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A Synchronization Problem for a Team of AUVs in the Max-Plus Algebra Framework ⋆

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Abstract: This paper presents a synchronization problem for a team of autonomous underwater vehicles (AUVs), represented by fish robots for reconnaissance purposes. The methodology and the underlying mathematical framework are developed to address a possible synchronization of such a team, defined as a fish robot shoal, by exploiting the max-plus algebra for modeling and controlling the devices. In detail, a switching max-plus linear system is considered to model each reconnaissance step. The synchronization problem consists of forcing the system's output to equal a given model's output so that the robots can perform their tasks according to a predefined schedule. A simulation test is also provided, confirming the feasibility of this strategy.

Keywords: Max-plus linear systems, model matching problem, switching systems, system's synchronization, algebraic/geometric methods.

1. INTRODUCTION

Max-plus algebra is an efficient tool to deal with modeling and control of synchronized tasks without concurrency, since it enables the representation of a special class of discrete algebraic systems. It is named after the operations of *max* and *plus* of conventional algebra, recast as tropical addition and multiplication, respectively. Introduced by Cohen et al. (1985), linear systems over this algebra have seen many advances over the years (Baccelli et al., 1992; Komenda et al., 2018). Control problems for these systems were solved in (De Schutter et al., 2019; Oke et al., 2017) and in (Di Loreto et al., 2010; Cardenas et al., 2015) with structural geometric methods.

Indeed, the geometric approach (Basile and Marro, 1992) was successfully applied to solve model matching problems, from which the synchronization problem dealt with herein stems, for various classes of dynamical systems (Perdon et al., 2016b,a; Zattoni et al., 2024). However, applying this approach to the max-plus semiring is not straightforward due to the unique properties of max-plus semimodules compared to vector spaces, while extensions to linear systems over rings were developed in (Hautus, 1982; Conte and Perdon, 1994). Building on these foundations, further investigations have recently emerged (Martinez et al., 2022). Animobono et al. (2023) have applied this approach to solving the model matching problem (MMP) for max-plus linear systems, and also for switching and periodic max-plus systems (Animobono et al., 2022, 2024a). A recent contribution (Animobono et al., 2024b) addresses structural solvability conditions for synchronization of switching max-plus systems, being the groundwork for this work.

This paper presents an application of the feedback system synchronization problem (FSSP) proposed in (Animobono et al., 2024b), focusing on coordinating a team of autonomous underwater vehicles (AUVs), engaged in reconnaissance missions. The main contribution lies in demonstrating how an FSSP can be formulated and solved in a realistic AUV deployment scenario. Vehicles' tasks are modeled by a switching max-plus linear system, whose modes represent the robot configurations. A configuration is a specific phase of the mission where the spatial arrangement of the AUVs, i.e., their assigned Points of Interest (POIs), is fixed. The use of a switching system to break down the activities is one of the novel aspects introduced, as such systems are usually used in other contexts, e.g., when a machine must process one raw part rather than another, according to external decisions (Animobono et al., 2023). The control of such a team is then addressed with the resolution of an FSSP, which is crucial to meet the real challenges of underwater reconnaissance, such as trajectory planning and vehicle coordination, made easier through a switching system.

To model a system as close to reality as possible, it is necessary to specify its constraints and characteristics. A paper on modeling teams of underwater robots is (Bartolucci et al., 2024), which details the process to obtain a non-switching max-plus linear system. Another work of Bartolucci (2024) addresses an MMP for such robots, with

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the test of the causal controlled invariance of its solution, through a novel algorithm (Bartolucci et al., 2023). The applicability of these methods to underwater robotics represents another novel contribution and although this work focuses on switching systems, other applications of max-plus linear systems, such as resource management in laboratories (Screpanti et al., 2024), highlight their potential in addressing synchronization and control problems.

Switching max-plus linear systems, introduced by Hillion and Proth (1989), can address changes in event order or alterations in system structure, e.g., when variable resource processing time or user allocation policies come into play. These systems offer a solution to the limitations of stationary max-plus linear systems, enabling concurrency to describe a plant with arbitrary structure variation. Expanding the max-plus formalism to event-varying scenarios paves the way for complex situations, such as conflicts solved depending on the current event's index or scenario. Various approaches have been proposed for modeling such situations. In (Van den Boom and De Schutter, 2006), switching max-plus linear systems are used to model systems accommodating changes in structure or event order. Different applications have also emerged, such as modeling the Dutch railway network (Kersbergen et al., 2016) and cube-packing systems (Majdzik, 2020).

The paper is organized as follows. Section 2 presents basic notions related to the max-plus algebra and the geometric approach, constituting the theoretical background of the work. In Section 3, the switching max-plus linear system is presented, which models the vehicle configurations. Section 4 includes the simulation of the obtained plant, and Section 5 presents an illustrative example, with the resolution of the synchronization problem. Finally, Section 6 contains conclusions and future developments.

2. MATHEMATICAL BACKGROUND

This section provides the essential theoretical notions used to formulate and solve the synchronization problem.

