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A new numerical scheme for Itô stochastic differential equations based on Wick-type Wong-Zakai arguments

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Abstract

The aim of this note is to propose a novel numerical scheme for drift-less one dimensional stochastic differential equations of Itô's type driven by standard Brownian motion. Our approximation method is equivalent to the well known Milstein scheme as long as the rate of convergence is concerned, i.e. it is strongly convergent with order one, but has the additional desirable property of being exact for linear diffusion coefficients. Our approach is inspired by Wick-type Wong-Zakai arguments in the sense that we only smooth the white noise through polygonal approximation of the Brownian motion while keep the equation in differential form. A first order Taylor expansion of the diffusion coefficient allows us to solve the resulting equation explicitly and hence to provide an implementable approximation scheme.

Key words and phrases: Brownian motion, stochastic differential equation, Wick product, Milstein scheme.

AMS 2000 classification: 60H10, 65C30, 60H25.

1 Statement of the main result

Let $\{X(t)\}_{t\in[0,1]}$ be the unique strong solution of the Itô stochastic differential equation (SDE)

$$\begin{cases} dX(t) = \sigma(X(t))dB(t), \quad t \in]0,1]; \\ X(0) = a, \end{cases} \tag{1.1}$$

where $\{B(t)\}_{t\in[0,1]}$ is a standard one dimensional Brownian motion, $\sigma : \mathbb{R} \to \mathbb{R}$ is a measurable function such that σ and $\sigma\sigma'$ are continuously differentiable with bounded derivatives while $a \in \mathbb{R}$ is a constant. We have the following result.

Theorem 1.1. *For $N \in \mathbb{N}$ let $\{x_N(k)\}_{k\in\{0,...,N\}}$ be defined recursively as*

$$\begin{cases} x_N(k+1) = x_N(k) + \frac{\sigma(x_N(k))}{\sigma'(x_N(k))} \left(\exp^{\diamond} \left\{ \sigma'(x_N(k)) \left(B\left(\frac{k+1}{N}\right) - B\left(\frac{k}{N}\right) \right) \right\} - 1 \right), \quad k \in \{0,...,N-1\}; \\ x_N(0) = a, \end{cases} \tag{1.2}$$

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where

$$\exp^{\diamond}\left\{\sigma'(x_N(k))\left(B\left(\frac{k+1}{N}\right)-B\left(\frac{k}{N}\right)\right)\right\}, \tag{1.3}$$

denotes the Wick exponential

$$\exp\left\{\sigma'(x_N(k))\left(B\left(\frac{k+1}{N}\right)-B\left(\frac{k}{N}\right)\right)-\frac{\sigma'(x_N(k))^2}{2N}\right\}.$$

Then, for all $N\in\mathbb{N}$ we have

$$\mathbb{E}[|X(1)-x_N(N)|]\leq\frac{\mathsf{c}}{N}, \tag{1.4}$$

where $\{X(t)\}_{t\in[0,1]}$ is the unique strong solution of the SDE (1.1) *while* c *is a positive constant independent of N. Moreover, when $\sigma:\mathbb{R}\to\mathbb{R}$ is linear, the sequence $\{x_N(k)\}_{k\in\{0,\dots,N\}}$ coincides with $\{X\left(\frac{k}{N}\right)\}_{k\in\{0,\dots,N\}}$.*
In words: scheme (1.2) *provides a strong approximation of order one for the unique strong solution of the SDE* (1.1)*; in addition, the scheme is exact when the diffusion coefficient σ is linear.*

Remark 1.2 (Vanishing derivative of σ)**.** *Possibly vanishing values of $\sigma'(x_N(k))$ in the ratio $\frac{\sigma(x_N(k))}{\sigma'(x_N(k))}$ from* (1.2) *do not cause any problem of ill-posedness. In fact, exploiting the Wick-series expansion of* (1.3) *we can write conditionally on $\mathcal{F}^B_{k/N}$ (here we denote by $\{\mathcal{F}^B_t\}_{t\in[0,1]}$ the augmented Brownian filtration) that*

$$\begin{aligned}
&\frac{\sigma(x_N(k))}{\sigma'(x_N(k))}\left(\exp^{\diamond}\left\{\sigma'(x_N(k))\left(B\left(\frac{k+1}{N}\right)-B\left(\frac{k}{N}\right)\right)\right\}-1\right)\\
&\quad=\frac{\sigma(x_N(k))}{\sigma'(x_N(k))}\sum_{j\geq1}\frac{\sigma'(x_N(k))^j}{j!}\left(B\left(\frac{k+1}{N}\right)-B\left(\frac{k}{N}\right)\right)^{\diamond j}\\
&\quad=\sigma(x_N(k))\sum_{j\geq1}\frac{\sigma'(x_N(k))^{j-1}}{j!}\left(B\left(\frac{k+1}{N}\right)-B\left(\frac{k}{N}\right)\right)^{\diamond j},
\end{aligned}$$

thus making perfectly sense for possibly vanishing values of $\sigma'(x_N(k))$. Here, the symbol $\cdot^{\diamond j}$ denotes the j-th Wick power.

