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General equilibrium with unhedgeable fundamentals and heterogeneous agents ☆

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ABSTRACT

This paper examines the implications of unhedgeable fundamental risk, combined with agents' heterogeneous preferences and wealth allocations, on dynamic asset pricing and portfolio choice. We solve in closed form a continuous-time general equilibrium model in which unhedgeable fundamental risk affects aggregate consumption dynamics, rendering the market incomplete. Several long-lived agents with heterogeneous risk-aversion and time-preference make consumption and investment decisions, trading risky assets and borrowing from and lending to each other. We find that a representative agent does not exist. Agents trade assets dynamically. Their consumption rates depend on the history of unhedgeable shocks. Consumption volatility is higher for agents with preferences and wealth allocations deviating more from the average. Unhedgeable risk reduces the equilibrium interest rate only through agents' heterogeneity and proportionally to the cross-sectional variance of agents' preferences and allocations.

1. Introduction

Not all risks can be perfectly shared among market participants: the cost and effort required for sharing may exceed its benefits, imperfect information may hinder certain insurance contracts, and credit constraints may limit market access (Geanakoplos, 1990). At the same time, economic agents that differ in preferences and endowments also differ in their consumption and investment responses to unhedgeable fundamental risks. Does the diversity of agents' responses mitigate or exacerbate the effects of market incompleteness?

We examine this question in a general equilibrium model of an exchange economy in which several agents, differing in both their risk aversion and time preference, borrow from or lend to each other, and trade stocks that pay dividends whose growth rates have small stochastic fluctuations. The market is dynamically incomplete, as stocks offer imperfect hedges for shocks to growth.

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Market completeness, which allows agents to plan for any contingency by trading with each other, has been the prevalent hypothesis in the general equilibrium literature. This idealized assumption guarantees the Pareto optimality of the equilibrium and the existence of a representative agent – central features to equilibrium models since Lucas Jr. (1978). Much of the asset pricing scholarship on incomplete markets has focused on agent-specific unhedgeable shocks, such as labor-income risk (e.g., Duffie et al. (1994), Heaton and Lucas (1996), Christensen et al. (2012), Weston and Žitković (2020)). But when all agents are exposed to the same shocks, such as unhedgeable macroeconomic fluctuations, little is known.[3] We aim at filling this gap.

This is the first paper to solve in closed-form a general-equilibrium model of a continuous-time exchange economy with unhedgeable fundamental risk and heterogeneous preferences. We use explicit formulas to examine in detail how different agents respond to the absence of markets for certain risks, and how their choices affect the dynamics of asset prices and interest rates.

We show that unhedgeable risk is crucial in shaping agents' consumption decisions. In fact, each agent's consumption rate depends on the history of unhedgeable shocks, hence is not Markov in current market prices alone. Market incompleteness induces dynamic trading, even in the absence of labor income, because agents constantly update their risk-sharing in response to unhedgeable shocks, thereby partially insuring each other. The equilibrium interest rate is reduced in proportion to the dispersion of agents' preferences and capital distribution, thereby violating the representative agent paradigm.

We consider an exchange economy with several agents, who differ in both capital endowments and preferences: each agent owns a personal share of market wealth, and has an individual idiosyncratic risk aversion and impatience rate. Market wealth consists in the ownership of stocks, which pay a continuous stream of dividends that are consumed immediately. Agents can trade such risky assets and lend to or borrow from each other. Asset prices, the interest rate, and agents' consumption-investment strategies are determined in equilibrium. The market is dynamically incomplete, as stocks offer imperfect hedges for the factors driving the dividend process. By introducing the notion of second-order equilibrium, we identify phenomena whose magnitude is proportional to unhedgeable shocks and their variance, and obtain closed-form expressions for stock prices, interest rates, as well as consumption and trading policies.

Our analysis studies the incomplete economy as a perturbation of a baseline market, in which dividends' expected growth is constant. This benchmark economy is complete and leads to constant interest rate and Sharpe ratio. A no-trade result holds in equilibrium, as agents own a constant number of shares of stocks, determined exclusively by their risk aversion relative to the representative agent, and individual consumption is affine in dividends and calendar time.

With unhedgeable risk, agents' behavior significantly departs from the benchmark, as individual preferences and capital allocations elicit different reactions to unhedgeable shocks. Agents would hold a financial instrument to share the risk from these shocks, were an additional risky asset that completes the market available for trading. Since the market is incomplete, agents mimic such an unattainable position by trading the safe asset instead. As unhedgeable shocks accumulate and safe holdings deviate from their target, agents consume the interest rate proceeds from excess lending to reduce their tracking error. Hence, consumption depends on the deviation of the agents' safe holdings from their target and therefore on the whole history of unhedgeable shocks. Agents with atypical preferences and initial wealth allocation have a stronger demand for risk-sharing of fundamental shocks, face higher tracking error volatility and therefore higher consumption uncertainty.

Similarly, dynamic trading appears in equilibrium from the agents' imperfect intertemporal hedging of common shocks. Agents dynamically trade the stock in order to finance a consumption stream that is as close as possible to the one that would be optimal in the same market completed with an additional risky asset. Hence, each agent trades according to personal preferences and their departure from the average, both in impatience and wealth, as measured by the same quantity that regulates consumption.

As consumption depends on the whole history of unhedgeable shocks, so does each agent's marginal utility. Hence, the dispersion of agents' preferences and their wealth distribution becomes a determinant of both the interest rate and stock prices. The additional consumption volatility stemming from unhedgeable shocks reduces the interest rate through the precautionary-saving channel, in proportion to the cross-sectional variance of agents' preferences and wealth allocations. In particular, unhedgeable risk does not reduce the interest rate in an economy with homogeneous agents. The effect on the interest rate in turn propagates to stock prices, which increase if agents are heterogeneous because future cash flows become more valuable.

Consumption policies also reveal a new higher-order effect: because each agent requires compensation for the consumption variance stemming from growth shocks, but in equilibrium aggregate consumption equals dividends, agents who take above-average consumption risk are rewarded with higher expected consumption, while others are penalized. As such risk is borne mainly by those agents that depart the most from the average, either in preferences or in wealth allocations, it is precisely these atypical agents who benefit from heterogeneity with increased consumption, while typical agents are the ones who bear the costs.

The results in this paper are obtained by assuming that unhedgeable risk is small, thereby defining second-order equilibria as sets of interest rates, prices, and policies such that each agent's utility is maximized up to the second order. The methodological advantage of this approach is to lead to explicit expressions for incomplete-market equilibria, which are rather scarce in the literature. The assumption of small unhedgeable fluctuations is also substantively relevant, as in reality market participants routinely devise securities that foster risk-sharing, which leaves those unhedgeable shocks that do not justify the introduction of additional contracts, in view of market frictions.

Understanding incomplete-market equilibria is a difficult enterprise, which has stimulated the development of several theoretical and computational approaches. Kubler and Schmedders (2003) and Kubler and Schmedders (2005) develop computational algorithms

[3] Anderson and Raimondo (2008) note that "with dynamic incompleteness, essentially nothing is known". Weston and Žitković (2020) identify the investigation of such markets as "one of the most interesting and important topics of future research in this area. Unfortunately, the formidable mathematical difficulties present in virtually all such problems leave them outside the scope of the techniques available to us today."

to numerically approximate market equilibria. Their method relies on the notion of ε-equilibrium, which requires that the errors in Euler equations are small, while insisting on the conditions that market clearing and budget constraints hold exactly. We also impose such conditions, but define second-order equilibria as those that achieve optimality up to an error of the second order for all the agents in the economy. This paper develops a small-noise approach to obtain closed-form asymptotics. In addition, it compares such asymptotic formulas with a numerical solution of the equilibrium and verifies their accuracy.

Our methodology is conceptually close to the perturbation literature for stochastic control problems initiated by Fleming (1971). Few papers have applied perturbation methods to general equilibrium models: Judd and Guu (2001) expand a one-period equilibrium model adding small noise to a deterministic market, Kim et al. (2008) and Mertens and Judd (2018) implement second-order expansions to numerically solve discrete DSGE models, Kogan and Uppal (2001) perturb the risk aversion parameters around the logarithmic utility case to analyze the implication of preference heterogeneity. We expand around a tractable complete benchmark by adding small unhedgeable noise to the model and find a second-order equilibrium for the ensuing incomplete market.

To obtain explicit formulas, we assume that agents have exponential utilities (i.e., CARA preferences) and that dividends are conditionally Gaussian. Notwithstanding their limitations, exponential utilities are commonly adopted in the incomplete-market equilibrium literature, as the absence of a representative agent makes other utility functions intractable.[4]

The rest of this paper is organized as follows: section 2 introduces the incomplete model and discusses the baseline model. Section 3 defines in detail the notion of second-order equilibrium, states its explicit form, and interprets its implications for consumption, investment, and asset prices. Section 4 discusses the model's asset pricing implications. Section 5 compares the market equilibrium from the closed-form expression with the one obtained numerically. Section 6 concludes the paper. Proofs are in the appendix.

2. The model

The economy consists of n agents maximizing their respective lifetime utility from consumption. The i-th agent has absolute risk-aversion α_i and time-preference rate β_i, hence seeks to maximize

$$E\left[\int_0^\infty e^{-\beta_i s} U_i(c_s^i)\,ds\right] \tag{2.1}$$

where $U_i(c) = \frac{e^{-\alpha_i c}}{-\alpha_i}$ and c_s^i denotes the i-th agent's consumption rate at time s. Agents finance their consumption by borrowing from and lending to each other at an endogenously determined safe rate, and by trading an asset that pays at time t a dividend flow D_t that follows the dynamics

$$dD_t = \mu_t dt + \sigma_D dW_t^D,$$
$$d\mu_t = a(\mu_D - \mu_t)dt + \sigma_\mu dW_t^\mu,$$

where W^D and W^μ are independent Brownian motions representing shocks to dividends and to fundamentals, respectively, and $\sigma_D, a > 0$. Such dividend dynamics are common in equilibrium models with stochastic fundamentals, such as Scheinkman and Xiong (2003), Dumas et al. (2009), Cujean and Hasler (2017). For notational convenience, we consider a single unhedgeable risk source. (The main findings in the multivariate case are discussed in Section 4.2.)

Denoting by X_t^i and θ_t^i respectively the wealth and the number of shares held by the i-th agent at time t, the agent's budget equation is

$$dX_t^i = \theta_t^i(dP_t + D_t dt) + r_t(X_t - \theta_t^i P_t)dt - c_t^i dt, \qquad X_0^i = x_i,$$

where r_t and P_t are respectively the safe rate and the asset price at time t, endogenously determined in equilibrium, while x_i denotes the initial wealth endowment of the i-th agent. Because the perishable dividend must be consumed immediately, the risky asset is available in unit net supply, and the (locally) safe asset is in zero net supply, the market-clearing conditions require that

$$\sum_{i=1}^n \theta_t^i = 1, \qquad \sum_{i=1}^n (X_t^i - \theta_t^i P_t) = 0, \qquad \sum_{i=1}^n c_t^i = D_t \qquad \text{a.s. for all } t > 0. \tag{2.2}$$

Note that the first two conditions imply that $\sum_{i=1}^n X_t^i = P_t$ and, in particular, that $\sum_{i=1}^n x_i = P_0$. Hence, the initial endowments $(x_i)_{i=1}^n$ are invariant to scaling, in that replacing them with $(kx_i)_{i=1}^n$ leads to the same setting, up to a change in numeraire. Put differently, only the relative endowments $x_i / \sum_{i=1}^n x_i$ have economic significance, while their sum $\sum_{i=1}^n x_i$ merely scales the asset price P_0.

In this market the only tradeable risky asset (the stock) is insufficient for hedging both the dividend and the fundamental shocks. As a result, the market is no longer complete, hence the equilibrium is not necessarily Pareto efficient, and a representative agent

[4] To study the impact of uninsurable income risk on asset prices, Calvet (2001) assumes investors with identical exponential utility; Christensen et al. (2012) and Christensen and Larsen (2014) assume heterogeneous CARA preferences in incomplete-markets models. Angeletos and Calvet (2006) assume CARA preferences in an incomplete production economy. Gromb and Vayanos (2002) assume CARA utilities in a model with market segmentation and financial constraints. Vayanos and Woolley (2013) adopt CARA preferences in a model with investment funds and flow costs.

may not exist. In addition, because not all contingent claims are tradeable, the marginal utilities of different agents do not have to be proportional to each other, complicating the search for the equilibrium further.

We study the incomplete model as a perturbation of a baseline case, in which the dividend's growth rate is constant. One expects, by continuity, that when the volatility of the fundamental state σ_μ is small, an equilibrium should exist and resemble a perturbation of such baseline equilibrium. Thus, consider an expansion of the volatility σ_μ around 0, setting $\mu_t := \mu_D + \varepsilon\sigma_z z_t$, where $\sigma_z := \frac{\sigma_\mu}{\varepsilon}$ and $z_t := \frac{\mu_t - \mu_D}{\varepsilon\sigma_z}$. With this notation, the dividend dynamics is

$$dD_t = (\mu_D + \varepsilon\sigma_z z_t)dt + \sigma_D dW_t^D,$$
$$dz_t = -az_t dt + dW_t^\mu.$$

For $\varepsilon = 0$, the model is complete. The next subsection discusses the equilibrium for this baseline case, while the remainder of the paper studies the incomplete-market equilibrium for $\varepsilon > 0$.

2.1. The baseline case

In the baseline case, i.e., when $\varepsilon = 0$, the dividend process dynamics is

$$dD_t = \mu_D dt + \sigma_D dW_t^D.$$

The next theorem characterizes the baseline equilibrium. Denote aggregate risk-aversion by $\bar{\alpha} = 1/\sum_{i=1}^n \frac{1}{\alpha_i}$ and aggregate time-preference by $\bar{\beta} = \sum_{i=1}^n \frac{\bar{\alpha}}{\alpha_i}\beta_i$.

