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Explicit option pricing with additive processes

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Abstract. We propose a fully explicit and empirically sound option pricing model based on additive processes. The model yields analytic closed formulae for pricing vanilla options, is easy to calibrate and simulate, and fits very well the market implied volatility at an extremely low computational cost. The explicit expressions for commonly traded products implicate that both large and small maturity leading orders of the implied volatility level and skew – as well as large strike asymptotics – can be determined exactly. A plethora of implied volatility shapes can thus be reproduced by just supplying appropriate term functions, leading to nearly complete control of the implied volatility surface. The methodological implication of our study is that, in contrast to widespread belief, realistic continuous time models allowing explicit option pricing do exist. We finally argue that explicit additive models need not to be only of the proposed Beta type, but by taking into account skew generation mechanisms, analiticity of vanilla prices can be potentially determined by other kinds of distributions.

Key words. Explicit option pricing; additive processes; implied volatility surface; implied volatility skew; large strike implied volatilities; infinitely-divisible distributions

MSC codes. 91G20, 60G51

1. Introduction.

The mathematical modelling of option markets took off as an independent strain of research in finance in the wake of the 1987 global equity market crash. The crisis marked a discontinuity in stock option markets, as the implied volatility surface moved away from an approximately flat shape – implying near Black-Scholes prices – and started exhibiting convexity and asymmetry (the “smile” and “skew”). According to many this reflected the change in the investors’ risk attitude and preferences in setting option premia [10]. In turn, this called for new dynamic models for asset prices, some that were able to reproduce this economic observed reality, and yet were (hopefully) simple enough to be used in the financial industry. As research expanded its scope, new stylized facts began to be observed and studied, such as volatility clustering and leverage effect, discontinuity of trajectories, volatility skew explosion, returns memory, slow decay of moments, and so on, up until more sophisticated interactions such as market microstructure roughness and the Zumbach effect [25, 13]. As a rule of thumb, the typical approach is that of being able to reproduce this or that market feature while trying to keep the mathematical complexity as low as required. After all, for a model to be a good one not only it must be capable to reproduce the investigated natural phenomenon, but also needs to be easy to implement and disseminate. This perhaps explains the reason of the successes of the early [44] and [23] models, that despite their by now well-known shortcomings are still widely used today.

In this respect the availability of closed or semi-closed formulae was traditionally regarded to be a particularly good feature in a model. Even though the staggering progresses of computer science in recent years has much eroded the computational advantage of an explicit model to one that requires numerical handling, it cannot be denied that representability of option prices retains an intrinsic value, to the extent to which the analytics of option prices share insight and help to explain some underlying financial stylized facts as those mentioned. Analytical option values thus represent today not so much a value on their own (although ease of implementation and peer communication in industry should not be discounted), but first and foremost an instrument for the investigation and discovery of the various market determinants. In this context, the implied volatility surface is a prime example. Moreover, closed or semi-closed formulae significantly speed-up the calibration of market data and ensure consistency of hedge ratios and

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other risk-management tools.

Hence, by the very beginning, mathematical model have thus appeared as a trade off between *simplicity* and *realism*. After all, Occam's principle stipulates that the best model is the simplest one that delivers an answer within the accepted error tolerance. On the other hand, at least from the academic perspective, it is difficult to deny that the race for sophistication has preponderance, which is only natural since, after all, tackling complexity is science's main purpose. This sometimes lead to the misconception that simple models are almost invariably toy models; the flip side of this inference is that a mathematical model needs to be complex if it is to be trusted.

In the present work we aim at bringing together these two apparently irreconcilable features. We present here what we believe to be a simple, explicit, and parsimonious model which is also able to capture well the option markets stylized facts and that performs very well empirically. Somehow this challenges the common wisdom – in some sense, an epistemological fallacy – that a reliable model has to be *as a necessity* complex, or, conversely, that a simple model is certainly never good enough.

The modelling idea is based on additive processes to model the log-price, i.e. the model itself is of exponential additive form. An additive process, in the sense of [52], is simply a stochastic process with independent increments. This of course includes the usual Lévy processes, widely popular in financial modelling, but it is a strictly larger class since returns stationarity is not enforced. Financial models based on additive processes first came to prominence with [16] who proposed to remedy to some shortcomings of Lévy models, especially to their inability to fit the implied volatility term structure, using a self-similar process of index H with independent increments introduced in [69], hence termed Sato processes. Such processes are easy to handle. Much like an infinitely divisible distributions uniquely determines in law a Lévy process, [69] shows that a self-decomposable distribution, together with a Hurst exponent H, determines in law an H self-similar process with independent increments. Therefore to go down this modelling route all one needs to supply is a unit-time self-decomposable marginal law, and the whole process term structure is fully determined and follows an H power law. Additive drivers in multi-factor OU processes for energy markets models were first proposed by [12]. Models of time-changed additive type were proposed later on in [56], [55] and [57] for applications to energy markets. Here, the additive process is taken to be a time change, and then is coupled with diffusive and sinusoidal components to capture price seasonality and stochastic volatility. The mathematics of such models is analyzed in more detail in [54], extending previously known results of [64].

More recently it has been recognized that additive processes retain distinctive features, and need not to borrow from the time-changed Lévy theory. Under certain monotonicity and continuity conditions, a continuous-time family of infinitely divisible distribution uniquely determines an additive process ([70], Theorem 9.8). Hence the term structure distributional rescaling does not need to be limited to the time dimension, as with the familiar time change technique, but by entering nonlinearly in the Lévy structure impacts the spatial component as well. Recently [61] observed that a space and time rescaling of a self-decomposable process yields to an additive model which can be used to accommodate a noncostant elasticity of variance; see also [60] for a similar approach based on integrals of Lamperti transforms of Sato processes, determining additive processes with bilateral Gamma distributions. In [5] the power law term structures are directly embedded in the Lévy characteristics of a normal tempered stable process, and it is shown how this additive modification calibrates almost perfectly option prices when comparing to Lévy and Sato processes. With regard to numerical simulation, critical for pricing financial products other than those already traded, [6] propose a fast algorithm based on a fast Fourier inversion (FFT) to obtain the increments' cumulative distribution function, a procedure which is as fast as standard algorithm for simulating Brownian motions. An earlier simulation approach, based on small-jump truncation and finite activity approximation for large jumps, was presented by [24].

In [17] a slightly different approach is taken. By starting from a no-arbitrage option valuation formula based on an ℓ^p norm, they show that the implied log-price distribution is supported by an infinitely divisible family of generalized logistic laws, and that such family determines an additive process. This idea of putting first valuation formula and then identifying a supporting process turns on its head the usual valuation paradigm according which the process comes before the state. The resulting model – whose marginal laws follow the Dagum distribution – is however of limited practical applicability, as it is unable to capture the volatility skew and convexity outside a small set of maturities and moneynesses.

In the present paper, we extend and develop this last idea into a realistic and financially viable pricing model. We posit directly a generalized logistic type IV distribution for the risk neutral price marginals and consider the additive process they determine. The resulting stock price will have thus generalized Beta of second kind dynamics. These probability distributions are well-suited to model option prices: they enjoy a remarkable number of properties, which are reflected in many useful parity and symmetry relations in option prices. Besides, the analytical theory of the Beta function is well established and plays a fundamental role for understanding the limits of option prices. By direct integration, we arrive to explicit option pricing formulae for vanilla and digital options, which we then exploit to completely describe the implied volatility surface.

It must be noted that the option pricing equations we end up with have been already discussed by [62], but without any real justification for the choices of the asset returns distribution, in the sense that no dynamic no-arbitrage asset pricing model supporting those distributions is exhibited. The derivations in the first part of this work thus make consistent the pricing formulae conjectured in [62] within the risk-neutral pricing theory.

We obtain simple and elegant expressions for the small and large maturity implied volatility level and skew, as well as for the large strike slope. As expected, the formulae are all determined by the relevant limiting behavior of the involved term structures. In particular, it is possible to specify the functions to simultaneously capture several implied volatility stylized facts, such as a positive at-the-money (ATM) small time level, the short time out-of-the-money (OTM) volatility explosion, the short time ATM skew explosion, and the large maturity persistence (slow decay) of volatility, to name a few. Since many of the parameters responsible for those individual features can be independently specified, control of global properties the surface is possible. For example, we are able to capture the possibility of a "torsion" between the small and large time segments of the surface, meaning that for a given moneyness, negative skews at early maturities may become positive at later ones, or perhaps conversely. Implied volatility surfaces in commodities market do sometimes show this peculiarity (e.g. [47]).

We show that the proposed additive model calibrates very well to the S&P 500 and the EURO STOXX 50 volatility surface. The calibration computational cost, thanks to the closed formula for option prices, is up to 50 times faster compared to other additive and Lévy models.

As a conclusion, we reinterpret the model construction through the lenses of a skewing mechanism popularized in [48], and argue that option price representability – with all of the attached modelling benefits – can possibly be determined by many different distribution classes, of which the model presented here is simply one of the many instantiations (i.e. the "logistic" one).

The paper is organized as follows. Section 2 recalls the logistic and beta distributions and illustrates some of their analytical theory of use. In Section 3 we introduce the additive model, compute the option pricing formulae, and find limits of the distribution cumulants. In Section 4 the entire volatility surface is analyzed, with particular attention to the large and small time asymptotic properties. In Section 5 we calibrate various specifications of the model to a set of traded option prices. Conclusions and the methodological discussion on the model, with an eye to further research, are found in the final section.

2. The model distributions and their infinite divisibility.

A logistic distribution is a member of the scale-location family of probability laws $\mathrm{L}(\mu,\sigma)$ with cumulative density function (CDF) and probability density function (PDF) given respectively by

$$F^L(x;\mu,\sigma) = \frac{1}{1+e^{-\frac{x-\mu}{\sigma}}}, \qquad f^L(x;\mu,\sigma) = \frac{e^{\frac{x-\mu}{\sigma}}}{\sigma(1+e^{\frac{x-\mu}{\sigma}})^2}, \qquad x,\mu\in\mathbb{R}, \sigma>0. \tag{2.1}$$

The logistic distribution has a long and distinguished history in statistics, demography and physics. It is a symmetric distribution with mean, mode and median all equal to μ, variance $\sigma^2\pi^2/3$, zero skewness, and kurtosis equal to $6/5$. It is thus often regarded as the "minimal" leptokurtic variation of the Gaussian distribution. For a comprehensive overview of the main properties of the logistic distribution, and its generalizations we will be shortly introducing, we refer the reader to [8].

The distribution class underpinning the stochastic model we are going to introduce is the so-called generalized logistic type IV $\mathrm{GL}(\mu,\sigma,\alpha,\beta)$ distribution family (also called log-F, or Z-distributions, see [48], [9]). Such laws are a skewed generalization of the logistic family, and its members have a PDF denoted f^{GL} given by

$$f^{GL}(x;\mu,\sigma,\alpha,\beta) = \frac{1}{\sigma B(\alpha,\beta)} \frac{e^{\alpha\frac{x-\mu}{\sigma}}}{(1+e^{\frac{x-\mu}{\sigma}})^{\alpha+\beta}}, \qquad x,\mu\in\mathbb{R}, \alpha,\beta,\sigma>0. \tag{2.2}$$

A $\mathrm{GL}(\mu,\sigma,\alpha,1)$ law is called a skew-logistic $\mathrm{SL}(\mu,\sigma,\alpha)$ distribution and is another type of generalized logistic distribution (type I). Clearly $\mathrm{SL}(\mu,\sigma,1)\equiv \mathrm{L}(\mu,\sigma)$.

The tails of a GL distribution are exponential, and taking the log-limits at $\mp\infty$ in (2.2) it is easy to see that the limiting log-slopes are respectively α/σ and $-\beta/\sigma$, indicating that a GL distribution is negatively skewed when $\alpha<\beta$, symmetric when $\alpha=\beta$, and positively skewed if $\alpha>\beta$.

Concerning the moments we have for $X\sim \mathrm{GL}(\mu,\sigma,\alpha,\beta)$ the following equations ([8], page 214)

$$\mathbb{E}[X] = \sigma(\psi(\alpha)-\psi(\beta))+\mu, \tag{2.3}$$

$$\mathrm{Var}[X] = \sigma^2(\psi'(\alpha)+\psi'(\beta)), \tag{2.4}$$

$$\mathrm{Skew}[X] = \frac{\psi''(\alpha)-\psi''(\beta)}{(\psi'(\alpha)+\psi'(\beta))^{3/2}}, \tag{2.5}$$

$$\mathrm{Kurt}[X] = \frac{\psi'''(\alpha)+\psi'''(\beta)}{(\psi'(\alpha)+\psi'(\beta))^{2}}, \tag{2.6}$$

where ψ and its n-th derivative $\psi^{(n)}$ are respectively the Digamma and Polygamma functions, the former being given as the logarithmic derivative of Euler's Gamma, i.e.

$$\psi(z) = \frac{d}{dz}\log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \qquad z\in\mathbb{C}\setminus\mathbb{Z}_{\le 0}. \tag{2.7}$$

The general n-th cumulant is also available, see [42], Corollary 2. For a real and positive argument, the Digamma function is increasing and convex, while the Polygamma function is positive and decreasing for n odd, and negative and increasing for n even. This confirms that the skewness of X is negative for $\alpha<\beta$, zero if $\alpha=\beta$ and positive otherwise. By direct differentiation is easy to see that the CDF F^{GL} of a $\mathrm{GL}(\mu,\sigma,\alpha,\beta)$ distribution is given by

$$F^{GL}(x;\mu,\sigma,\alpha,\beta) = \frac{1}{B(\alpha,\beta)}\int_0^{(1+e^{-\frac{x-\mu}{\sigma}})^{-1}} t^{\alpha-1}(1-t)^{\beta-1}dt = I_{F^L(x;\mu,\sigma)}(\alpha,\beta), \qquad x\in\mathbb{R}. \tag{2.8}$$

Here $I_y(a,b) = \frac{B_y(a,b)}{B(a,b)}$ is the Beta-regularized ratio, with $B(a,b)$ and $B_y(a,b)$ respectively the Beta and the upper incomplete Beta functions, i.e.

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad B_y(a,b) = \int_0^y t^{a-1}(1-t)^{b-1}dt,$$
$$\mathrm{Re}(a), \mathrm{Re}(b) > 0, y \in (0,1], \tag{2.9}$$

and $\Gamma(\cdot)$ indicates Euler's Gamma function. When $|y| < 1$ the following useful series representation of B_y is used (see [41], 8.392)

$$B_y(a,b) = \frac{y^a}{a}\, {}_2F_1(a, 1-b; a+1; y), \qquad y \in [-1,1], \tag{2.10}$$

where ${}_2F_1$ is Gauss' hypergeometric function which can be defined as the uniformly convergent series

$$ {}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)} \frac{z^k}{k!}, \qquad z \in \mathbb{C}, |z| \le 1, \mathrm{Re}(c) \notin \mathbb{Z}_{\le 0}, \tag{2.11}$$

so that (2.10) and (2.11) lead to the following series representation for the Beta regularized function

$$I_z(a,b) = \frac{z^a}{B(a,b)\Gamma(1-b)} \sum_{k=0}^{\infty} \frac{\Gamma(1-b+k)}{a+k} \frac{z^k}{k!}, \quad z \in [0,1], \mathrm{Re}(a) \notin \mathbb{Z}_{\le 0}, \tag{2.12}$$

which also serves as an analytic continuation of $I_z(a,b)$. An important relation of the Beta regularized function which we shall often exploit is the following symmetry property

$$I_z(a,b) = 1 - I_{1-z}(b,a), \quad z \in (0,1). \tag{2.13}$$

Notice that under some suitable assumptions on the parameters, satisfied by (2.10), (2.11) converges on the boundary of the unit radius disc, justifying the parameter range in (2.10). More compactly letting

$$\Phi(x;\alpha,\beta,\sigma) = I_{F^L(x;0,\sigma)}(\alpha,\beta) = F^{GL}(x;0,\sigma,\alpha,\beta), \tag{2.14}$$

we shall use the Φ notation, observing that

$$F^{GL}(x;\mu,\sigma,\alpha,\beta) = I_{F^L(x;\mu,\sigma)}(\alpha,\beta) = \Phi(x-\mu;\alpha,\beta,\sigma). \tag{2.15}$$

The characteristic function $\varphi_X(z)$ of a $\mathrm{GL}(\mu,\sigma,\alpha,\beta)$ random variable X is given as

$$\varphi_X(z) = \mathbb{E}[e^{izX}] = \left(\frac{B(\alpha+i\sigma z, \beta - i\sigma z)}{B(\alpha,\beta)}\right) e^{i\mu z}, \qquad z \in \mathbb{R}. \tag{2.16}$$

From the above and standard theory (see [58]), one sees that φ_X is analytic in the strip $S = \{z \in \mathbb{C} : \mathrm{Im}(z) < \sigma/\beta\}$, hence GL distributions admit moments of all orders (which also follows from the exponential tail decay of (2.2)), a moment generating function can be defined, and exponential moments of order less than σ/β do exist.

The Generalized Beta distribution of the second kind $\mathrm{GB}(a,b,p,q)$ is the continuous distribution family whose members' PDF f^{GB} is given as follows

$$f^{GB}(x;a,b,p,q) = \frac{1}{b^{ap}B(p,q)} \frac{a x^{ap-1}}{\left(1+\left(\frac{x}{b}\right)^a\right)^{p+q}}, \qquad x,a,b,p,q > 0. \tag{2.17}$$

The GB law generalizes many popular distributions, such as the Fisher F, and distributions classically used to model income such as the [71] distribution (or Burr type XII) and the [21] distribution. Its application to finance has been pioneered as far back as [14], who study estimation methods for such distributions in particular relation to mixing distributions.

As easily verified the GL and GB distributions are connected by an exponential relationship. If $X \sim \mathrm{GL}(\mu, \sigma, \alpha, \beta)$ then $Y = \exp(X) \sim \mathrm{GB}(1/\sigma, e^{\mu}, \alpha, \beta)$, so that the GB distribution is effectively a log-GL distribution. The variable Y just described is our candidate option price distribution. A first element of interest is thus the existence and expression of the moments of Y. Compatibly with the domain of analyticity of the GL function $X = \log Y$, we have that such moments exists only for $\beta > k\sigma$, and clearly

$$\mathbb{E}[Y^k] = \frac{B(\alpha + k\sigma, \beta - k\sigma)}{B(\alpha, \beta)} e^{\mu k}. \tag{2.18}$$

We recall that a real-valued random variable X is infinitely-divisible if for all $n \in \mathbb{N}_0$ there exists X_n^k, $k = 1, \dots, n$, i.i.d. random variables such that

$$X \overset{d}{=} X_n^1 + X_n^2 + \ldots + X_n^n. \tag{2.19}$$

The random variable X is instead said to be self-decomposable if for all $\alpha \in (0, 1)$ there exists a characteristic function φ_α such that

$$\varphi_X(z) = \varphi_X(\alpha z)\varphi_\alpha(z). \tag{2.20}$$

A self-decomposable law is also infinitely-divisible, but there are laws that are infinitely-divisible and not self-decomposable. For details on infinitely-divisible distributions, Lévy processes and additive processes, we refer to [70], Chapter 2.

The characteristic function φ of an infinitely-divisible distribution can be written as

$$\varphi_X(z) = \exp\left(izc - z^2\sigma/2 + \int_{\mathbb{R}} \left(e^{izx} - 1 - izx\mathbb{1}_{|x|<1}\right)\nu(dx) \right), \qquad z \in \mathbb{R} \tag{2.21}$$

where $c \in \mathbb{R}$, $\sigma \geq 0$ and ν is the so-called Lévy measure, a measure on $\mathbb{R}$ such that $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}}(x^2 \wedge 1)\nu(dx) < \infty$. Conversely any triplet (c, σ, ν) of the form just described is such that (2.21) is the characteristic function of some infinitely-divisible distribution. Furthermore, X is self-decomposable if and only if $\nu(dx) = k(x)/|x|dx$, with k increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. The triplet (c, σ, ν) is called the Lévy (or characteristic) triplet of X, and k the canonical function of the self-decomposable distribution.

A Lévy process $X = (X_t)_{t\geq 0}$, $X_0 = 0$ is a stochastic process supported on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ which is stochastically continuous and has stationary and independent increments. If X is a Lévy process then X_t is infinitely divisible for all $t > 0$; conversely, if $\mathcal{X}$ is an infinitely divisible law then there exists a unique in law Lévy process X such that X_1 has the same distribution of $\mathcal{X}$. One then says that X is the Lévy process associated to $\mathcal{X}$.

A stochastic process $Y = (Y_t)_{t\geq 0}$ started at 0 is instead said to be an additive process if it is stochastically continuous and has independent increments. Hence any Lévy process is also an additive process, but not the other way around. If Y is an additive process then its marginals Y_t are infinitely divisible for all $t > 0$. As is the case for Lévy processes, there are standard ways of constructing additive process by starting from a set of infinitely divisible distributions.

It is well-known that that the GL (and hence the SL and L) distribution classes are infinitely-divisible. In particular, the Lévy triplet is provided by [42], Proposition 1, and is of the form

$(a, 0, v(x)dx)$ with

$$a = \sigma \int_0^{1/\sigma} \frac{e^{-\beta x} - e^{-\alpha x}}{1 - e^{-x}} dx + \mu \tag{2.22}$$

$$v(x) = \frac{e^{-x\frac{\beta}{\sigma}}}{x(1 - e^{-\frac{x}{\sigma}})} \mathbb{1}_{\{x>0\}} + \frac{e^{x\frac{\alpha}{\sigma}}}{|x|(1 - e^{\frac{x}{\sigma}})} \mathbb{1}_{\{x<0\}}. \tag{2.23}$$

After letting $k(x) = |x|v(x)$ it is easy to see that k is increasing on the negative half line and decreasing on the positive half line, concluding that GL distributions are not only infinitely divisible, but also self-decomposable.

Having introduced the distribution classes and the Lévy framework we need, in the next section we proceed to the construction of martingale additive processes of generalized Beta type for the underlying security.

3. The Generalized Beta Additive model .

Additive processes can be generated starting from infinitely divisible distribution in a number of ways. Historically (see [16]), the first solution was to exploit self-decomposibility of certain distributions in order to determine Sato processes, and the self-similarity parameter arising was fitted to the term structure. More recently, in the context of commodities markets (see [56], [54]) time changes with respect to additive subordinators have been used to generate additive valuation models. This approach attempts at simultaneously capture the econometric reality of such markets and improve the implied volatility market fits of standard Lévy driven OU-processes.

However, there is a third and even simpler way of canonically constructing an additive process, that perhaps surprisingly, seems to have been overlooked until recently. Such construction consists in directly providing a time-dependent family of infinitely divisible distributions: under some time-consistency conditions the existence of a supporting additive process is automatically guaranteed.

We assume a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ representing a financial market that trades in a riskless money market account delivering a constant instantaneous interest rate r, and assume the standard setup that an equivalent martingale measure $\mathbb{Q}$ under which all the discounted risky securities are martingales exists.

Let $\alpha, \beta, \sigma : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous functions. In order to define an additive exponential martingale price process with GB marginal distributions we need to make the following assumptions.

(A1) $\alpha/\sigma,\ \beta/\sigma$ are non increasing;

(A2) $\sigma(0) = 0$ and σ is nondecreasing;

(A3) $0 < \alpha(0),\ \beta(0) < \infty$;

(A4) $\sigma(t) < \beta(t)\ \forall\, t > 0$.

The intuition behind (**A1**)–(**A4**) is as follows. Continuity, together with (**A1**)–(**A2**), makes it possible to apply the theorem of existence of additive processes supporting the given term structure of marginal laws. Specifically, continuity in time of the marginals translates into continuity in probability of the supporting process, the vanishing of $\sigma(0)$ reflects into the process being started at 0. The monotonicity assumptions ensure that the Lévy measure (see Corollary 3.2 below) increases with time on every given Borel set not containing zero, whose interpretation is that the expected number of jumps whose size belongs to such set grows with t, i.e. the process disperses. Assumption (**A3**) guarantees that the term structure is consistent with the parameters range of the GB distribution. Finally, by (2.18), (**A4**) ensures that the first exponential moment exists and thus it makes sense to consider an exponential martingale price process.

The function σ captures the term structure of the scale and the dispersion of the distribution, and α, β respectively that of the left/right distributional asymmetry, hence skewness. Financially, σ is a measure of the overall price dispersion and of the implied volatility term structure, and α and β of the returns asymmetry and hence the slope of the implied volatility. We shall make these heuristics precise throughout the paper.

