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RESEARCH

On the Harmonic Characterization Of The Spheres: A Sharp Stability Inequality

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Abstract

Let D be a bounded open subset of $\mathbb{R}^n$ with finite $(n-1)$-dimensional Hausdorff measure $|\partial D|$ and let x_0 be a point of D. We introduce a new harmonic invariant, that we call Kuran gap of ∂D w.r.t. x_0. To define this new invariant, denoted $\mathcal{K}(\partial D, x_0)$, we use a family of harmonic functions introduced by Kuran (Bull. London Math. Soc. **4**, 311-312, 1972). Our main stability result can be described as follows: if ∂D is sufficiently regular just in one of the points of ∂D nearest to x_0, then $\mathcal{K}(\partial D, x_0)$ is bounded from below by a kind of isoperimetric index, precisely the normalized difference between $|\partial D|$ and $|\partial B|$, being B the biggest ball contained in D and centered at x_0. This partially extends and improves a stability result by Preiss and Toro. By our stability result, we also obtain new rigidity results: *(i)* a characterization of the Euclidean spheres in terms of single-layer potentials, improving previous theorems by Fichera and by Shahgholian; *(ii)* a sufficient condition for a harmonic pseudosphere to be a Euclidean sphere, partially extending and improving rigidity results by Lewis and Vogel.

Keywords Surface gauss mean value formula · Stability · Harmonic functions · Rigidity

Mathematics Subject Classification (2010) Primary: 35B05 · Secondary: 31B05 · 35B06

Basic Notation

If D is an open set in $\mathbb{R}^n$, we write
$\overline{D}$:= closure of D
$|D|$:= Lebesgue measure of D in $\mathbb{R}^n$
$|\partial D| := H^{n-1}(\partial D)$= $(n-1)$-Hausdorff measure of ∂D
$d\sigma := dH^{n-1}$
$B(x_0, r)$:= Euclidean ball in $\mathbb{R}^n$ with center x_0 and radius r and we denote
$\omega_n := |B(0,1)|$
$\sigma_n := |\partial B(0,1)|$
Δ:= Laplace operator

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$u : D \to \mathbb{R}$ is harmonic in D:= u is smooth and $\Delta u = 0$ in D
If D is an open set in $\mathbb{R}^n$, we write
$\mathcal{H}(D)$:= set of the harmonic functions in D
$\mathcal{H}(\overline{D})$:= set of the harmonic functions in an open set containing $\overline{D}$
$C(A)$:=set of real continuous functions in A

1 Introduction and Main Results

1.1 Spheres and Harmonic Pseudospheres

Let $D \subseteq \mathbb{R}^n$, $n \geq 2$, be a bounded open set with $|\partial D| < \infty$ and let $x_0 \in D$. We say that ∂D *is a harmonic pseudosphere centered at* x_0 *if*

$$u(x_0) = \fint_{\partial D} u \, d\sigma \qquad \forall u \in \mathcal{H}(D) \cap C(\overline{D}). \tag{1.1}$$

Obviously, by Gauss Theorem, every Euclidean sphere is a harmonic pseudosphere. In general, the vice-versa is not true. Keldysch and Lavrentieff - in 1937 - proved the existence of a harmonic pseudosphere in $\mathbb{R}^2$, which is not a circle, see [10]. Many years later, in 1991, Lewis and Vogel proved that in every Euclidean space $\mathbb{R}^n$, $n \geq 2$, there exist harmonic preudospheres which are not spheres, see [12]. Actually, Lewis and Vogel proved something stronger: the existence of harmonic pseudospheres homeomorphic to $\partial B(0, 1)$ via Hölder continuous homeomorphisms.[1]

The following question naturally arises: *when is a harmonic pseudosphere a sphere*? Or, equivalently: is it possible to characterize the Euclidean spheres in terms of the surface mean value property for harmonic functions? The answer is yes. Here is a list of the main positive answers to this question.

A harmonic pseudosphere ∂D is a Euclidean sphere if

(i) ∂D is $C^{1,\alpha}$: Friedman and Littman (1962) [7],
(ii) ∂D is C^1: Fichera (1985) [5], see also [6],
(iii) ∂D is Lipschitz continuous: Lewis and Vogel (1989) [13],
(iv) D is Dirichlet regular and $H^{n-1} \llcorner \partial D$ has at most Euclidean growth: Lewis and Vogel (2002) [14].

We recall that, by definition, $H^{n-1} \llcorner \partial D$ *has at most Euclidean growth* if

$$\sup_{\substack{x \in \partial D \\ 0 < r < 1}} \frac{H^{n-1}(\partial D \cap B(x, r))}{r^{n-1}} < \infty. \tag{1.2}$$

Moreover, a bounded open set $D \subseteq \mathbb{R}^n$ is called *Dirichlet regular* if the boundary value problem

$$\begin{cases} \Delta u = 0 \text{ in } D \\ u|_{\partial D} = \varphi \end{cases}$$

has a classical solution $u \in \mathcal{H}(D) \cap C(\overline{D})$, for every $\varphi \in C(\partial D)$.

Up today, as far as we know, Lewis and Vogel's Theorem in (iv) is the most general and the deepest answer to the problem of the harmonic characterization of the Euclidean spheres.

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[1] The interested reader can find more details on the spherical mean value properties of harmonic functions and on the pseudospheres on the excellent survey by Netuka and Veselý [16].

An answer requiring very mild assumptions, but on the whole boundary of the domain. From the stability inequality that we prove in the present paper, it follows that a harmonic pseudosphere ∂D, with center at a point x_0, actually is a Euclidean sphere if it satisfies a stronger assumption, but only near to a single point, precisely near to one the point of ∂D closest to x_0; see Theorem 1.5.

1.2 Harmonic Stability of the Euclidean Spheres

The assumptions in Lewis-Vogel's Theorem [14] imply that ∂D has a Poisson kernel with respect to $x_0 \in D$. Precisely, there exists $h : \partial D \to \mathbb{R}$, $h \geq 0$, such that

$$u(x_0) = \int_{\partial D} hu \, d\sigma \qquad \forall u \in \mathcal{H}(D) \cap C(\overline{D}).$$

Then Lewis and Vogel's result can be rephrased as follows: let D be a bounded Dirichlet regular open set, such that $|\partial D| < \infty$ and $H^{n-1} \llcorner \partial D$ has at most Euclidean growth. If the Poisson kernel of D with respect to $x_0 \in D$ is constant, i.e.

$$h(x) = \frac{1}{|\partial D|} \qquad \forall x \in \partial D,$$

then ∂D is a Euclidean sphere centered at x_0.

In 2007 Preiss and Toro proved the stability of this Lewis and Vogel's rigidity theorem.

Theorem (Preiss and Toro [18]) *Let $D \subseteq \mathbb{R}^n$, x_0 and h be as in the Lewis and Vogel's result. Assume that*

$$\mathscr{H}(\partial D, x_0) := \sup_{\partial D} ||\partial D|h - 1|$$

is sufficiently small. Then, the following stability inequalities hold:

$$\mathscr{H}(\partial D, x_0) \geq c \, \frac{|\partial D| - |\partial B|}{|\partial D|} \tag{1.3}$$

and

$$\mathscr{H}(\partial D, x_0) \geq c \, \frac{|\partial B^*| - |\partial D|}{|\partial D|}, \tag{1.4}$$

where $c > 0$ is an absolute constant, B is the biggest ball centered at x_0 *contained in D, B^* is the* smallest ball centered at x_0 *containing D.*

In particular it follows that

$$\mathscr{H}(\partial D, x_0) \geq \frac{c}{2} \, \frac{|\partial B^*| - |\partial B|}{|\partial D|}. \tag{1.5}$$

Note 1.1 In case $n = 2$ and for $C^{1,\alpha}$-domains, an inequality as Eq. 1.5 has been proved by Agostiniani and Magnanini [1] without the smallness assumption on $\mathscr{H}(\partial D, x_0)$; see also the survey's paper by Magnanini [15].

We want to stress that the Preiss and Toro's smallness assumption on $\mathscr{H}(\partial D, x_0)$ obviously implies the boundedness of the Poisson kernel h, a condition which in general is not satisfied even for C^1-domains, see [9] and [17]. Then, assuming $\mathscr{H}(\partial D, x_0)$ small enough is implicitly like assuming regularity properties of ∂D. This is deeply analyzed in Preiss and Toro's paper: we directly refer to [18] for details.

1.3 The Closeness to a Sphere does Not Imply the Smallness of $\mathscr{H}$

Preiss and Toro's Theorem shows that the smallness of $\mathscr{H}$ is a sufficient condition for a domain to be geometrically close to a sphere. This condition, actually, is only sufficient, as the following theorem shows.

Theorem 1.2 *In $\mathbb{R}^n$, $n \geq 2$, there exists a family $(\hat{D}(\varepsilon))_{0<\varepsilon<\varepsilon_0}$ of Lipschitz-continuous domains containing the origin and such that, for every ε, $0 < \varepsilon < \varepsilon_0$,*

(i) $B(0,1) \subseteq \hat{D}(\varepsilon) \subseteq B(0, 1+\varepsilon)$

(ii) $\frac{1}{c}\varepsilon^{n-1} \leq \dfrac{|\partial \hat{D}(\varepsilon)| - |\partial B(0,1)|}{|\partial \hat{D}(\varepsilon)|} \leq c\,\varepsilon^{n-1}$

(iii) $\liminf\limits_{\varepsilon\to 0} \mathscr{H}(\partial \hat{D}(\varepsilon), 0) > 0,$

where c is an absolute constant. Moreover $B(0,1)$ is the biggest ball centered at 0 and contained in $\hat{D}(\varepsilon)$.

We will prove Theorem 1.2 in Sect. 3, see in particular Subsection 3.5.

We want to stress that the domains $\hat{D}(\varepsilon)$ are very small Lipschitz-continuous deformations of the unit ball in $\mathbb{R}^n$. Therefore, the cohexistence of (i) and (ii) with (iii) in Theorem 1.2 for the family $(\hat{D}(\varepsilon))_{0<\varepsilon<\varepsilon_0}$ highlights a pathological aspect of $\mathscr{H}(\,\cdot\,\cdot\,)$. To rule out this pathology is one of our motivations for introducing in next Subsection 1.4 a new harmonic invariant that we will call *Kuran gap* of the boundary of a domain D w.r.t. a point $x_0 \in D$.