2.1 Max-plus Algebra Framework

The max-plus semiring is the set $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$, equipped with the operations *max* and *plus*, denoted as addition ($\oplus$) and multiplication ($\otimes$). For any $x, y \in \mathbb{R}_{\max}$, such operations are defined as follows:

$$\begin{aligned} x \oplus y &= \max(x, y) \qquad \text{if } \; x, y \in \mathbb{R}_{\max} \\ x \otimes y &= x + y \qquad\qquad \text{if } \; x, y \in \mathbb{R}_{\max} \end{aligned}$$

The neutral elements for $\oplus$ and $\otimes$ are, respectively, $\epsilon = -\infty$, and $e = 0 \in \mathbb{R}$, playing the role of 0 and 1 of conventional algebra. In this structure, multiplication is distributive over finite sums, meaning that for any $a, b, c \in \mathbb{R}_{\max}$, $(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$ and $c \otimes (a \oplus b) = (c \otimes a) \oplus (c \otimes b)$. This confirms that $\mathbb{R}_{\max}$ has a semiring structure. The notations $\oplus$ and $\otimes$ extend to vectors and matrices, with $\mathbb{R}^{p\times q}_{\max}$ representing the set of $p \times q$ matrices with coefficients in $\mathbb{R}_{\max}$ and $p, q \in \mathbb{N}$. For two matrices $A, B \in \mathbb{R}^{m\times n}_{\max}$, the sum $A \oplus B$ is defined as $(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}$. If $A \in \mathbb{R}^{p\times n}_{\max}$ and $B \in \mathbb{R}^{n\times m}_{\max}$, the product $A \otimes B$, often denoted as AB, is defined as $(A \otimes B)_{ij} = \bigoplus_{k=1}^{n} A_{ik} \otimes B_{kj}$. The notation $\otimes$ is also used for the external product of a scalar $\lambda \in \mathbb{R}_{\max}$ and a matrix $A \in \mathbb{R}^{p\times n}_{\max}$, defined as $(\lambda \otimes A)_{ij} = \lambda \otimes A_{ij}$, for $i = 1$ to p and $j = 1$ to n. The null matrix is symbolized by ϵ, the $n \times n$ identity matrix is I_n, with diagonal elements as e and the other elements as ϵ, and the vector with all elements as e is denoted as 0. The interest is here focused on the sub-semi-modules of the Cartesian product $\mathbb{R}^n_{\max}$ and on finite-type modules generated by a finite family of vectors in $\mathbb{R}^n_{\max}$.

2.2 Feedback System Synchronization Problem

The theoretical geometric approach (Animobono et al., 2024b) allows the study of a feedback system synchronization problem (FSSP), which entails determining a control law that ensures a system (plant) behaves according to a model. In our case, the plant is a switching max-plus system representing the AUV team, while the model encodes the desired mission timing. This approach offers a new perspective on solving such an FSSP, wherein the applied control forces a plant, modeled as a max-plus linear system, to produce an output equal to that of a given model. To model the evolution of a system with the occurrence of n different events, an n-dimensional dater $d(.) : \mathbb{N} \to \mathbb{R}^n_{max}$ can be used, in which the value at $k \in \mathbb{N}$ is a vector whose i-th component is the time instant when an event of the i-th type occurs for the k-th time. To have physical meaning, daters must be non-decreasing $(d(k+1) \geq d(k) \, \forall \, k \in \mathbb{N})$. Considering now a switching max-plus linear system Σ_σ, it can be represented as a dynamical object defined by

$$\Sigma_\sigma \equiv \begin{cases} x(k) = A_{\sigma(k)} x(k-1) \oplus B_{\sigma(k)} u(k) \\ y(k) = C_{\sigma(k)} x(k) \\ x(0) = \epsilon \end{cases} \tag{1}$$

where $k \in \mathbb{N}$ is the event instance index, $x(.) : \mathbb{N} \to \mathcal{X} = \mathbb{R}^n_{max}$, $u(.) : \mathbb{N} \to \mathcal{U} = \mathbb{R}^m_{max}$ and $y(.) : \mathbb{N} \to \mathcal{Y} = \mathbb{R}^p_{max}$ are the daters of internal, input and output events, respectively. Moreover, the semi-modules $\mathcal{X}, \mathcal{U}$ and $\mathcal{Y}$ are called the state, input and output semimodules, respectively. Then, $\sigma(.) : \mathbb{N} \to \mathcal{I} = \{1, ..., I\}$ is the function that manages the switching behaviour and A_i, B_i and C_i for $i \in \mathcal{I}$ are matrices with entries in $\mathbb{R}_{\max}$.

The modes of Σ_σ are defined for each $i \in \mathcal{I}$ by the following max-plus linear system Σ_i:

$$\Sigma_i \equiv \begin{cases} x(k) = A_i x(k-1) \oplus B_i u(k) \\ y(k) = C_i x(k) \\ x(0) = \epsilon. \end{cases}$$

Consequently, $\sigma(k)$ represents the configuration assumed by the system at the k-th iteration.

