



Hypoellipticity and Loss of Derivatives of Sums of Squares of complex Vector Fields

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Abstract

In this paper I survey some recent results I obtained about the hypoellipticity of sums of squares of complex vector fields that allow for a generalization of Kohn's Thm. A, and give a new result about hypoellipticity with a loss of many derivatives that shows that Kohn's Thm. B holds in a more general setting.

Keywords Subellipticity · Hypoellipticity with a loss of many derivatives · Melin's inequality · Hörmander's condition · Sums of squares of complex vector fields

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1 Introduction

The C^∞ hypoellipticity of partial differential operators with non-constant coefficients is still far from being completely understood, especially that of degenerate operators of the kind sums of squares of *complex* vector fields (not to mention C^ω hypoellipticity, for sums of squares of real vector fields in connection with the celebrated Treves' Conjecture).

In recent times, in connection with Y. T. Siu's program [13] to use multipliers ideal sheaves for the $\bar{\partial}$ -Neumann problem to get an explicit construction of critical varieties that control the D'Angelo type, J. J. Kohn discovered in [4] *complex* vector fields (equivalently, first order homogeneous partial differential operators with smooth complex coefficients) that, despite the finite-generation of the *complexified* tangent space at any given point, are hypoelliptic and may lose many derivatives, in comparison

To the memory of Joe Kohn, a great mathematician and a wonderful person

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with the loss one has in the classical theorem of Hörmander (also in the sharp version due to Rothschild and Stein) on the hypoellipticity of sums of squares of *real* vector fields whose generated Lie algebra spans the *real* tangent space at any given point. The same kind of phenomenon was also discovered at the same time (but in a different context), and cast in a pretty general theory in the case of operators with transversal multiple characteristics, by Cesare Parenti and myself in [6].

In this paper, after reviewing Kohn's results, I will in the first place show how to generalize his Thm. A to include complex pseudodifferential operators and a setting in which the generation of the complexified tangent space appears in a weaker form, and then show an example of operator for which Kohn's Thm. B holds under a weaker complex "finite-type" condition (see Problem 5.2 below).

The plan of the paper is the following. After recalling, in the next section, Hörmander's hypoellipticity theorem [2], Kohn's Thm. A and Thm. B [4], and A. Melin's strong lower bound estimate [3] (a crucial result that allowed me to generalize Thm. A), I will describe my results (see [10] and [11]) and, in the last section, pose the problem of finding, if possible, complex vector fields satisfying a weaker generation condition and yet their related sum-of-squares operator is still hypoelliptic, possibly with a loss of many derivatives. There, I will give a generalization of Kohn's Thm. A and Thm. B, that in fact hold in a weaker "finite-type" setting.

It is important to observe (see Remark 2.6 below) that there is a main difference between the case Kohn considered and that considered by Hörmander or Rothschild and Stein: in the former one has the presence of first-order terms that may compromise hypoellipticity on which one does not have any a-priori information.

In what follows, I address the reader to the bibliography of [10] and of the books cited in the present paper to refer to some of the many important contributions.

2 Hörmander's and Kohn's Results

I recall in the first place the concept of hypoellipticity of a differential operator on an open set $\Omega \subset \mathbb{R}^n$.

Definition 2.1 One says that a differential operator P is C^∞ hypoelliptic if

$$\text{singsupp}(Pu) = \text{singsupp}(u), \quad \forall u \in \mathcal{D}'(\Omega).$$

Equivalently,

$$\forall u \in \mathcal{D}'(\Omega), \quad \forall V \subset \Omega \text{ open}, \quad Pu \in C^\infty(V) \implies u \in C^\infty(V).$$

Let next $X_1(x, D), \dots, X_N(x, D)$ ($D = -i\partial$) be first-order homogeneous partial differential operators (i.e. without zeroth-order terms) with smooth *real* coefficients. Note that, therefore, iX_j are *real* vector fields tangent to Ω . Let

$$P = \sum_{j=1}^N X_j^* X_j,$$

be the corresponding sum-of-squares operator. Here X_j^* is the formal adjoint of X_j .

For each $x \in \Omega$, consider the real vector space V_x^X spanned (over \mathbb{R}) by the vector fields iX_1, \dots, iX_N , and their repeated commutators $[iX_{j_1}, [iX_{j_2}, [\dots, [iX_{j_{h-1}}, iX_{j_h}]] \dots]]$, $1 \leq j_1, \dots, j_h \leq N$, frozen at the point x . Hörmander's celebrated hypoellipticity theorem for sums of squares ([2] or [3]) states the following.

Theorem 2.2 *Let $P = \sum_{j=1}^N X_j^* X_j$. Suppose that for every $x \in \Omega$ one has $V_x^X = T_x \Omega$ (i.e., one has that n among the vector fields and their iterated commutators are linearly independent and generate at each x). Then P is C^∞ -hypoelliptic.*

Recall that it is well-known from Fedii and Morimoto that the Lie-algebra condition $V_x^X = T_x \Omega$, for all $x \in \Omega$, is not necessary for the C^∞ -hypoellipticity to hold (see the references in [10]). However, when the coefficients are analytic, Derridj proved that the Lie-algebra condition is also necessary for the C^∞ -hypoellipticity. (The situation regarding *analytic-hypoellipticity* is completely different and to this day quite open.)

The main step in the proof of Theorem 2.2 is a subelliptic estimate, that is, an energy estimate of the following kind: There exists $\varepsilon > 0$ (necessarily smaller than or equal to 1, i.e. half the order of the operator which, in the above case, is 2) such that for any given compact $K \subset \Omega$ there is $C_K > 0$ such that

$$\|u\|_\varepsilon^2 \leq C_K \left(\operatorname{Re}(Pu, u) + \|u\|_0^2 \right), \quad \forall u \in C_c^\infty(K). \tag{SE_\varepsilon}$$

(Here (\cdot, \cdot) is the L^2 scalar product and $\|\cdot\|_\varepsilon$ is the H^ε Sobolev norm.)

When $P = P^*$ (i.e. P is formally self-adjoint), the real part in the inner product can be omitted. However, I prefer to keep it in the above estimate because I shall consider cases in which I add to P a not necessarily self-adjoint first-order operator Q .

