



# Nash Equilibria with Incomplete Preferences: A Strategic Completion Approach

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## Abstract

We introduce a strategic completion approach to deal with incomplete preferences relations in noncooperative games through appropriate value functions. We ensure that each Nash equilibrium of the original game with incomplete preferences is characterized as an equilibrium of a corresponding scalar game, obtaining both necessary and sufficient Nash equilibrium conditions. We compare our representation results with other approaches grounded on preference completion processes, including the special case where value functions are pseudo-utilities or utilities [in the sense of Peleg (Econom J Econom Soc 38:93–96, 1970)]. We identify minimal requirements for value functions to provide full characterization of Nash equilibria of a game with incomplete preferences through scalarization, with no convexity assumptions. As a special case, we characterize Nash equilibria in vector games by scalarization, extending to the non convex case the results originally developed by Shapley (Naval Res Logist Q 6:57–61, 1959), pointing out some weaknesses of the use of mixed extension in vector games. Finally, we illustrate our approach in vector games by means of special families of scalarizing functions. To conclude, we apply our method to study a bicriteria Cournot duopoly in the context of the managerial theory of firms.

**Keywords** Incomplete preferences · Noncooperative games · Preference completion · Vector games · Scalarization

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## 1 Introduction

Since the beginning, the assumption of completeness of preferences of the decision maker was considered questionable, since it does not allow occasional indecisiveness. A typical instance of indecisiveness can be found when multiple criteria are included within decision processes and optimality is defined by a Paretian principle.

In the seminal contribution by Peleg 1970 (see also Aumann 1962; Richter 1966), representation of preferences has been extended to the case of incomplete preferences. In the spirit of this approach, outcomes of decision processes are ranked on reals by means of a utility function. Each utility induces a completion of incomplete preferences, which allows to deal with scenarios of indecisiveness. Following this approach, sufficient optimality conditions can be obtained through utility representations. As noted in Majumdar and Sen 1976, this implies loss of information, since the original relation cannot be fully recovered by a single utility. As a consequence, necessary optimality conditions cannot be obtained through Peleg's utility representations. More recently, in Dubra et al. 2004 the authors consider families of utilities to represent incomplete relations in order to obtain necessary and sufficient optimality conditions in models with incomplete preferences (see also Evren 2014). In Ok et al. 2002, an alternative way to tackle incompleteness is developed. Here, a preference relation is represented by means of a vector-valued utility. As a consequence, relations among alternatives are described by a partial order in the outcomes' space, where the vector utility takes values. Another approach is presented in Bade 2005, where direct completions of an incomplete relation are considered, without passing through utility representations. The author in Bade 2005 applies this method in the context of game theory, where players have incomplete preferences on their alternatives, in order to provide necessary and sufficient Nash equilibrium conditions.

In the present work, we introduce a pointwise axiomatic approach to completion, originally considered in Miglierina and Molho 2002 in the special case of scalarization of vector optimization problems. Such scalarization techniques were subsequently extended to represent generic partial quasi orders, which typically encompass incomplete preferences as special instances (see e.g. Gutiérrez et al. 2015; Tammer and Weidner 2020; Gutiérrez et al. 2021). It is worth noting that some pointwise approaches have already been considered in decision theory, for instance through reference-dependent representations of incomplete preferences in prospect theory (see e.g. Bleichrodt 2007 and the reference therein).

Here, we extend the pointwise method introduced in Gutiérrez et al. 2015 to non-cooperative game theory with incomplete preferences. In detail, the incomplete preference relation  $\succeq_i$  of player  $i$  on a decision space  $X$  is considered with respect to a reference point  $\bar{x} \in X$ , coherently with the strategic behavior, in the following way

$$\forall x \in X, (x_i, \bar{x}_{-i}) \succeq_i \bar{x} \iff \varphi_i(x_i, \bar{x}_{-i}) \geq \varphi_i(\bar{x})$$

where  $\varphi_i : X \rightarrow \mathbb{R}$  is an appropriate value function for player  $i$ . We underline that this method neglects comparisons among alternatives that are not strategically relevant. As a consequence, this scalarization process does not require any coherence of the images of the value functions of those alternatives that are not involved in the determination of Nash equilibria. We prove that each Nash equilibrium of the original incomplete preference game is characterized as an equilibrium of a corresponding scalar game, where players evaluate the final results of their strategic choices through value functions satisfying some minimal requirements. We remark that we do not need any convexity assumption on the structure of the game.

In the second part of this work, we apply our approach to the special case of vector games, where players' payoffs are vector-valued and incomplete preferences naturally arise. The problem of finding equilibria in vector games through scalarization techniques dates back to the fundamental contribution by Shapley in 1959 (see Shapley and Rigby 1959). In that work, a family of scalar games is derived from a multiobjective finite game through a parameteric linear scalarization scheme. This method allows the author to find all the Nash equilibria of a convexified vector game, obtained from the original one as its mixed extension. A generalization of Shapley's result is presented in Bade 2005, where linear scalarizations have been exploited to characterize Nash equilibria in convex vector games with general conic preferences. It is well known that this kind of linear processes may fail to provide necessary Nash equilibrium conditions in absence of convexity properties. This drawback can be tackled by extending the strategy sets to mixed strategies. However, the mixed extension of a game with vector payoffs may not be fully coherent with the underlying pure strategy vector game. Indeed, pure Nash points may exist that do not correspond to any equilibrium in the mixed extension. In order to avoid any convexity assumption, we apply our pointwise approach to obtain a full characterization of Nash equilibria in games with vector payoffs. This result extends the characterization given in Bade 2005 to the non convex case. Moreover, we apply Gerstewitz scalarization technique in this context (see e.g. Tammer and Weidner 2020 and the reference therein). To our knowledge, there is a limited number of results relying on nonlinear scalarization techniques in game theory (see, for instance, Zapata et al. 2019; Novikova and Pospelova 2018, 2019; Song and Wang 2010 in the special case of multiobjective games). We compare with existing literature and improve some results.

As an exemplary application, we consider a model of a bicriteria Cournot duopoly in the context of managerial theory of firms, where the instances of two groups of stakeholders are considered: profit maximizing owners and revenue maximizing managers. We refer the reader to Vickers 1985, Fershtman and Judd 1987, Sklivas 1987, where players' objectives are modeled as linear combinations of profits and revenues. In a specific setup, we show the existence of a Nash equilibrium that cannot be found through linear scalarization. By contrast, we retrieve all Nash equilibria of the original bicriteria game through our pointwise approach, implemented by Gerstewitz function.

The work is organized as follows. In Section 2, preliminary notions are stated. In Section 3, games with incomplete preferences are introduced together with the notions of pointwise strategic completions. Then, Necessary and sufficient equilibrium conditions are obtained. Comparisons of pointwise approach with other

approaches are discussed. In Section 4.1, the pointwise approach is reframed in the context of vector games. We provide an application to a class of bicriteria Cournot competition models. Section 5 concludes.

## 2 Preliminary Notions and Notations

Let  $T$  be a topological vector space. Given a subset  $S \subseteq T$ , we will indicate by  $\text{cl}S$  the closure of  $S$ , by  $\text{int}S$  the topological interior of  $S$  and by  $\text{conv}S$  the convex hull of  $S$ . A nonempty subset  $K \subseteq T$  is said to be a cone when  $\lambda k \in K$ , for all  $\lambda \geq 0$  and  $k \in K$ . A cone  $K$  is proper when  $K \neq \{0\}$  and  $K \neq T$ , pointed when  $K \cap (-K) = \{0\}$ , solid when  $\text{int}K \neq \emptyset$ . For the proof of the following result see Lemma 1.12 in Jahn 2009.

**Lemma 1** *Let  $T$  be a topological vector space and let  $K \subseteq T$  be a convex cone. Then, the set  $C = \text{int}K \cup \{0\} \subseteq K$  is a convex cone.*

The set  $t + S$  denotes the algebraic sum between the singleton  $t \in T$  and the set,<sup>1</sup> namely  $t + S = \{t' \in T : t' = t + s, s \in S\}$ . We consider the ordering structure  $\succeq_K$  induced by the cone  $K$  on the space  $T$  as follows:

$$\begin{aligned} t \succeq_K t' &\iff t \in t' + K \\ t >_K t' &\iff t \in t' + K, t \neq t' \end{aligned}$$

where  $t, t' \in T$ . By  $T^*$  we denote the topological dual space of  $T$ . The dual and the strict dual cones are given, respectively, by

$$\begin{aligned} K^+ &:= \{\lambda \in T^* : \lambda(k) \geq 0, \forall k \in K\} \\ K^{++} &:= \{\lambda \in T^* : \lambda(k) > 0, \forall k \in K \setminus \{0\}\} \end{aligned}$$

Given any tuple  $\mathbf{v} = (v_1, \dots, v_N)$ , we will set  $\mathbf{v}_{-i} = (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_N)$  and, when convenient, we will write  $(v_i, \mathbf{v}_{-i})$  in place of  $\mathbf{v}$ . If  $N = 1$ , we set  $(v_1, \mathbf{v}_{-1}) = v_1$ .

## 3 Games with Incomplete Preferences

We consider the (finite) set  $\mathcal{N} = \{1, \dots, N\}$  of players.  $X_i$  will denote the set of alternatives of player  $i$  and  $X = \bigotimes_{j \in \mathcal{N}} X_j$  the cartesian product of all such sets. We will call a game finite when, for all  $i \in \mathcal{N}$ , the set  $X_i$  is finite. Player  $i$  compares tuples of alternatives by means of the reflexive and transitive relation  $\succeq_i$  on  $X$ , endowing it with a partial quasiorder structure. We underline that relation  $\succeq_i$  corresponds to the notion of preference-or-indifference considered in Aumann (1962) (see also Debreu 1959 where relation  $\succeq_i$  is considered in the special case of complete

<sup>1</sup>With abuse of notation we denote singleton  $\{t\}$  by  $t$ .

preferences and named preference preordering). Let  $x^1, x^2 \in X$ , we write  $x^1 \sim_i x^2$  when both  $x^1 \succeq_i x^2$  and  $x^2 \succeq_i x^1$  hold. Moreover,  $x^1 \succsim_i x^2$  stands for  $x^1 \succeq_i x^2$  and not  $x^1 \sim_i x^2$ . We underline that relation  $\succsim_i$  is irreflexive. A noncooperative game is fully described by the tuple  $\Gamma = (\mathcal{N}, \{X_i, \succeq_i\}_{i \in \mathcal{N}})$ . The concepts of Nash equilibrium of  $\Gamma$  can be defined as follows.

**Definition 2** The strategy profile  $x^* \in X$  is said to be

- i a strict Nash equilibrium for  $\Gamma$  when, for all  $i \in \mathcal{N}$ , no  $x_i \in X_i \setminus \{x_i^*\}$  exists such that  $(x_i, x_{-i}^*) \succeq_i x^*$ ;
- ii a Nash equilibrium for  $\Gamma$  when, for all  $i \in \mathcal{N}$ , no  $x_i \in X_i$  exists such that  $(x_i, x_{-i}^*) \succsim_i x^*$ .

The sets of strict Nash and Nash equilibria for  $\Gamma$  will be denoted by  $\mathcal{E}^s(\Gamma)$  and  $\mathcal{E}(\Gamma)$  respectively. According to Definition 2, a strict Nash equilibrium  $x^*$  is a Nash equilibrium to which no player remains indifferent by unilaterally changing her choice.

### 3.1 Strategic Completions of Incomplete Preferences

A typical approach to tackle economic models involving incomplete preferences consists in deriving a total order to compare alternatives that is coherent with the original partial order. Such a completion process allows to compare even those alternatives that are non comparable according to the original incomplete preference relation.

Here, we extend to noncooperative games with incomplete preferences the point-wise axiomatic scheme considered in Gutiérrez et al. 2015, where the original partial order is completed through scalarization. This method involves certain classes of value functions through which players evaluate their alternatives that are strategically relevant, coherently with the notion of Nash equilibrium. Properties of value functions that allow to find all the Nash points of a noncooperative game  $\Gamma$  with incomplete preferences by scalarization are here identified. Moreover, we will prove that such properties correspond to the minimal requirements that are needed to find all the equilibria of  $\Gamma$  through scalarization. We will show that preference preserving utilities, as those considered in Peleg 1970, Richter 1966 are included in our approach as special cases. For all  $i \in \mathcal{N}$ , let  $\varphi_i : X \rightarrow \mathbb{R}$  be the value function through which player  $i$  evaluates her alternatives. The following properties of functions  $\varphi_i$  at the reference point  $\bar{x}$  are of interest.