Coming back to the FSSP, a stricter version of the SSP, the control input $u_P(k)$ is here required to be, for each value of the switching signal $\sigma(k)$, a linear function of the state of the plant $x_P(k-1)$, the model $x_M(k-1)$) and the input of the model $u_M(k)$. In simpler terms, the FSSP sets up a system to follow the model, to achieve a specific behavior in the actual system, which is the plant.

The FSSP, its solution, and some basic definitions, formalized in Animobono et al. (2024b), are reported as follows.

Problem 1. (Feedback System Synchronization Problem). Given a plant $\Sigma_{P\sigma}$ of the form

$$\Sigma_{P\sigma} \equiv \begin{cases} x_P(k) = A_{P\sigma(k)} x_P(k-1) \oplus B_{P\sigma(k)} u_P(k) \\ y_P(k) = C_{P\sigma(k)} x_P(k) \\ x_P(0) = \epsilon \end{cases} \tag{2}$$

and a model

$$\Sigma_{M\sigma} \equiv \begin{cases} x_M(k) = A_{M\sigma(k)} x_M(k-1) \oplus B_{M\sigma(k)} u_M(k) \\ y_M(k) = C_{M\sigma(k)} x_M(k) \\ x_M(0) = \epsilon \end{cases} \tag{3}$$

with $x_P : \mathbb{N} \to \mathbb{R}^{n_P}_{max}$, $x_M : \mathbb{N} \to \mathbb{R}^{n_M}_{max}$, $u_P : \mathbb{N} \to \mathbb{R}^{m_P}_{max}$, $u_M : \mathbb{N} \to \mathbb{R}^{m_M}_{max}$ and $y_P, y_M : \mathbb{N} \to \mathbb{R}^{p}_{max}$, the FSSP consists in finding, for all possible non-decreasing input sequences $\{u_M(k)\}_{k\in\mathbb{N}}$ of the model and all possible switching signals $\sigma(.)$, two families of matrices $\{F_i\}_{i\in\mathcal{I}}$ and $\{G_i\}_{i\in\mathcal{I}}$, with $F_i \in \mathbb{R}^{m_P\times(n_P+n_M)}_{max}$ and $G_i \in \mathbb{R}^{m_P\times m_M}_{max}$, for all $i \in \mathcal{I}$, such that the control input sequence $\{u_P(k)\}_{k\in\mathbb{N}}$ defined by

$$u_P(k) = \begin{cases} F_{\sigma(1)} \begin{pmatrix} x_P(0) \\ x_M(0) \end{pmatrix} \oplus G_{\sigma(1)} u_M(1) \quad \text{for } k = 1 \\ F_{\sigma(k)} \begin{pmatrix} x_P(k-1) \\ x_M(k-1) \end{pmatrix} \oplus G_{\sigma(k)} u_M(k) \\ \qquad\qquad \oplus u_P(k-1), \ \text{for } k > 1 \end{cases} \tag{4}$$

is a solution for the corresponding SSP.

In order to look for a solution to the FSSP, the following extended system $\Sigma_{E\sigma}$ is considered, starting from the plant $\Sigma_{P\sigma}$ described by (2) and the model $\Sigma_{M\sigma}$, defined in (3):

$$\Sigma_{E\sigma} \equiv \begin{cases} x_E(k) = A_{E\sigma(k)} x_E(k-1) \oplus B_{1\sigma(k)} u_P(k) \\ \qquad\qquad \oplus B_{2\sigma(k)} u_M(k) \\ x_E(0) = \epsilon \end{cases} \tag{5}$$

where $x_E(.) = \begin{pmatrix} x_P(.) \\ x_M(.) \end{pmatrix} : \mathbb{N} \to \mathcal{X}_E = \mathbb{R}^{(n_P+n_M)}_{max}$ is the internal event dater, $A_{E\sigma(k)} = \begin{pmatrix} A_{P\sigma(k)} & \epsilon \\ \epsilon & A_{M\sigma(k)} \end{pmatrix}$, $B_{1\sigma(k)} = \begin{pmatrix} B_{P\sigma(k)} \\ \epsilon \end{pmatrix}$, and $B_{2\sigma(k)} = \begin{pmatrix} \epsilon \\ B_{M\sigma(k)} \end{pmatrix}$.

At this point, the concept of *output equalizer* family of semimodules $\mathcal{K}_\sigma \subseteq \mathcal{X}_E$ is needed, which is described as $\mathcal{K}_\sigma = \{\mathcal{K}_i\}_{i\in\mathcal{I}}$, with:

$$\mathcal{K}_i = \left\{ \begin{pmatrix} x_P \\ x_M \end{pmatrix} \in \mathcal{X}_E \text{ , such that } C_{Pi} x_P = C_{Mi} x_M \right\}. \tag{6}$$

The FSSP is then formulated as the problem of finding a control sequence $\{u_P(k)\}_{k\in\mathbb{N}}$ of the form (4), for suitable families $\{F_i\}_{i\in\mathcal{I}}$ and $\{G_i\}_{i\in\mathcal{I}}$ of matrices, that, for any input $\{u_M(k)\}_{k\in\mathbb{N}}$, forces $x_E(k)$ to evolve inside the output equalizer family $\mathcal{K}_\sigma$, for each possible switching signal $\sigma(.)$. Such control sequence is a solution for the SSP and, being of the form (4), it is also a solution for the FSSP.