Supposing that at each point commutators of length k span the tangent space at each point (iX_j has length 1, $[iX_j, iX_{j'}]$ has length 2 and so on), it was Rothschild and Stein [12] who obtained the sharp subelliptic exponent $\varepsilon = 1/k$. One has also results on the hypoellipticity of sums of squares of real vector fields by J. J. Kohn (whose proof is the one followed by F. Trèves in his book on pseudodifferential and Fourier integral operators, and by L. Hörmander in his third volume) and by F. Trèves in his study of hypoellipticity. Subsequent work by O. Oleinik and E. Radkevich vastly extended the result to more general operators with real coefficients. Hörmander's theorem was microlocalized by P. Bolley, J. Camus and J. Nourrigat, and in the case of polynomials in the operators X_j it was B. Helffer and J. Nourrigat [1] who obtained sharp microhypoelliptic results with optimal gain (maximal hypoelliptic estimates). On the side of the study of subelliptic operators, one has to mention the contributions of Egorov, Hörmander, Fefferman and Phong, but the list of fundamental contributions is still very long and I have mentioned just a few of them (I apologize for the necessary omissions in the mentioned list).

It is next convenient to introduce the following definition to precisely measure the loss of derivatives.

Definition 2.3 An operator P of order m is hypoelliptic with a loss of $r \geq 0$ derivatives at $x_0 \in \Omega$ if for any given $u \in \mathcal{D}'(\Omega)$ and any given $s \in \mathbb{R}$

$$Pu \in H_{\text{loc}}^s(x_0) \implies u \in H_{\text{loc}}^{s+m-r}(x_0).$$

See [3] for the definition of $H_{\text{loc}}^s(x_0)$ (it means, loosely speaking, that a distributions is locally near x_0 in the Sobolev space H^s). One may equivalently say that $u = u_1 + u_2$ where $u_1 \in H_{\text{loc}}^s$ and $u_2 \in C^\infty$ near x_0 .

For an even more precise measure of the loss of derivatives (we will be using it in the last section), one says that P is microlocally hypoelliptic with a loss of $r \geq 0$ derivatives at $\rho = (x_0, \xi_0) \in T^*\Omega \setminus 0$ if for all $u \in \mathcal{D}'(\Omega)$ and for all $s \in \mathbb{R}$

$$Pu \in H_{\text{ml}}^s(\rho) \implies u \in H_{\text{ml}}^{s+m-r}(\rho).$$

Again, I refer to [3] for the definition of the microlocal Sobolev space at a point in phase-space. One may equivalently say that $u \in H_{\text{ml}}^s(\rho)$ iff u may be written as $u = u_1 + u_2$, with $u_1 \in H_{\text{loc}}^s$ and $\rho \notin WF(u_2)$.

Note that when $r = 0$ (i.e. we have no loss of derivatives) the operator is elliptic near x_0 , respectivly microlocally elliptic near ρ .

At this point it is important to recall that there are already known examples of operators that are hypoelliptic and yet lose many derivatives. To my knowledge, it was E. Stein who first noticed this phenomenon. He considered the Kohn-Laplacian \square_b on the Heisenberg group \mathbb{H}^n on $(0, q)$ forms with $q = 0$ or $q = n$ and showed that, although in such a case \square_b cannot be hypoelliptic, $\square_b + c$ is indeed hypoelliptic with a loss of 2 derivatives whatever the complex number $c \neq 0$. C. Parenti and myself in [6] developed a theory (based on L. Boutet De Monvel’s concept of localized operator and subsequent work by Boutet De Monvel, Helffer and Grigis, and by Helffer) to understand this phenomenon for transversally elliptic operators. (I will in fact use that theory later on.)

For instance, from the theory in [6] one has that the operator in \mathbb{R}^2

$$P = (1 + x_1^{2k})(D_{x_1}^2 + \mu^2 x_1^2 D_{x_2}^2) + (\gamma + \mu x_1^{2k})D_{x_2} - 2ix_1^{2k-1}(D_{x_1} + i\mu x_1 D_{x_2}),$$

where $\mu > 0$, $k \geq 1$ is an integer and $\gamma \in \{\pm(2\ell + 1)\}$; $\ell \in \mathbb{Z}_+\}$, is C^∞ hypoelliptic with a loss of $k + 1$ derivatives.

As already mentioned in the Introduction, at about the same time Parenti and I were working on [6], Kohn in [4], motivated by Y. T. Siu’s program [13] to use multipliers for the $\bar{\partial}$ -Neumann problem to obtain an explicit construction of critical varieties that control the D’Angelo type, extended Hörmander’s hypoellipticity result to complex operators $Z_1(x, D), \dots, Z_N(x, D)$ (i.e. first-order partial differential operators with smooth complex coefficients and no zeroth-order terms) that span, along with their commutators of length 2, the complexified tangent space at every point (see Thm. A below). But he also showed that as soon as one needs commutators of length ≥ 3 to generate, there are sum-of-squares operators that cannot satisfy any subelliptic estimates and yet remain hypoelliptic, with a loss of many derivatives (see Thm. B

below). This striking phenomenon, which is different from Stein’s example since it occurs on the operator as a sum of squares without addition of zeroth order terms, is, in my opinion, yet to be fully understood. I will try, in the following, to clarify (and generalize) some aspects of it.

I next recall Kohn’s Thms. A and B from [4].

Let $Z_1(x, D), \dots, Z_N(x, D)$ be first-order partial differential operators with smooth complex coefficients and no zeroth-order terms on Ω . Hence, as before, the iZ_j may be regarded as vector fields on Ω with complex coefficients. Consider the corresponding sum-of-squares operator $P = \sum_{j=1}^N Z_j^* Z_j$. (Here, again, Z_j^* is the formal adjoint of Z_j .)

Theorem 2.4 ([4], Thm. A) *Suppose that*

$$\text{Span}_{\mathbb{C}}\{Z_j, [Z_j, Z_k]; 1 \leq j, k \leq N\}(x) = \mathbb{C}T_x\Omega, \quad \forall x \in \Omega. \tag{K}$$

Then the subelliptic estimate (SE_{1/2}) holds for P.

Therefore, as a consequence due to the sum-of-squares form, the operator P is hypoelliptic with a loss of one derivative.

However, as soon as more commutators are required to span the result is no longer true.

Theorem 2.5 ([4], Thm. B) *For any given $k \in \mathbb{Z}_+$ there are first-order complex operators Z_1, Z_{2k} (with no zeroth-order term) defined near $0 \in \mathbb{R}^3$ such that the complex vector fields Z_1 and Z_{2k} and their commutators of order $k + 1$ (i.e. of length $k + 2$) span $\mathbb{C}T_0\Omega$ and when $k \geq 1$ the subelliptic estimate no longer holds. However, the sum-of-squares operator $P_k = Z_1^* Z_1 + Z_{2k}^* Z_{2k}$ is hypoelliptic with a loss of $k + 1$ derivatives.*

Both theorems are remarkable, since in the conditions only the family of the Z_j , and their commutators, is considered and no assumption is made on the commutators, or iterated commutators, of the Z_j and the \bar{Z}_j .