Remark 1.3 (Comparison with Milstein scheme)**.** *If we Taylor expand the Wick exponential in* (1.2) *up to the second order, we recover the famous Milstein scheme [16, 8]. In fact, recalling the Wick-series expansion of* (1.3) *utilized in Remark 1.2 we can write*

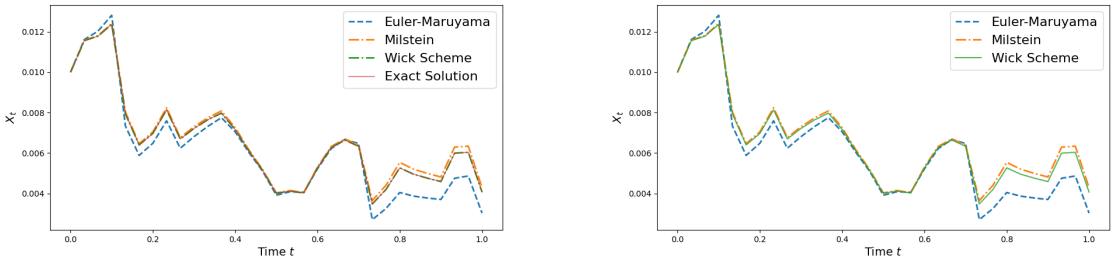
$$\begin{aligned}
&\frac{\sigma(x_N(k))}{\sigma'(x_N(k))}\left(\exp^{\diamond}\left\{\sigma'(x_N(k))\left(B\left(\frac{k+1}{N}\right)-B\left(\frac{k}{N}\right)\right)\right\}-1\right)\\
&\approx\frac{\sigma(x_N(k))}{\sigma'(x_N(k))}\left(\sigma'(x_N(k))\left(B\left(\frac{k+1}{N}\right)-B\left(\frac{k}{N}\right)\right)+\frac{\sigma'(x_N(k))^2}{2}\left(B\left(\frac{k+1}{N}\right)-B\left(\frac{k}{N}\right)\right)^{\diamond2}\right)\\
&=\sigma(x_N(k))\left(B\left(\frac{k+1}{N}\right)-B\left(\frac{k}{N}\right)\right)+\frac{\sigma(x_N(k))\sigma'(x_N(k))}{2}\left(B\left(\frac{k+1}{N}\right)-B\left(\frac{k}{N}\right)\right)^{\diamond2}\\
&=\sigma(x_N(k))\left(B\left(\frac{k+1}{N}\right)-B\left(\frac{k}{N}\right)\right)+\frac{\sigma(x_N(k))\sigma'(x_N(k))}{2}\left(\left(B\left(\frac{k+1}{N}\right)-B\left(\frac{k}{N}\right)\right)^2-\frac{1}{N}\right),
\end{aligned}$$

where in the last equality we exploit the identity between Wick powers and Hermite polynomials [7, 4, 5, 7]. Substituting the last member above into the right hand side of (1.2) *we obtain the Mistein scheme for the SDE* (1.1)*.*

Remark 1.4 (Equation with drift)**.** *Following the proof of Theorem* (1.1) *presented in Section 2 below, one can see how the introduction of a drift term in equation* (1.1) *causes a non trivial issue in the derivation of our scheme. In fact, while the Wick exponential possesses a Wick-free representation as pointed out in the statement of Theorem 1.1, the corresponding term for an equation with drift will fail in that direction. And since the scheme must be fully implementable, the presence of general terms involving Wick products may prevent to reach that goal. We devote to a future study the investigation of such an extension.*

Remark 1.5. *The statement of Theorem 1.1 and its proof exploit some standard tools of the so-called Wick calculus. While this framework allows to treat generalized random variables and processes (see for instance [6, 5]), here we limit its application to square integrable random objects due to the non linear nature of our problem.*

Remark 1.6. *Our main theorem is a further example of the beneficial effect of treating some aspects of the theory of Itô SDEs from a Wick calculus' perspective. We mention in passing that the role of Wick product as* natural *Gaussian convolution, shown in [2] and [15], has been identified for Poisson [14] and Gamma [13] distributions as well. The approach presented here (which essentially relies on the properties of chaos decompositions) together with those contributions may lead to analogous approximation results for stochastic equations driven by non-Gaussian noises.*



(a) $dX_t = X_t dB_t$, $X_0 = 0.01$.