Before presenting the final theoretical result, two further concepts are needed, which are the notion of (A_σ, B_σ)-invariance and (A_σ, B_σ)-invariance of feedback type.

Definition 1. Given a switching max-plus linear system Σ_σ of the form (1), a family of semimodules $\mathcal{V}_\sigma = \{\mathcal{V}_i\}_{i\in\mathcal{I}} \subseteq \mathcal{X}$ is said to be (A_σ, B_σ)-invariant if, for all $i, j \in \mathcal{I}$ and for all $v \in \mathcal{V}_j$, there exists $u \in \mathbb{R}^m_{max}$ such that $(A_i v \oplus B_i u)$ belongs to $\mathcal{V}_i$.

Definition 2. Given a switching max-plus linear system Σ_σ of the form (1), a family of semimodules $\mathcal{V}_\sigma \subseteq \mathbb{R}^n_{max}$ is said to be an (A_σ, B_σ)-invariant family of semimodules of feedback type for Σ_σ if there exists a family of matrices $\{F_i\}_{i\in\mathcal{I}}$, with $F_i \in \mathbb{R}^{m\times n}_{max}$ for all $i \in \mathcal{I}$, such that for all $i, j \in \mathcal{I}$ and for all $v \in \mathcal{V}_j$, $(A_i \oplus B_i F_i)v$ belongs to $\mathcal{V}_i$.

The theorem about the necessary and sufficient condition for the solvability of the FSSP is stated as follows, as presented in (Animobono et al., 2024b, Theorem 4), which also presents the proof of this theorem. Before that, it should also be reported that a switching max-plus linear system Σ_σ of the form (1) is strongly non-anticipative if it is non-anticipative (i.e. $A_i \geq I_n$ for all $i \in \mathcal{I}$) and $A_i B_j \geq B_i$ for all $i, j \in \mathcal{I}$ (Animobono et al., 2022, Definition 3).

Theorem 1. Given a strongly non-anticipative plant $\Sigma_{P\sigma}$ of the form (2) and a strongly non-anticipative model $\Sigma_{M\sigma}$ of the form (3), the system $\Sigma_{E\sigma}$ given by (5) is considered. Then, the related FSSP is solvable if and only if there exists an $(A_{E\sigma}, B_{1\sigma})$-invariant family of semimodules $\mathcal{V}_\sigma$ of feedback type contained in the output equalizer family of semimodules $\mathcal{K}_\sigma$ defined by (6) such that, for each $i \in \mathcal{I}$ and for each $x \in \operatorname{Im} B_{2i} = \operatorname{Im} \begin{pmatrix} \epsilon \\ B_{Mi} \end{pmatrix} \subseteq \mathcal{X}_E$, there exists $z \in \operatorname{Im} B_{1i} = \operatorname{Im} \begin{pmatrix} B_{Pi} \\ \epsilon \end{pmatrix} \subseteq \mathcal{X}_E$ such that $x \oplus z \in \mathcal{V}_i$.

3. PROBLEM FORMULATION

This section introduces the synchronization problem for the situation of Fig. 1, which considers a shoal of three autonomous underwater vehicles (AUVs), modeled as fish robots, for a structured reconnaissance mission. The aim is to design a control strategy that synchronizes their tasks across a sequence of mission phases, referred to as “configurations”. Each configuration corresponds to a different phase of the mission, where robots are assigned to specific Points of Interest (POIs) to be inspected, through predetermined routes. A configuration can thus be interpreted as a mode of the switching max-plus system, capturing a particular arrangement of task assignments. Transitions between configurations represent changes in the coordination pattern as the mission progresses. A POI is located at the end of each route step, and the final POI, labeled as POI n.4, is the end of the path. To simplify exposition, the paper focuses on a three-phase mission where the AUVs alternate their roles and positions across configurations. This structure allows the use of a switching max-plus linear system to encode the behavior of the full

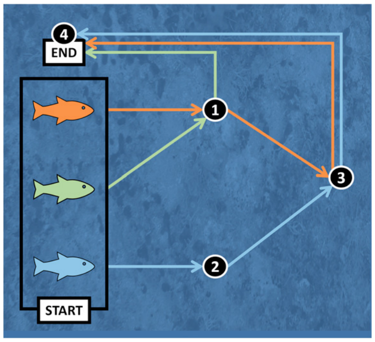

Fig. 1. General scheme for the reconnaissance

multi-agent system compactly and flexibly, to express the configurations of the shoal along the path. In this way, robots can be interpreted as working units, carrying out their task in a predefined sequence.