Theorem 2.1. *Assume $\bar{\beta} + \bar{\alpha}\mu_D - \frac{\bar{\alpha}^2}{2}\sigma_D^2 > 0$. The pair (r, P_t) defined by*

$$r = \bar{\beta} + \bar{\alpha}\mu_D - \frac{\bar{\alpha}^2}{2}\sigma_D^2, \qquad P_t = \frac{D_t}{r} + \frac{\mu_D - \bar{\alpha}\sigma_D^2}{r^2} \tag{2.3}$$

is a market equilibrium, and the agents' optimal policies are

$$\hat{\theta}_t^i = \frac{\bar{\alpha}}{\alpha_i}, \tag{2.4}$$

$$\hat{c}_t^i = \frac{\bar{\alpha}}{\alpha_i}D_t + \frac{\bar{\beta} - \beta_i}{\alpha_i}\left(t - \frac{1}{r}\right) + r\left(x_i - \frac{\bar{\alpha}}{\alpha_i}P_0\right). \tag{2.5}$$

The unique stochastic discount factor is $M_t = \exp\{-\bar{\beta}t - \bar{\alpha}(D_t - D_0)\}$.

The equilibrium in this baseline model has the following properties, which form the starting point of the discussion in the full-fledged model in the next section.

2.1.1. Equilibrium interest rate and stock price

The interest rate in (2.3) is constant and is the sum of the three familiar components of time-preference ($\bar{\beta}$), growth ($\bar{\alpha}\mu_D$), and precautionary savings ($-\bar{\alpha}^2\sigma_D^2/2$). The stock price is an affine function of the current dividend and equals the present value of future dividends $E_t[\int_t^\infty e^{-r(s-t)}D_s ds] = D_t/r + \mu_D/r^2$ minus a risk discount $\bar{\alpha}\sigma_D^2/r^2$ required by agents to hold an asset that pays variable dividends.

Equilibrium prices, summarized by r and P_t, depend on agents' preferences only through the aggregate risk-aversion $\bar{\alpha}$ and time-preference $\bar{\beta}$. Thus, the market is equivalent to a model in which all wealth is held by a single representative agent with the aggregate preferences parameters.

The equilibrium is Markov, in that both market prices r, P_t and the optimal consumption and investment policies depend only on the variable D_t that describes the state of the market. Put differently, the state D_t (along with calendar time t) is a sufficient statistic for all agents to make their optimal decisions.

2.1.2. Consumption and investment policies

Asset holdings are constant and inversely proportional to each agent's own risk-aversion, which means that *no trade* takes place among agents over time.

The consumption rate of each agent is the sum of: (i) the dividend $\frac{\bar{\alpha}}{\alpha_i}D_t$ on the shares held, (ii) the interest on wealth in excess of the stock holdings $r\left(x_i - \frac{\bar{\alpha}}{\alpha_i}P_0\right)$, and (iii) a time-dependent component $\frac{\bar{\beta}-\beta_i}{\alpha_i}\left(t - \frac{1}{r}\right)$ that describes how agents with different time-preference exchange consumption streams over time: agents with above-average impatience ($\beta_i > \bar{\beta}$) consume more in earlier periods ($t < 1/r$), while more patient agents defer additional consumption to later periods ($t > 1/r$). Note that such exchanges are settled through the money market, that is by lending and borrowing at the constant interest rate r.

Remark 2.2. An important feature of this equilibrium is that all agents coexist indefinitely, in that the economic weight of none of them vanishes over time, as detailed in Corollary 4.5. This result is in contrast to models based on isoelastic investors (cf. Kogan et al. (2006); Yan (2008); Cvitanić et al. (2012); Guasoni and Wong (2020)), in which one agent (typically the least risk averse, the most patient, or a combination thereof) eventually overtakes all others.

3. Incomplete-market equilibrium

Because the goal of this section is to define a second-order equilibrium, a strategy is in fact represented as a *family* of strategies, defined for ε small enough. A second-order equilibrium consists of asymptotic expansions for the asset price, the interest rate, and consumption-investment strategies, for which the market-clearing and self-financing conditions hold exactly, while optimality holds at the second-order.

It is essential to include effects proportional to the variance of the unhedgeable shock to understand its asset pricing implications, as the precautionary savings effect depends on the second moment. To understand the impact of moments of higher order, the definition can be adjusted to higher order, at the expense of obtaining more cumbersome expressions for the equilibrium.

Definition 3.1. A second-order equilibrium is a family $\{r_t(\varepsilon), P_t(\varepsilon), (\hat{c}^i_t(\varepsilon))_{1\le i\le n}, (\hat{\theta}^i_t(\varepsilon))_{1\le i\le n}\}_{\varepsilon\in[0,\bar{\varepsilon})}$ such that (henceforth, unless ambiguity arises, the dependence on ε is omitted):

(i) $P_t(\varepsilon)_{0\le\varepsilon\le\bar{\varepsilon}}$ is a continuous semimartingale and $r_t(\varepsilon)_{0\le\varepsilon\le\bar{\varepsilon}}$ an adapted, integrable process;
(ii) $\{\hat{c}^i_t, \hat{\theta}^i_t\}_{0\le\varepsilon\le\bar{\varepsilon}}$ satisfies the admissibility conditions in Definition C.1 for all $1\le i\le n$;
(iii) For every ε, the market clearing conditions hold:

$$\sum_{i=1}^n \hat{\theta}^i_t = 1, \qquad \sum_{i=1}^n (X^i_t - \hat{\theta}^i_t P_t) = 0, \qquad \sum_{i=1}^n \hat{c}^i_t = D_t;$$

(iv) For every agent i, the family of strategies $\{\hat{c}^i_t, \hat{\theta}^i_t\}_{0\le\varepsilon\le\bar{\varepsilon}}$ is second-order optimal in ε, i.e.,

$$E\left[\int_0^\infty \frac{1}{-\alpha_i} e^{-\beta_i t - \alpha_i \hat{c}^i_t} dt\right] \ge \sup_{(c_t,\theta_t)\in\mathcal{A}} E\left[\int_0^\infty \frac{1}{-\alpha_i} e^{-\beta_i t - \alpha_i c_t} dt\right] + o(\varepsilon^2).$$

3.1. Homogeneous time-preference

The description of the second-order equilibrium is technically simpler when agents have the same time-preference, i.e., $\beta_i = \bar{\beta}$ for all $1\le i\le n$, leaving heterogeneity in risk aversions α_i and initial endowments x_i. Most of the effects of incompleteness are already visible with homogeneous time-preference, motivating the initial exposition in this setting. The next subsection describes the general case with time-preference heterogeneity and how it affects the model's implications.

Theorem 3.2. *Assume $\bar{r} := \bar{\beta} + \bar{\alpha}\mu_D - \frac{\bar{\alpha}^2}{2}\sigma_D^2 > 0$. The interest rate r_t, price P_t, consumption $\hat{c}^i_t$ and investment $\hat{\theta}^i_t$ strategies yield a second-order equilibrium. Define*[5]*:*

$$r_t = \bar{r} + \varepsilon\bar{\alpha}\sigma_z z_t - \frac{\varepsilon^2}{2} H(x), \tag{3.1}$$

$$P_t = D_t\tilde{C}_t + (\mu_D - \bar{\alpha}\sigma_D^2)\tilde{L}_t + \varepsilon\frac{1}{\bar{r}(a+\bar{r})}\sigma_z z_t - \varepsilon^2 \frac{2\bar{\alpha}\sigma_z^2}{\bar{r}(a+\bar{r})(2a+\bar{r})}\left(z_t^2 + \frac{1}{\bar{r}}\right), \tag{3.2}$$

$$\hat{\theta}^i_t = \frac{\bar{\alpha}}{\alpha_i} + \varepsilon^2\hat{\theta}^{i,2}_t, \tag{3.3}$$

$$\hat{c}^i_t = \frac{\bar{\alpha}}{\alpha_i} D_t + \frac{1}{C^i_0}\left(x_i - \frac{\bar{\alpha}}{\alpha_i}P_0\right) + \varepsilon^2\frac{1}{\alpha_i\bar{r}}\left(\frac{1}{2}(m^{i,1})^2 + \frac{1}{2}H(x) - \frac{\bar{\alpha}\sigma_z}{a+\bar{r}}m^{i,1}\right) \tag{3.4}$$

$$- \frac{\varepsilon}{\alpha_i} m^{i,1} W^\mu_t - \frac{\varepsilon^2}{\alpha_i}\left(\int_0^t l^{i,2}_s ds + \int_0^t m^{i,2}_s dW^\mu_s + \int_0^t n^{i,2}_s dW^D_s\right), \tag{3.5}$$

$$\log M^i_t = -\bar{\beta}t - \bar{\alpha}(D_t - D_0) + \varepsilon m^{i,1} W^\mu_t + \varepsilon^2\left(\int_0^t l^{i,2}_s ds + \int_0^t m^{i,2}_s dW^\mu_s + \int_0^t n^{i,2}_s dW^D_s\right), \tag{3.6}$$

where

[5] Observe that $H(x)$ depends on the entire vector of all endowments $x = (x_1, \dots, x_n)$.

$$m^{i,1} = -\alpha_i \frac{\bar{\alpha}\sigma_z}{a+\bar{r}}\bar{r}\left(x_i - \frac{\bar{\alpha}}{\alpha_i}P_0\right), \quad H(x) = \left(\frac{\bar{\alpha}\sigma_z\bar{r}}{a+\bar{r}}\right)^2 \sum_{j=1}^{n} \bar{\alpha}\alpha_j \left(x_j - \frac{\bar{\alpha}}{\alpha_j}P_0\right)^2 = \sum_{j=1}^{n} \frac{\bar{\alpha}}{\alpha_j}(m^{j,1})^2, \tag{3.7}$$

$$C_t^i = \frac{1}{\bar{r}} - \varepsilon\frac{\bar{\alpha}\sigma_z}{\bar{r}(a+\bar{r})}z_t + \varepsilon^2\left[\frac{\bar{\alpha}^2\sigma_z^2}{\bar{r}(a+\bar{r})(2a+\bar{r})}\left(z_t^2 + \frac{1}{\bar{r}}\right) + \frac{H(x)}{2\bar{r}^2} - \frac{\bar{\alpha}\sigma_z}{\bar{r}^2(a+\bar{r})}m^{i,1}\right], \quad \tilde{C}_t = \sum_{j=1}^{N}\frac{\bar{\alpha}}{\alpha_j}C_t^j,$$

$$\tilde{L}_t = \frac{1}{\bar{r}^2} - \varepsilon\left(\frac{1}{\bar{r}} + \frac{1}{a+\bar{r}}\right)\frac{\bar{\alpha}\sigma_z}{\bar{r}(a+\bar{r})}z_t + \frac{\varepsilon^2}{\bar{r}^2}\left[\frac{\bar{\alpha}^2\sigma_z^2}{(a+\bar{r})(2a+\bar{r})}\left(\frac{1}{\bar{r}} + \frac{2a^2+6a\bar{r}+3\bar{r}^2}{(a+\bar{r})(2a+\bar{r})}\left(z_t^2 + \frac{1}{\bar{r}}\right)\right) + \frac{1}{\bar{r}}H(x)\right],$$

$$\hat{\theta}_t^{i,2} = -\frac{\sigma_z}{\sigma_D^2}\frac{\bar{\alpha}}{\alpha_i}\frac{m^{i,1}}{(a+\bar{r})}\left(D_t + \frac{a+2\bar{r}}{\bar{r}(a+\bar{r})}(\mu_D - \bar{\alpha}\sigma_D^2) - \frac{1}{\bar{\alpha}}\right),$$

$$l_t^{i,2} = -\alpha_i\bar{\alpha}\sigma_D^2\hat{\theta}_t^{i,2} - \frac{1}{2}(m^{i,1})^2 + \frac{1}{2}H(x), \tag{3.8}$$

$$m_t^{i,2} = -\bar{\alpha}\sigma_z m^{i,1}\left[\frac{\bar{r}}{(2a+\bar{r})(a+\bar{r})}z_t - \frac{1}{a+\bar{r}}(W_t^{\mu} + z_0)\right], \quad n_t^{i,2} = -\alpha_i\sigma_D\hat{\theta}_t^{i,2}. \tag{3.9}$$

The above result highlights several features, differences, and analogies with the baseline setting in the previous section. Because the unit of account in the model is the single consumption good, to render the model complete it would be necessary to add real (i.e., inflation indexed) long-term bonds to the menu of tradeable assets. The rationale for excluding such bonds is twofold: First, real bonds with indefinite maturities are available only in limited, inelastic supply and are issued by governments, while the model would require such bonds to be collectively issued by some agents and bought by others in perfectly elastic amounts.[6] Second, real bonds, indexed by some aggregate consumer price index, would be of limited use to hedge the consumption basket of an individual consumer: in this sense, the absence of long-term bonds in the model mitigates the simplification of assuming a single consumption good. The crucial model feature is that some risks are unhedgeable, and the exclusion of long-term bond achieves this goal while also preserving tractability.[7]

Theorem F.1 in the Appendix identifies the equilibrium in an economy that allows the trading of long-term bonds. The discussion below uses this complete benchmark to explain the effects of market incompleteness.

3.1.1. Wealth distribution and market equilibrium

In view of the second term in equation (3.2), the interest rate is stochastic. Equilibrium asset prices depend on the distribution of agents' wealth even if their time-preferences and risk-aversions are identical because interest-rate risk is unhedgeable. To understand how wealth distribution affects asset prices, consider first the complete market where long-term bonds are traded (Theorem F.1 in the Appendix computes the equilibrium in this setting).

In this economy, agents hedge interest-rate risk by holding long-term bonds. Exponential-utility preferences imply that each agent holds a constant number of shares. Assuming homogeneous time-preference, any excess wealth $x_i - \frac{\bar{\alpha}}{\alpha_i}$ is held in long-term bonds because such bonds allow to make a consumption plan that is insensitive to interest-rate shocks, unlike money-market holdings. A stochastic interest rate is crucial for this consideration: when interest rates are deterministic, long-term bonds are redundant and are indeed replicable through the money market.