(b) $dX_t = \sin(X_t) dB_t$, $X_0 = 0.01$.

Figure 1: In Figure 1a we compare Euler-Maruyama and Milstein schemes for the SDE $dX_t = X_t dB_t$, $X_0 = 0.01$ with its exact solution. Notice that in this case the numerical scheme described in Theorem 1.1, named *Wick scheme* in the figure, coincides with the exact solution. In Figure 1b we consider the SDE $dX_t = \sin(X_t) dB_t$, $X_0 = 0.01$ and plot the approximations provided by Euler-Maruyama, Milstein and Wick schemes. In both figure N is set equal to 30.

The similarity of the sample paths in Figure 1a and 1b hints the advantage of using the scheme provided in (1.2) when the diffusion term is almost linear.

1.1 Heuristic derivation of the scheme

In this section we describe some basic ideas that lead to the derivation of scheme (1.2). This is inspired by Wong-Zakai approximation results [17, 18] for Itô's type SDEs firstly obtained in [6] (see also [5] and the references quoted there) and then extended to different contexts in [1, 3, 10, 11, 12]. The key observation guiding the approach has to be found in the correspondence between Itô-Skorohod integration and Gaussian Wick product; for a detailed discussion on this issue we refer the reader to [4]. Exploiting this fact we can indeed rewrite the Cauchy problem (1.1) as

$$\begin{cases} \dot{X}(t) = \sigma(X(t)) \diamond \dot{B}(t), \quad t \in]0, 1]; \\ X(0) = a, \end{cases} \tag{1.5}$$

where $\{\dot{B}(t)\}_{t\in[0,1]}$ denotes Gaussian White Noise. Now, for $N \in \mathbb{N}$ we consider the partition of the interval $[0,1]$ given by $\{0, 1/N, 2/N, ..., 1\}$ and denote by $\{B_N(t)\}_{t\in[0,1]}$ the associated polygonal approximation of $\{B(t)\}_{t\in[0,1]}$; this means that

$$B_N(t) := B(k/N) + (t - k/N)\frac{B((k+1)/N) - B(k/N)}{1/N}, \quad t \in [k/N, (k+1)/N[\,, k \in \{0, ..., N-1\},$$

and replacing $\{B(t)\}_{t\in[0,1]}$ with $\{B_N(t)\}_{t\in[0,1]}$ we get

$$\begin{cases} \dot{X}_N(t) = \sigma(X_N(t)) \diamond \dot{B}_N(t), \quad t \in]0,1]; \\ X_0 = a. \end{cases} \tag{1.6}$$

Notice that the function $t \mapsto \dot{B}_N(t)$ is piece-wise constant; therefore, we can rewrite equation (1.6) more accurately as the collection of the following nested Cauchy problems

$$\begin{cases} \dot{x}_{N,k+1}(t) = \sigma(x_{N,k+1}(t)) \diamond \frac{B((k+1)/N) - B(k/N)}{1/N}, \quad t \in]k/N, (k+1)/N]; \\ x_{N,k+1}(k/N) = x_{N,k}(k/N), \end{cases} \tag{1.7}$$

for $k \in \{0, ..., N-1\}$ and with the initial condition: $x_{N,1}(0) = x_{N,0}(0) = a$. We now Taylor-expand up to the first order the function σ in (1.7) around the initial condition $x_{N,k}(k/N)$ to get

$$\begin{aligned} \dot{x}_{N,k+1}(t) \approx & (\sigma(x_{N,k}(k/N)) + \sigma'(x_{N,k}(k/N))(x_{N,k+1}(t) - x_{N,k}(k/N))) \diamond \frac{B((k+1)/N) - B(k/N)}{1/N} \\ = & \sigma'(x_{N,k}(k/N)) x_{N,k+1}(t) \diamond \frac{B((k+1)/N) - B(k/N)}{1/N} \\ & + (\sigma(x_{N,k}(k/N)) - \sigma'(x_{N,k}(k/N)) x_{N,k}(k/N))) \diamond \frac{B((k+1)/N) - B(k/N)}{1/N} \end{aligned} \tag{1.8}$$