1.4 The Kuran Gap and our Stability Result

We define our new harmonic invariant by using the functions introduced by Kuran in [11].

- THE KURAN FUNCTIONS. For $\alpha \in \mathbb{R}^n$, we call *α-Kuran function* the map $k_\alpha : \mathbb{R}^n \setminus \{\alpha\} \to \mathbb{R}$,

  $$k_\alpha = 1 + h_\alpha, \tag{1.6}$$

  where

  $$h_\alpha(x) := |\alpha|^{n-2}\frac{|x|^2 - |\alpha|^2}{|x-\alpha|^n}, \qquad x \neq \alpha. \tag{1.7}$$

  Up to a multiplicative constant, h_α is the Poisson kernel of $B(0, |\alpha|)$.
  It is quite well known that

  $$k_\alpha \in \mathcal{H}(\mathbb{R}^n \setminus \{\alpha\}) \quad \text{for every } \alpha \in \mathbb{R}^n.$$

  We note that, being $h_\alpha(0) = -1$, we have

  $$k_\alpha(0) = 0 \quad \text{for every } \alpha \neq 0.$$

- THE REGULAR TOUCHING SET. We say that an open D is of class C^1 at $z \in \partial D$ if there exists a neighborhood V of z such that

  $$\partial D \cap V \text{ is a } C^1\text{-manifold} \qquad \text{and} \qquad V \cap D = \operatorname{int}(V \cap \overline{D}).$$

  This assumption simply means that, in a neighborhood V of z, the boundary of D is the graph of a C^1-function of $(n-1)$-real variables and that D lies on one side of its boundary.

If $x_0 \in D$ and B is the biggest ball centered at x_0 and contained in D; i.e.

$$B := B(x_0, r_{x_0}), \qquad r_{x_0} = \operatorname{dist}(x_0, \partial D),$$

then we define

$$\begin{aligned} T(\partial D, x_0) &= \textit{regular touching set of } \partial D \textit{ w.r.t. } B \\ &:= \{z \in \partial D \cap \partial B \ : \ D \text{ is of class } C^1 \text{ at } z\}. \end{aligned}$$

We note that $z \in T(\partial D, x_0)$ is orthogonally accessible from $\mathbb{R}^n \setminus \overline{D}$. This aspect will play a crucial role in the sequel.

- Definition of Kuran gap. Let $D \subseteq \mathbb{R}^n$ be bounded and open with $|\partial D| < \infty$. We preliminarily assume $x_0 = 0 \in D$.
  If $T(\partial D, 0) = \emptyset$ we put
  $$\mathcal{K}(\partial D, 0) = \infty. \tag{1.8}$$
  If $T(\partial D, 0) \neq \emptyset$ we define
  $$\mathcal{K}(\partial D, 0) := \inf_{z \in T(\partial D, 0)} \liminf_{\substack{\alpha \searrow z \\ \alpha \notin \overline{D}}} \left| k_\alpha(0) - \fint_{\partial D} k_\alpha \, d\sigma \right|$$

$$= \inf_{z\in T(\partial D,0)} \liminf_{\substack{\alpha\searrow z\\ \alpha\notin\overline{D}}} \left| \fint_{\partial D} k_\alpha \, d\sigma \right| ; \tag{1.9}$$

with the notation $\alpha \searrow z$, $\alpha \notin \overline{D}$, we mean that α *radially goes to* z from outside of $\overline{D}$; i.e.

$$\alpha = tz, \quad t > 1, \quad \alpha \notin \overline{D}, \quad t \to 1.$$

If $x_0 \in D$, $x_0 \neq 0$, we define

$$\mathcal{K}(\partial D, x_0) := \mathcal{K}(\partial(-x_0 + D), 0).$$

We call

$$\mathcal{K}(\partial D, x_0) \textit{ the Kuran gap of } D \textit{ w.r.t. } x_0.$$

- BASIC KURAN GAP'S PROPERTIES. It is quite easy to prove that $\mathcal{K}(\partial D, x_0)$ is invariant w.r.t. Euclidean

  translations, rotations and *dilations*.

  Moreover,

  $$\text{if } T(\partial D, x_0) \neq \emptyset \text{ and } \partial D \text{ is a } \textit{harmonic pseudosphere} \text{ centered at } x_0, \text{ then } \mathcal{K}(\partial D, x_0) = 0. \tag{1.10}$$

  Indeed, due to the translation invariance of the Kuran gap, it is enough to prove this claim when $x_0 = 0$. In this case, if $T(\partial D, 0) \neq \emptyset$, then $\mathcal{K}(\partial D, 0)$ is defined in Eq. 1.9. On the other hand, if ∂D is a harmonic pseudosphere centered at 0, then

  $$k_\alpha(0) - \fint_{\partial D} k_\alpha \, d\sigma = 0 \qquad \text{for every } \alpha \in \mathbb{R}^n \setminus \overline{D},$$

  so that $\mathcal{K}(\partial D, 0) = 0$.
- OUR STABILITY INEQUALITY. The main result of this paper, is the following stability theorem.

Theorem 1.3 *Let D be a bounded open subset of $\mathbb{R}^n$ containing x_0 and with $|\partial D| < \infty$. Let B be the* biggest ball centered at x_0 and contained in *D; i.e.*

$$B := B(x_0, r_{x_0}), \quad r_{x_0} := \operatorname{dist}(x_0, \partial D).$$

Then

$$\mathcal{K}(\partial D, x_0) \geq \frac{|\partial D| - |\partial B|}{|\partial D|}. \tag{1.11}$$

Of course, inequality Eq. 1.11 is not trivial only if $T(\partial D, x_0) \neq \emptyset$.

As a first consequence of Theorem 1.3, we have the following *solid* stability inequality.

Corollary 1.4 *Let D, x_0 and B be as in Theorem 1.3. Then*

$$\mathcal{K}(\partial D, x_0) \geq \frac{(n-1)\omega_n^{\frac{1}{n}}}{|D|^{\frac{1}{n}}} \frac{|D \setminus B|}{|\partial D|}. \tag{1.12}$$

Corollary 1.4 comes from Theorem 1.3 and the isoperimetric inequality. Indeed, since $B \subseteq D$ from the isoperimetric inequality we obtain

$$|\partial D| - |\partial B| \geq n\omega_n^{\frac{1}{n}} \left(|D|^{\frac{n-1}{n}} - |B|^{\frac{n-1}{n}} \right) \geq \frac{(n-1)\omega_n^{\frac{1}{n}}}{|D|^{\frac{1}{n}}} |D \setminus B|, \tag{1.13}$$

where in the last inequality we applied the Lagrange mean value theorem to $t \mapsto t^{\frac{n-1}{n}}$. From Eqs. 1.11 and 1.13 we get Eq. 1.12.

Inequalities Eqs. 1.11 and 1.12 allow us to say that if $\mathcal{K}(\partial D, x_0)$ is *small* then

D is *close* to the ball B

and

$|\partial D|$ is *close* to $|\partial B|$.

We want to stress that inequality Eq. 1.11 and its consequence Eq. 1.12 do not require the smallness of the Kuran gap.

Our method to prove Theorem 1.3 (see Sect. 2) is direct and does not require the profound real and harmonic analysis techniques used by Lewis and Vogel, and by Preiss and Toro, in their papers quoted above.

- ON OUR REGULARITY ASSUMPTION ON THE DOMAINS.
  We point out that, for the inequalities Eqs. 1.11 and 1.12 to be non-trivial, we require a quite strong assumption on the boundary *at a single point*, i.e., the C^1-regularity of ∂D in a neighborhood of just one point of its touching set. On the other hand, we do not require any *global* - even weak - regularity property of the boundary. In particular, differently than in Lewis-Vogel's and Preiss-Toro's papers, we do not assume neither the *Dirichlet regularity* of D nor the *at most Euclidean growth* Eq. 1.2.
  Moreover, we have to remark that the C^1-regularity of ∂D at a single point actually seems not a strong assumption: as a matter of fact, it is not even enough to guarantee the boundedness of the Kuran gap. Indeed, if $\mathcal{K}(\partial D, x_0) < \infty$ then ∂D has to be *more than* C^1 in at least one point of its touching set. See Remark 2.2 for the precise statement.
- ON THE SHARPNESS OF OUR STABILITY INEQUALITY.
  In Theorem 1.2 we prove that in $\mathbb{R}^n$, $n \geq 2$, there exists a family $(\hat{D}(\varepsilon))_{0<\varepsilon<\varepsilon_0}$ of Lipschitz-continuous domains containing the origin, such that, for every ε, $0 < \varepsilon < \varepsilon_0$, $B(0,1)$ is the biggest ball centered at 0 and contained in $\hat{D}(\varepsilon)$,

$$B(0,1) \subseteq \hat{D}(\varepsilon) \subseteq B(0, 1+\varepsilon)$$

  and

$$\frac{1}{c}\varepsilon^{n-1} \leq \frac{|\partial \hat{D}(\varepsilon)| - |\partial B(0,1)|}{|\partial \hat{D}(\varepsilon)|} \leq c\,\varepsilon^{n-1} \tag{1.14}$$

  where c is an absolute constant.
  On the other hand, see Eq. 3.22, we have

$$\frac{1}{c}\varepsilon^{n-1} \leq \mathcal{K}(\partial \hat{D}(\varepsilon), 0) \leq c\,\varepsilon^{n-1}. \tag{1.15}$$

  These inequalities, together with Eqs. 1.11 and 1.14, imply

$$\frac{|\partial \hat{D}(\varepsilon)| - |\partial B(0,1)|}{|\partial \hat{D}(\varepsilon)|} \leq \mathcal{K}(\partial \hat{D}(\varepsilon), 0) \leq c\frac{|\partial \hat{D}(\varepsilon)| - |\partial B(0,1)|}{|\partial \hat{D}(\varepsilon)|}. \tag{1.16}$$

  Estimates Eqs. 1.15 and 1.16 prove that our stability inequality is sharp, in the following sense. The right-hand side of inequality Eq. 1.11 is less than 1, so that it implies

$$\mathcal{K}(\partial D, x_0) \geq \left(\frac{|\partial D| - |\partial B|}{|\partial D|}\right)^{\alpha} \tag{1.17}$$

for every exponent $\alpha \geq 1$. Then a natural question, quite standard in analogous stability settings, is the following one: is there some $\alpha \in]0,1[$ such that inequality Eq. 1.17 holds true for every domain D satisfying the assumptions of Theorem 1.3? Inequalities Eqs. 1.15 and 1.16 show that the answer to this question is no! Hence the sharp exponent in Eq. 1.17 is $\alpha = 1$, the exponent appearing in Eq. 1.11.