Proposition 3.1. *Let α, β, σ be functions satisfying* (**A1**)-(**A4**)*, and $S_0 > 0$. Then under $\mathbb{Q}$ there exists a unique in law process $S = (S_t)_{t\geq 0}$ with $S_t = S_0 \exp(rt + X_t)$ where $X = (X_t)_{t\geq 0}$ is an additive process with $X_0 = 0$ and such that $X_t \sim \mathrm{GL}(-\mu^*(t), \sigma(t), \alpha(t), \beta(t))$ $t > 0$, where*

$$\mu^*(t) := \log\left(\frac{B(\alpha(t)+\sigma(t), \beta(t)-\sigma(t))}{B(\alpha(t), \beta(t))}\right). \tag{3.1}$$

Furthermore $(e^{-rt}S_t)_{t\geq 0}$ is a $\mathbb{Q}$-martingale.

Of course by the discussion in Section 2 it also follows that the asset pricing model distribution is GB and namely

$$S_t \sim \mathrm{GB}(1/\sigma(t), S_0 e^{rt-\mu^*(t)}, \alpha(t), \beta(t)), \qquad t > 0. \tag{3.2}$$

We have thus succeeded in finding a $\mathbb{Q}$-martingale underlying asset pricing process supporting the GL and GB distributions as respectively log-prices and prices. We shall refer to the asset pricing model S in Proposition 3.1 as the generalized-beta additive (GBA) option pricing model.

Once the Lévy triplet of an additive process is known it is possible to recover the semimartingale characteristic of the GL log-price process X by direct differentiation. For a definition of characteristics we refer to [45].

Corollary 3.2. *Assume that $\alpha, \beta, \sigma \in C^1(\mathbb{R}_+)$. The process X in Proposition 3.1 has triplet of semimartingale characteristics $(a_t, 0, v(t,x)dt\,dx)$ given by $a_0 = 0 = v(0,x)$, for all x in $\mathbb{R}\setminus\{0\}$, with*

$$a_t = \sigma(t)\int_0^{1/\sigma(t)} \frac{e^{-\beta(t)x} - e^{-\alpha(t)x}}{1-e^{-x}}dx - \mu^*(t), \tag{3.3}$$

$$\begin{aligned} v(t,x) = {} & \frac{e^{-x\beta(t)/\sigma(t)}}{\sigma(t)^2}\Big(F^L(x,0,\sigma(t))(\sigma'(t)\beta(t) - \beta'(t)\sigma(t)) + f^L(x,0,\sigma(t))\sigma(t)\sigma'(t)\Big)\mathbb{1}_{\{x>0\}} + \\ & \frac{e^{x\,\alpha(t)/\sigma(t)}}{\sigma(t)^2}\Big(F^L(-x,0,\sigma(t))(\sigma'(t)\alpha(t) - \alpha'(t)\sigma(t)) + f^L(x,0,\sigma(t))\sigma(t)\sigma'(t)\Big)\mathbb{1}_{\{x<0\}}. \end{aligned} \tag{3.4}$$

Remark 3.1. *From Corollary 3.2, or even from* (2.23)*, we see that $\int_0^t \int_{\mathbb{R}} |x| v(t,x)dtdx = \infty$ which implies that the* GBA *process is of infinite variation.*

3.1. Option prices.

Determining vanilla and digital option prices in the GBA model is a standard probability exercise. As well-known (see [15]), digital option values can be interpreted as the supporting distribution's CDF or survival function, whereas vanilla options are determined by a straightforward integration, identical in spirit to that involved in the Black-Scholes formula.

Proposition 3.3. *Given a spot price S_0 a strike price K and an expiry τ, the initial values $C(S_0,K,\tau)$, $P(S_0,K,\tau)$, $DC(S_0,K,\tau)$, $DP(S_0,K,\tau)$ of respectively a call option, a put option, a digital call option and a digital put option of the given parameters in the* GBA *model are*

$$\begin{aligned} C(S_0,K,\tau) = {} & S_0\Phi\big(d_{S_0,K,\tau}; \beta(\tau)-\sigma(\tau), \alpha(\tau)+\sigma(\tau), \sigma(\tau)\big) \\ & - e^{-r\tau}K\Phi\big(d_{S_0,K,\tau}; \beta(\tau), \alpha(\tau), \sigma(\tau)\big), \end{aligned} \tag{3.5}$$

$$\begin{aligned} P(S_0,K,\tau) = {} & e^{-r\tau}K\Phi\big(-d_{S_0,K,\tau}; \alpha(\tau), \beta(\tau), \sigma(\tau)\big) \\ & - S_0\Phi\big(-d_{S_0,K,\tau}; \alpha(\tau)+\sigma(\tau), \beta(\tau)-\sigma(\tau), \sigma(\tau)\big), \end{aligned} \tag{3.6}$$

$$DC(S_0,K,\tau) = e^{-r\tau}\Phi\big(d_{S_0,K,\tau}; \beta(\tau), \alpha(\tau), \sigma(\tau)\big), \tag{3.7}$$

$$DP(S_0,K,\tau) = e^{-r\tau}\Phi\big(-d_{S_0,K,\tau}; \alpha(\tau), \beta(\tau), \sigma(\tau)\big), \tag{3.8}$$

with

$$d_{S_0,K,\tau} = \log \frac{S_0}{K} - \mu^*(\tau) + r\tau. \tag{3.9}$$

These simple formulae are akin to those of the Samuelson-Black-Scholes model, where a standard GL distribution replaces the standard normal distribution, after supplying the two skewing term structures α, β. As mentioned in the introduction these expressions were given in a different but equivalent form in [62], but were not supported by a martingale underlying model.

We now list a number of important model properties as a series of remarks.

Remark 3.2. *The valuation equations can be re-interpreted through the familiar "change of numeraire" technique, where the first partial moment of the equity prices at the moneyness level represents itself a probability in a new* GL *law induced by changing from the risk-neutral probability measure to the so-called "share measure" (the measure associated with the normalized stock-price numeraire [39]). Indeed [42] shows that* $X \sim \mathrm{GL}\ (\mu, \sigma, \alpha, \beta)$ *and* $Y \sim \mathrm{GL}\ (\mu', \sigma', \alpha', \beta')$ *are equivalent if and only if* $\mu' = \mu$, $\sigma' = \sigma$, $\alpha' = \alpha + \theta\sigma$, $\beta' = \beta - \theta\sigma$, $-\alpha/\sigma < \theta < \beta/\sigma$, *with density transform to a measure* $\mathbb{Q}^* \sim \mathbb{Q}$, *and* $t \in [0, T^*]$, $T^* > 0$

$$\left.\frac{d\mathbb{Q}^*}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = Z_t := \exp(\theta X_t - \Psi_t^X(-i\theta)), \tag{3.10}$$

where Ψ_t^X *is the cumulant process of* X *(see the Appendix). The density* $Z = (Z_t)_{t\geq 0}$ *is an example of the so-called Esscher transform, popular for changes of measure in the realm of Lévy processes, but that can also be defined for additive ones, as here. In our case taking* $\theta = 1$, *which is allowed by* (**A1**), *and we obtain* $Z_t = e^{-rt} S_t/S_0$, *from which the term "share measure". Notice that* (**A1**)–(**A3**) *still hold under the measure* Q^* *and* S *is the exponential of an additive process on* $[0, T^*]$ *under such measure. Notice also that under* Q^* *the process is integrable (and hence* (**A4**) *holds) as long as we take* $T^* < \sup\{t > 0 : \beta(t) - 2\sigma(t) > 0\}$.

Remark 3.3. Homogeneity, stickiness by moneyness. *It is easy to verify that the* GBA *option pricing model is homogeneous of degree 1, i.e. for all* $\lambda > 0$ *it holds*

$$C(\lambda S_0, \lambda K, \tau) = \lambda C(S_0, K, \tau). \tag{3.11}$$

After applying a chain rule arguments, one sees that homogeneity is sufficient to ensure that the GBA *model is sticky-by-moneyness (or sticky-by-delta). Stickiness-by-moneyness means that the implied volatility surface is a function of the option moneyness alone, and not of separately strike and spot. This is an important and useful property of option prices, see e.g. [20].*

Remark 3.4. The Dagum model. *The* CPDA *(conjugate power Dagum additive) model proposed in [17] based on the Dagum distribution is included in the* GBA *model class as the particular case* $\alpha(t) = 1 - \sigma(t)$, $\beta(t) = 1$, *with the constraint* $\sigma(t) < 1$. *In such case* $\mu^*(\tau) = 0$ *and hence, because of the basic relations* $I_x(\alpha, 1) = x^\alpha$, $I_x(1, \beta) = 1 - (1 - x)^\beta$ *it holds for the* CPDA *call option price* C^D

$$\begin{aligned} C^D(S_0, K, \tau) &= S_0 \Phi\left(d_{S_0,K,\tau}; 1 - \sigma(\tau), 1, \sigma(\tau)\right) - e^{-r\tau} \Phi\left(d_{S_0,K,\tau}; 1, 1 - \sigma(\tau), \sigma(\tau)\right) \\ &= S_0 \left(1 + \left(\frac{K}{S_0 e^{r\tau}}\right)^{1/\sigma(\tau)}\right)^{\sigma(\tau)-1} - e^{-r\tau} K \left(1 + \left(\frac{S_0 e^{r\tau}}{K}\right)^{1/\sigma(\tau)}\right)^{\sigma(\tau)-1} - e^{-r\tau} K \\ &= e^{-r\tau} \left(\left(S_0 e^{r\tau}\right)^{1/\sigma(\tau)} + K^{1/\sigma(\tau)}\right)^{\sigma(\tau)} - K\right). \end{aligned} \tag{3.12}$$

Consider now [17], Equation (2.2): by adding a positive interest rate and rewriting the formula in terms of the forward price, the value $M(S_0, K, \tau)$ *of a portfolio long a call option struck at* K

maturing at τ, and long K units of cash, promising at τ the payoff $\max\{K, S_\tau\}$*, can be written as*

$$M(S_0, K, \tau) = e^{-r\tau}\left((S_0 e^{r\tau})^{1/\sigma(\tau)} + K^{1/\sigma(\tau)}\right)^{\sigma(\tau)}. \tag{3.13}$$

Since $M(S_0, K, \tau) = C(S_0, K, \tau) + Ke^{-r\tau}$ we have just found that the new GBA *pricing formulae, for appropriate parameters specify to the max option value of the Dagum model. This is of course expected, since as we mentioned the Dagum distribution is embedded in the* GB *distribution class.*

Remark 3.5. Increments, Greeks and simulation. *A key property of additive processes making them amenable not just for computing initial values of vanilla options, but also for valuation after contract inception and pricing of exotic products, is that the characteristic function of the increments is analytically known. By independence of the increments, for all $t, h > 0$ we have that*

$$\varphi_{X_{t+h}-X_t}(z) = \frac{\varphi_{X_{t+h}}(z)}{\varphi_{X_t}(z)} \tag{3.14}$$

and φ_{X_t} is explicit for all t from (2.16)*. As well known since [53], knowledge of the characteristic function allows to implement a time $t > 0$ integral valuation formulae of a financial product. Therefore* (3.14)*, conditional on t, with the laws from* (2.16) *is all that is needed to obtain a seasoned option price. Greeks can be then naturally computed, by just differentiating the resulting integral representation with respect to the relevant state variable. For more detail see the aforementioned reference. With regards to the simulation of the process path, necessary to obtain prices of path-dependent products such as barrier or Asian options, one can follow a recent suggestion by [6] based on the inversion of the known characteristic function* (3.14) *to obtain the* CDF *of the increments.*

3.2. Cumulants term structure. A first indication of how the GBA model term structure is influenced by the term functions is offered by the cumulants analysis of the log-price, which as apparent from (2.3)– (2.6) are completely determined by α and β. It is commonly accepted that the implied volatility term structure is related to the distributional properties and time spread of the price implied distribution, which is in turn captured, to a first approximation, by the cumulants term structure. We are especially interested in the small and large times cumulants asymptotic as this will further on connect with the small and large maturity asymptotics of the implied volatility surface. Since assumptions (**A2**) and (**A4**) together rule out a vanishing $\beta(t)$ at ∞, there are six possible cases for the long-time asymptotics of the skewness and kurtosis, which are classified in the following proposition.

Proposition 3.4. *Let X_t be any of the* GL *random variables appearing in Proposition 3.1. We, have as $t \to \infty$, the following asymptotic results:*

(i) if $\alpha(t) \to \alpha_\infty > 0$ and $\beta(t) \to \beta_\infty > 0$, then

$$\mathrm{Skew}[X_t] \to \frac{\psi''(\alpha_\infty) - \psi''(\beta_\infty)}{(\psi'(\alpha_\infty) + \psi'(\beta_\infty))^{3/2}}, \quad \mathrm{Kurt}[X_t] \to \frac{\psi'''(\alpha_\infty) + \psi'''(\beta_\infty)}{(\psi'(\alpha_\infty) + \psi'(\beta_\infty))^2}; \tag{3.15}$$

(ii) if $\alpha(t) \to \alpha_\infty > 0$ and $\beta(t) \to \infty$, then

$$\mathrm{Skew}[X_t] \to \frac{\psi''(\alpha_\infty)}{\psi'(\alpha_\infty)^{3/2}}, \quad \mathrm{Kurt}[X_t] \to \frac{\psi'''(\alpha_\infty)}{\psi'(\alpha_\infty)^2}; \tag{3.16}$$

(iii) if $\alpha(t) \to \infty$ and $\beta(t) \to \beta_\infty$, then

$$\mathrm{Skew}[X_t] \to -\frac{\psi''(\beta_\infty)}{\psi'(\beta_\infty)^{3/2}}, \quad \mathrm{Kurt}[X_t] \to \frac{\psi'''(\beta_\infty)}{\psi'(\beta_\infty)^2}; \tag{3.17}$$

(iv) if $\alpha(t) \to 0$ *and* $\beta(t) \to \infty$ *or* $\beta(t) \to \beta_\infty > 0$*, then*

$$\text{Skew}[X_t] \to -2, \qquad \text{Kurt}[X_t] \to 6; \tag{3.18}$$

(v) if $\beta(t) \to 0$ *and* $\alpha(t) \to \infty$ *or* $\alpha(t) \to \alpha_\infty > 0$*, then*

$$\text{Skew}[X_t] \to 2, \qquad \text{Kurt}[X_t] \to 6; \tag{3.19}$$

(vi) if $\alpha(t), \beta(t) \to \infty$ *then*

$$\text{Skew}[X_t] \sim 2\frac{-\alpha(t)^{-2} + \beta(t)^{-2}}{(\alpha(t)^{-1} + \beta(t)^{-1})^{3/2}}, \quad \text{Kurt}[X_t] \sim 6\frac{\alpha(t)^{-3} + \beta(t)^{-3}}{(\alpha(t)^{-1} + \beta(t)^{-1})^{2}}. \tag{3.20}$$

Furthermore, in all cases $\text{Var}[X_t]$ *is nondecreasing.*

We thus observe asymptotic finite levels for the normalized cumulants unless both the skew term functions tend to ∞, in which case the cumulants tend to zero. The precise rate at which this happens depends on the asymptotic rate of α and β. These five possible limiting cases can be used as a guiding principle on how to select a parametrization for α and β in order to obtain a plausible term structure of the price implied distributions. In particular, we notice that cases (i) and (vi) seem to be overparametrized. Case (iv) and (v) have fixed limiting asymptotic negative skewness and kurtosis, so are not ideal for calibration purposes. Choice (iii) is viable, but because of (**A4**), it enforces a bounded scale term function σ, which is somewhat undesirable for reasons we will explain later on. The most natural choice seems then to us (ii), implicating a negative skewness and cumulants driven asymptotically by a single value. This is the condition we will enforce in Subsection 4.4. The final statement provides better statistical insight on assumption (**A1**): essentially, enforcing (**A1**) avoids a nonsensical decreasing price variance.

We point out that thanks to the continuity of α and β both skewness and kurtosis tend to a finite constant for small values (the limit skewness is zero if and only if $\alpha_0 = \beta_0$). [49] observed that the equity market implied skewness and kurtosis appear to be finite both at long and at short term and non-decreasing in time. The flexibility of the GBA process allows us to replicate this behavior and other possible term structures: by selecting suitable model parameters we can replicate two different short and long term finite limits for both cumulants.

4. The implied volatility surface.

The explicit formulae for option prices in the GBA model, other than being a precious tool in the financial practice, lead to explicit characterizations of the implied volatility surface. It is well-recognized that both long term and short term implied volatility levels are determined to the first order by the prices of traded options Results on the short term at the money implied volatility level date back to [18] and [68]. The case of jump diffusions was first taken into account in [63]. A comprehensive analysis for Lévy processes is provided by [3] and for the diffusive Heston model see [29], [30] and [31]. The relation between the at-the-money implied volatility implied volatility skew has been explored by [22], [28], [40] among others. Results for the long term implied volatility level and skew are scarcer. See [73] for general results, the cited [3] and [31], as well as the survey [72] for Lévy models. A model-free framework for the asymptotics of the implied volatility in terms of various combinations of strike/maturity parameters is obtained by [36].

Throughout the whole Section 4 we assume that $r = 0$, which leads to no loss of generality, up to rewriting the pricing equations of Proposition 3.3 in terms of the forward price and discounting. For an expiry τ, a spot price S_0 and a maturity K we denote the Black-Scholes implied volatility $\sigma_{BS}(S_0, K, \tau)$ which we recall is defined as the unique value such that

$$C_{BS}(S_0, K, \tau, \sigma_{BS}(S_0, K, \tau)) = C(S_0, K, \tau), \tag{4.1}$$

where $C_{BS}(S_0, K, \tau, \sigma)$ is the Black-Scholes option value of given parameters, with σ the volatility coefficient. Notice that in our GBA setup, because of Remark 3.3 we can define $\sigma_{BS}(k, \tau)$, where

$k := \log K/S_0$ as

$$C_{BS}(S_0, K, \tau, \sigma_{BS}(k, \tau)) = C(S_0, K, \tau). \tag{4.2}$$

The reason for which an explicit option model is so beneficial for modelling the volatility surface is that literature has made clear that implied volatility asymptotics enjoys probabilistic representations which can be further related to traded option prices. As we illustrate in this Section this means that for the GBA model we can achieve near full control of the volatility surface by specifying suitable term functions α, β and σ.

4.1. Short term implied volatility level. In the option pricing literature, the short term implied volatility is investigated thoroughly. Short term asymptotics have a prominent role since risk management of option portfolios and digital risk becomes increasingly important as expiry approaches.

Before starting out the analysis, it is necessary to slightly strengthen (**A1**) as follows:

(**A1'**) $\sigma(0) = 0$ and σ is strictly increasing.

Hereafter, for the sake of simplicity, we let $\alpha(0) = \alpha_0$, $\beta(0) = \beta_0$. We begin by describing the short term asympotics of the GL distribution, and in particular of the digital option prices.

Proposition 4.1. *Assume $\mathbb{P}'$ is a measure such that $\mathbb{P}' \sim \mathbb{P}$. According to Remark 3.2 let S be an exponential additive model with $\mathbb{P}'$ law S_t given by (3.2) with parameters satisfying requirements (**A1'**), (**A2**)-(**A4**) and let $k = \log(K/S_0)$. Then, as $\tau \to 0$ the following asymptotic equivalences hold*

$$\mathbb{P}'(S_\tau \geq K) \sim \begin{cases} \Phi(-D_{\alpha,\beta}, \beta_0, \alpha_0, 1), & \text{if } S_0 = K, \\ \dfrac{e^{-\beta_0 D_{\alpha,\beta}}}{\beta_0 B(\alpha_0, \beta_0)} e^{-k\,\beta_0/\sigma(\tau)}, & \text{if } S_0 < K, \\ 1 - \dfrac{e^{\alpha_0 D_{\alpha,\beta}}}{\alpha_0 B(\alpha_0, \beta_0)} e^{k\,\alpha_0/\sigma(\tau)}, & \text{if } S_0 > K, \end{cases} \tag{4.3}$$

and

$$\mathbb{P}'(S_\tau \leq K) \sim \begin{cases} \Phi(D_{\alpha,\beta}, \alpha_0, \beta_0, 1), & \text{if } S_0 = K, \\ 1 - \dfrac{e^{-\beta_0 D_{\alpha,\beta}}}{\beta_0 B(\alpha_0, \beta_0)} e^{-k\,\beta_0/\sigma(\tau)}, & \text{if } S_0 < K, \\ \dfrac{e^{\alpha_0 D_{\alpha,\beta}}}{\alpha_0 B(\alpha_0, \beta_0)} e^{k\,\alpha_0/\sigma(\tau)}, & \text{if } S_0 > K, \end{cases} \tag{4.4}$$

where

$$D_{\alpha,\beta} = \psi(\alpha_0) - \psi(\beta_0) \tag{4.5}$$

is the expectation of a $\mathrm{GL}(0, 1, \alpha_0, \beta_0)$ random variable. In particular, if $\mathbb{P}' = \mathbb{Q}$, then (4.3) and (4.4) provides the $\tau \sim 0$ behavior of $DC(S_0, K, \tau)$ and $DP(S_0, K, \tau)$ respectively.

Under (**A1'**) let $\hat{\alpha}$ and $\hat{\beta}$ be the skew term functions α, β written in the σ coordinate (see the proof of Proposition 4.1). In order to transfer the results in Proposition 4.1 to the corresponding ones for vanilla options, we need to make another technical assumption on the regularity of $\hat{\alpha}$ and $\hat{\beta}$.

(**AN**) $\hat{\alpha}$ and $\hat{\beta}$ are analytical in a neighborhood of $\sigma = 0$.

Once this is assumed we can derive the asymptotics for the small maturities at-the-money (ATM) call options.

Proposition 4.2. *Assume that in the* GBA *option pricing model* (**A1'**), (**A2**)–(**A4**) *and* (**AN**) *hold, and let $C(S_0, K, \tau)$ be given by* (3.5)*. We have, as $\tau \to 0$, that*

$$C(S_0, S_0, \tau) = S_0 c_{\alpha,\beta} \sigma(\tau) + O(\sigma(\tau)^2), \tag{4.6}$$

where

$$c_{\alpha,\beta} = \frac{d}{d\sigma}\hat{C}(0, \alpha_0, \beta_0) > 0 \tag{4.7}$$

and

$$\hat{C}(\sigma, \hat{\alpha}(\sigma), \hat{\beta}(\sigma)) := C(1, 1, \tau(\sigma)) \tag{4.8}$$

is the unit-spot ATM *call price in the σ coordinate, and $\tau(\sigma)$ is the inverse function of σ.*

From standard theory (see e.g., [68]) we know that the ATM small maturities option prices easily translate into the short time asymptotics of the ATM implied volatility, which then follows as a corollary.

Corollary 4.3. *Under the assumptions and notation of Proposition 4.2 for the* GBA *model we have that, as $\tau \sim 0$*

$$\sigma_{BS}(0, \tau) \sim \sqrt{2\pi} c_{\alpha,\beta} \sigma(\tau) \tau^{-1/2}. \tag{4.9}$$

This is the first important result highlighting the flexibility of the GBA volatility surface. By selecting a suitable the local rate of σ around zero it is possible to generate different ATM implied volatility levels to match those observed in the market. We can have a vanishing, positive, or exploding ATM implied volatility level according to whether, respectively, $\sigma(\tau) = o(\sqrt{\tau})$, $\sigma(\tau) \sim \sqrt{\tau}$ or $\sqrt{\tau} \sim o(\sigma(\tau))$ as $\tau \to 0$. This flexibility is unlike Lévy models which necessarily have either a positive or vanishing volatility level depending on the presence or absence of a Brownian motion part (see [68, 40]), irrespective of the model parameters. In particular, a market feature that is observed in most option markets is a finite implied volatility level which suggests the rate $\sigma(\tau) \sim c\sqrt{\tau}$, $c > 0$. In Figure 1, we plot the ATM implied volatility obtained numerically and with the short-term approximation in (4.9).

The rate of convergence to the intrinsic value of option prices outside the ATM point is given in the following proposition.