Equation (1.8) is linear in $\dot{x}_{N,k+1}(t)$ and can be easily solved either using the reduction method proposed in [6] or by means of (conditional on $\mathcal{F}^B_{k/N}$) $\mathcal{S}$-transform [9], i.e.

$$X \mapsto \mathbb{E}\left[X \exp\left\{ \int_{k/N}^1 f(s) dB_s - \frac{1}{2}\int_{k/N}^1 f^2(s) ds \right\} \middle| \mathcal{F}^B_{k/N} \right], \quad f \in L^2([k/N, 1]),$$

which will convert the previous stochastic equation into a linear deterministic one. In particular, one will find that the value of the solution to (1.8) at the node $(k+1)/N$ is given by

$$\begin{aligned} x_{N,k+1}((k+1)/N) = & x_{N,k}(k/N) \\ & + \frac{\sigma(x_{N,k}(k/N))}{\sigma'(x_{N,k}(k/N))} \left(\exp^{\diamond}\left\{ \sigma'(x_{N,k}(k/N)) \left(B\left(\frac{k+1}{N}\right) - B\left(\frac{k}{N}\right)\right)\right\} - 1 \right). \end{aligned}$$

If we now set $x_N(k) := x_{N,k}(k/N)$ for $k \in \{0, ..., N\}$ we see that the previous identity coincides with scheme (1.2).

2 Proof of the main result

To ease the notation, for $k \in \{1, ..., N\}$ we set

$$Z_k := \sqrt{N}\left(B\left(\frac{k}{N}\right) - B\left(\frac{k-1}{N}\right)\right),$$

so that $\{Z_k\}_{k\in\{1,...,N\}}$ is a collection of independent standard Gaussian random variables; therefore, scheme (1.2) now reads

$$\begin{cases} x_N(k+1) = x_N(k) + \frac{\sigma(x_N(k))}{\sigma'(x_N(k))}\left(\exp^{\diamond}\left\{\sigma'(x_N(k))Z_{k+1}/\sqrt{N}\right\} - 1\right), \quad k \in \{0,...,N-1\}; \\ x_N(0) = a. \end{cases} \tag{2.1}$$

We start proving the following upper bound for the second moment of $x_N(k+1)$. In the sequel we will write $\mathcal{F}_k$ for the sigma algebra generated by $\{Z_1,...,Z_k\}$.

Lemma 2.1. *For all $N \in \mathbb{N}$ and $k \in \{0,...,N\}$ we have*

$$\mathbb{E}[x_N(k)^2] \leq \left(1 + M_1\frac{\exp\{L_1^2/N\}}{N}\right)^k (a^2+1) - 1, \tag{2.2}$$

where L_1 stands for the Lipschitz constant of the function σ while M_1 is a positive constant with which the inequality

$$\sigma(x)^2 \leq M_1(1+x^2), \quad x \in \mathbb{R} \tag{2.3}$$

holds true (the existence of such M_1 follows from the global Lipschitz continuity of σ).

Proof. Notice that for all $k \in \{1,...,N\}$ the random variable $x_N(k)$ is $\mathcal{F}_k$-measurable and hence taking the square of both sides in (2.1) and computing conditional expectations with respect to the sigma-algebra $\mathcal{F}_k$ we get

$$\mathbb{E}[x_N(k+1)^2|\mathcal{F}_k] = x_N(k)^2 + \frac{\sigma(x_N(k))^2}{\sigma'(x_N(k))^2}\mathbb{E}\left[\left(\exp^{\diamond}\left\{\sigma'(x_N(k))Z_{k+1}/\sqrt{N}\right\} - 1\right)^2 \middle| \mathcal{F}_k\right].$$

Here, the mixed product of $\frac{\sigma(x_N(k))}{\sigma'(x_N(k))}$ with $\exp^{\diamond}\left\{\sigma'(x_N(k))Z_{k+1}/\sqrt{N}\right\} - 1$ vanishes since the first term is $\mathcal{F}_k$-measurable while the second one has null conditional expectation. Moreover, a simple computation gives

$$\mathbb{E}\left[\left(\exp^{\diamond}\left\{\sigma'(x_N(k))Z_{k+1}/\sqrt{N}\right\} - 1\right)^2 \middle| \mathcal{F}_k\right] = \exp\left\{\sigma'(x_N(k))^2/N\right\} - 1;$$

hence,

$$\mathbb{E}[x_N(k+1)^2|\mathcal{F}_k] = x_N(k)^2 + \frac{\sigma(x_N(k))^2}{\sigma'(x_N(k))^2}\left(\exp\left\{\sigma'(x_N(k))^2/N\right\} - 1\right),$$

and taking expectation of both sides we get

$$\mathbb{E}[x_N(k+1)^2] = \mathbb{E}[x_N(k)^2] + \mathbb{E}\left[\sigma(x_N(k))^2\frac{\exp\left\{\sigma'(x_N(k))^2/N\right\} - 1}{\sigma'(x_N(k))^2}\right].$$

