1}{2} \langle \hat{a}^\dagger \hat{a} \rangle + \frac{1}{4} \langle \hat{a}^2 - \hat{a}^{\dagger 2} \rangle \\ &= \left\langle -\frac{1}{2} \hat{\theta}^2 - \frac{1}{8} \hat{\eta}^2 - \frac{i}{4} [\hat{\eta}, \hat{\theta}] \right\rangle + \frac{i}{4} \langle \hat{\eta} \hat{\theta} + \hat{\theta} \hat{\eta} \rangle \\ &= \left\langle -\frac{1}{2} \hat{\theta}^2 - \frac{1}{8} \hat{\eta}^2 + \frac{1}{4} \frac{\delta(\mathbf{0})}{\rho_{\text{GP}}} \right\rangle + \frac{i}{2} \text{Re} \langle \hat{\eta} \hat{\theta} \rangle \\ &= -\frac{1}{2} \left\langle \hat{\theta}^2 - \frac{1}{4} \frac{\delta(\mathbf{0})}{\rho_{\text{GP}}} \right\rangle - \frac{1}{8} g^{(2)} + \frac{i}{2} \text{Re} \langle \hat{\eta} \hat{\theta} \rangle, \end{aligned} \quad (47)$$

where Eq. (41) has been employed to obtain the last expression. The term between brackets in the third line of (47) is identical to the first average in the right-hand side of the first line. It is known as the depletion of the condensate. It is ultraviolet convergent in dimensions 1, 2, and 3, but phase fluctuations make it infrared diverging in one dimension at zero temperature. It is this effect which forbids *bona fide* Bose-Einstein condensation in 1D. However, it has been shown in [45] that this divergence is position independent and thus killed by the spatial derivative acting on  $\langle \hat{A} \rangle$  in Eq. (54) below.

Explicit expressions of quantities such as those appearing in the last line of Eq. (47) are given in Appendix C in terms of a Bogoliubov expansion in the presence of a sonic horizon.

#### D. Second order: Backreaction for the classical fields

Once the Bogoliubov equation is solved, we are able, thanks to the expressions derived in the previous section, to explicitly compute the averages of quantum fields appearing in the backreaction equation (22). It is convenient here to write  $\delta\Phi = \varphi_{\text{GP}} \delta\Phi / \varphi_{\text{GP}}$  and to use a formula exactly analogous to Eq. (29):

$$(i\hbar\partial_t - L)\delta\Phi = \varphi_{\text{GP}}(-g|\varphi_{\text{GP}}|^2 + i\hbar\partial_t - \mathcal{L})\frac{\delta\Phi}{\varphi_{\text{GP}}}, \quad (48)$$

where  $L$  and  $\mathcal{L}$  are defined by Eqs. (23) and (32), respectively. Then, the use of Eq. (21) and simple manipulations make it possible to eliminate  $\delta\Phi$  from (22) and to recast this equation under the form

$$\begin{aligned} (i\hbar\partial_t - \mathcal{L})\left(\frac{\delta\varphi}{\varphi_{\text{GP}}} + \langle \hat{A} \rangle\right) - g\rho_{\text{GP}}\left(\frac{\delta\varphi}{\varphi_{\text{GP}}} + \frac{\delta\varphi^*}{\varphi_{\text{GP}}^*}\right) \\ = g\rho_{\text{GP}}(2\langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a} \hat{a} \rangle + \langle \hat{A} \rangle + \langle \hat{A}^\dagger \rangle) \\ = g\rho_{\text{GP}}(\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}). \end{aligned} \quad (49)$$

In the above expression the second line has been simplified, resulting in the final line, by observing that  $\langle \hat{A} \rangle + \langle \hat{A}^\dagger \rangle = -\langle \hat{a}^\dagger \hat{a} \rangle$ , as can be directly checked from expressions (14). This is an important step because in 1D the averages  $\langle \hat{a}^\dagger \hat{a} \rangle$  and  $\langle \hat{A} \rangle$  which appear in the second line of Eq. (49) are ill defined due to infrared-divergent phase fluctuations, whereas the term

$\langle \hat{a}^\dagger \hat{a} + \hat{a} \hat{a} \rangle$  is infrared regular. In 2D and 3D this source term is correctly taken into account by a proper renormalization of the coupling constant which cancels an ultraviolet divergence in the anomalous average  $\langle \hat{a} \hat{a} \rangle$  (see Sec. IV below or also, e.g., Refs. [30,39]). Hence, whereas Eq. (22) contains troublesome divergent terms, the only possible divergence in (49) comes from the  $\langle \hat{A} \rangle$  term in the left-hand side. However, it is acted upon by space and time derivatives which kill its diverging part as shown in [45] and verified below.

Note that the external potential  $U(\mathbf{x})$  does not appear in Eq. (49). It is nonetheless implicitly included through  $\varphi_{\text{GP}}(\mathbf{x}, t)$ . Indeed, the fact that  $\varphi_{\text{GP}}$  is a solution of the Gross-Pitaevskii equation in the presence of  $U$  has been used in Eq. (48) for removing the external potential from the equations. The same remark holds for Eqs. (34), (35), and (38).

In the same way as we have written  $\varphi = \varphi_{\text{GP}} + \delta\varphi$  in Eq. (20), we may expand the density  $\rho$  and the phase  $\Theta$  defined in (12) as

$$\rho = \rho_{\text{GP}} + \delta\rho, \quad \Theta = \Theta_{\text{GP}} + \delta\Theta, \quad (50)$$

where  $\rho_{\text{GP}}$  and  $\Theta_{\text{GP}}$  are defined in (25). This leads to

$$\frac{\delta\varphi}{\varphi_{\text{GP}}} = \frac{1}{2} \frac{\delta\rho}{\rho_{\text{GP}}} + i\delta\Theta. \quad (51)$$

The quantities  $\delta\rho$  and  $\delta\Theta$  are the modifications of the density and the phase of the classical field induced by the quantum fluctuations. In order to write explicitly the equations they obey, it suffices to write the imaginary and real parts of (49).

For properly following the different steps of the computation, we use Eqs. (45) and (51) for rewriting (49) as

$$\begin{aligned} (i\hbar\partial_t - \mathcal{L})\left(\frac{1}{2} \frac{\delta\rho}{\rho_{\text{GP}}} + i\delta\Theta + \langle \hat{A} \rangle\right) - g\delta\rho \\ = \frac{1}{2} g\rho_{\text{GP}}(g^{(2)} + i\langle \hat{\eta} \hat{\theta} + \hat{\theta} \hat{\eta} \rangle). \end{aligned} \quad (52)$$

Let us first consider the real part of (52). From Eq. (37) it reads as

$$\begin{aligned} \mathcal{T}\delta\Theta - \mathcal{X} \frac{\delta\rho}{2\rho_{\text{GP}}} + \frac{g\delta\rho}{\hbar} = \mathcal{X} \text{Re} \langle \hat{A} \rangle - \mathcal{T} \text{Im} \langle \hat{A} \rangle \\ - \frac{g\rho_{\text{GP}}}{2\hbar} g^{(2)}. \end{aligned} \quad (53)$$

Using the explicit expressions (36) and (47) this reads as

$$\begin{aligned} \hbar(\partial_t + \mathbf{V}_{\text{GP}} \cdot \nabla) \left( \delta\Theta - \frac{1}{2} \text{Re} \langle \hat{\eta} \hat{\theta} \rangle \right) + g \left( \delta\rho + \frac{1}{2} \rho_{\text{GP}} g^{(2)} \right) \\ = \frac{-\hbar^2}{4m\rho_{\text{GP}}} \nabla \cdot \left[ \rho_{\text{GP}} \nabla \left( \frac{\delta\rho}{\rho_{\text{GP}}} - \langle \hat{\theta}^2 \rangle + \frac{\delta(\mathbf{0})}{4\rho_{\text{GP}}} - \frac{g^{(2)}}{4} \right) \right]. \end{aligned} \quad (54)$$

Equation (54) is an unsteady Bernoulli equation. Its classical version is usually obtained by performing a Madelung transform on the Gross-Pitaevskii equation. We consider here the next order which involves quadratic quantum contributions and the corresponding modifications of the density and phase of the classical field. In the first line it involves the classical modification of the phase (the velocity potential) together with a quantum fluctuation term ( $\text{Re} \langle \hat{\eta} \hat{\theta} \rangle$ ) and the interacting term resulting from the nonlinear dependence of pressure on density, also affected by quantum fluctuations (the  $g^{(2)}$  factor).

The second line is the contribution of the so-called quantum pressure term which involves the modification  $\delta\rho$  of the classical density and a quantum correction corresponding to the “quantum depletion”  $\langle \hat{a}^\dagger \hat{a} \rangle = \langle \hat{\theta}^2 \rangle - \frac{1}{4} \delta(\mathbf{0}) / \rho_{\text{GP}} + \frac{1}{4} g^{(2)}$  [see Eq. (47)]. At this point it is important to stress that the infrared divergence of  $\langle \hat{a}^\dagger \hat{a} \rangle$  in one dimension is killed by the spatial derivative in (54), a result which has been proven in Ref. [45]. Thus, Eq. (54) can *a priori* be solved in any dimension, as illustrated below.

It is shown in Appendix D that the imaginary part of (52) can be cast in the form of a continuity equation

$$\partial_t \delta\rho + \nabla \cdot (\mathbf{V}_{\text{GP}} \delta\rho + \rho_{\text{GP}} \delta\mathbf{V} + \text{Re}\langle \rho_{\text{GP}} \hat{\eta} \hat{\mathbf{v}} \rangle) = 0, \quad (55)$$

where

$$\delta\mathbf{V} = \frac{\hbar}{m} \nabla \delta\Theta \quad \text{and} \quad \hat{\mathbf{v}} = \frac{\hbar}{m} \nabla \hat{\theta}. \quad (56)$$

Equation (55) is of the form

$$\partial_t \delta\rho + \nabla \cdot \delta\mathbf{J} = 0, \quad (57)$$

where

$$\delta\mathbf{J}(\mathbf{x}, t) = \mathbf{V}_{\text{GP}} \delta\rho + \rho_{\text{GP}} \delta\mathbf{V} + \text{Re}\langle \rho_{\text{GP}} \hat{\eta} \hat{\mathbf{v}} \rangle \quad (58)$$

represents the modification of the classical conserved current induced by backreaction effects. The various contributions to  $\delta\mathbf{J}$  illustrate that the classical current is influenced not only by the variations  $\delta\rho$  and  $\delta\mathbf{V}$  in the classical density and velocity fields, but also by the term  $\text{Re}\langle \rho_{\text{GP}} \hat{\eta} \hat{\mathbf{v}} \rangle$ , which arises from quantum corrections. This contribution corresponds to the expectation value  $\langle \hat{\mathbf{j}} \rangle$  of the quantum current operator

$$\hat{\mathbf{j}}(\mathbf{x}, t) = \frac{1}{2} (\hat{\rho} \hat{\mathbf{v}} + \hat{\mathbf{v}} \hat{\rho}) \quad (59)$$

evaluated at the order appropriate to our second-order expansion.