Proposition 4.4. *In a* GBA *model satisfying* (**A1**)–(**A4**) *as $\tau \to 0$ we have*

$$C(S_0, K, \tau) - (S_0 - K)^+ \sim \begin{cases} \dfrac{K e^{-D_{\alpha,\beta}\beta_0}}{\beta_0^2 B(\alpha_0, \beta_0)} \sigma(\tau) \left(\dfrac{S_0}{K}\right)^{\beta_0/\sigma(\tau)}, & \text{if } K > S_0, \\[2ex] \dfrac{K e^{D_{\alpha,\beta}\alpha_0}}{\alpha_0^2 B(\alpha_0, \beta_0)} \sigma(\tau) \left(\dfrac{K}{S_0}\right)^{\alpha_0/\sigma(\tau)}, & \text{if } K < S_0. \end{cases} \tag{4.10}$$

with $D_{\alpha,\beta}$ given in (4.5)*.*

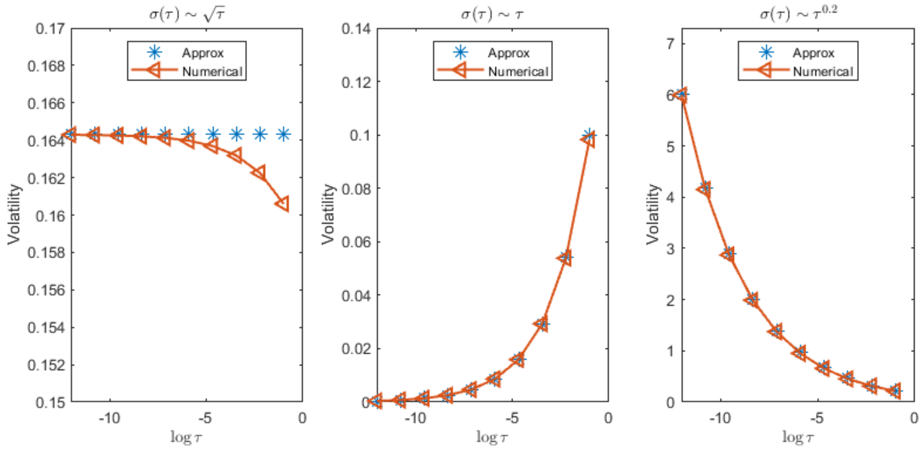

Figure 1. ATM *implied volatility obtained numerically and with the short-term approximation in* (4.9). *We consider* $\alpha_0 = 0.8$, $\beta_0 = 2$, $\sigma(\tau) \sim 0.1\,\tau^H$, *where* $H = 0.5$ *on the left,* $H = 1$ *in the middle and* $H = 0.2$ *on the left.*

Observe that by put-call parity $C(S_0, K, \tau) - (S_0 - K)^+$ is just the value of the OTM option, call if $K > S_0$, put if $K < S_0$. As in the ATM case, it is known theory (see [68, 7]) that the asymptotics in Proposition 4.2 determine the corresponding small maturity level of the implied volatility surface. The GBA small maturity implied volatility limits outside the ATM point are then as follows.

Corollary 4.5. *Under the assumptions and in the notation of Proposition 4.4, as* $\tau \to 0$ *it holds*

$$\sigma_{BS}(k,\tau) \sim \begin{cases} \left(\dfrac{k}{2\beta_0}\dfrac{\sigma(\tau)}{\tau}\right)^{1/2}\left(1 - \dfrac{3}{2}\dfrac{\log\left(\frac{\sigma(\tau)}{k\,\beta_0}\right)}{k\beta_0}\sigma(\tau)\right)^{1/2}, & \text{if } k > 0, \\ \left(\dfrac{|k|}{2\alpha_0}\dfrac{\sigma(\tau)}{\tau}\right)^{1/2}\left(1 - \dfrac{3}{2}\dfrac{\log\left(\frac{\sigma(\tau)}{|k|\alpha_0}\right)}{|k|\alpha_0}\sigma(\tau)\right)^{1/2}, & \text{if } k < 0. \end{cases} \tag{4.11}$$

A generally accepted stylized fact in the small maturity segment of the implied volatility surface is that of an exploding out-of-the-money (OTM) and in-the-money (ITM) implied volatility level. The GBA model captures such stylized fact if we specify $\sigma(\tau) = o(\tau)$. Instead $\sigma(\tau) \sim c\tau$, $c > 0$ and $\tau = o(\sigma(\tau))$ respectively correspond to a limiting level – as typically observed in stochastic volatility models (see e.g., [29]) – and a vanishing implied volatility, not supported by empirical evidence. Another feature of the GBA model is that the asymmetry of the smile can persist in the limit, since such limit does depend on α_0 and β_0, whereas the asymmetry vanishes for e.g. Lévy models (see e.g. [72], Proposition 4). In Figure 2, we plot the one-day implied volatility smile obtained numerically. We consider both the full approximations in (4.11) and, for comparison, its first order truncation by considering only the factor outside parentheses. The improvement in using the full expression is apparent.

4.2. Short term skew.

The implied volatility skew is defined as the sensitivity of the implied volatility in the variable K (see e.g., [37] pages 35-38). Such quantity is a measure of implied volatility – and thus of option prices – variability for some given strike and valuation time. Generally, for small maturity the ATM skew is considered the most important risk factor, since liquidity concentrates ATM for short expiries. However, the ATM skew typically blows up and

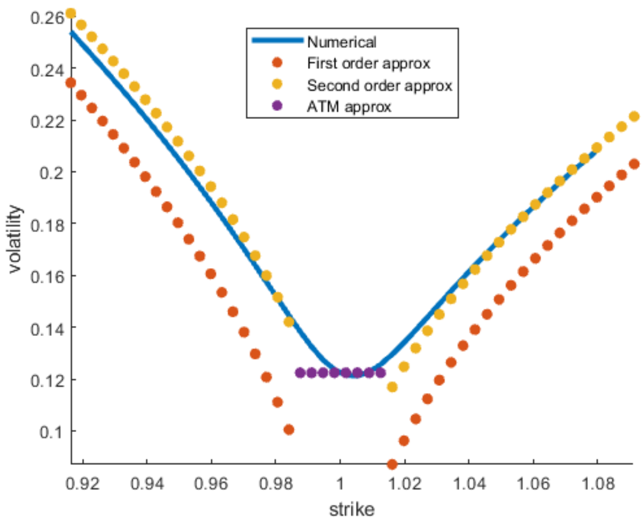

Figure 2. *One day implied volatility smile obtained numerically and with the short-term first and second order approximations in* (4.11)*, with* $\alpha_0 = 1.5$*,* $\beta_0 = 2$*,* $\sigma(\tau) \sim 0.1\sqrt{\tau}$*.*

other moneynesses can be important, especially as these can also exhibit different time rates than that in the ATM point. We thus look at the function $\mathcal{S}$ given by

$$\mathcal{S}(S_0, K, \tau) = \frac{\partial \sigma_{BS}}{\partial K}(S_0, K, \tau) \tag{4.12}$$

in the maturity variable τ. As a consequence of the results in [28], [40] and [7] the implied volatility skew depends on the total implied volatility $\sigma_{BS}(k,\tau)\sqrt{\tau}$ and the digital and (possibly) vanilla call values only. Since in view of propositions 4.1 and 4.4 and corollaries 4.3 and 4.5 we have explicit expressions for the leading orders of all these quantities, we can derive the short time skew of the GBA process for any given moneyness.

Proposition 4.6. *Let S be a* GBA *model satisfying* (**AN**)*,* (**A1'**) *and* (**A2**)*–*(**A4**)*. For any given S_0, K, as $\tau \to 0$ we have*

$$\mathcal{S}(S_0, K, \tau) = \begin{cases} \sqrt{\dfrac{2\pi}{\tau}}\Big(1/2 - \Phi(D_{\alpha,\beta}, \beta_0, \alpha_0, 1) + o(1)\Big), & \text{if } K = S_0, \\[2ex] \dfrac{1}{2K\sqrt{2\beta_0 \log(K/S_0)}}\left(\dfrac{\sigma(\tau)}{\tau}\right)^{1/2} + o(1), & \text{if } K > S_0, \\[2ex] -\dfrac{1}{2K\sqrt{2\alpha_0 \log(S_0/K)}}\left(\dfrac{\sigma(\tau)}{\tau}\right)^{1/2} + o(1), & \text{if } K < S_0, \end{cases} \tag{4.13}$$

with $D_{\alpha,\beta}$ given in (4.5)*.*

From Proposition 4.6 we derive the following fundamental properties of the ATM short term skew in the GBA model.

Corollary 4.7. *Under the assumptions of Proposition 4.6, for $K = S_0$, we have the following:*
(i) as $\tau \sim 0$ the ATM *skew satisfies*

$$\mathcal{S}(S_0, S_0, \tau) \sim \sqrt{2\pi}\big(1/2 - \Phi(D_{\alpha,\beta}, \beta_0, \alpha_0, 1)\big)\tau^{-1/2} \tag{4.14}$$

if and only if $\alpha_0 \neq \beta_0$;

(ii) if $\alpha_0 = \beta_0$ then as $\tau \sim 0$ it holds

$$\mathcal{S}(S_0, S_0, \tau) = -\frac{\sqrt{2\pi}}{2} a_\alpha \sigma(\tau)\tau^{-1/2} + O\left(\frac{\sigma(\tau)^2}{\sqrt{\tau}}\right), \tag{4.15}$$

where $a_\alpha \geq 0$;

(iii) there exists τ_0 such that for all $\tau < \tau_0$ we have $\mathcal{S}(S_0, S_0, \tau) < 0$ (resp. $\mathcal{S}(S_0, S_0, \tau) > 0$) if $\alpha_0 < \beta_0$ (resp. $\alpha_0 > \beta_0$).

The model always determines two distinct rates of skew asymptotics for ATM and OTM volatilities. Such discontinuity produces the typical ATM kink, a structural feature of the short term implied volatility. In particular, under the specification $\sigma(\tau) \sim c\sqrt{\tau}$, $c > 0$, which is simultaneously consistent with a finite ATM small time level and an OTM blowup, leads to an OTM skew explosion of order $\tau^{-1/4}$. A comparison of Corollary 4.7, (ii) with (2.5), confirms that the sign of the limiting skew is the same as the distributional skewness, as one would expect. By (ii) in Corollary 4.7, assuming asymptotic distributional symmetry with $\alpha_0 = \beta_0$ in the GBA model – so long as $a_\alpha > 0$ – produces an asymptotic ATM order of the skew which is different (and necessarily slower, consistently with the model free bounds of [51]) than $\tau^{-1/2}$. In such case the skew may or may not explode, depending upon the asymptotic of σ, and the rate of such behavior is the same as that of the ATM level (Corollary 4.3).

Remark 4.1. *The symmetric case is of some interest in relation to the now popular literature of "rough volatility" models (e.g. [38], [26]), using as underlying asset dynamics a stochastic variance following non-Markovian dynamics involving a fractional Brownian motion or other Volterra-type stochastic integrals. From the perspective of fitting the implied volatility such models are generally supported by an empirically observed rate of skew blow-up of $\tau^{H-1/2}$, $H < 1/2$ which these models are able to capture (see [37], [32]). Now, assuming $\sigma(\tau) \sim \tau^H$, $H < 1/2$ the (asymptotically) symmetric* GBA *model also reproduces a sub-inverse square root* ATM *exploding skew. In other words, the stylized fact of a slowly exploding* ATM *skew as maturity tends to the initial valuation time is perfectly consistent with a one-dimensional Markovian model such as the* GBA *(although rough models can capture this phenomenon also at later valuation dates, see [34], [35]). A similar slow* ATM *skew blow-up is observed in stable and tempered stable exponential Lévy models ([72], Proposition 5).*

In Figure 3, we report the ATM implied volatility skew obtained numerically and with the short-term approximation in (4.14) for different values of α_0 and β_0 and $\sigma(\tau) \sim \sqrt{\tau}/10$, as $\tau \sim 0$. In the left panel, where $\alpha_0 < \beta_0$, we observe a negative power scaling skew in the short term. In the central panel, with $\alpha_0 > \beta_0$, we have instead a positive power scaling skew. In the right panel, where $\alpha_0 = \beta_0$, we obtain a zero skew.

4.3. Large strike asymptotics.

A direct link between the risk-neutral price distribution S_t of an underlying asset price process with the asymptotic slope of the implied variance for large strikes is established by [50] and [11], i.e the large strike steepness of the "arms" of the time cross sections of the implied volatility surface. In [50] it is established that by defining the critical exponents $p_+ \geq 1$ and $p_- \leq 0$ of S_t, that is

$$p_+ := \sup\{p : \mathbb{E}[S_t^p] < \infty\}, \quad p_- := \inf\{p : \mathbb{E}[S_t^p] < \infty\}; \tag{4.16}$$

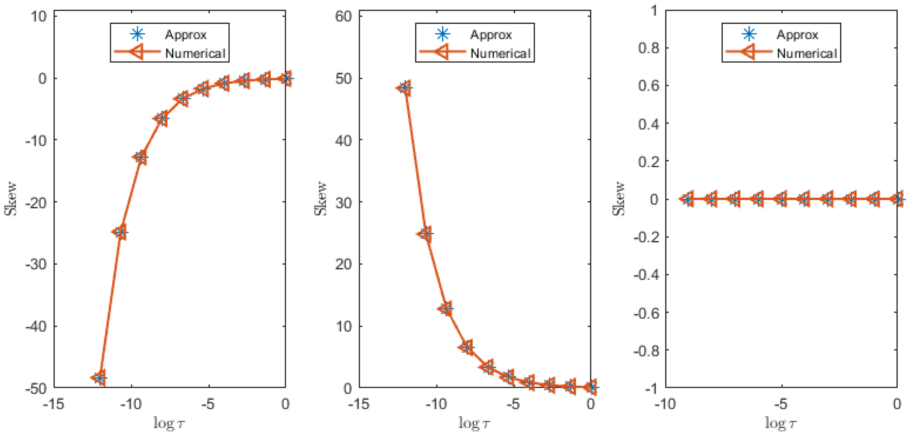

Figure 3. *ATM implied volatility skew obtained numerically and with the short-term approximation in (4.14) for different values of α_0 and β_0. In all cases we consider $\sigma(\tau) \sim 0.1\sqrt{\tau}$. Left panel $\alpha_0 =$ 0.8, $\beta_0 = 2$, central panel $\alpha_0 = 2$, $\beta_0 = 0.8$, right panel $\alpha_0 = \beta_0 = 2$.*

the limiting slopes of the Black-Scholes implied variances in log coordinates, are determined as[1]

$$\limsup_{k\to\infty} \frac{\sigma_{BS}(k,\tau)^2}{k} = 2\tau^{-1}(\sqrt{p_+} - \sqrt{p_+ - 1})^2 \tag{4.17}$$

$$\limsup_{k\to-\infty} \frac{\sigma_{BS}(k,\tau)^2}{-k} = 2\tau^{-1}(\sqrt{1-p_-} - \sqrt{-p_-})^2. \tag{4.18}$$

Under the conditions that the log-tails of the PDF p of an implied price distribution are regularly varying[2] at $\pm\infty$ for some index α [11], sharpen these results to actual limiting behavior. In our situation we can apply this theory since we do have explicit PDFs available: actually, as we observed at the very beginning, the parameters α and β dictate the slopes of the log-tails of the GL PDF. Unsurprisingly, the critical exponents are dictated by the left and right skew functions in conjunction with the variance rate.

Proposition 4.8. *In a* GBA *model satisfying* (**A1**)–(**A4**) *we have that* (4.17) *and* (4.18) *hold with limits replacing limsups, and*

$$p_+ = \frac{\beta(\tau)}{\sigma(\tau)}, \qquad p_- = -\frac{\alpha(\tau)}{\sigma(\tau)}. \tag{4.19}$$

This means that in the GBA model the left and right skewness parameters α and β are directly responsible for the steepness of the implied volatility wings at large strikes. We have the following corollary.

Corollary 4.9. *Under the assumptions* (**A1**)–(**A4**) *the inequality*

$$\lim_{k\to-\infty} \frac{\sigma_{BS}(k,\tau)^2}{-k} \geq \lim_{\kappa\to\infty} \frac{\sigma_{BS}(k,\tau)^2}{k} \tag{4.20}$$

is satisfied if and only if $\beta(\tau) - \alpha(\tau) \geq \sigma(\tau)$. The corresponding relationship holds true replacing the inequalities with equalities.

[1] We thank an anonymous reviewer for noticing these expressions, which we find much simpler and easier to handle than the original ones of [50] and [11], appearing in a previous version of this paper.

[2] A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is regularly varying of order $\alpha \in \mathbb{R}$ at ∞ if for all $\lambda > 0$ it holds

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \lambda^\alpha.$$

As $\tau \to 0$ by (**A2**) and (**A3**) the statement of this corollary holds under the asymptotic requirement $\alpha_0 \leq \beta_0$. Such condition includes the characterization of a negative skew in Corollary 4.7, (ii). This clarifies that the relative steepness of the arms at short maturity in the GBA model follows the sign of the skew, at least when $\alpha_0 \neq \beta_0$. This is a desirable, but not universal, feature of asset pricing models, and it does indeed fail for some Lévy models. For a thorough discussion see [40], Section 7.

An important consequence of the corollary is that for appropriate choices of term structures, large strike slopes can change relative steepness ordering, a phenomenon that relates to the "torsion" of the shape of the volatility surface as maturity increases, which the GBA model can capture.

4.4. Implied volatility far from maturity and the torsion of the skew. The large maturity asymptotics of the implied volatility surface are a less investigated topic compared to the short maturity properties. This may be the inherent difficulties in the financial practice to calibrate to the long term segment of the implied volatility due to a lack of liquidity or of reliable option prices. The large maturity skew and level are studied by [3] using the Fourier-Plancharel option price representation and the steepest descent method. Some general results are provided in [72], and formulae for the Heston model are given by [31]. In [46] is presented a specific time-changed model that is able to part from the typical $O(\tau^{-1})$ order of the long term implied volatility skew. In a beautiful piece of mathematics, [73], extending prior research of [67], illustrates that the leading order of the long term implied volatility is model-free, and just as the small maturity skew and level leading orders are determined by option prices (vanilla and digital respectively), it is also fully determined by a financial value, namely that of a covered call. A covered call is a portfolio short a call option and long the underlying asset. Its payoff is the minimum between the underlying at maturity and the strike price. In the GBA model, when the interest rates are 0, we have the covered call value $CC(S_0, K, \tau)$ given by

$$\begin{aligned} CC(S_0,K,\tau) &= \mathbb{E}[S_\tau - (S_\tau - K)^+] = \mathbb{E}[\min\{S_\tau, K\}] = S_0 - C(S_0,K,\tau) = \\ &= S_0\Phi\left(-d_{S_0,K,\tau}; \alpha(\tau)+\sigma(\tau), \beta(\tau)-\sigma(\tau), \sigma(\tau)\right) \\ &\quad + K\Phi\left(d_{S_0,K,\tau}; \beta(\tau), \alpha(\tau), \sigma(\tau)\right), \end{aligned} \tag{4.21}$$

having used in (3.5) the usual symmetry property (2.13).

According to the results of [73], Theorem 3.1, under the condition that the limit in probability of the process S_τ as τ tends to infinity is zero, and that S_τ has an absolutely continuous law, it holds

$$\sigma_{BS}(S_0,K,\tau) \sim \left(\frac{-8\log CC(S_0,K,\tau)}{\tau}\right)^{1/2}, \qquad \tau \to \infty. \tag{4.22}$$

Furthermore, from Corollary 3.2 in [73], after an easy change of variables, we obtain the leading order of the long term skew for fixed[3] $K, S_0 > 0$ as $\tau \to \infty$

$$\mathcal{S}(S_0,K,\tau) = \frac{2}{\tau\sigma_{BS}(S_0,K,\tau)}\left(\frac{1}{K} - 2\frac{DC(S_0,K,\tau)}{CC(S_0,K,\tau)}\right), \quad \tau \to \infty. \tag{4.23}$$

Therefore, exactly as is the case for the small maturity rate, once the leading order of digital and covered call are determined we have a complete asymptotic description of the large maturity GBA implied volatility surface level and skew, for all strikes and maturities.

[3]Relationships (4.22) and (4.23) are actually uniform in K on compact sets. However for the sake of brevity we will not make use of this additional property, and we will leave the search for uniform bounds in the GBA model for further research.

In order to make further progress we must enforce some minimal asymptotic requirements on the term structure. Our choice is the following:

$$\textbf{(LTA)} \qquad \lim_{\tau\to\infty}\sigma(\tau)=\infty, \quad \lim_{\tau\to\infty}\beta(\tau)-\sigma(\tau)=\beta_\infty>0, \quad \lim_{\tau\to\infty}\alpha(\tau)=\alpha_\infty>0.$$

The first requirement is sufficient[4] to ensure that the stock price tends to 0 in probability as τ gets larger. The conditions on α,β are based on the discussion ensuing Proposition 3.4, where we argued that the limiting regime (ii) is the most natural. This finds confirmation here, as (**LTA**) determines extremely natural asymptotic formulae for the skew.

The large maturity asymptotic rate of the covered call in the GBA model is given by the following proposition.

Proposition 4.10. *In a* GBA *model satisfying assumptions* (**A1**)–(**A4**) *and* (**LTA**)*, for all given K and S_0, we have as $\tau\to\infty$ the asymptotic equivalence*

$$CC(S_0,K,\tau)\sim\left(\frac{1}{2}\right)^{\sigma(\tau)+\alpha_\infty+\beta_\infty-2}\sqrt{S_0\,K}\,\frac{\sigma(\tau)^{(\alpha_\infty+\beta_\infty)/2-1}}{\sqrt{\Gamma(\alpha_\infty)\Gamma(\beta_\infty)}}. \tag{4.24}$$

This proposition not only provides us with the required asymptotic for the financial value to be supplied in (4.22) and (4.23), but also reveals that the GBA model underlying price distribution tends to 0 in probability as $\tau\to\infty$ (see [73], p.631). We have the corresponding corollary for the large maturity asymptotic implied volatility.

Corollary 4.11. *In a* GBA *model, under the assumptions of Proposition 4.10, as $\tau\to\infty$ we have for all fixed $k=\log(K/S_0)$ the following implied volatility asymptotic order*

$$\sigma_{BS}(k,\tau)=\sqrt{8\log 2}\left(\frac{\sigma(\tau)}{\tau}\right)^{1/2}+O\left(\left(\frac{\log\sigma(\tau)}{\tau}\right)^{1/2}\right). \tag{4.25}$$

From (4.11) we deduce that the implied volatility has a positive large maturity limiting level if and only if $\sigma(\tau)\sim c\,\tau$, $c>0$. If that is not the case it either vanishes or blows up depending on whether $\sigma(\tau)=o(\tau)$, $\tau=o(\sigma(\tau))$. The latter case is inconsistent with market data, but does not apparently lead to no-arbitrage violations.

It is instructive to compare Corollary 4.11 to its short term counterparts corollaries 4.3 and 4.5. When $\tau\to 0$ we have two different level rates, namely $O(\sigma(\tau)/\sqrt{\tau})$ ATM and $O(\sqrt{\sigma(\tau)/\tau})$ OTM. Instead as $\tau\to\infty$ we have a single decay rate $O(\sqrt{\sigma(\tau)/\tau})$ for all strikes. The level and skew rates, both at small and large expiry, are governed by σ, and this gives the flexibility of modelling different configurations of short and large maturity regimes, by *independently* specifying suitable rates of σ at 0 and ∞.

In view of (4.23), Proposition 4.10 together with Corollary 4.11 leads to the decay rate of the implied volatility skew.

Proposition 4.12. *In a* GBA *model, assuming* (**A1**)–(**A4**) *and* (**LTA**) *the implied volatility skew satisfies, as $\tau\to\infty$, the following leading order*

$$\mathcal{S}(S_0,K,\tau)\sim\frac{\sigma(\tau)^{-3/2}\tau^{-1/2}}{\sqrt{8\log 2}\,K}\left(\log\sigma(\tau)(\alpha_\infty-\beta_\infty)+\log\frac{K\Gamma(\beta_\infty)}{S_0\Gamma(\alpha_\infty)}\right), \tag{4.26}$$

for all $S_0,K>0$.

[4]Asking $\sigma(\tau)\to\infty$, $\tau\to\infty$, is not *necessary* for $S_\tau\to 0$ to hold in probability. The Dagum model is a counterexample where (**LTA**) does not hold and yet the Dagum call value C^D satisfy

$$\lim_{\tau\to\infty}C^D(S_0,K,\tau)=\lim_{\tau\to\infty}(S_0^{1/\sigma(\tau)}+K^{1/\sigma(\tau)})^{\sigma(\tau)}-K=S_0$$

since $\sigma(\tau)\to 1$, which implies $S_\tau\to 0$ in probability.