The operators that Kohn constructs in Thm. B are related to (a version of) the Lewy operator: set

$$\bar{L} = \frac{\partial}{\partial \bar{z}_1} - iz_1 \frac{\partial}{\partial x_3}, \quad z_1 = x_1 + ix_2,$$

take

$$Z_1 = -i\bar{L}, \quad Z_{2k} = -i\bar{z}_1^k L,$$

and consider

$$P_k = -(L\bar{L} + \bar{L}|z_1|^{2k}L).$$

Soon after Kohn’s paper appeared, M. Christ came out with a simplified version of P_k and Parenti and myself with a general class of simplified examples [7]. The latter

class can be described as follows. With $x \in \mathbb{R}^n, y \in \mathbb{R}$, let $k \geq 1$, let $\mu_1, \dots, \mu_n > 0$ be rationally independent, and let $\gamma \in \mathbb{R}$. Consider then the polynomial

$$Q(x) = \sum_{|\alpha|=k} c_\alpha x^{2\alpha}, \quad x \in \mathbb{R}^n, \quad \text{with} \quad \sum_{|\alpha|=k} c_\alpha > 0.$$

Let $Z_j = D_{x_j} - i\mu_j x_j D_y, 1 \leq j \leq n$, and consider

$$P = \sum_{j=1}^n Z_j^* Z_j + \sum_{j=1}^n Z_j(Q(x)Z_j^*) + \left(\gamma + \sum_{j=1}^n \mu_j \right) D_y.$$

Then P is hypoelliptic with a loss of exactly 1 derivative iff

$$\gamma \notin S := \left\{ \pm \left(\sum_{j=1}^n h_j \mu_j + \sum_{j=1}^n \mu_j \right); h_1, \dots, h_n \in \mathbb{Z}_+ \right\}.$$

When $\gamma \in S$ then P is hypoelliptic with a loss of exactly $k + 1$ derivatives. Note that in this case, depending on the polynomial $Q(x)$, P may fail to be a sum-of-squares operator.

Let me mention that an important issue in the hypoellipticity theory of operators of the kind sums of squares of homogeneous first-order differential operators is the *stability* of hypoellipticity by perturbations of order ≤ 1 . That is a very delicate issue, because of the degeneracy of the operators considered (one may look at [9] and [8] in the C^∞ setting, and in the references in [10] at some recent work of P. Cordaro and his collaborators in the C^ω setting). An example is given by $\square_b + c$ discussed earlier.

Going back to Kohn’s theorems, I will first consider Theorem 2.4 (Thm. A), and show an alternative approach which goes through Melin’s inequality (see [5], and [3] for the strong form used here). This allows one also to deal with the mentioned kind of perturbations and with further generalizations that are based on a weakening of Kohn’s condition (K) above.

Before recalling Melin’s inequality and its setting in the next section, I finish the present one by explaining the kind of lower order terms one has to control in, respectively, Hörmander’s, Kohn’s and Rothschild-Stein’s operators. This will show the main differences between Hörmander’s and Rothschild-Stein’s sums of squares on one side, and Kohn’s one on the other.

Remark 2.6 Denote by $Y = Y(x, \partial)$ a real vector field.

- For Y_0, Y_1, \dots, Y_N real vector fields, Hörmander’s sum-of-squares operator is of the form

$$P = - \sum_{j=1}^N Y_j^2 + Y_0,$$

and the condition on the generation of the tangent space at each point is required on the Y_0, \dots, Y_N and their (iterated) commutators.

- Write $iZ_j = Y_{2j-1} + iY_{2j}$, $1 \leq j \leq N$. Kohn’s sum-of-squares operator is of the form

$$P = \sum_{j=1}^N Z_j^* Z_j = - \sum_{j=1}^N (Y_{2j-1}^2 + Y_{2j}^2) - i \sum_{j=1}^N [Y_{2j-1}, Y_{2j}] + Q_1,$$

where Q_1 is a linear combination with smooth coefficients of the Z_1, \dots, Z_N plus a zeroth order term. The first-order part of the operator Q_1 is harmless since it is automatically under control (the principal symbol of Q_1 vanishes on the characteristic set given by the Z_j). The condition on generation concerns **only** the complexified tangent space at each point and is required **only** on the Z_1, \dots, Z_N and their (iterated) commutators, and *not* on the Y_1, \dots, Y_{2N} and their (iterated) commutators. In this setting, it turns out that also the zeroth-order part of Q_1 is harmless.

- For Y_1, \dots, Y_N smooth real vector-fields that satisfy Hörmander’s condition, Rothschild and Stein considered the operator

$$P = - \sum_{j=1}^N Y_j^2 + \sum_{1 \leq j < k \leq N} c_{jk} [Y_j, Y_k],$$

where

$$c_{jk} = -c_{kj}, \quad \|\text{Im}[c_{jk}]\|_{\mathbb{C}^N \rightarrow \mathbb{C}^N} < 1.$$

By the approach of Rothschild and Stein, the generating condition on the Y_1, \dots, Y_N controls the commutator term.

Hence, only in Kohn’s setting are the assumptions on the commutator terms **not** giving an a priori control on the commutator terms that appear in the operator.

3 Melin’s Strong Inequality

Suppose $P = P^*$ is an m th-order properly supported classical pseudodifferential operator on an open set Ω . Let p_m be its principal symbol, which is a function (positively homogeneous of degree m) defined on $T^*\Omega \setminus 0$. Let $\Sigma = p_m^{-1}(0) \subset T^*\Omega \setminus 0$ be the characteristic set of P , which in general is not a smooth manifold. Let $\Sigma_2 \subset \Sigma$ be the subset of Σ on which p_m vanishes at least to second order. On Σ_2 one has another invariant, which is the *subprincipal* symbol, given by

$$p_{m-1}^s(x, \xi) = p_{m-1}(x, \xi) + \frac{i}{2} \sum_{j=1}^n \partial_{x_j} \partial_{\xi_j} p_m(x, \xi).$$

Notice that when P is formally self-adjoint then p_{m-1}^s is real.