1.5 A Sufficient Condition for a Harmonic Pseudosphere to be a Sphere

From Corollary 1.4 one easily obtains the following theorem.

Theorem 1.5 *Let ∂D be a harmonic pseudosphere centered at x_0. If $T(\partial D, x_0) \neq \emptyset$ then ∂D is a Euclidean sphere centered at x_0.*

Proof We have already observed in Eq. 1.10 that the assumptions of this theorem imply $\mathcal{K}(\partial D, x_0) = 0$. Then, by inequality Eq. 1.12, $|D \setminus B| = 0$, where B is the biggest ball centered at x_0 and contained in D. Since D is open and B is a ball, this implies $D = B$. ◻

1.6 A Characterization of the Sphere in Terms of Single-Layer Potentials

Let B be a Euclidean ball centered at x_0 and let Γ be the fundamental solution of the Laplace equation. Then $x \mapsto \Gamma(x-y)$ is harmonic in $\overline{B}$ for every $y \notin \overline{B}$. As a consequence, by the Gauss Theorem

$$\fint_{\partial B} \Gamma(x-y)\, d\sigma(x) = \Gamma(x_0 - y) \qquad \forall y \notin \overline{B}.$$

This identity is a *rigidity* property of the Euclidean sphere. Indeed, as a by-product of Theorem 1.3, we get the following result.

Theorem 1.6 *Let $D \subseteq \mathbb{R}^n$ be a bounded open set containing a point x_0 and such that $|\partial D| < \infty$. Assume that, for a suitable constant $c > 0$,*

$$\int_{\partial D} \Gamma(x-y)\, d\sigma(x) = c\, \Gamma(x_0 - y) \qquad \forall y \notin \overline{D}. \tag{1.18}$$

Then $c = |\partial D|$ and, if $T(\partial D, x_0) \neq \emptyset$, D is a Euclidean ball centered at x_0.

We will prove this theorem in Sect. 4. Here we only want to stress that a simple and elegant proof of Theorem 1.6 for $C^{1,\alpha}$-domains is due to Shahgholian, see [19]. However, as far as we know, the first proof of Theorem 1.6 for C^1-domains is basically already contained in Fichera's paper [5]. Indeed the main theorem in [5] can be rephrased as follows: *"A harmonic pseudosphere of class C^1 is a Euclidean sphere"*, but, by carefully reading Fichera's proof, one easily realizes that, actually, Fichera proves the following theorem: *"Let D be a bounded open set containing a point x_0 and such that ∂D is of class C^1. If identity Eq. 1.18 is satisfied with $c = |\partial D|$, then ∂D is a Euclidean sphere centered at x_0."*

The plan of the paper is the following: in Sect. 2 we prove Theorem 1.3, in Sect. 3 we prove Theorem 1.2, in Sect. 4 we prove Theorem 1.6.

2 Proof of Theorem 1.3

Let D be a bounded open subset of $\mathbb{R}^n$ containing x_0 and with $|\partial D| < \infty$. It suffices to prove the theorem under the assumptions $x_0 = 0$.

Our proof of Theorem 1.3 requires several preliminaries, which we will develop in the subsections below.

STEP 1. The boundary of D near a point $z \in T(\partial D, 0)$ Throughout the proof we split $\mathbb{R}^n$ as $\mathbb{R}^{n-1} \times \mathbb{R}$ and denote a point $x \in \mathbb{R}^n$ as

$$x = (y, t) \quad \text{with } y \in \mathbb{R}^{n-1},\ t \in \mathbb{R}.$$

Fix $z \in T(\partial D, 0)$. Since $\mathcal{K}(\partial D, 0)$ is rotation invariant, we can assume $z = (0, r_0) \in \mathbb{R}^{n-1} \times \mathbb{R}$, where $r_0 := \operatorname{dist}(\partial D, 0)$.
For $R > 0$ we let

$$\hat{B}_R := \{y \in \mathbb{R}^{n-1} : |y| < R\}$$

and, if $-\infty < a < b < \infty$,

$$U_R(a, b) := \hat{B}_R \times]a, b[.$$

Since $z \in T(\partial D, 0)$, there exist positive constants a, b, with $a < r_0 < b$, $R_0 > 0$, and a function $\Phi \in C^1(\hat{B}_{R_0},]a, b[)$ such that

(a) $\partial D \cap U_{R_0}(a, b) = \{(y, \Phi(y)) : y \in \hat{B}_{R_0}\}$;
(b) $D \cap U_{R_0}(a, b) = \{(y, t) : y \in \hat{B}_{R_0},\ a < t < \Phi(y) < b\}$;
(c) $z = (0, r_0) = (0, \Phi(0))$;
(d) $\nabla\Phi(0) = 0$;
(e) $\sup_{y \in \hat{B}_{R_0}} |\nabla\Phi(y)| < \infty$.

We point out that $\nabla\Phi(0) = 0$ since the ball $B = B(0, r_0)$ is tangent to ∂D at z.
From now on, for $0 < R < R_0$ we simply write U_R instead of $U_R(a, b)$.
Moreover, we consider $\alpha \in \mathbb{R}^n$,

$$\alpha = (0, r) \quad \text{with } r_0 < r < b. \tag{2.1}$$

In particular, $\alpha \in U_R \setminus \overline{D}$, so that k_α and h_α are harmonic in a neighborhood of $\overline{D}$.
Moreover,

$$(\alpha \searrow z \text{ with } \alpha \notin \overline{D}) \quad \text{iff} \quad r \to r_0. \tag{2.2}$$

STEP 2. The function $(x, \alpha) \mapsto k_\alpha(x)$ is smooth in $(\mathbb{R}^n \times \mathbb{R}^n) \setminus \{x = \alpha\}$ so that

$$\sup_{\partial D \setminus U_R} |k_\alpha| < C(R) \tag{2.3}$$

for a constant $C(R)$ independent of α if α is sufficiently close to z. Then, by the dominated convergence theorem,

$$\lim_{\substack{\alpha \searrow z \\ \alpha \notin \overline{D}}} \int_{\partial D \setminus U_R} k_\alpha \, d\sigma = \int_{\partial D \setminus U_R} \lim_{\substack{\alpha \searrow z \\ \alpha \notin \overline{D}}} k_\alpha \, d\sigma = \int_{\partial D \setminus U_R} k_z \, d\sigma.$$

On the other hand if $x \in \partial D \setminus U_R$ then $x \notin B(0, r_0)$ $(= B(0, |z|))$ and $x \neq z$, so that $|x| \geq |z|$ and

$$k_z(x) = 1 + |z|^{n-2} \frac{|x|^2 - |z|^2}{|x - z|^n} \geq 1.$$

Then, using also Eq. 2.3,

$$\infty > \lim_{\substack{\alpha \searrow z \\ \alpha \notin \overline{D}}} \int_{\partial D \setminus U_R} k_\alpha \, d\sigma \geq |\partial D \setminus U_R|. \tag{2.4}$$

STEP 3. In order to estimate

$$\int_{\partial D \cap U_R} k_\alpha \, d\sigma, \tag{2.5}$$

for α close to z, we first remark that, by definition of k_α,

$$\int_{\partial D \cap U_R} k_\alpha \, d\sigma = |\partial D \cap U_R| + \int_{\partial D \cap U_R} h_\alpha \, d\sigma. \tag{2.6}$$

Moreover, keeping in mind that $\partial D \cap U_R$ is the graph of the function Φ,

$$\int_{\partial D \cap U_R} h_\alpha \, d\sigma = \int_{\{|y|<R\}} h_\alpha(y, \Phi(y)) \, N(y) \, dy, \tag{2.7}$$

where

$$N(y) = \sqrt{1 + |\nabla \Phi(y)|^2}. \tag{2.8}$$

We recall that every $x \in \partial D \cap U_R$ can be written as $x = (y, \Phi(y))$, with $y \in \mathbb{R}^{n-1}$, $|y| < R$, so that, since $\alpha = (0, r)$,

$$x - \alpha = (y, \Phi(y) - r).$$

Therefore, keeping in mind Eqs. 1.7 and 2.7,

$$\int_{\partial D \cap U_R} h_\alpha \, d\sigma = -r^{n-2} \int_{\{|y|<R\}} \frac{r^2 - \left(|y|^2 + (\Phi(y))^2\right)}{\left(|y|^2 + (\Phi(y) - r)^2\right)^{\frac{n}{2}}} N(y) \, dy. \tag{2.9}$$

STEP 4. We now consider the last integral in Eq. 2.9.
By using the polar coordinates that integral is equal to

$$\int_0^R \left(\int_{\{|y|=\rho\}} \frac{\left(r^2 - (\Phi(y))^2\right) - \rho^2}{\left(\rho^2 + (\Phi(y) - r)^2\right)^{\frac{n}{2}}} N(y) \, d\sigma(y) \right) d\rho$$
$$= (\text{letting } y = \rho\eta, \ \eta \in \mathbb{R}^{n-1}, \ |\eta| = 1)$$
$$\int_0^R \rho^{n-2} \left(\int_{\{|\eta|=1\}} \frac{\left(r^2 - (\Phi(\rho\eta))^2\right) - \rho^2}{\left(\rho^2 + (\Phi(\rho\eta) - r)^2\right)^{\frac{n}{2}}} N(\rho\eta) \, d\sigma(\eta) \right) d\rho.$$