Equations (54) and (55) are the main result of this paper. They describe the dynamics of the leading-order modification of the classical background flow induced by quantum fluctuations. They are valid even in a time-dependent flow for which the zeroth-order Gross-Pitaevskii equation (19) and the first-order Bogoliubov equations (24) (together with their solutions) are explicitly position and time dependent. They constitute an extension of the results obtained by Mora and Castin in Ref. [30] to a possibly nonstationary flow in the presence of a background current. The amplitude-phase formalism makes it possible, without using a number-conserving approach, to circumvent the infrared divergence problem encountered in other studies [22,23]. Our treatment pertains to the so-called time-dependent Hartree-Fock-Bogoliubov scheme (see, e.g., discussions in Ref. [46]). Note, however, that we do not push the development as far as to include perturbatively coupled time-dependent equations for the condensate and noncondensate components of the Bose gas, as done by Giorgini in Ref. [42] for instance.

#### IV. STATIONARY CONFIGURATIONS

In the following we will consider a situation where the zeroth-order Gross-Pitaevskii flow is stationary, described by time-independent density  $\rho_{\text{GP}}(\mathbf{x})$  and velocity  $\mathbf{V}_{\text{GP}}(\mathbf{x})$ . We look for a stationary solution of the backreaction equations (54)

and (55) for which the quantities  $\delta\rho$  and  $\delta\mathbf{V}$  are also time independent. In such a configuration  $\delta\Theta$  is nonetheless typically time dependent and we introduce the quantity

$$\delta\mu = -\hbar \partial_t \delta\Theta. \quad (60)$$

Then Eqs. (54) and (55) simplify to

$$\begin{aligned} m\mathbf{V}_{\text{GP}} \cdot \delta\mathbf{V} - \frac{\hbar^2}{4m\rho_{\text{GP}}} \nabla \cdot \left( \rho_{\text{GP}} \nabla \frac{\delta\rho}{\rho_{\text{GP}}} \right) + g\delta\rho \\ = \delta\mu - \frac{1}{2} g \rho_{\text{GP}} g^{(2)} - \frac{\hbar}{2} \mathbf{V}_{\text{GP}} \cdot \nabla \text{Re}\langle \hat{\eta} \hat{\theta} \rangle \\ - \frac{\hbar^2}{4m\rho_{\text{GP}}} \nabla \cdot \left[ \rho_{\text{GP}} \nabla \left( \langle \hat{\theta}^2 \rangle - \frac{1}{4} \frac{\delta(\mathbf{0})}{\rho_{\text{GP}}} + \frac{1}{4} g^{(2)} \right) \right], \end{aligned} \quad (61)$$

and

$$\nabla \cdot (\mathbf{V}_{\text{GP}} \delta\rho + \rho_{\text{GP}} \delta\mathbf{V} + \text{Re}\langle \rho_{\text{GP}} \hat{\eta} \hat{\mathbf{v}} \rangle) = 0. \quad (62)$$

It is easily verified that  $\delta\mu$  is position independent for such stationary solutions since, from the definitions (56) and (60),  $\nabla \delta\mu = -m \partial_t \delta\mathbf{V}$ . Besides, Eq. (61) proves that  $\delta\mu$  is also time independent since all the other contributions in this equation are.  $\delta\mu$  is thus a constant, it is the modification of the chemical potential induced by the backreaction equations.

#### A. Uniform and stationary configurations

As a simple test of the validity of the backreaction equations (61) and (62), we first consider a stationary and uniform system [with  $U(\mathbf{x}) = 0$ ] at zero temperature in the absence of current in dimension  $d$ . In such a system  $\mathbf{V}_{\text{GP}} = 0 = \langle \hat{\mathbf{j}} \rangle$  and all the ingredients of Eq. (61) (such as  $\rho_{\text{GP}}$ ,  $g^{(2)}$ ,  $\delta\rho$ , ...) are time and position independent. Equation (19) reads as  $\mu_{\text{GP}} = g\rho_{\text{GP}}$  where  $\mu_{\text{GP}}$  is the value of the chemical potential of a system of constant density  $\rho_{\text{GP}}$  according to the Gross-Pitaevskii approach. Likewise, Eq. (61) reduces to  $\delta\mu = g\delta\rho + \frac{1}{2} g \rho_{\text{GP}} g^{(2)}$ . Combining these two results yields (noting  $\mu = \mu_{\text{GP}} + \delta\mu$  and  $\rho = \rho_{\text{GP}} + \delta\rho$ )

$$\mu = g\rho + \frac{1}{2} g \rho_{\text{GP}} g^{(2)}. \quad (63)$$

This is known as the Hartree-Fock-Bogoliubov result for the chemical potential [37]. This expression differs from the Hugenholtz-Pines result [47] indicating that our approach, if pursued at higher order, would yield a gapped spectrum: in terms of the Hohenberg-Martin classification the present approach is conserving and not gapless, see discussions in [37,43]. However, it is perfectly legitimate for our purpose, as illustrated in Secs. IV A 1 and IV A 2 (see also Refs. [30,39]). Moreover, the violation of the Hugenholtz-Pines theorem is not necessarily prohibitive. A possible approach to recover a gapless spectrum, along with a corrected phonon velocity and damping rate, is to analyze the linear response of the classical field within a perturbative framework, as demonstrated by Giorgini in Ref. [42].

#### 1. The three-dimensional case

In three dimensions, the last term of the right-hand side of expression (63) has an ultraviolet divergence, associated to the anomalous average  $\langle \hat{a} \hat{a} \rangle$  in (46). But, expanding the coupling constant to second order in the scattering length regularizes

this divergence (see, e.g., [48,49]), leading to the Lee-Huang-Yang expression [7,50]

$$\mu = g\rho + \frac{4g}{3\pi^2\xi^3} = g\rho \left[ 1 + \frac{32}{3\sqrt{\pi}} \sqrt{a^3\rho} \right], \quad (64)$$

where

$$\xi = \frac{\hbar}{\sqrt{mg\rho}} \quad (65)$$

is the healing length and  $a$  the 3D scattering length with, at leading order,  $g = 4\pi\hbar^2 a/m$ .

## 2. The one-dimensional case

We are mainly interested in 1D configurations where the situation is actually simpler because in this case  $g^{(2)}$  given from the Bogoliubov expression (46) is regular. We get [51]

$$g^{(2)} = -\frac{2}{\pi\rho_{\text{GP}}\xi_{\text{GP}}}, \quad (66)$$

leading from (63) to

$$\mu = g\rho - \frac{g}{\pi\xi}. \quad (67)$$

In this expression we replaced in the corrective term the Gross-Pitaevskii healing length  $\xi_{\text{GP}}$  by  $\xi$ , which is legitimate. The result (67) has already been obtained in Ref. [30] and corresponds to the weak interaction expansion of the exact Lieb-Liniger result [52]. Inserted in the thermodynamic relation  $mc^2 = \rho(\partial\mu/\partial\rho)$  it yields

$$c = \frac{1}{\sqrt{m}} \sqrt{g\rho - \frac{g}{2\pi\xi}}. \quad (68)$$

This formula defines the speed of sound  $c$  as a function of the density  $\rho$  in a 1D homogeneous system at zero temperature. It corrects the Gross-Pitaevskii formula  $c_{\text{GP}} = (g\rho_{\text{GP}}/m)^{1/2}$ . It is clear from (67) that in 1D the dimensionless small parameter of our approach is  $(\rho\xi)^{-1}$ , implying that our results are expected to be valid only in the limit  $\rho\xi \gg 1$ . However, it was observed by Lieb [53] that expression (68) agrees very well with the exact result down to  $\rho\xi \sim 0.3$ . In the case of interest for us, a typical experimental order of magnitude is  $\rho\xi \approx 30 \leftrightarrow 60$  [54], i.e., quite far from the regime where (68) becomes incorrect.

### B. Analogous black hole in a 1D stationary transonic flow

We now come to the main interest of our study: backreaction effects in a 1D flow mimicking a black hole. Different black-hole configurations have been proposed in the past and we focus in this work on the ones presented in Appendix A, which we denote as waterfall,  $\delta$  peak, and flat profile.

Since we work in 1D, from now on we do not consider vectors, but their unique Cartesian coordinate. For instance we no longer use  $\nabla$  and  $\delta\mathbf{V}$  but  $\partial_x$  and  $\delta V$  instead. Then Eqs. (61)

and (62) read as

$$\begin{aligned} mV_{\text{GP}}\delta V - \frac{\hbar^2}{4m\rho_{\text{GP}}}\partial_x\left(\rho_{\text{GP}}\partial_x\frac{\delta\rho}{\rho_{\text{GP}}}\right) + g\delta\rho \\ = \delta\mu - \frac{1}{2}g\rho_{\text{GP}}g^{(2)} - \frac{\hbar}{2}V_{\text{GP}}\partial_x\text{Re}(\hat{\eta}\hat{\theta}) \\ - \frac{\hbar^2}{4m\rho_{\text{GP}}}\partial_x\left[\rho_{\text{GP}}\partial_x\left(\langle\hat{\theta}^2\rangle - \frac{1}{4}\frac{\delta(0)}{\rho_{\text{GP}}} + \frac{1}{4}g^{(2)}\right)\right] \end{aligned} \quad (69)$$

and

$$V_{\text{GP}}\delta\rho + \rho_{\text{GP}}\delta V + \text{Re}\langle\rho_{\text{GP}}\hat{\eta}\hat{v}\rangle = \delta J. \quad (70)$$

Whereas all the quantities in (69) and (70) [such as  $\delta\rho(x)$ ,  $\rho_{\text{GP}}(x)$ ,  $g^{(2)}(x)$ , ...] are position-dependent fields,  $\delta\mu$  and  $\delta J$  are time and position independent.