Moreover, at a point $\rho \in \Sigma_2$ one has also that $dp_m(\rho) = 0$ whence the Hessian of p_m is invariantly defined and one may invariantly define the *Hamilton map* $F(\rho): T_\rho T^*\Omega \rightarrow T_\rho T^*\Omega$ (also called *fundamental matrix*) of p_m at ρ as

$$\sigma(w, F(\rho)w') = \frac{1}{2} \langle \text{Hess}(p_m)(\rho)w, w' \rangle, \quad w, w' \in T_\rho T^*\Omega.$$

Recall that $\sigma = \sum_{j=1}^n d\xi_j \wedge dx_j$ is the canonical symplectic form of $T^*\Omega$. Note that $F(\rho)$ is the linearization of the Hamilton vector field

$$H_{p_m}(\rho) = \sum_{j=1}^n \left(\frac{\partial p_m}{\partial \xi_j}(\rho) \frac{\partial}{\partial x_j} - \frac{\partial p_m}{\partial x_j}(\rho) \frac{\partial}{\partial \xi_j} \right)$$

at ρ . Note also that F is skew-symmetric with respect to σ .

When $p_m \geq 0$ on $T^*\Omega \setminus 0$ one has that $\Sigma_2 = \Sigma$ and that $F(\rho)$, $\rho \in \Sigma$, has the following spectral structure:

- $\text{Ker } F(\rho) \subset \text{Ker}(F(\rho)^2) = \text{Ker}(F(\rho)^3)$, 0 is the only generalized eigenvalue while all the others (when $F \neq 0$) are semisimple and of the form $\pm i\mu_j$ so that, with repetitions according to multiplicities, for some integer ν one has

$$\text{Spec}(F(\rho)) = \{0\} \cup \{\pm i\mu_j; \mu_j > 0, 1 \leq j \leq \nu\};$$

- One has

$$T_\rho T^*\Omega = \text{Ker}(F(\rho)^2) \oplus \text{Range}(F(\rho)^2);$$

- The *positive trace* of $F(\rho)$ is defined to be

$$\text{Tr}^+ F(\rho) = \sum_{\substack{\mu > 0 \\ i\mu \in \text{Spec}(F(\rho))}} \mu.$$

It turns out that $\text{Tr}^+ F$ is positively homogeneous of degree $m - 1$. Of course, the positive trace is symplectically invariant.

One has the following fundamental a priori estimate (see [5], and [3] for the form of the estimate used here).

Theorem 3.1 (*Melin’s strong inequality*) *Let $P = P^*$ be a properly supported m th-order classical pseudodifferential operator on Ω such that $p_m \geq 0$ and*

$$p_m(\rho) = 0 \implies p_{m-1}^s(\rho) + \text{Tr}^+ F(\rho) > 0. \tag{3.1}$$

Then for all compact $K \subset \Omega$ there are $c_K, C_K > 0$ such that

$$(Pu, u) \geq c_K \|u\|_{(m-1)/2}^2 - C_K \|u\|_{(m-2)/2}^2, \quad \forall u \in C_c^\infty(K). \tag{3.2}$$

Note that the characteristic set Σ is not required to have any sort of smoothness.

Hence, when $m = 2$ and Melin’s condition (3.1) is satisfied, we have a subelliptic estimate with a loss of one derivative, i.e. we have $(SE_{1/2})$.

Then a natural idea is that there should be a connection between Kohn’s condition (K) above and Melin’s condition. The problem is, given the generality of the statement: *How to control the positive trace of F when P is the Kohn sum of squares?*

In the next section I will give a positive answer to that question.

4 Another Proof of Kohn’s Thm. A and Generalizations.

From Remark 2.6 it is important to observe that in the case of Hörmander’s Theorem 2.2 (say, with generation of length 2), the subprincipal part vanishes on the characteristic manifold $\Sigma_X = \bigcap_{j=1}^N X_j^{-1}(0)$, whereas in the case of complex coefficients, that is, in the case of Kohn’s Theorem 2.4, the subprincipal term on the characteristic manifold

$$\Sigma = \bigcap_{j=1}^N Z_j^{-1}(0)$$

in general does not vanish identically and hence may destroy the subelliptic estimate. In fact, writing in what follows $Z_j = Z_j(x, \xi)$ for the symbol of $Z_j(x, D)$, and recalling that the symbol of a commutator starts with the Poisson bracket $\{\cdot, \cdot\}/i$, one has for the principal symbol p_2 and subprincipal symbol p_1^s of Kohn’s sum of squares

$$p_2(x, \xi) = \sum_{j=1}^N |Z_j(x, \xi)|^2, \quad p_1^s|_{\Sigma} = -\frac{i}{2} \sum_{j=1}^N \{\bar{Z}_j, Z_j\}|_{\Sigma}.$$

Therefore, when $\rho \in \Sigma$ and $w \in T_{\rho}T^*\Omega$, for the Hamilton map of Kohn’s operator we have

$$F(\rho)w = \sum_{j=1}^N \left(\sigma(w, H_{2j-1}(\rho))H_{2j-1}(\rho) + \sigma(w, H_{2j}(\rho))H_{2j}(\rho) \right), \quad (4.1)$$

where $H_{Z_j} := H_{2j-1} + iH_{2j}$, $1 \leq j \leq N$.

So, to fulfill Melin’s condition (3.1) we need to show that the subprincipal symbol on the characteristic set is not so negative compared to the negative of the positive trace of the Hamilton map. This requires a study of $\text{Spec}(F(\rho))$ when $\rho \in \Sigma$, and hence the study the eigenvalue equation $F(\rho)w = i\mu w$, $\mu > 0$, with $0 \neq w \in \mathbb{C}\text{Range}(F(\rho)^2)$.

Kohn’s condition (K), although not an information on the “mixed” commutator terms $\{\bar{Z}_j, Z_k\}$, should allow us to have such a control. But (K) is the only structural information we have on the $Z_j(x, D)$, while $F(\rho)$ in (4.1) has a quite general expression.

One has the following, to me remarkable, result on the positive trace of the Hamilton map F of any given operator P which is a sum of squares of complex vector fields.

Theorem 4.1 ([10]) *Let $P = \sum_{j=1}^N Z_j^* Z_j$. The sum-of-squares form of P yields always*

$$\text{Tr}^+ F(\rho) \geq \left(p_1^s(\rho)^2 + \max_{1 \leq j < k \leq N} |\{Z_j, Z_k\}(\rho)|^2 \right)^{1/2}, \quad \forall \rho \in \Sigma. \tag{4.2}$$

To exploit condition (K) one has the following immediate, and fundamental, observation ($\pi : T^*\Omega \setminus 0 \rightarrow \Omega$ will always be denoting the canonical projection).

Proposition 4.2 *Consider $x \in \Omega$ with $\pi^{-1}(x) \cap \Sigma \neq \emptyset$. Condition (K) at $x \in \Omega$ implies that for all $0 \neq \xi \in T_x^*\Omega$ with $(x, \xi) \in \Sigma$, there exist j, k with $1 \leq j < k \leq N$ such that*

$$\{Z_j, Z_k\}(x, \xi) \neq 0.$$

Therefore, combining Proposition 4.2 and Theorem 4.1 one has the following corollary, which gives Thm. A.