By changing the integration order and letting $\rho = (r - r_0)s$, the right hand side is equal to

$$\int_{\{|\eta|=1\}} \left(\int_0^{\frac{R}{r-r_0}} \frac{s^{n-2}}{\left(s^2 + \left(\frac{\Phi - r}{r - r_0}\right)^2\right)^{\frac{n}{2}}} \left(\frac{r^2 - \Phi^2}{r - r_0} - s^2 (r - r_0) \right) N \, ds \right) d\sigma(\eta)$$

where

$$\Phi := \Phi((r - r_0)s\eta), \qquad N := N((r - r_0)s\eta).$$

Letting

$$I_\alpha := -r^{n-2}\int_{\{|\eta|=1\}}\Big(\int_0^{\frac{R}{r-r_0}}\frac{s^{n-2}}{\Big(s^2+\big(\frac{\Phi-r}{r-r_0}\big)^2\Big)^{\frac{n}{2}}}\frac{r^2-\Phi^2}{r-r_0}N\,ds\Big)\,d\sigma(\eta) \tag{2.10}$$

and

$$J_\alpha := r^{n-2}\int_{\{|\eta|=1\}}\Big(\int_0^{\frac{R}{r-r_0}}\frac{s^{n}}{\Big(s^2+\big(\frac{\Phi-r}{r-r_0}\big)^2\Big)^{\frac{n}{2}}}(r-r_0)N\,ds\Big)\,d\sigma(\eta), \tag{2.11}$$

by Eq. 2.9 we have

$$\int_{\partial D\cap U_R} h_\alpha\,d\sigma = I_\alpha + J_\alpha. \tag{2.12}$$

STEP 5. To study the behavior of I_α as $\alpha \to z$ (i.e. as $r \to r_0$), in this step, recalling that

$$\Phi = \Phi((r-r_0)s\eta),$$

we analyze the behavior of

$$\frac{r^2-\Phi^2}{r-r_0} \quad \text{and} \quad \frac{\Phi-r}{r-r_0}$$

as $r \to r_0$.

We have

$$\frac{r^2-\Phi^2}{r-r_0} = \frac{r-\Phi}{r-r_0}(r+\Phi), \qquad \frac{r-\Phi}{r-r_0} = 1-\frac{\Phi-r_0}{r-r_0}$$

We let

$$\omega(y) := \frac{\Phi(y)-\Phi(0)}{|y|}, \qquad y \neq 0. \tag{2.13}$$

We explicitly notice that, since Φ is differentiable at 0 and $\nabla\Phi(0) = 0$, then $\omega(y) \to 0$ as $y \to 0$ and ω is bounded on $\hat{B}_R$, because Φ is bounded.

By Eq. 2.13 it follows

$$\Phi(y) - r_0 = \Phi(y) - \Phi(0) = |y|\omega(y), \tag{2.14}$$

so that

$$\frac{r-\Phi(y)}{r-r_0} = 1-\frac{|y|\omega(y)}{r-r_0},$$

and, being $y = (r-r_0)s\eta$ with $|\eta| = 1$,

$$\frac{r-\Phi}{r-r_0} = 1 - s\omega((r-r_0)s\eta).$$

As a consequence,

$$\frac{r^2-\Phi^2}{r-r_0} = (r+\Phi) - (r+\Phi)s\omega((r-r_0)s\eta).$$

Recalling that

$$N = N(s(r-r_0)\eta) = \sqrt{1+|\nabla\Phi(s(r-r_0)\eta)|^2},$$

we get

$$\sup_{s\in[0,\frac{R}{r-r_0}]} N(s(r-r_0)\eta) \le \sqrt{1+\|\nabla\Phi\|^2_{L^\infty(\hat{B}_R)}} \tag{2.15}$$

and, by $\nabla\Phi(0)=0$,

$$N = N(s(r-r_0)\eta) = 1 + o(1) \quad \text{as } r \to r_0,$$

uniformly w.r.t. $\eta \in \mathbb{R}^{n-1}$ with $|\eta|=1$, and $s\in[0,a]$ for any fixed $a>0$.
With these estimates at hand, it is convenient to split I_α as follows:

$$I_\alpha = I_\alpha^{(1)} + I_\alpha^{(2)}, \tag{2.16}$$

with

$$I_\alpha^{(1)} := -r^{n-2}\int_{\{|\eta|=1\}}\Big(\int_0^{\frac{R}{r-r_0}} \frac{s^{n-2}}{\big(s^2+(1-s\omega)^2\big)^{\frac{n}{2}}}(r+\Phi)N\,ds\Big)\,d\sigma(\eta) \tag{2.17}$$

and

$$I_\alpha^{(2)} := r^{n-2}\int_{\{|\eta|=1\}}\Big(\int_0^{\frac{R}{r-r_0}} \frac{s^{n-1}}{\big(s^2+(1-s\omega)^2\big)^{\frac{n}{2}}}(r+\Phi)\,\omega\,N\,ds\Big)\,d\sigma(\eta), \tag{2.18}$$

where, similarly to the notations Φ and N,

$$\omega := \omega((r-r_0)s\eta). \tag{2.19}$$

Using the notation Eq. 2.16 in Eq. 2.12, we obtain

$$\int_{\partial D\cap U_R} h_\alpha\,d\sigma = I_\alpha^{(1)} + I_\alpha^{(2)} + J_\alpha. \tag{2.20}$$

STEP 6. In this step we show that

$$\lim_{\substack{\alpha\searrow z\\ \alpha\notin\overline{D}}} I_\alpha^{(1)} = -|\partial B|, \tag{2.21}$$

where $B=B(0,r_0)$ is the Euclidean ball in $\mathbb{R}^n$ with center 0 and radius r_0.
This is obtained by passing to the limit under the integral sign. Let us show that this is possible.
By Eq. 2.1 and the boundedness of ω there exists $0<\delta<\frac{R}{b-r_0}<\frac{R}{r-r_0}$ such that

$$|s\omega| \le \frac{1}{2} \qquad \forall s\in[0,\delta].$$

Since the absolute value of the function $r\mapsto(r+\Phi)N$, for every r close to r_0, for instance $r_0<r<2r_0$, is bounded by the positive constant C

$$C = C(R) = \sup_{|y|\le R}\big(2r_0+|\Phi|\big)N, \tag{2.22}$$

we get

$$\frac{s^{n-2}}{\big(s^2+(1-s\omega)^2\big)^{\frac{n}{2}}}|r+\Phi|N \le \left[\begin{array}{ll} \dfrac{s^{n-2}}{(1-s\omega)^n}|r+\Phi|N \le C2^n s^{n-2} & \text{if } s\in[0,\delta] \\[2ex] \dfrac{s^{n-2}}{s^n}|r+\Phi|N \le \dfrac{C}{s^2} & \text{if } s\in\left]\delta,\frac{R}{r-r_0}\right[. \end{array}\right.$$

This estimate allows to pass to the limit under the integral sign in the left hand side in Eq. 2.21.

By the boundedness of ω on $\hat{B}_R$ and $\omega(y) \to 0$ as $y \to 0$, the definition of N, see Eq. 2.8, and recalling that $\Phi(0) = r_0$, $\nabla\Phi(0) = 0$, we get

$$\omega \to 0, \quad N \to 1, \quad (r + \Phi) \to 2r_0 \quad \text{as } r \to r_0.$$

Therefore

$$\lim_{\substack{\alpha \searrow z \\ \alpha \notin \overline{D}}} I_\alpha^{(1)} = -2r_0^{n-1}\sigma_{n-1} \int_0^\infty \frac{s^{n-2}}{(s^2+1)^{\frac{n}{2}}}\, ds$$

where σ_{n-1} is the area of the unit sphere of $\mathbb{R}^{n-1}$.

On the other hand,

$$2\sigma_{n-1} \int_0^\infty \frac{s^{n-2}}{(s^2+1)^{\frac{n}{2}}}\, ds = \sigma_n, \tag{2.23}$$

where σ_n is the area of the unit sphere of $\mathbb{R}^n$ (see Remark 2.1 below). Then

$$\lim_{\substack{\alpha \searrow z \\ \alpha \notin \overline{D}}} I_\alpha^{(1)} = -\sigma_n r_0^{n-1} = -|\partial B|.$$

Remark 2.1 For readers convenience we give the proof of Eq. 2.23.

It is known that for every dimension p, the surface measure of the unit spheres in $\mathbb{R}^p$ is

$$\sigma_p = \frac{2\pi^{\frac{p}{2}}}{\Gamma\left(\frac{p}{2}\right)},$$

where here Γ is the Euler's gamma function. Taking into account that $\Gamma\left(\frac{1}{2}\right) = \pi^{\frac{1}{2}}$ and using the Euler's beta function, we have

$$\begin{aligned}\frac{\sigma_n}{\sigma_{n-1}} &= \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\pi^{\frac{n-1}{2}}} = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \\ &= \beta\left(\frac{n-1}{2}, \frac{1}{2}\right) = \int_0^\infty \frac{t^{\frac{n-1}{2}-1}}{(1+t)^{\frac{n}{2}}}\, dt = 2\int_0^\infty \frac{s^{n-2}}{(1+s^2)^{\frac{n}{2}}}\, ds.\end{aligned}$$

STEP 7. In this step we study the behavior of $I_\alpha^{(2)}$ in Eq. 2.20. Keeping in mind Eq. 2.18 we first split ω as follows

$$\omega = \omega^+ - \omega^- := \max\{0, \omega\} - \max\{0, -\omega\}$$

and define

$$(I_\alpha^{(2)})^+ := r^{n-2} \int_{\{|\eta|=1\}} \left(\int_0^{\frac{R}{r-r_0}} \frac{s^{n-1}}{\left(s^2 + (1-s\omega)^2\right)^{\frac{n}{2}}} (r+\Phi)\, \omega^+ \, N\, ds \right) d\sigma(\eta), \tag{2.24}$$

and

$$(I_\alpha^{(2)})^- := r^{n-2} \int_{\{|\eta|=1\}} \left(\int_0^{\frac{R}{r-r_0}} \frac{s^{n-1}}{\left(s^2 + (1-s\omega)^2\right)^{\frac{n}{2}}} (r+\Phi)\, \omega^- \, N\, ds \right) d\sigma(\eta); \tag{2.25}$$

so that

$$I_\alpha^{(2)} = (I_\alpha^{(2)})^+ - (I_\alpha^{(2)})^-. \tag{2.26}$$

We observe that, by Eq. 2.14,

$$\Phi(y) - \Phi(0) = (\Phi(y) - \Phi(0))^+ - (\Phi(y) - \Phi(0))^- = |y|(\omega^+(y) - \omega^-(y)).$$