In the configurations we consider (waterfall,  $\delta$  peak, or flat profile, see Sec. A), there are two asymptotic regions, the far upstream and the far downstream one, for which  $\rho_{\text{GP}}$ ,  $V_{\text{GP}}$  and all the source terms in (69) and (70) are not only time, but also position independent. The relevant quantities being

$$\begin{aligned} \rho_{\text{GP}}^\alpha = \lim_{x\rightarrow\pm\infty}\rho_{\text{GP}}(x), \quad g_\alpha^{(2)} = \lim_{x\rightarrow\pm\infty}g^{(2)}(x), \\ V_{\text{GP}}^\alpha = \lim_{x\rightarrow\pm\infty}V_{\text{GP}}(x), \quad j_\alpha = \lim_{x\rightarrow\pm\infty}\text{Re}\langle\rho_{\text{GP}}\hat{\eta}\hat{v}\rangle(x), \end{aligned} \quad (71)$$

where the index  $\alpha = d$  ( $u$ ) in the limit  $x \rightarrow +\infty$  ( $-\infty$ ). Finally, the asymptotic Gross-Pitaevskii speed of sound and healing length are denoted as  $c_{\text{GP}}^\alpha = (g\rho_{\text{GP}}^\alpha/m)^{1/2}$  and  $\xi_{\text{GP}}^\alpha = \hbar/mc_{\text{GP}}^\alpha$ , respectively.

In the asymptotic regions, the backreaction equations assume a quite simple form. In particular the modification of the density is governed by the following equation:

$$-\frac{\hbar^2}{4m}\partial_x^2\delta\rho + m[(c_{\text{GP}}^\alpha)^2 - (V_{\text{GP}}^\alpha)^2]\delta\rho = S_\alpha, \quad (72)$$

where

$$S_\alpha = \rho_{\text{GP}}^\alpha\left[\delta\mu - \frac{1}{2}m(c_{\text{GP}}^\alpha)^2g_\alpha^{(2)}\right] + mV_{\text{GP}}^\alpha[j_\alpha - \delta J] \quad (73)$$

is a constant source term. The corresponding solutions are of the form

$$\delta\rho(x) = \begin{cases} \delta\rho_u + \mathcal{A}_u \exp(\kappa_u x) & \text{when } x \rightarrow -\infty, \\ \delta\rho_d + \mathcal{A}_d \sin(\kappa_d x + \phi) & \text{when } x \rightarrow +\infty, \end{cases} \quad (74)$$

where

$$\delta\rho_\alpha = \frac{S_\alpha}{m[(c_{\text{GP}}^\alpha)^2 - (V_{\text{GP}}^\alpha)^2]} \quad (75)$$

and

$$\kappa_\alpha = \frac{2m}{\hbar} |(c_{\text{GP}}^\alpha)^2 - (V_{\text{GP}}^\alpha)^2|^{1/2}. \quad (76)$$

Expression (75) faces the risk of divergence when the Mach number  $M_{\text{GP}}^\alpha = V_{\text{GP}}^\alpha/c_{\text{GP}}^\alpha$  tends to 1. However, it holds also in the homogeneous case for which no such velocity-dependent divergence should occur since, by a Galilean transform it is always possible to work in a reference frame where the flow velocity cancels, which is the situation studied in Sec. IV A 2. This implies that, in the specific situations considered below, the source term (73) should also cancel when  $M_{\text{GP}}^\alpha$  tends to unity. We will make this check in due time [cf. Eqs. (81) and (82)].

The asymptotic modifications of the velocity are of a form similar to that of the density

$$\delta V(x) = \begin{cases} \delta V_u + \mathcal{B}_u \exp(\kappa_u x) & \text{when } x \rightarrow -\infty, \\ \delta V_d + \mathcal{B}_d \sin(\kappa_d x + \phi) & \text{when } x \rightarrow +\infty, \end{cases} \quad (77)$$

where  $\mathcal{B}_\alpha = -\mathcal{A}_\alpha V_{\text{GP}}^\alpha / \rho_{\text{GP}}^\alpha$  and

$$\delta V_\alpha = \frac{1}{\rho_{\text{GP}}^\alpha} (\delta J - j_\alpha - V_{\text{GP}}^\alpha \delta \rho_\alpha). \quad (78)$$

The asymptotic expressions (74) and (77) show that, whereas the upstream density and velocity modifications tend to constant values, the downstream asymptotic profile typically displays undulations with a wave vector  $\kappa_d$  associated to a zero-energy channel (identified in Fig. 3 in Appendix B). Such stationary undulations have been predicted to appear in the supersonic region of a white-hole configuration nonlinearly stimulated by an external seed impinging from the subsonic region [55]. A similar stimulated process is not possible in a black-hole configuration because the group velocity of the corresponding zero mode is directed outward the supersonic region,<sup>2</sup> and thus cannot be excited by a source located outside the black hole. In our case, although we consider a black-hole configuration, the appearance of undulations is, however, perfectly legitimate because it results from a *spontaneous* process, for which the effective “seed” (the source term  $S_d$  which accounts for quantum fluctuations) exists throughout the whole supersonic region.

The precise asymptotic behavior of  $\delta \rho(x)$  and  $\delta V(x)$  depends on the constants  $\mathcal{A}_\alpha$  and  $\phi$  in (74) and (77) which can only be determined by a full solution of the backreaction equations (69) and (70). This, in turn, necessitates a determination of the source terms in the whole physical space, which can be accurately done only by correctly taking account of zero-mode solutions of the Bogoliubov equations [30]. It has been shown in Ref. [56] that this is indeed crucial for accurately determining the local density-density correlation function  $g^{(2)}(x)$  in the vicinity of the horizon. However, the contributions of the zero modes are not necessary when considering the upstream and downstream asymptotic regions, far away from the horizon. This noticeably simplifies the computation of the contribution of the quantum fluctuations to the asymptotic local density-density correlation function  $g_\alpha^{(2)}$  and to the asymptotic current  $j_\alpha$ . We show in Appendix C that they are of the form

$$g_\alpha^{(2)} = \frac{1}{\rho_{\text{GP}}^\alpha \xi_{\text{GP}}^\alpha} \left( -\frac{2}{\pi} + \mathcal{G}_\alpha^{(H)} \right), \quad (79a)$$

$$j_\alpha = \frac{c_{\text{GP}}^\alpha}{\xi_{\text{GP}}^\alpha} \mathcal{J}_\alpha^{(H)}. \quad (79b)$$

These expressions encompass the standard quantum fluctuation (66) of  $g_\alpha^{(2)}$  together with additional terms denoted as  $\mathcal{G}_\alpha^{(H)}$  and  $\mathcal{J}_\alpha^{(H)}$  which are induced by the quantum Hawking radiation and cancel in the absence of sonic horizon. These terms are determined numerically as detailed in Eqs. (C8), (C9), (C13), and (C14).

<sup>2</sup>In the terminology of Appendix B, the zero-energy channel is not outgoing (see Fig. 3).

In the present pilot study we focus on the mean values  $\delta \rho_\alpha$  and  $\delta V_\alpha$  of the asymptotic behaviors (74) and (77) and defer a full numerical solution of the backreaction equations to a future work. It should be kept in mind that, because of the downstream undulations, the modifications  $\delta \rho_d$  and  $\delta V_d$  are average quantities. The situation is simpler in the upstream region where the asymptotic profile is flat. Hence, it is more appropriate to solve Eqs. (75) and (78) fixing boundary conditions in the upstream region where the modified flow pattern is asymptotically featureless (constant density and velocity). In an equilibrium situation, in the absence of background flow velocity, it was argued in Ref. [30] that the appropriate condition is  $\delta \mu = 0$ , which corresponds to a grand canonical situation. In a homogeneous system, in the presence of a constant background flow, Galilean invariance furthermore imposes that the beyond-mean-field effects do not modify the flow velocity.<sup>3</sup> Because of the asymptotic homogeneity of the upstream flow we thus impose the similar condition  $\delta V_u = 0$ . From Eqs. (73), (75), and (78) these two conditions determine  $\delta J$ ,  $\delta \rho_u$ ,  $\delta \rho_d$ , and  $\delta V_d$ .

In the framework of the Gross-Pitaevskii approximation, it has been shown that the main characteristics of the analog black hole and of the associated Hawking radiation are determined by the values of the asymptotic Mach numbers  $M_{\text{GP}}^\alpha = V_{\text{GP}}^\alpha / c_{\text{GP}}^\alpha$  (see, e.g., Ref. [57]). It is thus important to evaluate to which extent these quantities are affected by the modifications  $\delta \rho_\alpha$  and  $\delta V_\alpha$  of the asymptotic densities. Backreaction effects modify the Gross-Pitaevskii result to  $M_\alpha = M_{\text{GP}}^\alpha + \delta M_\alpha$  with

$$\frac{\delta M_\alpha}{M_{\text{GP}}^\alpha} = \frac{\delta V_\alpha}{V_{\text{GP}}^\alpha} - \frac{\delta c_\alpha}{c_{\text{GP}}^\alpha} = \frac{\delta V_\alpha}{V_{\text{GP}}^\alpha} - \frac{1}{2} \frac{\delta \rho_\alpha}{\rho_{\text{GP}}^\alpha} + \frac{1}{4\pi \xi_{\text{GP}}^\alpha \rho_{\text{GP}}^\alpha}, \quad (80)$$

where use has been made of expression (68) for the one-dimensional speed of sound. In the upper panel of Fig. 1 we display the relative modifications  $\delta M_\alpha / M_{\text{GP}}^\alpha$  for the waterfall configuration (the results for other configurations are presented in Appendix E). In region  $\alpha$  ( $\alpha = u$  or  $d$ ) this modification is proportional to the small parameter  $(\rho_{\text{GP}}^\alpha \xi_{\text{GP}}^\alpha)^{-1}$ , the value of which depends of the experimental situation, and is typically of order 0.02 in Steinhauer’s experiment [54]. This is the reason why  $\delta M_\alpha / M_{\text{GP}}^\alpha$  is rescaled by  $\rho_{\text{GP}}^\alpha \xi_{\text{GP}}^\alpha$  in the upper panel of this figure. In the same plot, the result obtained when discarding the contribution of Hawking radiation [i.e., by removing the contributions  $\mathcal{G}_\alpha^{(H)}$  and  $\mathcal{J}_\alpha^{(H)}$  in Eq. (79)] are represented by dashed lines. The blue horizontal dashed line corresponds to the value  $-1/4\pi$  which would be the rescaled relative modification of the Gross-Pitaevskii Mach number of a homogeneous condensate (constant density  $\rho_{\text{GP}}^\alpha$  and constant velocity  $V_{\text{GP}}^\alpha$ ) induced by beyond-mean-field effects. Concerning the downstream modification  $\delta M_d$  of the Mach number, we stress that it is evaluated from the average quantities  $\delta V_d$  and  $\delta \rho_d$ . When the period  $2\pi/\kappa_d$  of the undulations of the downstream density is small compared to  $\xi_{\text{GP}}^d$  (i.e., when

<sup>3</sup>Consider a uniform flow of constant density and velocity in the laboratory frame. In the comoving frame of the fluid, the Gross-Pitaevskii background velocity vanishes. Since backreaction, i.e., beyond-mean-field corrections, cannot generate a spontaneous flow in this frame, it follows that in the laboratory frame the initial velocity remains unaffected by backreaction.