In the incomplete market, agents use stocks and the money market as imperfect substitute for long-term bonds. Unable to hedge interest-rate shocks, agents face increased risk, prompting additional precautionary savings and driving the interest rate lower. Significant wealth inequality leads to increased lending and borrowing among agents, increasing aggregate exposure to interest-rate risk and amplifying precautionary savings. Vice versa, less inequality leads to more autarchic holdings, lower interest-rate exposure, hence a higher interest rate. The interest rate is maximum in absolute equality, which results in absolute autarchy.

The following subsections discuss the precise mechanism through which wealth dispersion and interest rate risk increase consumption volatility in the incomplete market, explaining the interest-rate gap $\frac{\varepsilon^2}{2}H(x)$ in (3.1) from the complete equilibrium of Theorem F.1.

3.1.2. Consumption policy

At order ε, the consumption policy includes both the first two terms in (3.4), which do not depend on time-varying dividend growth and coincide with the optimal consumption (2.5) in the baseline model, and the novel first term in (3.5), which reflects the combined effect of random growth and market incompleteness.

To understand this novel component, it is useful to compare it to the consumption in (F.6) that would be optimal in a hypothetical complete market where the consol bond C_t is traded, that is the bond that pays a constant coupon flow equal to 1. Assuming homogeneous time-preference, the consumption policy (3.4)-(3.5) in the incomplete market differs from that in the complete one at order ε by the stochastic term

[6] By elastic, we mean an amount that can be adjusted instantly and without frictions, even in response to economic shocks.

[7] An alternative approach would be to include both stocks and long-term bonds as tradeable assets, while achieving market incompleteness through additional shocks and multiple consumption goods. Such a model would be significantly less tractable and would not necessarily yield additional insights.

$$-\frac{\varepsilon}{\alpha_i} m^{i,1} W_t^{\mu} = \varepsilon \frac{\bar{\alpha}\sigma_z}{a+\bar{r}} \left(x_i - \frac{\bar{\alpha}}{\alpha_i} P_0 \right) \bar{r} W_t^{\mu}.$$

In such a complete model, the consumption stream (F.6) is financed by fixed amounts of stocks and consol bonds, i.e., by the wealth process $\frac{\bar{\alpha}}{\alpha_i} P_t^{comp} + \frac{1}{C_0^{comp}} \left(x_i - \frac{\bar{\alpha}}{\alpha_i} P_0^{comp} \right) C_t^{comp}$, where P_t^{comp} and C_t^{comp} denote the equilibrium prices in the complete model of the stock and consol bond, respectively. However, in the incomplete market under consideration these long-term bonds are not traded, as the only securities are the dividend-paying asset and the short-term bond with rate r_t.

Now, denote by $Q_t^i := \frac{1}{C_0^i} \left(x_i - \frac{\bar{\alpha}}{\alpha_i} P_0 \right) C_t^i$ the *subjective* value of the bond component of the wealth process above. This quantity is subjective because it is based on the marginal values of the consol bonds for the i-th agent, which are not market prices, but rather the hypothetical prices that would make the i-th agent indifferent to trading small amounts of these securities.

Even though holding the position Q_t^i is not possible in the incomplete market, because long-term bonds are not traded, the i-th agent can mimic this position with an appropriate investment in the safe asset. In fact,

$$X_t^i - \hat{\theta}^i P_t = Q_t^i - \int_0^t \frac{d\langle Q^i, W^{\mu}\rangle_s}{ds} dW_s^{\mu} + O(\varepsilon^2).$$

In other words, the agent's position in the safe asset is the closest to Q_t^i attainable in the incomplete market. Since the first-order dynamics of Q_t^i is

$$dQ_t^i = (\dots)dt + \varepsilon \frac{m^{i,1}}{\alpha_i \bar{r}} dW_t^{\mu},$$

we obtain $\frac{d\langle Q^i, W^{\mu}\rangle_t}{dt} = \varepsilon \frac{m_t^{i,1}}{\alpha_i \bar{r}} + O(\varepsilon^2)$.

In order for the safe position to track Q_t^i over time, the i-th agent has to consume the interest-rate proceeds on excess lending, that is

$$\hat{c}_t^i = \hat{c}_t^{i,comp} - \bar{r} \int_0^t \frac{d\langle Q^i, W^{\mu}\rangle_s}{ds} dW_s^{\mu} + O(\varepsilon^2),$$

where $\hat{c}_t^{i,comp}$ is the optimal consumption in the complete market. Agents adjust their consumption to mimic, albeit imperfectly, the position in long-term bonds they would hold in the complete economy. The noisy tracking error in the replication leads to the additional stochastic term in the consumption policy.

Agents with an initial wealth endowment equal to their natural stock position ($x_i = \frac{\bar{\alpha}}{\alpha_i} P_0$) would have zero demand for long-term bonds, even if they were available. Hence, as there is no bond position to mimic with the safe asset, they do not adjust their consumption. Instead, agents with high initial endowment ($x_i > \frac{\bar{\alpha}}{\alpha_i} P_0$) are natural lenders. An unexpected increase in dividend growth ($dW_t^{\mu} > 0$) leads to a higher interest rate, lower indifference prices for long-term bonds, hence to lower Q_t^i for lenders. Hence, lenders – who receive higher interest income – respond by increasing current consumption to reduce the deviation of their safe position from Q_t^i.

Agents with atypical initial wealth endowments ($x_i \ll \frac{\bar{\alpha}}{\alpha_i} P_0$ or $x_i \gg \frac{\bar{\alpha}}{\alpha_i} P_0$) hold a larger position, either positive or negative, in the safe asset and therefore their deviation from their desired bond position Q_t^i is larger after an unexpected shock dW_t^{μ}. Since the deviation size affects the extent of the adjustment in consumption, atypical agents face more consumption uncertainty relative to average agents.

3.1.3. Dynamic trading

The second-order equilibrium implies dynamic trading:

$$\hat{\theta}_t^i = \frac{\bar{\alpha}}{\alpha_i} - \varepsilon^2 \frac{\sigma_z}{\sigma_D^2} \frac{\bar{\alpha}}{\alpha_i} \frac{m^{i,1}}{(a+\bar{r})} \left(D_t + \frac{a+2\bar{r}}{\bar{r}(a+\bar{r})} (\mu_D - \bar{\alpha}\sigma_D^2) - \frac{1}{\bar{\alpha}} \right).$$

To understand the trading motive, consider some of the mechanisms identified in the literature. In the overlapping generations model of Vayanos (1998) and Vayanos and Vila (1999), trading is a life-cycle phenomenon, as young agents buy securities and sell them in their old age. The infinitely-lived agents in this paper are clearly not susceptible to such motive.

In the models of Heaton and Lucas (1996), Lo et al. (2004), Christensen et al. (2012), Herdegen and Muhle-Karbe (2018), and Buss and Dumas (2019), dynamic trading arises from a response to fluctuations in agents' personal labor incomes. This motive also does not apply to the present setting, in which labor incomes are absent.

Instead, in this market agents trade for the purpose of consumption smoothing by sharing dividend-growth risk. Note that trading originates from the deviation of the initial endowment x_i from the first-order autarchy level $\frac{\bar{\alpha}}{\alpha_i} P_0$: that is, any agent with $x_i = \frac{\bar{\alpha}}{\alpha_i} P_0$ does not trade.

Different agents respond in opposite ways to dividend growth shocks, depending on their wealth. (The aggregate response is null, as the asset's supply is fixed.) A positive dividend growth shock forecasts higher future dividends, thereby eliciting two countervailing effects: The income effect encourages natural borrowers ($x_i < \frac{\bar{\alpha}}{\alpha_i} P_0$) to consume today some of the expected future dividends, hence to sell. The substitution effect, which prevails for natural lenders, focuses instead on the increased attractiveness of the security, hence encouraging to buy.

To link dynamic trading with hedging, consider again the *subjective* value Q^i_t of the i-th agent's desired position in consol bonds (if they were available). Recall that $\frac{d\langle Q^i, W^\mu\rangle_t}{dt} = \varepsilon \frac{m^{i,1}_t}{\alpha_i \bar{r}} + O(\varepsilon^2)$, which, combined with $d\langle P\rangle_t = \frac{\sigma_D^2}{\bar{r}^2} + O(\varepsilon)$, yields

$$\hat{\theta}^i_t = \frac{\bar{\alpha}}{\alpha_i} + \frac{d\langle Q^i, P\rangle_t}{d\langle P\rangle_t} + O(\varepsilon^3). \tag{3.10}$$

This expression sheds light on agents' motive for trading: the i-th agent aims to mimic as closely as possible the shocks to the *subjective* wealth process that would replicate the optimal consumption stream in a complete-market.

The dynamic trading component in $\hat{\theta}^i_t$ is the projection of the desired position in long-term bonds Q^i_t on the space of positions attainable by exclusively trading the stock, therefore it depends on the covariation between P_t and Q^i_t and is proportional to the variance of the unhedgeable shock, implying that it is of order ε^2.

Note also that agents are trading despite sharing the same prior beliefs and the same information, ostensibly in contrast to classical no-trade results (Milgrom and Stokey, 1982) and to the dynamic no-trade theorems of Constantinides and Duffie (1996) and Judd et al. (2003a). Constantinides and Duffie (1996) consider an equilibrium model in which agents have idiosyncratic shocks to labor income but homogeneous preferences, and find that agents do not trade in equilibrium because hedging demand is null. In a model with heterogeneous risk aversion and a dividend driven by a Markov chain dynamics, Judd et al. (2003a) show that agents' holdings of long-lived securities remain constant over time, unperturbed by the flow of information. Unlike Constantinides and Duffie (1996), in the present model agents do hedge by exchanging their respective exposures to growth shocks because they have different preferences for both time and risk. Unlike Judd et al. (2003a), who complete their market through a set of short-lived hedging instruments, the present market is incomplete because agents face two sources of randomness W^D and W^μ but can trade only one risky asset, hence are unable to perfectly hedge shocks to the growth rate μ.

The key trading motive is thus hedging, and in the model hedging requires trading because available securities are insufficient to set up contingent plans through a static position. Such a trading motive is not in contrast with extant no-trade theorems, in which new information does not generate trading in view of agents' agreement on the value of securities. Here agents willingly trade while agreeing on securities' values by providing each other with actuarially fair insurance, and each of them has different insurance needs that are dictated by individual preferences.

3.1.4. *The heterogeneity premium*

The quantities $l^{i,2}$, $m^{i,2}$ and $n^{i,2}$ in equation (3.5) contain second-order effects on consumption. As dynamic trading implies that the number of shares changes over time, the contribution of dividends to consumption changes accordingly, as attested by the last term (reported in (3.9)), in which the dividend shock dW^D_t is precisely proportional to the number of shares $\hat{\theta}^i_t$ held at time t. Dynamic trading also accounts for the first term in (3.8), while the remaining terms

$$\frac{\varepsilon^2}{2\alpha_i}\left((m^{i,1})^2 - \sum_{j=1}^{N} \frac{\bar{\alpha}}{\alpha_j}(m^{j,1})^2\right)t \tag{3.11}$$

reflect the effect of heterogeneity on consumption sharing.

Heterogeneity leads atypical agents (large $(m^{i,1})^2$) to consume more than typical ones (small $(m^{i,1})^2$), a phenomenon that can be understood as the reward for respective consumption risk. Indeed, recall that the i-th agent's first-order consumption exposure to state shocks W^μ_t is $-\varepsilon m^{i,1}/\alpha_i$, which contributes an additional consumption variance of $\varepsilon^2((m^{i,1})^2/\alpha_i^2)t$. In equilibrium, such consumption variance elicits the reward in (3.11), so that the aggregate reward is null (in equilibrium aggregate consumption must equal dividends) while each agent receives an incremental reward proportional to the consumption variance accepted. As atypical agents take most of this risk, they also receive most of its rewards.

3.1.5. *Interest rates*

In a complete economy with additional risky assets, equation (3.1) implies an interest rate that reverts around the mean $\bar{r}$ prevailing in the baseline. In particular, as the state z_t (or, equivalently, $\mu_t = \mu_D + \varepsilon\sigma_z z_t$) follows an Ornstein-Uhlenbeck process, the equilibrium implies that the endogenous interest rate r_t follows a Vasicek model, in which short-rate volatility is proportional to the aggregate risk aversion $\bar{\alpha}$ and the volatility of the expected growth of dividends $\varepsilon\sigma_z$, as in the single-agent model by Goldstein and Zapatero (1996). Note also that the interest rate is procyclical because it increases with dividend growth.

In the complete equilibrium, agents' preferences appear only through the aggregate risk aversion $\bar{\alpha}$ and time-preference $\bar{\beta}$, which implies that the market is indistinguishable from another one populated with a single representative agent with these preferences. Put differently, the actual dispersion of $(\alpha_i)_{i=1}^n$ and $(\beta_i)_{i=1}^n$ is irrelevant, insofar as it leads to the same $\bar{\alpha}$ and $\bar{\beta}$.

The incomplete-market equilibrium uncovers the impact of such dispersion. The interest rate (3.1) is reduced by $H(x) = \frac{\varepsilon^2}{2}\sum_{j=1}^n \frac{\bar{\alpha}}{\alpha_j}(m^{j,1})^2$, where $m^{j,1}$ (defined in (3.7)) captures the combined sensitivity of consumption to both preferences and initial endowments. The key to understanding such reduction lies in the term of order ε in (3.5), which reflects the response of consumption to growth-rate shocks. The individual precautionary savings component is proportional to the quadratic variation $d\langle c^j\rangle_t$ and the consumption response to growth shocks is proportional to $m^{j,1}$. Each agent contributes to the equilibrium interest rate with the respective weight $\bar{\alpha}/\alpha_j$, resulting in the aggregate correction $\frac{\varepsilon^2}{2}\sum_{j=1}^n \frac{\bar{\alpha}}{\alpha_j}(m^{j,1})^2$.

As $m^{j,1}$ measures the j-th agent's overall deviation from autarchy, the quantity $\sum_{j=1}^n \frac{\bar{\alpha}}{\alpha_j}(m^{j,1})^2$ reflects the heterogeneity of agents in endowments. The interest rate decrease is proportional to

$$\sum_{i=1}^n \alpha_i \left(x_i - \frac{\bar{\alpha}}{\alpha_i} P_0 \right)^2, \tag{3.12}$$

hence to the variance of the deviations of initial endowments from their autarchic values, weighted according to risk aversions. Ceteris paribus, an increase in dispersion of endowments depresses the interest rate by increasing aggregate precautionary savings against growth shocks. Because each agent's precautionary savings effect is proportional to the square deviation of the endowment from autarchy, it follows that the aggregate precautionary savings effect is proportional to the variance of autarchy deviation across agents.