Let us start by detailing and modeling the robots' coordinated behavior after some hypotheses. The assumptions are based on the fact that each fish robot has different characteristics in terms of sensor equipment and mobility features. Similarly, POIs may have differences in terms of extension, topology, and/or depth. Consequently, each POI has to be explored by the robots with the most suitable sensors. The type of analysis to be performed on each POI can be visual, acoustic, or environmental (such as temperature). This problem is modeled through the max-plus algebra framework, considering these assumptions:

- Each POI has to be surveyed by one robot at a time.
- Each fish will leave its current POI only after reaching it from the previous exploration or the start and receiving the start command input.
- Three configurations are considered, the last of which is associated with the end of the reconnaissance.

In this situation, the index k represents the index of the robots' mission on the k-th point it has to explore, thus indicating the order of the tasks to be performed by the vehicles. Then, $\sigma(k)$ defines the configuration assumed by the shoal at the k-th series of its operations.

At this point, it is necessary to define the variables. For each configuration, there are three input events, $u_1(k)$, $u_2(k)$ and $u_3(k)$, three internal events, $x_1(k)$, $x_2(k)$ and $x_3(k)$ and one output, $y(k)$. Input events need to be provided by an external source and represent the time when each robot (of type 1, 2, and 3) is commanded to start from its current POI. As regards the internal events, calculated with proper rules, they represent the exploration completion time by each fish in its POIs and also include the travel time. Finally, the output event represents the global completion time of each part of the route. In summary, variables are defined as follows:

- $u_i(k)$ is the time when the fish i is commanded to start from its current POI, for its k-th mission
- $x_i(k)$ is the exploration completion time, including the travel time, of the fish of type i in its currently assigned POI for its k-th mission
- $y(k)$ is the global completion time of each part of the route, for the k-th mission

Some constants for the travel and exploration times for the vehicles to reach and inspect POIs must also be defined. Those times are identified by T_{ij}, representing the time for the fish of type i to reach and explore the POI j. The three systems are now presented, remembering that each fish starts from its POI once it has reached it from the previous exploration (or the start) at time $x_i(k)$, and once it receives the start command at time $u_i(k+1)$. The configurations are shown in Fig. 2 a), b), and c). For the first one of Fig.2 a), the system obtained is as follows:

$$\begin{aligned} x_1(1) &= T_{11} \otimes u_1(1) \\ x_2(1) &= T_{21} \otimes u_2(1) \\ x_3(1) &= T_{32} \otimes u_3(1) \\ y(1) &= x_1(1) \oplus x_2(1) \oplus x_3(1) . \end{aligned} \tag{7}$$

Using a matrix notation, as in (1), we can write with

$$A_{\sigma(1)} = \begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon \end{pmatrix} \quad B_{\sigma(1)} = \begin{pmatrix} T_{11} & \epsilon & \epsilon \\ \epsilon & T_{21} & \epsilon \\ \epsilon & \epsilon & T_{32} \end{pmatrix}$$

$$C_{\sigma(1)} = (e \ e \ e) .$$

A similar approach is considered for the other configurations. The second system, represented by Fig. 2 b) is

$$\begin{aligned} x_1(2) &= T_{13} \otimes u_1(2) \oplus x_1(1) \\ x_2(2) &= T_{24} \otimes u_2(2) \oplus x_2(1) \\ x_3(2) &= T_{33} \otimes u_3(2) \oplus x_3(1) \\ y(2) &= x_1(2) \oplus x_2(2) \oplus x_3(2) \end{aligned} \tag{8}$$

with matrices $A_{\sigma(2)}$, $B_{\sigma(2)}$ and $C_{\sigma(2)}$ as follows:

$$A_{\sigma(2)} = B_{\sigma(2)} = \begin{pmatrix} T_{13} & \epsilon & \epsilon \\ \epsilon & T_{24} & \epsilon \\ \epsilon & \epsilon & T_{33} \end{pmatrix} , \ C_{\sigma(2)} = (e \ e \ e) .$$

Finally, the third configuration of Fig. 2 c) refers to the last part of the route, associated with the global end of the reconnaissance, and is expressed by

$$\begin{aligned} x_1(3) &= T_{14} \otimes u_1(3) \oplus x_1(2) \\ x_2(3) &= x_2(2) \\ x_3(3) &= T_{34} \otimes u_3(3) \oplus x_3(2) \\ y(3) &= x_1(3) \oplus x_2(3) \oplus x_3(3) \end{aligned} \tag{9}$$

with

$$A_{\sigma(3)} = \begin{pmatrix} T_{14} & \epsilon & \epsilon \\ \epsilon & e & \epsilon \\ \epsilon & \epsilon & T_{34} \end{pmatrix} \quad B_{\sigma(3)} = \begin{pmatrix} T_{14} & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & T_{34} \end{pmatrix}$$

$$C_{\sigma(3)} = (e \ e \ e) .$$

The representation of the plant as in (1) will be used and considered for the FSSP presented in the next section.