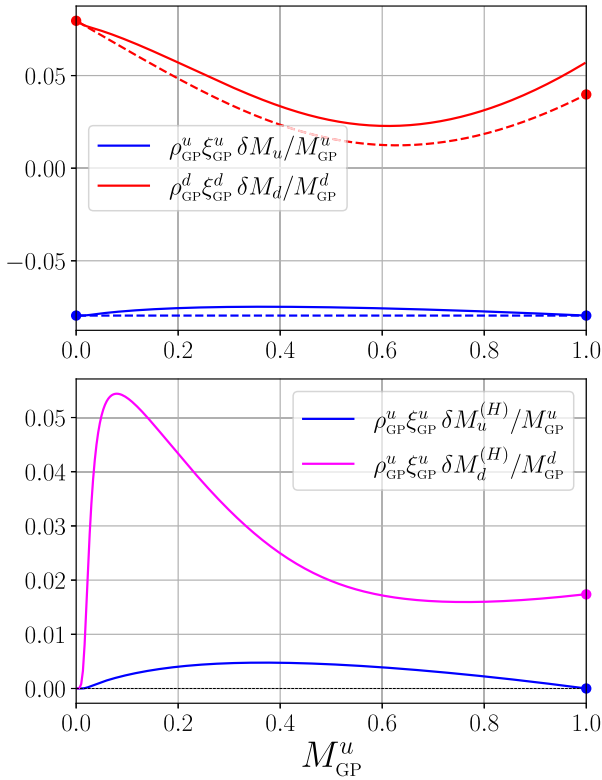


FIG. 1. Effect of backreaction on the asymptotic Mach numbers for the waterfall configuration. The results are plotted as functions of the upstream asymptotic Mach number (computed within the Gross-Pitaevskii approximation)  $M_{\text{GP}}^u$ . Upper plot: the solid lines represent the rescaled relative modifications  $\delta M_u / M_{\text{GP}}^u$  and  $\delta M_d / M_{\text{GP}}^d$  of the upstream and downstream Mach numbers. The dashed lines terminated by circles present the result obtained when discarding the contribution of the Hawking radiation. Lower plot: relative variation of the Mach numbers computed by removing the usual beyond-mean-field contributions and incorporating only the backreaction induced by Hawking radiation. The values of the functions for  $M_{\text{GP}}^u = 1$  are marked with full circles and correspond to expressions (81) and (82).

$M_{\text{GP}}^d \gg 1$ ) it is *a priori* not possible to define a local speed of sound: in this limit  $\delta M_d$  defined by (80) is not the spatial average of an hypothetical local downstream Mach number.

The primary observation drawn from the upper panel of Fig. 1 is that the relative modifications in  $M_u$  and  $M_d$  are small. The upstream Mach number weakly decreases and the downstream one weakly increases. It thus appears that, for the chosen boundary conditions, the transonic character of the flow is not modified by quantum fluctuations and backreaction. The flows are poorly affected in the whole range  $0 < M_{\text{GP}}^u < 1$ , which legitimates our perturbative approach. Another clear feature is that the solid lines depart weakly from the dashed ones. This means that Hawking radiation has a lesser impact on the Mach number than the standard beyond-mean-field quantum fluctuations. However, one should keep in mind that the beyond-mean-field corrections are present even in the absence of an acoustic horizon, i.e., they should be taken into account already in a homogeneous system, before the formation of the analog black hole. It is thus appropriate to distinguish the beyond-mean-field terms from the

backreaction effects truly induced by Hawking radiation. This is the reason why we represent in the lower panel of Fig. 1 the quantities  $\delta M_\alpha^{(H)}$  ( $\alpha = u$  or  $d$ ) which are the difference between the modification of the Mach number computed with and without the Hawking contribution. For being able to compare the relative upstream and downstream influence of Hawking radiation, in the lower panels we rescale  $\delta M_u^{(H)} / M_{\text{GP}}^u$  and  $\delta M_d^{(H)} / M_{\text{GP}}^d$  by the same quantity  $\rho_{\text{GP}}^u \xi_{\text{GP}}^u$ . It appears that Hawking radiation induces positive modification of the upstream and downstream Mach numbers, of roughly the same order. We show in Appendix E that, while the positivity of  $\delta M_u^{(H)}$  observed in the lower panel of Fig. 1 is a general feature due to our specific choice of boundary conditions,  $\delta M_d^{(H)}$  may be positive or negative in other configurations.

It is also important to stress that in the limit  $M_{\text{GP}}^u \rightarrow 1$  [which also imposes  $M_{\text{GP}}^d \rightarrow 1$  from (72)] the source term in (75) cancels and thus  $\delta \rho_\alpha$ ,  $\delta V_\alpha$ , and  $\delta M_\alpha$  do not diverge. Simple algebraic manipulations and the use of expression (C17) show that

$$\lim_{M_{\text{GP}}^u \rightarrow 1} \rho_{\text{GP}}^u \xi_{\text{GP}}^u \frac{\delta M_u}{M_{\text{GP}}^u} = -\frac{1}{4\pi}, \quad (81)$$

and that for the waterfall configuration

$$\lim_{M_{\text{GP}}^u \rightarrow 1} \rho_{\text{GP}}^d \xi_{\text{GP}}^d \frac{\delta M_d}{M_{\text{GP}}^d} = \frac{1}{8\pi} + \frac{9}{8}(\mathcal{H}_u - \mathcal{H}_d), \quad (82)$$

where  $\mathcal{H}_u$  and  $\mathcal{H}_d$  are defined in (C17). This regular behavior can be attributed to two factors: the cancellation of the source term (73) in the limit of unit Mach number [with the Hawking contribution likewise canceling, as noted in (C17)], and the fact that  $M_{\text{GP}}^d$  approaches 1 as  $M_{\text{GP}}^u$  does.

## V. CONCLUSION AND PERSPECTIVES

In this work we have obtained backreaction equations [Eqs. (54) and (55)] describing the effect of quantum fluctuations onto the background flow in a Bose-Einstein condensate. These equations have been derived in arbitrary dimension  $d$ , for a possibly current-carrying, or nonstationary background, in the limit where the quantity  $\rho \xi^d$  is large compared to unity.

Our main interest lies in 1D situations with a transonic flow realizing a sonic horizon. In this case, we solved the stationary backreaction equations far from the acoustic horizon (deep in the upstream and downstream regions). Whereas the asymptotic upstream flow tends to a homogeneous limit, the existence of downstream channels of zero energy (themselves resulting from the transonic character of the flow) induces, through backreaction effects, stationary density and velocity undulations in the downstream region. The existence of undulations in the interior of the analog black hole is a nonlinear effect of quantum backreaction, not expected within a Gross-Pitaevskii approach.

A natural extension of our work is to consider a generic situation, such as the ones experimentally realized in Refs. [3,54], and to determine a stationary solution of the backreaction equations, not only in the asymptotic regions, but in the whole physical space. In this case, the theoretical expansion of the quantum fluctuations should explicitly account for the existence of zero modes of Bogoliubov equations

[58]. In such generic situations, on the basis of the orders of magnitudes we obtained in the asymptotic regions, we expect that the backreaction effects should be small, and would not drastically affect the leading-order Gross-Pitaevskii flow profile.

It should be noted that the detailed form of the stationary solutions and of the modifications of the upstream and downstream Mach numbers depend on the boundary conditions imposed at infinity. Simple physical arguments lead to impose  $\delta\mu = 0 = \delta V_u$ , but other more elaborate conditions may be imposed which modify the specifics of the results presented in Figs. 1 and 4–6 (such as the positivity of  $\delta M_u^{(H)}$  for instance). Since the backreaction equations we derived are valid for a time-dependent background flow, a natural way to circumvent this issue would be to study the dynamics of formation of an acoustic horizon in a quasi-1D BEC, as already studied in Refs. [59–65], with account of quantum backreaction effects. Work in this direction is in progress.

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### DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

### APPENDIX A: DIFFERENT BLACK-HOLE CONFIGURATIONS

In this study, we examine three different types of analog black holes, referred to as “waterfall,” “ $\delta$  peak,” and “flat profile” in Ref. [57]. They all correspond to the flow of a one-dimensional BEC which is subsonic in the upstream region and supersonic downstream. The associated order parameter is a stationary solution of the Gross-Pitaevskii equation (19). According to the convention adopted in this work, the corresponding fields and characteristic quantities should be denoted with a subscript “GP.” However, to facilitate readability, we omit this index in this appendix.

In these configurations the background classical field is of the form

$$\Phi(x) = \sqrt{\rho_\alpha} \phi_\alpha(x) \exp(ik_\alpha x), \quad (\text{A1})$$

where the index  $\alpha$  takes the value  $\alpha = u$  for  $x < 0$  (upstream region) and  $\alpha = d$  for  $x > 0$  (downstream region).  $k_\alpha = mV_\alpha/\hbar$  where  $V_\alpha > 0$  is the asymptotic velocity of the flow (in the limit  $x \rightarrow -\infty$  if  $\alpha = u$  and for  $x > 0$  if  $\alpha = d$ ). We have  $\lim_{x \rightarrow \pm\infty} |\phi_\alpha| = 1$  and  $\rho_\alpha$  is thus the asymptotic density

in region  $\alpha$ . The asymptotic speed of sound, Mach number, and healing length are  $c_\alpha = (g\rho_\alpha/m)^{1/2}$ ,  $M_\alpha = V_\alpha/c_\alpha$ , and  $\xi_\alpha = \hbar/mc_\alpha$ , respectively.