We now remark that

$$|y|\omega^-(y) = (\Phi(y) - \Phi(0))^- = \begin{cases} 0 & \text{if } \Phi(y) \geq \Phi(0) \\ \Phi(0) - \Phi(y) & \text{if } \Phi(y) \leq \Phi(0). \end{cases}$$

Notice that the graph of Φ is outside B; therefore if $\Phi(y) \leq \Phi(0)$ we have

$$0 \leq \Phi(0) - \Phi(y) \leq r_0 - \sqrt{r_0^2 - |y|^2} = \frac{|y|^2}{r_0 + \sqrt{r_0^2 - |y|^2}} \leq \frac{|y|^2}{r_0},$$

that is

$$0 \leq \omega^-(y) = \frac{(\Phi(y) - \Phi(0))^-}{|y|} \leq \frac{|y|}{r_0}.$$

By this estimate, Eq. 2.15 and recalling that $y = (r - r_0)s\eta$ with $|\eta| = 1$, we get that if $r_0 < r < r_0 + b$

$$0 \leq (I_\alpha^{(2)})^- \leq \frac{r^{n-2}}{r_0} \int_{\{|\eta|=1\}} \left(\int_0^{\frac{R}{r-r_0}} \frac{s^{n-1}}{\left(s^2 + (1 - s\omega)^2\right)^{\frac{n}{2}}} (r + \Phi)\, |(r - r_0)s\eta|\, N\, ds \right) d\sigma(\eta)$$

$$\leq c \int_0^{\frac{R}{r-r_0}} \frac{s^n}{s^n} (r - r_0)\, ds = c\, R,$$

with a constant c depending on n, r_0, b and $\|\nabla\Phi\|_\infty$. We have so proved that

$$0 \leq (I_\alpha^{(2)})^- \leq c\, R \tag{2.27}$$

for every $r_0 < r < b$. This inequality, together with Eq. 2.26 and the positivity of $(I_\alpha^{(2)})^+$, implies

$$I_\alpha^{(2)} \geq (I_\alpha^{(2)})^+ - cR \geq -cR. \tag{2.28}$$

STEP 8. By proceeding as in the previous step, we get an estimate of J_α for every α close to z.
By the definition of J_α, see Eq. 2.11, we obtain

$$|J_\alpha| \leq c \int_0^{\frac{R}{r-r_0}} (r - r_0)\, ds = c\, R \tag{2.29}$$

where c is a constant depending only on $n, r_0, b, \|\nabla\Phi\|_\infty$.

STEP 9. From Eqs. 2.6 and 2.20 we get

$$\int_{\partial D} k_\alpha\, d\sigma = \int_{\partial D \setminus U_R} k_\alpha\, d\sigma + \int_{\partial D \cap U_R} k_\alpha\, d\sigma$$

$$= \int_{\partial D \setminus U_R} k_\alpha\, d\sigma + |\partial D \cap U_R| + I_\alpha^{(1)} + I_\alpha^{(2)} + J_\alpha. \tag{2.30}$$

Therefore, by Eqs. 2.28 and 2.29, we obtain

$$\int_{\partial D} k_\alpha \, d\sigma \geq \int_{\partial D\setminus U_R} k_\alpha \, d\sigma + |\partial D \cap U_R| + I_\alpha^{(1)} - 2cR.$$

It follows by Eqs. 2.4 and 2.21 that

$$\begin{aligned}\liminf_{\substack{\alpha\searrow z\\ \alpha\notin\overline{D}}} \int_{\partial D} k_\alpha \, d\sigma \geq & |\partial D \setminus U_R| + |\partial D \cap U_R| - |\partial B| - 2cR \\ = & |\partial D| - |\partial B| - 2cR.\end{aligned}$$

Since this estimate holds for any $R \in]0, R_0]$, letting R go to zero, we conclude that

$$\liminf_{\substack{\alpha\searrow z\\ \alpha\notin\overline{D}}} \int_{\partial D} k_\alpha \, d\sigma \geq |\partial D| - |\partial B|. \tag{2.31}$$

STEP 10. Taking into account that $k_\alpha(0) = 0$, we have

$$\left| k_\alpha(0) - \fint_{\partial D} k_\alpha \, d\sigma \right| \geq \frac{1}{|\partial D|} \int_{\partial D} k_\alpha \, d\sigma;$$

therefore, using Eq. 2.31, we get

$$\liminf_{\substack{\alpha\searrow z\\ \alpha\notin\overline{D}}} \left| k_\alpha(0) - \fint_{\partial D} k_\alpha \, d\sigma \right| \geq \frac{|\partial D| - |\partial B|}{|\partial D|}. \tag{2.32}$$

Since this estimate holds for every $z \in T(\partial D, 0)$ and keeping in mind the definition of $\mathcal{K}(\partial D, 0)$, see Eq. 1.9, we conclude that

$$\mathcal{K}(\partial D, 0) \geq \frac{|\partial D| - |\partial B|}{|\partial D|}.$$

Remark 2.2 A careful estimate of the term $(I_\alpha^{(2)})^+$ in Eq. 2.24 allows to improve inequality Eq. 2.32 as follows:

$$\liminf_{\substack{\alpha\searrow z\\ \alpha\notin\overline{D}}} \left| k_\alpha(0) - \fint_{\partial D} k_\alpha \, d\sigma \right| \geq \frac{|\partial D| - |\partial B|}{|\partial D|} + c\frac{1}{|\partial D|} \lim_{R\to 0^+} \omega_R(\partial D, z), \tag{2.33}$$

where $c = c(z)$ is a real positive constant independent of $R \in]0, R_0[$ and

$$\omega_R(\partial D, z) := \int_{\{\xi\in\mathbb{R}^{n-1}\,:\,|\xi|\leq R\}} \frac{(\Phi(\xi) - \Phi(0))^+}{|\xi|^n} \, d\xi. \tag{2.34}$$

For readers convenience we postpone to the Appendix the proof of Eq. 2.33. Here we observe that, being $|\partial D| - |\partial B| \geq 0$, inequality Eq. 2.33 implies

$$\liminf_{\substack{\alpha\searrow z\\ \alpha\notin\overline{D}}} \left| k_\alpha(0) - \fint_{\partial D} k_\alpha \, d\sigma \right| \geq c\frac{1}{|\partial D|} \lim_{R\to 0^+} \omega_R(\partial D, z). \tag{2.35}$$

Therefore if the left-hand side of this inequality is finite, then

$$\omega_R(\partial D, z) < \infty \quad \text{for } 0 < R < R_0.$$

This condition is not satisfied, in general, if Φ is merely C^1.

We agree to say that ∂D is *more than* C^1 at a point $z \in T(\partial D, 0)$ if $\omega_R(\partial D, z)$ is finite for some $R > 0$. Thus, keeping in mind the very definitions of harmonic pseudosphere and of Kuran gap, from Eq. 2.35 we may conclude that:

(i) if ∂D is a harmonic pseudosphere centered at x_0 and $T(\partial D, x_0) \neq \emptyset$, then ∂D is more than C^1 at any point of $T(\partial D, x_0)$;
(ii) if $\mathcal{K}(\partial D, x_0) < \infty$ and $T(\partial D, x_0) \neq \emptyset$, then ∂D is more than C^1 in at least one point $z \in T(\partial D, x_0)$.

3 Proof of Theorem 1.2 and Sharpness of the Stability Inequality

3.1 The Beaked Sphere

In $\mathbb{R}^n$, $n \geq 2$, let us denote a vector $x \in \mathbb{R}^n$ as $x = (x_1, \hat{x}) \in \mathbb{R} \times \mathbb{R}^{n-1}$.

Fixed $\varepsilon > 0$, let $B(\varepsilon)$ be the ball centered at $x(\varepsilon) := (1 + \varepsilon, \hat{0})$ and radius 1; i.e.,

$$B(\varepsilon) := B(x(\varepsilon), 1).$$

Let us consider the cone

$$K := \left\{ x \in \mathbb{R}^n \ : \ \frac{x_1}{|x|} > \frac{1}{\sqrt{n}} \right\} = \left\{ (x_1, \hat{x}) \in \mathbb{R}^n \ : \ \frac{|\hat{x}|}{\sqrt{n-1}} < x_1 \right\}.$$

Define the function $u : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$,

$$u(x) := \frac{1}{|x|^n} \left(\frac{x_1^2}{|x|^2} - \frac{1}{n} \right). \tag{3.1}$$

Notice that $u \in \mathcal{H}(\mathbb{R}^n \setminus \{0\})$, since

$$u = c_n \frac{\partial^2 \Gamma}{\partial x_1^2},$$

where Γ is the fundamental solution of the Laplace operator with pole at 0 and c_n is a dimensional constant. Moreover,

$$u > 0 \quad \text{in } K$$

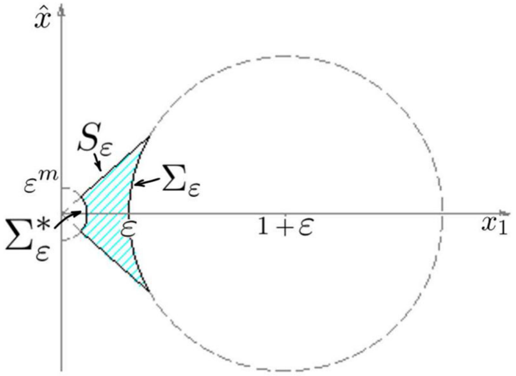

Fig. 1 The set $K(\varepsilon)$

and

$$u = 0 \quad \text{on } \partial K.$$

For every $0 < \varepsilon < \varepsilon_1$, with ε_1 suitable absolute constant sufficiently small, the set $K \cap \partial B(\varepsilon)$ has two non-empty connected components: one containing $(\varepsilon, \hat{0})$ and the other one containing $(2 + \varepsilon, \hat{0})$. We denote Σ_ε the first of these two components.