**Definition 3** Given the point  $\bar{x} \in X$ , the function  $\varphi_i : X \rightarrow \mathbb{R}$  is said to be

**P**  $\succeq_i$ -preserving at  $\bar{x}$  when

$$(x_i, \bar{x}_{-i}) \in X, (x_i, \bar{x}_{-i}) \succeq_i \bar{x} \Rightarrow \varphi_i(x_i, \bar{x}_{-i}) \geq \varphi_i(\bar{x})$$

**sP** strictly  $\succeq_i$ -preserving at  $\bar{x}$  when

$$(x_i, \bar{x}_{-i}) \in X, (x_i, \bar{x}_{-i}) \succ_i \bar{x} \Rightarrow \varphi_i(x_i, \bar{x}_{-i}) > \varphi_i(\bar{x})$$

**R**  $\succeq_i$ -representing at  $\bar{x}$  when

$$(x_i, \bar{x}_{-i}) \in X, \varphi_i(x_i, \bar{x}_{-i}) > \varphi_i(\bar{x}) \Rightarrow (x_i, \bar{x}_{-i}) \succ_i \bar{x}$$

**sR** strictly  $\succeq_i$ -representing at  $\bar{x}$  when

$$(x_i, \bar{x}_{-i}) \in X, \varphi_i(x_i, \bar{x}_{-i}) \geq \varphi_i(\bar{x}) \Rightarrow (x_i, \bar{x}_{-i}) \succeq_i \bar{x}$$

**Remark 1** If property **sP** (resp. **sR**) holds for  $\varphi_i$  at a point  $\bar{x}$ , then property **P** (resp. **R**) holds for  $\varphi_i$  at  $\bar{x}$  as well, whenever  $(x_i, \bar{x}_{-i}) \sim_i \bar{x}$  implies  $\varphi_i(x_i, \bar{x}_{-i}) = \varphi_i(\bar{x})$ .

We note that the properties in Definition (3) correspond to those given in Gutiérrez et al. 2015 when  $\mathcal{N} = \{1\}$ .

Two distinct strategic completion processes of the original incomplete relation  $\succeq_i$  at a point  $\bar{x} \in X$  can be defined through specific combinations of properties introduced in Definition 3. In detail, let  $\varphi_i$  be the value function of player  $i$ . When properties **P** and **sR** hold for  $\varphi_i$  at point  $\bar{x}$ , then  $\varphi_i$  refines relation  $\succeq_i$  at the reference point  $\bar{x} \in X$  as follows

$$(x'_i, \bar{x}_{-i}) \in X, (x'_i, \bar{x}_{-i}) \succeq_i \bar{x} \iff \varphi_i(x'_i, \bar{x}_{-i}) \geq \varphi_i(\bar{x}) \tag{1}$$

We underline that equivalence (1) involves only comparisons among alternatives with respect to  $\bar{x}$  that are strategically relevant for player  $i$  in choosing her best reply to her competitors' choices  $\bar{x}_{-i}$ . We refer to a refinements induced on  $\succeq_i$  by a function  $\varphi_i$  for which properties **P** and **sR** hold at  $\bar{x}$  as *strategic  $\succeq_i$ -completion at  $\bar{x}$* .

A similar interpretation can be considered when properties **sP** and **R** hold for  $\varphi_i$  at point  $\bar{x}$  instead. In this case, relation  $\succ_i$  are refined at the reference point  $\bar{x} \in X$  as follows

$$(x'_i, \bar{x}_{-i}) \in X, (x'_i, \bar{x}_{-i}) \succ_i \bar{x} \iff \varphi_i(x'_i, \bar{x}_{-i}) > \varphi_i(\bar{x}) \tag{2}$$

We refer to a refinement induced on  $\succ_i$  by function  $\varphi_i$  for which properties **sP** and **R** hold at  $\bar{x}$  as *strategic  $\succ_i$ -completion at  $\bar{x}$* .

If no additional regularity properties are required, it is easy to find examples of value functions enjoying all properties given in Definition 3.

**Example 4** For all  $i \in \mathcal{N}$ , let  $X$  be a set endowed with reflexive and transitive relation  $\succeq_i$  and let the function  $\psi_i^{\bar{x}} : X \rightarrow \mathbb{R}$  be defined as

$$\psi_i^{\bar{x}}(\mathbf{x}) = \begin{cases} +1 & \text{if } \mathbf{x} = (x_i, \bar{x}_{-i}) \succ_i \bar{x} \\ 0 & \text{if } \mathbf{x} = (x_i, \bar{x}_{-i}) \sim_i \bar{x} \\ -1 & \text{otherwise} \end{cases}$$

where  $\bar{x} \in X$  is a given parameter. Taking into account that  $\psi_i^{\bar{x}}(\bar{x}) = 0$ , it is immediate to verify that properties **P**, **sP**, **R** and **sR** hold for function  $\psi_i^{\bar{x}}$  at  $\bar{x}$ . Moreover, property **P** holds for  $\psi_i^{\bar{x}}$  at  $x$ , for all  $x \in X$ .

From now on, we will denote by  $\varphi$  any tuple of value functions, namely  $\varphi := (\varphi_1, \dots, \varphi_i, \dots, \varphi_N)$ , where  $\varphi_i : X \rightarrow \mathbb{R}$ , for all  $i \in \mathcal{N}$ . With a given choice of  $\varphi$ , the scalar game obtained from  $\Gamma$  by evaluating  $i$ 's alternatives through  $\varphi_i$  will be denoted by  $S_\varphi - \Gamma$ . The scalar game so obtained is fully described by the tuple  $S_\varphi - \Gamma = (\mathcal{N}, \{X_i, \varphi_i\}_{i \in \mathcal{N}})$ . The sets of strict Nash and Nash equilibria of the scalar game  $S_\varphi - \Gamma$  will be denoted by  $\mathcal{E}^s(S_\varphi - \Gamma)$  and  $\mathcal{E}(S_\varphi - \Gamma)$ , respectively.

**Remark 2** We assume that the incomplete relations  $\succeq_i$  as well as the whole structure of the game  $\Gamma$  are common knowledge. As a consequence, the strategic upper contour set of the relation  $\succeq_i$  at  $\bar{x}$

$$U^{\succeq_i}(\bar{x}) := \{x \in X : x = (x_i, \bar{x}_{-i}) \succeq_i \bar{x}\}$$

is also common knowledge. Moreover, we can define the strategic upper level set for function  $\varphi_i$  at  $\bar{x}$  as

$$U^{\varphi_i}(\bar{x}) := \{x \in X : \varphi_i(x_i, \bar{x}_{-i}) \geq \varphi_i(\bar{x})\}$$

When properties **P** and **sR** hold for  $\varphi_i$  at  $\bar{x}$ , then it holds  $U^{\succeq_i}(\bar{x}) = U^{\varphi_i}(\bar{x})$ . Hence, the strategic upper level set  $U^{\varphi_i}(\bar{x})$  of value function  $\varphi_i$  at  $\bar{x}$  is common knowledge, even if the specific functional form of  $\varphi_i$  may remain unknown to competitors of player  $i$ .

The following proposition provides sufficient equilibrium conditions through scalarization, establishing when an equilibrium point  $x^*$  of the scalar game  $S_\varphi - \Gamma$  is an equilibrium of  $\Gamma$ .

**Theorem 5** Let  $x^* \in X$  and let  $S_\varphi - \Gamma$  be the scalar game obtained from  $\Gamma$ , where  $\varphi = (\varphi_1, \dots, \varphi_N)$ .

**S1** If, for all  $i \in \mathcal{N}$ , **P** holds for  $\varphi_i$  at  $x^*$ , then

$$x^* \in \mathcal{E}^s(S_\varphi - \Gamma) \Rightarrow x^* \in \mathcal{E}^s(\Gamma)$$

**S2** if, for all  $i \in \mathcal{N}$ , **sP** holds for  $\varphi_i$  at  $x^*$ , then

$$x^* \in \mathcal{E}(S_\varphi - \Gamma) \Rightarrow x^* \in \mathcal{E}(\Gamma)$$

**Proof S1.** Let  $x^* \in \mathcal{E}^s(S_\varphi - \Gamma)$ . Then, for all  $i \in \mathcal{N}$ , the relation

$$\varphi_i(x_i, x_{-i}^*) < \varphi_i(x^*)$$

holds for all  $x_i \in X_i \setminus \{x_i^*\}$ . Since  $\varphi_i$  is  $\succeq_i$ -preserving at  $\mathbf{x}^*$ , no  $x_i \in X_i \setminus \{x_i^*\}$  exists such that  $(x_i, \mathbf{x}_{-i}^*) \succeq_i \mathbf{x}^*$ . Hence,  $\mathbf{x}^* \in \mathcal{E}^s(\Gamma)$ .

**S2.** Let  $\mathbf{x}^* \in \mathcal{E}(S_\varphi - \Gamma)$ . Then, for all  $i \in \mathcal{N}$ , the relation

$$\varphi_i(x_i, \mathbf{x}_{-i}^*) \not\succeq_i \varphi_i(\mathbf{x}^*)$$

holds for all  $x_i \in X_i$ . Since  $\varphi_i$  is strictly  $\succeq_i$ -preserving at  $\mathbf{x}^*$ , then either  $(x_i, \mathbf{x}_{-i}^*) \not\succeq_i \mathbf{x}^*$  or  $(x_i, \mathbf{x}_{-i}^*) \sim_i \mathbf{x}^*$  holds for all  $x_i \in X_i$ . Hence,  $\mathbf{x}^* \in \mathcal{E}(\Gamma)$ .  $\square$

The following proposition provides necessary equilibrium conditions, establishing when an equilibrium point  $\mathbf{x}^*$  of the game  $\Gamma$  is an equilibrium of the scalar game  $S_\varphi - \Gamma$ .

**Theorem 6** *Let  $\mathbf{x}^* \in X$  and let  $S_\varphi - \Gamma$  be the scalar game obtained from  $\Gamma$ , where  $\varphi = (\varphi_1, \dots, \varphi_N)$ .*

**N1** If, for all  $i \in \mathcal{N}$ , **sR** holds for  $\varphi_i$  at  $\mathbf{x}^*$ , then

$$\mathbf{x}^* \in \mathcal{E}^s(\Gamma) \Rightarrow \mathbf{x}^* \in \mathcal{E}^s(S_\varphi - \Gamma)$$

**N2** if, for all  $i \in \mathcal{N}$ , **R** holds for  $\varphi_i$  at  $\mathbf{x}^*$ , then

$$\mathbf{x}^* \in \mathcal{E}(\Gamma) \Rightarrow \mathbf{x}^* \in \mathcal{E}(S_\varphi - \Gamma)$$

**Proof N1.** Since  $\mathbf{x}^* \in \mathcal{E}^s(\Gamma)$ , then for all  $i \in \mathcal{N}$  the relation

$$(x_i, \mathbf{x}_{-i}^*) \not\succeq_i \mathbf{x}^*$$

holds for all  $x_i \in X_i \setminus \{x_i^*\}$ , implying  $\varphi_i(x_i, \mathbf{x}_{-i}^*) < \varphi_i(\mathbf{x}^*)$  for all  $x_i \in X_i \setminus \{x_i^*\}$ , since property **sR** holds for  $\varphi_i$  at  $\mathbf{x}^*$ . Hence,  $\mathbf{x}^* \in \mathcal{E}^s(S_\varphi - \Gamma)$ .

**N2.** Let  $\mathbf{x}^* \in \mathcal{E}(\Gamma)$  and suppose, by contradiction, that, for some  $i \in \mathcal{N}$ , an element  $x_i \in X_i$  exists such that

$$\varphi_i(x_i, \mathbf{x}_{-i}^*) > \varphi_i(\mathbf{x}^*)$$

Since property **R** holds for  $\varphi_i$  at  $\mathbf{x}^*$ , then  $(x_i, \mathbf{x}_{-i}^*) \succ_i \mathbf{x}^*$  follows, a contradiction. Hence,  $\mathbf{x}^* \in \mathcal{E}(S_\varphi - \Gamma)$ .  $\square$

**Remark 3** When  $\mathcal{N} = \{1\}$  the game  $\Gamma$  reduces to the optimization problem considered in Gutiérrez et al. 2015, where strategic interdependence among players is not considered. The results given in Theorems 5 and 6 reduce to Propositions 3.5, 3.6, 3.9, and 3.10 in Gutiérrez et al. 2015.

We also underline that the results in Theorems 5 and 6 crucially rely on the assumption that the same strategy profile  $\mathbf{x}^*$  is considered as a reference point for

the pointwise completion process based on the scalarizing function  $\varphi_i$  of player  $i$ , for all  $i \in \mathcal{N}$ .