The precautionary savings effect due to common unhedgeable risk is more subtle than the one elicited by uninsurable labor income. In the model by Christensen et al. (2012), agents receive a partially unhedgeable income stream and the volatility of the unhedgeable idiosyncratic shocks to the j-th agent's income is $(1-\rho_j)\sigma_{Y_j}$. The interest rate is then reduced by the risk-aversion weighted average of unhedgeable income variances $\frac{1}{2}\bar{\alpha}^2\sum_i \frac{\alpha_i}{\bar{\alpha}}(1-\rho_i)^2\sigma_{Y_i}^2$. Crucially, such a reduction occurs even if agents are homogeneous, therefore it is ascribed to their individual shocks, not to the heterogeneity of their preferences.

Instead, in the present model with common unhedgeable risk, it is agents' heterogeneity that drives the demand for the non-tradeable asset, increasing consumption volatility and consequently reducing the interest rate. If agents are homogeneous, there is no demand for sharing unhedgeable risk, hence there is no additional consumption uncertainty, and the volatility of the unhedgeable shock does not reduce the interest rate.

3.1.6. *Equilibrium prices*

Recall that in this market agents can only (i) trade the stock and (ii) borrow and lend at the endogenous safe rate. In particular, they cannot trade other fixed-income securities such as consol bonds. Thus, each agent has a different *shadow price* for such securities, determined by their individual marginal utility. In particular, the shadow price of the consol bond equals

$$C_t^i = \frac{1}{\bar{r}} - \varepsilon \frac{\bar{\alpha}\sigma_z}{\bar{r}(a+\bar{r})} z_t + \varepsilon^2 \left[\frac{\bar{\alpha}^2\sigma_z^2}{\bar{r}(a+\bar{r})(2a+\bar{r})} \left(z_t^2 + \frac{1}{\bar{r}} \right) + \frac{H(x)}{2\bar{r}^2} - \frac{\bar{\alpha}\sigma_z}{\bar{r}^2(a+\bar{r})} m^{i,1} \right].$$

The first two terms imply that the consol's shadow price reverts around its baseline value of $1/\bar{r}$, with the proportionally higher volatility $\varepsilon\sigma_z\bar{\alpha}/(r(r+a))$, which reflects the combined effects of scaling $(1/\bar{r})$, risk aversion $(\bar{\alpha})$, and mean-reverting dividend shocks $(1/(a+\bar{r}))$. As bond prices move in the opposite direction as interest rates, bond prices are thus counter-cyclical.

The ε^2-correction to the consol bond is divided into three parts: the first term, which is also present in the complete economy in Theorem F.1, can be understood as a convexity effect, as it implies that the shadow price of the consol is convex in the state z_t and hence in the interest rate in (3.1), which is affine in the state z_t. The second term captures the effect of overall agents' heterogeneity – it vanishes with autarchic holdings, in particular with a single agent. The third and last term is agent-specific, and reveals the individual marginal utility from the consol's cash flow. The shadow price is higher for lenders ($x_i > \frac{\bar{\alpha}}{\alpha_i}P_0$) because shocks in z_t have an asymmetric effect on lenders and borrowers. A positive shock to z_t increases the dividend growth rate, benefiting all agents, but also raises the interest rate, benefiting only lenders. Thus, the same shock to z_t has a larger impact on a lender, who is therefore willing to pay more to hedge it.

Note that idiosyncratic corrections do not affect the prices of traded assets. Denote by L_t^i each agent's shadow price for the linear bond, that is the bond that pays at time t a coupon flow equal to t. The stock price in (3.2) is

$$P_t = D_t\tilde{C}_t + (\mu_D - \bar{\alpha}\sigma_D^2)\tilde{L}_t + \varepsilon \frac{1}{\bar{r}(a+\bar{r})}\sigma_z z_t - \varepsilon^2 \frac{2\bar{\alpha}\sigma_z^2}{\bar{r}(a+\bar{r})(2a+\bar{r})} \left(z_t^2 + \frac{1}{\bar{r}} \right),$$

and depends on $\tilde{C}_t$ and $\tilde{L}_t$, which are the averages of the shadow prices of the consol and linear bond in the cross-section of agents, with the same weights $\bar{\alpha}/\alpha_i$ as in the interest rate correction. Importantly, in these averages agents' dependence enters only through the cross-sectional variance $H(x)$, which also reflects heterogeneity in the interest-rate. By contrast, the components of the shadow prices that are linear in $m^{i,1}$ cancel out in the averages and therefore do not affect the stock price.

The stock price in (3.2) includes four terms: The first one $(D_t\tilde{C}_t)$ equals the current dividend multiplied by the average consol bond, hence reflects the change in the present value of a permanent cash flow equal to the current dividend. The second term $(\mu_D - \bar{\alpha}\sigma_D^2)\tilde{L}_t$ is the risk-adjusted long-term growth-rate of the dividend, multiplied by the average linear bond, and represents the change in the present value of the linear increase in cash flow above the current dividend. Together, they capture the present value of future dividends at present interest rates.

The third and fourth terms reflect the cyclical corrections due to fluctuations in dividend growth. The linear term $\varepsilon\frac{1}{\bar{r}(a+\bar{r})}\sigma_z z_t$ entails that, at the first order, stock prices increase as dividend growth increases. The fourth term captures second-order effects, which include a constant and a quadratic component, leading to two implications. First, the equilibrium prices' response to dividend growth is asymmetric: the increase in prices for above-average growth ($z_t > 0$) is smaller than the decrease for below-average growth

($z_t < 0$) of the same magnitude. Second, the effect of dividend-growth variability σ_z is negative on average. This effect does not follow from risk-aversion per se, but from exponential utility's positive intertemporal hedging demand, hence aversion to investment-opportunity variability. Thus, stock prices differ from the present value of their future dividends due to both the constant discount for risk $\bar{\alpha}\sigma_D^2\tilde{L}_t$ and the time-varying hedging premium $\varepsilon^2 \frac{2\bar{\alpha}\sigma_z^2}{\bar{r}(a+\bar{r})(2a+\bar{r})}\left(z_t^2 + \frac{1}{\bar{r}}\right)$.

The next proposition examines the stock price's dynamics and sheds light on the relation between risk and return:

Proposition 3.3. *The stock price satisfies*

$$dP_t = (P_t r_t + \bar{\alpha}\sigma_D^2\tilde{C}_t - D_t)dt + \tilde{C}_t\sigma_D dW_t^D + \sigma_{P,\mu}(z_t, D_t)dW_t^\mu, \tag{3.13}$$

where

$$\begin{aligned}\sigma_{P,\mu}(z_t, D_t) = D_t&\left(-\varepsilon\frac{\bar{\alpha}\sigma_z}{\bar{r}(a+\bar{r})} + \varepsilon^2\frac{2\bar{\alpha}^2\sigma_z^2}{\bar{r}(a+\bar{r})(2a+\bar{r})}z_t\right)\\ &+(\mu_D - \bar{\alpha}\sigma_D^2)\left(-\varepsilon\left(\frac{1}{\bar{r}}+\frac{1}{a+\bar{r}}\right)\frac{\bar{\alpha}\sigma_z}{\bar{r}(a+\bar{r})} + \varepsilon^2\frac{\bar{\alpha}^2\sigma_z^2(2a^2+6ar+3r^2)}{\bar{r}^2(a+\bar{r})^2(2a+\bar{r})^2}2z_t\right)\\ &+\varepsilon\frac{1}{\bar{r}(a+\bar{r})}\sigma_z - \varepsilon^2\frac{4\bar{\alpha}\sigma_z^2}{\bar{r}(a+\bar{r})(2a+\bar{r})}z_t.\end{aligned}$$

At order ε, volatility is proportional to[8]

$$\sigma_D\left(1 - \varepsilon\frac{\bar{\alpha}\sigma_z}{a+\bar{r}}z_t\right)$$

and therefore is counter-cyclical: it is higher in bad times ($z_t < 0$) and lower in good times ($z_t > 0$). This phenomenon can be understood as a discount-rate effect: stock prices reflect the present value of future dividends, which becomes more sensitive to changes in the interest rate when rates are lower (similar to how a bond's duration increases as interest rates decline). Thus, a procyclical interest rate leads to countercyclical volatility.

The stock price dynamics in (3.13) reveals which source of risk is rewarded – and which one is not. Indeed, the excess return is $\bar{\alpha}\sigma_D^2\tilde{C}_t$, which means that the only component of stock-price shocks that is rewarded is the present value of future dividends – and only in relation to dividend risk. In particular, state shocks dW_t^μ do not carry a reward in equilibrium, despite affecting the consumption of agents, who could use exposure to such shocks for hedging. Crucially, the effect of state risk on aggregate consumption is null: aggregate demand for hedging positive and negative shocks are exactly the same and offset each other, resulting in actuarially fair risk-sharing in equilibrium. Note also that the volatility of the excess return follows that of the consol bond's average shadow price $\tilde{C}_t$.

Because state shocks are the same as interest-rate shocks (in view of (3.1)), it follows that interest-rate risk is also unrewarded in equilibrium. Although long-term bonds are not tradeable, interest-rate risk is in fact present in stocks.

3.1.7. Non-Markovian equilibrium

As noted earlier, asset prices are affected by the distribution of wealth, as captured by the coefficient $H(x) = \bar{\alpha}\left(\frac{\bar{\alpha}\sigma_z\bar{r}}{a+\bar{r}}\right)^2 \times \sum_{j=1}^n \alpha_j\left(x_j - \frac{\bar{\alpha}}{\alpha_j}P_0\right)^2$. In the second-order equilibrium outlined in Theorem 3.2, equilibrium prices are expressed in terms of the *initial* rather than *current* wealth distribution. Once the initial wealth distribution is fixed, the interest rate r_t and stock price P_t at second order are determined exclusively by the state variables (t, D_t, z_t). However, Proposition 3.4 reveals that when considering the market equilibrium at a higher order, asset prices no longer depend solely on the initial wealth distribution and the state variables (t, D_t, z_t). Additional state variables must be introduced, indicating that the wealth distribution at time t cannot be fully inferred from the initial wealth distribution and the variables (t, D_t, z_t) alone.

Proposition 3.4. *At the third order, the equilibrium interest rate is*

$$r_t = \bar{r} + \varepsilon\bar{\alpha}\sigma_z z_t - \frac{\varepsilon^2}{2}H(x) - \varepsilon^3\frac{\sigma_z}{a+\bar{r}}\left(\frac{\bar{r}}{2a+\bar{r}}z_t - z_0 - W_t^\mu\right)H(x) + o(\varepsilon^3).$$

In particular, it does not depend on the natural state variables (t, D_t, z_t) alone, but also on W_t^μ.

The dynamics of the interest rate becomes Markovian if one introduces tradeable long-term real bonds, which are not spanned by the assets in the model above (the stock and the money-market). However, such additional securities would also make the market

[8] Note that $\sqrt{\tilde{C}_t^2\sigma_D^2 + \sigma_{P,\mu}(z_t, D_t)^2} = \sigma_D\left(1 - \varepsilon\frac{\bar{\alpha}\sigma_z}{a+\bar{r}}z_t\right) + o(\varepsilon)$.

complete, thereby forfeiting heterogeneity, dynamic trading in stocks, and all the other implications of incompleteness (cf. Theorem F.1). Put differently, non-Markovianity appears to be part and parcel of an incomplete dynamic equilibrium, even starting from a dividend dynamics defined by a simple Markov process.

Path-dependence also arises in the consumption component in (3.5), which is not a function of D_t and z_t. Intuitively, this consumption term represents the tracking error in mimicking a long-term bond position with the safe asset, and hence depends on the whole history of unhedgeable shocks dW_t^{μ}: a sequence of positive shocks in dividend growth results in a wider deviation from the desired bond position and therefore in higher consumption. Of course, and consistently with Duffie et al. (1994), the Markov property holds if one expands state variables to include all agents' wealth processes. In fact, it follows from the first-order condition that

$$\hat{c}_t^i = \frac{\bar{\alpha}}{\alpha_i} D_t + \frac{1}{C_t^i}\left(X_t^i - \frac{\bar{\alpha}}{\alpha_i} P_t\right) + \frac{1}{\alpha_i}\frac{A_t^i}{C_t^i},$$

where $A_t^i := E_t \int_t^\infty (Y_s^i - Y_t^i)\frac{M_s^i}{M_t^i} ds$ and $M_t^i = \exp\{-\bar{\beta}t - \bar{\alpha}(D_t - D_0) + Y_t^i\}$.

3.2. Heterogeneous time-preference

The following theorem describes the characteristics of the second-order equilibrium in the general case of heterogeneous endowments $(x_i)_{i=1}^n$, risk-aversions $(\alpha_i)_{i=1}^n$, and time-preferences $(\beta_i)_{i=1}^n$. With heterogeneous time-preferences, the asymptotic expansions need to be truncated after a long horizon T^ε, as detailed in Appendix C. Such truncation ensures that the welfare effect of approximation errors, which accumulate over time with heterogeneous time-preferences, remains bounded also at very long horizons.