4. SIMULATIONS

This section aims to validate a synchronization strategy by applying it to the scenario described. The simulation uses time values of a hypothetical yet credible reconnaissance mission, incorporating both travel and exploration times for each vehicle to reach and survey each POI. The term "hypothetical" refers to a synthetic case study built upon reasonable assumptions about AUV behavior and mission objectives. While not based on real-world data, it is constructed to test the feasibility of the synchronization problem under non-trivial conditions.

For the simulation, the following time units are considered:

- 1^{st} configuration: $T_{11} = 9$, $T_{21} = 11$, $T_{32} = 11$
- 2^{nd} configuration: $T_{13} = 12$, $T_{24} = 12$, $T_{33} = 8$
- 3^{rd} configuration: $T_{14} = 18$, $T_{34} = 20$.

The numerical computations and symbolic operations were carried out using ScicosLab (2007), a software package well suited for modeling discrete-event systems and performing algebraic manipulations over max-plus semirings.

For the plant simulation, the following $u(k)$ are used, leading to the results presented, with $u(1), u(2), u(3)$ respectively referred to the first, second, and third configuration.

$$u(1) = \begin{pmatrix} 25 \\ 0 \\ 0 \end{pmatrix} \rightarrow x(1) = \begin{pmatrix} 34 \\ 11 \\ 11 \end{pmatrix} \rightarrow y(1) = 34$$

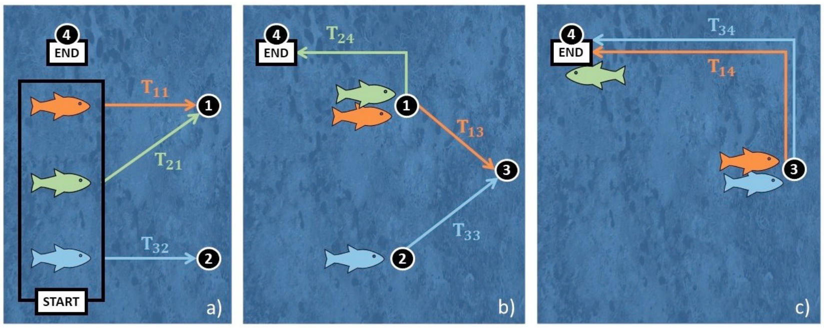

Fig. 2. First (a), second (b), and third (c) configuration of the AUV Team

$$u(2) = \begin{pmatrix} 35 \\ 25 \\ 25 \end{pmatrix} \rightarrow x(2) = \begin{pmatrix} 47 \\ 37 \\ 33 \end{pmatrix} \rightarrow y(2) = 47$$

$$u(3) = \begin{pmatrix} 45 \\ 45 \\ 35 \end{pmatrix} \rightarrow x(3) = \begin{pmatrix} 65 \\ 37 \\ 55 \end{pmatrix} \rightarrow y(3) = 65$$

From the simulation of the system, it emerges that the first mission takes 34 time units, the second route ends after 47 time units (this value also includes the output of the first mission), and the third configuration, representing the last phase of the survey, has as output 65 time units, which represent the total amount of time for the whole survey, which also involves the two previous missions.

Note that in the current simulation, $u(3)_1 = 45$ while $x(2)_1 = 47$, meaning that the first AUV is commanded to start the third task before completing the previous one. This situation may occur in practice, e.g., when a vehicle is instructed to proceed without interruption, or when there is temporarily no information about the robots' position and its completion time is underestimated, causing the command to be sent too early. For this reason, the schedule is modeled as the maximum between the task completion times (contribution of $A_{\sigma(k)}x(k-1)$) and the commands (contributions of $B_{\sigma(k)}u(k)$).

5. PROBLEM SOLUTION

We can now state the feedback system synchronization problem, following the approach of Animobono et al. (2024b), to check with the model whether the mission can be made faster, through a different sequence of $u_i(k)$. The following max-plus system of the form (3) is the model:

$$\Sigma_M \equiv \begin{cases} x_M(k) = 20 x_M(k-1) \oplus 15 u_M(k) \\ y_M(k) = x_M(k) \\ x_M(0) = \epsilon. \end{cases} \tag{10}$$

The objective is that each part of the route, i.e., each mission, should end exactly 15 time units after the reception of start commands or 20 time units after the completion of the previous mission. By simulating this model and assuming the following $u_M(k)$, the output produced is:

$$u_M(1) = (0) \rightarrow y(1) = 15$$
$$u_M(2) = (20) \rightarrow y(2) = 35$$
$$u_M(3) = (40) \rightarrow y(3) = 55.$$