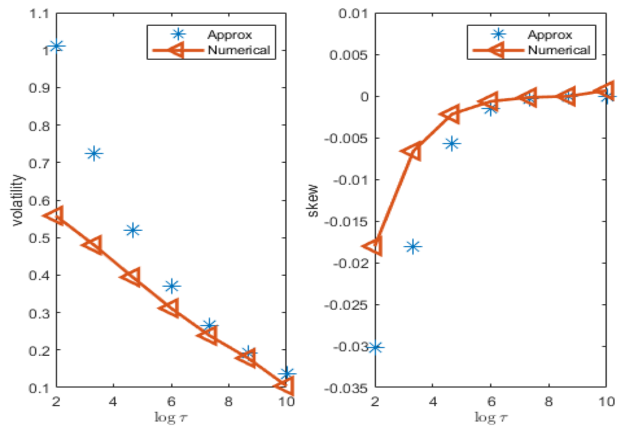

Figure 4. *ATM implied volatility (left panel) and skew (right panel) obtained numerically and with the long term approximations respectively in (4.25) and (4.26), with* $\sigma(\tau) \sim 0.5\sqrt{\tau}$, $\alpha_\infty = 0.8$, $\beta_\infty = 2$.

The above proposition confirms that the long term skew sign and steepness depend on the limits of the skewness term structure α_∞ and β_∞, as well as from the strike K and moneyness $k = \log(K/S_0)$. The precise relationships are as follows.

Corollary 4.13. *In a* GBA *model let* (**A1'**), (**A2**)–(**A4**) *and* (**LTA**) *be in force. Then for all* S_0, K *we have that*

(*i*) *as* $\tau \to \infty$ *the implied volatility skew satisfies*

$$\mathcal{S}(S_0, K, \tau) \sim \frac{1}{\sqrt{8\log 2K}} \frac{\log\sigma(\tau)}{\sigma(\tau)^{3/2}\sqrt{\tau}} \tag{4.27}$$

if and only if $\alpha_\infty \neq \beta_\infty$, *otherwise* $\mathcal{S}(S_0, K, \tau) = O(\sigma(\tau)^{-3/2}\tau^{-1/2})$;

(*ii*) *there exists* τ_∞ *such that for all* $\tau > \tau_\infty$ *we have* $\mathcal{S}(S_0, K, \tau) > 0$ *if* $\alpha_\infty > \beta_\infty$, *and* $\mathcal{S}(S_0, K, \tau) < 0$ *if* $\alpha_\infty < \beta_\infty$.

Corollary 4.13 mirrors Corollary 4.7. The time decay of the skew is governed by $\sigma(\tau)$ and the sign of the limiting slope by α_∞ and β_∞. In particular, in part (*ii*) of Proposition 4.6, we established that the sign of the skew is provided by the ordering of α_0 and β_0, and in (*ii*) of the corollary above we see that the analogous relationship holds for α_∞ and β_∞, with the exact same ordering. A higher limiting value – at both 0 and ∞ – of α than that of β implies a positive skew, whereas a lower value a negative one. In Figure 4, we plot an example of the long term approximations discussed in this section. In the left panel, we have the ATM implied volatility obtained numerically and with the long term approximation in (4.25) while in the right panel, we report the skew and the long term approximation in (4.26).

Regarding the skew decay, we can reproduce any rate, as opposed to standard Lévy (and other) models. Unlike the short term case, where we argued that the specification $\sigma(\tau) \sim c\sqrt{\tau}$, $c > 0$ is consistent with the market stylized fact, this same choice would make $S(S_0, K, \tau) \sim C\tau^{-5/4}\log(c\sqrt{\tau})$ for some constant $C > 0$ as $\tau \to \infty$, provided that $\alpha_\infty \neq \beta_\infty$. This is faster than Lévy and does not capture the market stylized fact of a persistent skew. Letting $\sigma(\tau) \sim c\tau^H$, $c, H > 0$, as $\tau \to \infty$, then $S(S_0, K, \tau) \sim C\tau^{-3H/2-1/2}\log\tau$ for some $C > 0$, provided that $\alpha_\infty \neq \beta_\infty$. So in order to impose a long term skew with a decay slower than the Lévy rate, which is $O(\tau^{-1})$ as shown in [27], Proposition 4.1, we need to set $H < 1/3$. In Section 5, we will define some subcases of the generalized beta additive with these properties.

Similarly to the rate of σ at 0 and ∞, in the GBA framework we have the freedom to specify also the limiting constants $\alpha_0, \beta_0, \alpha_\infty$ and β_∞ according to the market observed surface. In

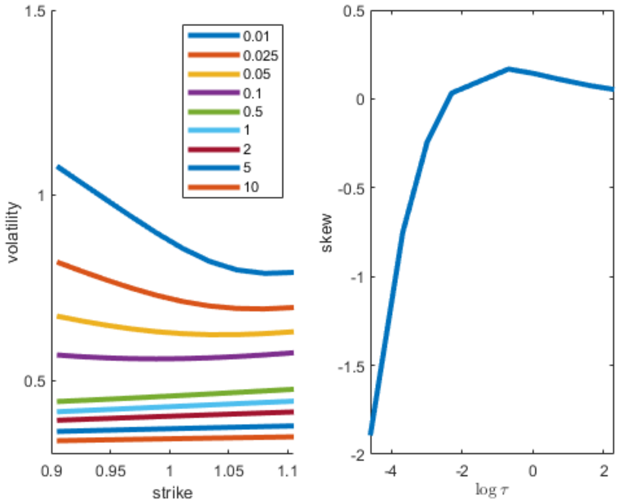

Figure 5. *Implied volatility smile (left panel) and skew (right panel) for different times to maturity obtained numerically. The change of sign of the skew is evident.*

	Level		Skew		
	ATM	O/ITM	ATM		O/ITM
			$\alpha_0 \neq \beta_0$	$\alpha_0 = \beta_0$	
Short	$\frac{\sigma(\tau)}{\tau^{1/2}}$	$\left(\frac{\sigma(\tau)}{\tau}\right)^{1/2}$	$\tau^{-1/2}$	$\frac{\sigma(\tau)}{\tau^{1/2}}$	$\left(\frac{\sigma(\tau)}{\tau}\right)^{1/2}$
Long	$\left(\frac{\sigma(\tau)}{\tau}\right)^{1/2}$		$\frac{\log\sigma(\tau)}{\sqrt{\tau}\sigma(\tau)^{3/2}}$		

Table 1

Absolute values of the implied volatility level and skew leading order asymptotics, short and long maturities. Constants are neglected.

a standard equity index volatility surface we would expect a negative short term skew to be maintained at longer dates, and therefore the specification $\alpha_0 < \beta_0$, $\alpha_\infty < \beta_\infty$ makes sense. However, in certain asset classes, such as commodities, it is sometimes observed empirically that as maturity progresses, the skew changes sign. This has been recently documented by [47] for natural gas and gold markets.

We dub this stylized fact the *implied volatility torsion*. The GBA model can capture a degree of torsion, and that is achieved by specifying $\alpha_0 < \beta_0$ and $\alpha_\infty > \beta_\infty$ (negative to positive), or the other way around. We also remark that the torsion is not just limited to neighborhood of the ATM point. As we saw in Section 4.3, the change of ordering of α and β also impacts the large strike log-moneyness slopes, suggesting that the torsion in the GBA model involves globally the surface, which is financially consistent. In Figure 5, we plot an example of torsion: notice how the skew changes sign as time to maturity increases and finally flattens to zero as no-arbitrage imposes (see e.g. [67]).

In Table 1, we report a summary of the asymptotics obtained in this section. Moreover, in Table 2 we detail how the volatility surface behaves depending upon the parametric choices of the power scaling index, both at infinity and around 0. We assume thus two exponents H_0 and H_∞ such that $\sigma(\tau) \sim \sigma_0 \tau^{H_0}$, $\tau \sim 0$ and $\sigma(\tau) = \sigma_\infty \tau^{H_\infty}$, $\tau \to \infty$. The choices $H_0 = 1/2$ and $H_\infty < 1/3$ seem to be the most appropriate to reproduce the volatility surface stylized facts. In the next section we will introduce actual specifications of σ achieving this.

5. Model specification and calibration.

$H = H_0, H_\infty$	Short					Long	
	Level		Skew			Level	Skew
	ATM	O/ITM	ATM		O/ITM		
			$\alpha_0 \neq \beta_0$	$\alpha_0 = \beta_0$			
$0 < H < 1/3$	∞	∞	∞	∞	∞	0	slow
$H = 1/3$	∞	∞	∞	∞	∞	0	Lévy
$1/3 < H < 1/2$	∞	∞	∞	∞	∞	0	fast
$H = 1/2$	$c > 0$	∞	∞	$c > 0$	∞	0	fast
$1/2 < H < 1$	0	∞	∞	0	∞	0	fast
$H = 1$	0	$c > 0$	∞	0	$c > 0$	$c > 0$	fast
$H > 1$	0	0	∞	0	0	∞	fast

Table 2

Asymptotics for level and skew for short and long maturities in relation to the power scaling, assumed of the form $\sigma(\tau) \sim \sigma_0 \tau^{H_0}$, $\tau \sim 0$, $\sigma(\tau) \sim \sigma_\infty \tau^{H_\infty}$, $\tau \to \infty$, $\sigma_0, \sigma_\infty, H_0, H_\infty > 0$. The short and long term sections of the table are independent. "Slow" and "fast" are meant in comparison to Lévy models. We are also assuming $\alpha_0 \neq \beta_0$.

5.1. The dataset. We consider all quoted monthly S&P 500 and the EURO STOXX 50 option closing prices: these are the two most liquid equity indices worldwide. We consider the CBOE European options (option prices are reported by the U.S.A. Options Price Reporting Authority).[5] The most liquid options expire on the third Friday of the first six months after the value date and then on March, June, September, and December in the front year and June and December in the next year. Contracts expiring on June and December of the following years are available for the EURO STOXX 50. In sections 5.2 and 5.3, we present the calibration results on the 24$^{\text{th}}$ of October 2018, a month after the September quadruple witching Friday. The name quadruple witching Friday refers to the simultaneous expiration of stock options, index futures, and index futures options derivatives contracts. To show that our calibration results are robust and that calibrated parameters are stable we include also an analysis on all business days of the 3 months that follow the 24$^{\text{th}}$ of October 2018 in Section 5.4. The dataset provides call/put bid and ask prices for each available maturity. We implement three standard data pre-processing liquidity criteria. First, we discard the so-called "penny options", i.e. options at a very low price: all options whose value is below 0.1 S&P 500 or EURO STOXX 50 index points fall within this class. Second, we filter out options with a wide bid-ask spread i.e. options with a ratio ask-bid/ask larger than 60%. This second criterion excludes also all strikes for which either bid or ask prices for call and put options are not available. Finally, we consider only options with delta in the range 10%-90%. Options outside this range are usually not traded with a significant volume. In tables 3 and 4, we report some descriptive statistic of the option dataset for the S&P 500 and the EURO STOXX 50 surfaces respectively. In particular, we report, for every expiry date, the number of options, the average, minimum and maximum strike, the average and variance of OTM options prices. We report the statistics for OTM option prices because we calibrate only on OTM calls and puts.

We derive the risk-free interest rate and forward prices following the methodology described in [4]. This approach allows us to obtain the implicit interest rates and forward prices using only option prices and a no-arbitrage condition on an option portfolio known as *synthetic* forward: a synthetic forward is the price of a call option minus the price of a put option with the same strike. Synthetic forwards are well synchronized with option prices and usually more reliable than quoted futures (see e.g., [65, 43]).

[5] Eikon Reuters option chain is *0#SPX*.U* for the S&P 500 and *STXE*.EX* for the EURO STOXX 50.

Expiry	# options	avg strike	min strike	max strike	avg OTM price	var OTM price
16-Nov-2018	15	2608.3	2400	2800	26.5	261.6
21-Dec-2018	24	2564.6	2250	2875	33.5	455
18-Jan-2019	28	2537.5	2175	2900	39.9	604.2
15-Feb-2019	33	2524.2	2100	2950	45.1	793.5
15-Mar-2019	37	2498	2025	2975	49.5	966.1
21-Jun-2019	48	2459.4	1850	3075	61.9	1565.3
20-Sep-2019	56	2442.9	1725	3150	72.7	2148.5
20-Dec-2019	64	2416.4	1600	3225	78.8	2686.4
17-Jan-2020	66	2416.3	1575	3250	80.9	2854.5
19-Jun-2020	74	2390.5	1450	3350	95.1	3854.1
18-Dec-2020	78	2392.3	1375	3500	113.3	4976.8

Table 3

Descriptive statistic of the option dataset for the S&P 500 *surface.*

Expiry	# options	avg strike	min strike	max strike	avg OTM price	var OTM price
16-Nov-2018	16	3037.5	2850	3225	29.7	261.9
21-Dec-2018	21	3050	2700	3350	46.2	538.5
18-Jan-2019	25	2989	2575	3375	49.8	833.1
15-Feb-2019	33	2953	2475	3425	51.1	1096.4
15-Mar-2019	38	2917.8	2400	3450	54.9	1306
19-Apr-2019	13	3034.6	2875	3200	106.4	476.3
21-Jun-2019	19	2871.1	2150	3450	72.3	2297.9
20-Sep-2019	30	2816.7	2000	3550	71.8	2583.9
20-Dec-2019	28	2819.6	2000	3600	86.9	3278
19-Jun-2020	28	2821.4	2000	3650	108.4	3956.3
18-Dec-2020	18	2961.1	2450	3550	166.3	3825.6
18-Jun-2021	21	2978.6	2400	3550	169.6	3989.8
17-Dec-2021	13	2969.2	2400	3600	186.3	3979

Table 4

Descriptive statistic of the option dataset for the EURO STOXX 50 *surface.*

5.2. Calibration. We calibrate the GBA model on OTM calls and puts following the procedure reported by [19], Ch.14, pages 464-465. We divide the volatility surface into maturity cross-sections, each one containing options with the same maturity, and calibrate each slice separately. We calibrate on OTM put and call.

In Algorithm 5.1, we report the slice-by-slice calibration procedure for the GBA. For each different maturity, the three time-dependent parameters $\alpha(\tau), \beta(\tau), \sigma(\tau)$ are obtained by minimizing numerically the squared differences between model and market prices. The calibration is performed imposing conditions (**A1**)–(**A4**). In particular, we need to check that $\alpha(\tau)/\sigma(\tau)$ and $\beta(\tau)/\sigma(\tau)$ are non increasing and $\sigma(\tau)$ is non decreasing. Notice that we start to verify these monotonicity conditions only from the second maturity because for the first maturity we can always assume that they are verified. We compute option prices in the GBA framework with the closed formulae in (3.5)–(3.6). We point out that, thanks to these closed formulae, the calibration of the GBA process is very fast, up to 50 times faster than the alternatives.

Alongside the GBA process, we consider the calibration of two exponential Lévy models: the normal inverse Gaussian (NIG) and the generalized beta exponential Lévy model (GB). Furthermore, we calibrate four additive models: the ATS NIG model and the ATS VG model

Algorithm 5.1 Slice-by-slice calibration algorithm of the GBA model

M number of maturities, N_s number of OTM options in the s^{th} maturity.
O_{sj}^{mkt}, O_{sj}^{mdl} market and model OTM option prices for the j^{th} strike and the s^{th} maturity.
Calibrate $\alpha(\tau_1), \beta(\tau_1), \sigma(\tau_1)$ minimizing $\sum_{j=1}^{N_1} \frac{(O_{1j}^{mkt} - O_{1j}^{mdl})^2}{N_1}$ conditional to (**A3**)–(**A4**).
for $s = 2 : M$ **do**
  Calibrate $\alpha(\tau_s), \beta(\tau_s), \sigma(\tau_s)$ minimizing $\sum_{j=1}^{N_s} \frac{(O_{sj}^{mkt} - O_{sj}^{mdl})^2}{N_s}$ conditional to (**A1**)–(**A4**). (**A3**)–(**A4**) are self-explicative while (**A1**)–(**A2**) are equivalent to $\alpha(\tau_s)/\sigma(\tau_s) \leq \alpha(\tau_{s-1})/\sigma(\tau_{s-1})$, $\beta(\tau_s)/\sigma(\tau_s) \leq \beta(\tau_{s-1})/\sigma(\tau_{s-1})$ and $\sigma(\tau_s) \geq \sigma(\tau_{s-1})$. The minimization is performed with the MATLAB function `fmincon`.
end for

in [5], the four-parameter Sato model proposed by [16], and the CPDA model of [17]. The ATS model is regarded as a benchmark since, up to our knowledge, it is the additive process with the best calibration performance in equity market (see e.g., [5, 60]). For LTS and Sato models, the parameters are obtained through a global calibration of the whole volatility surface. For these models, other than for CPDA for which a closed form formula exists, call option prices are determined using the formula of [53].

$$C(S_0, K, \tau) = S_0 - \frac{\sqrt{S_0\,K} e^{-\tau r/2}}{2\pi} \int_{-\infty}^{\infty} e^{-iz\,k} \phi^c\left(z - \frac{i}{2}\right) \frac{1}{z^2 + \frac{1}{4}} dz, \tag{5.1}$$

where $\phi^c(u)$ is the characteristic function of the log-process. For the models above the computation of option prices on the grid of market strikes has been accelerated using the Fast Fourier Transform technique.

The calibration performance for the date of interest is reported in Table 5 in terms of Mean Squared Error (MSE) and Mean Absolute Percentage Error (MAPE). We define the MSE and the MAPE below for OTM put and call:

$$\text{MSE} := \frac{1}{M} \sum_{s=1}^{M} \frac{1}{N_s} \sum_{j=1}^{N_s} (O_{sj}^{mkt} - O_{sj}^{mdl})^2, \qquad \text{MAPE} := \frac{1}{M} \sum_{s=1}^{M} \frac{1}{N_s} \sum_{j=1}^{N_s} \frac{|O_{sj}^{mkt} - O_{sj}^{mdl}|}{O_{sj}^{mkt}}, \tag{5.2}$$

where M is the number of maturities, N_s is the number of options (OTM puts and calls) for maturity s, O_{sj}^{mkt} is the mid market price of the j^{th} OTM option of the s^{th} maturity and O_{sj}^{mdl} is the corresponding model price. The GBA model outperforms the Lévy processes by two orders of magnitude in terms of MSE and one order of magnitude in terms of MAPE both in the case of the S&P 500 and the EURO STOXX 50 volatility surface. GBA also performs better than the selfsimilar NIG, ATS VG and CPDA models and is outperformed only by the ATS NIG model. We again stress that thanks to the closed formulae for option prices, calibrating the GBA model is significantly faster than all the alternatives (from 1/10 to 1/50 of the ATS calibration time). The GBA and ATS mean absolute percentage errors are below the average bid-ask in the option prices in the day of interest (4.6% in the S&P 500 case and 6.36% in the EURO STOXX 50 case).

In Figure 6, we plot the market implied volatility and the implied volatility extracted from the GBA, CPDA, ATS, Lévy GBA, NIG, and ATS NIG models for 3 weeks (left panel) and 2 years (right panel) maturities for the S&P 500 index. In Figure 7 the same comparison is drawn for the EURO STOXX 50 index.

We observe that the GBA model calibrates very well the implied volatility level and skew with an especially marked improvement in the short time. Let us notice that the short maturity skew

Surface	S&P 500		EURO STOXX 50	
	MSE	MAPE	MSE	MAPE
GBA	**9.53**	**3.96%**	**9.87**	**2.00%**
CPDA	513.22	30.43%	294.77	18.24%
ATS NIG	0.20	0.68%	6.69	1.31%
ATS VG	10.68	4.32%	11.15	2.32%
Lévy GB	175.19	21.93%	137.20	15.56%
Lévy NIG	130.75	21.52%	137.20	16.35%
Selfsimilar NIG	10.45	4.55%	16.28	4.55%

Table 5

Calibration performance for the S&P 500 *and the* EURO STOXX 50 *indices in terms of* MSE *and* MAPE.

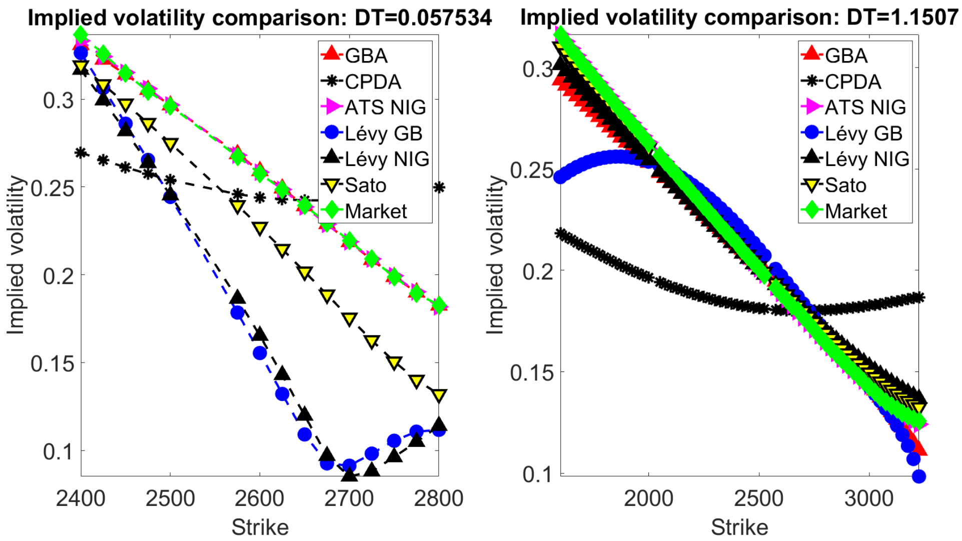

Figure 6. *Implied volatility smile of* S&P 500 *for a given time to maturity: three weeks (left panel) and 2 years (right panel).*

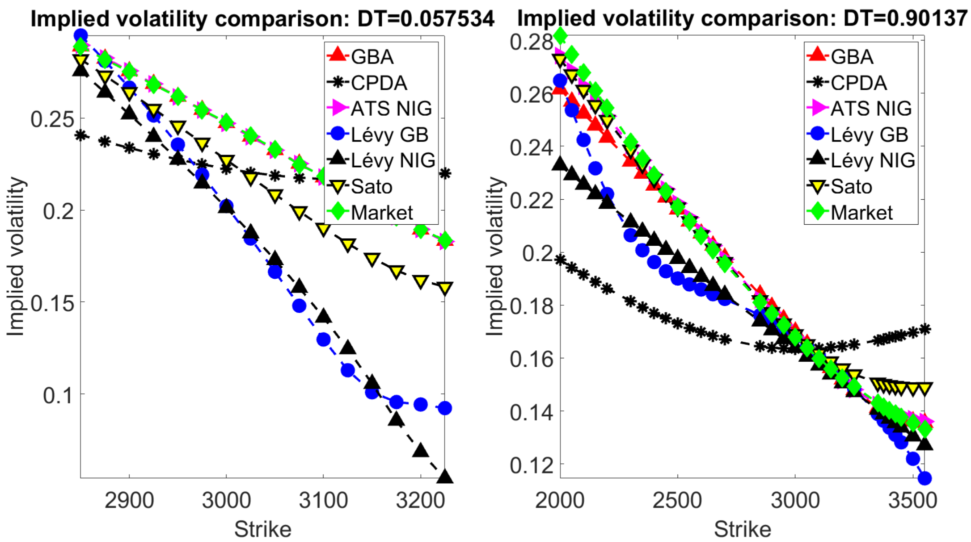

Figure 7. *Implied volatility smile of* EURO STOXX 50 *for a given time to maturity: three weeks (left panel) and 3.5 years (right panel).*

of volatility surface is significantly more evident in the short time and that the level of volatility does not seem to go to zero. Overall ATS and GBA models outperform the other considered models at both short and long maturities.

5.3. Term structure specification. Slice-by-slice calibration achieves a nearly perfect match of the implied volatility surface, but the number of parameters increases with the number of option maturities: on average, we have 13 liquid maturities, and this implies around 39 parameters. For most applications, which include pricing and hedging European and discretely monitored products (whose monitoring dates coincide with traded option maturities) slice-by-slice calibration is a reliable choice. It reproduces very well market prices and leaves the trader with just 3 parameters (or risk factors) for every maturity that can be easily interpreted with the level and the left and right distributional asymmetries. However, when using the calibrated model as an input for pricing and simulation on maturities for which options are not available or for continuously monitored products the problem becomes under-specified, as it is left undetermined which actual term structures are being used. A reasonable alternative is instead to specify functional forms α, β, σ and calibrate the free parameters, so that those parametrized functions can then be passed to pricing and hedging routines. This can be done consistently with all the various assumptions needed in the previous sections for the theoretical results to hold.

We introduce here four GBA model class specifications. One reduced-form model, with the lowest possible number of parameters (reduced form GBA, R-GBA), a model which captures the different time decay/explosions of volatility skew and level (double scaling GBA, DS-GBA), another one which can exhibit the skew torsion (GBA with torsion, T-GBA), and, finally, a fourth model combining both torsion and double time scaling (DST-GBA).