Let us first present the specifics of the  $\delta$ -peak and waterfall configurations. In both configurations the chemical potential is  $\mu = \frac{1}{2}mV_u^2 + g\rho_u$  and upstream and downstream quantities are related through the formulas

$$\frac{V_d}{V_u} = \frac{\rho_u}{\rho_d} = \left(\frac{c_u}{c_d}\right)^2 = \left(\frac{\xi_d}{\xi_u}\right)^2. \quad (\text{A2})$$

The first equality in the above relations reflects the conservation of the current. The second stems from the expression of the speed of sound in the Gross-Pitaevskii approximation and the last one from the fact that, still in the Gross-Pitaevskii framework,  $c_\alpha \xi_\alpha = \hbar/m$ .

We have in both configurations

$$\phi_d(x) = \exp(i\beta_d x), \quad (\text{A3})$$

where  $\beta_d$  is a constant, indicating that the order parameter (A1) is a plane wave in the whole downstream region  $x \geq 0$ . In the upstream region ( $x \leq 0$ ) we have instead

$$\phi_u(x) = \cos\theta \tanh\left(\frac{x-x_0}{\xi_u} \cos\theta\right) - i \sin\theta, \quad (\text{A4})$$

indicating that the upstream flow pattern corresponds to a fraction of a dark soliton, which can be assimilated to a plane wave only in the limit  $x \rightarrow -\infty$ . In (A4),  $\theta \in [0, \pi/2]$  with  $\sin\theta = M_u$ .

After having listed properties valid in both configurations, let us now consider the specifics of each of them. For the waterfall configuration the external potential in (19) is a step function  $U(x) = -U_0 \Upsilon(x)$ , where  $\Upsilon$  is the Heaviside function. In this case, the order parameter defined by (A1), (A3), and (A4) is a solution of the Gross-Pitaevskii equation provided

$$x_0 = 0, \quad \beta_d = \pi, \quad \frac{U_0}{g\rho_u} = \frac{M_u^2}{2} + \frac{1}{2M_u^2} - 1, \quad (\text{A5})$$

and

$$\frac{V_d}{V_u} = \frac{1}{M_u^2} = M_d. \quad (\text{A6})$$

The analog black hole realized in the 2019 Technion experiment [3] is close to the waterfall configuration with  $M_d = 2.9$  (cf. the discussion in [56]).

For the  $\delta$ -peak configuration, the external potential in (19) is a  $\delta$  peak  $U(x) = \kappa \delta(x)$ . In this case, denoting as

$$y = -\frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{8}{M_u^2}}, \quad (\text{A7})$$

the order parameter (A1) is a solution of the Gross-Pitaevskii equation provided

$$\begin{aligned} \tanh\left(\frac{x_0}{\xi_u} \cos\theta\right) &= \sqrt{\frac{y-1}{2}} \tan\theta, \\ \sin\beta_d &= -M_u \sqrt{y}, \quad \kappa = \frac{\hbar^2 M_u}{m \xi_u} \left(\frac{y-1}{2}\right)^{3/2}, \end{aligned} \quad (\text{A8})$$

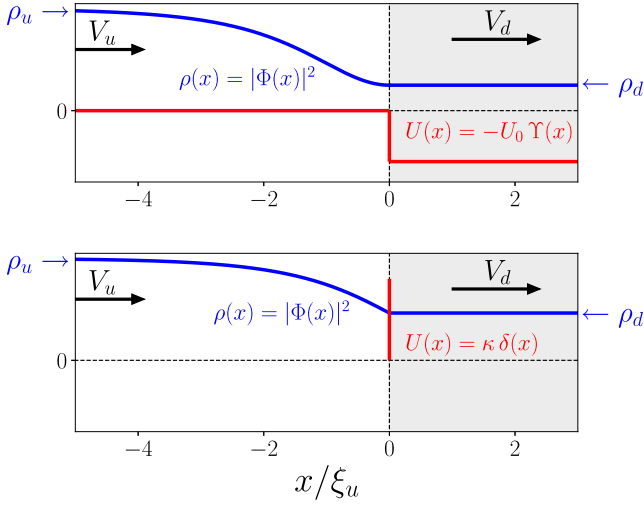


FIG. 2. Upper plot: density profile of the waterfall configuration. Lower plot: density profile of the  $\delta$ -peak configuration. In both cases the order parameter takes the form of a plane wave downstream ( $x > 0$ ), and of a fraction of a gray soliton upstream ( $x < 0$ ). The downstream region is shaded to underline that it corresponds to the interior of the analog black hole. Note that, according to the convention used in the main text, all the quantities in this plot should be written with an index “GP” since they correspond to a (stationary) solution of Gross-Pitaevskii equation (19). These indices are omitted here for legibility.

and

$$\frac{V_d}{V_u} = \left( \frac{M_d}{M_u} \right)^{2/3} = y. \quad (\text{A9})$$

The density profiles of the waterfall and  $\delta$ -peak configurations are sketched in Fig. 2. In both configurations  $M_u \leq 1 \leq M_d$  and the knowledge of  $M_u$  determines  $M_d$  and all the ratios between the upstream and downstream values of the relevant parameters of the flow, as exposed in Eqs. (A2), (72), and (A9).

The flat profile configuration, introduced in Refs. [4,59], is an idealized setting in which the Gross-Pitaevskii density and velocity are constant:  $\rho(x) = \rho_0$  and  $V(x) = V_0$ . An acoustic horizon can be implemented in such a configuration by means of a steplike nonlinear constant  $g(x)$  combined with a steplike external potential  $U(x)$ :

$$g(x) = \begin{cases} g_u & \text{when } x < 0, \\ g_d & \text{when } x > 0. \end{cases} \quad (\text{A10})$$

$$U(x) = \begin{cases} U_u & \text{when } x < 0, \\ U_d & \text{when } x > 0. \end{cases} \quad (\text{A11})$$

The constancy of the density and of the velocity corresponds to a background order parameter of the form (A1) with  $\rho_u = \rho_d = \rho_0$ ,  $k_u = k_d = k_0 = mV_0/\hbar$ , and  $\phi_u(x) = \phi_d(x) = 1$ . The corresponding  $\Phi(x)$  is made a solution of the Gross-Pitaevskii equation by enforcing the relation

$$g_u \rho_0 + U_u = g_d \rho_0 + U_d. \quad (\text{A12})$$

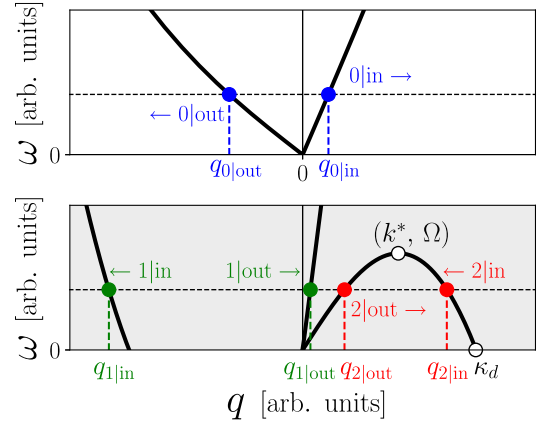


FIG. 3. Upper plot: dispersion relation (B1) in the asymptotic upstream subsonic region. Lower plot: dispersion relation in the downstream supersonic region. This plot is shaded to emphasize that this region corresponds to the interior of the analog black hole. The horizontal dashed line is the angular frequency  $\omega$  of a given excitation. The colored points mark the corresponding excitation channels, of wave vectors  $q_{0|out}(\omega)$ ,  $q_{0|in}(\omega)$ ,  $q_{1|in}(\omega)$ , etc. The arrows indicate the direction of propagation of the different channels. The wave vector  $\kappa_d$  corresponds to a zero-energy channel (see Sec. IV B).

In this configuration the whole of relation (A2) is not verified, but we have separately

$$\frac{V_d}{V_u} = \frac{\rho_u}{\rho_d} = 1 \quad \text{and} \quad \frac{c_u}{c_d} = \frac{\xi_d}{\xi_u} = \frac{M_d}{M_u}. \quad (\text{A13})$$

Note that, at variance with the waterfall and  $\delta$ -peak configurations, for the flat profile configuration the values of  $M_u$  and  $M_d$  are independent one from the other: any choice with  $M_u < 1$  and  $M_d > 1$  is acceptable.

## APPENDIX B: ELEMENTARY EXCITATIONS AND QUANTUM MODES

In an analog black configuration the flow is upstream subsonic and downstream supersonic. The Doppler effect is thus different in these regions, from which it stems that the asymptotic dispersion relations are also different. In both asymptotic regions the flow is uniform (with density  $\rho_\alpha$  and velocity  $V_\alpha$ , where  $\alpha = u$  in the upstream asymptotic region and  $\alpha = d$  downstream, see Appendix A). In these regions the elementary excitations are thus plane waves. Denoting their angular frequency as  $\omega$  and their wave vector as  $q$  we have<sup>4</sup>

$$(\omega - qV_\alpha)^2 = \omega_{\text{B},\alpha}^2(q), \quad (\text{B1})$$

where

$$\omega_{\text{B},\alpha}(q) = c_\alpha q \sqrt{1 + q^2 \xi_\alpha^2 / 4} \quad (\text{B2})$$

is the usual Bogoliubov dispersion relation. In (B2)  $c_\alpha$  and  $\xi_\alpha$  are the speed of sound and healing length in region  $\alpha$ . The dispersion relations (B1) are represented in Fig. 3. There

<sup>4</sup>As in Appendix A, throughout this appendix we omit for legibility all the subscripts “GP” and write  $V_\alpha$  instead of  $V_{\text{GP}}^\alpha$  for instance.

are two upstream propagation channels, which we denote as 0|in and 0|out. The corresponding wave-vectors are denoted as  $q_{0|in}(\omega)$  and  $q_{0|out}(\omega)$ . In the downstream region, below an angular frequency which we denote as  $\Omega$ , there are four propagation channels: 1|in, 1|out, 2|in and 2|out, associated to wave vectors  $q_{1|in}(\omega)$ ,  $q_{1|out}(\omega)$ ,  $q_{2|in}(\omega)$  and  $q_{2|out}(\omega)$ , respectively. Only 1|in and 1|out survive when  $\omega > \Omega$ . The propagation channels with note with a label “in” propagate towards the horizon, the ones with label “out” propagate away from the horizon.