Theorem 3.5. *Assume $\bar{r} := \bar{\beta} + \bar{\alpha}\mu_D - \frac{\bar{\alpha}^2}{2}\sigma_D^2 > 0$. The interest rate r_t, price P_t, consumption $\hat{c}_t^i$ and investment $\hat{\theta}_t^i$ strategies yield a second-order equilibrium. For $t \le T^\varepsilon := -\frac{3}{\bar{r}}\log\varepsilon$, set*

$$r_t = \bar{r} + \varepsilon\bar{\alpha}\sigma_z z_t - \frac{\varepsilon^2}{2}\sum_{j=1}^n \frac{\bar{\alpha}}{\alpha_j}(m_t^{j,1})^2, \tag{3.14}$$

$$P_t = D_t\tilde{C}_t + (\mu_D - \bar{\alpha}\sigma_D^2)\tilde{L}_t + \varepsilon\frac{1}{\bar{r}(a+\bar{r})}\sigma_z z_t - \varepsilon^2\frac{2\bar{\alpha}\sigma_z^2}{\bar{r}(a+\bar{r})(2a+\bar{r})}\left(z_t^2 + \frac{1}{\bar{r}}\right), \tag{3.15}$$

$$\hat{\theta}_t^i = \frac{\bar{\alpha}}{\alpha_i} + \varepsilon^2\frac{\bar{r}^2}{\sigma_D^2} pm_t^1 \frac{m_t^{i,1}}{\alpha_i\bar{r}}, \tag{3.16}$$

$$\hat{c}_t^i = \frac{\bar{\alpha}}{\alpha_i}D_t + \frac{1}{C_0^i}\left(x_i - \frac{\bar{\alpha}}{\alpha_i}P_0\right) + \frac{\bar{\beta}-\beta_i}{\alpha_i}\left(t - \frac{L_0^i}{C_0^i}\right) \tag{3.17}$$

$$-\frac{\varepsilon}{\alpha_i}\int_0^t m_s^{i,1} dW_s^{\mu} - \frac{\varepsilon^2}{\alpha_i}\left(\int_0^t l_s^{i,2} ds + \int_0^t m_s^{i,2} dW_s^{\mu} + \int_0^t n_s^{i,2} dW_s^D\right) + \frac{1}{\alpha_i}\frac{K_0^i}{C_0^i}, \tag{3.18}$$

where C_t^i (resp. L_t^i), is the i-th agent's indifference price at time t of a consol bond that pays a constant coupon flow equal to 1 (resp. of a linear bond that pays at time $s \ge t$ a coupon flow equal to s), defined in closed form in (D.7) (resp. (D.8)), $l_t^{i,2}, m_t^{i,2}, n_t^{i,2}, K_t^i$ are defined in (D.18), (D.19), (D.20), (D.11), $\tilde{C}_t := \sum_{j=1}^N \frac{\bar{\alpha}}{\alpha_j}C_t^{j,2}$, $\tilde{L}_t := \sum_{j=1}^N \frac{\bar{\alpha}}{\alpha_j}(L_t^{j,2} - t\cdot C_t^{j,2})$, and

$$m_t^{i,1} = -\alpha_i\frac{\bar{\alpha}\sigma_z}{a+\bar{r}}\left(\frac{\bar{\beta}-\beta_i}{\alpha_i}\left(t + \frac{1}{a+\bar{r}}\right) + \left(x_i - \frac{\bar{\alpha}}{\alpha_i}P_0\right)\bar{r}\right), \tag{3.19}$$

$$pm_t^1 = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\frac{d\langle P, W^{\mu}\rangle_t}{dt} = -\frac{\sigma_z}{\bar{r}(a+\bar{r})}\left(D_t\bar{\alpha} + \frac{(a+2\bar{r})\bar{\alpha}(\mu_D - \bar{\alpha}\sigma_D^2) - \bar{r}(a+\bar{r})}{\bar{r}(a+\bar{r})}\right). \tag{3.20}$$

In addition, the discounted marginal utility M_t^i of the i-th agent satisfies

$$\log M_t^i = -\bar{\beta}t - \bar{\alpha}(D_t - D_0) + \varepsilon\int_0^t m_s^{i,1} dW_s^{\mu} + \varepsilon^2\left(\int_0^t l_s^{i,2} ds + \int_0^t m_s^{i,2} dW_s^{\mu} + \int_0^t n_s^{i,2} dW_s^D\right). \tag{3.21}$$

This statement generalizes Theorem 3.2, describing the effect of agents' heterogeneous time-preferences on their consumption-investment policies and on asset prices. The main differences are driven by the expression of $m_t^{i,1}$ in (3.19), which in general combines an agent's deviation of the initial endowment from autarchy with the deviation of time-preference from the average. Consequently, the interpretation of the effects described above is updated as follows.

3.2.1. Consumption

The deviation of consumption from the complete market setting (where a consol bond is traded) becomes

$$-\frac{\varepsilon}{\alpha_i}\int_0^t m_s^{i,1}\,dW_s^\mu = \varepsilon\int_0^t \frac{\bar\alpha\sigma_z}{a+\bar r}\left(\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right)\bar r+\frac{\bar\beta-\beta_i}{\alpha_i}\left(t+\frac{1}{a+\bar r}\right)\right)dW_s^\mu.$$

This formula has the following meaning: agents that are more patient relative to the average ($\beta_i<\bar\beta$) initially consume less than their share of dividends, lending the remainder. Thus, more patience is akin to an endowment above autarchy ($x_i>\frac{\bar\alpha}{\alpha_i}P_0$), in that it tilts an agent's behavior towards lending.

Similarly, an atypical time-preference ($\beta_i\ll\bar\beta$ or $\beta_i\gg\bar\beta$) is observationally similar to an atypical endowment ($x_i\ll\frac{\bar\alpha}{\alpha_i}P_0$ or $x_i\gg\frac{\bar\alpha}{\alpha_i}P_0$) in generating a larger deviation from an agent's desired bond position Q_t^i after an unexpected shock dW_t^μ, resulting in higher consumption uncertainty relative to average agents.

Finally, note that the magnitude of time-preference deviations, unlike autarchy deviations, increases over time: atypical agents see their exposure to unhedgeable shocks grow, even though these deviations aggregate to zero, resulting in agents with different time-preference insuring one another.

3.2.2. Dynamic trading

The general expression of the dynamic trading policy is:

$$\hat\theta_t^i=\frac{\bar\alpha}{\alpha_i}+\varepsilon^2\frac{\bar\alpha^2\sigma_z^2}{(a+\bar r)^2\sigma_D^2}\left(\frac{\bar\beta-\beta_i}{\alpha_i}\left(t+\frac{1}{a+\bar r}\right)+\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right)\bar r\right)\left(D_t+\frac{(a+2\bar r)(\mu_D-\bar\alpha\sigma_D^2)}{\bar r(a+\bar r)}-\frac{1}{\bar\alpha}\right).\tag{3.22}$$

This formula shows that the deviation of the time-preference rate β_i from its aggregate value $\bar\beta$ is also a motive for trading, in addition to the autarchy deviation discussed above. Agents with more or less patience respond in opposite ways to dividend growth shocks: The income effect prevails for impatient agents ($\beta_i>\bar\beta$), who respond to a positive dividend growth shock by increasing consumption and selling some of their holdings, similar to natural borrowers ($x_i<\frac{\bar\alpha}{\alpha_i}P_0$). More patient agents are the natural buyers in this trade, as the substitution effect leads them to increase holdings.

Overall, the intuition is still that higher patience has qualitatively the same effect of a higher endowment. The exact quantitative relationship is in (3.19), which combines both effects with risk-aversion and other market parameters. Thus, the trading motive for agents in this market remains hedging, though the magnitude of the hedging motive is jointly determined by deviations from both autarchy and average time-preference.

3.2.3. Interest rates

With heterogeneous time-preference, the interest rate (3.14) is still reduced by $\frac{\varepsilon^2}{2}\sum_{j=1}^n\frac{\bar\alpha}{\alpha_j}(m_t^{j,1})^2$, but the expression of $m_t^{j,1}$ in (3.19) reflects the joint effect of preferences and endowments. With autarchic endowments ($x_i=\frac{\bar\alpha}{\alpha_i}P_0$), the decrease in the interest rate is proportional to

$$\left(t+\frac{1}{a+\bar r}\right)^2\sum_{i=1}^n\frac{\bar\alpha}{\alpha_i}(\bar\beta-\beta_i)^2.\tag{3.23}$$

The last factor $\sum_{i=1}^n\frac{\bar\alpha}{\alpha_i}(\bar\beta-\beta_i)^2$ is precisely the variance of the time-preference rates $(\beta_i)_{i=1}^n$ in the cross-section of agents, each of them weighted inversely with risk aversion (recall that $\sum_{i=1}^n\frac{\bar\alpha}{\alpha_i}=1$). Ceteris paribus, an increase in dispersion of time-preference depresses the interest rate because it increases aggregate precautionary savings against growth shocks. Because each agent's precautionary savings effect is proportional to the square distance of time preference from average time preference, it follows that the aggregate precautionary savings effect is proportional to the variance of time-preference across agents.

The first factor $\left(t+\frac{1}{a+\bar r}\right)^2$ reveals how temporal dependence in the interest rate emerges from time-preference heterogeneity among agents. In contrast to the homogeneous case, where interest rates follow a time-homogeneous Vasicek process, heterogeneity in agent patience generates a Hull-White extension of the Vasicek model characterized by negative, time-dependent drift. The intuition is that heterogeneity in time-preference induces growing imbalances in borrowing and lending: over time, patient agents become increasingly rich and eager to lend, while impatient agents increasingly poor and hesitant to borrow. As a result, the equilibrium interest rate declines.

The time-preference and autarchy-deviation effects can either offset or reinforce each other in (3.19), depending on their joint distribution. For example, if natural lenders are more patient than borrowers, the savings imbalance is stronger, hence the combined effect higher. Vice versa, if natural lenders are less patient, then the effects offset each other.

3.2.4. Equilibrium prices

Asset pricing implications are similar to the homogeneous time-preference case, with the difference that heterogeneity has a time-varying effect. For example, the shadow price of the consol bond equals

$$C_t^i = \frac{1}{\bar r} - \varepsilon \frac{\bar\alpha}{\bar r(a+\bar r)}\sigma_z z_t + \varepsilon^2 \left[\frac{\bar\alpha^2\sigma_z^2}{\bar r(a+\bar r)(2a+\bar r)}\left(z_t^2 + \frac{1}{\bar r}\right) + \frac{1}{2}\sum_{j=1}^N \frac{\bar\alpha}{\alpha_j} M_t^{j,C} - \bar\alpha \frac{\sigma_z}{a}\int_t^\infty \frac{a e^{-r(u-t)}}{r(a+r)} m_u^{i,1} du, \right]$$

Here, the effect of heterogeneity is captured by the term $\sum_{j=1}^N \frac{\bar\alpha}{\alpha_j} M_t^{j,C}$ (respectively, $\sum_{j=1}^N \frac{\bar\alpha}{\alpha_j} M_t^{j,L}$ for the linear bond), which aggregates over time the discounted correction $\sum_{j=1}^n \frac{\bar\alpha}{\alpha_j}(m_t^{j,1})^2$ of the instantaneous interest-rate. As with homogeneous time-preference, the components of the shadow prices that are linear in $m^{i,1}$ cancel out in the average, and therefore do not affect prices. Thus, the impact of heterogeneity on the stock price in (3.15) is driven only by its discounting implications through $\tilde C_t$ and $\tilde L_t$.

3.2.5. Relation to idiosyncratic income risk models

In the present model, market incompleteness arises from unhedgeable shocks to the dividend's growth rate, whereas the most commonly studied type of incompleteness in the literature stems from idiosyncratic income risk. Christensen et al. (2012) derive closed-form expressions for the market equilibrium with CARA agents and Gaussian dynamics. This shared combination of preferences and market dynamics enables a clear comparison between the two models: both types of incompleteness lead to more volatile consumption and lower interest rates through the precautionary-savings channel. However, their implications for asset prices and the required mathematical methods differ substantially.

The idiosyncratic endowment shocks in Christensen et al. (2012) do not affect dividends' dynamics, but change agents' demand for stocks for endowment hedging, thereby affecting their equilibrium expected return but leaving the covariance matrix of returns proportional to the covariance matrix of dividends. In addition, the resulting equilibrium interest rate is deterministic, hence stock prices do not fluctuate through the discount-rate channel. Thus, shocks to individual endowments that are uncorrelated with dividends remain unhedgeable in the equilibrium and are not priced.

In contrast, the present model links shocks to dividends' growth rate with the dynamics of dividends themselves, resulting in a stochastic interest rate. This stochastic rate influences stock prices, turning them into hedging instruments against interest rate shocks. (In Proposition 3.3, both W^D and W^μ drive the stock return.) In the equilibrium, dividend risk and interest-rate risk cannot be disentangled and are priced in relation to agents' heterogeneity in preferences and wealth.

These differences have significant conceptual and technical implications. Idiosyncratic income risk leads to the separation between individual agents' optimization and equilibrium identification. With asset prices independent of endogenous variables, optimal investment and consumption strategies are Markovian and amenable to stochastic control methods. On the other hand, in our model, equilibrium quantities are not Markovian in exogenous state variables alone, rendering control methods infeasible, and requiring duality techniques instead to identify the equilibrium.

4. Asset pricing implications

This section examines the asset pricing implications of our model. We first analyze how return predictability varies with the investment horizon, then study the cross-section of expected returns and correlations in an economy with multiple assets.

4.1. Price predictability

The empirical literature documents that the predictability of stock prices increases with the horizon: while short-term returns are virtually unpredictable, long-term returns are significantly correlated with the price-dividend ratio (see Cochrane (2008) for a recent discussion).