As it appears clear from the various discussions in section 4 the main parameters are the power law for σ and the asymptotic values of α and β at 0 and ∞. The following choices for σ, α, β achieve the features discussed.

(**R-GBA**)

$$\alpha(t) = \alpha_0, \qquad \beta(t) = \beta_0 + \sigma(t), \quad \sigma(t) = \sigma_0 t^{H_0},$$

(**DS-GBA**)

$$\alpha(t) = \alpha_0, \qquad \beta(t) = \beta_0 + \sigma(t), \qquad \sigma(t) = \frac{\arctan\left(\sigma_0 \frac{\pi}{2} t^{H_0}\right)}{\arctan\left(\frac{\pi}{2\sigma_\infty} t^{-H_\infty}\right)},$$

(**T-GBA**)

$$\alpha(t) = \alpha_\infty + \frac{\alpha_0 - \alpha_\infty}{1+\sigma(t)}, \qquad \beta(t) = \beta_\infty + \frac{\beta_0 - \beta_\infty}{1+\sigma(t)} + \sigma(t), \qquad \sigma(t) = \sigma_0 t^{H_0},$$

(**DST-GBA**)

$$\alpha(t) = \alpha_\infty + \frac{\alpha_0 - \alpha_\infty}{1+\sigma(t)}, \qquad \beta(t) = \beta_\infty + \frac{\beta_0 - \beta_\infty}{1+\sigma(t)} + \sigma(t) \quad \sigma(t) = \frac{\arctan\left(\sigma_0 \frac{\pi}{2} t^{H_0}\right)}{\arctan\left(\frac{\pi}{2\sigma_\infty} t^{-H_\infty}\right)}.$$

The parameter ranges are $\alpha_0, \beta_0, \alpha_\infty, \beta_\infty, \sigma_0, \sigma_\infty, H_0, H_\infty > 0$.

Proposition 5.1. *All the models above satisfy assumptions* (**A1'**), (**A2**)–(**A3**), (**AN**) *and* (**LTA**). *We have $\alpha_0 > 0$, $\beta_0 > 0$ and $\sigma(t) \sim \sigma_0 t^{H_0}$ as $t \sim 0$ for all the models. In the* R-GBA *and* DS-GBA *model we have the asymptotics*

$$\alpha(t) \to \alpha_0, \qquad \beta(t) - \sigma(t) \to \beta_0, \qquad t \to \infty, \tag{5.3}$$

whereas in the T-GBA *and the* DST-GBA *models it holds*

$$\alpha(t) \to \alpha_\infty, \qquad \beta(t) - \sigma(t) \to \beta_\infty, \qquad t \to \infty. \tag{5.4}$$

Moreover in the R-GBA *and* T-GBA *models it is*

$$\sigma(t) \sim \sigma_0 t^{H_0}, \qquad t \to \infty, \tag{5.5}$$

while in the DS-GBA *and the* DST-GBA *models we have*

$$\sigma(t) \sim \sigma_\infty t^{H_\infty}, \qquad t \to \infty. \tag{5.6}$$

By detaching the long and short term limits of the term structures we are able to differentiate the asymptotics of the long and short term section of the volatility structure and corollaries 4.3, 4.5, 4.11, and 4.13 all apply independently to separate leading orders.

The number of parameters for the models above are respectively 4, 6, 6 and 8. Generally speaking the power law parameters are model selection parameters as these determine the term structure decay/explosion, and since these are characteristics observed in markets and asset classes, H needs to be selected accordingly. For example as discussed in Section 4.1, $H_0 = 1/2$ in equity markets is the only possible choice consistent with an explosive OTM and positive ATM implied volatility levels. Likewise, we argued in Section 4.4 that $H_\infty < 1/3$ is consistent with a long term skew decay slower than Lévy (and other) models (see also Table 2). When these choices are made the number of parameters drops to 3 for the R-GBA model, 4 for the DS-GBA model, 5 for the T-GBA model and 6 for the DTS-GBA model.

We also consider a further specification of the GBA model, namely a generic term structure for σ which is linked (at least in the short time) with the term structure of the implied volatility and same β and α as R-GBA. For this reason, we call this instantiation the R-GBA model with level (R-GBA-L). The parameter functions are then given as follows

$$\text{(R-GBA-L)} \qquad \alpha(t) = \alpha_0, \qquad \beta(t) = \beta_0 + \sigma(t)$$

with σ satisfying the general GBA assumptions, but otherwise unspecified. A similar specification can be introduced for the other parametric term structure choices in this section, but we will not discuss those. Once the term structure of volatility has been taken into account this model specification has only two free parameters (α_0 and β_0) with a clear financial interpretation (the left and right skew).

We calibrate the GBA model instantiations on the S&P 500 and the EURO STOXX 50 volatility surfaces. In Table 6, we report MSE and MAPE for the GBA calibrated slice-by-slice the generalized beta Lévy, R-GBA, R-GBA-L, DS-GBA, T-GBA and DST-GBA. In Figure 8, we plot the market implied volatility and the volatility replicated via the models reported in the table for the 3 weeks (on the left) and 2 years (on the right) maturities for the S&P 500. It seems that the five GBA specifications, even if slightly less accurate than the GBA calibrated-slice-slice, are replicating very well the implied volatility smile with fewer parameters. As expected, we lose some accuracy with respect to the slice-by-slice calibration but we still replicate very well the implied volatility and skew.

Among the proposed GBA subcases, the reduced form R-GBA shows the worst performance, while still calibrating much better than the GB-based Lévy model. Adding torsion parameters (T-GBA) produces only a negligible improvement, which is expected since the data considered do not exhibit skew torsion. Such specification is thus not warranted for the data set analyzed. A sizeable error reduction is instead observed when taking into account a term structure for σ, i.e. the R-GBA-L model, as an effect of essentially proceeding slice-by-slice for $\sigma(t)$. What we find interesting though is that the DS-GBA model is not that far behind the R-GBA-L model, which is a clear indication that the double scaling functional specification introduced improves the fit of the volatility term structure. Again, the torsion parameters in the DST-GBA do not significantly improve on the DS-GBA calibration. All in all, introducing double scaling (DS, DS-T) reduces the error of the corresponding single scaling models (R, R-T) by 30% up to 50%.

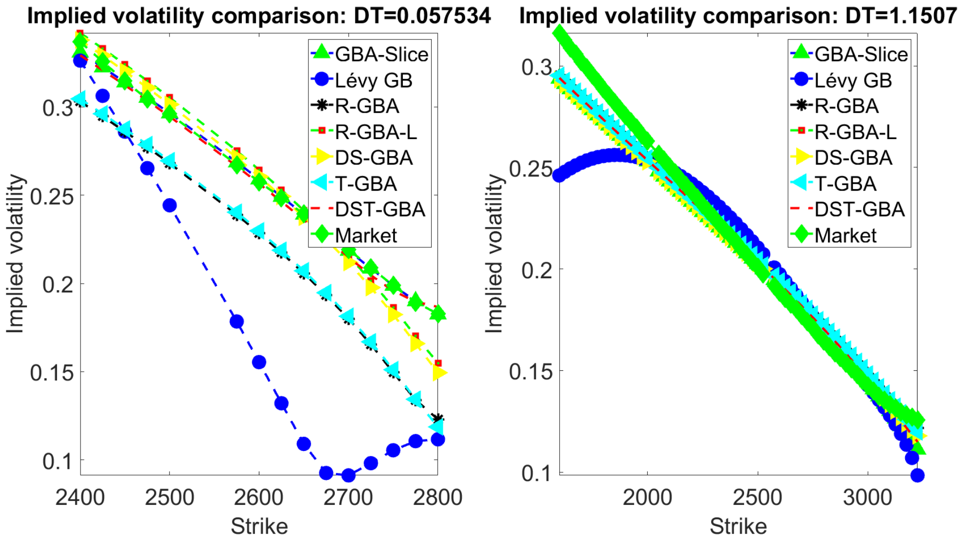

Figure 8. *Implied volatility smile of* S&P 500 *for a given time to maturity: three weeks (left panel) and 2.5 years (right panel).*

Surface	S&P 500		EURO STOXX 50	
	MSE	MAPE	MSE	MAPE
GBA Slice	**9.53**	**3.96%**	**9.78**	**2.00%**
Lévy GB	175.19	21.93%	137.20	15.56%
R-GBA	18.29	6.8%	19.42	6.01%
R-GBA-L	10.42	4.57%	19.83	3.28%
DS-GBA	11.25	4.77%	20.58	3.51%
T-GBA	17.87	6.72%	27.81	5.81%
DST-GBA	10.84	4.26%	12.66	2.93%

Table 6

Calibration performance for the S&P 500 *and the* EURO STOXX 50 *indices in terms of* MSE *and* MAPE*.*

The small cost in terms of calibration error while using parametric specifications instead of slice-by-slice calibration is outweighed by the drastic parameter reduction and increased model clarity. The parameter term structures introduced retain a clear financial interpretation, namely the time rates of the volatility level and of the right and left skew asymmetry. The specifications introduced provide an accurate, flexible and parsimonious alternative to the classic slice-by-slice additive calibration.

In Table 7, we report the calibrated parameters of the 5 GBA specifications (for the R-GBA-L we do not report the term structure of volatility) for both volatility surfaces. Notably, in all models $\beta_0 > \alpha_0$ and (where present) $\beta_\infty > \alpha_\infty$ consistently with the equity negative skew. We also point out that H_0 is reasonably close to 0.5, as implied by the foregoing discussion and consistent with the equity markets. Another empirical test we offer (Figure 9) is a comparison between market skewness and excess kurtosis and the corresponding quantities from the GBA slice-by-slice, R-GBA and DST-GBA calibrations. Market higher moments are extracted by calibrating a GB distribution on the single maturities without imposing additivity constraints. This methodology has been previously applied by [49] to the Variance Gamma process (pages 89-91). Both market skewnesss and excess kurtosis appear to be oscillating around a constant value. The GBA model calibrated slice-by-slice matches very well the market moments while the R-GBA and the DST-GBA appear very similar between each other and stable at the average values of the observed moments.

Surface	S&P 500								EURO STOXX 50							
	α_0	α_∞	β_0	β_∞	σ_0	σ_∞	H_0	H_∞	α_0	α_∞	β_0	β_∞	σ_0	σ_∞	H_0	H_∞
R-GBA	0.02	–	0.12	–	0.01	–	0.49	–	0.24	–	0.71	–	0.04	–	0.49	–
R-GBA-L	0.02	–	0.12	–	–	–	–	–	0.21	–	0.59	–	–	–	–	–
DS-GBA	0.07	–	0.54	–	0.01	2.11	0.22	0.40	0.21	–	0.58	–	0.02	0.93	0.34	0.38
T-GBA	0.08	0.00	0.83	0.00	0.02	–	0.49	–	0.26	0.00	0.78	0.00	0.02	–	0.47	–
DST-GBA	0.08	0.00	47.75	12.66	0.01	2.41	0.23	0.37	0.12	0.00	0.72	1.73	0.02	2.29	0.35	0.57

Table 7

Calibrated GBA *parameters for the* S&P 500 *and the* EURO STOXX 50 *indices.*

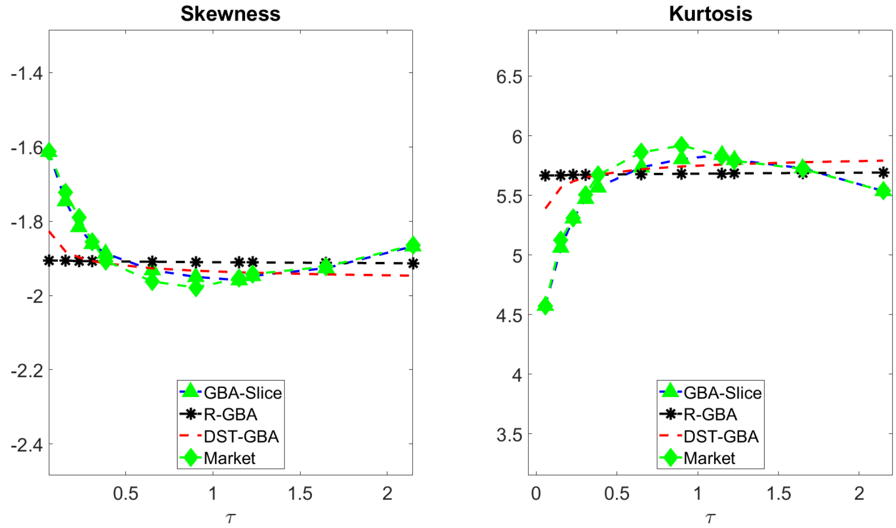

Figure 9. *Skewness and excess kurtosis for* S&P 500 *in terms of the time-to-maturity.*

5.4. Robustness analysis. To verify the robustness of the calibration results and the parameters stability of the GBA specifications we recalibrate all the considered models in all business days of a three-month dataset that goes from the 24$^{\text{th}}$ of October 2018 to the 24$^{\text{th}}$ of January 2019. We consider the same calibration methodologies described in Section 5.2.

In Table 8, we present the average calibration performance for the S&P 500 and the EURO STOXX 50 indices in terms of MSE and MAPE for the three-month period from the 24$^{\text{th}}$ of October 2018 to the 24$^{\text{th}}$ of January 2019. We report the calibration results for the GBA calibrated slice-by-slice, the CPDA, the ATS NIG and VG, the Lévy NIG, and the selfsimilar NIG. The calibration results, in terms of order of magnitudes, are very similar to those presented in Table 5: the GBA model substantially outperforms the Lévy processes in terms of MSE and MAPE and performs better than the selfsimilar NIG, ATS VG and CPDA models while is outperformed only by the ATS NIG model.

In Table 9, we report the average calibration performance in the three-month period for all the GBA specifications introduced in Section 5.3. Also in this case, results are comparable to those of Table 6. The reduced form R-GBA shows the worst performance, while still calibrating much better than the Lévy model. DS-GBA and R-GBA-L perform better than the T-GBA model (in terms of RMSE and MAPE for the EURO STOXX 50 and in terms of MAPE for the S&P 500). Moreover, we confirm that introducing double scaling (DS, DS-T) reduces the error of the corresponding single scaling models (R, R-T) by 30% up to 40% in terms of MAPE.

In figures 10-13, we plot the time-evolution of the R-GBA, T-GBA, DS-GBA and DST-GBA parameters for the three-month period that goes from the 24$^{\text{th}}$ of October 2018 to the 24$^{\text{th}}$ of January 2019. For every model, we report the S&P 500 parameters on the left and the EURO STOXX 50 ones on the right. In all cases accounted the parameters α_0, β_0, σ_0 and H_0 are rather stable. In particular, α_0 is always below β_0, which reflects into a small maturity negative

Surface	S&P 500		EURO STOXX 50	
	MSE	MAPE	MSE	MAPE
GBA	**7.74**	**2.22%**	**4.30**	**1.76%**
CPDA	407.76	26.50%	289.62	18.32%
ATS NIG	2.91	1.13%	4.20	1.34%
ATS VG	21.57	4.43%	13.34	3.03%
Lévy GB	113.40	18.40%	104.15	14.34%
Lévy NIG	92.30	16.67%	95.97	13.81%
Selfsimilar NIG	5.99	4.26%	9.13	3.93%

Table 8

Average calibration performance for the S&P 500 and the EURO STOXX 50 *indices in terms of* MSE *and* MAPE *for the three-month period from the* 24^{th} *of October 2018 to the* 24^{th} *of January 2019.*

Surface	S&P 500		EURO STOXX 50	
	MSE	MAPE	MSE	MAPE
GBA Slice	**7.74**	**2.22%**	**4.30**	**1.76%**
Lévy GB	113.40	18.40%	137.20	15.56%
R-GBA	19.43	4.44%	17.05	5.20%
R-GBA-L	15.23	3.96%	12.39	3.16%
DS-GBA	12.04	3.60%	13.22	3.45%
T-GBA	8.22	4.63%	15.50	5.07%
DST-GBA	9.47	3.31%	6.55	3.02%

Table 9

Average calibration performance for the S&P 500 *and the* EURO STOXX 50 *indices in terms of* MSE *and* MAPE *for the three-month period from the* 24^{th} *of October 2018 to the* 24^{th} *of January 2019.*

skew. The parameter H_0 oscillates around 0.5, which is consistent with a finite and positive short time ATM implied volatility. As mentioned, the torsion parameters α_∞ and β_∞ are not particularly relevant in this framework because the equity implied volatility surface does not normally exhibit a torsion of the skew. They are typically close to zero. Finally, the double scaling parameters σ_∞ and H_∞ are still relatively stable but oscillate more than their short time counterparts α_0 and H_0. However, in DS-GBA and DST-GBA, these two parameters have a low impact on model prices for the available range of maturities. For this reason, the higher variance of σ_∞ and H_∞ has a negligible impact on the model Greeks and thus should not affect hedging costs. Regarding H_∞, the values can be either higher or lower 0.33, and hence in accordance to the discussion after Corollary 4.13, our study does not find evidence of a slow decay of the implied volatility skew, as mentioned e.g. by the important study [33]. This might be evidence of difficulties with calibrating to possibly illiquid large maturity option prices, or suggest that the square root decay of large maturity skew deduced in [33] may not be a universal stylized fact.

Interestingly, the short time parameters α_0, β_0, σ_0 and H_0 do not vary too much from the R-GBA ones, when considering the T-GBA, DS-GBA and DST-GBA specifications, which is further evidence that these are the most relevant for model calibration.

6. Conclusion: model or methodology?.

Inspired by the original idea of [17], in this paper we have proposed a simple and yet realistic closed-form option pricing model, challenging the decades old popular belief that the Samuelson-Black-Scholes model is the only model supporting analytical option prices. The supporting dynamics for the pricing formula follow an exponential of an additive process with GL type IV marginal laws. Additive processes are statistically sound as returns processes and can be efficiently simulated. Since option prices are explicit, the full

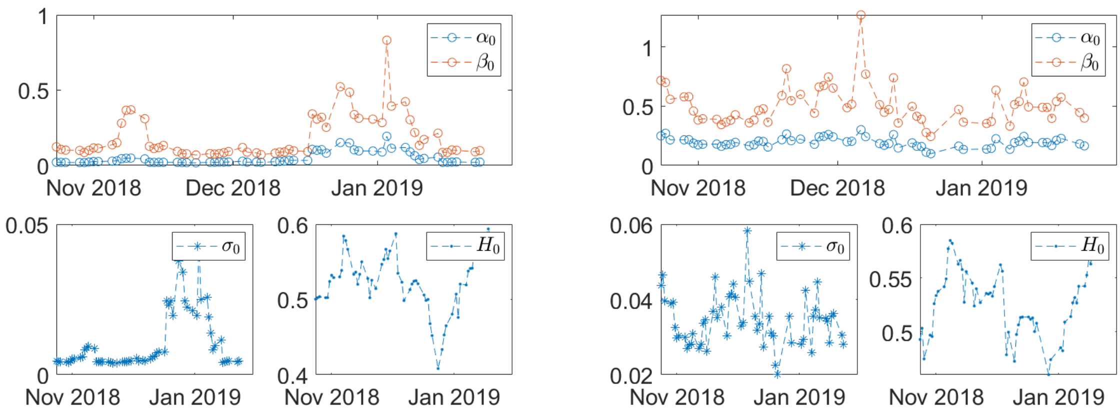

Figure 10. *Time-evolution of the* R-GBA *parameters for the three-month period from the* 24^{th} *of October 2018 to the* 24^{th} *of January 2019. We report the* S&P 500 *parameters on the left and the* EURO STOXX 50 *ones on the right.*

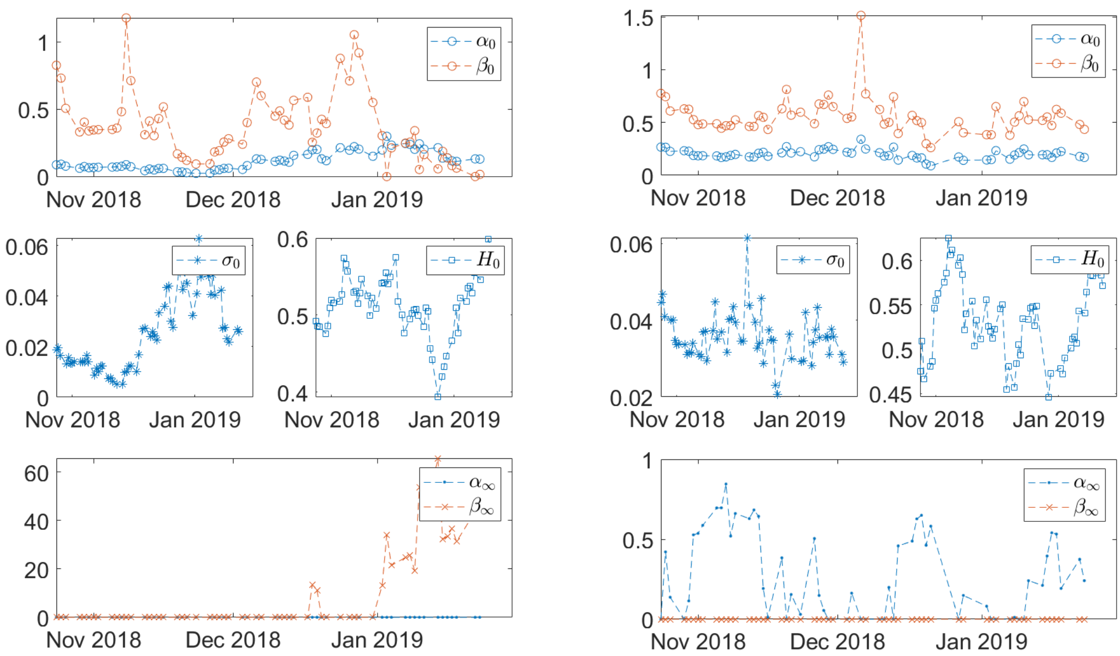

Figure 11. *Time-evolution of the* T-GBA *parameters for the three-month period from the* 24^{th} *of October 2018 to the* 24^{th} *of January 2019. We report the* S&P 500 *parameters on the left and the* EURO STOXX 50 *ones on the right.*

range of asymptotics of the implied volatility surface can be characterized: in particular, the small and large maturity implied volatility level and skew for any range of moneyness, as well as the large strike slopes. The additive specification results in a full analytical theory of the implied volatility surface, and makes the role of the parameters clear and intuitive, with one dispersion time rate and two skewness term functions. The GBA model allows great flexibility for model calibration: the term functions can be either pre-specified in parametric form and then the parameters fitted to option prices, or calibrated slice-by-slice and the resulting values interpolated. Moreover, the various asymptotic results can readily substitute numerical methods at extreme maturity ranges, where instabilities may occur. The model calibrates accurately at

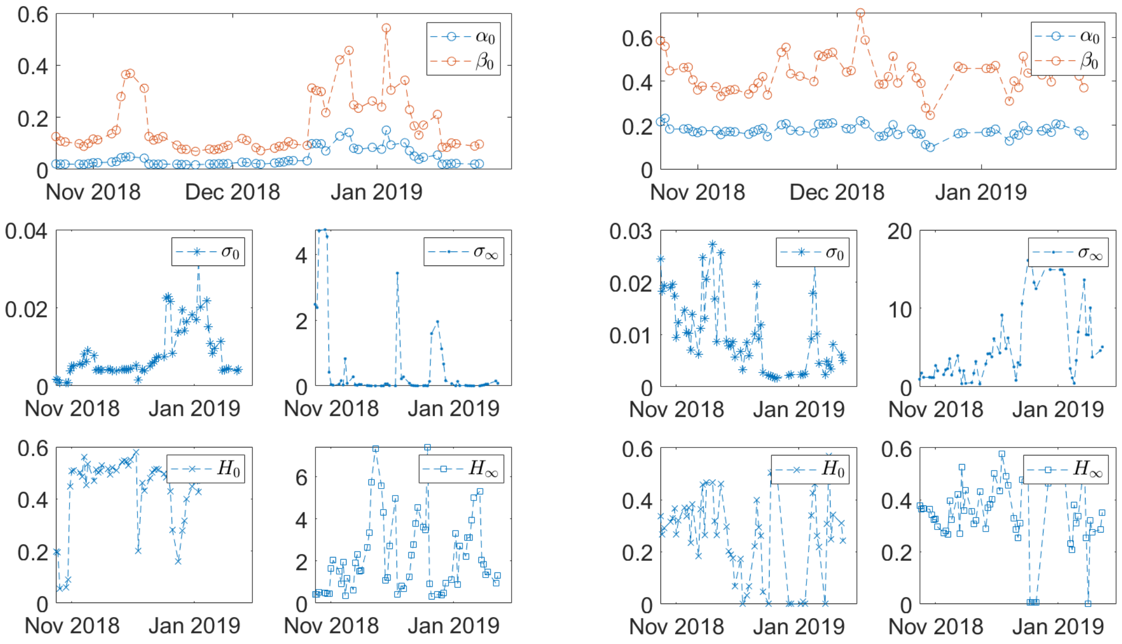

Figure 12. *Time-evolution of the* DS-GBA *parameters for the three-month period from the* 24^{th} *of October 2018 to the* 24^{th} *of January 2019. We report the* S&P 500 *parameter on the left and the* EURO STOXX 50 *ones on the right.*

near Black-Scholes computational time, far outperforming both Lévy models and models based on Sato (self-similar with independent increments) processes.