Several propagation modes correspond to the channels identified in Fig. 3. For instance, a mode with we denote as “ingoing” and associate to a quantum operator  $\hat{b}_0(\omega)$  corresponds to a wave incident in channel 0|in at energy  $\hbar\omega$ , scattered onto the horizon to the exit channels 0|out, 1|out, and 2|out with amplitudes  $S_{0,0}(\omega)$ ,  $S_{0,1}(\omega)$ , and  $S_{0,2}(\omega)$ , respectively. There are two other ingoing modes which are associated to operators we denote as  $\hat{b}_1(\omega)$  and  $\hat{b}_2(\omega)$ . They correspond to modes initiated by a wave incident in channels 1|in and 2|in, respectively. There are also three outgoing modes associated to scattering processes resulting in the emission of a single wave along one of the three “out” channels 0|out, 1|out, and 2|out. We denote the corresponding quantum operators as  $\hat{c}_0(\omega)$ ,  $\hat{c}_1(\omega)$ , and  $\hat{c}_2(\omega)$ . In conclusion, the modes, be they ingoing or outgoing, are identified by an index  $L \in \{0, 1, 2\}$  and the channels by an index  $\ell \in \{0|in, 0|out, 1|in, 1|out, 2|in, 2|out\}$ . We also note that the outgoing modes  $\hat{c}_0$  and  $\hat{c}_2$  are analogous to the modes denoted in the General Relativity context as the Hawking and partner modes, respectively.

The outgoing modes relate to the incoming ones via

$$\begin{pmatrix} \hat{c}_0 \\ \hat{c}_1 \\ \hat{c}_2^\dagger \end{pmatrix} = \begin{pmatrix} S_{00} & S_{01} & S_{02} \\ S_{10} & S_{11} & S_{12} \\ S_{20} & S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} \hat{b}_0 \\ \hat{b}_1 \\ \hat{b}_2^\dagger \end{pmatrix}, \quad (\text{B3})$$

where for legibility we omit the  $\omega$  dependence of all the terms. The appearance of annihilation operators  $\hat{b}_2^\dagger$  and  $\hat{c}_2^\dagger$  in (B3) reflects the fact that the modes  $L = 2$  have a negative norm and should accordingly be quantized inverting the usual role of the creation and annihilation operators in order that the elementary excitations satisfy the standard Bose commutation relations [58].

The  $3 \times 3$  scattering matrix  $S(\omega)$  defined in (B3) obeys a skew-unitarity relation [66]

$$S^\dagger \eta S = \eta = S \eta S^\dagger, \quad \text{where } \eta = \text{diag}(1, 1, -1). \quad (\text{B4})$$

For  $\omega > \Omega$  the channels 2|in and 2|out disappear (cf. Fig. 3), as well as the modes  $\hat{b}_2(\omega)$  and  $\hat{c}_2(\omega)$ . In this case the  $S$  matrix becomes  $2 \times 2$  and unitary.

### APPENDIX C: ASYMPTOTIC SOURCE TERMS IN A 1D BLACK HOLE

The quantum fluctuation field  $\hat{\psi}$  defined in (2) can be expanded of the basis of ingoing modes defined in Appendix B supplemented by the contribution of zero modes. As explained in the main text, we will evaluate the source terms in Eqs. (69) and (70) only in the asymptotic regions ( $x \rightarrow \pm\infty$ ) where the contribution of the zero modes can be omitted. In this case we

have

$$\begin{aligned} \hat{\psi}(x, t) = & e^{ik_\alpha x} \int_0^\infty \frac{d\omega}{\sqrt{2\pi}} \sum_{L=0}^1 [u_L(x, \omega) e^{-i\omega t} \hat{b}_L(\omega) \\ & + v_L^*(x, \omega) e^{i\omega t} \hat{b}_L^\dagger(\omega)] \\ & + e^{ik_\alpha x} \int_0^\Omega \frac{d\omega}{\sqrt{2\pi}} [u_2(x, \omega) e^{-i\omega t} \hat{b}_2^\dagger(\omega) \\ & + v_2^*(x, \omega) e^{i\omega t} \hat{b}_2(\omega)], \end{aligned} \quad (\text{C1})$$

where<sup>5</sup>  $k_\alpha = k_u = mV_u/\hbar$  is  $x < 0$  and  $k_\alpha = k_d = mV_d/\hbar$  if  $x > 0$ . In this expression the  $u_L(x, \omega)$ 's and  $v_L(x, \omega)$ 's are linear combinations of the usual Bogoliubov coefficients involving the coefficient of the scattering matrix (B3) [see Eqs. (C4) and (C5) below].

At zero temperature the different contributions to the source terms which appear in Eqs. (69) and (70) without derivative read as [67]

$$\begin{aligned} g^{(2)}(x) = & \frac{2}{\rho_\alpha |\phi_\alpha|^4} \int_0^\infty \frac{d\omega}{2\pi} \sum_{L=0}^1 [|\tilde{v}_L|^2 + \text{Re}(\tilde{u}_L \tilde{v}_L^*)] \\ & + \frac{2}{\rho_\alpha |\phi_\alpha|^4} \int_0^\Omega \frac{d\omega}{2\pi} [|\tilde{u}_2|^2 + \text{Re}(\tilde{u}_2 \tilde{v}_2^*)] \end{aligned} \quad (\text{C2})$$

and

$$\begin{aligned} \text{Re}(\hat{\eta} \partial_x \hat{\theta}) = & \frac{1}{2\rho_\alpha |\phi_\alpha(x)|^2} \int_0^\infty \frac{d\omega}{2\pi} \sum_{L=0}^2 \text{Im} \\ & \times \left[ (\tilde{u}_L^* + \tilde{v}_L^*) \partial_x \left( \frac{\tilde{u}_L - \tilde{v}_L}{|\phi_\alpha|^2} \right) \right], \end{aligned} \quad (\text{C3})$$

where  $\phi_\alpha(x)$  is defined in Appendix A,  $\tilde{u}_L(x, \omega) = u_L(x, \omega) \phi_\alpha^*(x)$ , and  $\tilde{v}_L(x, \omega) = v_L(x, \omega) \phi_\alpha(x)$ , with  $\alpha = u$  if  $x < 0$  and  $\alpha = d$  if  $x > 0$ . In the asymptotic regions, the  $\tilde{u}_L$ 's and  $\tilde{v}_L$ 's are combinations of plane waves. More precisely, deep in the upstream subsonic region, i.e., when  $x < 0$ ,  $x \ll -\xi_u$ ,

$$\begin{aligned} \begin{pmatrix} \tilde{u}_0 \\ \tilde{v}_0 \end{pmatrix} &= S_{0,0} \begin{pmatrix} \tilde{\mathcal{U}}_{0|out} \\ \tilde{\mathcal{V}}_{0|out} \end{pmatrix} e^{iq_{0|out}x} + \begin{pmatrix} \tilde{\mathcal{U}}_{0|in} \\ \tilde{\mathcal{V}}_{0|in} \end{pmatrix} e^{iq_{0|in}x}, \\ \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} &= S_{0,1} \begin{pmatrix} \tilde{\mathcal{U}}_{0|out} \\ \tilde{\mathcal{V}}_{0|out} \end{pmatrix} e^{iq_{0|out}x}, \\ \begin{pmatrix} \tilde{u}_2 \\ \tilde{v}_2 \end{pmatrix} &= S_{0,2} \begin{pmatrix} \tilde{\mathcal{U}}_{0|out} \\ \tilde{\mathcal{V}}_{0|out} \end{pmatrix} e^{iq_{0|out}x}, \end{aligned} \quad (\text{C4})$$

<sup>5</sup>As in Appendixes A and B, throughout this appendix we omit for legibility all the subscripts “GP” and write  $V_\alpha$  instead of  $V_{GP}^\alpha$  for instance.

and deep in the downstream supersonic region, i.e., when  $x > 0$ ,  $x \gg \xi_d$ :

$$\begin{aligned} \begin{pmatrix} \tilde{u}_0 \\ \tilde{v}_0 \end{pmatrix} &= S_{1,0} \begin{pmatrix} \tilde{U}_{d1|out} \\ \tilde{V}_{d1|out} \end{pmatrix} e^{iq_{d1|out}x} + S_{2,0} \begin{pmatrix} \tilde{U}_{2|out} \\ \tilde{V}_{2|out} \end{pmatrix} e^{iq_{2|out}x}, \\ \begin{pmatrix} \tilde{u}_1 \\ \tilde{v}_1 \end{pmatrix} &= S_{1,1} \begin{pmatrix} \tilde{U}_{1|out} \\ \tilde{V}_{1|out} \end{pmatrix} e^{iq_{1|out}x} \\ &\quad + S_{2,1} \begin{pmatrix} \tilde{U}_{2|out} \\ \tilde{V}_{2|out} \end{pmatrix} e^{iq_{2|out}x} + \begin{pmatrix} \tilde{U}_{1|in} \\ \tilde{V}_{1|in} \end{pmatrix} e^{iq_{1|in}x}, \\ \begin{pmatrix} \tilde{u}_2 \\ \tilde{v}_2 \end{pmatrix} &= S_{1,2} \begin{pmatrix} \tilde{U}_{1|out} \\ \tilde{V}_{1|out} \end{pmatrix} e^{iq_{1|out}x} + S_{2,2} \begin{pmatrix} \tilde{U}_{2|out} \\ \tilde{V}_{2|out} \end{pmatrix} e^{iq_{2|out}x} \\ &\quad + \begin{pmatrix} \tilde{U}_{2|in} \\ \tilde{V}_{2|in} \end{pmatrix} e^{iq_{2|in}x}. \end{aligned} \quad (C5)$$

The Bogoliubov coefficients  $\tilde{U}_\ell(\omega)$  and  $\tilde{V}_\ell(\omega)$  are real; their explicit expression is given in Ref. [57]. Note that the  $q_\ell$ 's and  $S_{i,j}$ 's in Eqs. (C2) and (C3) all depend on  $\omega$  (see Appendix B).