Indeed, the equilibrium shows that fluctuations in dividend growth generate predictability in expected excess returns: in contrast to the baseline case, in which the risk-premium merely depends on the horizon, here it also depends on the state variable z_t. Such predictability is negligible at short horizons, but significant at longer horizons. First, define by $E_{s,t}P$ the expected present value of the cash flow from time s to time t above the current price (i.e., including both dividends and the final price):

$$E_{s,t}P := E\left[P_t e^{-\int_s^t r_u du} + \int_s^t D_u e^{-\int_s^u r_l dl} du \middle| \mathcal{F}_s \right] - P_s. \tag{4.1}$$

In other words, the expected total return from s to t is $E_{s,t}P/P_s$ and is predictable if and only if $E_{s,t}P$ is predictable. Likewise, define the (hypothetical) expected excess cash flow on a consol bond as $E_{s,t}C = E[C_t e^{-\int_s^t r_u du} + \int_s^t 1 e^{-\int_s^u r_l dl} du|\mathcal{F}_s] - C_s$. The following proposition obtains $E_{s,t}P$ and $E_{s,t}C$ at order ε:

Proposition 4.1. *Let P_t and r_t be as in Theorem 3.5. Then*

$$E_{s,t}P = \frac{\bar\alpha\sigma_D^2}{\bar r^2}(1-e^{-\bar r(t-s)}) + \varepsilon \frac{\bar\alpha^2\sigma_D^2\sigma_z z_s}{a\bar r^2(a+\bar r)^2}\left(e^{-\bar r(t-s)}(a+\bar r)^2 - e^{-(a+\bar r)(t-s)}\bar r^2 - a^2 - 2a\bar r\right) + o(\varepsilon), \tag{4.2}$$

$$E_{s,t}C = o(\varepsilon). \tag{4.3}$$

To understand how predictability increases with the horizon, it is useful to consider the first-order expansion of (4.2) for $\delta = t - s$ small, which is

$$E_{s,s+\delta} P = \frac{\bar{\alpha}\sigma_D^2}{\bar{r}} \left(1 - \frac{\bar{\alpha}}{a+\bar{r}} \varepsilon \sigma_z z_s\right) \delta + o(\varepsilon) + o(\delta). \tag{4.4}$$

Vice versa, for a long horizon ($\delta = \infty$), this quantity is

$$E_{s,\infty} P = \frac{\bar{\alpha}\sigma_D^2}{\bar{r}^2} \left(1 - \frac{\bar{\alpha}(a+2\bar{r})}{(a+\bar{r})^2} \varepsilon \sigma_z z_s\right) + o(\varepsilon). \tag{4.5}$$

Compare now the sensitivity of (4.4) and (4.5) with respect to a change in the state z_s:

$$\frac{\partial E_{s,s+\delta} P}{\partial z_s} = (-\delta)\varepsilon\sigma_z \frac{\bar{\alpha}^2\sigma_D^2}{\bar{r}(a+\bar{r})} \qquad \frac{\partial E_{s,\infty} P}{\partial z_s} = \left(-\frac{a+2\bar{r}}{\bar{r}(a+\bar{r})}\right)\varepsilon\sigma_z \frac{\bar{\alpha}^2\sigma_D^2}{\bar{r}(a+\bar{r})} \tag{4.6}$$

and note that the factor outside parentheses is common to both sensitivities. These expressions indicate that a positive shock in the state z_s reduces both short- and long-term expected excess returns, due to a discounting effect (future cash flows are more heavily discounted). However, the impact on the short-term expected excess cash flows is proportional to the horizon δ, hence negligible as δ vanishes. Vice versa, the long-term effect is large: in the typical case of $\bar{r}$ small, $\frac{a+2\bar{r}}{\bar{r}(a+\bar{r})} \sim \frac{1}{\bar{r}}$, which means that the long-term impact is approximately $\frac{1}{\bar{r}}$ larger than the one-year impact.

Price predictability in the model is specific to stocks, and not a general feature of all asset prices. Indeed, (4.3) confirms that the hypothetical price of the consol bond has no predictability at the first order, and in fact no excess return. Put differently, the time-varying expected excess returns of stocks do not mimic those of bonds, as the latter are zero.

Finally, even if shocks to dividends and the growth rate are correlated, return predictability remains negligible at short horizons. Proposition 4.2 below computes the expected total stock return under correlation ρ between the Brownian motions W^D and W^μ (affecting dividends and their growth rate).

Proposition 4.2. *Assume W^D and W^μ have correlation ρ, and let $P_{\rho,t}$ be the stock price in equilibrium. Then*

$$\begin{aligned} E_{s,t} P_{\rho,t} &= E_{s,t} P_{0,t} \\ &\quad + \rho\varepsilon \frac{\sigma_D \sigma_z \bar{\alpha}^2}{\bar{r}^2(a+\bar{r})} \left(\mu_D e^{-\bar{r}(t-s)}(t-s) - (1-e^{-\bar{r}(t-s)})\left(D_s - \frac{1}{\bar{\alpha}}\right) - \frac{a(2a+3\bar{r})}{\bar{r}(a+\bar{r})}(1-e^{-\bar{r}(t-s)})(\mu_D - \bar{\alpha}\sigma_D^2)\right) + o(\varepsilon), \\ E_{s,t} C_{\rho,t} &= -\varepsilon \bar{\alpha}^2 \sigma_D \sigma_z \frac{1}{\bar{r}^2(a+\bar{r})} \rho (1 - e^{-\bar{r}(t-s)}) + o(\varepsilon), \end{aligned}$$

where $P_{0,t}$ is the equilibrium stock price for $\rho = 0$.

Note that for $\delta = t - s$ small,

$$E_{s,s+\delta} P_\rho = E_{s,s+\delta} P_0 + \rho\varepsilon \frac{\sigma_D \sigma_z \bar{\alpha}^2}{\bar{r}^2(a+\bar{r})} \left(\mu_D - \bar{r}\left(D_s - \frac{1}{\bar{\alpha}}\right) - \frac{a(2a+3\bar{r})}{a+\bar{r}}(\mu_D - \bar{\alpha}\sigma_D^2)\right)\delta + o(\varepsilon) + o(\delta).$$

Instead, for a long horizon ($\delta = \infty$), the expected total return is

$$E_{s,\infty} P_\rho = E_{s,\infty} P_0 + \rho\varepsilon \frac{\sigma_D \sigma_z \bar{\alpha}^2}{\bar{r}^2(a+\bar{r})} \left(-D_s + \frac{1}{\bar{\alpha}} - \frac{a(2a+3\bar{r})}{\bar{r}(a+\bar{r})}(\mu_D - \bar{\alpha}\sigma_D^2)\right) + o(\varepsilon).$$

With positive correlation $\rho > 0$ between dividends' and state variable' shocks, a positive dividend shock corresponds to a positive shock to the state z_t, leading again to lower expected excess returns at both short and long horizons. While this effect is negligible in the short term ($\delta \approx 0$), it becomes substantial at longer horizons.

4.2. Cross-sectional implications

The results in the previous section extend to a market with multiple assets, and such an extension is useful to understand the cross-sectional asset pricing implications of the model. Assume that the market includes k stocks, each of them available in unit supply. The dynamics of the k-dimensional dividend process is

$$\begin{aligned} dD_t &= (\boldsymbol{\mu_D} + \varepsilon \boldsymbol{\sigma_z} \mathbf{z}_t)dt + \boldsymbol{\sigma_D}\, dW_t^D, \\ d\mathbf{z}_t &= -A\mathbf{z}_t\, dt + dW_t^\mu, \end{aligned}$$

where W^D and W^μ are k-dimensional independent Brownian motions, $\boldsymbol{\mu_D}$ is a k-vector, and $\boldsymbol{\sigma_D}, \boldsymbol{\sigma_z}, A$ are nonsingular $k \times k$ matrices.

4.2.1. Baseline case

For $\varepsilon = 0$, the dynamics of the dividend process reduce to

$$dD_t = \boldsymbol{\mu_D}\, dt + \boldsymbol{\sigma_D} dW_t^D.$$

Henceforth, the column vector σ_D^j denotes the transpose of the j-th row of $\boldsymbol{\sigma_D}$, μ_D^j the j-th element of $\boldsymbol{\mu_D}$ and $\mathbf{1}$ the column vector with all components equal to one.

Theorem 4.3. *Assume* $\bar\beta + \bar\alpha \mathbf{1}^\top \boldsymbol{\mu_D} - \frac{1}{2}\bar\alpha^2 \mathbf{1}^\top \boldsymbol{\sigma_D}\boldsymbol{\sigma_D}^\top \mathbf{1} > 0$. *The pair* $(r, (P^i_\cdot)_{i=1}^k)$ *defined by*

$$r = \bar\beta + \bar\alpha \mathbf{1}^\top \boldsymbol{\mu_D} - \frac{1}{2}\bar\alpha^2 \mathbf{1}^\top \boldsymbol{\sigma_D}\boldsymbol{\sigma_D}^\top \mathbf{1}, \qquad P_t^j = \frac{D_t^j}{r} + \frac{\mu_D^j - \bar\alpha \mathbf{1}^\top \boldsymbol{\sigma_D} \sigma_D^j}{r^2}$$

is a market equilibrium, and the agents' optimal policies are

$$\theta_t^i = \frac{\bar\alpha}{\alpha_i}\mathbf{1},$$

$$\hat c_t^i = \frac{\bar\alpha}{\alpha_i}\mathbf{1}^\top D_t + \frac{\bar\beta - \beta_i}{\alpha_i}\left(t - \frac{1}{r}\right) + r\left(x_i - \frac{\bar\alpha}{\alpha_i}\mathbf{1}^\top P_0\right).$$

The corresponding stochastic discount factor is $M_t = \exp\{-\bar\beta t - \bar\alpha \mathbf{1}^\top (D_t - D_0)\}$.

This result, which mirrors its scalar counterpart, Theorem 2.1, yields the following long-term version of the capital asset pricing model (CAPM): the expected price change in an asset i is proportional to its market beta, defined as the covariance of the asset's price change with the market's, divided by the variance of the market's price change.

Corollary 4.4. *Define* S_t^i *as the expected present value of future dividends and by* R_t^i *the expected present value of gains (on* $[t, \infty]$*):*

$$S_t^i := E_t\left[\int_t^\infty e^{-r(s-t)} D_s^i ds\right] = \frac{D_t^i}{r} + \frac{\mu_D^i}{r^2}, \qquad R_t^i := S_t^i - P_t^i = \frac{1}{r^2}\bar\alpha (\sigma_D^i)^\top \boldsymbol{\sigma_D}^\top \mathbf{1}. \tag{4.7}$$

Denote also the total market price as $P_t^m = \sum_{i=1}^n P_t^i$, *the expected total market gains as* $R_t^m := \sum_{i=1}^n (S_t^i - P_t^i)$, *and the market beta of the* i*-th asset as*

$$Beta_i := \frac{d\langle P^i, P^m\rangle_t}{d\langle P^m\rangle_t} = \frac{(\sigma_D^i)^\top \boldsymbol{\sigma_D}^\top \mathbf{1}}{\mathbf{1}^\top \boldsymbol{\sigma_D}\boldsymbol{\sigma_D}^\top \mathbf{1}}. \tag{4.8}$$

Then asset prices satisfy the following version of the Capital Asset Pricing Model:

$$R_t^i = Beta_i R_t^m. \tag{4.9}$$

The next result examines the long-term composition of market's wealth and agents' portfolios. No agent vanishes, and neither does any agent's stock exposure.

Corollary 4.5. *In the long run (as* $t \to \infty$*):*

(i) The share of total wealth $\frac{X_t^i}{\sum_{j=1}^k P_t^j}$ *owned by the* i*-th agent converges a.s. to* $\frac{\bar\alpha}{\alpha_i} + \frac{\bar\beta - \beta_i}{\alpha_i \mathbf{1}^\top \boldsymbol{\mu_D}}$.

(ii) The share $\frac{\theta_t^{i,j} P_t^j}{X_t^i}$ *of* i*-th agent's wealth in the* j*-th asset converges a.s. to* $\frac{\mu_D^j}{\frac{\bar\beta - \beta_i}{\bar\alpha} + \mathbf{1}^\top \boldsymbol{\mu_D}}$.

In the long run, agents' shares of wealth revert to asymptotic values that are determined by their preferences and are independent of the initial conditions. Specifically, the "natural" share of wealth of the i-th agent is the sum of the relative risk tolerance $\bar\alpha/\alpha_i$ and the excess patience $\bar\beta - \beta_i$, scaled by growth ($\mathbf{1}^\top \boldsymbol{\mu_D}$) and risk-aversion ($\alpha_i$).

In the long run, also the portfolio weight of the i-th agent in the j-th asset converges to a constant, similar to partial-equilibrium portfolio choice with isoelastic utility (Merton, 1969), and in contrast to models with exponential utility and exogenous safe rate, in which the number of shares is constant, while the portfolio weights decline to zero. The endogenous safe rate in the model adjusts to make aggregate savings null, thereby keeping the number of shares constant for all agents while letting their portfolio weight converge.

The asymptotic portfolio weight of each agent in the j-th asset is $\pi_i^j = (\frac{\bar{\beta}-\beta_i}{\bar{\alpha}\mu_D^j} + \frac{\mathbf{1}^\top \boldsymbol{\mu_D}}{\mu_D^j})^{-1}$, which reduces to $\mu_D^j / \mathbf{1}^\top \boldsymbol{\mu_D}$ for an agent with average time-preference $\bar{\beta}$, meaning that each asset is held in proportion to the growth rate of its dividend. For more patient agents ($\beta_i < \bar{\beta}$), portfolio weights are lower and closer to each other than their relative dividend growth rates, and vice versa.

4.2.2. General case

The intertemporal capital asset pricing model of Merton (1973) links the presence of state variables to multifactor asset pricing in a partial equilibrium setting, thereby raising the natural question of whether such a link arises in the present general equilibrium setting.

Similar arguments to the previous section lead to the following result

Theorem 4.6. *Expected cash flows R^i and market cash flows R^m, defined as in Corollary 4.4, are*

$$R_t^i = \left(\frac{\bar{\alpha}}{\bar{r}^2} - \varepsilon\bar{\alpha}^2 \mathbf{1}^T \boldsymbol{\sigma_z}\left(\frac{1}{\bar{r}^2} I - \int_0^\infty s e^{-(\bar{r}I+A)s} ds\right) A^{-1} \mathbf{z}_t\right) e^{iT} \boldsymbol{\sigma_D} \boldsymbol{\sigma_D}^T \mathbf{1} + o(\varepsilon),$$

$$R_t^m = \left(\frac{\bar{\alpha}}{\bar{r}^2} - \varepsilon\bar{\alpha}^2 \mathbf{1}^T \boldsymbol{\sigma_z}\left(\frac{1}{\bar{r}^2} I - \int_0^\infty s e^{-(\bar{r}I+A)s} ds\right) A^{-1} \mathbf{z}_t\right) \mathbf{1}^T \boldsymbol{\sigma_D} \boldsymbol{\sigma_D}^T \mathbf{1} + o(\varepsilon),$$

and the market beta of the i-th asset is $Beta_i = \frac{(\sigma_D^i)^\top \boldsymbol{\sigma_D}^\top \mathbf{1}}{\mathbf{1}^\top \boldsymbol{\sigma_D}\boldsymbol{\sigma_D}^\top \mathbf{1}} + o(\varepsilon)$. Therefore the CAPM holds at the first order, in that $R_t^i = Beta_i R_t^m + o(\varepsilon)$.