Fixed $m \in \mathbb{N}$, $m > n$, we define

$$\Sigma_\varepsilon^* := \{x \in K \ : \ |x| = \varepsilon^m\}$$

and

$$K(\varepsilon) := \{\lambda x + (1 - \lambda) y \ : \ x \in \Sigma_\varepsilon^*, \ y \in \Sigma_\varepsilon, \ \lambda \in [0, 1[\,\},$$

see Fig. 1. If we consider the open set

$$D(\varepsilon) := B(\varepsilon) \cup K(\varepsilon),$$

see Fig. 2, we will call *beaked sphere* its boundary $\partial D(\varepsilon)$. We have

$$\partial D(\varepsilon) = (\partial B(\varepsilon) \setminus \Sigma_\varepsilon) \cup \Sigma_\varepsilon^* \cup S_\varepsilon, \tag{3.2}$$

where

$$S_\varepsilon := \partial K \cap \partial K(\varepsilon).$$

We remark that

$$u \in \mathcal{H}(\overline{D(\varepsilon)}), \qquad u = 0 \ \text{ in } S_\varepsilon, \qquad u > 0 \text{ in } K(\varepsilon).$$

We observe that $D(\varepsilon)$ is geometrically close to a ball, since

$$B(\varepsilon) \subseteq D(\varepsilon) \subseteq B^*(\varepsilon), \tag{3.3}$$

where

$$B^*(\varepsilon) := B(x(\varepsilon), 1 + \varepsilon).$$

We stress that $B(\varepsilon)$ is the biggest ball centered at $x(\varepsilon)$ and contained in $D(\varepsilon)$.

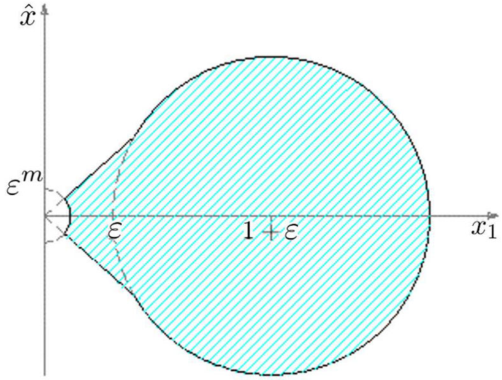

Fig. 2 The set $D(\varepsilon)$

3.2 The Poisson Kernel's Estimate of the Beaked Sphere

Since the beaked spheres are Lipschitz domains, then they have a Poisson kernel; hence $\mathscr{H}(\partial D(\varepsilon), x(\varepsilon))$ is well defined.

Aim of this subsection is to prove that

$$\liminf_{\varepsilon \to 0} \mathscr{H}(\partial D(\varepsilon), x(\varepsilon)) > 0. \tag{3.4}$$

Since the function u in Eq. 3.1 is harmonic in an open set containing $\overline{D(\varepsilon)}$, we have

$$\frac{\left| u(x(\varepsilon)) - ⨍_{\partial D(\varepsilon)} u(x)\, d\sigma \right|}{⨍_{\partial D(\varepsilon)} |u(x)|\, d\sigma} \le \mathscr{H}(\partial D(\varepsilon), x(\varepsilon)).$$

Therefore inequality Eq. 3.4 is a consequence of the following claim: there exist $\varepsilon_0, c > 0$ such that

$$\frac{\left| u(x(\varepsilon)) - ⨍_{\partial D(\varepsilon)} u(x)\, d\sigma \right|}{⨍_{\partial D(\varepsilon)} |u(x)|\, d\sigma} > c \qquad \forall\, \varepsilon \in]0, \varepsilon_0[. \tag{3.5}$$

Let us prove this claim.

Since $u \in \mathcal{H}(\overline{D(\varepsilon)})$ and $B(\varepsilon) \subseteq D(\varepsilon)$, then by the surface mean value formula

$$u(x(\varepsilon)) - ⨍_{\partial D(\varepsilon)} u(x)\, d\sigma = ⨍_{\partial B(\varepsilon)} u(x)\, d\sigma - ⨍_{\partial D(\varepsilon)} u(x)\, d\sigma.$$

By Eq. 3.2 and recalling that $u = 0$ on S_ε we get

$$u(x(\varepsilon)) - ⨍_{\partial D(\varepsilon)} u(x)\, d\sigma =$$

$$
\begin{aligned}
&=\frac{1}{|\partial B(\varepsilon)|}\left(\left(1-\frac{|\partial B(\varepsilon)|}{|\partial D(\varepsilon)|}\right)\int_{\partial B(\varepsilon)\setminus\Sigma_\varepsilon} u(x)\,d\sigma+\int_{\Sigma_\varepsilon} u(x)\,d\sigma-\frac{|\partial B(\varepsilon)|}{|\partial D(\varepsilon)|}\int_{\Sigma_\varepsilon^*} u(x)\,d\sigma\right)\\
&=:\frac{1}{|\partial B(\varepsilon)|}\left(\left(1-\frac{|\partial B(\varepsilon)|}{|\partial D(\varepsilon)|}\right)I_\varepsilon^{(1)}+I_\varepsilon^{(2)}-\frac{|\partial B(\varepsilon)|}{|\partial D(\varepsilon)|}I_\varepsilon^{(3)}\right)\\
&= k_\varepsilon^{(1)}I_\varepsilon^{(1)}+k_\varepsilon^{(2)}I_\varepsilon^{(2)}-k_\varepsilon^{(3)}I_\varepsilon^{(3)}, \qquad (3.6)
\end{aligned}
$$

where

$$
k_\varepsilon^{(1)}=\frac{1}{|\partial B(\varepsilon)|}\left(1-\frac{|\partial B(\varepsilon)|}{|\partial D(\varepsilon)|}\right),\quad k_\varepsilon^{(2)}=\frac{1}{|\partial B(\varepsilon)|},\quad k_\varepsilon^{(3)}=\frac{1}{|\partial D(\varepsilon)|}.
$$

It is easy to recognize the existence of $\varepsilon_2\in]0,\varepsilon_1[$ and of $c_1>0$, independent of $\varepsilon\in]0,\varepsilon_2[$, such that

$$
0<k_\varepsilon^{(i)}\le c_1,\ \ i=1,2,\quad \text{and}\quad k_\varepsilon^{(3)}\ge\frac{1}{c_1}\qquad \forall\,\varepsilon\in]0,\varepsilon_2[. \qquad (3.7)
$$

We now estimate $I_\varepsilon^{(3)}$. By the change of variable $x=\varepsilon^m y$ we have

$$
\begin{aligned}
I_\varepsilon^{(3)}&=\int_{\substack{x\in K\\ |x|=\varepsilon^m}} u(x)\,d\sigma(x)\\
&=\int_{\substack{y\in K\\ |y|=1}}\frac{1}{\varepsilon^{mn}}\frac{1}{|y|^n}\left(\frac{y_1^2}{|y|^2}-\frac{1}{n}\right)\varepsilon^{m(n-1)}\,d\sigma(y)\\
&=\int_{\substack{y\in K\\ |y|=1}}\left(\frac{y_1^2}{|y|^2}-\frac{1}{n}\right)d\sigma(y)\frac{\varepsilon^{m(n-1)}}{\varepsilon^{mn}}=:c_2\frac{1}{\varepsilon^m}. \qquad (3.8)
\end{aligned}
$$

Since

$$
\frac{y_1^2}{|y|^2}-\frac{1}{n}>0\qquad \forall y\in K
$$

then

$$
c_2:=\int_{\substack{y\in K\\ |y|=1}}\left(\frac{y_1^2}{|y|^2}-\frac{1}{n}\right)d\sigma(y)
$$

is a strictly positive constant only depending on n.

Let us now consider $I_\varepsilon^{(2)}$. By the change of variable $x=\varepsilon y$ we have

$$
\begin{aligned}
|I_\varepsilon^{(2)}|&=\left|\int_{\Sigma_\varepsilon}\frac{1}{|x|^n}\left(\frac{x_1^2}{|x|^2}-\frac{1}{n}\right)d\sigma\right|\\
&=\frac{1}{\varepsilon}\int_{\frac{1}{\varepsilon}\Sigma_\varepsilon}\frac{1}{|y|^n}\left|\frac{y_1^2}{|y|^2}-\frac{1}{n}\right|d\sigma\\
&\le\frac{1}{\varepsilon}\int_{\frac{1}{\varepsilon}\Sigma_\varepsilon}\frac{1}{|y|^n}\,d\sigma\\
&\le\frac{1}{\varepsilon}\int_{\frac{1}{\varepsilon}\partial B(\varepsilon)}\frac{1}{|y|^n}\,d\sigma,
\end{aligned}
$$

where we used $\Sigma_\varepsilon \subseteq K \cap B(\varepsilon)$ and

$$\left|\frac{y_1^2}{|y|^2} - \frac{1}{n}\right| = \frac{y_1^2}{|y|^2} - \frac{1}{n} \le \frac{y_1^2}{|y|^2} \le 1 \qquad \forall y \in \frac{1}{\varepsilon}\Sigma_\varepsilon.$$

Therefore, since $|y| \ge 1$ on $\partial B\left(\frac{x(\varepsilon)}{\varepsilon}, \frac{1}{\varepsilon}\right)$,

$$\begin{aligned} |I_\varepsilon^{(2)}| \le & \frac{1}{\varepsilon} \int_{\partial B\left(\frac{x(\varepsilon)}{\varepsilon}, \frac{1}{\varepsilon}\right)} \frac{1}{|y|^n}\, d\sigma \\ \le & \frac{1}{\varepsilon} \left|\partial B\left(\frac{x(\varepsilon)}{\varepsilon}, \frac{1}{\varepsilon}\right)\right| = \frac{n\omega_n}{\varepsilon^n}. \end{aligned} \tag{3.9}$$

Finally let us consider $I_\varepsilon^{(1)}$. By the change of variable $x = \varepsilon y$ we have

$$\begin{aligned} |I_\varepsilon^{(1)}| = & \left|\int_{\partial B(\varepsilon)\setminus\Sigma_\varepsilon} u(x)\, d\sigma\right| \\ \le & \int_{\frac{1}{\varepsilon}\partial B(\varepsilon)\setminus\frac{1}{\varepsilon}\Sigma_\varepsilon} \frac{1}{|y|^n \varepsilon^n} \left|\frac{y_1^2}{|y|^2} - \frac{1}{n}\right| \varepsilon^{n-1}\, d\sigma \\ \le & \int_{\partial B\left(\frac{x(\varepsilon)}{\varepsilon}, \frac{1}{\varepsilon}\right)} \frac{1}{|y|^n}\, d\sigma \frac{1}{\varepsilon} \\ \le & \frac{1}{\varepsilon} n\omega_n \frac{1}{\varepsilon^{n-1}} = \frac{n\omega_n}{\varepsilon^n}. \end{aligned} \tag{3.10}$$