From expressions (C2), (C4), and (C5) we get the following expression of the asymptotic density-density correlation function  $g_\alpha^{(2)}$  defined in (71):

$$g_u^{(2)} = -\frac{2}{\pi \xi_u \rho_u} + \frac{2}{\rho_u} \int_0^{\Omega_u} \frac{d\omega}{2\pi} |S_{0,2}|^2 \tilde{\mathcal{R}}_{0|out}^2, \quad (C6)$$

$$\begin{aligned} g_d^{(2)} &= -\frac{2}{\pi \xi_d \rho_d} + \frac{2}{\rho_d} \int_0^{\Omega_d} \frac{d\omega}{2\pi} [ |S_{1,2}|^2 \tilde{\mathcal{R}}_{1|out}^2 \\ &\quad + (|S_{2,2}|^2 - 1) \tilde{\mathcal{R}}_{2|out}^2 ], \end{aligned} \quad (C7)$$

where  $\tilde{\mathcal{R}}_\ell(\omega) = \tilde{U}_\ell(\omega) + \tilde{V}_\ell(\omega)$ . Note that the skew unitarity (B4) allows to write, in the last contribution to the integrand of (C7):  $|S_{2,2}|^2 - 1 = |S_{2,0}|^2 + |S_{2,1}|^2 = |S_{0,2}|^2 + |S_{1,2}|^2$ . As a result, the Hawking contribution additional to the standard term  $-2/\pi \xi_\alpha \rho_\alpha$  in  $g_\alpha^{(2)}$  [cf. Eq. (66)] is positive both in (C6) and (C7).

According to (C6) and (C7) the explicit expression of the dimensionless quantities  $\mathcal{G}_\alpha^{(H)}$  defined in (79a) is

$$\mathcal{G}_u^{(H)} = \int_0^{\Omega_u} \frac{d\varepsilon_u}{2\pi} |S_{0,2}|^2 N(Q_{0|out}, M_u) \quad (C8)$$

and

$$\begin{aligned} \mathcal{G}_d^{(H)} &= \int_0^{\Omega_d} \frac{d\varepsilon_d}{2\pi} \{ |S_{1,2}|^2 N(Q_{1|out}, -M_d) \\ &\quad - (|S_{2,2}|^2 - 1) N(Q_{2|out}, M_d) \}, \end{aligned} \quad (C9)$$

where the  $Q_\ell(\omega)$  are the dimensionless wave vectors:  $Q_\ell = q_\ell \xi_u$  for  $\ell = 0|out$  and  $Q_\ell = q_\ell \xi_d$  for  $\ell = 1|out$  and  $\ell = 2|out$  [see the definition of the  $q_\ell(\omega)$ 's in Appendix B]. We have also  $\varepsilon_\alpha = \hbar\omega/(mc_\alpha^2)$ ,  $\Omega_\alpha = \hbar\Omega/(mc_\alpha^2)$ , where  $\Omega$  is defined in Fig. 3, and

$$N(Q, M) = \frac{Q}{1 + Q^2/2 - M\sqrt{1 + Q^2/4}}. \quad (C10)$$

Similarly to what has just been done for the density-density correlation function, we now describe the steps enabling to compute the quantum contribution to the average current. From expressions (C3), (C4), and (C5) we get the following expression for the asymptotic quantum contribution  $j_\alpha$  defined

in (71):

$$j_u = \frac{\hbar}{m} \int_0^{\Omega} \frac{d\omega}{2\pi} \frac{q_{0|out}}{|\partial\omega/\partial q_{0|out}|} |S_{0,2}|^2 \quad (C11)$$

and

$$\begin{aligned} j_d &= \frac{\hbar}{m} \int_0^{\Omega} \frac{d\omega}{2\pi} \left[ \frac{q_{1|out}}{|\partial\omega/\partial q_{1|out}|} |S_{1,2}|^2 \right. \\ &\quad \left. - \frac{q_{2|out}}{|\partial\omega/\partial q_{2|out}|} (|S_{2,2}|^2 - 1) \right]. \end{aligned} \quad (C12)$$

We recall that  $q_{0|out}(\omega) < 0$ , whereas  $q_{1|out}(\omega)$  and  $q_{2|out}(\omega)$  are both positive (see Fig. 3). Hence,  $j_u < 0$  whereas the integrand of  $j_d$  contains two contributions, one positive and one negative. In practice we find numerically that the second is dominant and that  $j_d$  is negative. Accordingly to Eqs. (C11) and (C12) the dimensionless quantities  $\mathcal{J}_\alpha^{(H)}$  defined in Eq. (79b) read as

$$\mathcal{J}_u^{(H)} = \int_0^{\Omega_u} \frac{d\varepsilon_u}{2\pi} |S_{0,2}|^2 T(Q_{0|out}, M_u) \quad (C13)$$

and

$$\begin{aligned} \mathcal{J}_d^{(H)} &= \int_0^{\Omega_d} \frac{d\varepsilon_d}{2\pi} \{ |S_{1,2}|^2 T(Q_{1|out}, -M_d) \\ &\quad + (|S_{2,2}|^2 - 1) T(Q_{2|out}, M_d) \}, \end{aligned} \quad (C14)$$

where

$$T(Q, M) = N(Q, M) \times \sqrt{1 + Q^2/4}. \quad (C15)$$

We determine numerically the four quantities  $\mathcal{G}_\alpha^{(H)}$  and  $\mathcal{J}_\alpha^{(H)}$  ( $\alpha = u$  and  $d$ ) from expressions (C8), (C9), (C13), and (C14) after a numerical computation of the elements of the  $S$  matrix [57]. It may be shown that these quantities cancel in the limit  $M_d \rightarrow 1$ . This should be expected since in this limit the negative norm channels  $2|out$  and  $2|in$  vanish, resulting in a disappearance of Hawking radiation. Also, since in this limit the upper integration point in integrals (C8), (C9), (C10), and (C13) tend to zero, it is easy to show that

$$\mathcal{J}_u^{(H)} \underset{M_d \rightarrow 1}{\simeq} -\mathcal{G}_u^{(H)} \quad \text{and} \quad \mathcal{J}_d^{(H)} \underset{M_d \rightarrow 1}{\simeq} -\mathcal{G}_d^{(H)}. \quad (C16)$$

The first equality follows directly from comparing expressions (C8) and (C13) and noticing that  $T(Q, M) \simeq N(Q, M)$  when  $Q \rightarrow 0$  which is the appropriate limit to consider when  $\Omega \rightarrow 0$ . The second one also follows from this property complemented by the fact that, in the limit  $M_d \rightarrow 1$ , the second terms of the integrands of (C9) and (C14) become dominant.

It follows from these remarks that in the limit  $M_d \rightarrow 1$  the leading terms of the series expansion of  $\mathcal{G}_\alpha^{(H)}$  and  $\mathcal{J}_\alpha^{(H)}$  are of the form

$$\mathcal{G}_u^{(H)} \simeq -\mathcal{J}_u^{(H)} \simeq \mathcal{H}_u(M_d - 1), \quad (C17a)$$

$$\mathcal{G}_d^{(H)} \simeq -\mathcal{J}_d^{(H)} \simeq \mathcal{H}_d(M_d - 1), \quad (C17b)$$

where  $\mathcal{H}_u$  and  $\mathcal{H}_d$  are positive constants which we determine numerically. In the waterfall and  $\delta$ -peak configurations  $M_u$  also tends to 1 when  $M_d$  does, and  $\mathcal{H}_u$  and  $\mathcal{H}_d$  are thus universal constants. We find  $\mathcal{H}_u = 2.48 \times 10^{-2}$  and  $\mathcal{H}_d = 9.36 \times 10^{-3}$  in the waterfall configuration,  $\mathcal{H}_u = 3.91 \times 10^{-2}$  and  $\mathcal{H}_d = 5.34 \times 10^{-2}$  in the  $\delta$ -peak configuration. On the

other hand, in the flat profile configuration  $\mathcal{H}_u$  and  $\mathcal{H}_d$  both depend on  $M_u$ .

#### APPENDIX D: DERIVATION OF EQ. (55)

In this appendix we make use of the Bogoliubov equations (38) to cast the imaginary part of Eq. (52) under the form (55). From Eq. (37) this imaginary part reads as

$$\mathcal{T} \left( \frac{\delta \rho}{2\rho_{\text{GP}}} \right) + \mathcal{X} \delta \Theta = -\mathcal{T} \text{Re} \langle \hat{A} \rangle - \mathcal{X} \text{Im} \langle \hat{A} \rangle + \frac{g\rho_{\text{GP}}}{2\hbar} (\hat{\eta} \hat{\theta} + \hat{\theta} \hat{\eta}). \quad (\text{D1})$$

To evaluate the source terms in the above we make use of the explicit expression (47). We first note that the action of  $\mathcal{T}$  and  $\mathcal{X}$  [defined in Eqs. (36)] on a product of possibly noncommuting fields  $\hat{B}(\mathbf{x}, t)$  and  $\hat{C}(\mathbf{x}, t)$  reads as

$$\mathcal{T} \hat{B} \hat{C} = (\mathcal{T} \hat{B}) \hat{C} + \hat{B} (\mathcal{T} \hat{C}), \quad (\text{D2a})$$

$$\mathcal{X} \hat{B} \hat{C} = (\mathcal{X} \hat{B}) \hat{C} + \hat{B} (\mathcal{X} \hat{C}) + \frac{\hbar}{m} \nabla \hat{B} \cdot \nabla \hat{C}. \quad (\text{D2b})$$

Use of these relations and of Bogoliubov Eq. (38) yields

$$\mathcal{T} \hat{\theta}^2 = \frac{1}{2} \hat{\theta} (\mathcal{X} \hat{\eta}) + \frac{1}{2} (\mathcal{X} \hat{\eta}) \hat{\theta} - \frac{g\rho_{\text{GP}}}{\hbar} (\hat{\theta} \hat{\eta} + \hat{\eta} \hat{\theta}), \quad (\text{D3})$$

$$\mathcal{T} \hat{\eta}^2 = -2\hat{\eta} (\mathcal{X} \hat{\theta}) - 2(\mathcal{X} \hat{\theta}) \hat{\eta}, \quad (\text{D4})$$

and

$$\mathcal{T} (\hat{\eta} \hat{\theta} - \hat{\theta} \hat{\eta}) = 0. \quad (\text{D5})$$

In this last equation use has been made of the fact that the equal-time commutator of a quantum field and its spatial derivative cancels: thus,  $\hat{\eta} (\mathcal{X} \hat{\eta}) = (\mathcal{X} \hat{\eta}) \hat{\eta}$  and  $\hat{\theta} (\mathcal{X} \hat{\theta}) = (\mathcal{X} \hat{\theta}) \hat{\theta}$ . We also have

$$\mathcal{X} (\hat{\theta} \hat{\eta} + \hat{\eta} \hat{\theta}) = (\mathcal{X} \hat{\theta}) \hat{\eta} + \hat{\theta} (\mathcal{X} \hat{\eta}) + \frac{\hbar}{m} \nabla \hat{\theta} \cdot \nabla \hat{\eta} + (\mathcal{X} \hat{\eta}) \hat{\theta} + \hat{\eta} (\mathcal{X} \hat{\theta}) + \frac{\hbar}{m} \nabla \hat{\eta} \cdot \nabla \hat{\theta}. \quad (\text{D6})$$