Further relevance to properties in Definition 3 is given by the fact that these properties are also necessary to obtain equilibrium conditions. In detail, for all  $i \in \mathcal{N}$ , property **P** (resp. **sP**) of value functions is necessary in order to obtain sufficient strict equilibrium (resp. equilibrium) conditions through scalarization. Moreover, for all  $i \in \mathcal{N}$ , property **R** (resp. **sR**) is necessary in order to obtain necessary equilibrium (resp. strict equilibrium) conditions through scalarization.

**Theorem 7** *Let  $\mathbf{x}^* \in X$  and  $\varphi_i : X \rightarrow \mathbb{R}$  be value functions, for all  $i \in \mathcal{N}$ .*

- i* If  $\mathbf{x}^* \in \mathcal{E}^s(\Gamma)$ , then either  $\mathbf{x}^* \notin \mathcal{E}^s(S_\varphi - \Gamma)$  or property **P** holds for  $\varphi_i$  at  $\mathbf{x}^*$ , for all  $i \in \mathcal{N}$ .
- ii* If  $\mathbf{x}^* \in \mathcal{E}(\Gamma)$ , then either  $\mathbf{x}^* \notin \mathcal{E}(S_\varphi - \Gamma)$  or property **sP** holds for  $\varphi_i$  at  $\mathbf{x}^*$ , for all  $i \in \mathcal{N}$ .
- iii* If  $\mathbf{x}^* \in \mathcal{E}(S_\varphi - \Gamma)$ , then either  $\mathbf{x}^* \notin \mathcal{E}(\Gamma)$  or property **R** holds for  $\varphi_i$  at  $\mathbf{x}^*$ , for all  $i \in \mathcal{N}$ .
- iv* If  $\mathbf{x}^* \in \mathcal{E}^s(S_\varphi - \Gamma)$ , then either  $\mathbf{x}^* \notin \mathcal{E}^s(\Gamma)$  or property **sR** holds for  $\varphi_i$  at  $\mathbf{x}^*$ , for all  $i \in \mathcal{N}$ .

**Proof** *i.* Let  $\mathbf{x}^* \in \mathcal{E}^s(S_\varphi - \Gamma)$  and property **P** does not hold for  $\varphi_i$  at  $\mathbf{x}^*$ . Then,  $\varphi_i(x_i, \mathbf{x}_{-i}^*) < \varphi_i(\mathbf{x}^*)$  holds for all  $x_i \in X_i \setminus \{x_i^*\}$  and an element  $\tilde{x}_i \in X \setminus \{x_i^*\}$  exists such that  $(\tilde{x}_i, \mathbf{x}_{-i}^*) \succeq_i \mathbf{x}^*$ . This implies  $\mathbf{x}^* \notin \mathcal{E}^s(\Gamma)$ .

*ii.* is analogously proved.

*iii.* Let  $\mathbf{x}^* \in \mathcal{E}(\Gamma)$  and property **R** does not hold for  $\varphi_i$  at  $\mathbf{x}^*$ . Then,  $(x_i, \mathbf{x}_{-i}^*) \not\prec_i \mathbf{x}^*$  holds for all  $x_i \in X_i$  and an element  $\tilde{x}_i \in X_i$  exists such that  $\varphi_i(\tilde{x}_i, \mathbf{x}_{-i}^*) > \varphi_i(\mathbf{x}^*)$ . This implies  $\mathbf{x}^* \notin \mathcal{E}(S_\varphi - \Gamma)$ .

*iv.* is analogously proved. □

**Remark 4** Theorem 7 shows that properties of value functions in Definition 3 cannot be omitted when a Nash equilibrium of a game with incomplete preferences is characterized as a Nash equilibrium of a corresponding scalar game. In detail, point *i* states that whenever the strict Nash point  $\mathbf{x}^*$  of  $\Gamma$  can be obtained as a strict Nash equilibrium of the scalarized game  $S_\varphi - \Gamma$ , then property **P** holds for  $\varphi_i$  at  $\mathbf{x}^*$ , for all  $i \in \mathcal{N}$ . Similarly, point *iii* states that whenever the Nash point  $\mathbf{x}^*$  of the scalarized game  $S_\varphi - \Gamma$  corresponds to a Nash point of game  $\Gamma$ , then property **R** holds for  $\varphi_i$  at  $\mathbf{x}^*$ , for all  $i \in \mathcal{N}$ . Analogous observations can be given for point *ii* and *iv*.

### 3.2 A Comparison with other Approaches to Preference Completion

Completion of incomplete preferences based on utility functions has been considered in the seminal work Peleg 1970 in the context of decision making (see also Aumann 1962; Richter 1966). The author defines a utility function as a continuous function  $u : X \rightarrow \mathbb{R}$  that is strictly monotone with respect to relation  $\succ$ , that is such that

$$x^1 \succ x^2 \implies u(x^1) > u(x^2) \tag{3}$$

for all  $x^1, x^2 \in X$ .

**Remark 5** If function  $\varphi_i : X \rightarrow \mathbb{R}$  completes  $i$ 's preferences as a Peleg utility function, then property **sP** holds for  $\varphi_i$  at  $x \in X$ , for all  $x \in X$ . Moreover, if  $\varphi_i$  completes  $i$ 's preferences as a *pseudo-utility* (see Peleg 1970) then property **P** holds for  $\varphi_i$  at  $x \in X$ , for all  $x \in X$ . The converse implication does not hold. Indeed, property **sP** (resp. **P**) holding for  $\varphi_i$  at  $x' \in X$ , for all  $x' \in X$ , does not imply strict monotonicity (resp. monotonicity) on  $X$  with respect to relation  $\succ_i$  (resp  $\succeq_i$ ). Nevertheless, in the case of a single decision maker ( $\mathcal{N} = \{1\}$ ), if  $X$  is a topological space,  $\varphi_1 : X \rightarrow \mathbb{R}$  is continuous on  $X$ , and property **sP** holds for  $\varphi_1$  everywhere on  $X$ , then  $\varphi_1$  completes 1's preferences as a Peleg utility function. Moreover, if the less restrictive property **P** holds for  $\varphi_1$  everywhere on  $X$ , then  $\varphi_1$  completes 1's preferences as a *pseudo-utility* (see Peleg 1970).

Here we extend the approach originally devised by Peleg in the context of utility theory to noncooperative games with incomplete preferences. Specifically, we consider properties **P** and **sP** everywhere on  $X$ . This allows us to obtain sufficient equilibrium conditions through scalarization with no need of on an a priori knowledge of Nash points of the original game  $\Gamma$  with incomplete preferences. Full characterizations of both strict Nash and Nash equilibria in games with incomplete preferences will be provided by means of strategic  $\succeq_i$ - and  $\succ_i$ -completions. In detail, we identify two sets of value functions that allow us to represent respectively all Nash and all strict Nash points of a game with incomplete preferences, with no need of additional assumptions on the convex structure of the game. Let  $\varphi_i : X \rightarrow \mathbb{R}$  and let  $\varphi$  be the tuple with components  $(\varphi)_i = \varphi_i$ , for all  $i \in \mathcal{N}$ . We will consider the following collections of tuples

$$\Phi^s(A) = \bigcup_{x \in A} \left\{ \mathbf{u} : \mathbf{P} \text{ holds at } x' \in X, \forall x' \in X, \text{ and } \mathbf{sR} \text{ holds at } x \right. \\ \left. \text{for } u_i = (\mathbf{u})_i, \forall i \in \mathcal{N} \right\}$$

$$\Phi(A) = \bigcup_{x \in A} \left\{ \mathbf{u} : \mathbf{sP} \text{ holds at } x' \in X, \forall x' \in X, \text{ and } \mathbf{R} \text{ holds at } x \right. \\ \left. \text{for } u_i = (\mathbf{u})_i, \forall i \in \mathcal{N} \right\}$$

where  $A \subseteq X$ . The set  $\Phi^s(A)$  is nonempty, including at least the functions  $\psi^x$  illustrated in example 4. By contrast, the set  $\Phi(A)$  can be empty in absence of additional hypotheses on the nature of preferences. An example is the case of lexicographic preferences, which cannot be represented by a value function satisfying property **sP** everywhere on  $X$ , as shown in Debreu 1959 when  $\mathcal{N} = \{1\}$ .

Full characterizations of the sets of strict Nash and Nash equilibria of a game  $\Gamma$  with incomplete preferences through strategic completions are given in the following Proposition.

**Proposition 8** *Let  $\Gamma$  be a game with incomplete preferences and let  $A \subseteq X$  be nonempty.*

*i* If  $\mathcal{E}^s(\Gamma) \subseteq A$ , then

$$\mathcal{E}^s(\Gamma) = \bigcup_{\varphi \in \Phi^s(A)} \mathcal{E}^s(S_\varphi - \Gamma) \tag{4}$$

*ii* Let  $\Phi(A)$  be nonempty. If  $\mathcal{E}(\Gamma) \subseteq A$ , then

$$\mathcal{E}(\Gamma) = \bigcup_{\varphi \in \Phi(A)} \mathcal{E}(S_\varphi - \Gamma) \tag{5}$$

**Proof** *i.* Since  $\mathcal{E}^s(\Gamma) \subseteq A$ , for all  $\mathbf{x}^* \in \mathcal{E}^s(\Gamma)$  a tuple  $\varphi \in \Phi^s(A)$  exists such that, for all  $i \in \mathcal{N}$ , its component  $(\varphi)_i = u_i$  satisfies property **sR** at  $\mathbf{x}^*$ . Then, the relation  $\mathbf{x}^* \in \mathcal{E}^s(S_\varphi - \Gamma)$  follows by point **N1** in Theorem 6. Hence, the inclusion

$$\mathcal{E}^s(\Gamma) \subseteq \bigcup_{\mathbf{u} \in \Phi^s(A)} \mathcal{E}^s(S_\mathbf{u} - \Gamma) \tag{6}$$

holds. Conversely, if  $\mathbf{x} \notin \mathcal{E}^s(\Gamma)$ , then an element  $x'_i \in X_i$  exists such that

$$(x'_i, \mathbf{x}_{-i}) \succeq_i \mathbf{x} \tag{7}$$

Since, for all  $\mathbf{u} \in \Phi^s(A)$ , property **P** holds for  $(\mathbf{u})_i = u_i$  at any  $\mathbf{x} \in X$ , relation (7) implies

$$u_i(x'_i, \mathbf{x}_{-i}) \geq u_i(\mathbf{x})$$

for all  $i \in \mathcal{N}$ . Hence,  $\mathbf{x} \notin \mathcal{E}^s(S_\mathbf{u} - \Gamma)$  holds for all  $\mathbf{u} \in \Phi^s(A)$ . This shows that  $\mathbf{x} \in \mathcal{E}^s(S_\mathbf{u} - \Gamma)$  for some  $\mathbf{u} \in \Phi^s(A)$  implies  $\mathbf{x} \in \mathcal{E}^s(\Gamma)$ , that is

$$\mathcal{E}^s(\Gamma) \supseteq \bigcup_{\mathbf{u} \in \Phi^s(A)} \mathcal{E}^s(S_\mathbf{u} - \Gamma) \tag{8}$$

Inclusions (6) and (8) prove (4).

ii. The proof of the second equality (5) is analogous, exploiting point **N2** in Theorem 6 and property **sP** holding for functions  $(u)_i = u_i$  at any  $x$  in  $X$ , for all  $i \in \mathcal{N}$  and all  $u \in \Phi(A)$ . □

**Remark 6** Both the collections  $\Phi^s(A)$  and  $\Phi(A)$  can be restricted by choosing  $A = \mathcal{E}^s(\Gamma)$  and  $A = \mathcal{E}(\Gamma)$  respectively. By virtue of points *ii* and *iv* respectively in Theorem 7, the requirements on such collections cannot be reduced any further. Nevertheless, the possibility to arbitrarily choose the set  $A$  avoids the need of a preliminary knowledge of the set of all Nash or strict Nash equilibria of  $\Gamma$ .