Corollary 4.3 *When the sum of squares $P = \sum_{j=1}^N Z_j^* Z_j$ satisfies condition (K) we have*

$$\text{Tr}^+ F(\rho) > |p_1^s(\rho)|, \quad \forall \rho \in \Sigma.$$

Therefore Melin’s inequality holds, and hence also Kohn’s Thm. A.

Defining

$$\kappa(\rho) := \left(p_1^s(\rho)^2 + \max_{1 \leq j < k \leq N} |\{Z_j, Z_k\}(\rho)|^2 \right)^{1/2}, \quad \rho \in \Sigma,$$

one has that $\kappa : \Sigma \rightarrow [0, +\infty)$ is continuous and positively homogeneous of degree 1. Therefore the maps

$$-p_1^s \pm \kappa : \Sigma \rightarrow \mathbb{R},$$

are continuous and positively homogeneous of degree 1, and one always has $-p_1^s - \kappa \leq 0 \leq -p_1^s + \kappa$ on Σ , with strict inequalities when (K) holds. This yields a result that shows to what extent perturbations of a Kohn’s sum of squares satisfying (K) keeps satisfying the subelliptic estimate (SE_{1/2}).

In what follows, I consider the open interval $I(\rho) := (-\kappa(\rho), \kappa(\rho))$ centered at 0 and radius $\kappa(\rho)$.

Theorem 4.4 ([10]) *Suppose condition (K) for P . Then one has the subelliptic estimate (SE_{1/2}). Moreover, if Q is a first-order properly supported classical pseudodifferential operator on Ω , the operator $P + Q$ keeps satisfying (SE_{1/2}) provided the real part q_1 of the principal symbol of Q fulfills*

$$q_1(\rho) \in -p_1^s(\rho) + I(\rho), \quad \forall \rho \in \Sigma.$$

In particular, in case Q is a partial differential operator, when

$$|q_1(\rho)| < \min\{p_1^s(\rho) + \kappa(\rho), -p_1^s(\rho) + \kappa(\rho)\} = \kappa(\rho) - |p_1^s(\rho)|, \quad \forall \rho \in \Sigma.$$

Note once more that no smoothness assumption on Σ is required.

Melin’s inequality holds for pseudodifferential operators, and Theorem 4.1 is based on the form of the principal symbol of a sum of squares. Therefore there is no obstruction to making the above result pseudodifferential.

Let P_1, \dots, P_N be properly supported, classical m th-order pseudodifferential operators with complex symbols on $\Omega \subset \mathbb{R}^n$. Let $P = \sum_{j=1}^N P_j^* P_j$, and let p_j be the principal symbol of P_j .

Definition 4.5 I say that the system (P_1, \dots, P_N) satisfies condition (K_Σ) if the Poisson brackets $\{p_j, p_k\}$ of the complex symbols p_1, \dots, p_N satisfy

$$\sum_{1 \leq j < k \leq N} |\{p_j, p_k\}(\rho)| > 0, \quad \forall \rho = (x, \xi) \in \Sigma, \quad \text{with } |\xi| = 1, \quad (4.3)$$

where $\Sigma = \bigcap_{j=1}^N p_j^{-1}(0)$ is the characteristic set of P .

In this case the function κ in (4.2) is positively homogeneous of degree $2m - 1$.

One has therefore the following result.

Theorem 4.6 *If the system (P_1, \dots, P_N) satisfies condition (K_Σ) , then the sum-of-squares operator P , of order $2m$, fulfills the subelliptic estimate $(SE_{(2m-1)/2})$: For any given compact $K \subset \Omega$ there is $C_K > 0$ such that*

$$\|u\|_{m-1/2}^2 \leq C_K (\text{Re}(Pu, u) + \|u\|_{m-1}^2), \quad \forall u \in C_c^\infty(K).$$

Moreover, if Q is a $(2m - 1)$ st-order properly supported classical pseudodifferential operator on Ω , the operator $P + Q$ keeps satisfying the above subelliptic estimate provided the real part q_{2m-1} of the principal symbol of Q fulfills

$$q_{2m-1}(\rho) \in -p_{2m-1}^s(\rho) + I(\rho), \quad \forall \rho \in \Sigma.$$

In particular, in case Q is a partial differential operator, when

$$|q_{2m-1}(\rho)| < \kappa(\rho) - |p_{2m-1}^s(\rho)|, \quad \forall \rho \in \Sigma.$$

Again, no smoothness assumption on Σ is required.

5 Other extensions of Thm. A and Thm. B

Let Z_1, \dots, Z_N be first-order partial differential operator with smooth complex coefficients on $\Omega \subset \mathbb{R}^n$ (and no zeroth-order term). Let Σ be the characteristic set of the system (Z_1, \dots, Z_N) . We think of them as complex vector fields on $\Omega \subset \mathbb{R}^n$.

Definition 5.1 Define, for $k \geq 1$ and $x \in \Omega$,

$$V_x^{(k)} = \text{Span}_{\mathbb{C}} \{Z_j, 1 \leq j \leq N, \text{ and their commutators up to length } k\} (x) \tag{5.1}$$

Hence the family $(V_x^{(j)})_{j \geq 1}$ forms an ascending sequence of complex vector subspaces of \mathbb{C}^n .

We have, for instance,

$$V_x^{(1)} = \text{Span}_{\mathbb{C}} \{Z_j, 1 \leq j \leq N\} (x),$$

and

$$V_x^{(2)} = \text{Span}_{\mathbb{C}} \{Z_j, [Z_j, Z_k]; 1 \leq j, k \leq N\} (x).$$

In this final section I wish to consider the following problem:

Problem 5.2 Construct examples of vectors fields $Z_j(x, D)$ such that for the ascending sequence $(V_x^{(j)})_{j \geq 1}$, $x \in \Omega$, of Definition 5.1, there exists a smallest $k \geq 1$ for which

$$\dim_{\mathbb{C}} V_x^{(k)} \leq n - 1 \quad \text{and} \quad V_x^{(k)} + \overline{V_x^{(k)}} = \mathbb{C}T_x\Omega, \quad \forall x \in \pi(\Sigma), \tag{5.2}$$

and the corresponding sum-of-squares operator P is hypoelliptic, possibly with a loss of many derivatives.