Although much more empirical testing is certainly required to assess the quality of the model, it is our hope to have presented a full picture of how additive models can give rise to simple, explicit, and yet powerful solutions for option pricing and volatility surface calibration.

As a concluding comment we briefly discuss the possibility that the route followed for the introduction of the GBA model is not specific to beta/logistic type laws but can indeed be generalized to many other distributions. It is in fact possible to recast the modelling idea expressed thus far in a slightly different way, that might uncover its general character.

Assume we have a scale-location (μ, σ) probability distribution class with CDF $F_{\mu,\sigma}$ and density function $f_{\mu,\sigma}$. A skew-transformation of a distribution is mechanism for generating a skewed distribution by functionally manipulating either $F_{\mu,\sigma}$ or $f_{\mu,\sigma}$ in such a way as to obtain a skewed CDF/PDF . The simplest and most common way of doing so is using a distortion mapping $x \mapsto x^\alpha$, $\alpha > 0$, in which case $G_{\alpha,\mu,\sigma} = (F_{\mu,\sigma})^\alpha$ is clearly again a CDF and it determines a skewed probability law. However, more sophisticated types of skew-transformation exist. Another one is Beta skew-generation, or order-statistic based skewing, popularized by [48], which amounts to composing the distribution given by $f_{\mu,\sigma}$ with a Beta(α, β) PDF. The generated distribution will have density $g_{\alpha,\beta,\mu,\sigma}$ given by

$$g_{\alpha,\beta,\mu,\sigma}(x) = \frac{1}{B(\alpha,\beta)} f_{\mu,\sigma}(x) F_{\mu,\sigma}(x)^{\alpha-1}(1 - F_{\mu,\sigma}(x))^{\beta-1}, \qquad x \in \mathbb{R}. \tag{6.1}$$

or, even more simply, in terms of CDF

$$G_{\alpha,\beta,\mu,\sigma}(x) = I_{F_{\mu,\sigma}(x)}(\alpha, \beta), \qquad x \in \mathbb{R}. \tag{6.2}$$

In the case $f_{\mu,\sigma} = f^L$ it is easy to verify that $g_{\mu,\sigma} = f^{GL}$. In other words the GL distribution can be looked at as the Beta-skew transform of the logistic distribution. We can thus reinterpret the

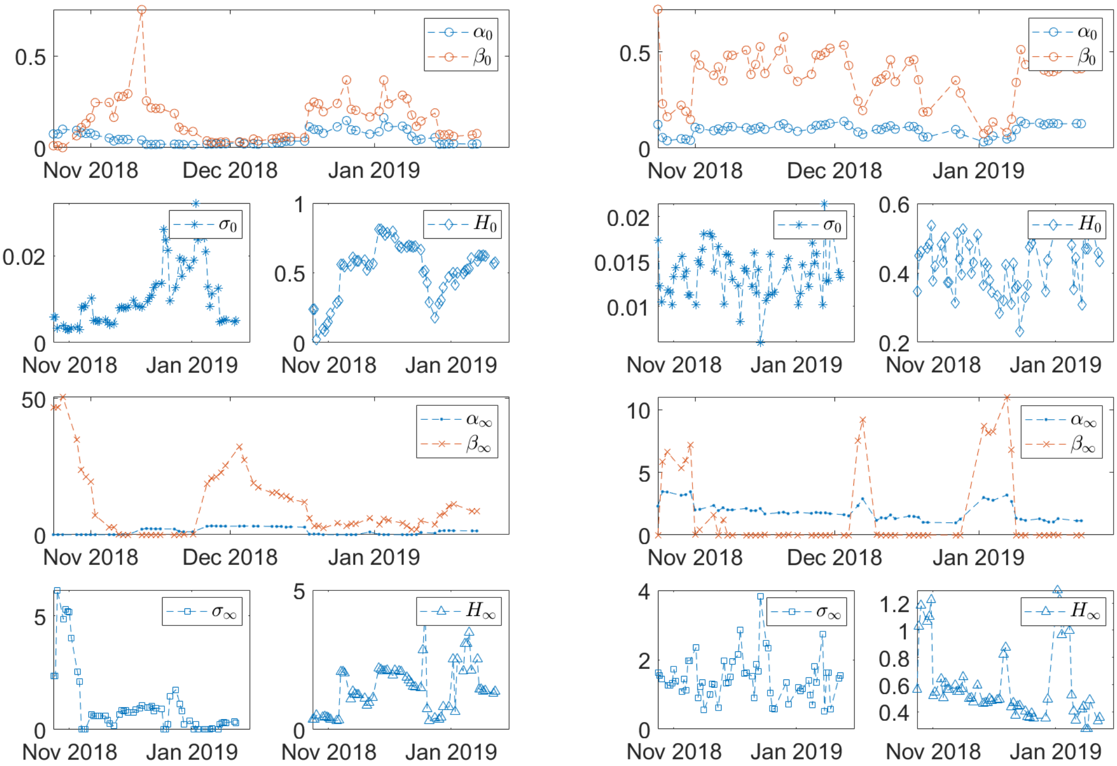

Figure 13. *Time-evolution of the* DST-GBA *parameters for the three-month period from the* 24^{th} *of October 2018 to the* 24^{th} *of January 2019. We report the* S&P 500 *parameter on the left and the* EURO STOXX 50 *ones on the right.*

GBA model (upon the usual verification using [70], Theorem 9.8) as one obtained by skewing a symmetric logistic process with scale $\sigma(t)$ with a time dependent $B(\alpha(t), \beta(t))$ function. This view of the GBA model clearly separates the model in two components: a variability driver, represented by an infinitely-divisible underlying law $F_{\mu,\sigma}$, being fed to a skewing mechanism. It is now apparent that by replacing F^L with some other infinitely-divisible family of scale-location symmetric laws other than the logistic distribution, new candidate asset pricing dynamics can be introduced within the additive paradigm we followed throughout. As in the GBA model, such putative marginal log-price laws will have a variance/term structure rate σ and two skew functions α, β, while the location will be fixed by martingale constraints. For example we could take as underlying generating laws the Normal or Student-t laws (see [48]).

However compelling, such construction faces two challenges. In first place, one must verify that the Beta-skewed variable is still infinitely divisible. For the GL case this has been essentially established by [42], as we made clear. However, there are instances in which the Beta-skewed function fails to be infinitely divisible, for example in the Normal case for some range of values for α and β (see [66]). Proving infinite-divisibility of a probability law can be very difficult. Secondly, one must still verify the conditions for the existence of a supporting additive process to be verified, although this is conceivably a less complex task given the freedom we have in choosing the term functions.

The Beta-skewing argument reveals that the set of techniques used to build the GBA model can be potentially adapted for generating several other additive models with an explicit option pricing formula and implied volatility surface. This is simply achieved by replacing the logistic distribution with some other symmetric, leptokurtic, infinitely-divisible law, performing a scale-location change, and finally Beta-skewing the resulting law to generate the log-price distribution. In the authors' view this methodological idea represents a promising way forward in option pricing research.

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Appendix: proofs.

Proof of Proposition 3.1. Assumptions (**A1**)–(**A3**), make it possible to apply Proposition 4.1[6] of [17], establishing the existence of X. Then S_t has finite expectation for all t because of (**A4**). Recall that for a semimartingale $Z = (Z_t)_{t\geq 0}$ and $z \in D \subseteq \mathbb{C}$, the Fourier cumulant process $\Psi^Z(z) = (\Psi^Z_t(z))_{t\geq 0}$ of Z is the unique predictable process such that $\exp(izZ - \Psi^Z(z))_{t\geq 0}$ is a local martingale, for all values $z \in D$. When Z is additive then Ψ^Z is deterministic and $\Psi^Z_t(z) = \log(\varphi_{Z_t}(z))$. Let now $\tilde{X}_t \sim \mathrm{GL}(0, \sigma(t), \alpha(t), \beta(t))$ so that $\mu^*(t) = \Psi^{\tilde{X}}_t(-i) < \infty$ and $X_t = \tilde{X}_t - \mu^*(t)$, which implies that $S_0 \exp(X) = (e^{-rt}S_t)_{t\geq 0}$ is a local martingale. But since $\Psi^{\tilde{X}}$ is deterministic then $\mathbb{E}[\exp(\tilde{X}_t)] = e^{\mu^*(t)}$ and thus $(e^{-rt}S_t)_{t\geq 0}$ is a positive local martingale with constant expectation S_0, hence a true martingale. See [45] for details.

Proof of Corollary 3.2. Follows from Proposition 3.1 after differentiating (2.23) with respect to t, according to what is explained in [70], Remark 9.9.

Proof of Proposition 3.3. Let us start from (3.6).
Let $X_\tau = \log S_\tau$, $\tilde{X}_\tau \sim \mathrm{GL}(0, \sigma(\tau), \alpha(\tau), \beta(\tau))$. We have from the Fundamental Theorem of Asset Pricing, and straightforward manipulations with the GL PDF, that

$$
\begin{aligned}
P(S_0, K, \tau) &= \mathbb{E}[e^{-r\tau}(K - S_\tau)^+] = \mathbb{E}\left[e^{-r\tau}(K - S_0 e^{X_\tau})\mathbb{1}_{\{X_\tau < \log \frac{K}{S_0}\}}\right] \\
&= e^{-r\tau} K \mathbb{Q}\left(\tilde{X}_\tau < \log \frac{K}{S_0} + \mu^*(\tau) - r\tau\right) - S_0 \mathbb{E}\left[e^{\tilde{X}_\tau}\mathbb{1}_{\{\tilde{X}_\tau < \log \frac{K}{S_0} + \mu^*(\tau) - r\tau\}}\right] \\
&= e^{-r\tau} K \Phi\left(-d_{S_0,K,\tau}, \alpha(\tau), \beta(\tau), \sigma(\tau)\right) \\
&\qquad - S_0 \int_{-\infty}^{F^L(-d_{S_0,K,\tau};0,\sigma(\tau))} e^x f^{GL}(x; 0, \sigma(\tau), \alpha(\tau), \beta(\tau)) dx \\
&= e^{-r\tau} K \Phi\left(-d_{S_0,K,\tau}; \alpha(\tau), \beta(\tau), \sigma(\tau)\right) - \\
&\qquad \frac{S_0}{\sigma(t) B(\alpha(\tau), \beta(t))} \int_{-\infty}^{F^L(-d_{S_0,K,\tau};0,\sigma(\tau))} \frac{e^{x + \frac{\alpha(\tau)}{\sigma(\tau)} x}}{\left(1 + e^{\frac{x(\tau)}{\sigma(\tau)}}\right)^{\alpha(\tau)+\beta(\tau)}} dx
\end{aligned}
\tag{6.3}
$$

and (3.5) follows after recognizing $f^{GL}(x; 0, \sigma(\tau), \alpha(\tau) + \sigma(\tau), \beta(\tau) - \sigma(\tau))$ in the integrand of the right hand side of the last line of (6.3). Using put-call parity and the symmetry property

[6]Equation (4.2) in the statement of such result misses a term $\mu(t)$ also not considered in the proof as it should. Compare with (2.22).

(2.13) of $I_z(a,b)$ together with $1-(1+e^{-x})^{-1}=(1+e^{x})^{-1}$ we obtain (3.5). Formulae (3.8) and (3.7) follow from the analogous probabilistic manipulations applied to the payoffs $\mathbb{1}_{\{S_T>K\}}$ and $\mathbb{1}_{\{S_T<K\}}$ respectively. ∎

Proof of Proposition 3.4. Points (i), (ii) and (iii) are obvious from (2.5) and (2.6) and the continuity of the Digamma function. For (iv) we recall that the Polygamma functions have series expansion

$$\psi^{(n)}(z)=(-1)^{n+1}n!\sum_{k=0}^{\infty}\frac{1}{(z+k)^{n+1}},\qquad z\notin\mathbb{Z}_{\le 0} \tag{6.4}$$

and first order asymptotic expansion

$$\psi^{(n)}(z)\sim(-1)^{n-1}\frac{(n-1)!}{z^n},\qquad |\arg(z)|<\pi \tag{6.5}$$

(see [2], 6.4.10 and 6.4.11). Using the leading orders in (6.4) and (6.5) in respectively (2.5) and (2.6) we obtain instead

$$\mathrm{Skew}[X_t]=\frac{-2\alpha(t)^{-3}-O(1)}{(\alpha(t)^{-2}+O(1))^{3/2}}\to-2,\quad \mathrm{Kurt}[X_t]=\frac{6\alpha(t)^{-4}+O(1)}{(\alpha(t)^{-2}+O(1))^{2}}\to 6, \tag{6.6}$$

i.e. (iv). For (v) the conclusion is similar, except with the sign of the skewness switched. The case (vi) follows directly replacing (6.5) in (2.5) and (2.6). Regarding the last statement, (2.4) and (6.5) lead to

$$\mathrm{Var}[X_t]\sim\sigma(t)\left(\frac{1}{\alpha(t)}+\frac{1}{\beta(t)}\right), \tag{6.7}$$ ∎

which is nondecreasing because of (**A1**).

Proof of Proposition 4.1. Assume first $K=S_0$. By virtue of (**A1'**), with abuse of notation, let τ be the inverse function to σ, we can introduce the functions $\hat\alpha(\sigma):=\alpha(\tau(\sigma))$, $\hat\beta(\sigma):=\beta(\tau(\sigma))$ and $\hat\mu^*(\sigma)=\mu^*(\tau(\sigma))$. Using de l'Hôpital rule and continuity of ψ we obtain

$$\begin{aligned}
\lim_{\tau\to0}\frac{\mu^*(\tau)}{\sigma(\tau)}&=\lim_{\sigma\to0}\frac{\hat\mu^*(\sigma)}{\sigma}=\lim_{\sigma\to0}\frac{d}{d\sigma}\mu^*(\sigma)=\lim_{\sigma\to0}\frac{d}{d\sigma}\log\frac{B(\hat\alpha(\sigma)+\sigma,\hat\beta(\sigma)-\sigma)}{B(\hat\alpha(\sigma),\hat\beta(\sigma))}\\
&=\lim_{\sigma\to0}\frac{d}{d\sigma}\log\frac{\Gamma(\hat\alpha(\sigma)+\sigma)\Gamma(\hat\beta(\sigma)-\sigma)}{\Gamma(\hat\alpha(\sigma))\Gamma(\hat\beta(\sigma))}\\
&=\lim_{\sigma\to0}(\hat\alpha'(\sigma)+1)\psi(\hat\alpha(\sigma)+\sigma)-\hat\alpha'(\sigma)\psi(\hat\alpha(\sigma))\\
&\qquad+(\hat\beta'(\sigma)-1)\psi(\hat\beta(\sigma)-\sigma)-\hat\beta'(\sigma)\psi(\hat\beta(\sigma))=\psi(\alpha_0)-\psi(\beta_0),
\end{aligned} \tag{6.8}$$

having also used $\hat\alpha(0)=\alpha_0$, $\hat\beta(0)=\beta_0$ (recall that ψ is the Digamma function). We have

$$\lim_{\tau\to0}F^L(d_{S_0,S_0,\tau};0;\sigma(\tau))=\lim_{\sigma\to0}\frac{1}{1+e^{\hat\mu^*(\sigma)/\sigma}}=F^L(\psi(\beta_0)-\psi(\alpha_0);0,1). \tag{6.9}$$

Hence, using $X_\tau=\log(S_\tau/S_0)\sim\mathrm{GL}(-\mu^*(\tau),\sigma(\tau),\alpha(\tau),\beta(\tau))$ under $\mathbb{P}'$, it follows

$$\lim_{\tau\to0}\mathbb{P}'(S_\tau>S_0)=\lim_{\tau\to0}\mathbb{P}'(X_\tau\ge0)=\lim_{\tau\to0}\Phi(d_{S_0,S_0,\tau};\alpha(\tau),\beta(\tau),\sigma(\tau)) \tag{6.10}$$

and by (6.9) and the continuity of $I_z(a,b)$ in the first three arguments the first line of (4.3) is proved.

Assuming $K > S_0$, i.e $k > 0$, we see that $\frac{\mu^*(\tau)+k}{\sigma(\tau)} \to \infty$ as $\tau \to 0$. Then,

$$F^L(d_{K,S_0,\tau}; 0; \sigma(\tau)) = \frac{1}{1 + e^{\frac{\mu^*(\tau)+k}{\sigma(\tau)}}} \sim e^{-\frac{\mu^*(\tau)+k}{\sigma(\tau)}}, \tag{6.11}$$

having used $(1 + e^x)^{-1} = (1 + e^{-x})^{-1} e^{-x} \sim e^{-x}$, $x \to \infty$. Now based on (2.12), (2.14), (2.15), and recalling (2.13), we can write explicitly the following survival function

$$\mathbb{P}'(S_\tau \geq K) = \frac{F^L(d_{K,S_0,\tau}; 0; \sigma(\tau))^{\beta(\sigma(\tau))}}{B(\alpha(\sigma(\tau)), \beta(\sigma(\tau)))\Gamma(1 - \alpha(\sigma(\tau)))} \cdot \sum_{h=0}^{\infty} \frac{\Gamma(1 - \alpha(\sigma(\tau)) + h)}{\beta(\sigma(\tau)) + h} \frac{F^L(d_{K,S_0,\tau}; 0; \sigma(\tau))^h}{h!}. \tag{6.12}$$

Considering the first term of (6.12) we would have the asymptotic equivalence

$$\mathbb{P}'(S_t \geq K) \sim \frac{F^L(d_{K,S_0,\tau}; 0; \sigma(\tau))^{\beta(\sigma(\tau))}}{\beta(\sigma(\tau))B(\alpha(\sigma(\tau)), \beta(\sigma(\tau)))} \sim \frac{F^L(d_{K,S_0,\tau}; 0; \sigma(\tau))^{\beta_0}}{\beta_0 B(\alpha_0, \beta_0))}, \qquad \tau \sim 0, \tag{6.13}$$

and thus ultimately – by virtue of (6.11) and (6.8) – the second line of (4.3), if we can show that

$$\lim_{\sigma \to 0} \frac{1}{\Gamma(1 - \hat{\alpha}(\sigma))} \sum_{h=1}^{\infty} \frac{\Gamma(1 - \hat{\alpha}(\sigma) + h)}{\hat{\beta}(\sigma) + h} \frac{F^L(d_{K,S_0,\tau}; 0; \sigma)^h}{h!} = 0. \tag{6.14}$$

The terms under the series on the left hand side all tend to zero again by (6.11), and thus (6.14) holds if we can apply the dominated convergence theorem. In order to see that this is the case, fix $\bar{\sigma}$ such that $F^L(d_{K,S_0,\tau}; 0; \sigma) < c < 1$ and let $m_\alpha, m_\beta > 0$ respectively the minimum of $\hat{\alpha}$ and $\hat{\beta}$ on $[0, \overline{\sigma}]$. Choose then h_0 large enough so that $\Gamma(1 - \hat{\alpha}(\sigma) + h)$ is positive and increasing as a function of σ for all $h > h_0$. For all h as just stated and $\sigma \in (0, \overline{\sigma}]$ we then have

$$\frac{\Gamma(1 - \hat{\alpha}(\sigma) + h)}{\hat{\beta}(\sigma) + h} \frac{F^L(d_{K,S_0,\tau}; 0; \sigma)^h}{h!} < \frac{\Gamma(1 - m_\alpha + h)}{m_\beta + h} \frac{c^h}{h!}. \tag{6.15}$$

■

The right hand side is a summable sequence so that dominated convergence does apply. The first asymptotic equivalence in (4.3) is thus proven.

When $K < S_0$, we have $\frac{\mu^*(\tau)+k}{\sigma(\tau)} \to -\infty$. Furthermore, we recall $F^L(d_{K,S_0,\tau}; 0; \sigma(\tau)) = 1 - F^L(-d_{K,S_0,\tau}; 0; \sigma(\tau))$ then use (2.13) and repeat the argument of the $S_0 < K$ case to obtain the last line of (4.3). Equation (4.4) then follows from $\mathbb{P}'(S_\tau \leq K) = 1 - \mathbb{P}'(S_\tau \geq K)$.

Proof of Proposition 4.2. By (**A1'**) and (**A3**) we have $\hat{\alpha}(0) = \alpha_0 > 0$ and $\hat{\beta}(0) = \beta_0 > 0$. For notational convenience, in the following evaluation at $\sigma = 0$ or $\tau = 0$ will be understood as a left limit. Looking at (3.5) we have the explicit expression

$$\begin{aligned} C(S_0, K, \tau(\sigma)) =& S_0 \Big(\Phi(-\hat{\mu}^*(\sigma), \hat{\beta}(\sigma) - \sigma, \hat{\alpha}(\sigma) + \sigma, \sigma) - \Phi(-\hat{\mu}^*(\sigma), \hat{\beta}(\sigma), \hat{\alpha}(\sigma), \sigma) \Big) \\ =& S_0 \hat{C}(\sigma, \hat{\alpha}(\sigma), \hat{\beta}(\sigma)) \end{aligned} \tag{6.16}$$

for all $\sigma = \sigma(\tau)$. Observe further that $F_L(x; 0; 0) = 1$ so that also $\Phi(x, \mu, \alpha, \beta, 0) = 1$ and hence $C(S_0, S_0, 0) = \hat{C}(0, \alpha(0), \beta(0)) = 0$. Also by (**AN**), we have that $\hat{C}$ is a composition of analytical functions (recall that Φ is holomorphic in the complex unit disc) in a neighbourhood of 0. The Maclaurin expansion of $\hat{C}$ around $\sigma = 0$ thus reads

$$C(S_0, S_0, 0) = \hat{C}(\sigma, \hat{\alpha}(\sigma), \hat{\beta}(\sigma)) = S_0 c_{\alpha,\beta} \sigma + O(\sigma^2). \tag{6.17}$$

After substituting $\sigma = \sigma(\tau)$ we would have (4.6) once we show $c_{\alpha,\beta} > 0$, i.e. $c_{\alpha,\beta}$ is strictly positive as opposed to just nonnegative, which is not obvious. By applying the chain rule

$$\begin{aligned} c_{\alpha,\beta} = \frac{d}{d\sigma}\hat{C}(0,\alpha_0,\beta_0) =& \frac{\partial \hat{C}}{\partial \sigma}(0,\alpha_0,\beta_0) + \frac{\partial \hat{C}}{\partial \hat{\alpha}}(0,\alpha_0,\beta_0)\frac{d\hat{\alpha}}{d\sigma}(0) \\ &+ \frac{\partial \hat{C}}{\partial \hat{\beta}}(0,\alpha_0,\beta_0)\frac{d\hat{\beta}}{d\sigma}(0). \end{aligned} \tag{6.18}$$

First we show

$$\frac{\partial \hat{C}}{\partial \sigma}(0,\alpha_0,\beta_0) > 0. \tag{6.19}$$

To lighten notation we set $p(\sigma) = F^L(-\hat{\mu}^*(\sigma);0,\sigma)$, and rewrite

$$\begin{aligned} &\hat{C}(\sigma,\hat{\alpha}(\sigma),\hat{\beta}(\sigma)) = \frac{1}{B(\hat{\beta}(\sigma)-\sigma,\hat{\alpha}(\sigma)+\sigma)}\int_0^{p(\sigma)} z^{\hat{\beta}(\sigma)-\sigma-1}(1-z)^{\hat{\alpha}(\sigma)+\sigma-1}dz- \\ &\frac{1}{B(\hat{\beta}(\sigma),\hat{\alpha}(\sigma))}\int_0^{p(\sigma)} z^{\hat{\beta}(\sigma)-1}(1-z)^{\hat{\alpha}(\sigma)-1}dz = \Gamma(\hat{\beta}(\sigma)+\hat{\alpha}(\sigma))\int_0^{p(\sigma)} z^{\hat{\beta}(\sigma)-1}(1-z)^{\hat{\alpha}(\sigma)-1} \\ &\left(\left(\frac{1-z}{z}\right)^{\sigma}\frac{1}{\Gamma(\hat{\alpha}(\sigma)+\sigma)\Gamma(\hat{\beta}(\sigma)-\sigma)} - \frac{1}{\Gamma(\hat{\alpha}(\sigma))\Gamma(\hat{\beta}(\sigma))}\right)dz. \end{aligned} \tag{6.20}$$