We can now compare our characterization in equation (5) with the one proved in Bade 2005, where a characterization of Nash equilibria in games with incomplete preferences has been proved without passing through utility functions. In order to better discuss this result, let us introduce the following notations. We will say that a complete relation  $\succeq'_i$  is a  $\succeq_i$ -completion of preference relation  $\succeq_i$  of player  $i$  when  $x_1 \succeq_i x_2$  implies  $x_1 \succeq'_i x_2$ . When, in addition,  $x_1 \succ_i x_2$  implies  $x_1 \succ'_i x_2$ , we say that  $\succeq'_i$  is a  $\succ_i$ -completion of  $\succeq_i$ . Moreover, we say that a game  $\Gamma' = (\mathcal{N}, \{X_i, \succeq'_i\}_{i \in \mathcal{N}})$  is a  $\succeq$ -completion (resp.  $\succ$ -completion) of  $\Gamma = (\mathcal{N}, \{X_i, \succeq_i\}_{i \in \mathcal{N}})$  when  $\succeq'_i$  is a  $\succeq_i$ -completion (resp.  $\succ_i$ -completion) of  $\succeq_i$ , for all  $i \in \mathcal{N}$ . With this terminology, Theorem 1 in Bade 2005 states that all the equilibria of a game  $\Gamma$  with incomplete preferences can be characterized as Nash equilibria of all  $\succ$ -completions of  $\Gamma$ , that is

$$\mathcal{E}(\Gamma) = \{\mathcal{E}(\Gamma') : \Gamma' \text{ is a } \succ\text{-completion of } \Gamma\} \tag{9}$$

Relation (9) provides an alternative representation of Nash equilibria of a game with incomplete preferences with respect to result in (5). On one hand, property **sP** of  $\varphi_i$  involved in (5) does not require any coherence of the original preference  $\succeq_i$  between strategy tuples  $x$  and  $x'$  and the ordering of the images  $\varphi_i(x)$  and  $\varphi_i(x')$ , whenever  $x_{-i} \neq x'_{-i}$ . By contrast, any  $\succ$ -completion of  $\Gamma$  considered in Bade 2005 preserves the comparisons between alternatives expressed by each player through the original preference  $\succeq_i$ . On the other hand, we obtain necessary equilibrium conditions through scalarization by means of  $\succeq_i$ -representing property **R**. Such property has no counterpart in the approach considered in Bade 2005, which is not grounded on value function representations.

### 4 Application to Games with Vector Payoffs

A special class of games characterized by incompleteness of players' preferences is described by vector games, where players' payoffs are vector-valued functions (see e.g. the early contributions Shapley and Rigby 1959; Corley 1985; Blackwell 1956, among others). Vector payoffs naturally arise in many strategic contexts, where players deal with multiple conflicting objectives. Moreover, recent results on representation of incomplete preferences by means of vector-valued utilities (see Ok et al. 2002) provide further relevance to vector games as reformulations of games with incomplete preferences.

In vector games, incomplete preferences are represented by conic partial orders defined in the image spaces of vector-valued payoffs. In this framework, scalarizations of vector outcomes can be interpreted as strategic completions or completions of such incomplete preferences. In this Section, we reframe our pointwise axiomatic approach in vector games in order to provide necessary and sufficient equilibrium conditions, without any convexity requirements. In order to study equilibrium strategy profiles in vector games, scalarization methods have been considered in literature. In 1959, L. Shapley obtained a representation result for mixed Nash equilibria through linear scalarizations of a zero sum finite game where players have multiple payoffs (see Shapley and Rigby 1959). Under appropriate assumptions, the author proved necessary and sufficient equilibrium conditions through linear scalarization, after convexifying the game by passing to its mixed extension. Some refinements of Shapley’s representation theorem have been proved in Bade 2005, under appropriate convexity assumptions. Our pointwise approach allows us to extend the results in Bade 2005 to the non convex case. Finally, we specialize our results by choosing the Gerstewitz scalarizing functions (see Gerth and Weidner 1990) as value functions and we compare our results with some existing literature (see e.g. Zapata et al. 2019 and the reference therein).

For each  $i \in \mathcal{N}$ , the vector function  $u_i : X \rightarrow Y_i$  represents  $i$ ’s payoff and maps tuples of alternatives in  $i$ ’s outcomes space  $Y_i$ . When not differently specified,  $Y_i$  will denote a topological vector space and  $K_i \subseteq Y_i$  a proper, pointed and convex cone. As usual, we set  $Y := \bigotimes_{i \in \mathcal{N}} Y_i$ . Incomplete preferences of player  $i$  are represented on  $Y_i$  by the partial (conic) order induced by the reflexive and transitive (preference) relation  $\geq_{K_i}$ . With this, a noncooperative game with vector payoffs is completely specified by the tuple  $\Gamma = (\mathcal{N}, \{X_i, u_i, (Y_i, \geq_{K_i})\}_{i \in \mathcal{N}})$ . For all  $i \in \mathcal{N}$ , the set  $\mathcal{C}_i(\mathbf{x}_{-i}) \subseteq Y_i$  includes all  $i$ ’s payoffs at given strategy  $\mathbf{x}_{-i} \in X_{-i}$  of her opponents, namely

$$\mathcal{C}_i(\mathbf{x}_{-i}) := \{u_i(x_i, \mathbf{x}_{-i}) : x_i \in X_i\}$$

Notions of strict Nash and Nash equilibria of a vector game  $\Gamma$  are derived from Definition 2 as special cases. The additional notion of weak Nash equilibria can be considered in this context.

**Definition 9** Let  $K_i$  be solid for all  $i \in \mathcal{N}$ . The point  $\mathbf{x}^* \in X$  is a weak Nash equilibrium of a game  $\Gamma$  when, for all  $i \in \mathcal{N}$ , no  $x_i \in X_i$  exists such that  $u_i(x_i, \mathbf{x}_{-i}^*) \in u_i(\mathbf{x}^*) + \text{int}K_i$ . The set of weak Nash equilibria of  $\Gamma$  will be denoted by  $\mathcal{E}_W(\Gamma)$ .

It is immediate to verify that

$$\mathcal{E}(\Gamma) \subseteq \mathcal{E}_W(\Gamma)$$

The following Lemma emphasizes that the weak Nash equilibria of a vector game are Nash equilibria of another vector game with appropriate modifications of conic preferences. We omit the proof.

**Lemma 10** For all  $i \in \mathcal{N}$ , let  $K_i$  be solid,  $C_i = \text{cl}K_i$ ,  $D_i = \text{int}K_i \cup \{0\}$ , and the games  $\Gamma^{(1)} = (\mathcal{N}, \{X_i, u_i, (Y_i, \geq C_i)\}_{i \in \mathcal{N}})$  and  $\Gamma^{(2)} = (\mathcal{N}, \{X_i, u_i, (Y_i, \geq D_i)\}_{i \in \mathcal{N}})$  be given. Then,  $\mathcal{E}_W(\Gamma^{(1)}) = \mathcal{E}(\Gamma^{(2)})$ .

**Remark 7** In the case of multicriteria games, where incomplete preferences are given by the classical componentwise ordering, two particular specifications of the notion of Nash equilibrium were introduced in Shapley and Rigby 1959, under the name of strong and weak Nash equilibria (respectively SEP and WEP). Notions of SEP and WEP are reformulations of the Nash equilibrium in Definition 2, provided that certain cone preferences of players are given. In detail, consider the game  $\Gamma = (\mathcal{N}, \{X_i, u_i, (Y_i, \geq K_i)\}_{i \in \mathcal{N}})$  where, for all  $i \in \mathcal{N}$ ,  $Y_i = \mathbb{R}^{m_i}$  and let  $x^* \in X$  be a Nash equilibrium of  $\Gamma$ . If, for all  $i \in \mathcal{N}$ ,  $K_i = \mathbb{R}_+^{m_i}$ , then  $x^*$  is a SEP. If, for all  $i \in \mathcal{N}$ ,  $K_i = \text{int}\mathbb{R}_+^{m_i} \cup \{0\}$ , then  $x^* \in X$  is a WEP. We also mention that a vector game reduces to a scalar game when payoff functions are real valued and preferences are described, for each player, by the usual order relation in  $\mathbb{R}$ . Strict Nash and Nash equilibrium concepts, given in Definition 2, reduce respectively to the classic notions of strict Nash and Nash equilibrium (see e.g. Fudenberg and Tirole 1991 or any standard game theory textbook).

### 4.1 Scalarization in Vector Games

Let  $\Gamma$  be a vector game. The scalar game obtained from  $\Gamma$  by completion of cone preferences through scalarizations of players' outcomes by means of value functions  $\varphi_i : Y_i \rightarrow \mathbb{R}$ , for all  $i \in \mathcal{N}$ , is fully described by the tuple  $S_\varphi - \Gamma = (\mathcal{N}, \{X_i, \varphi_i \circ u_i, \mathbb{R}\}_{i \in \mathcal{N}})$ . We will use the notation  $\lambda_i$  in place of  $\varphi_i$  when  $i$ 's preferences are linearly completed, namely when  $\varphi_i$  is linear. When each component of  $\varphi$  is linear, we will use the notation  $\lambda$  instead. For all  $i \in \mathcal{N}$ , properties in Definition 3 can be reframed in the context of vector games, focusing on the image space  $Y_i$  of the vector-valued payoff function  $u_i$ .

**Definition 11** Let the points  $\bar{x} \in X$  and  $\bar{y} \in C_i(\bar{x}_{-i})$  be given. The function  $\varphi_i : Y_i \rightarrow \mathbb{R}$  is said to be

$\widehat{\mathbf{P}}$   $\geq_{K_i}$ -preserving at  $\bar{y}$  when

$$y \in \mathcal{C}(\bar{x}_{-i}), y \geq_{K_i} \bar{y} \implies \varphi_i(y) \geq \varphi_i(\bar{y})$$

$\mathbf{sP}$  strictly  $\geq_{K_i}$ -preserving at  $\bar{y}$  when

$$y \in \mathcal{C}(\bar{x}_{-i}), y >_{K_i} \bar{y} \implies \varphi_i(y) > \varphi_i(\bar{y})$$

$\widehat{\mathbf{R}}$   $\geq_{K_i}$ -representing at  $\bar{y}$  when

$$y \in \mathcal{C}(\bar{x}_{-i}), \varphi_i(y) > \varphi_i(\bar{y}) \implies y >_{K_i} \bar{y}$$

$\widehat{\mathbf{sR}}$  strictly  $\geq_{K_i}$ -representing at  $\bar{y}$  when

$$y \in \mathcal{C}(\bar{x}_{-i}), \varphi_i(y) \geq \varphi_i(\bar{y}) \implies y \geq_{K_i} \bar{y}$$

**Remark 8** For all  $i \in \mathcal{N}$ , properties of  $\varphi_i : Y_i \rightarrow \mathbb{R}$  in Definition 11 coincide with the corresponding properties in Definition 3 of function  $\varphi_i \circ u_i : X \rightarrow \mathbb{R}$ , when the preference relation  $\succeq_i$  is induced on  $X$  as

$$\mathbf{x}', \mathbf{x} \in X, \mathbf{x}' \succeq_i \mathbf{x} \iff u_i(\mathbf{x}') \geq_{K_i} u_i(\mathbf{x})$$

We say that function  $\varphi_i$  provides a pointwise strategic  $\geq_{K_i}$ -completion of conic preferences at  $\bar{y} \in \mathcal{C}_i(\mathbf{x}_{-i})$  for player  $i$  when properties  $\widehat{\mathbf{P}}$  and  $\widehat{\mathbf{sR}}$  hold for  $\varphi_i$  at  $\bar{y}$ . In this case, it results

$$y \in \mathcal{C}_i(\mathbf{x}_{-i}), y \geq_{K_i} \bar{y} \iff \varphi_i(y) \geq \varphi_i(\bar{y})$$

Similarly, We say that function  $\varphi_i$  provides a pointwise strategic  $>_{K_i}$ -completion of conic preferences at  $\bar{y} \in \mathcal{C}_i(\mathbf{x}_{-i})$  for player  $i$  when properties  $\widehat{\mathbf{sP}}$  and  $\widehat{\mathbf{R}}$  hold for  $\varphi_i$  at  $\bar{y}$ . In this case, it results

$$y \in \mathcal{C}_i(\mathbf{x}_{-i}), y >_{K_i} \bar{y} \iff \varphi_i(y) > \varphi_i(\bar{y})$$

We remark that all the results presented in Sect. 3 directly allow us to characterize Nash equilibria of vector games, as Nash equilibria of the corresponding scalar games derived by appropriate pointwise completions of conic partial orders. In detail, we consider the following collections of tuples

$$\widehat{\Phi}^s(A) = \bigcup_{\mathbf{x} \in A} \left\{ \varphi : \widehat{\mathbf{P}} \text{ holds at } y, \text{ for all } y \in \mathcal{C}_i(\mathbf{x}_{-i}), \text{ and } \widehat{\mathbf{sR}} \text{ holds} \right. \\ \left. \text{at } u_i(\mathbf{x}) \text{ for } \varphi_i = (\varphi)_i, \forall i \in \mathcal{N} \right\}$$

$$\widehat{\Phi}(A) = \bigcup_{\mathbf{x} \in A} \left\{ \varphi : \widehat{\mathbf{sP}} \text{ holds at } y, \text{ for all } y \in \mathcal{C}_i(\mathbf{x}_{-i}), \text{ and } \widehat{\mathbf{R}} \text{ holds} \right. \\ \left. \text{at } u_i(\mathbf{x}) \text{ for } \varphi_i = (\varphi)_i, \forall i \in \mathcal{N} \right\}$$

where  $A \subseteq X$ . The following result is an immediate reformulation of Proposition 8.