Moreover,

$$\begin{aligned} \frac{\exp\left\{\sigma'(x_N(k))^2/N\right\} - 1}{\sigma'(x_N(k))^2} &= \sum_{j\geq 1}\frac{\sigma'(x_N(k))^{2(j-1)}}{j!N^j} \\ &= \frac{1}{N}\sum_{j\geq 1}\frac{\sigma'(x_N(k))^{2(j-1)}}{j!N^{j-1}} \\ &\leq \frac{1}{N}\sum_{j\geq 1}\frac{\sigma'(x_N(k))^{2(j-1)}}{(j-1)!N^{j-1}} \end{aligned}$$

$$= \frac{\exp\{\sigma'(x_N(k))^2/N\}}{N}$$
$$\leq \frac{\exp\{L_1^2/N\}}{N}.$$

Therefore,

$$\begin{aligned}\mathbb{E}[x_N(k+1)^2] &= \mathbb{E}[x_N(k)^2] + \mathbb{E}\left[\sigma(x_N(k))^2 \frac{\exp\left\{\sigma'(x_N(k))^2/N\right\} - 1}{\sigma'(x_N(k))^2}\right] \\ &\leq \mathbb{E}[x_N(k)^2] + \frac{\exp\{L_1^2/N\}}{N}\mathbb{E}\left[\sigma(x_N(k))^2\right] \\ &\leq \mathbb{E}[x_N(k)^2] + \frac{\exp\{L_1^2/N\}}{N}\mathbb{E}\left[M_1(1+x_N(k)^2)\right] \\ &= \mathbb{E}[x_N(k)^2]\left(1 + M_1\frac{\exp\{L_1^2/N\}}{N}\right) + M_1\frac{\exp\{L_1^2/N\}}{N};\end{aligned}$$

here, we utilized the inequality

$$\sigma(x)^2 \leq M_1(1+x^2), \quad x \in \mathbb{R}.$$

The previous computation shows that the sequence of second moments $k \mapsto \mathbb{E}[x_N(k)^2]$ satisfies the recursive inequality

$$\mathbb{E}[x_N(k+1)^2] \leq \mathbb{E}[x_N(k)^2]\left(1 + M_1\frac{\exp\{L_1^2/N\}}{N}\right) + M_1\frac{\exp\{L_1^2/N\}}{N},$$

with $\mathbb{E}[x_N(0)^2] = a^2$, which gives via a simple algebraic manipulation

$$\begin{aligned}\mathbb{E}[x_N(k)^2] &\leq a^2\left(1 + M_1\frac{\exp\{L_1^2/N\}}{N}\right)^k + M_1\frac{\exp\{L_1^2/N\}}{N}\frac{\left(1 + M_1\frac{\exp\{L_1^2/N\}}{N}\right)^k - 1}{\left(1 + M_1\frac{\exp\{L_1^2/N\}}{N}\right) - 1} \\ &= a^2\left(1 + M_1\frac{\exp\{L_1^2/N\}}{N}\right)^k + \left(1 + M_1\frac{\exp\{L_1^2/N\}}{N}\right)^k - 1 \\ &= \left(1 + M_1\frac{\exp\{L_1^2/N\}}{N}\right)^k (a^2+1) - 1.\end{aligned}$$

The proof is complete. □

We are now ready to prove Theorem 1.1; we will compare our scheme with the Milstein's one and show that they are of the same order. In the sequel we will denote the Milstein scheme with $\{y_N(k)\}_{k\in\{0,\dots,N\}}$, namely

$$\begin{cases} y_N(k+1) = y_N(k) + \sigma(y_N(k))Z_{k+1}/\sqrt{N} + \sigma(y_N(k))\sigma'(y_N(k))\left(Z_{k+1}^2 - 1\right)/2N \\ y_N(0) = a. \end{cases} \tag{2.4}$$