Computing the derivative by the Leibniz rule yields

$$\begin{aligned} \frac{\partial \hat{C}}{\partial \sigma}(\sigma,\hat{\alpha}(\sigma),\hat{\beta}(\sigma)) = & \frac{\partial p(\sigma)}{\partial \sigma}\Gamma(\hat{\beta}(\sigma)+\hat{\alpha}(\sigma))p(\sigma)^{\hat{\beta}(\sigma)-1}(1-p(\sigma))^{\hat{\alpha}(\sigma)-1} \\ & \left(\left(\frac{1-p(\sigma)}{p(\sigma)}\right)^{\sigma}\frac{1}{\Gamma(\hat{\alpha}(\sigma)+\sigma)\Gamma(\hat{\beta}(\sigma)-\sigma)} - \frac{1}{\Gamma(\hat{\alpha}(\sigma))\Gamma(\hat{\beta}(\sigma))}\right) \\ +\Gamma(\hat{\beta}(\sigma)+\hat{\alpha}(\sigma)) & \int_0^{p(\sigma)} z^{\hat{\beta}(\sigma)-1}(1-z)^{\hat{\alpha}(\sigma)-1}\left(\frac{1-z}{z}\right)^{\sigma}\frac{1}{\Gamma(\hat{\alpha}(\sigma)+\sigma)\Gamma(\hat{\beta}(\sigma)-\sigma)} \\ & \Big(\log(1-z) - \log(z) + \psi(\hat{\beta}(\sigma)-\sigma) - \psi(\hat{\alpha}(\sigma)+\sigma)\Big)\,dz, \end{aligned} \tag{6.21}$$

where we recall ψ is the Digamma function. By assumptions (**A3**) and (**A4**) and continuity of the Beta and ψ functions, it follows that it exists $\bar{\sigma}, c_1, c_2, D_1, D_2 > 0$ such that for all $\sigma < \bar{\sigma}$ we have $\hat{\beta}(\sigma) - \sigma > c_1 > 0$ and $\hat{\alpha}(\sigma) + \sigma > c_2 > 0$ and $\frac{1}{\Gamma(\hat{\alpha}(\sigma)+\sigma)\Gamma(\hat{\beta}(\sigma)-\sigma)} < D_1$, $|\psi(\hat{\beta}(\sigma)-\sigma) - \psi(\hat{\alpha}(\sigma)+\sigma)| < D_2$. Now for all such σ and $z \in (0,1)$ we have

$$\begin{aligned} &\Bigg| z^{\hat{\beta}(\sigma)-1}(1-z)^{\hat{\alpha}(\sigma)-1}\left(\frac{1-z}{z}\right)^{\sigma}\frac{1}{\Gamma(\hat{\alpha}(\sigma)+\sigma)\Gamma(\hat{\beta}(\sigma)-\sigma)} \\ &\qquad \Big(\log(1-z) - \log z + \psi(\hat{\beta}(\sigma)-\sigma) - \psi(\hat{\alpha}(\sigma)+\sigma)\Big)\Bigg| \\ &< z^{c_1-1}(1-z)^{c_2-1}D_1(|\log z - \log(1-z)| + D_2), \end{aligned} \tag{6.22}$$

and the last expression in (6.22) is integrable in $[0,1]$. Let $p(0) := \lim_{\sigma\to 0} p(\sigma) = F^L(\psi(\beta_0) - \psi(\alpha_0);0,1)$, with the second equality following by 6.9. We can then apply the Dominated Convergence Theorem to conclude

$$\begin{aligned} \lim_{\sigma\to 0}\frac{\partial \hat{C}}{\partial \sigma}(\sigma,\hat{\alpha}(\sigma),\hat{\beta}(\sigma)) =& \frac{1}{B(\beta_0,\alpha_0)}\int_0^{p(0)} z^{\beta_0-1}(1-z)^{\alpha_0-1} \\ & (\log(1-z) - \log z + \psi(\beta_0) - \psi(\alpha_0))\,dz. \end{aligned} \tag{6.23}$$

We still need to show that the limit in (6.23) is positive which amounts to determining if $(\log(1-z)-\log(z)+\psi(\beta_0)-\psi(\alpha_0))>0$. To this end notice that $\log(1-z)-\log z$ is decreasing in $(0,p(0))$ and crucially $\log(1-p(0))-\log p(0)=\psi(\alpha_0)-\psi(\beta_0)$. Hence, the integral in (6.23) is strictly positive because it is the integral of a positive quantity on a finite interval. We have thus shown (6.19).

Because of (**AN**) to conclude the proof it is enough to show that

$$\lim_{\sigma\to 0}\frac{\partial \hat{C}}{\partial \hat{\alpha}}(\sigma,\hat{\alpha}(\sigma),\hat{\beta}(\sigma))=\lim_{\sigma\to 0}\frac{\partial \hat{C}}{\partial \hat{\beta}}(\sigma,\hat{\alpha}(\sigma),\hat{\beta}(\sigma))=0. \tag{6.24}$$

By the Leibniz rule

$$\begin{aligned}\frac{\partial \hat{C}}{\partial \hat{\alpha}}(\sigma,\hat{\alpha}(\sigma),\hat{\beta}(\sigma)) &= \frac{\partial p(\sigma)}{\partial\hat{\alpha}(\sigma)}\Gamma(\hat{\beta}(\sigma)+\hat{\alpha}(\sigma))p(\sigma)^{\hat{\beta}(\sigma)-1}(1-p(\sigma))^{\hat{\alpha}(\sigma)-1}\\
&\quad\left(\left(\frac{1-p(\sigma)}{p(\sigma)}\right)^{\sigma}\frac{1}{\Gamma(\hat{\alpha}(\sigma)+\sigma)\Gamma(\hat{\beta}(\sigma)-\sigma)}-\frac{1}{\Gamma(\hat{\alpha}(\sigma))\Gamma(\hat{\beta}(\sigma))}\right)\\
&+\int_0^{p(\sigma)} z^{\hat{\beta}(\sigma)-1}(1-z)^{\hat{\alpha}(\sigma)-1}\\
&\left[\left(\frac{1-z}{z}\right)^{\sigma}\frac{\Gamma'(\hat{\beta}(\sigma)+\hat{\alpha}(\sigma))-\psi(\hat{\alpha}(\sigma)+\sigma)}{\Gamma(\hat{\alpha}(\sigma)+\sigma)\Gamma(\hat{\beta}(\sigma)-\sigma)}-\frac{\Gamma'(\hat{\beta}(\sigma)+\hat{\alpha}(\sigma))-\psi(\hat{\alpha}(\sigma)+\sigma)}{\Gamma(\hat{\alpha}(\sigma))\Gamma(\hat{\beta}(\sigma))}\right.\\
&\left.+\log(1-z)\left(\frac{\Gamma(\hat{\beta}(\sigma)+\hat{\alpha}(\sigma))}{\Gamma(\hat{\alpha}(\sigma)+\sigma)\Gamma(\hat{\beta}(\sigma)-\sigma)}-\frac{\Gamma(\hat{\beta}(\sigma)+\hat{\alpha}(\sigma))}{\Gamma(\hat{\alpha}(\sigma))\Gamma(\hat{\beta}(\sigma))}\right)\right]dz.\end{aligned} \tag{6.25}$$

We can apply the Dominated Convergence Theorem using a bound similar to the one in (6.22). Taking the limit inside the integral we immediately obtain

$$\lim_{\sigma\to 0}\frac{\partial \hat{C}}{\partial \hat{\alpha}}(\sigma,\hat{\alpha}(\sigma),\hat{\beta}(\sigma))=0. \tag{6.26}$$

■

The same steps show that the partial derivative of $\partial\hat{C}/\partial\hat{\beta}$ vanishes in $\sigma=0$, finishing the proof.

Proof of Corollary 4.3. It is well-known (e.g. [68], Theorem 5.1) that as $\tau\sim 0$

$$\sigma_{BS}(0,\tau)\sim\frac{\sqrt{2\pi}C(S_0,S_0,\tau)}{S_0\sqrt{\tau}}. \tag{6.27}$$

■

Equation (4.9) follows by substituting (4.6) in (6.27).

Proof of Proposition 4.4. Assume the call is struck OTM i.e., introducing log-moneynesses, that $k:=\log(K/S_0)>0$. Equation (3.5) enjoys the familiar decomposition as spot-strike linear combination of digital-option values under the risk-neutral and share measure (compare with Remark 3.2) i.e.

$$C(S_0,K,\tau)=S_0\mathbb{Q}^*(S_\tau\geq K)-KDC(S_0,K,\tau), \tag{6.28}$$

where under $\mathbb{Q}^*$ we have $S_\tau\sim\mathrm{GB}(1/\sigma(t),S_0^{-\mu^*(t)},\alpha(t)+\sigma(t),\beta(t)-\sigma(t))$. Applying (4.3), with $\mathbb{P}'$ being equal to $\mathbb{Q}^*$ and $\mathbb{Q}$, we have

$$S_0\mathbb{Q}^*(S_\tau\geq K)\sim\frac{e^{-\frac{\beta(\tau)}{\sigma(\tau)}(\mu^*(\tau)+k)+\mu^*(\tau)}}{(\beta(\tau)-\sigma(\tau))B(\alpha(\tau)+\sigma(\tau),\beta(\tau)-\sigma(\tau))} \tag{6.29}$$

$$KDC(S_0,K,\tau)\sim\frac{e^{-\frac{\beta(\tau)}{\sigma(\tau)}(\mu^*(\tau)+k)}}{\beta(\tau)B(\alpha(\tau),\beta(\tau))}. \tag{6.30}$$

Moreover, let us notice that, by a dominated convergence theorem argument similar to the one applied in (6.14),

$$S_0\mathbb{Q}^*(S_\tau \geq K) - \frac{e^{-\frac{\beta(\tau)}{\sigma(\tau)}(\mu^*(\tau)+k)+\mu^*(\tau)}}{(\beta(\tau)-\sigma(\tau))B(\alpha(\tau)+\sigma(\tau),\beta(\tau)-\sigma(\tau))} \sim O\left(e^{-\frac{\beta_0+1}{\sigma(\tau)}(\mu^*(\tau)+k)}\right) \tag{6.31}$$

$$KDC(S_0,K,\tau) - \frac{e^{-\frac{\beta(\tau)}{\sigma(\tau)}(\mu^*(\tau)+k)}}{\beta(\tau)B(\alpha(\tau),\beta(\tau))} \sim O\left(e^{-\frac{\beta_0+1}{\sigma(\tau)}(\mu^*(\tau)+k)}\right). \tag{6.32}$$

Hence, in order to produce the asymptotics of $C(S_0,K,\tau)$ we can substitute (6.29)–(6.30) in the individual terms of (6.28) to yield the asymptotic equivalence, provided that $e^{-\frac{\beta_0+1}{\sigma(\tau)}(\mu^*(\tau)+k)}$ is of higher order compared to the linear combination of the asymptotics. Doing the algebra results in

$$\begin{aligned}
&C(S_0,K,\tau) \sim \\
&Ke^{-\frac{\beta(\tau)}{\sigma(\tau)}(\mu^*(\tau)+k)}\left(\frac{e^{\mu^*(\tau)}}{(\beta(\tau)-\sigma(\tau))B(\alpha(\tau)+\sigma(\tau),\beta(\tau)-\sigma(\tau))} - \frac{1}{\beta(\tau)B(\alpha(\tau),\beta(\tau))}\right) \\
&\sim Ke^{-\frac{\beta(\tau)}{\sigma(\tau)}(\mu^*(\tau)+k)}\left(\frac{\sigma(\tau)}{\beta(\tau)(\beta(\tau)-\sigma(\tau))B(\alpha(\tau),\beta(\tau))}\right) \sim \sigma(\tau)K\frac{e^{-\frac{\beta_0}{\sigma(\tau)}(\mu^*(\tau)+k)}}{\beta_0^2 B(\alpha_0,\beta_0)},
\end{aligned} \tag{6.33}$$

which is indeed of lower order compared to (6.31)–(6.32). After using the limit in (6.8), the expression above yields the first line of (4.10).

To obtain the OTM prices, apply put-call parity to write $C(S_0,K,\tau) = P(S_0,K,\tau)+S_0-K$ and repeat the same analysis of (6.33) using the decomposition in (3.6) as

$$P(S_0,K,\tau) = KDP(S_0,K,\tau) - S_0\mathbb{Q}^*(S_\tau \leq K) \tag{6.34}$$

and use (4.4) for the digital put price asymptotics. ■

Proof of Corollary 4.5. From [7], Proposition 1, improving [68], Theorem 5.1, we obtain, letting $O(S_0,K,\tau) = C(S_0,K,\tau) - (S_0-K)^+$ and $k := \log(K/S_0)$

$$\sigma_{BS}(S_0,K,\tau) \sim \frac{|k|}{\sqrt{-2\tau\log O(S_0,K,\tau)}}\left(1 - \frac{\log\left(c_{S_0,K}(-\log\left(c_{S_0,K}O(S_0,K,\tau)\right))^{3/2}\right)}{\log O(S_0,K,\tau)}\right)^{1/2}. \tag{6.35}$$

for some positive constant $c_{S_0,K}$ depending on S_0 and K. Recalling the first line of (4.10) we have for $K > S_0$

$$\begin{aligned}
-\log(c_{S_0,K}O(S_0,K,\tau)) &\sim -\log O(S_0,K,\tau) \\
&= \frac{k\,\beta_0}{\sigma(\tau)} - \log\sigma(\tau) - \log\left(\frac{Ke^{-D_{\alpha,\beta}\beta_0}}{\beta_0^2 B(\alpha_0,\beta_0)}\right) \sim \frac{k\,\beta_0}{\sigma(\tau)}
\end{aligned} \tag{6.36}$$

and thus by replacing (6.36) in (6.35) we see that it holds

$$\frac{|k|}{\sqrt{-2\tau\log O(S_0,K,\tau)}} \sim \left(\frac{k}{2\tau\beta_0\sigma(\tau)^{-1}}\right)^{1/2}. \tag{6.37}$$

Moreover, again from (6.36) we have

$$\log(c_{S_0,K}(-\log(c_{S_0,K}O(S_0,K,\tau)))^{3/2}) \sim \frac{3}{2}\log\left(\frac{\sigma(\tau)}{k\,\beta_0}\right). \tag{6.38}$$

■

Using (6.37) and (6.38) in (6.35) we arrive at the first determination of (4.11). The case $K < S_0$ is analogous, using the second line of (4.10).

Proof of Proposition 4.6. We assume first $K \neq S_0$. In [7], Equation (2.21), it is established that the leading order of the skew follows

$$\begin{aligned}\mathcal{S}(S_0, K, \tau) \sim & \frac{\operatorname{sgn}\Big(\log(K/S_0)\Big)}{K\sqrt{\tau}\,(-2\log O(S_0, K, \tau))^{1/2}} \\ & - \log(K/S_0)\frac{DO(S_0, K, \tau)}{\sqrt{\tau}O(S_0, K, \tau)(-2\log(c_{S_0,K}O(S_0, K, \tau)))^{3/2}},\end{aligned} \tag{6.39}$$

where $DO(S_0, K, \tau) = DC(S_0, K, \tau)\mathbb{1}_{\{K>S_0\}} + DP(S_0, K, \tau)\mathbb{1}_{\{K<S_0\}}$ is the OTM digital option value. Denote $\mathcal{S}_1(S_0, K, \tau)$ and $\mathcal{S}_2(S_0, K\,\tau)$ the two summands in (6.39). Let first be $K > S_0$. Using (6.36) in (6.39) it holds

$$\mathcal{S}_1(S_0, K, \tau) \sim \frac{1}{K}\left(\frac{\sigma(\tau)}{2\tau\log(K/S_0)\beta_0}\right)^{1/2}. \tag{6.40}$$

From the OTM call option asymptotics leading order (4.10) and the digital option in (4.3) we notice

$$\frac{DC(S_0, K, \tau)}{C(S_0, K, \tau)} \sim \frac{\beta_0}{K\sigma(\tau)} \tag{6.41}$$

and by substituting (6.41) and (6.36) in (6.39) we obtain

$$\mathcal{S}_2(S_0, K, \tau) \sim -\frac{1}{2K}\left(\frac{\sigma(\tau)}{2\tau\log(K/S_0)\beta_0}\right)^{1/2}. \tag{6.42}$$

Adding $\mathcal{S}_1$ and $\mathcal{S}_2$ the claim follows. The case $S_0 > K$ is identical upon replacing α_0 to β_0, put options to call options, and using the absolute log-moneyness. When $K = S_0$ we make use of the well-known ATM short maturity skew expansion, e.g. [40] Equation (3.2), i.e.

$$\mathcal{S}(S_0, S_0, \tau) = \sqrt{\frac{2\pi}{\tau}}(1/2 - DC(S_0, S_0, \tau)) - \frac{\sigma_{BS}(0, \tau)}{2} + O(\sigma_{BS}(0, \tau)^2\sqrt{\tau}) \tag{6.43}$$ ■

and the claim follows again directly from Proposition 4.1.

Proof of Corollary 4.7. Clearly $\Phi(D_{\alpha,\alpha}, \alpha_0, \alpha_0, 1) = I_{1/2}(\alpha_0, \alpha_0) = 1/2$ and therefore the only if part of (i) follow immediately from (4.13).

We show the if part of (i) and (iii) together. Negative skewness for an unimodal absolutely continuous random variable Y "usually" (counterexamples for general random variables are well-known, see e.g. [1]) implies the inequality

$$\mathbb{E}[Y] < m(Y) \leq M(Y) \tag{6.44}$$

where $m(Y)$ and $M(Y)$ are respectively the median and mode of Y. The works [74] and [59] make clear that for (6.44) to hold it is sufficient that the density f of Y is such that $f(a+x)-f(a-x)$, with $a = \mathbb{E}[Y]$ or $a = m(Y)$, changes sign at most once in $x > 0$. The direction of the sign change reflects the sign of the skewness, positive to negative implying negative skewness.

Let us consider $X \sim \mathrm{GL}(0, 1, \alpha, \beta)$. Assuming negative skewness, i.e. by (2.5) letting $\alpha < \beta$, we want to show that X satisfies (6.44) through the [59] criterion. Now, for all $a \in \mathbb{R}$ changes of sign of $f(a + x) - f(a - x)$ correspond to those of $\log f(a + x) - \log f(a - x)$. If f is the PDF of X, recalling (2.2) we see that

$$\log f(a + x) - \log f(a - x) = 2\alpha x + (\alpha + \beta)\log\left(\frac{1 + e^{a-x}}{1 + e^{a+x}}\right). \tag{6.45}$$

Observe now , for all $a \in \mathbb{R}$ that

$$\frac{d}{dx} \log\left(\frac{1+e^{a-x}}{1+e^{a+x}}\right) = -e^{a} \frac{2e^{2x}+e^{2x}+1}{(e^{a}+e^{x})(e^{a+x}+1)} < 0, \quad x > 0. \tag{6.46}$$

and

$$\frac{d^2}{dx^2} \log\left(\frac{1+e^{a-x}}{1+e^{a+x}}\right) = \frac{\sinh(a)\sinh(x)}{(\cosh(a)+\cosh(x))^2}, \quad x > 0, \tag{6.47}$$

which has the same sign of a. Hence the logarithm in (6.45) is strictly decreasing; moreover $\log\left(\frac{1+e^{a-x}}{1+e^{a+x}}\right)|_{x=0} = 0$. Therefore (6.45) can be seen as the difference of two positive functions, a straight line and a strictly increasing and convex function on $x > 0$ if $a < 0$ (or a concave function if $a > 0$). Moreover, the two functions are both 0 in $x = 0$. Therefore, their difference changes sign at most once. By the aforementioned results, we conclude that (6.44) is satisfied for X.

Now observe that

$$\Phi(D_{\alpha,\beta}, \beta_0, \alpha_0, 1) = \mathbb{Q}(X \geq \mathbb{E}[X]) \tag{6.48}$$

with X being a $\mathrm{GL}(0,1,\alpha_0,\beta_0)$ random variable, and $\alpha_0 < \beta_0$. The if part of (i) now follows, since if for some $\alpha^* \neq \beta^*$ we had that $\Phi(D_{\alpha,\beta}, \beta^*, \alpha^*, 1) = 1/2$ then by (6.48) would imply $\mathbb{E}[X] = D_{\alpha,\beta} = m(X)$, in contradiction with (6.44). Finally, by the inequality

$$\Phi(D_{\alpha,\beta}, \beta_0, \alpha_0, 1) = \mathbb{Q}(X \geq \mathbb{E}[X]) > \mathbb{Q}(X \geq m(X)) = 1/2, \tag{6.49}$$

continuity, and (4.13), we deduce that there must exist some τ_0 such that for all $\tau < \tau_0$, $\mathcal{S}(S_0, K, \tau) < 0$. The corresponding inequality when $\alpha_0 > \beta_0$ follows by reversing (6.44) for positively skewed random variables, and reasoning as above. This finishes the proof of (iii).

To show (ii), following the notation of Proposition 4.2 let $\widehat{DC}(\sigma, \hat{\alpha}(\sigma), \hat{\beta}(\sigma))$ be the ATM digital call price in σ coordinate. By (i) it is $\widehat{DC}(0, \alpha_0, \alpha_0) = 1/2$. Then expanding this function in Maclaurin series in σ and substituting it in (6.43) together with (4.9) when $\alpha = \beta$ in (4.7), we have, as $\sigma \sim 0$

$$\begin{aligned}
&S(S_0, S_0, \tau(\sigma)) \\
&= -\sqrt{\frac{2\pi}{\tau(\sigma)}}\sigma\left(\frac{d}{d\sigma}\widehat{DC}(0,\alpha_0,\alpha_0) + \frac{1}{2}\frac{d}{d\sigma}\hat{C}(0,\alpha_0,\alpha_0)\right) + O\left(\frac{\sigma^2}{\sqrt{\tau(\sigma)}}\right) \\
&= -\sqrt{\frac{2\pi}{\tau(\sigma)}}\sigma\left(\frac{d}{d\sigma}\widehat{DC}(0,\alpha_0,\alpha_0) + \frac{1}{2}\frac{d}{d\sigma}\hat{\mathbb{Q}}^*(S_\tau > 1) - \frac{1}{2}\frac{d}{d\sigma}\widehat{DC}(0,\alpha_0,\alpha_0)\right) \\
&\qquad + O\left(\frac{\sigma^2}{\sqrt{\tau(\sigma)}}\right)
\end{aligned}$$

$$= -\sqrt{\frac{2\pi}{\tau(\sigma)}}\frac{\sigma}{2}\frac{d}{d\sigma}\widehat{CC}(0,\alpha_0,\alpha_0) + O\left(\frac{\sigma^2}{\sqrt{\tau(\sigma)}}\right), \tag{6.50}$$

with $\widehat{CC}$ denoting the covered call price in σ coordinates (see (4.21)). Now (ii) follows by setting $a_\sigma = \frac{d}{d\sigma}\widehat{CC}(0,\alpha_0,\alpha_0)$. We notice that a_σ cannot be negative, otherwise we would have a calendar-arbitrageable covered call. However, we could not prove its strict positivity for all choices of α: it is not possible to reason as in Proposition 4.2, as the partial derivatives in $\hat{\alpha}, \hat{\beta}$ of $\widehat{CC}$ do not vanish in 0.