This result demonstrates that, when dividend shocks and growth shocks are uncorrelated, then the CAPM holds at first order. The next result shows that correlation between these shocks can lead to first-order deviations from the CAPM, and explains the determinants of such deviations. The central theoretical question is to understand the relation between the resulting pricing formulas and multifactor models such as Fama and French (1993) and its numerous variants.

To ease notation, and because the resulting formulas are already complex, the result is derived in the case of a scalar state z_t, that is in the model

$$\begin{aligned} dD_t &= (\boldsymbol{\mu_D} + \varepsilon \boldsymbol{\sigma_z} z_t)dt + \boldsymbol{\sigma_D}\, dW_t^D, \\ dz_t &= -az_t dt + \boldsymbol{\rho}\, dW_t^D + \sqrt{1-|\boldsymbol{\rho}|^2} d\tilde{W}_t^\mu, \end{aligned}$$

where $\tilde{W}^\mu$ is a Brownian motion independent of the k-dimensional Brownian motion W^D, and $\boldsymbol{\rho} \in \mathbb{R}^k$.

In this setting, the asset pricing result is as follows:

Theorem 4.7. *Asset cash flows R^i and the market cash flows R^m, defined as in Corollary 4.4, are*

$$\begin{aligned} R_t^i =& \frac{\bar{\alpha}}{\bar{r}^2}\left(1 - \varepsilon\bar{\alpha}\mathbf{1}^T \boldsymbol{\sigma_z} \frac{a+2\bar{r}}{(a+\bar{r})^2} z_t\right)(\sigma_D^i \boldsymbol{\sigma_D}^T 1) + \\ & \left(-\bar{\alpha}(\mathbf{1}^T \boldsymbol{\sigma_z})D_t^i + \sigma_z^i - \frac{2a+3\bar{r}}{\bar{r}(a+\bar{r})}\bar{\alpha}(\mathbf{1}^T \boldsymbol{\sigma_z})(\mu_D^i - \bar{\alpha}\sigma_D^i \boldsymbol{\sigma_D}^T 1)\right)\frac{\varepsilon\bar{\alpha}}{\bar{r}^2(a+\bar{r})}(\mathbf{1}^T \boldsymbol{\sigma_D}\boldsymbol{\rho}) + o(\varepsilon), \\ R_t^m =& \frac{\bar{\alpha}}{\bar{r}^2}\left(1 - \varepsilon\bar{\alpha}\mathbf{1}^T \boldsymbol{\sigma_z} \frac{a+2\bar{r}}{(a+\bar{r})^2} z_t\right)(\mathbf{1}^T \boldsymbol{\sigma_D}\boldsymbol{\sigma_D}^T 1) + \\ & \left(-\bar{\alpha}\mathbf{1}^T D_t + 1 - \frac{2a+3\bar{r}}{\bar{r}(a+\bar{r})}(\mathbf{1}^T \boldsymbol{\mu_D} - \bar{\alpha}\mathbf{1}^T \boldsymbol{\sigma_D}\boldsymbol{\sigma_D}^T 1)\right)(\mathbf{1}^T \boldsymbol{\sigma_z})\frac{\varepsilon\bar{\alpha}}{\bar{r}^2(a+\bar{r})}(\mathbf{1}^T \boldsymbol{\sigma_D}\boldsymbol{\rho}) + o(\varepsilon), \end{aligned}$$

and therefore

$$R^i = Beta_i R^m \tag{4.10}$$

$$+ \frac{\varepsilon\bar{\alpha}(\mathbf{1}^T \boldsymbol{\sigma_D}\boldsymbol{\rho})}{\bar{r}^2(a+\bar{r})}(\mathbf{1}^T \boldsymbol{\sigma_z})\left(-\bar{\alpha}\left(D_t^i - Beta_i(\mathbf{1}^T D_t)\right) + \left(\frac{\sigma_z^i}{\mathbf{1}^T \boldsymbol{\sigma_z}} - Beta_i\right) - \frac{2a+3\bar{r}}{\bar{r}(a+\bar{r})}\bar{\alpha}\left(\mu_D^i - Beta_i(\mathbf{1}^T \boldsymbol{\mu_D})\right)\right) \tag{4.11}$$

$$+\, o(\varepsilon).$$

This result shows that, in the presence of correlation, the CAPM relation $R^i = Beta_i R^m$ does not hold at first order, in view of the additional terms in (4.11), which reflect the extent to which an asset's characteristics deviate from their aggregate counterparts, relative to the asset's market beta: (i) the last term in (4.11) is proportional to $\mu_D^i/(\mathbf{1}^T \boldsymbol{\mu_D}) - Beta_i$, the amount by which the growth in the i-th asset, relative to aggregate growth, exceeds the asset's own beta $Beta_i$; (ii) the second term is proportional to $\frac{\sigma_z^i}{\mathbf{1}^T \boldsymbol{\sigma_z}} - Beta_i$, the amount by which the i-th asset's relative sensitivity exceeds $Beta_i$; (iii) the first term is proportional to $\frac{D_t^i}{\mathbf{1}^T D_t} - Beta_i$, the deviation of the dividend share of i-th asset from the asset's beta. Note that this term is stochastic, unlike the others, as it depends on the asset's dividend share $D_t^i/\mathbf{1}^T D_t$. Thus, all dividend shares are necessary to explain the cross-section of asset returns, which excludes the possibility of a low-dimensional factor structure (that is, a fixed number of factors but an arbitrary number of assets n).

More broadly, when the dividend-state correlation $\mathbf{1}^T \boldsymbol{\sigma_D} \boldsymbol{\rho}$ and the market sensitivity $\mathbf{1}^T \boldsymbol{\sigma_z}$ are concordant (i.e., either both positive or both negative), the term $\frac{D_t^i}{\mathbf{1}^T D_t} - Beta_i$ generates a "size" effect (Banz, 1981), whereby assets whose dividends (hence prices) are abnormally low command a higher return, and vice versa. In this model, such an effect is not a pricing anomaly, but an integral part of the equilibrium. Likewise, the last two terms in (4.11) correspond to nonzero "alphas" in the regression of asset's excess returns on the market's return, and convey information on the relative growth rates of different assets. Again in the case of concordant $\mathbf{1}^T \boldsymbol{\sigma_D} \boldsymbol{\rho}$ and $\mathbf{1}^T \boldsymbol{\sigma_z}$, the term $\mu_D^i - Beta_i(\mathbf{1}^T \boldsymbol{\mu_D})$ mimics a "value" effect, in that an asset whose relative growth rate $\mu_D^i/\mathbf{1}^T \boldsymbol{\mu_D}$ is lower than its beta is linked to a higher expected return than predicted by the CAPM. Unlike the "size" effect, which in this model varies with the dividend share, the "value" effect is fixed for each asset.

4.2.3. Price comovements

This subsection examines the dynamics of stock prices, in particular on the extent to which their volatility and correlations depart from their respective dividends (Shiller et al., 1981).

Proposition 4.8 below extends Proposition 3.3 to the multiple-asset setting, demonstrating that the excess return $\bar{\alpha}\sigma_{P,D}(D_t, z_t)\boldsymbol{\sigma_D}^T \mathbf{1}$ is proportional to the volatility component $\sigma_{P,D}(D_t, z_t)$ driven by the multivariate Brownian motion W^D. The volatility matrix $\sigma_{P,D}(D_t, z_t)$ includes a term proportional to the dividends' volatility matrix $\boldsymbol{\sigma_D}$ and, if state-dividends correlations $\boldsymbol{\rho}$ are nonzero, a contribution from shocks to z_t, due to both discounting and dividend growth. Note that the excess return is time-varying and counter-cyclical, as it increases when consumption growth decreases ($z_t < 0$).

Proposition 4.8. *Define $\boldsymbol{\Sigma_D} := \boldsymbol{\sigma_D}\boldsymbol{\sigma_D}^T$. At the first order, stock prices have the dynamics*

$$dP_t = (P_t r_t - D_t + \bar{\alpha}\sigma_{P,D}(D_t, z_t)\boldsymbol{\sigma_D}^T \mathbf{1})dt + \sigma_{P,D}(D_t, z_t)dW_t^D + \sigma_{P,\mu}(D_t)d\tilde{W}_t^{\mu},$$

where

$$\begin{aligned}
\sigma_{P,D}(D_t, z_t) :=& \tilde{C}_t \boldsymbol{\sigma_D} - \varepsilon\bar{\alpha}\mathbf{1}^T \boldsymbol{\sigma_z} \frac{1}{\bar{r}(a+\bar{r})} D_t \boldsymbol{\rho}^T \\
&+ \varepsilon \frac{1}{\bar{r}(a+\bar{r})} \left[\boldsymbol{\sigma_z} - \bar{\alpha}\mathbf{1}^T \boldsymbol{\sigma_z} \left(\frac{1}{\bar{r}} + \frac{1}{a+\bar{r}}\right)(\boldsymbol{\mu_D} - \bar{\alpha}\boldsymbol{\Sigma_D}\mathbf{1})\right]\boldsymbol{\rho}^T, \\
\sigma_{P,\mu}(D_t) :=& -\varepsilon\bar{\alpha}\mathbf{1}^T \boldsymbol{\sigma_z} \frac{1}{\bar{r}(a+\bar{r})} D_t \sqrt{1-|\boldsymbol{\rho}|^2} \\
&+ \varepsilon \frac{1}{\bar{r}(a+\bar{r})} \left[\boldsymbol{\sigma_z} - \bar{\alpha}\mathbf{1}^T \boldsymbol{\sigma_z} \left(\frac{1}{\bar{r}} + \frac{1}{a+\bar{r}}\right)(\boldsymbol{\mu_D} - \bar{\alpha}\boldsymbol{\Sigma_D}\mathbf{1})\right]\sqrt{1-|\boldsymbol{\rho}|^2}, \\
\tilde{C}_t :=& \frac{1}{\bar{r}} - \varepsilon\bar{\alpha}\mathbf{1}^T \boldsymbol{\sigma_z} \frac{1}{\bar{r}(a+\bar{r})} z_t + \bar{\alpha}\varepsilon\bar{\alpha}\mathbf{1}^T \boldsymbol{\sigma_z} \frac{1}{\bar{r}^2(a+\bar{r})} \mathbf{1}^T \boldsymbol{\sigma_D}\boldsymbol{\rho}.
\end{aligned}$$

Corollary 4.9 below computes the covariances and correlations of assets' returns. The covariation matrix (4.13) equals the baseline $\frac{1}{\bar{r}^2}\boldsymbol{\Sigma_D}$, which reflects covariance among dividends, plus first-order terms that either increase or decrease from the baseline. The covariation is also counter-cyclical – higher in bad times ($z_t < 0$) than in good times ($z_t > 0$) – due to discounting: interest rates are lower when dividend growth is lower, thereby amplifying the impact of dividend shocks on stock prices.

Corollary 4.9. *Define $\rho_{i,j} = \frac{d\langle P^i, P^j\rangle_t/dt}{\sqrt{d\langle P^i, P^i\rangle_t/dt}\sqrt{d\langle P^j, P^j\rangle_t/dt}}$ and*

$$\boldsymbol{w}_t := -\bar{\alpha}(\mathbf{1}^T \boldsymbol{\sigma_z})D_t - \bar{\alpha}\mathbf{1}^T \boldsymbol{\sigma_z}\left(\frac{1}{\bar{r}} + \frac{1}{a+\bar{r}}\right)(\boldsymbol{\mu_D} - \bar{\alpha}\boldsymbol{\Sigma_D}\mathbf{1}) + \boldsymbol{\sigma_z}, \quad A_t := \boldsymbol{\sigma_D}\boldsymbol{\rho}\boldsymbol{w}_t^T + \boldsymbol{w}_t\boldsymbol{\rho}^T\boldsymbol{\sigma_D}^T. \tag{4.12}$$

The covariation matrix of stock prices is

$$\frac{d\langle P, P\rangle_t}{dt} = \frac{1}{\bar{r}^2}\boldsymbol{\Sigma_D} + 2\varepsilon\frac{\bar{\alpha}}{\bar{r}^2(a+\bar{r})}\mathbf{1}^T \boldsymbol{\sigma_z}\left(\frac{\bar{\alpha}}{\bar{r}}\mathbf{1}^T \boldsymbol{\sigma_D}\boldsymbol{\rho} - z_t\right)\boldsymbol{\Sigma_D} + \frac{\varepsilon}{\bar{r}^2(a+\bar{r})}A_t + o(\varepsilon), \tag{4.13}$$

and the correlation between the i-th and j-th assets is[9]

$$\rho_{i,j} = \frac{\Sigma_D^{ij}}{\sqrt{\Sigma_D^{ii}\Sigma_D^{jj}}} + \varepsilon\frac{1}{\sqrt{\Sigma_D^{ii}\Sigma_D^{jj}}}\frac{1}{a+\bar{r}} \cdot \left(A_t^{ij} - \frac{\Sigma_D^{ij}}{\Sigma_D^{ii}}\frac{A_t^{ii}}{2} - \frac{\Sigma_D^{ij}}{\Sigma_D^{jj}}\frac{A_t^{jj}}{2}\right) + o(\varepsilon), \tag{4.14}$$

where Σ_D^{hk} (resp., A_t^{hk}) is the (h,k)-th element of the matrix $\boldsymbol{\Sigma_D}$ (resp., A_t).

[9] While the exact correlation $\rho_{i,j}$ lies within the range $[-1, 1]$, in Corollary 4.9 and Proposition 4.10 we compute asymptotic approximations of $\rho_{i,j}$ that may fall outside this range when ε is not sufficiently small.