We start with

$$\begin{aligned}\mathbb{E}[|X(1) - x_N(N)|] &\leq \mathbb{E}[|X(1) - y_N(N)|] + \mathbb{E}[|y_N(N) - x_N(N)|] \\ &\leq \mathbb{E}[|X(1) - y_N(N)|] + \mathbb{E}[|y_N(N) - x_N(N)|^2]^{\frac{1}{2}}.\end{aligned}$$

It is well known (see [8]) that there exists a constant C_1 such that

$$\mathbb{E}[|X(1) - y_N(N)|] \leq \frac{C_1}{N}; \tag{2.5}$$

our aim is to prove that there exists a constant C_2 such that

$$\mathbb{E}[|y_N(N) - x_N(N)|^2]^{\frac{1}{2}} \leq \frac{C_2}{N},$$

which in combination with (2.5) is equivalent to (1.4).
For all $N \in \mathbb{N}$ and $k \in \{0, ..., N-1\}$ we have (recall Remark 1.3)

$$\begin{aligned} x_N(k+1) - y_N(k+1) =& x_N(k) - y_N(k) + (\sigma(x_N(k)) - \sigma(y_N(k)))Z_{k+1}/\sqrt{N} \\ &+ (\sigma(x_N(k))\sigma'(x_N(k)) - \sigma(y_N(k))\sigma'(y_N(k)))\left(Z_{k+1}^2 - 1\right)/2N \\ &+ \frac{\sigma(x_N(k))}{\sigma'(x_N(k))} \sum_{j\geq 3} \frac{\sigma'(x_N(k))^j}{j!} Z_{k+1}^{\diamond j}/N^{\frac{j}{2}}. \end{aligned}$$

We now take the square of both sides in the equality above and compute conditional expectations with respect to the sigma-algebra $\mathcal{F}_k$; recall that the Wick powers $Z_{k+1}^{\diamond j}$ are orthogonal in $\mathbb{L}^2(\Omega)$ for different j's (see [7]) and hence all the mixed products generated with the square of the right hand side above will vanish in conditional expectation. Moreover, the squared $\mathbb{L}^2(\Omega)$-norm of $Z_{k+1}^{\diamond j}$ is $j!$. Therefore,

$$\begin{aligned} \mathbb{E}[(x_N(k+1) - y_N(k+1))^2|\mathcal{F}_k] =& (x_N(k) - y_N(k))^2 + (\sigma(x_N(k)) - \sigma(y_N(k)))^2/N \\ &+ (\sigma(x_N(k))\sigma'(x_N(k)) - \sigma(y_N(k))\sigma'(y_N(k)))^2/2N^2 \\ &+ \frac{\sigma(x_N(k))^2}{\sigma'(x_N(k))^2} \sum_{j\geq 3} \frac{\sigma'(x_N(k))^{2j}}{j!^2} j!/N^j \\ =& (x_N(k) - y_N(k))^2 + (\sigma(x_N(k)) - \sigma(y_N(k)))^2/N \\ &+ (\sigma(x_N(k))\sigma'(x_N(k)) - \sigma(y_N(k))\sigma'(y_N(k)))^2/2N^2 \\ &+ \frac{\sigma(x_N(k))^2}{\sigma'(x_N(k))^2} \sum_{j\geq 3} \frac{\sigma'(x_N(k))^{2j}}{j!N^j}. \end{aligned}$$

Let us focus on the last term above:

$$\begin{aligned} \frac{\sigma(x_N(k))^2}{\sigma'(x_N(k))^2} \sum_{j\geq 3} \frac{\sigma'(x_N(k))^{2j}}{j!N^j} &= \frac{\sigma(x_N(k))^2}{N} \sum_{j\geq 3} \frac{\sigma'(x_N(k))^{2(j-1)}}{j!N^{j-1}} \\ &\leq \frac{\sigma(x_N(k))^2}{N} \sum_{j\geq 3} \frac{\sigma'(x_N(k))^{2(j-1)}}{(j-1)!N^{j-1}} \\ &\leq \frac{\sigma(x_N(k))^2}{N} \sum_{j\geq 3} \frac{L_1^{2(j-1)}}{(j-1)!N^{j-1}} \\ &= \frac{\sigma(x_N(k))^2}{N} \left(e^{L_1^2/N} - 1 - L_1^2/N\right). \end{aligned}$$