**Corollary 12** *Let  $\Gamma$  be a game with vector payoffs and let  $A \subseteq X$  be nonempty. If  $\mathcal{E}^s(\Gamma) \subseteq A$  holds, then*

$$\mathcal{E}^s(\Gamma) = \bigcup_{\varphi \in \widehat{\Phi}^s(A)} \mathcal{E}^s(S_\varphi - \Gamma) \tag{10}$$

Moreover, if  $\mathcal{E}(\Gamma) \subseteq A$  holds and  $\widehat{\Phi}(A)$  is nonempty, then

$$\mathcal{E}(\Gamma) = \bigcup_{\varphi \in \widehat{\Phi}(A)} \mathcal{E}(S_\varphi - \Gamma) \tag{11}$$

Equation (11) provides the extension of *Theorem 3* in Bade 2005 to the non convex case. Representations of equilibria given in (10) and (11) have special reformulations depending on the specific choices of scalarization methods. We will consider some well known linear and non-linear approaches to scalarization in vector optimization that can be reframed in our axiomatic approach and extended to characterize Nash equilibria in vector-valued game theory.

### 4.1.1 Linear Scalarization in Vector Games

Linear scalarizations are effective in the study of Nash equilibria in vector games only under convexity assumptions on the structure of the game, see e.g. Shapley and Rigby 1959 or Bade 2005. When this is the case, the following Proposition shows that linear scalarizations can be reframed in our axiomatic approach.

- Proposition 13** **L1** Property  $\widehat{\mathbf{P}}$  holds for  $\lambda_i \in K_i^+$  at  $y \in Y_i$ , for all  $y \in Y_i$ .  
**L2** Property  $s\widehat{\mathbf{P}}$  holds for  $\lambda_i \in K_i^{++}$  at  $y \in Y_i$  for all  $y \in Y_i$ .  
**L3** Let  $C \subseteq Y_i$  be such that  $C - K_i$  is convex with  $\text{int}(C - K_i) \neq \emptyset$  and let  $\bar{y} \in C$ . If  $y \not\prec_{K_i} \bar{y}$  for all  $y \in C \setminus \{\bar{y}\}$ , then there exists  $\lambda_i \in K_i^+$  satisfying property  $\widehat{\mathbf{R}}$  at  $\bar{y}$ .

**Proof** Points **L1** and **L2** are immediate consequences of the definitions of dual cones  $K_i^+$  and  $K_i^{++}$ . To show **L3**, consider that  $\bar{y} \in C$  and  $K_i$  is pointed. Hence, relation  $y \not\prec_{K_i} \bar{y}$  holding for all  $y \in C \setminus \{\bar{y}\}$  implies  $(\bar{y} + K_i) \cap C = \bar{y}$ . By Theorem 5.4 in Jahn 2009, there exists  $\lambda_i \in K_i^+$  such that the relation  $\lambda_i(y) \leq \lambda_i(\bar{y})$  holds for all  $y \in C - K_i$ . Then, property  $\widehat{\mathbf{R}}$  holds for such linear function  $\lambda_i$  at  $\bar{y}$ .  $\square$

Proposition 13 immediately implies the following Corollary to Theorems 5 and 6.

**Corollary 14** Let  $\Gamma$  be a vector game and let the scalar game  $S_\lambda - \Gamma$  be obtained from  $\Gamma$  by linear scalarization through  $\lambda = (\lambda_1, \dots, \lambda_i, \dots, \lambda_N)$ .

**LS** If, for all  $i \in \mathcal{N}$ ,  $\lambda_i \in K_i^{++}$ , then

$$\mathbf{x}^* \in \mathcal{E}(S_\lambda - \Gamma) \implies \mathbf{x}^* \in \mathcal{E}(\Gamma)$$

**LN** If, for all  $\mathbf{x}^* \in \mathcal{E}(\Gamma)$ , the set  $\mathcal{C}_i(\mathbf{x}_{-i}^*) - K_i$  is convex and  $\text{int}(\mathcal{C}_i(\mathbf{x}_i^*) - K_i) \neq \emptyset$  for all  $i \in \mathcal{N}$ , then there exists a tuple  $\lambda$  of linear functions, with  $(\lambda)_i = \lambda_i \in K_i^+$ , such that

$$x^* \in \mathcal{E}(S_\lambda - \Gamma) \iff x^* \in \mathcal{E}(\Gamma)$$

**Remark 9** It easy to verify that the assumption of convexity, involved in point **LN** of Proposition 14, is fulfilled assuming the concavity of vector-valued payoffs. By contrast, the concavity of payoffs is not necessary for the convexity of the set  $\mathcal{C}_i(x_{-i}) - K_i$ . For example, this occurs in the case of concave-like functions (see Theorem 2.1 in Jahn 2009). As a consequence, Corollary 14 extends Theorem 2 in Bade 2005 (and also Theorem 2.5 in Mármol et al. 2017), allowing an infinite dimensional setting, where players’ outcomes are ordered by general proper, convex and pointed cones, and vector-valued payoffs are not necessarily concave. Moreover, we remark that the assumption of convexity of the set  $\mathcal{C}_i(x_{-i}) - K_i$  is implicitly satisfied in Shapley and Rigby 1959, where the mixed extension of an original finite game is considered. Indeed, in that case, the set  $\mathcal{C}_i(x_{-i})$  is convex, for all  $x_{-i} \in X_{-i}$  and for all  $i \in \mathcal{N}$ .

The following example shows that, when the convexity assumptions in Proposition 14 are not satisfied, pure strategy Nash equilibria may exist that are not Nash points of any scalar game obtained from  $\Gamma$  through linear scalarization.

**Example 15** Let  $\Gamma = \left( \{1, 2\}, \{X_i, u_i, (\mathbb{R}^2, \geq_{\mathbb{R}_+^2})\}_{i \in \{1,2\}} \right)$  be a zero sum game with payoff maximizing players, where  $\mathbb{R}_+^2$  is the non negative orthant<sup>2</sup> and with payoff matrix (with vector entries) given in the Table 1.

The set of pure Nash equilibria of  $\Gamma$  is  $\mathcal{E} = \{(x_1^1, x_2^1), (x_1^2, x_2^2), (x_1^3, x_2^1)\}$ . We will show that the pure Nash equilibrium  $(x_1^1, x_2^1)$  cannot be found through linear scalarization. The scalar game  $S_\lambda - \Gamma$  obtained from  $\Gamma$  through scalarization

<sup>2</sup>That is  $\mathbb{R}_+^2 := \{(v_1, v_2) \in \mathbb{R}^2 : v_1 \geq 0, v_2 \geq 0\}$ .

**Table 1** Payoff matrix of  $\Gamma$

		Player 2	
		$x_2^1$	$x_2^2$
Player 1	$x_1^1$	$(-4, -4)$ $(4, 4)$	$(-6, -6)$ $(6, 6)$
	$x_1^2$	$(-3, -6)$ $(3, 6)$	$(-4, -7)$ $(4, 7)$
	$x_1^3$	$(-6, -3)$ $(6, 3)$	$(-7, -4)$ $(7, 4)$

by  $\lambda = (\lambda_1, \lambda_2)$  is characterized by the payoff matrix provided in Table 2, where  $\lambda_i = (\lambda_i^1, \lambda_i^2)$ ,  $\lambda_i^1 \in [0, 1]$ , and  $\lambda_i^2 = 1 - \lambda_i^1$ , with  $i = 1, 2$ .

Given any choice of player 1 in  $S_\lambda - \Gamma$ , player 2 will select strategy  $x_2^1$ , this corresponding to 2's minimum profit losses. At the same time, player 1 avoids the choice of  $x_1^1$ , never corresponding to 1's best profit gain, for all choices of linear function  $\lambda$ . Indeed, the system

$$\begin{cases} \lambda_1(u_1(x_1^1, x_2^1)) \geq \lambda_1(u_1(x_1^2, x_2^1)) \\ \lambda_1(u_1(x_1^1, x_2^1)) \geq \lambda_1(u_1(x_1^3, x_2^1)) \end{cases} \implies \begin{cases} 4\lambda_1^1 + 4\lambda_1^2 \geq 3\lambda_1^1 + 6\lambda_1^2 \\ 4\lambda_1^1 + 4\lambda_1^2 \geq 6\lambda_1^1 + 3\lambda_1^2 \end{cases} \\ \implies \begin{cases} \lambda_1^1 \geq 2\lambda_1^2 \\ \lambda_1^1 \leq \lambda_1^2/2 \end{cases}$$

is unfeasible. Hence  $(x_1^1, x_2^1) \notin \mathcal{E}(S_\lambda - \Gamma)$  for all  $\lambda_{1,2} \in \mathbb{R}_+^2$ .

### 4.1.2 Mixed Extension in Finite Games and Characterization of Extendable Nash Equilibria

Extension to mixed strategies plays a fundamental role in game theory by ensuring existence of Nash equilibria through convexification. Let us formulate the mixed extension of a game  $\Gamma$  with vector payoffs. For the sake of simplicity, we limit ourselves to the case of finite games. For all  $i \in \mathcal{N}$ , a mixed strategy  $\sigma_i : X_i \rightarrow [0, 1]$  of player  $i$  is any probability distribution over  $X_i$ . By  $\Sigma_i(X_i)$  we will denote the mixed extension of  $X_i$ , namely the set of all possible lotteries over  $X_i$ . We will set  $\Sigma(X) := \otimes_{j \in \mathcal{N}} \Sigma_j(X_j)$ . Given the alternative  $x_i \in X_i$ , we will denote by  $\delta(x_i) \in \Sigma_i$  the (degenerate) mixed strategy corresponding to  $x_i$  in the sense that  $\delta(x_i)$  has support  $x_i$ . Given the profile  $x \in X$ , we will denote by  $\delta(x) \in \Sigma$  the profile of (degenerate) mixed strategies whose  $i$ -th component  $\delta(x_i)$  corresponds to  $x_i = (x)_i$  in the above specified sense. The expected payoff of  $i$  is the vector function  $U_i : \Sigma(X) \rightarrow Y_i$  given by  $U_i(\sigma) = \mathbb{E}_\sigma[u_i]$ . The mixed extension of game  $\Gamma$  will be denoted by  $\Gamma_{ME}$ , which is fully specified by the tuple

**Table 2** Payoff matrix of  $S_\lambda - \Gamma$

		Player 2	
		$x_2^1$	$x_2^2$
Player 1	$x_1^1$	$-4\lambda_2^1 - 4\lambda_2^2$ $4\lambda_1^1 + 4\lambda_1^2$	$-6\lambda_2^1 - 6\lambda_2^2$ $6\lambda_1^1 + 6\lambda_1^2$
	$x_1^2$	$-3\lambda_2^1 - 6\lambda_2^2$ $3\lambda_1^1 + 6\lambda_1^2$	$-4\lambda_2^1 - 7\lambda_2^2$ $4\lambda_1^1 + 7\lambda_1^2$
	$x_1^3$	$-6\lambda_2^1 - 3\lambda_2^2$ $6\lambda_1^1 + 3\lambda_1^2$	$-7\lambda_2^1 - 4\lambda_2^2$ $7\lambda_1^1 + 4\lambda_1^2$

$\Gamma_{ME} = (\mathcal{N}, \{\Sigma_i(X_i), U_i, (Y_i, \leq_{K_i})\}_{i \in \mathcal{N}})$ . For all  $i \in \mathcal{N}$ , the set  $\mathcal{C}_i(\sigma_{-i})$  includes all  $i$ 's expected payoffs at given strategy  $\sigma_{-i} \in \Sigma_{-i}(X_{-i})$  of his opponents, namely

$$\mathcal{C}_i(\sigma_{-i}) := \{U_i(\sigma_i, \sigma_{-i}) : \sigma_i \in \Sigma_i(X_i)\}$$

**Remark 10** For all  $i \in \mathcal{N}$ , the set  $\mathcal{C}_i(\sigma_{-i})$  is convex for all  $\sigma_{-i} \in \Sigma_{-i}$ , since

$$\mathcal{C}_i(\sigma_{-i}) = \text{conv} \{y \in Y_i : y = U_i(\delta(x_i), \sigma_{-i}), x_i \in X_i\} \tag{12}$$

The extension of pure strategies to mixed strategies preserves pure equilibria in scalar games. A pure strategy profile  $\mathbf{x}^*$  is, indeed, a Nash equilibrium of a scalar game if and only if the corresponding (degenerate) mixed strategy profile  $\delta(\mathbf{x}^*)$  is a Nash equilibrium of the mixed extension of the game (see e.g. Proposition 7.3 in Narahari 2014). Differently, the mixed extension of a game with vector payoffs may not be fully coherent with the underlying pure strategy vector game. Pure Nash points may exist indeed that do not correspond to any mixed strategy equilibrium in the mixed extension. This event is shown in the following Example.