Definition 5.3 When (5.2) holds, I will say that $V_x^{(k)}$ is **nontrivially generating** and say that k is the **step of nontrivial generation**. When we do not have such a k in the construction of the $V_x^{(j)}$ and have instead that $V_x^{(k)} = \mathbb{C}T_x\Omega$ for all $x \in \pi(\Sigma)$, I will just say that $V_x^{(k)}$ **spans**.

Remark 5.4 Of course, the above problem is meaningful when P is nonelliptic, i.e. $\Sigma \neq \emptyset$. That is, however, the case we are interested in.

When $V_x^{(2)}$ is nontrivially generating, Theorem 4.1 and Theorem 4.6 give that P is hypoelliptic with a loss of 1 derivative, since $(SE_{1/2})$ holds. One has, in fact, the following result (see also [11]) that further generalizes Thm. A.

Theorem 5.5 Let Z_1, \dots, Z_N be first-order partial differential operator with smooth complex coefficients on $\Omega \subset \mathbb{R}^n$ (and no zeroth-order term). Suppose that $V_x^{(2)}$ is nontrivially generating for all $x \in \pi(\Sigma)$. Then $P = \sum_{j=1}^N Z_j^* Z_j$ satisfies $(SE_{1/2})$ and therefore P is hypoelliptic with a loss of 1 derivative.

Proof I show that condition (K_{Σ}) (see (4.3)) is fulfilled. For $x \in \pi(\Sigma)$ consider the system

$$\begin{cases} Z_j(x, \xi) = 0, & 1 \leq j \leq N, \\ \{Z_j, Z_k\}(x, \xi) = 0, & 1 \leq j < k \leq N, \end{cases} \tag{5.3}$$

in the unknown $\xi \in T_x^* \Omega$, $\xi \neq 0$. Since ξ is *real*, the system (5.3) has a solution ξ_0 iff the complex conjugate system $\overline{(5.3)}$ has the same solution ξ_0 . Therefore, one must have $(x, \xi_0) \in \Sigma \subset T^* \Omega \setminus 0$ and at the same time

$$\xi_0 \in V_x^{(2)\perp} \cap \overline{V_x^{(2)}}^\perp = (V_x^{(2)} + \overline{V_x^{(2)}})^\perp = \{0\}.$$

Therefore, for all $x \in \pi(\Sigma)$ we have that

$$(x, \xi) \in \Sigma \implies \{Z_j, Z_k\}(x, \xi) \neq 0$$

for some j, k with $1 \leq j < k \leq N$, whence condition (K_Σ) is satisfied. □

Example 5.6 A meaningful example of Theorem 5.5 is given by $Z_1(x, \xi) = \xi_1 + ix_1\xi_2$, $Z_2(x, \xi) = x_1(\xi_2 + i\xi_3)$, with $(x, \xi) \in T^*\mathbb{R}^3$. Then

$$\Sigma = \{(x, \xi); \xi_1 = x_1\xi_2 = x_1\xi_3 = 0, (\xi_2, \xi_3) \neq (0, 0)\},$$

and

$$\{Z_1, Z_2\}(x, \xi) = \xi_2 + i\xi_3, \quad \{Z_1, \{Z_1, Z_2\}\}(x, \xi) = 0, \quad \{Z_2, \{Z_1, Z_2\}\}(x, \xi) = 0.$$

Therefore $k = 2$,

$$V_x^{(2)} = \text{Span}_{\mathbb{C}} \left\{ \begin{bmatrix} 1 \\ ix_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x_1 \\ ix_1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix} \right\}, \quad \forall x \in \mathbb{R}^3,$$

and $\dim_{\mathbb{C}} V_x^{(2)} = 2$ everywhere. Hence Kohn’s condition (K) *does not hold*. However,

$$V_x^{(2)} + \overline{V_x^{(2)}} = \mathbb{C}^3, \quad \forall x \in \mathbb{R}^3,$$

whence by Theorem 5.5, the operator $P = Z_1^* Z_1 + Z_2^* Z_2$ is hypoelliptic with a loss of 1 derivative.

The natural question related to Problem 5.2 is then:

- Is there an example similar to that in Thm. B, for which I do **not** have the full condition that the vector fields and their commutators of length $k \geq 3$ span the complexified tangent space, and yet the operator P is hypoelliptic with a loss of derivatives related to k ?

In other words, *I would like to find complex vector fields Z_1 and Z_2 in \mathbb{R}^3 for which $V_x^{(k+1)}$, $k \geq 2$, is nontrivially generating and P is hypoelliptic with a loss of k derivatives.*

Let us look at two examples before proceeding further.

- **The example by Kohn in Thm. B.** He took (recall, $D = -i\partial$) in \mathbb{R}^3 , setting $z_1 = x_1 + ix_2$

$$Z_1(x, D) = D_{\bar{z}_1} - iz_1 D_{x_3}, \quad Z_{2k}(x, D) = \bar{z}_1^k (D_{z_1} + i\bar{z}_1 D_{x_3}) = -\bar{z}_1^k \overline{Z_1(x, D)},$$

and $P_k = Z_1^* Z_1 + Z_{2k}^* Z_{2k}$. Writing, as usual, $\text{ad}(X)Y = [X, Y]$, one has

$$\text{ad}(Z_1)^k Z_{2k} = i^{k+1} (-ik! \bar{Z}_1 - 2k k! \bar{z}_1 D_{x_3}), \quad \text{ad}(Z_1)^{k+1} Z_{2k} = -2i^k (k + 1)! D_{x_3}$$

(which have, respectively, length $k + 1$ and $k + 2$), so that

$$\begin{aligned} V_x^{(k+1)} &= \text{Span}_{\mathbb{C}} \{ Z_1, \bar{Z}_1 + (\text{const.}) \bar{z}_1 D_{x_3} \} (x), \\ V_x^{(k+2)} &= \text{Span}_{\mathbb{C}} \{ Z_1, \bar{Z}_1, D_{x_3} \} (x) = \mathbb{C}^3. \end{aligned}$$

Hence, since

$$\dim_{\mathbb{C}} V_x^{(k+1)} = 2, \quad V_x^{(k+1)} = \overline{V_x^{(k+1)}} \subsetneq V_x^{(k+2)} = \mathbb{C}^3,$$

we have that $V_x^{(k+1)}$ is not nontrivially generating, and we have to consider the further space $V_x^{(k+2)} = \mathbb{C}^3$. In this case, Kohn’s theorem says that P_k is hypoelliptic with a loss of $k + 1$ derivatives (which is the step $k + 2$ minus 1). Notice that in this case the characteristic set of P_k is

$$\Sigma = \{(x, \xi); x_1 = x_2 = \xi_1 = \xi_2 = 0, \xi_3 \neq 0\}$$

which is symplectic, with 1-dimensional fibers that allow a microlocalization in the directions $\xi_3 > 0$ and $\xi_3 < 0$, respectively, which is crucial in Kohn’s proof of hypoellipticity in Thm. B.