This gives

$$\begin{aligned} \mathbb{E}[(x_N(k+1) - y_N(k+1))^2|\mathcal{F}_k] \leq& (x_N(k) - y_N(k))^2 + (\sigma(x_N(k)) - \sigma(y_N(k)))^2/N \\ &+ (\sigma(x_N(k))\sigma'(x_N(k)) - \sigma(y_N(k))\sigma'(y_N(k)))^2/2N^2 \\ &+ \frac{\sigma(x_N(k))^2}{N} \left(e^{L_1^2/N} - 1 - L_1^2/N\right) \end{aligned}$$

and taking the expectation of both sides yields

$$\mathbb{E}[(x_N(k+1) - y_N(k+1))^2] \leq \mathbb{E}[(x_N(k) - y_N(k))^2] + \mathbb{E}[(\sigma(x_N(k)) - \sigma(y_N(k)))^2]/N$$

$$
\begin{aligned}
&+ \mathbb{E}[(\sigma(x_N(k))\sigma'(x_N(k)) - \sigma(y_N(k))\sigma'(y_N(k)))^2]/2N^2 \\
&+ \frac{\mathbb{E}[\sigma(x_N(k))^2]}{N}\left(e^{L_1^2/N} - 1 - L_1^2/N\right) \\
\leq &\left(1 + \frac{L_1^2}{N} + \frac{L_2^2}{2N^2}\right)\mathbb{E}[(x_N(k) - y_N(k))^2] \\
&+ \frac{\mathbb{E}[\sigma(x_N(k))^2]}{N}\left(e^{L_1^2/N} - 1 - L_1^2/N\right) \\
\leq &\left(1 + \frac{L_1^2}{N} + \frac{L_2^2}{2N^2}\right)\mathbb{E}[(x_N(k) - y_N(k))^2] \\
&+ \frac{M_1}{N}\left(e^{L_1^2/N} - 1 - L_1^2/N\right) \\
&+ \frac{M_1\left(\left(1 + M_1\frac{\exp\{L_1^2/N\}}{N}\right)^N(a^2+1) - 1\right)}{N}\left(e^{L_1^2/N} - 1 - L_1^2/N\right) \\
= &\left(1 + \frac{L_1^2}{N} + \frac{L_2^2}{2N^2}\right)\mathbb{E}[(x_N(k) - y_N(k))^2] + \Lambda_N. \qquad (2.6)
\end{aligned}
$$

Here, in the second inequality we exploited the Lipschitz continuity of σ (with constant L_1) and of $\sigma\sigma'$ (with constant L_2); in the third inequality we utilized the upper bound (2.3) together with Lemma (2.1) (with k replaced by N in the right hand side of (2.2)). In addition, we set

$$
\Lambda_N := \frac{M_1\left(1 + M_1\frac{\exp\{L_1^2/N\}}{N}\right)^N(a^2+1)}{N}\left(e^{L_1^2/N} - 1 - L_1^2/N\right).
$$

Notice that $\Lambda_N \sim \frac{1}{N^3}$, as N tends to infinity. If we now compare the first and last members in (2.6) we deduce that

$$
\mathbb{E}[(x_N(k+1) - y_N(k+1))^2] \leq \left(1 + \frac{L_1^2}{N} + \frac{L_2^2}{2N^2}\right)\mathbb{E}[(x_N(k) - y_N(k))^2] + \Lambda_N
$$

and hence

$$
\mathbb{E}[(x_N(N) - y_N(N))^2] \leq \Lambda_N \frac{\left(1 + \frac{L_1^2}{N} + \frac{L_2^2}{2N^2}\right)^N - 1}{\frac{L_1^2}{N} + \frac{L_2^2}{2N^2}}; \qquad (2.7)
$$

recall that $\mathbb{E}[(x_N(0) - y_N(0))^2] = 0$. Observing that the ratio in (2.7) goes like N as N tends to infinity we conclude saying that

$$
\mathbb{E}[(x_N(N) - y_N(N))^2] \sim \frac{1}{N^2}
$$

and hence

$$
\mathbb{E}[(x_N(N) - y_N(N))^2]^{\frac{1}{2}} \sim \frac{1}{N}
$$

which is precisely what we wanted to show.