Through an algorithm developed on ScicosLab (2007), the output equalizer family of semimodules $\mathcal{K}_\sigma$ and the maximal $(A_{E\sigma}, B_{1\sigma})$-invariant families of semimodules $\mathcal{V}^*_\sigma(\mathcal{K})$ contained in them are computed. The sequence of semimodules of (Animobono et al., 2024b, Theorem 1), with $\mathcal{V}^0_\sigma = \mathcal{K}_\sigma$ converges after one iteration ($\mathcal{V}^1_\sigma = \mathcal{V}^*_\sigma(\mathcal{K}_\sigma)$):

$$\mathcal{K}_\sigma = \mathcal{V}^*_\sigma(\mathcal{K}_\sigma) = \mathrm{Im} \begin{pmatrix} e & \epsilon & \epsilon \\ \epsilon & e & \epsilon \\ \epsilon & \epsilon & e \\ e & e & e \end{pmatrix} \quad \text{for } \sigma = 1, 2, 3$$

Therefore, the SSP is solvable and since the condition of Theorem 1 is satisfied with $\mathcal{V}^*_\sigma(\mathcal{K}_\sigma)$ of feedback type, the FSSP is also solvable. A control signal $u_P(k)$ of the form (4) that solves that problem is composed of:

$$F_{\sigma(1)} = \begin{pmatrix} \epsilon & 12 & \epsilon & \epsilon \\ \epsilon & \epsilon & 10 & \epsilon \\ \epsilon & \epsilon & \epsilon & 10 \end{pmatrix}, \quad F_{\sigma(2)} = \begin{pmatrix} \epsilon & \epsilon & 9 & \epsilon \\ 13 & \epsilon & \epsilon & \epsilon \\ \epsilon & 9 & \epsilon & \epsilon \end{pmatrix}$$

$$F_{\sigma(3)} = \begin{pmatrix} \epsilon & 3 & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & 1 \\ \epsilon & \epsilon & \epsilon & \epsilon \end{pmatrix}$$

$$G_{\sigma(1)} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}, \quad G_{\sigma(2)} = \begin{pmatrix} -2 \\ 2 \\ 7 \end{pmatrix}, \quad G_{\sigma(3)} = \begin{pmatrix} -3 \\ -5 \\ -5 \end{pmatrix}.$$

6. CONCLUSION

This work provides a concrete case study of how recent theoretical developments in max-plus algebra can be applied to real-world synchronization challenges in underwater robotics. The contribution validates the framework of feedback system synchronization on a structured mission involving a team of autonomous underwater vehicles (AUVs), modeled through a switching max-plus linear system. The system output is successfully forced to equal that of a model, representing the required completion time. The proposed application demonstrates the feasibility of using switching max-plus linear systems to model mission phases and to compute synchronization control laws that guarantee timely coordination. The methodology is general and scalable, and can be adapted to more complex tasks involving heterogeneity, resource sharing, or dynamic reallocation. The work highlights the implementability of the solution, unifying theoretical max-plus control and its practical deployment. This represents a meaningful step toward the integration of algebraic methods in AUV coordination and discrete-event control, with potential impact on real underwater reconnaissance challenges, such as planning trajectories, coordinating multiple AUVs efficiently and accurately, and environmental monitoring. Since multiple solutions could be obtained, this topic opens avenues for future developments, to optimize choices or refine models for individual vehicle trajectories. Exploring priority decision algorithms for non-unique solutions could improve flexibility, while integrating state-space models over time for each vehicle (agent) and specifying depths may enable a more thorough evaluation of trajectories.

REFERENCES

Animobono, D., Scaradozzi, D., Zattoni, E., Perdon, A.M., and Conte, G. (2022). The model matching problem for switching max-plus systems: a geometric approach. *IFAC-PapersOnLine*, 55(40), 7–12. doi:10.1016/j.ifacol.2023.01.040.

Animobono, D., Scaradozzi, D., Zattoni, E., Perdon, A.M., and Conte, G. (2023). The model matching problem for max-plus linear systems: A geometric approach. *IEEE Transactions on Automatic Control*, 68(6), 3581–3587. doi:10.1109/TAC.2022.3191362.

Animobono, D., Zattoni, E., Scaradozzi, D., Perdon, A.M., and Conte, G. (2024a). The model matching problem for periodic max-plus systems. *63rd IEEE Conference on Decision and Control, Milano, Italy, December 16–19.*

Animobono, D., Zattoni, E., Scaradozzi, D., Perdon, A.M., and Conte, G. (2024b). Synchronization and subsynchronization problems for switching max-plus systems: Structural solvability conditions. *IEEE Transactions on Automatic Control*, 69(8), 5613–5619. doi:10.1109/tac.2024.3368298.

Baccelli, F., Cohen, G., Olsder, G.J., and Quadrat, J.P. (1992). *Synchronization and linearity*, volume 1. Wiley New York.

Bartolucci, V. (2024). *Modelling a Shoal of Marine Biomimetic Vehicles: a Max-Plus Algebra Approach.* Ph.D. thesis, Università Politecnica delle Marche. URL `https://hdl.handle.net/11566/326882`.