Proof of Proposition 4.8. Observe that p_+, p_- are well-defined because of (2.18) and (**A4**); crucially, the inequality in (**A4**) is strict. We now prove that $\log f^{GL}(\cdot)$, $\log f^{GL}(-\cdot)$ are regularly varying of index 1. For all $\lambda, \sigma, \alpha, \beta > 0$, using de l'Hôpital rule, we have

$$\begin{aligned} \lim_{x\to\infty} \frac{\log f^{GL}(\lambda x; \mu,\sigma,\alpha,\beta)}{f^{GL}(x;\mu,\sigma,\alpha,\beta)} &= \lim_{x\to\infty} \frac{-\log(B(\alpha,\beta)\sigma) + \alpha\frac{\lambda x-\mu}{\sigma} - (\alpha+\beta)\log(1+e^{\frac{\lambda x-\mu}{\sigma}})}{-\log(B(\alpha,\beta)\sigma) + \alpha\frac{x-\mu}{\sigma} - (\alpha+\beta)\log(1+e^{\frac{x-\mu}{\sigma}})} \\ &= \frac{\lambda\alpha/\sigma - \frac{\lambda(\alpha+\beta)}{\sigma}}{\alpha/\sigma - \frac{\alpha+\beta}{\sigma}} = \frac{\lambda\beta/\sigma}{\beta/\sigma} = \lambda. \end{aligned} \tag{6.51}$$

so the GL PDF is log-regularly varying of index 1 at ∞. The same arguments applies to show that $\lim_{x\to-\infty} \frac{\log f^{GL}(\lambda x;\mu,\sigma,\alpha,\beta)}{f^{GL}(x;\mu,\sigma,\alpha,\beta)} = \lambda$. Then, from [11], Theorems 1.1-1.2 and using $\log(1+e^x) \sim x$ as $x \to \infty$, $\log(1+e^x) \to 0$ as $x \to -\infty$ we obtain

$$\begin{aligned} p_+ &= \lim_{k\to\infty} -\frac{\log f^{GL}(k; -\mu^*(\tau), \sigma(\tau), \alpha(\tau), \beta(\tau))}{k} \\ &= \lim_{k\to\infty} \frac{\log(B(\alpha,\beta)\sigma)}{k} - \alpha(\tau)\frac{1+\mu^*(\tau)k^{-1}}{\sigma(\tau)} \\ &\quad + (\alpha(\tau)+\beta(\tau))\frac{1+\mu^*(\tau)k^{-1}}{\sigma(\tau)} = \frac{\beta(t)}{\sigma(\tau)} \end{aligned} \tag{6.52}$$

$$\begin{aligned} p_- &= \lim_{k\to-\infty} -\frac{\log f^{GL}(k; -\mu^*(\tau), \sigma(\tau), \alpha(\tau), \beta(\tau))}{k} \\ &= \lim_{k\to\infty} \frac{\log(B(\alpha,\beta)\sigma)}{k} - \alpha(\tau)\frac{1+\mu^*(\tau)k^{-1}}{\sigma(\tau)} \\ &= -\frac{\alpha(\tau)}{\sigma(\tau)}. \end{aligned} \tag{6.53}$$

■

Proof of Corollary 4.9. It is easily checked that (4.20) holds if and only if $p_+ + p_- \geq 1$. By Proposition 4.8 this is the case if and only if $\beta(\tau) - \alpha(\tau) \geq \sigma(\tau)$.

The following Lemma is auxiliary to the proof of Proposition 4.10.

Lemma 6.1. *Let $k = \log(K/S_0)$ and*

$$p_\pm(k,\tau) := F^L(\mp d_{S_0,K,\tau}; 0, \sigma(\tau)) = F^L(\mp(k+\mu^*(\tau)); 0, \sigma(\tau)). \tag{6.54}$$

*In a GBA model satisfying (**A1**)-(**A4**) and (**LTA**) we have that for all $K > 0$, $c \in \mathbb{R}$:*

$$\begin{aligned} p_\pm(k,\tau)^{\sigma(\tau)+c} &\sim \left(\frac{1}{2}\right)^{\sigma(\tau)+c} (e^{k+\mu^*(\tau)})^{\mp 1/2} \\ &\sim \left(\frac{1}{2}\right)^{\sigma(\tau)+c} \left(e^k \frac{\Gamma(\beta_\infty)}{\Gamma(\alpha_\infty)} \sigma(\tau)^{\alpha_\infty-\beta_\infty}\right)^{\pm 1/2} \end{aligned} \tag{6.55}$$

as $\tau \to \infty$, and thus, in particular

$$p_\pm(k,\tau) \to 1/2, \qquad \tau \to \infty. \tag{6.56}$$

Proof of Lemma 6.1. First, we show that under (**LTA**)

$$\lim_{\tau\to\infty} \frac{\mu^*(\tau)+k}{\sigma(\tau)} = 0. \tag{6.57}$$

By [2], equation (6.147), for $a, b > 0$ as $z \to \infty$

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left(1 + O(z^{-1})\right), \tag{6.58}$$

hence in view of (3.1) and (**LTA**) and applying (6.58) and continuity of the Euler's Gamma

$$\lim_{\tau\to\infty}\frac{\mu^*(\tau)+k}{\sigma(\tau)}=\lim_{\tau\to\infty}\frac{\mu^*(\tau)}{\sigma(\tau)}=\lim_{\tau\to\infty}\frac{1}{\sigma(\tau)}\log\left(\frac{\Gamma(\sigma(\tau)+\alpha(\tau))\Gamma(\beta(\tau)-\sigma(\tau))}{\Gamma(\alpha(\tau))\Gamma(\beta(\tau))}\right)=$$

$$=\lim_{\tau\to\infty}\frac{1}{\sigma(\tau)}\log\left(\frac{\Gamma(\beta_\infty)}{\Gamma(\alpha_\infty)}\frac{\Gamma(\sigma(\tau)+\alpha_\infty)}{\Gamma(\sigma(\tau)+\beta_\infty)}\right)$$

$$=\lim_{\tau\to\infty}\frac{1}{\sigma(\tau)}\log\left(\frac{\Gamma(\beta_\infty)}{\Gamma(\alpha_\infty)}\sigma(\tau)^{\alpha_\infty-\beta_\infty}\right)=0. \tag{6.59}$$

Consider now

$$\log p_\pm(k,\tau)^{\sigma(\tau)+c}=-(\sigma(\tau)+c)\log\left(1+\exp\left(\pm\frac{\mu^*(\tau)+k}{\sigma(\tau)}\right)\right) \tag{6.60}$$

and let $f(x)=\log(1+e^x)$. Observe $f'(x)=F^L(x;0;1)$ and expand f in its Maclaurin series at the first order. We have that $f(0)=\log 2$, $f'(0)=1/2$ and hence as $\tau>>0$ by the previous part we deduce

$$\log p_\pm(K,t)^{\sigma(\tau)+c}=-(\sigma(\tau)+c)\left(\log 2\pm\frac{\mu^*(\tau)+k}{2\sigma(\tau)}+O(\sigma(\tau)^{-2})\right)$$

$$=-(\sigma(\tau)+c)\log 2\mp\frac{\mu^*(\tau)+k}{2}+O(\sigma(\tau)^{-1}). \tag{6.61}$$

By exponentiating the equation above the first asymptotic relation in (6.55) follows.

Now from (3.1) and applying (**LTA**) and (6.58) produces

$$e^{\mu^*(\tau)}=\frac{\Gamma(\alpha(\tau)+\sigma(\tau)\Gamma(\beta(\tau)-\sigma(\tau))}{\Gamma(\alpha(\tau))\Gamma(\beta(\tau))}\sim\frac{\Gamma(\beta_\infty)}{\Gamma(\alpha_\infty)}\frac{\Gamma(\sigma(\tau)+\alpha_\infty)}{\Gamma(\sigma(\tau)+\beta_\infty)}$$

$$=\frac{\Gamma(\beta_\infty)}{\Gamma(\alpha_\infty)}\sigma(\tau)^{\alpha_\infty-\beta_\infty}+O(\sigma(\tau)^{-1}) \tag{6.62}$$

∎

having used (6.58) in the second asymptotic equivalence. Substituting in (6.61) yields the second equivalence (6.55).

Proof of Proposition 4.10. Recall $\Phi(\mp d_{S_0,K,\tau};a,b,c)=I_{p_\pm(k,\tau)}(a,b,c)$ for all $a,b,c>0$. Using expression (6.54) in (2.12) we have

$$I_{p_-(k,\tau)}(\beta(\tau),\alpha(\tau))$$

$$=p_-(k,\tau)^{\beta(\tau)}\frac{\Gamma(\alpha(\tau)+\beta(\tau))}{\Gamma(\beta(\tau))\Gamma(1-\alpha(\tau))\Gamma(\alpha(\tau))}\sum_{j=0}^{\infty}\frac{\Gamma(1-\alpha(\tau)+j)}{\beta(\tau)+j}\frac{p_-(k,\tau)^j}{j!}. \tag{6.63}$$

Based on the limit (6.56), let $\tau_0>0$ be such that for all $\tau>\tau_0$ it is $p_-(k,\tau)<2/3$. Moreover, let us define $\bar{j}=\lceil 1+\sup_{\mathbb{R}_+}\alpha(\tau)\rceil$ the smallest integer exceeding the supremum of α, which is finite by (**LTA**). Then, for all $\tau>\tau_0$ and $j>\bar{j}$ it holds

$$\left|\Gamma(1-\alpha(\tau)+j)\frac{p_-(k,\tau)^{j+\beta(\tau)}}{(\beta(\tau)+j)j!}\right|\le\frac{\Gamma(1+j)}{j!}\left(\frac{2}{3}\right)^j=\left(\frac{2}{3}\right)^j. \tag{6.64}$$

The first inequality comes from the positivity of $\beta(\tau)$ and the fact that $\Gamma(z)$ is increasing for all $z>2$, which is the case whenever $j>\bar{j}$. Besides, recalling that $\Gamma(z+j)/\Gamma(z)=\prod_{i=0}^{j-1}(z+i)$ reveals that for all j the left hand side of (6.64) is bounded as a function of τ, and thus in particular for $j\le\bar{j}$. Hence denote with $g(\tau,j)$ the left hand side of (6.64) and set $M=\sup_{\mathbb{R}_+\times\{j<\bar{j}\}}g(\tau,j)$: for all $\tau>\tau_0$ we have

$$g(\tau,j)<M\mathbb{1}_{\{j<\bar{j}\}}+\left(\frac{2}{3}\right)^j\mathbb{1}_{\{j\ge\bar{j}\}} \tag{6.65}$$

and the right hand side is a summable sequence. Therefore, we can apply the Dominated Convergence Theorem and take the limit on τ inside the series in (6.63). Recall now that the binomial series for $a>0$, $|z|<1$, is given by

$$\sum_{i=0}^{\infty}\frac{\Gamma(a+i)}{\Gamma(a)}\frac{z^i}{i!}=(1-z)^{-a}. \tag{6.66}$$

Thus,

$$\begin{aligned}
&\lim_{\tau\to\infty} I_{p_-(k,\tau)}(\beta(\tau),\alpha(\tau))\\
&=\lim_{\tau\to\infty}\frac{\Gamma(\alpha_\infty+\beta(\tau))}{\Gamma(\alpha_\infty)\Gamma(\beta(\tau))}\sum_{j=0}^{\infty}\lim_{\tau\to\infty}\frac{\Gamma(1-\alpha_\infty+j)}{\Gamma(1-\alpha_\infty)}\frac{p_-(k,\tau)^{j+\beta(\tau)}}{j!(\beta(\tau)+j)}\\
&=\lim_{\tau\to\infty}\frac{\sigma(\tau)^{\alpha_\infty}}{\Gamma(\alpha_\infty)}\sum_{j=0}^{\infty}\lim_{\tau\to\infty}\frac{\Gamma(1-\alpha_\infty+j)}{\sigma(\tau)\Gamma(1-\alpha_\infty)}\frac{p_-(k,\tau)^{j+\beta_\infty+\sigma(\tau)}}{j!}\\
&=\lim_{\tau\to\infty}p_-(k,\tau)^{\beta_\infty+\sigma(\tau)}\left(1-p_-(k,\tau)\right)^{\alpha_\infty-1}\frac{\sigma(\tau)^{\alpha_\infty-1}}{\Gamma(\alpha_\infty)}.
\end{aligned} \tag{6.67}$$

In the last equality, we took the limit outside the summation before calculating the binomial series limit, which is again justified by dominated convergence.

Similarly for the term $I_{p_+(k,\tau)}(\beta(\tau),\alpha(\tau))$ we deduce

$$\begin{aligned}
&\lim_{\tau\to\infty} I_{p_+(k,\tau)}(\alpha(\tau)+\sigma(\tau),\beta(\tau)-\sigma(\tau))\\
&=\lim_{\tau\to\infty}p_+(k,\tau)^{\alpha_\infty+\sigma(\tau)}\left(1-p_+(k,\tau)\right)^{\beta_\infty-1}\frac{\sigma(\tau)^{\beta_\infty-1}}{\Gamma(\beta_\infty)}.
\end{aligned} \tag{6.68}$$

Combining equations (4.21), (6.67), (6.68) and Lemma 6.1 it holds

$$\begin{aligned}
CC(S_0,K,\tau)&\sim S_0p_+(k,\tau)^{\alpha_\infty+\sigma(\tau)}\left(1-p_+(k,\tau)\right)^{\beta_\infty-1}\frac{\sigma(\tau)^{\beta_\infty-1}}{\Gamma(\beta_\infty)}\\
&\qquad+Kp_-(k,\tau)^{\beta_\infty+\sigma(\tau)}\left(1-p_-(k,\tau)\right)^{\alpha_\infty-1}\frac{\sigma(\tau)^{\alpha_\infty-1}}{\Gamma(\alpha_\infty)}\\
&\sim S_0\left(\frac{1}{2}\right)^{\sigma(\tau)+\alpha_\infty+\beta_\infty-1}\left(\frac{K}{S_0}\frac{\Gamma(\beta_\infty)}{\Gamma(\alpha_\infty)}\sigma(\tau)^{\alpha_\infty-\beta_\infty}\right)^{1/2}\frac{\sigma(\tau)^{\beta_\infty-1}}{\Gamma(\beta_\infty)}\\
&+K\left(\frac{1}{2}\right)^{\sigma(\tau)+\alpha_\infty+\beta_\infty-1}\left(\frac{S_0}{K}\frac{\Gamma(\alpha_\infty)}{\Gamma(\beta_\infty)}\sigma(\tau)^{\beta_\infty-\alpha_\infty}\right)^{1/2}\frac{\sigma(\tau)^{\alpha_\infty-1}}{\Gamma(\alpha_\infty)}
\end{aligned} \tag{6.69}$$

■

and after rearranging (4.24) is proven.

Proof of Corollary 4.11. It follows from (4.24) that $CC(S_0,K,\tau)\to 0$ as $\tau\to\infty$ which implies $S_\tau\to 0$ in probability as $\tau\to\infty$ (again [73]). Hence substituting (4.24) in (4.22) leads to

$$\begin{aligned}
\sigma_{BS}(k,\tau)\sim\frac{\sqrt{8}}{\sqrt{\tau}}\Big(&\log 2\left(\sigma(\tau)+\alpha_\infty+\beta_\infty-2\right)\\
&-\left(\frac{1}{2}(\alpha_\infty+\beta_\infty)-1\right)\log\sigma(\tau)\Big)^{1/2}
\end{aligned} \tag{6.70}$$

■

which implies (4.25).

Proof of Proposition 4.12. Keeping in mind (3.7) and (4.21) we have

$$\frac{DC(S_0,K,\tau)}{CC(S_0,K,\tau)} = \left(S_0 \frac{\Phi\left(-d_{K,S_0,\tau};\alpha(\tau)+\sigma(\tau),\beta(\tau)-\sigma(\tau),\sigma(\tau)\right)}{\Phi\left(d_{K,S_0,\tau};\beta(\tau),\alpha(\tau),\sigma(\tau)\right)} + K\right)^{-1}. \tag{6.71}$$

We need to study the ratio

$$\frac{\Phi\left(k+\mu^*(\tau);\alpha(\tau)+\sigma(\tau),\beta(\tau)-\sigma(\tau),\sigma(\tau)\right)}{\Phi\left(-k-\mu^*(\tau);\beta(\tau),\alpha(\tau),\sigma(\tau)\right)}. \tag{6.72}$$

With $p_\pm(k,\tau)$ as in (6.54) we observe the relationship

$$\left(\frac{p_+(k,\tau)}{p_-(k,\tau)}\right)^{\sigma(\tau)} = \left(\frac{1+e^{(\mu^*(\tau)+k)/\sigma(\tau)}}{1+e^{-(\mu^*(\tau)+k)/\sigma(\tau)}}\right)^{\sigma(\tau)} = e^{\mu^*(\tau)+k}. \tag{6.73}$$

Substituting (6.63) in (6.72) and taking into account (6.73) leads to

$$\begin{aligned}
&\frac{\Phi\left(k+\mu^*(\tau);\alpha(\tau)+\sigma(\tau),\beta(\tau)-\sigma(\tau),\sigma(\tau)\right)}{\Phi\left(-k-\mu^*(\tau);\beta(\tau),\alpha(\tau),\sigma(\tau)\right)} = \frac{\Gamma(\beta(\tau))\Gamma(\alpha(\tau))}{\Gamma(\alpha(\tau)+\sigma(\tau))\Gamma(\beta(\tau)-\sigma(\tau))} \\
&\cdot \sum_{j=0}^{\infty} \frac{\Gamma(1-\beta(\tau)+\sigma(\tau)+j)}{\Gamma(1-\beta(\tau)+\sigma(\tau))} \frac{p_+(k,\tau)^{j+\alpha(\tau)+\sigma(\tau)}}{j!(\alpha(\tau)+\sigma(\tau)+j)} \left(\sum_{j=0}^{\infty} \frac{\Gamma(1-\alpha(\tau)+j)}{\Gamma(1-\alpha(\tau))} \frac{p_-(k,\tau)^{j+\beta(\tau)}}{j!(\beta(\tau)+j)}\right)^{-1} \\
&= e^{-\mu^*(\tau)} \frac{p_+(k,\tau)^{\sigma(\tau)+\alpha(\tau)}}{p_-(k,\tau)^{\beta(\tau)}} \sum_{j=0}^{\infty} \frac{\Gamma(1-\beta(\tau)+\sigma(\tau)+j)}{\Gamma(1-\beta(\tau)+\sigma(\tau))} \frac{p_+(k,\tau)^j}{j!(\alpha(\tau)+\sigma(\tau)+j)} \\
&\qquad \cdot \left(\sum_{j=0}^{\infty} \frac{\Gamma(1-\alpha(\tau)+j)}{\Gamma(1-\alpha(\tau))} \frac{p_-(k,\tau)^j}{j!(\beta(\tau)+j)}\right)^{-1} \\
&= e^{k} \frac{p_+(k,\tau)^{\alpha(\tau)}}{p_-(k,\tau)^{\beta(\tau)-\sigma(\tau)}} \sum_{j=0}^{\infty} \frac{\Gamma(1-\beta(\tau)+\sigma(\tau)+j)}{\Gamma(1-\beta(\tau)+\sigma(\tau))} \frac{p_+(k,\tau)^j}{j!(\alpha(\tau)+\sigma(\tau)+j)} \\
&\qquad \cdot \left(\sum_{j=0}^{\infty} \frac{\Gamma(1-\alpha(\tau)+j)}{\Gamma(1-\alpha(\tau))} \frac{p_-(k,\tau)^j}{j!(\beta(\tau)+j)}\right)^{-1}.
\end{aligned} \tag{6.74}$$

By the dominated convergence argument of Proposition 4.10, it is possible to interchange the limit on τ and the series in the expression above. After taking the limit on τ term-by-term, we observe that for all $j>0$ we have by (**LTA**) that $\beta(\tau)+j \sim \alpha(\tau)+\sigma(\tau)+j \sim \sigma(\tau)$ as $\tau\to\infty$. Further using the binomial series (6.66) produces

$$\begin{aligned}
&\lim_{\tau\to\infty} \frac{\Phi\left(k+\mu^*(\tau);\alpha(\tau)+\sigma(\tau),\beta(\tau)-\sigma(\tau),\sigma(\tau)\right)}{\Phi\left(-k-\mu^*(\tau);\beta(\tau),\alpha(\tau),\sigma(\tau)\right)} \\
&\quad = \lim_{\tau\to\infty} e^{k} \frac{p_+(k,\tau)^{\alpha_\infty}}{p_-(k,\tau)^{\beta_\infty}} \sum_{j=0}^{\infty} \frac{\Gamma(1-\beta_\infty+j)}{\Gamma(1-\beta_\infty)} \frac{p_+(k,\tau)^j}{j!\sigma(\tau)} \left(\sum_{j=0}^{\infty} \frac{\Gamma(1-\alpha_\infty+j)}{\Gamma(1-\alpha_\infty)} \frac{p_-(k,\tau)^j}{j!(\sigma(\tau))}\right)^{-1} \\
&\quad = \lim_{\tau\to\infty} e^{k} \frac{p_+(k,\tau)^{\alpha_\infty}}{p_-(k,\tau)^{\beta_\infty}} \frac{(1-p_-(k,\tau))^{1-\alpha_\infty}}{(1-p_+(k,\tau))^{1-\beta_\infty}} = \lim_{\tau\to\infty} e^{k} \frac{p_+(k,\tau)}{p_-(k,\tau)} = \lim_{\tau\to\infty} K e^{\frac{k+\mu^*(\tau)}{\sigma(\tau)}},
\end{aligned} \tag{6.75}$$

where in the last equality we also used (6.73).

That S_τ tends to 0 in probability has been shown in Corollary 4.11. Substituting (6.75) in (6.71) and the resulting expression in (4.25) together with (4.23), we obtain, since $e^x - 1 \sim x$ as $x \sim 0$, the leading order

$$
\begin{aligned}
\mathcal{S}(S_0, K, \tau) &\sim \frac{1}{\sqrt{8 \log 2}\, \sqrt{\sigma(\tau)\tau}} \left(\frac{1}{K} - \frac{2}{K} \frac{1}{e^{(\log(K/S_0)+\mu^*(\tau))/\sigma(\tau)} + 1} \right) \\
&\sim \frac{1}{\sqrt{8 \log 2}\, \sqrt{\sigma(\tau)\tau}} \frac{\log(K/S_0) + \mu^*(\tau)}{K\sigma(\tau)} \\
&\sim \frac{log(K/S_0) + \log \sigma(\tau)(\alpha_\infty - \beta_\infty) + \log(\Gamma(\beta_\infty)/\Gamma(\alpha_\infty))}{\sqrt{8 \log 2}\, \sqrt{\tau}\sigma(\tau)^{3/2} K}, \qquad (6.76)
\end{aligned}
$$
■

having used the asymptotic rate of $\mu^*(\tau)/\sigma(\tau)$ provided by (6.59) in the last equivalence. Upon rearranging the proof is finished.

Proof of Corollary 4.13. Point (i) is clear from (4.26). To show (ii) for any fixed K and S_0, define

$$
s(k, \tau) = \log \sigma(\tau)(\alpha_\infty - \beta_\infty) + \log \frac{\Gamma(\beta_\infty)}{\Gamma(\alpha_\infty)} + k. \qquad (6.77)
$$

Based on (4.26) there exists τ_1 such that for all $\tau > \tau_1$ we have $\mathcal{S}(S_0, K, \tau) > 0$ (< 0) whenever $s(k, \tau) > 0$ (< 0). Observe that since σ is strictly increasing (by condition (**A1'**)), for fixed k there is a unique solution τ_k to the equation $s(k, \tau) = 0$, which satisfies

$$
\sigma(\tau_k) = \left(-\log k \cdot \frac{\Gamma(\alpha_\infty)}{\Gamma(\beta_\infty)} \right)^{\frac{1}{\alpha_\infty - \beta_\infty}}. \qquad (6.78)
$$
■

Therefore letting $\tau_\infty = \max\{\tau_1, \tau_k\}$, for all $\tau > \tau_\infty$ we have $s(k, \tau) > 0$ if $\alpha_\infty > \beta_\infty$ and $s(k, \tau) < 0$ if $\beta_\infty > \alpha_\infty$, proving (ii).

Proof of Proposition 5.1. The verifications are straightforward for the R-GBA and the T-GBA. For the DS-GBA and the DST-GBA $\sigma(t)$ is non negative and continuous due to the fact that arctan is positive and continuous, and is increasing in t because

$$
\begin{aligned}
\frac{d}{dt}\sigma(t) = &\frac{\pi \sigma_0 H_0 t^{H_0 - 1}}{2(\sigma_0^2 \pi^2 t^{2H_0}/4 + 1) \arctan(\pi/(2\sigma_\infty) t^{-H_\infty})} \\
&+ \frac{\pi H_\infty t^{-H_\infty - 1} \arctan(\pi \sigma_0/2) t^{H_0})}{2\sigma_\infty((\pi/(2\sigma_\infty))^2 t^{-2H_\infty} + 1) \arctan(\pi/(2\sigma_\infty) t^{-H_\infty})^2} > 0. \qquad (6.79)
\end{aligned}
$$

Moreover, recall that $\arctan(x) \sim x$ as $x \sim 0$ and $\lim_{x\to\infty} \arctan(x) = \pi/2$, so that, for both specifications, $\sigma(t) \sim \sigma_0 t^{H_0}$ as $t \sim 0$ and $\sigma(t) \sim \sigma_\infty t^{H_\infty}$ when $t \to \infty$.

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