To prove the second part of the statement of Theorem 1.1 we now assume σ to be of the form $\sigma(x) = \alpha x$, $x \in \mathbb{R}$, for some real constant $\alpha \neq 0$. Then, scheme (1.2) simplifies to

$$
x_N(k+1) = x_N(k) + x_N(k)\left(\exp^{\diamond}\left\{\alpha\left(B\left(\frac{k+1}{N}\right) - B\left(\frac{k}{N}\right)\right)\right\} - 1\right)
$$

$$= x_N(k) \exp^{\diamond} \left\{ \alpha \left(B\left(\frac{k+1}{N} \right) - B\left(\frac{k}{N} \right) \right) \right\}, \quad k \in \{0, ..., N-1\}.$$

It is easy to see using standard Wick calculus [4] and the independence of Brownian increments that the solution to the previous recursive equation is given by

$$\begin{aligned} x_N(k) &= a \exp^{\diamond} \left\{ \alpha B\left(\frac{1}{N} \right) \right\} \cdots \exp^{\diamond} \left\{ \alpha \left(B\left(\frac{k}{N} \right) - B\left(\frac{k-1}{N} \right) \right) \right\} \\ &= a \exp^{\diamond} \left\{ \alpha B\left(\frac{k}{N} \right) \right\}, \quad k \in \{0, ..., N\}. \end{aligned}$$

Recalling the correspondence between Wick and ordinary exponentials mentioned in the statement of the theorem we find:

$$x_N(k) = a \exp \left\{ \alpha B\left(\frac{k}{N} \right) - \frac{\alpha^2 k}{2N} \right\} \tag{2.8}$$

so the $x_N(k)$ written above in expression (2.8) coincides with $X\left(\frac{k}{N}\right)$, the strong solution of (1.1), with $\sigma(x) = \alpha x$, evaluated at $\frac{k}{N}$ (compare with Figure (1a)). The proof is complete.

References

[1] B. K. Ben Ammou and A. Lanconelli. Rate of convergence for Wong-Zakai-type approximations of Itô stochastic differential equations. *J. Theoret. Probab.*, 32(4):1780–1803, 2019.

[2] P. Da Pelo, A. Lanconelli, and A. I. Stan. A Hölder-Young-Lieb inequality for norms of Gaussian Wick products. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 14(3):375–407, 2011.

[3] P. Da Pelo, A. Lanconelli, and A. I. Stan. An Itô formula for a family of stochastic integrals and related Wong-Zakai theorems. *Stochastic Process. Appl.*, 123(8):3183–3200, 2013.

[4] H. Holden, B. Øksendal, J. Ubø e, and T. Zhang. *Stochastic partial differential equations*. Universitext. Springer, New York, second edition, 2010. A modeling, white noise functional approach.

[5] Y. Hu. *Analysis on Gaussian spaces*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017.

[6] Y. Hu and B. Øksendal. Wick approximation of quasilinear stochastic differential equations. In *Stochastic analysis and related topics, V (Silivri, 1994)*, volume 38 of *Progr. Probab.*, pages 203–231. Birkhäuser Boston, Boston, MA, 1996.

[7] S. Janson. *Gaussian Hilbert spaces*, volume 129 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1997.

[8] P. E. Kloeden and E. Platen. *Numerical solution of stochastic differential equations*, volume 23 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, 1992.

[9] H.-H. Kuo. *White noise distribution theory*. Probability and Stochastics Series. CRC Press, Boca Raton, FL, 1996.

[10] A. Lanconelli. Absolute continuity and Fokker-Planck equation for the law of Wong-Zakai approximations of Itô's stochastic differential equations. *J. Math. Anal. Appl.*, 482(2):123557, 19, 2020.

[11] A. Lanconelli and R. Scorolli. Wong-Zakai approximations for quasilinear systems of Itô's type stochastic differential equations. *Stochastic Process. Appl.*, 141:57–78, 2021.

[12] A. Lanconelli and R. Scorolli. A small time approximation for the solution to the Zakai equation. *Potential Anal.*, 58(3):561–571, 2023.

[13] A. Lanconelli and L. Sportelli. Wick calculus for the square of a Gaussian random variable with application to Young and hypercontractive inequalities. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 15(3):1250018, 16, 2012.

[14] A. Lanconelli and A. I. Stan. A Hölder inequality for norms of Poissonian Wick products. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 16(3):1350022, 39, 2013.

[15] A. Lanconelli and A. I. Stan. A note on a local limit theorem for Wiener space valued random variables. *Bernoulli*, 22(4):2101–2112, 2016.

[16] G. N. Milstein. Approximate integration of stochastic differential equations. *Teor. Verojatnost. i Primenen.*, 19:583–588, 1974.

[17] E. Wong and M. Zakai. On the convergence of ordinary integrals to stochastic integrals. *Ann. Math. Statist.*, 36:1560–1564, 1965.

[18] E. Wong and M. Zakai. On the relation between ordinary and stochastic differential equations. *Internat. J. Engrg. Sci.*, 3:213–229, 1965.