**Example 16** The strategy profile  $(x_1^1, x_2^1)$  considered in Example 15, which is not found as an equilibrium by linear scalarization, is not an equilibrium in the mixed extension of the game, where the strategy payoff space is convexified by considering probability distributions on the strategy sets. This is a direct consequence of Shapley's characterization of mixed Nash equilibria in multicriteria finite games (see Shapley and Rigby 1959).

**Remark 11** Similar observations concerning the exclusion of Nash equilibria under randomization in vector games have been made in Evren 2014 (see also Nehring 1997; Heller 2012 in the context of decision making under incomplete preferences).

The previous observations motivate the following Definition.

**Definition 17** Let  $\Gamma$  be a game with vector payoffs. The pure Nash equilibrium  $\mathbf{x}^*$  of  $\Gamma$  is said to be extendable when the (degenerate) mixed strategy profile  $\delta(\mathbf{x}^*)$  corresponding to it is a mixed strategy Nash equilibrium of the mixed extension  $\Gamma_{ME}$ .

The relation between non extendable Nash points and Nash points that cannot be found through linear scalarization in finite vector games is established in the following Proposition. It shows that Nash points that cannot be found through linear scalarizations are exactly the non extendable ones.

**Proposition 18** Let  $\Gamma$  be a finite vector game and let the scalar game  $S_\lambda - \Gamma$  be obtained from  $\Gamma$  by linear scalarization through  $\lambda = (\lambda_1, \dots, \lambda_i, \dots, \lambda_N)$ .

**LS'** If, for all  $i \in \mathcal{N}$ ,  $\lambda_i \in K_i^{++}$  and  $\mathbf{x}^* \in \mathcal{E}(S_\lambda - \Gamma)$ , then  $\mathbf{x}^*$  is an extendable Nash point of  $\Gamma$ .

**NS'** If  $\mathbf{x}^*$  is an extendable Nash point of  $\Gamma$ , then there exists a tuple  $\lambda$  of linear

functions with  $(\lambda)_i = \lambda_i \in K_i^+$  such that  $\mathbf{x}^* \in \mathcal{E}(S_\lambda - \Gamma)$ .

**Proof LS’.** Since  $\mathbf{x}^* \in \mathcal{E}(S_\lambda - \Gamma)$ , then, for all  $i \in \mathcal{N}$ , the relation

$$\lambda_i(u_i(x_i, \mathbf{x}_{-i}^*) - u_i(\mathbf{x}^*)) \geq 0$$

holds for all  $x_i \in X_i$ . Let  $H_i^+ = \{y \in Y_i : \lambda_i(y) \geq 0\}$ ; then  $u_i(x_i, \mathbf{x}_{-i}^*) \in H_i^+ + u_i(\mathbf{x}^*)$ , for all  $x_i \in X_i$ . Hence

$$\text{conv} \{y \in Y_i : y = u_i(x_i, \mathbf{x}_{-i}^*), x_i \in X_i\} = C_i(\delta(\mathbf{x}_{-i}^*)) \subseteq H_i^+ + u_i(\mathbf{x}^*)$$

Therefore,  $\lambda_i(U_i(\sigma_i, \delta(\mathbf{x}_{-i}^*)) - u_i(\mathbf{x}^*)) \geq 0$  holds for all  $\sigma_i \in \Sigma_i$ . Taking into account that  $u_i(\mathbf{x}^*) = U_i(\delta(\mathbf{x}^*))$  and  $\lambda_i \in K_i^{++}$ , it follows that

$$U_i(\sigma_i, \delta(\mathbf{x}_{-i}^*)) - U_i(\delta(\mathbf{x}^*)) \notin -K \setminus \{0\}$$

for all  $\sigma_i \in \Sigma_i$ . Hence,  $\delta(\mathbf{x}^*) \in \mathcal{E}(\Gamma_{ME})$ .

**NS’.** Let  $\mathbf{x}^*$  be an extendable Nash point of  $\Gamma$ . Hence,  $\delta(\mathbf{x}^*) \in C_i(\delta(\mathbf{x}_{-i}^*))$  is a Nash point of  $\Gamma_{ME}$ . Since the set  $C_i(\delta(\mathbf{x}_{-i}^*))$  is convex (see 12), by Proposition 14 a tuple  $\lambda$  exists such that, for all  $i \in \mathcal{N}$ ,  $\lambda_i \in K_i^+$  and  $\delta(\mathbf{x}^*) \in \mathcal{E}(S_\lambda - \Gamma_{ME})$ . This, in turn, implies  $\mathbf{x}^* \in \mathcal{E}(S_\lambda - \Gamma)$ . Indeed, by relation  $\delta(\mathbf{x}^*) \in \mathcal{E}(S_\lambda - \Gamma_{ME})$ , it follows that

$$\lambda_i(u_i(\mathbf{x}^*)) = \lambda_i(U_i(\delta(x_i^*), \delta(\mathbf{x}_{-i}^*))) \leq \lambda_i(U_i(\sigma_i, \delta(\mathbf{x}_{-i}^*)))$$

holds for all  $\sigma_i \in \Sigma_i(X_i)$ . Then, for any  $x_i \in X_i$ , the relation

$$\lambda_i(u_i(\mathbf{x}^*)) \leq \lambda_i(U_i(\delta(x_i), \delta(\mathbf{x}_{-i}^*))) = \lambda_i(u_i(x_i, \mathbf{x}_{-i}^*))$$

holds whenever the degenerate distribution  $\sigma_i = \delta(x_i)$  is chosen. This proves the thesis. □

### 4.1.3 A Nonlinear Scalarization in Vector Games: The Gerstewitz Approach

The axiomatic approach to scalarization outlined in Section 3 can provide complete characterizations of Nash equilibria in vector games, even in absence of convexity. In this event, nonlinear value functions cannot be avoided. As an example, we present the so called Gerstewitz function, whose properties make it appropriate to provide pointwise completions. Let  $i \in \mathcal{N}$ . The Gerstewitz function  $\phi_i^{e_i, \bar{y}} : Y_i \rightarrow \mathbb{R}$  is defined as

$$\phi_i^{e_i, \bar{y}}(y) = \max\{t \in \mathbb{R} : y \in te_i + \bar{y} + \text{cl}K_i\} \tag{13}$$

where  $K_i$  is convex,  $\text{cl}K_i$  is pointed, solid and elements  $\bar{y} \in Y_i$  and  $e_i \in \text{int}K_i$  are given (see Gerth and Weidner 1990). For the convenience of the reader, we reframe some known properties of Gerstewitz function in our axiomatic approach.

**Proposition 19** *If  $K_i$  is closed, function  $\phi_i^{e_i, \bar{y}}$  provides a  $\geq_{K_i}$ -pointwise completion of  $i$ 's cone preferences at  $\bar{y}$ .*

*If  $K_i = \text{int}K_i \cup \{0\}$ , function  $\phi_i^{e_i, \bar{y}}$  provides a  $>_{K_i}$ -pointwise completion of  $i$ 's cone preferences at  $\bar{y}$ .*

**Proof** Let  $K_i$  be closed. Then, properties

**G1** property  $\widehat{\mathbf{P}}$  holds for  $\phi_i^{e_i, \bar{y}}$  at  $y$ , for all  $y \in Y_i$ , and

**G2** property  $\widehat{\mathbf{R}}$  holds for  $\phi_i^{e_i, \bar{y}}$  at  $\bar{y}$

hold. **G1** is immediate. Now, let  $y \not\geq_{K_i} \bar{y}$  hold. This is equivalent to  $y \notin \bar{y} + K_i$ . It follows  $\phi_i^{e_i, \bar{y}}(y) < 0$  (see point v) of Lemma 2.17 in Hernández and Rodríguez-Marín 2007). Since  $\phi_i^{e_i, \bar{y}}(\bar{y}) = 0$ , then  $\phi_i^{e_i, \bar{y}}(y) < \phi_i^{e_i, \bar{y}}(\bar{y})$ . This shows **G2**. Now, let  $K_i = \text{int}K_i \cup \{0\}$ . Then, properties

**G3** property  $\widehat{\mathbf{P}}$  holds for  $\phi_i^{e_i, \bar{y}}$  at  $y$ , for all  $y \in Y_i$ , and

**G4** property  $\widehat{\mathbf{R}}$  holds for  $\phi_i^{e_i, \bar{y}}$  at  $\bar{y}$ .

hold. **G3** is immediate. Now, let  $y \not>_{K_i} \bar{y}$  hold. This is equivalent to  $y \notin \bar{y} + \text{int}K_i$ . It follows  $\phi_i^{e_i, \bar{y}}(y) \leq 0$  (see point iii) of Lemma 2.17 in Hernández and Rodríguez-Marín 2007). Since  $\phi_i^{e_i, \bar{y}}(\bar{y}) = 0$ , then  $\phi_i^{e_i, \bar{y}}(y) \leq \phi_i^{e_i, \bar{y}}(\bar{y})$ . This shows **G4**.  $\square$

By virtue of Proposition 19, the Gerstewitz function can be considered to provide representations of Nash equilibria. To this end, let us consider the following set of scalarizing tuples

$$\widehat{\Phi}_G(A) = \bigcup_{\mathbf{x} \in A} \left\{ \varphi : (\varphi)_i = \phi_i^{e_i, u_i(\mathbf{x})}, \forall i \in \mathcal{N} \right\}$$

where  $A \subseteq X$ . If we consider the game  $\Gamma^{(1)} = (\mathcal{N}, \{X_i, u_i, (Y_i, \geq_{C_i})\}_{i \in \mathcal{N}})$ , where  $C_i = \text{cl}K_i$  holds for all  $i \in \mathcal{N}$ , then the set  $\widehat{\Phi}_G(A)$  corresponds to the set  $\widehat{\Phi}^s(A)$  as defined above and formula (10) takes the form

$$\mathcal{E}^s(\Gamma^{(1)}) = \bigcup_{\varphi \in \widehat{\Phi}_G(A)} \mathcal{E}^s(S_\varphi - \Gamma^{(1)}) \tag{14}$$

If we consider the game  $\Gamma^{(2)} = (\mathcal{N}, \{X_i, u_i, (Y_i, \geq_{D_i})\}_{i \in \mathcal{N}})$ , where  $K_i$  is solid and  $D_i = \text{int}K_i \cup \{0\}$  for all  $i \in \mathcal{N}$ , then the set  $\widehat{\Phi}_G(A)$  corresponds to the set  $\widehat{\Phi}(A)$  as defined above and formula (11) takes the form

$$\mathcal{E}(\Gamma^{(2)}) = \bigcup_{\varphi \in \widehat{\Phi}_G(A)} \mathcal{E}(S_\varphi - \Gamma^{(2)}) \tag{15}$$

In the following example we make an appropriate use of the Gerstewitz function to find, by scalarization, the Nash point of the multicriteria game in Example 15 that is missed by linear methods.

**Example 20** Consider the game introduced in Example 15 and let  $S_\phi - \Gamma$  be obtained from  $\Gamma$  through scalarization by  $\phi = \left( \phi_1^{\mathbb{1}, \bar{y}_1}, \phi_2^{\mathbb{1}, \bar{y}_2} \right)$ , where  $\bar{y}_1 = -\bar{y}_2 = (4, 4)$ , and

$\mathbb{1} = (1, 1)$ . The pair  $(x_1^1, x_2^1)$  is a Nash equilibrium of scalar game  $S_\phi - \Gamma$ , whose payoff matrix is shown in Table 3.

We remark that the Gerstewitz function is widely used as a scalarizing function to obtain optimality conditions in vector optimization problems (see e.g. Gerth and Weidner 1990; Hernández and Rodríguez-Marín 2007; Khan et al. 2015; Tammer and Weidner 2020 and the references therein). We also remark that, in the special case of multiobjective optimization, several well known scalarization methods, such as Pascoletti-Serafini,  $\epsilon$ -constraint, Chebyshev scalarization, among others, can be traced back to the use of the Gerstewitz scalarizing function (see e.g. Section 2.5 in Eichfelder 2008).