Bartolucci, V., Loiseau, J.J., Martinez, C., and Scaradozzi, D. (2023). Invariance causale contrôlée pour les systèmes max-plus linéaires. In *Modélisation des Systèmes Réactifs (MSR '23)*. Toulouse, France. URL `https://msr2023.sciencesconf.org/504078/document`.

Bartolucci, V., Scaradozzi, D., Zattoni, E., Loiseau, J.J., Martinez, C., and Conte, G. (2024). A max-plus algebra-based approach for modelling shoals of fish robots during underwater exploration. In *The 34th International Ocean and Polar Engineering Conference, Rhodes, Greece, June 16-21*, 1966–1973.

Basile, G. and Marro, G. (1992). *Controlled and Conditioned Invariants in Linear System Theory.* Prentice Hall, Englewood Cliffs, NJ.

Cardenas, C., Loiseau, J.J., and Martinez, C. (2015). Controlled invariance and dynamic feedback for systems over semirings. In *SIAM Conference on Control and its Applications*. Paris, France. doi:10.1137/1.9781611974072.1.

Cohen, G., Dubois, D., Quadrat, J., and Viot, M. (1985). A linear-system-theoretic view of discrete-event processes and its use for performance evaluation in manufacturing. *IEEE transactions on Automatic Control*, 30(3), 210–220.

Conte, G. and Perdon, A.M. (1994). Problems and results in a geometric approach to the theory of systems over rings. In *Linear algebra for control theory*, 61–74. Springer.

De Schutter, B., Van Den Boom, T.J.J., Xu, J., and Farahani, S.S. (2019). Analysis and control of max-plus linear discrete-event systems: An introduction. *Discrete Event Dynamic Systems*, 30(1), 25–54.

Di Loreto, M., Gaubert, S., Katz, R.D., and Loiseau, J.J. (2010). Duality between invariant spaces for max-plus linear discrete event systems. *SIAM Journal on Control and Optimization*, 48(8), 5606–5628.

Hautus, M.L.J. (1982). Controlled invariance in systems over rings. In D. Hinrichsen and A. Isidori (eds.), *Feedback Control of Linear and Nonlinear Systems: Proceedings of the Joint Workshop on Feedback and Synthesis of Linear and Nonlinear Systems*, volume 39 of *Lecture Notes in Control and Information Sciences*, 107–122. Springer Berlin Heidelberg.

Hillion, H.P. and Proth, J.M. (1989). Performance evaluation of job-shop systems using timed event-graphs. *IEEE Transactions on Automatic Control*, 34(1), 3–9.

Kersbergen, B., Rudan, J., van den Boom, T., and De Schutter, B. (2016). Towards railway traffic management using switching max-plus-linear systems. *Discrete Event Dynamic Systems*, 26(2), 183–223.

Komenda, J., Lahaye, S., Boimond, J.L., and an den Boom, T. (2018). Max-plus algebra in the history of discrete event systems. *Annual Reviews in Control*, 45, 240–249.

Majdzik, P. (2020). Feasible schedule under faults in the assembly system. In *2020 16th International Conference on Control, Automation, Robotics and Vision (ICARCV)*, 1049–1054. IEEE.

Martinez, C., Kara, R., Abdesselam, A.N., and Loiseau, J.J. (2022). Systems synchronisation in Max-Plus algebra: a controlled invariance perspective. In memoriam Édouard Wagneur. *IFAC-PapersOnLine*, 55(40), 1–6. doi:10.1016/j.ifacol.2023.01.039.

Oke, A., Hardouin, L., Lhommeau, M., and Shang, Y. (2017). Observer-based controller for disturbance decoupling of max-plus linear systems with applications to a high throughput screening system in drug discovery. In *56th IEEE Conference on Decision and Control*, 4242–4247. Melbourne, Australia.

Perdon, A.M., Zattoni, E., and Conte, G. (2016a). Model matching with strong stability in switched linear systems. *Systems and Control Letters*, 97, 98–107. doi:10.1016/j.sysconle.2016.09.009.

Perdon, A.M., Conte, G., and Zattoni, E. (2016b). Necessary and sufficient conditions for asymptotic model matching of switching linear systems. *Automatica*, 64, 294–304. doi:10.1016/j.automatica.2015.11.017.

ScicosLab (2007). ScicosLab. `http://scicoslab.org`. Accessed: 2024-12-17.

Screpanti, L., Bartolucci, V., Scaradozzi, D., and Zattoni, E. (2024). A max-plus algebra approach to model laboratory experiences in blended control engineering education. *IFAC-PapersOnLine*, 58(26), 100–105. doi:10.1016/j.ifacol.2024.10.278.

Van den Boom, T.J. and De Schutter, B. (2006). Modelling and control of discrete event systems using switching max-plus-linear systems. *Control engineering practice*, 14(10), 1199–1211.

Zattoni, E., Otsuka, N., Perdon, A.M., and Conte, G. (2024). Model matching problems for impulsive linear systems with polytopic uncertainties. *Nonlinear Analysis: Hybrid Systems*, 52. doi:10.1016/j.nahs.2024.101465.