- **A Hörmander type operator.** In \mathbb{R}^3 take

$$Z_1(x, \xi) = \xi_1, \quad Z_2(x, \xi) = x_1^k (\xi_2 + i\xi_3).$$

Then for the principal and subprincipal symbols, respectively, one has

$$p_2(x, \xi) = \xi_1^2 + x_1^{2k} (\xi_2^2 + \xi_3^2), \quad p_1^s(x, \xi) \equiv 0,$$

with characteristic set

$$\Sigma = \{(x, \xi); x_1 = \xi_1 = 0, (\xi_2, \xi_3) \neq (0, 0)\}$$

which is symplectic. Since for $1 \leq j \leq k - 1$

$$\underbrace{\{Z_1, \dots, \{Z_1, \{Z_1, Z_2\}\} \dots\}}_{j\text{-times, i.e. length } j+1}(x, \xi) = k(k - 1) \dots (k - j + 1) x_1^{k-j} (\xi_2 + i\xi_3) \equiv 0 \text{ on } \Sigma,$$

while

$$\underbrace{\{Z_1, \dots, \{Z_1, \{Z_1, Z_2\}\} \dots\}}_{k\text{-times, i.e. length } k+1}(x, \xi) = k!(\xi_2 + i\xi_3) \neq 0 \text{ on } \Sigma,$$

one has

$$V_x^{(k+1)} = \text{Span}_{\mathbb{C}} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix} \right\}, \text{ and } V_x^{(k+1)} + \overline{V_x^{(k+1)}} = \mathbb{C}^3.$$

Therefore $V_x^{(k+1)}$ is nontrivially generating. However,

$$P = Z_1^* Z_1 + Z_2^* Z_2 = D_{x_1}^2 + x_1^{2k} D_{x_2}^2 + x_1^{2k} D_{x_3}^2$$

(no first-order term), whence one has that P is actually a sum of squares of real vector-fields that span at step $k + 1$. Therefore Rothschild-Stein result gives a hypoellipticity with a loss of $2 - 2/(k + 1) = 2k/(k + 1)$ derivatives.

I am finally in a position to give my generalization of Thm. B.

In \mathbb{R}^3 for $k \geq 2$, consider $P = Z_1^* Z_1 + Z_2^* Z_2$, where

$$Z_1(x, \xi) = \xi_1 + ix_1 \xi_2, \quad Z_2(x, \xi) = x_1^k (\xi_2 + i\xi_3).$$

I therefore have, for p_2 and p_1^s , the principal and subprincipal symbols respectively,

$$p_2(x, \xi) = \xi_1^2 + x_1^2 \xi_2^2 + x_1^{2k} (\xi_2^2 + \xi_3^2), \quad p_1^s(x, \xi) = \xi_2.$$

Here

$$\Sigma = \{(x, \xi); \xi_1 = x_1 = 0, (\xi_2, \xi_3) \neq 0\}$$

is symplectic, with 2-dimensional fibers. Notice that

$$P = D_1^2 + (x_1^2 + x_1^{2k}) D_2^2 + x_1^{2k} D_3^2 + D_2, \tag{5.4}$$

so that neither Hörmander’s nor Rothschild-Stein’s result may be applied (as D_2 is a complex vector field). In this case

$$V_x^{(k+1)} = \text{Span}_{\mathbb{C}} \left\{ \begin{bmatrix} 1 \\ ix_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix} \right\}$$

is nontrivially generating, since

$$\dim_{\mathbb{C}} V_x^{(k+1)} = 2, \quad V_x^{(k+1)} = V_x^{(k+2)}, \quad V_x^{(k+1)} + \overline{V_x^{(k+1)}} = \mathbb{C}^3, \quad \forall x \in \mathbb{R}^3.$$

For $\delta \in (0, 1/2)$, consider the cone in $T^*\mathbb{R}^3 \setminus 0$

$$\Gamma_\delta := \{(x, \xi) \in T^*\mathbb{R}^3 \setminus 0; |x_1| < \delta, (\xi_1^2 + \xi_3^2)^{1/2} < \delta|\xi_2|\}.$$

Then $p_2|_{\Gamma_\delta}$ is transversally elliptic with respect to $\Sigma \cap \Gamma_\delta$, since (with $(\xi_2, \xi_3) = \xi'$)

$$|\xi|^2 \text{dist}_\Sigma(x, \xi)^2 \approx \xi_1^2 + x_1^2|\xi'|^2 \lesssim p_2(x, \xi) \lesssim \xi_1^2 + x_1^2|\xi'|^2, \quad (x, \xi) \in \Gamma_\delta.$$

One has the following result for P (see (5.4) above).

Theorem 5.7 *When $\rho = (0, x'_0, 0, \xi'_0 \neq 0) \in \Sigma \cap \Gamma_\delta$ is such that $\xi_2^0 > 0$, for all $s \in \mathbb{R}$*

$$u \in \mathcal{D}'(\mathbb{R}^3), \quad Pu \in H_{\text{ml}}^s(\rho) \implies u \in H_{\text{ml}}^{s+1}(\rho).$$

When $\rho \in \Sigma \cap \Gamma_\delta$ is such that $\xi_2^0 < 0$, for all $s \in \mathbb{R}$

$$u \in \mathcal{D}'(\mathbb{R}^3), \quad Pu \in H_{\text{ml}}^s(\rho) \implies u \in H_{\text{ml}}^{s+2-k}(\rho).$$

Therefore, in Γ_δ the operator P is microlocally hypoelliptic with a loss of k derivatives.

Proof The proof is based on the machinery ([6] and [7]) that Cesare Parenti and I developed (after Boutet De Monvel, Boutet De Monvel-Grigis-Helffer and Helffer) for studying the hypoellipticity with a loss of many derivatives for operators that are of transversal symplectic type, that is, having a symplectic characteristic manifold Σ and a principal symbol p_2 vanishing on Σ exactly to second order. Since the approach is microlocal, considering our operator on Γ_δ is perfectly fine. I will use the notation from the quoted papers.