**Remark 12** Other examples of nonlinear scalarization methods fitting the axiomatic approach here presented can be found in Miglierina and Molho 2002; Zaffaroni 2003; Gutiérrez et al. 2015.

Some nonlinear scalarization methods have been considered to tackle multicriteria games in a finite dimensional setting, where, for all  $i \in \mathcal{N}$ ,  $Y_i = \mathbb{R}^{m_i}$ ,  $m_i \in \mathbb{N}$  with  $m_i \geq 2$ , and  $K_i = \mathbb{R}_+^{m_i}$ . This is the case of the so called Rawlsian function, considered in Zapata et al. 2019 (see also Novikova and Pospelova 2019; Kreines et al. 2020; Novikova and Pospelova 2018, where it appears under the name of Germeyer function). The Rawlsian function  $w_i^{\gamma_i} : \mathbb{R}^{m_i} \rightarrow \mathbb{R}$  is defined, for all  $i \in \mathcal{N}$ , by

$$w_i^{\gamma_i}(y) = \min \left\{ \frac{y^j}{(\gamma_i)_j} : (\gamma_i)_j \neq 0 \text{ and } j = 1, \dots, m_i \right\} \tag{16}$$

where  $\gamma_i \in \mathbb{R}_+^{m_i} \setminus \{0\}$  is a vector of weights. Remarkably, the Rawlsian function can be traced back to the Gerstewitz function in a finite dimensional framework. Indeed the equality

**Table 3** Payoff matrix of  $S_\phi - \Gamma$   
*Player 2*

		$x_2^1$	$x_2^2$
<i>Player 1</i>	$x_1^1$	0	-2
	$x_1^2$	-1	0
	$x_1^3$	-1	0

$$w_i^{\gamma_i}(y) = \phi_i^{\gamma_i,0}(y) := \max\{t' \in \mathbb{R} : y \in t'\gamma_i + \mathbb{R}_+^{m_i}\} \tag{17}$$

holds<sup>3</sup> (see e.g. Eichfelder 2008).

In Zapata et al. 2019, Rawlsian functions (16) have been used to complete preferences in multicriteria games, where multiobjective utilities are componentwise non negative for all players. We point out that formula (14) and (15) extend some results given in Zapata et al. 2019 to an infinite dimensional framework with general conic preferences. Moreover, for a game  $\Gamma = (\mathcal{N}, \{X_i, u_i, (Y_i, \geq_{K_i})\}_{i \in \mathcal{N}})$  where, for all  $i \in \mathcal{N}$ ,  $K_i$  is closed, inclusions in Proposition 3.1 in Zapata et al. 2019 can be refined using (15) as

$$\mathcal{E}(\Gamma) \subseteq \bigcup_{\varphi \in \widehat{\Phi}(A)} \mathcal{E}(S_\varphi - \Gamma) = \mathcal{E}_W(\Gamma)$$

The first inclusion is obtained by Theorem 6, point N2, point G2 in Proposition 19, and Remark 1. The second equality follows by 15 and Lemma 10. This equality refines an inclusion in Zapata et al. 2019, where only sufficient weak Nash equilibrium conditions are given<sup>4</sup>.

**Remark 13** In Zapata et al. 2019, the authors prove Proposition 3.1 considering the scalarizing function  $w_i^{\gamma_i}(y) = \phi_i^{\gamma_i,0}(y)$ , where  $\gamma_i \in \mathbb{R}_+^{m_i} \setminus \{0\}$  with  $\gamma_i = s_i \bar{y}_i$ , where  $\bar{y}_i \in \mathbb{R}_+^{m_i} \setminus \{0\}$ , and  $s_i > 0$ . Whenever  $\gamma_i \notin \text{int}\mathbb{R}_+^{m_i}$ , property  $\widehat{\mathbf{R}}$  at  $\bar{y}$  can fail. Hence, the inclusion

$$\bigcup_{\varphi \in \widehat{\Phi}(A)} \mathcal{E}(S_\varphi - \Gamma) \supseteq \mathcal{E}_W(\Gamma)$$

does not hold.

We also mention that in Proposition 3.3 in Zapata et al. 2019 the Nash equilibria with unique best response for each player are considered, obtaining only sufficient Nash equilibrium condition by scalarization. In our framework, such points are named strict Nash equilibria. Hence, by using formula (14), we can refine Proposition

<sup>3</sup>The equality holds even if  $\gamma_i \in \mathbb{R}_+^{m_i} \setminus \{0\}$  is given.

<sup>4</sup>The decision problem considered in Zapata et al. (2019) in Example 3.2 can be tackled with our approach, through which all weakly efficient points of the set  $A = \{(1, 5), (2, 2), (3.5), (4, 3), (6, 2)\} \cup \{(x, 0) : 6 \leq x \leq 10\}$  can be found by appropriate choices of parameters. For an illustrative purpose, let us show that the point  $\bar{y} := (8 \ 0)^T$  can be found by scalarization through the Gersewitz approach choosing

$$\phi^{\mathbb{1}, \bar{y}}(y) = \max \left\{ t \in \mathbb{R} : y \in \begin{pmatrix} 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 8 \\ 0 \end{pmatrix} + \mathbb{R}_+^2 \right\}$$

It results  $\phi^{\mathbb{1}, \bar{y}}(y) = 0$  for all  $y \in L_{\bar{y}} := \{(x \ 0)^T : x \geq 8\}$  and  $\phi^{\mathbb{1}, \bar{y}}(y) < 0$  for all  $y \in A \setminus L_{\bar{y}}$ . Hence,  $\bar{y} \in \arg \max_{y \in A} \phi^{\mathbb{1}, \bar{y}}(y)$ .

3.3 in Zapata et al. 2019 by proving necessary and sufficient strict Nash equilibrium conditions.

With a slightly different approach, Rawlsian functions appear also in Song and Wang 2010. Here, for all  $i \in \mathcal{N}$ , player  $i$  evaluates her vector utility at  $x \in X$  with respect to a reference point  $\bar{x}$  through  $w_i^{\mathbb{1}_i}(u_i(x) - u_i(\bar{x}))$ , where  $\mathbb{1}_i \in \mathbb{R}_+^{m_i}$  is a vector with unitary entries. Even this last approach can be traced back to the Gerstewitz function. It is easy to verify that

$$w_i^{\mathbb{1}_i}(y - \bar{y}) = \phi_i^{\mathbb{1}_i, \bar{y}}(y)$$

holds for all  $y \in Y_i$ , where  $\bar{y} \in Y_i$ .

### 4.2 A Bicriteria Cournot Duopoly

An example of a strategic context where players have multiple objectives can be found in the framework of oligopolistic competitions. In literature on industrial organizations, strategic motivations to consider additional criteria other than profits go back to the pioneering work Schelling 1980 (see also Hay and Morris 1979; Williamson 1971), where the separation of the roles of management and ownership is discussed. Along this line, multiple criteria in Cournot models, where profits and revenues are considered as objectives of quantity setting firms, can be found, for instance, in the relevant contributions Vickers 1985; Fershtman and Judd 1987; Sklivas 1987. In these models, equilibrium solutions are found by aggregating multiple criteria in weighted sums, that is by the use of linear scalarization techniques. Here, parameters are weighting coefficients, which can be interpreted as indexes of criteria relative importance. The main drawback of linear scalarizing methods is the need of convexity properties of the structure of the game, which are required to obtain a full characterization of Nash equilibria of the original vector game.

Let us consider here a Cournot duopoly of quantity setting players that produce homogeneous goods. When the strategy profile  $(x_1, x_2) \in X_1 \times X_2$  is selected by both players,  $i$ 's profits and revenues are  $\pi_i = P(x_1 + x_2)x_i - c_i(x_i)$  and  $r_i = P(x_1 + x_2)x_i$  respectively, where  $P$  is the inverse demand and  $c_i$  is  $i$ 's cost function on  $X_i$ . The payoff of  $i$  is the vector-valued function  $u_i : X_1 \times X_2 \rightarrow \mathbb{R}^2$ , with  $i$ 's profits and sales as first and second component respectively, that is

$$u_i(x_1, x_2) = \begin{pmatrix} P(x_1 + x_2)x_i - c_i(x_i) \\ P(x_1 + x_2)x_i \end{pmatrix} \tag{18}$$

The resulting game is described by  $\Gamma = \left( \{1, 2\}, \left\{ X_i, u_i, \left( \mathbb{R}_+^2, \geq_{\mathbb{R}_+^2} \right) \right\}_{i=1,2} \right)$ . We remark that the components of  $u_i$  may not be concave, as well as strategy sets  $X_i$ , with  $i = 1, 2$ , may not be convex.

In the following Example, we show the existence of a (strict) Nash equilibrium of a bicriteria duopoly that cannot be found when players' objectives are modeled as linear combinations of profits and revenues.

**Table 4** Price matrix

	$L^{(2)}$	$M^{(2)}$	$H^{(2)}$
$L^{(1)}$	$P(L^{(1)} + L^{(2)}) = 7$	$P(L^{(1)} + M^{(2)}) = 3.6$	$P(L^{(1)} + H^{(2)}) = 2.9$
$M^{(1)}$	$P(M^{(1)} + L^{(2)}) = 6$	$P(M^{(1)} + M^{(2)}) = 3.2$	$P(M^{(1)} + H^{(2)}) = 2.5$
$H^{(1)}$	$P(H^{(1)} + L^{(2)}) = 4.8$	$P(H^{(1)} + M^{(2)}) = 2.5$	$P(H^{(1)} + H^{(2)}) = 2$

**Table 5** Payoff matrix

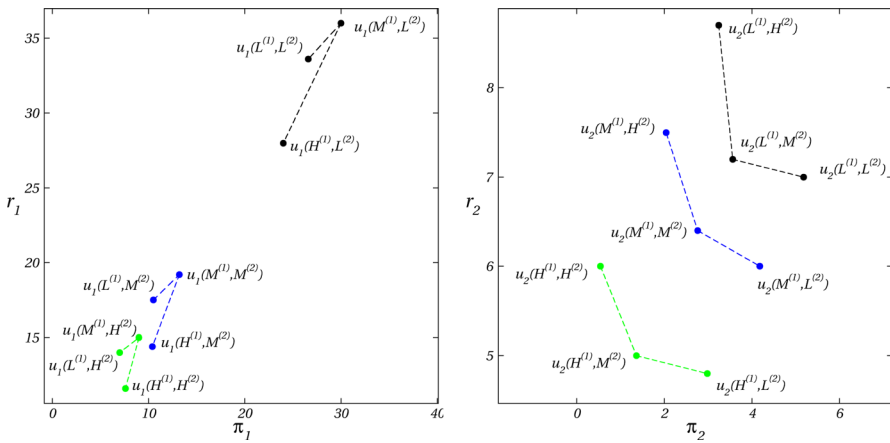
		Player 2		
		$L^{(2)}$	$M^{(2)}$	$H^{(2)}$
Player 1	$L^{(1)}$	$\begin{pmatrix} 5.18 \\ 7 \end{pmatrix}$ $\begin{pmatrix} 24 \\ 28 \end{pmatrix}$	$\begin{pmatrix} 3.56 \\ 7.2 \end{pmatrix}$ $\begin{pmatrix} 10.4 \\ 14.4 \end{pmatrix}$	$\begin{pmatrix} 3.24 \\ 8.7 \end{pmatrix}$ $\begin{pmatrix} 7.6 \\ 11.6 \end{pmatrix}$
	$M^{(1)}$	$\begin{pmatrix} 4.18 \\ 6 \end{pmatrix}$ $\begin{pmatrix} 30 \\ 36 \end{pmatrix}$	$\begin{pmatrix} 2.76 \\ 6.4 \end{pmatrix}$ $\begin{pmatrix} 13.2 \\ 19.2 \end{pmatrix}$	$\begin{pmatrix} 2.04 \\ 7.5 \end{pmatrix}$ $\begin{pmatrix} 9 \\ 15 \end{pmatrix}$
	$H^{(1)}$	$\begin{pmatrix} 2.98 \\ 4.8 \end{pmatrix}$ $\begin{pmatrix} 26.6 \\ 33.6 \end{pmatrix}$	$\begin{pmatrix} 1.36 \\ 5 \end{pmatrix}$ $\begin{pmatrix} 10.5 \\ 17.5 \end{pmatrix}$	$\begin{pmatrix} 0.54 \\ 6 \end{pmatrix}$ $\begin{pmatrix} 7 \\ 14 \end{pmatrix}$

**Example 21** We consider two players producing homogeneous goods choosing among low (L), medium (M) and high (H) levels of production and facing linear costs  $c_i(x) = c_i x$ . The set of strategies of player  $i$  is  $X_i = \{L^{(i)}, M^{(i)}, H^{(i)}\}$ . Let  $c_1 = 1$ ,  $c_2 = 1.82$ ,  $X_1 = \{4, 6, 7\}$ ,  $X_2 = \{1, 2, 3\}$ , and the price matrix given in Table 4. Resulting vector payoffs, with profits and revenues as components, are obtained according to formula (18). The related matrix is given in Table 5 (see Fig. 1).