Then, by Eqs. 3.6, 3.7, 3.8, 3.9, 3.10, we get

$$\begin{aligned} \left|u(x(\varepsilon)) - \fint_{\partial D(\varepsilon)} u(x)\, d\sigma\right| \ge & k_\varepsilon^{(3)} I_\varepsilon^{(3)} - k_\varepsilon^{(1)}|I_\varepsilon^{(1)}| - k_\varepsilon^{(2)}|I_\varepsilon^{(2)}| \\ \ge & \frac{c_2}{c_1}\frac{1}{\varepsilon^m} - 2c_1\frac{n\omega_n}{\varepsilon^n} \qquad \forall \varepsilon \in]0, \varepsilon_2[. \end{aligned}$$

Remembering that $m > n$ we conclude that there exists $\varepsilon_3 \in]0, \varepsilon_2[$ and a positive constant c_3, independent of $\varepsilon \in]0, \varepsilon_3[$, such that

$$\left|u(x(\varepsilon)) - \fint_{\partial D(\varepsilon)} u(x)\, d\sigma\right| \ge c_3 \frac{1}{\varepsilon^m} \qquad \forall \varepsilon \in]0, \varepsilon_3[. \tag{3.11}$$

By using the same procedure used to get the estimate Eq. 3.11, we can prove that there exists $\varepsilon_0 \in]0, \varepsilon_3[$ and a positive constant c_4, independent of $\varepsilon \in]0, \varepsilon_0[$, such that

$$\fint_{\partial D(\varepsilon)} |u(x)|\, d\sigma \le c_4 \frac{1}{\varepsilon^m} \qquad \forall \varepsilon \in]0, \varepsilon_0[.$$

Denoting $c := \frac{c_3}{c_4}$ we conclude that

$$\frac{\left|u(x(\varepsilon)) - \fint_{\partial D(\varepsilon)} u(x)\, d\sigma\right|}{\fint_{\partial D(\varepsilon)} |u(x)|\, d\sigma} \ge c \frac{\frac{1}{\varepsilon^m}}{\frac{1}{\varepsilon^m}} = c > 0 \quad \forall \varepsilon \in]0, \varepsilon_0[$$

and the claim Eq. 3.5 is proved.

3.3 The Kuran Gap Estimate from Below of the Beaked Sphere

By Theorem 1.3 we have that

$$\mathcal{K}(\partial D(\varepsilon), x(\varepsilon)) \geq \frac{|\partial D(\varepsilon)| - |\partial B(\varepsilon)|}{|\partial D(\varepsilon)|}. \tag{3.12}$$

By using the previous notation,

$$|\partial D(\varepsilon)| - |\partial B(\varepsilon)| = |S_\varepsilon| + |\Sigma_\varepsilon^*| - |\Sigma_\varepsilon|, \tag{3.13}$$

where in this context $|\cdot|$ stands for the $(n-1)$-dimensional Hausdorff measure.

We have that

$$|S_\varepsilon| = \varepsilon^{n-1} \left| \frac{1}{\varepsilon} S_\varepsilon \right|$$

Letting ε go to 0 we get

$$\frac{|S_\varepsilon|}{\varepsilon^{n-1}} \underset{\varepsilon \to 0}{\to} |S_0|,$$

where the set S_0 is

$$S_0 := \partial K \cap \{x_1 \leq 1\},$$

whose measure is a positive constant independent of ε. Therefore,

$$|S_\varepsilon| = |S_0| \varepsilon^{n-1}(1 + o(1)). \tag{3.14}$$

Analogously,

$$|\Sigma_\varepsilon| = \varepsilon^{n-1} \big| \frac{1}{\varepsilon} \Sigma_\varepsilon \big|.$$

Letting ε go to 0 we get

$$\frac{|\Sigma_\varepsilon|}{\varepsilon^{n-1}} \underset{\varepsilon \to 0}{\to} |\Sigma_0|,$$

where

$$\Sigma_0 := K \cap \{x_1 = 1\},$$

whose measure is a positive constant independent of ε. Therefore,

$$|\Sigma_\varepsilon| = |\Sigma_0| \varepsilon^{n-1}(1 + o(1)). \tag{3.15}$$

As far as Σ_ε^* is concerned,

$$|\Sigma_\varepsilon^*| = \varepsilon^{n-1} \big| \frac{1}{\varepsilon} \Sigma_\varepsilon^* \big|.$$

Since

$$\frac{1}{\varepsilon} \Sigma_\varepsilon^* = K \cap B(0, \varepsilon^{m-1}),$$

by a rescaling argument we get

$$\big| \frac{1}{\varepsilon} \Sigma_\varepsilon^* \big| = \varepsilon^{(m-1)(n-1)} |\Sigma_0^*|,$$

with

$$\Sigma_0^* := K \cap B(0, 1).$$

Therefore,

$$\big| \Sigma_\varepsilon^* \big| = \varepsilon^{m(n-1)} |\Sigma_0^*|. \tag{3.16}$$

Collecting Eqs. 3.13–3.16 we obtain that

$$|\partial D(\varepsilon)| - |\partial B(\varepsilon)| = (|S_0| - |\Sigma_0|)\varepsilon^{n-1}(1 + o(1)) + \varepsilon^{m(n-1)}|\Sigma_0^*|. \tag{3.17}$$

Since $m > 1$ and

$$\alpha_0 := |S_0| - |\Sigma_0| > 0,$$

taking into account that

$$|\partial D(\varepsilon)| = |\partial B(\varepsilon)| + o(1) \qquad \text{for } \varepsilon \to 0$$

and that $|\partial B(\varepsilon)|$ is the measure of a ball of radius 1 in $\mathbb{R}^n$, we get that there exists ε_1, with $0 < \varepsilon_1 \le \varepsilon_0$, such that

$$\frac{\alpha_0}{2}\varepsilon^{n-1} \le \frac{|\partial D(\varepsilon)| - |\partial B(\varepsilon)|}{|\partial D(\varepsilon)|} \le 2\alpha_0\varepsilon^{n-1} \qquad \forall \varepsilon \in]0, \varepsilon_1[. \tag{3.18}$$

Therefore, by Eq. 3.12 we conclude that

$$\mathcal{K}(\partial D(\varepsilon), x(\varepsilon)) \ge \alpha^* \varepsilon^{n-1}, \tag{3.19}$$

for every $\varepsilon << 1$, with $\alpha^* > 0$ depending only on n and m.

3.4 The Kuran Gap Estimate from Above of the Beaked Sphere

Recalling that the Kuran gap is translation invariant, to prove an estimate from above of $\mathcal{K}(\partial D(\varepsilon), x(\varepsilon))$ it is convenient to translate the domain $D(\varepsilon)$ in the $e_1 = (1, 0, \dots, 0)$ direction, as follows:

$$\hat{D}(\varepsilon) := -x(\varepsilon) + D(\varepsilon);$$

we act accordingly for the sets $B(\varepsilon)$, S_ε, Σ_ε and Σ_ε^*. In particular, $\hat{B}(\varepsilon)$ turns out to be the unit ball of $\mathbb{R}^n$ centered at the origin. Moreover, since the Kuran gap is translation invariant,

$$\mathcal{K}(\partial D(\varepsilon), x(\varepsilon)) = \mathcal{K}(\partial \hat{D}(\varepsilon), 0). \tag{3.20}$$

Obviously the point e_1 is a touching point, i.e.

$$z := e_1 \in T(\partial \hat{D}(\varepsilon), 0),$$

therefore, by definition of Kuran gap, see Eq. 1.9,

$$\mathcal{K}(\partial \hat{D}(\varepsilon), 0) \le \liminf_{t \searrow 1} \left| \fint_{\partial \hat{D}(\varepsilon)} k_{\alpha(t)}\, d\sigma \right|,$$

where $\alpha(t) := te_1$.

Taking into account that for $t > 1$ the Kuran function $x \mapsto k_{\alpha(t)}(x)$ is harmonic in $\overline{B(0, 1)}$, and $k_{\alpha(t)}(0) = 0$, by the mean value formula we obtain

$$\fint_{\partial \hat{B}(\varepsilon)} k_{\alpha(t)}\, d\sigma = \fint_{\partial B(0,1)} k_{\alpha(t)}\, d\sigma = 0.$$

Therefore

$$\mathcal{K}(\partial \hat{D}(\varepsilon), 0) \le \liminf_{t \searrow 1} \frac{1}{|\partial \hat{D}(\varepsilon)|} \left| -\int_{\hat{\Sigma}_\varepsilon} k_{\alpha(t)}\, d\sigma + \int_{\hat{S}_\varepsilon} k_{\alpha(t)}\, d\sigma + \int_{\hat{\Sigma}_\varepsilon^*} k_{\alpha(t)}\, d\sigma \right|.$$

Since the integration domains are far from the pole of the functions $h_{\alpha(t)}$ for $t \geq 1$, the lim inf above is a limit and we can pass the limit under the integral signs, so obtaining

$$\mathcal{K}(\partial \hat{D}(\varepsilon), 0) \leq \frac{1}{|\partial \hat{D}(\varepsilon)|} \left| - \int_{\hat{\Sigma}_\varepsilon} k_z \, d\sigma + \int_{\hat{S}_\varepsilon} k_z \, d\sigma + \int_{\hat{\Sigma}^*_\varepsilon} k_z \, d\sigma \right|.$$

By Eqs. 1.6 and 1.7,

$$\int_{\hat{\Sigma}_\varepsilon} k_z \, d\sigma = |\hat{\Sigma}_\varepsilon| + \int_{\hat{\Sigma}_\varepsilon} h_z \, d\sigma,$$

$h_z = 0$ on $\partial B(0, 1)$ and $\hat{\Sigma}_\varepsilon \subseteq \partial B(0, 1)$, therefore

$$\int_{\hat{\Sigma}_\varepsilon} k_z \, d\sigma = |\hat{\Sigma}_\varepsilon|.$$

Since k_z is uniformly bounded w.r.t. ε on the integration domains we conclude that there exists a positive constant c such that

$$\mathcal{K}(\partial \hat{D}(\varepsilon), 0) \leq c\big(|\hat{\Sigma}_\varepsilon| + |\hat{S}_\varepsilon| + |\hat{\Sigma}^*_\varepsilon|\big).$$

Taking into account the estimates Eqs. 3.14–3.16 we conclude that there exists a constant $c^* > 0$ such that

$$\mathcal{K}(\partial \hat{D}(\varepsilon), 0) \leq c^* \varepsilon^{n-1} \tag{3.21}$$

for every $\varepsilon << 1$.