Combining the results (D3), (D4), (D5), and (D6) enables us to express the source term in Eq. (D1) as

$$\begin{aligned} & -\mathcal{T} \text{Re} \langle \hat{A} \rangle - \mathcal{X} \text{Im} \langle \hat{A} \rangle + \frac{g\rho_{\text{GP}}}{2\hbar} (\hat{\eta} \hat{\theta} + \hat{\theta} \hat{\eta}) \\ &= -\frac{1}{2} \langle (\mathcal{X} \hat{\theta}) \hat{\eta} + \hat{\eta} (\mathcal{X} \hat{\theta}) \rangle - \frac{\hbar}{4m} \langle \nabla \hat{\theta} \cdot \nabla \hat{\eta} + \nabla \hat{\eta} \cdot \nabla \hat{\theta} \rangle. \end{aligned} \quad (\text{D7})$$

From the definition (56) of  $\hat{\mathbf{v}}$  and the explicit expression (36b) we may write

$$\mathcal{X} \hat{\theta} = \frac{1}{2\rho_{\text{GP}}} \nabla \cdot (\rho_{\text{GP}} \hat{\mathbf{v}}), \quad (\text{D8})$$

and thus Eq. (D7) reads as

$$\begin{aligned} & -\mathcal{T} \text{Re} \langle \hat{A} \rangle - \mathcal{X} \text{Im} \langle \hat{A} \rangle + \frac{g\rho_{\text{GP}}}{2\hbar} (\hat{\eta} \hat{\theta} + \hat{\theta} \hat{\eta}) \\ &= -\frac{1}{4\rho_{\text{GP}}} \nabla \cdot \langle \rho_{\text{GP}} \hat{\eta} \hat{\mathbf{v}} + \rho_{\text{GP}} \hat{\mathbf{v}} \hat{\eta} \rangle \\ &= -\frac{1}{2\rho_{\text{GP}}} \nabla \cdot \text{Re} \langle \rho_{\text{GP}} \hat{\eta} \hat{\mathbf{v}} \rangle. \end{aligned} \quad (\text{D9})$$

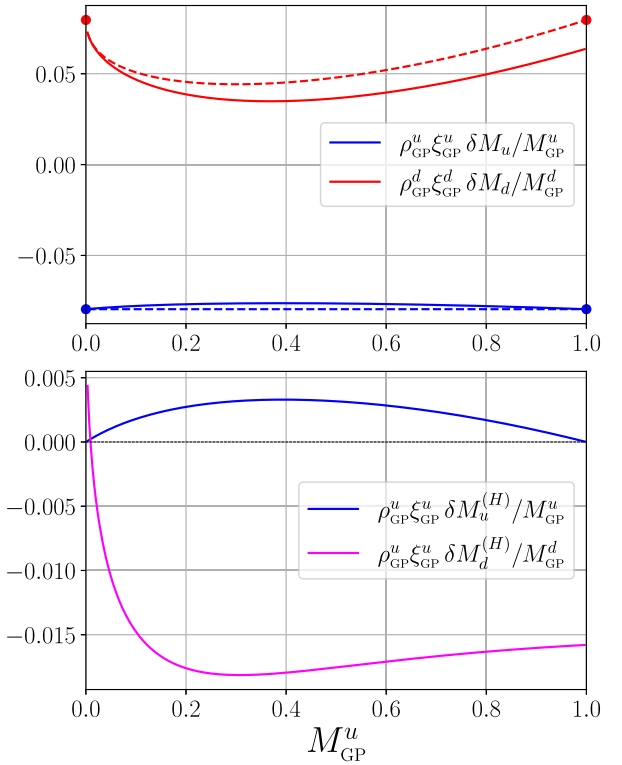


FIG. 4. Same as Fig. 1 for the  $\delta$ -peak configuration.

The left-hand side term of Eq. (D1) can also be simplified. To this end, let us first remark that, from current conservation

$$\partial_t \rho_{\text{GP}} + \nabla \cdot (\rho_{\text{GP}} \mathbf{V}_{\text{GP}}) = 0. \quad (\text{D10})$$

From this relation and from the explicit definition (36a) of operator  $\mathcal{T}$ , it results that for any scalar quantity  $Y(\mathbf{x}, t)$  we have

$$\mathcal{T} \left( \frac{Y}{\rho_{\text{GP}}} \right) = \frac{1}{\rho_{\text{GP}}} \partial_t Y + \frac{1}{\rho_{\text{GP}}} \nabla \cdot (Y \mathbf{V}_{\text{GP}}). \quad (\text{D11})$$

Using relation (D11) and the definition (56) of  $\delta V$  makes it possible to write the left-hand side of Eq. (D1) as

$$\mathcal{T} \left( \frac{\delta \rho}{2\rho_{\text{GP}}} \right) + \mathcal{X} \delta \Theta = \frac{1}{2\rho_{\text{GP}}} \partial_t \delta \rho + \frac{1}{2\rho_{\text{GP}}} \nabla \cdot (\delta \rho \mathbf{V}_{\text{GP}} + \rho_{\text{GP}} \delta \mathbf{V}). \quad (\text{D12})$$

Inserting expressions (D9) and (D12) in (D1) directly yields Eq. (55).

#### APPENDIX E: ASYMPTOTIC BACKREACTION FOR $\delta$ -PEAK AND FLAT PROFILE CONFIGURATIONS

In the main text we present the modification of the asymptotic Mach numbers for a waterfall configuration. The reason why we put the emphasis on this configuration is that this is the only one which has been realized experimentally so far [3,64]. In this appendix we consider two other configurations (dubbed  $\delta$ -peak and flat profile in Appendix A) in order to check to what extent the waterfall configuration is typical.

In Fig. 4 we represent the modification of the upstream and downstream Mach numbers due to quantum backreaction in

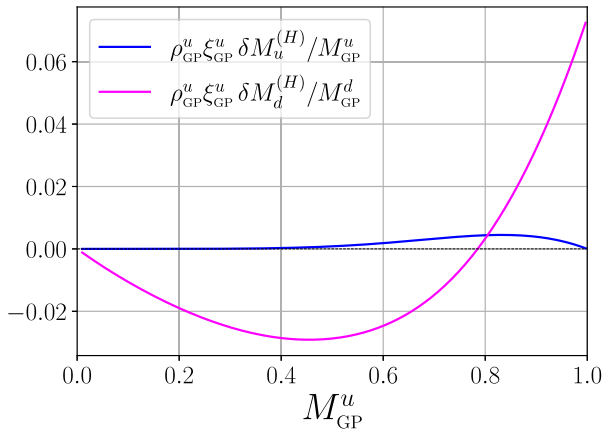


FIG. 5. Hawking induced modifications  $\delta M_\alpha^{(H)}$  of the asymptotic Mach numbers in a flat profile partially imitating a waterfall configuration.

a  $\delta$ -peak configuration. The gross features of the results are the same as those obtained for the waterfall configuration and presented in Fig. 1. A noticeable difference between the two figures is that the part  $\delta M_d^{(H)}$  of the downstream modifications caused by Hawking radiation, while being roughly of the same order in both settings, are positive for the waterfall configuration and negative for the  $\delta$ -peak configuration (compare the lower panels of Figs. 1 and 4). In order to assess if these different signs are signatures of specific physical features we also determine  $\delta M_\alpha^{(H)}$  for flat profile configurations. As explained in Appendix A, for these configurations the values of the upstream and downstream Mach numbers can be chosen independently one from the other. In order to compare with the results of Figs. 1 and 4 we choose either a flat profile setting for which  $M_{\text{GP}}^u$  and  $M_{\text{GP}}^d$  are related through  $M_{\text{GP}}^d = (M_{\text{GP}}^u)^{-2}$  [thus partially imitating a waterfall configuration, cf. Eq. (72)], or through  $M_{\text{GP}}^d/M_{\text{GP}}^u = y^{3/2}$  where  $y$  is defined in (A7) [thus partially imitating a  $\delta$ -peak configuration, cf. Eq. (A9)]. The corresponding results are presented in Figs. 5 and 6, respectively.

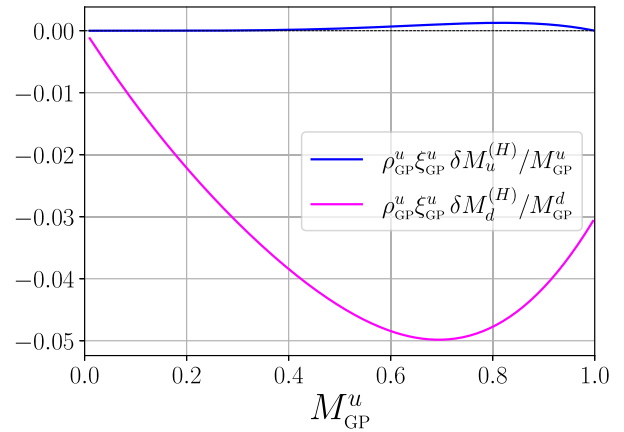


FIG. 6. Same as Fig. 5 in a flat profile partially imitating a  $\delta$ -peak configuration.

It appears that the Hawking contribution  $\delta M_u^{(H)}$  to  $\delta M_u$  is positive in Figs. 1 and 4–6. This can be understood from expression (80) which implies that, for the boundary conditions  $\delta\mu = 0 = \delta V_u$ , we have

$$\frac{\delta M_u^{(H)}}{M_{\text{GP}}^u} = -\frac{1}{2} \frac{\delta \rho_u^{(H)}}{\rho_{\text{GP}}^u}, \quad (\text{E1})$$

where

$$\delta \rho_u^{(H)} = -\frac{\mathcal{G}_u^{(H)}}{2 \xi_{\text{GP}}^u} \quad (\text{E2})$$

is the part of  $\delta \rho_u$  due to the Hawking backreaction. It is shown in Appendix C that  $\mathcal{G}_\alpha^{(H)}$  is always positive, which explains, via (E1) and (E2), why  $\delta M_u^{(H)}$  is necessarily positive. In contrast, no such simple relation exists in the downstream region, where  $\delta M_d^{(H)}$  can be either positive or negative. Nevertheless, despite some differences, Figs. 4–6 support the discussion of the waterfall configuration presented in the main text.

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