In the first place, taking $\delta \in (0, 1/2)$, I extend P to an operator \tilde{P} which is transversally elliptic with respect to Σ everywhere. Let χ be a zeroth-order symbol, positively homogeneous of degree 0 in ξ such that $0 \leq \chi \leq 1$ and

$$\chi(x, \xi) = \begin{cases} 1, & (x, \xi) \in \Gamma_\delta, \\ 0, & (x, \xi) \notin \Gamma_{2\delta}. \end{cases}$$

Put $Q = D_1^2 + x_1^2|D'|^2$. Then the operator \tilde{P} acts for all $\varphi \in C_c^\infty(\mathbb{R}^3)$ as

$$\tilde{P}\varphi = P\chi(x, D)\varphi + Q(1 - \chi(x, D))\varphi,$$

it is transversally elliptic with respect to Σ and satisfies, in $\Sigma \cap \Gamma_\delta$, the hypotheses of [6] whence the hypoellipticity of P at $\rho \in \Sigma \cap \Gamma_\delta$ is equivalent to that of \tilde{P} at ρ .

Following the approach in that paper, we next have to look at the localized symbols of P at $\rho \in \Sigma \cap \Gamma_\delta$ (actually, of \tilde{P} , but the two coincide there) on $N_\rho\Sigma = T_\rho\Sigma^\sigma$ (the symplectic orthogonal of $T_\rho\Sigma$). Let (t, τ) be coordinates in $N_\rho\Sigma$. With $\rho = (0, x'_0, 0, \xi'_0) \in \Sigma$, I put $\rho' = (x'_0, \xi'_0)$. Since the symbol of P is given by

$$p(x, \xi) = p_2(x, \xi) + p_1(x, \xi) = \xi_1^2 + (x_1^2 + x_1^{2k})\xi_2^2 + x_1^{2k}\xi_3^2 + \xi_2,$$

one has that the localized symbol of P at $\rho \in \Sigma$ is given by

$$p_\rho(t, \tau) = \sum_{\alpha+\beta+2j=2} \frac{1}{\alpha!\beta!} (\partial_{x_1}^\alpha \partial_{\xi_1}^\beta p_{2-j})(\rho) t^\alpha \tau^\beta = \tau^2 + \xi_2^2 t^2 + \xi_2,$$

and the higher localized symbols by

$$p_\rho^{(2+\ell)}(t, \tau) = \sum_{\alpha+\beta+2j=2+\ell} \frac{1}{\alpha!\beta!} (\partial_{x_1}^\alpha \partial_{\xi_1}^\beta p_{2-j})(\rho) t^\alpha \tau^\beta,$$

where $\ell \geq 0$. Using then the spectrum and eigenfunctions of $P_\rho^{(2)} := p_\rho^w(t, D_t)$ (which is self-adjoint as an unbounded operator in $L^2(N_\rho \Sigma)$ on its maximal domain) and the operators $P_\rho^{(2+\ell)} := (p_\rho^{(2+\ell)})^w(t, D_t)$, one (microlocally) constructs the symbol $\Lambda(x', \xi')$ of a first order ψ do $\Lambda(x', D')$ in the variables (x', ξ') for which (see [6] and [7]) one has that

- P is hypoelliptic at ρ with a loss of $r + 1$ derivatives iff Λ is hypoelliptic at ρ' with a loss of r derivatives.

In our case, it turns out that conically near ρ' the symbol $\Lambda(x', \xi')$ is *semiregular* and independent of x' , that is, $\Lambda(x', \xi') = \Lambda(\xi') \sim \sum_{j \geq 0} \Lambda_{1-j/2}(\xi')$. Of course, $\text{Spec}(P_\rho^{(2)}) = \{|\xi_2|(2h + 1) + \xi_2; h \in \mathbb{Z}_+\}$ whence the lowest eigenvalue (the one we care of), which is the principal symbol of $\Lambda(x', D')$ near ρ' , is given by

$$\Lambda_1(\xi') = |\xi_2| + \xi_2$$

(recall that since $\rho \in \Gamma_\rho \cap \Sigma$, necessarily $\xi_2 \neq 0$). The corresponding normalized eigenfunction is

$$\phi_0(\xi'; t) = \phi_0(\xi_2; t) = (\pi|\xi_2|)^{-1/4} e^{-|\xi_2|t^2/2}$$

(so that $\|\phi_0\|_{L^2} = 1$). Since

$$P_\rho^{(2+\ell)} = 0, \quad \forall 1 \leq \ell \leq 2(k - 1) - 1, \quad \text{and} \quad P_\rho^{(2k)} = |\xi'|^2 t^{2k},$$

it follows that the first $2(k - 1) + 1$ terms ($j = 0, \dots, 2(k - 1)$) of the symbol of $\Lambda(x', D')$ (recall, it is semiregular) are given by

$$\begin{cases} \Lambda_1(\xi') = |\xi_2| + \xi_2, \\ \Lambda_{1-j/2}(\xi') = 0, \quad \text{for } j = 1, \dots, 2(k - 1) - 1, \\ \Lambda_{1-(k-1)}(\xi') = |\xi'|^2 \|t^k \phi_0(\xi'; \cdot)\|_{L^2}^2 > 0. \end{cases}$$

Therefore, by [6] (see also [7]), if $\rho \in \Gamma_\delta \cap \Sigma$ is such that $\xi_2^0 > 0$, the operator $\Lambda(x', D')$ is elliptic at ρ' , whence it loses 0 derivatives so that P is microlocally hypoelliptic with a loss of 1 derivatives, that is,

$$Pu \in H_{\text{ml}}^s(\rho) \implies u \in H_{\text{ml}}^{s+1}(\rho).$$

When $\rho \in \Gamma_\delta \cap \Sigma$ is such that $\xi_2^0 < 0$, $\Lambda(x', D')$ is no longer elliptic at ρ' , but since $\Lambda_{2-k}(\xi') > 0$ in a (smaller) conic neighborhood of ρ' , we have that $\Lambda(x', D')$ is microlocally hypoelliptic with a loss of $k-1$ derivatives there, so that P is microlocally hypoelliptic with a loss of k derivatives, that is,

$$Pu \in H_{\text{ml}}^s(\rho) \implies u \in H_{\text{ml}}^{s+2-k}(\rho).$$

This concludes the proof. \square

Remark 5.8 Operator P in Theorem 5.7 loses k derivatives, that is, the loss of derivatives $k = k + 1 - 1$ is the step $k + 1$ of nontrivial generation (i.e. $V_x^{(k+1)}$ is nontrivially generating) minus 1. This is in perfect accord with Thm. B by Kohn, where, although we did not have a nontrivial generating $V_x^{(k+2)}$, we had that $V_x^{(k+2)} = \overline{V_x^{(k+2)}} = \mathbb{C}^3$ spans and had that P was hypoelliptic with a loss of $k + 1$ derivatives.

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