3.5 Conclusion

By what proved in the subsections above we conclude that there exists $\varepsilon_0 > 0$ and an absolute constant $c > 0$, such that the family of the *translated beaked spheres* $(\partial \hat{D}(\varepsilon))_{0<\varepsilon<\varepsilon_0}$ satisfies, for every $\varepsilon \in]0, \varepsilon_0[$, the following properties:

by Eq. 3.3,

$$B(0, 1) \subseteq \hat{D}(\varepsilon) \subseteq B(0, 1 + \varepsilon),$$

and $B(0, 1)$ is the biggest ball centered at 0 contained in $\hat{D}(\varepsilon)$;

by Eq. 3.18,

$$\frac{1}{c}\varepsilon^{n-1} \leq \frac{|\partial \hat{D}(\varepsilon)| - |\partial B(0, 1)|}{|\partial \hat{D}(\varepsilon)|} \leq c\, \varepsilon^{n-1};$$

by Eq. 3.4,

$$\liminf_{\varepsilon \to 0} \mathscr{H}(\partial \hat{D}(\varepsilon), 0) > 0;$$

by Eqs. 3.19 and 3.21,

$$\frac{1}{c}\varepsilon^{n-1} \leq \mathcal{K}(\partial \hat{D}(\varepsilon), 0) \leq c\, \varepsilon^{n-1}. \tag{3.22}$$

4 Proof of Theorem 1.6

We begin by stating the following lemma, whose quite standard proof is included for readers convenience.

Lemma 4.1 *Let $D \subseteq \mathbb{R}^n$ be a bounded open set containing a point x_0 and such that $|\partial D| < \infty$.Then the following properties are equivalent.*

(i) $\fint_{\partial D} \Gamma(x-y)\,d\sigma(x) = \Gamma(x_0-y)$ *for every* $y \notin \overline{D}$*;*

(ii) $\fint_{\partial D} u(x)\,d\sigma(x) = u(x_0)$ *for every* $u \in \mathcal{H}(\overline{D})$.

Proof The implication (ii) ⇒ (i) is trivial.

Let us prove (i) ⇒ (ii).

Consider $u \in \mathcal{H}(\overline{D})$ and an open set O containing $\overline{D}$ such that $u \in \mathcal{H}(O)$. Let $\varphi \in C_0^\infty(O,\mathbb{R})$ such that $\varphi \equiv 1$ in O_1, with O_1 open set satisfying

$$\overline{D} \subseteq O_1 \subseteq \overline{O}_1 \subseteq O.$$

Then, since Γ is the fundamental solution of the Laplace operator,

$$\begin{aligned}\fint_{\partial D} u(x)\,d\sigma(x) =& \frac{1}{|\partial D|}\int_{\partial D}(u\varphi)(x)\,d\sigma(x)\\ =& -\frac{1}{|\partial D|}\int_{\partial D}\left(\int_{\mathbb{R}^n}\Delta(u\varphi)(y)\Gamma(x-y)\,dy\right)d\sigma(x)\\ =& -\frac{1}{|\partial D|}\int_{\mathbb{R}^n}\Delta(u\varphi)(y)\left(\int_{\partial D}\Gamma(x-y)\,d\sigma(x)\right)dy.\end{aligned}$$

Since $\Delta(u\varphi) \equiv 0$ in O_1, by using (i) we conclude that

$$\begin{aligned}\fint_{\partial D} u(x)\,d\sigma(x) =& \int_{\mathbb{R}^n\setminus O_1}\Delta(u\varphi)(y)\left(\fint_{\partial D}\Gamma(x-y)\,d\sigma(x)\right)dy\\ =& -\int_{\mathbb{R}^n}\Delta(u\varphi)(y)\Gamma(x_0-y)\,dy\\ =& \, u(x_0)\varphi(x_0) = u(x_0).\end{aligned}$$

□

We are now ready to prove Theorem 1.6.

Proof of Theorem 1.6 By translation's invariance of identity Eq. 1.18, we can assume $x_0 = 0$, so obtaining that

$$\int_{\partial D}\frac{\Gamma(x-y)}{\Gamma(-y)}\,d\sigma(x) = c \qquad \forall y \notin \overline{D}.$$

Since D is bounded,

$$\lim_{\substack{|y|\to\infty\\ y\notin D}}\frac{\Gamma(x-y)}{\Gamma(-y)} = 1$$

uniformly w.r.t. $x \in \partial D$. As a consequence

$$c = \lim_{|y|\to\infty}\int_{\partial D}\frac{\Gamma(x-y)}{\Gamma(-y)}\,d\sigma(x) = |\partial D|.$$

This proves that $c = |\partial D|$, so identity Eq. 1.18 can be written as follows

$$\fint_{\partial D}\Gamma(x-y)\,d\sigma(x) = \Gamma(-y) \qquad \forall y \notin \overline{D}.$$

Then, by Lemma 4.1,

$$\fint_{\partial D} u(x)\,d\sigma(x) = u(0) \qquad \forall y \in \mathcal{H}(\overline{D})$$

and, in particular,

$$\fint_{\partial D} k_\alpha(x)\, d\sigma = k_\alpha(0) \qquad \forall \alpha \notin \overline{D}.$$

Hence, by the very definition of $\mathcal{K}(\partial D, 0)$, the assumption $T(\partial D, 0) \neq \emptyset$ implies $\mathcal{K}(\partial D, 0) = 0$. Therefore, by Corollary 1.4, $|D \setminus B| = 0$, where B is the biggest ball centered at 0 and contained in D. This gives $D = B$. □

Appendix

Here we prove inequality Eq. 2.33 using the notations established in Sect. 2.

From Eq. 2.24, recalling that $\omega^+ = \omega^+((r-r_0)s\eta)$ (see Eq. 2.19) and using the change of variable $t = (r-r_0)s$, we obtain

$$\begin{aligned}(I_\alpha^{(2)})^+ =& \frac{r^{n-2}}{r-r_0} \int_{\{|\eta|=1\}} \left(\int_0^R \frac{t^{n-1}(r-r_0)^{-(n-1)}(r+\Phi)N\omega^+(t\eta)}{\left(\left(\frac{t}{r-r_0}\right)^2 + \left(1 - \frac{t}{r-r_0}\omega(t\eta)\right)^2\right)^{\frac{n}{2}}}\, dt \right) d\sigma(\eta) \\ =& r^{n-2} \int_{\{|\eta|=1\}} \left(\int_0^R \frac{t^{n-1}(r+\Phi)N\omega^+(t\eta)}{\left(t^2 + \left(r - r_0 - t\omega(t\eta)\right)^2\right)^{\frac{n}{2}}}\, dt \right) d\sigma(\eta).\end{aligned}$$

By Fatou lemma and recalling that $N \geq 1$, we obtain

$$\begin{aligned}\liminf_{r\to r_0^+} (I_\alpha^{(2)})^+ \geq & 2r_0^{n-1} \int_{\{|\eta|=1\}} \left(\int_0^R \frac{t^{n-1}\omega^+(t\eta)}{\left(t^2 + \left(t\omega(t\eta)\right)^2\right)^{\frac{n}{2}}}\, dt \right) d\sigma(\eta) \\ =& 2r_0^{n-1} \int_{\{|\eta|=1\}} \left(\int_0^R \frac{\omega^+(t\eta)}{t\left(1 + \omega^2(t\eta)\right)^{\frac{n}{2}}}\, dt \right) d\sigma(\eta) \\ \geq & c(z) \int_{\{|\eta|=1\}} \left(\int_0^R \frac{\omega^+(t\eta)}{t}\, dt \right) d\sigma(\eta) = c(z) \int_{\hat{B}(0,R)} \frac{(\Phi(\xi) - \Phi(0))^+}{|\xi|^n}\, d\xi,\end{aligned}$$

where $c(z)$ is a real positive constant depending on n, r_0 and $\sup|\omega|$, but independent of $R \in]0, R_0[$. Hence, see Eq. 2.34,

$$\liminf_{r\to r_0^+} (I_\alpha^{(2)})^+ \geq c(z)\, \omega_R(\partial D, z). \tag{A.1}$$

We now slightly modify some computations in Step 9 of Sect. 2.

From Eqs. 2.30 and 2.26 we get

$$\begin{aligned}\liminf_{\substack{\alpha \searrow z \\ \alpha \notin \overline{D}}} \int_{\partial D} k_\alpha\, d\sigma \geq & \liminf_{\substack{\alpha \searrow z \\ \alpha \notin \overline{D}}} \left(\int_{\partial D \setminus U_R} k_\alpha\, d\sigma + |\partial D \cap U_R| + I_\alpha^{(1)} - (I_\alpha^{(2)})^- + J_\alpha \right) \\ & + \liminf_{\substack{\alpha \searrow z \\ \alpha \notin \overline{D}}} (I_\alpha^{(2)})^+.\end{aligned}$$

Then, from Eqs. 2.4, 2.21, 2.27, 2.29 and A.1, we obtain

$$\liminf_{\substack{\alpha \searrow z \\ \alpha \notin \overline{D}}} \int_{\partial D} k_\alpha\, d\sigma \geq |\partial D| - |\partial B| - 2cR + c(z)\, \omega_R(\partial D, z).$$

Letting R go to zero in this inequality, we get

$$\liminf_{\substack{\alpha \searrow z \\ \alpha \notin \overline{D}}} \int_{\partial D} k_\alpha \, d\sigma \geq |\partial D| - |\partial B| + c(z) \lim_{R \to 0} \omega_R(\partial D, z),$$

from which, keeping in mind that $k_\alpha(0) = 0$, inequality Eq. 2.33 immediately follows.

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