Strategy  $M^{(1)}$  of player 1 is dominant. Therefore, (strict) Nash equilibria of the game are

$$\mathcal{E}(\Gamma) = \mathcal{E}^s(\Gamma) = \{(M^{(1)}, L^{(2)}); (M^{(1)}, M^{(2)}); (M^{(1)}, H^{(2)})\}$$

Nevertheless, the Nash equilibrium  $(M^{(1)}, M^{(2)})$  cannot be found through any linear scalarization of the game. Indeed, let us consider the scalar game  $S_\lambda - \Gamma$  obtained from  $\Gamma$  with the scalarizing tuple of linear functions  $\lambda = (\lambda_1, \lambda_2)$  with  $\lambda_{1,2} \in \mathbb{R}_+^2$ . It



**Fig. 1** Payoffs of player 1 (left) and 2 (right). Points of the same colors denote utilities of a player at a given action of her opponent

is easy to verify that strategy  $M^{(1)}$  is in the best reply of player 1 in any scalar game  $S_\lambda - \Gamma$ .<sup>5</sup> By contrast, strategy  $M^{(2)}$  is never the best reply of 2. Indeed, the system

$$\begin{cases} \lambda_2(u_2(M^{(1)}, M^{(2)})) \geq \lambda_2(u_2(M^{(1)}, L^{(2)})) \\ \lambda_2(u_2(M^{(1)}, M^{(2)})) \geq \lambda_2(u_2(M^{(1)}, H^{(2)})) \end{cases}$$

which is equivalent to

$$\begin{cases} 2.76\lambda_2^1 + 6.4\lambda_2^2 \geq 4.18\lambda_2^1 + 6\lambda_2^2 \\ 2.76\lambda_2^1 + 6.4\lambda_2^2 \geq 2.04\lambda_2^1 + 7.5\lambda_2^2 \end{cases} \Leftrightarrow \begin{cases} \lambda_2^2 \geq 3.55\lambda_2^1 \\ \lambda_2^2 \leq 0.65\lambda_2^1 \end{cases}$$

is unfeasible whenever  $\lambda_2 \in \mathbb{R}_+^2$ . Hence,  $(M^{(1)}, M^{(2)}) \notin \mathcal{E}(S_\lambda - \Gamma)$  holds for all  $\lambda = (\lambda_1, \lambda_2)$  with  $\lambda_{1,2} \in \mathbb{R}_+^2$ .

In order to avoid the drawbacks deriving from the use of linear scalarization, we propose to solve the bicriteria Cournot model formulated above by considering a preference completion method based on a scalarizing function that is coherent with our pointwise approach. We formulate the scalar game  $S_\phi - \Gamma$  obtained from  $\Gamma$  by using

<sup>5</sup>Indeed, the following systems

$$\begin{cases} \lambda_1(u_1(M^{(1)}, L^{(2)})) \geq \lambda_1(u_1(L^{(1)}, L^{(2)})) \\ \lambda_1(u_1(M^{(1)}, L^{(2)})) \geq \lambda_1(u_1(H^{(1)}, L^{(2)})) \end{cases} \Leftrightarrow \begin{cases} 3\lambda_1^1 + 4\lambda_1^2 \geq 0 \\ 3.4\lambda_1^1 + 2.4\lambda_1^2 \geq 0 \end{cases}$$

$$\begin{cases} \lambda_1(u_1(M^{(1)}, M^{(2)})) \geq \lambda_1(u_1(L^{(1)}, M^{(2)})) \\ \lambda_1(u_1(M^{(1)}, M^{(2)})) \geq \lambda_1(u_1(H^{(1)}, M^{(2)})) \end{cases} \Leftrightarrow \begin{cases} 2.8\lambda_1^1 + 4.8\lambda_1^2 \geq 0 \\ 2.7\lambda_1^1 + 1.7\lambda_1^2 \geq 0 \end{cases}$$

$$\begin{cases} \lambda_1(u_1(M^{(1)}, H^{(2)})) \geq \lambda_1(u_1(L^{(1)}, H^{(2)})) \\ \lambda_1(u_1(M^{(1)}, H^{(2)})) \geq \lambda_1(u_1(H^{(1)}, H^{(2)})) \end{cases} \Leftrightarrow \begin{cases} 1.4\lambda_1^1 + 3.4\lambda_1^2 \geq 0 \\ 2\lambda_1^1 + \lambda_1^2 \geq 0 \end{cases}$$

are satisfied for all  $\lambda_1 \in \mathbb{R}_+^2$ .

the pair of Gerstewitz functions  $\phi = (\phi_1^{\mathbb{1}, \bar{y}_1}, \phi_2^{\mathbb{1}, \bar{y}_2})$ , where  $e_1 = e_2 = \mathbb{1} = (1, 1)$  and  $\bar{y}_i$  is the reference point of  $i$ . If we denote the reference point  $\bar{y}_i$  as

$$\bar{y}_i = \begin{pmatrix} \bar{\pi}_i \\ \bar{r}_i \end{pmatrix}$$

by (17) the value function  $\phi_i^{\mathbb{1}, \bar{y}_i}$  of player  $i$  can be rewritten as

$$\phi_i^{\mathbb{1}, \bar{y}_i}(u_i) = \phi_i^{\mathbb{1}, 0}(u_i - \bar{y}_i) = \min\{\pi_i - \bar{\pi}_i, r_i - \bar{r}_i\} \tag{19}$$

Expression (19) shows that the behavior of player  $i$ , when using value function  $\phi_i^{\mathbb{1}, \bar{y}_i}$  with chosen reference point  $\bar{y}_i$ , is to focus either on profits or on revenues, depending on which criterion provides the worst performance with respect to the related reference level. Formula (19) can be rewritten equivalently as

$$\phi_i^{\mathbb{1}, \bar{y}_i}(u_i) = \begin{cases} \pi_i - \bar{\pi}_i & \text{if } c_i \geq \bar{r}_i - \bar{\pi}_i \\ r_i - \bar{r}_i & \text{if } c_i < \bar{r}_i - \bar{\pi}_i \end{cases}$$

This reformulation highlights that, on one hand, player  $i$  is interested in profits when her costs are perceived to be heavy, namely beyond the level  $\bar{c}_i := \bar{r}_i - \bar{\pi}_i$  that can be interpreted as a subjective reference cost value for player  $i$ . On the other hand, player  $i$  is interested in revenues when her costs are below  $\bar{c}_i$ . The resulting scalar game is described as  $S_\phi - \Gamma = (\{1, 2\}, \{X_i, \phi_i^{\mathbb{1}, \bar{y}_i}(u_i), \mathbb{R}\}_{i=1,2})$ . A Nash equilibrium of  $S_\phi - \Gamma$  is a pair  $x^* = (x_1^*, x_2^*)$  such that

$$\begin{aligned} x_1^* &\in \arg \max_{x \in X_1} \begin{cases} \pi_1(x, x_2) & \text{if } c_1(x) \geq \bar{r}_1 - \bar{\pi}_1 \\ r_1(x, x_2) & \text{if } c_1(x) < \bar{r}_1 - \bar{\pi}_1 \end{cases} \\ x_2^* &\in \arg \max_{x \in X_2} \begin{cases} \pi_1(x_1, x) & \text{if } c_2(x) \geq \bar{r}_2 - \bar{\pi}_2 \\ r_2(x_1, x) & \text{if } c_2(x) < \bar{r}_2 - \bar{\pi}_2 \end{cases} \end{aligned}$$

Independently from the choice of references objectives  $\bar{y}_i$ , sufficient strict equilibrium conditions can be obtained from scalar game  $S_\phi - \Gamma$ . Indeed, each strict Nash equilibrium of game  $S_\phi - \Gamma$  identifies a strict Nash equilibrium of the original game  $\Gamma$ , which follows from point **S1** of Theorem 5 and Proposition 19. Moreover, by appropriate choices of the references  $\bar{y}_i$ , all strict Nash equilibria of  $\Gamma$  can be characterized as strict Nash equilibria of scalar game  $S_\phi - \Gamma$ , which follows by point **N1** of Theorem 6 and Proposition 19.

The following Example shows that the Nash equilibria that is not found by linear scalarization in Example 21 can be obtained when the scalarization method is reframed within our pointwise approach.

**Example 22** Let  $\Gamma$  be the bicriteria Cournot game introduced in Example 21. Let  $S_\phi - \Gamma$  be the scalar game derived from  $\Gamma$  by using the couple of Gerstewitz function  $\phi = (\phi_1^{\mathbb{1}, \bar{y}_1}, \phi_2^{\mathbb{1}, \bar{y}_2})$ , where  $\mathbb{1} = (1, 1)$ . The solution  $(M^{(1)}, M^{(2)})$  can be recovered

as a Nash equilibrium of the scalar game  $S_\phi - \Gamma$  by choosing reference points  $\bar{y}_1$  and  $\bar{y}_2$  as follows

$$\bar{y}_1 = \begin{pmatrix} \bar{\pi}_1 \\ \bar{r}_1 \end{pmatrix} = \begin{pmatrix} \pi_1(M^{(1)}, M^{(2)}) \\ r_1(M^{(1)}, M^{(2)}) \end{pmatrix}, \quad \bar{y}_2 = \begin{pmatrix} \bar{\pi}_2 \\ \bar{r}_2 \end{pmatrix} = \begin{pmatrix} \pi_2(M^{(1)}, M^{(2)}) \\ r_2(M^{(1)}, M^{(2)}) \end{pmatrix}$$

With this choice it results

$$\begin{aligned} \phi_1^{\mathbb{1}, \bar{y}_1} \left( u_1(L^{(1)}, M^{(2)}) \right) < 0 & \quad \phi_2^{\mathbb{1}, \bar{y}_2} \left( u_2(M^{(1)}, L^{(2)}) \right) < 0 \\ \phi_1^{\mathbb{1}, \bar{y}_1} \left( u_1(M^{(1)}, M^{(2)}) \right) = 0 & \quad \phi_2^{\mathbb{1}, \bar{y}_2} \left( u_2(M^{(1)}, M^{(2)}) \right) = 0 \\ \phi_1^{\mathbb{1}, \bar{y}_1} \left( u_1(H^{(1)}, M^{(2)}) \right) < 0 & \quad \phi_2^{\mathbb{1}, \bar{y}_2} \left( u_2(M^{(1)}, H^{(2)}) \right) < 0 \end{aligned}$$

Hence,  $(M^{(1)}, M^{(2)}) \in \mathcal{E}(S_\phi - \Gamma)$ .

## 5 Conclusions

In this paper we develop a strategic pointwise completion approach to deal with games where players show some indecisiveness when comparing outcomes. Our approach is derived from an axiomatic pointwise scalarization method, originally introduced in vector optimization, and subsequently extended to a more general framework, where scalarization processes by means of value functions can be interpreted as incomplete preference completions. The use of scalarization techniques in noncooperative game theory, where the notion of Nash equilibrium plays a fundamental role, involves the formulation of some coherence requirements only with respect to strategically relevant comparisons. Without any convexity assumptions, we identify a set of minimal properties of the value functions that allow us to characterize each Nash equilibrium of the original game with incomplete preferences as a Nash point of a corresponding scalar game, thus obtaining necessary and/or sufficient Nash equilibrium conditions. We try to analyze the role of our approach in the context of some existing models of completion of incomplete preferences, originally developed in decision theory.

As a relevant application, we reframe our pointwise approach to the special case of games with vector payoffs. Even if linear scalarization methods are widely used in the context of multicriteria games, where convexification is often achieved through the mixed extension, we point out that some drawbacks may arise in the non convex case. Our approach allows us to use appropriate nonlinear scalarizations, avoiding any convexity assumptions. We also provide a general framework to some existing results on scalarization in vector games.

As an illustrative example, we apply our approach to a bicriteria Cournot duopoly, where firms consider both profits and revenues as objectives and incomplete preferences naturally arise from the interaction between management and ownership.

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**Data Availability** No datasets were generated or analysed during the current study.

## Declarations

**Conflict of interest** The authors declare no conflict of interest.

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