Empirical work that documents time-varying departures of the correlations in assets' returns from the correlations in their cash flows includes Longin and Solnik (1995) and Erb et al. (1994). In particular, cross-assets correlations tend to rise during economic downturns. Dumas et al. (2003) explore why international stock market correlations are higher than those between countries' economic fundamentals. In a model with two Lucas trees, Cochrane et al. (2008) show that assets can be correlated even when their underlying dividend processes are not. In a habit-formation and heterogeneous risk-preferences setting, Ehling and Heyerdahl-Larsen (2017) show how correlations arise endogenously through investors' portfolio rebalancing decisions, with both volatilities and correlations increasing in bad times.

An important difference between covariances in (4.13) and correlations in (4.14) is that the latter do not depend on z_t at the first order because the effects of a shock in z_t on covariances and variances offset each other. Yet, the correlations of returns depart from those of dividends, with differences driven by the state-dividend correlations $\boldsymbol{\rho}$.

Equation (4.14) shows that the returns of two stock prices may be correlated even if their dividends' shocks are uncorrelated ($\Sigma^{ij} = 0$). Indeed, their correlation is

$$\varepsilon \frac{1}{a+\bar{r}}(w_{i,t}\rho_{z,j} + w_{j,t}\rho_{z,i}) / \sqrt{\Sigma_D^{ii}\Sigma_D^{jj}},$$

where $\mathbf{w}_t$ is defined in (4.12). If $\rho_{z,j} > 0$, a positive shock to D^j tends to increase z_t, which has a dual effect on the price of the i-th stock, decreasing its value through a higher interest rate (the first two terms of $w_{j,t}$ in (4.12)), but increasing its value through a higher dividends' growth rate (the third term of $w_{j,t}$ in (4.12)). The overall effect may be either positive or negative correlation, depending on parameter values.

Equation (4.14) demonstrates that, at the first order in ε, correlation is time-varying, as the first term of $\mathbf{w}_t$ in (4.12) depends on D_t. However, it remains unaffected by z_t. The next proposition examines the correlation at the second order, and in particular how z_t affects the correlation.

Proposition 4.10. *Define*

$$\boldsymbol{v}_t := \frac{\bar{\alpha}(1^T \boldsymbol{\sigma_z})}{\bar{r}(a+\bar{r})^2(2a+\bar{r})} \left(\bar{\alpha}(1^T \boldsymbol{\sigma_z}) D_t + \bar{\alpha}(1^T \boldsymbol{\sigma_z}) \left(\frac{2}{\bar{r}} + \frac{1}{a+\bar{r}} + \frac{1}{2a+\bar{r}} \right) (\boldsymbol{\mu_D} - \bar{\alpha}\, \boldsymbol{\Sigma_D}\, 1) - \frac{2a+3\bar{r}}{\bar{r}} \boldsymbol{\sigma_z} \right)$$

$$B_t := \boldsymbol{\sigma_D}\, \boldsymbol{\rho}\, \boldsymbol{v}_t^T + \boldsymbol{v}_t\, \boldsymbol{\rho}^T \boldsymbol{\sigma_D}^T .$$

The second-order (ε^2) term in the expansion of $\rho_{i,j}$, denoted by $\rho_{i,j}^{(2)}$ is linear in z_t and has slope

$$\frac{\bar{r}^2}{\sqrt{\Sigma_D^{ii}\Sigma_D^{jj}}} \left(B_t^{ij} - \frac{\Sigma_D^{ij}}{\Sigma_D^{ii}} \frac{B_t^{ii}}{2} - \frac{\Sigma_D^{ij}}{\Sigma_D^{jj}} \frac{B_t^{jj}}{2} \right), \tag{4.15}$$

where Σ_D^{hk} (resp., B_t^{hk}) is the (h,k)-th element of the matrix $\boldsymbol{\Sigma_D}$ (resp., B_t).

Proposition 4.10 shows that, at the second order, correlation is linear in z_t: it may either increase or decrease, depending on the sign of the expression in the equation, and in particular on $\boldsymbol{\rho}$.

5. Numerical computation of the equilibrium

This section validates the accuracy of the closed-form expressions for the incomplete-market equilibrium by comparing them with a numerical solution of the equilibrium, through a recursive algorithm conceptually similar to those of Judd et al. (2000), Judd et al. (2003b), Dumas and Lyasoff (2012). Our numerical implementation follows the steps outlined in Dumas and Lyasoff (2012).[10]

The economy solved numerically has two agents, Ann ($i = 1$) and Bob ($i = 2$), with identical absolute risk-aversion $\alpha = 2\bar{\alpha}$ and time-preference $\beta = \bar{\beta}$. (Recall that if two agents have identical risk-aversion α, then $\bar{\alpha} = \alpha/2$.) The two agents differ exclusively in their initial wealth x_i. Parameter values are chosen as to reproduce the average interest rate, equity returns, price-dividend ratio, and aggregate consumption growth rate in Beeler and Campbell (2012, Yearly data in Table 2). Setting $\sigma_z = 1$ and $z_0 = 0$, the parameters $\bar{\alpha}$, $\bar{\beta}$, D_0, μ_D, σ_D, ε, and a are calibrated so that the interest rate, its standard deviation and autocorrelation, the equity premium, equity volatility, and price-dividend ratio in the equilibrium model coincide with their respective historical values, while minimizing the difference between historical aggregate consumption growth rate and the model value.[11]

[10] The code used to obtain the results in the paper is available at https://github.com/marko-hans-weber/General-Equilibrium-with-Unhedgeable-Fundamentals.

[11] Specifically, first-order equilibrium quantities are aligned with the historical values in Table 1. The interest rate, its standard deviation and mean reversion, i.e., r_0, $\frac{\bar{\alpha}}{\sqrt{2a}}\sigma_z\varepsilon$ and e^{-a}, match the values $E[r_f]$, $\sigma(r_f)$ and $AC1(r_f)$, respectively. The equity premium and equity volatility, i.e., $\frac{\bar{\alpha}\sigma_D^2\tilde{C}_0}{P_0}$ and $\frac{1}{P_0}\sqrt{\sigma_D^2\tilde{C}_0^2 + \sigma_{P,\mu}(z_0, D_0)^2}$ (where $\sigma_{P,\mu}(z_0, D_0)$ is defined in (3.13)) match $E[r_e]$ and $\sigma(r_e)$. Since Beeler and Campbell (2012) consider the log price-dividend ratio, $\frac{P_0}{D_0}$ is matched to $e^{E[p-d]+\frac{\sigma(p-d)^2}{2}}$. Finally, the distance between the model's aggregate consumption growth rate, i.e., $\frac{\mu_D}{D_0}$, and $E[\Delta c]$ is minimized.

Table 1
The left side of the table presents the moments for interest rates, equity premia, and aggregate consumption growth rates that are used for model calibration, based on Table 2 in Beeler and Campbell (2012). The right side of the table lists the calibrated model parameters.

Moment	Data	Model
$E[r_f]$	0.56%	0.56%
$\sigma(r_f)$	2.89%	2.89%
$AC1(r_f)$	0.65	0.65
$E[r_e]$	5.47%	5.47%
$\sigma(r_e)$	20.17%	20.17%
$E[p-d]$	3.36	3.36
$E[\Delta c]$	1.93%	0.52%

Parameter	Calibrated Value
$\bar{\alpha}$	1.6561
$\bar{\beta}$	0.0035
D_0	4.5509
μ_D	0.0235
σ_D	0.1638
ε	0.0162
a	0.4308

5.1. Numerical procedure

The subsection summarizes the main steps of the numerical scheme to compute the equilibrium. Mathematical details are in Appendix G.

1. First, approximate the dynamics of the processes D_t and z_t on the finite time interval $[0,T]$ with a two-dimensional recombining binomial tree. Let D_t^* and z_t^* be the processes with discrete dynamics
$$\begin{bmatrix} D^*_{t+\Delta t} \\ z^*_{t+\Delta t} \end{bmatrix} = \begin{bmatrix} D^*_t + \mu_D \Delta t \\ z^*_t \end{bmatrix} + \xi_t \cdot \sqrt{\Delta t}, \qquad \begin{bmatrix} D^*_0 \\ z^*_0 \end{bmatrix} = \begin{bmatrix} D_0 \\ z_0 \end{bmatrix},$$
where ξ_t is a random variable (independent of D_t^* and z_t^*) taking the four possible values $\begin{bmatrix} \pm\sigma_D \\ \pm 1 \end{bmatrix}$. The four corresponding transition probabilities are chosen so that for a small time step Δt the processes D_t^* and z_t^* approximate the joint distribution of D_t and z_t.
2. Note that the model is not Markovian in the exogenous state variables D_t^* and z_t^* alone, therefore Ann's consumption share $\frac{c_t^1}{D_t^*}$ is chosen as endogenous state variable (cf. Dumas and Lyasoff (2012)).[12] Let Ξ be a grid of values covering the interval $(0,1)$ representing Ann's consumption share. For each terminal node of the binomial tree, initialize the stock price $P_T(\omega)$ and Ann's wealth $X_T^1(\omega)$ – as functions of Ann's consumption share $\omega \in (0,1)$ – equal to their values in the complete equilibrium in Theorem F.1.
3. Fix a (non-terminal) node of the binomial tree (D_t^*, z_t^*) and a value $\omega \in \Xi$, representing Ann's consumption share at the chosen node, and let η index the four subsequent nodes in the tree. Assume that $P_{t+\Delta t,\eta}(\cdot)$ and $X^1_{t+\Delta t,\eta}(\cdot)$, i.e., the stock price and Ann's wealth in the subsequent nodes, are known. Solve a system of six equations in the six variables $\{c^1_{t+\Delta t,\eta}\}_\eta$, θ_t^1 and $\theta_t^{rf,1}$, i.e., Ann's optimal consumption rates at time $t+\Delta t$ in the four subsequent nodes, Ann's optimal stock holdings at time t, and the dollar value (at the next time step) of Ann's optimal position in the safe asset at time t. The corresponding quantities for Bob follow from the market clearing conditions.
The system of equations comprises Ann's budget constraints at time $t+\Delta t$ for each subsequent node η and the pricing constraints on the two financial assets, i.e., agents must agree on the stock price and the interest rate at time t (see equations (G.1), (G.2) and (G.3) in the Appendix).
The solution of the system of equations yields Ann's marginal utility and the values $P_t(\omega)$ and $X_t^1(\omega)$. The stock price and Ann's wealth are computed for each $\omega \in \Xi$. These values are then interpolated to obtain the functions $P_t(\cdot)$ and $X_t^1(\cdot)$ on the interval $(0,1)$. The system of equations is solved by backwards iteration, starting from the nodes at time $T-\Delta t$ and finishing with the root node at time 0.
4. The last iteration of the recursive procedure yields the function $X_0^1(\cdot)$. Let $\bar{\omega}$ be the solution to the equation $X_0^1(\omega) = x_i$, where x_i is Ann's initial wealth. Ann's optimal consumption rate at time 0 is then $\hat{c}_0^1 = \bar{\omega} \cdot D_0$. From Ann's marginal utility, equilibrium asset prices at time 0 follow.

Once the above quantities are computed by backward recursion, optimal consumption rates and trading strategies at each time t are obtained by forward propagation, as follows. Recall that, for each node at time t of the binomial tree and for each $\omega \in \Xi$, the numerical scheme yields a solution to a system of equations that determines the consumption rate $c_{t+\Delta t,\eta}(\omega)$ at a subsequent node of the tree. These values are then interpolated over ω to obtain a function $c_{t+\Delta t,\eta}(\cdot)$ over $(0,1)$.

Consider a path through the binomial tree, represented by the sequence of nodes $\{\eta_t\}_{t=0}^T$, starting from the root node η_0. Using the value $\bar{\omega}$ computed in the final step of the backward recursion, the optimal consumption rate at node η_1 is obtained as $c_{\Delta t,\eta_1}(\bar{\omega})$. Next, define $\omega_{\Delta t,\eta_1} = \frac{c_{\Delta t,\eta_1}(\bar{\omega})}{D^*_{\Delta t,\eta_1}}$, where $D^*_{\Delta t,\eta_1}$ is the dividend at node η_1. Using these quantities, one can calculate the optimal consumption

[12] In the continuous-time model, dividends can become negative. With the chosen parameter values, dividends remain always positive within the horizon considered.

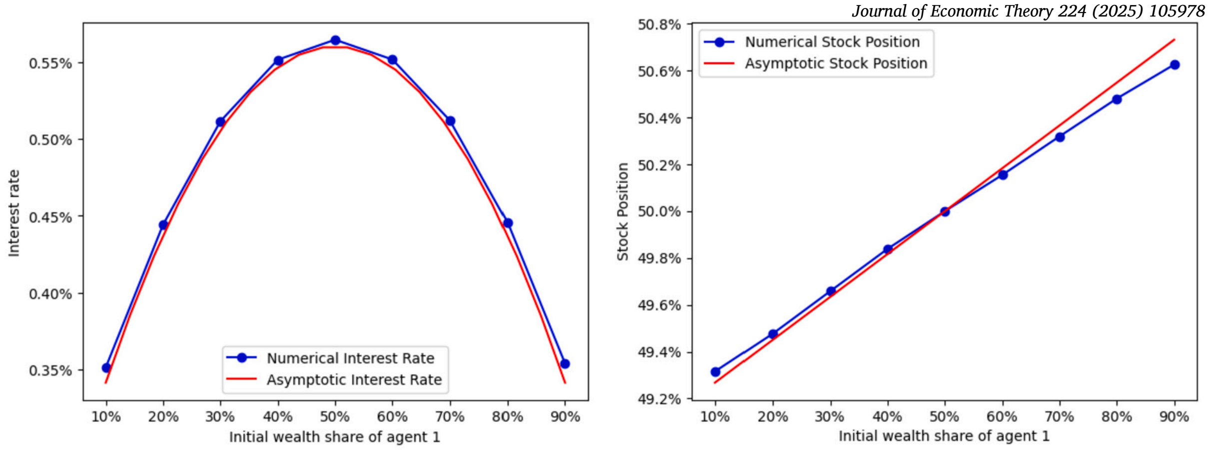

Fig. 1. Interest rate (left) and Ann's stock holding (right) computed numerically (marked) and from the closed-form expressions (solid) in the second-order equilibrium.

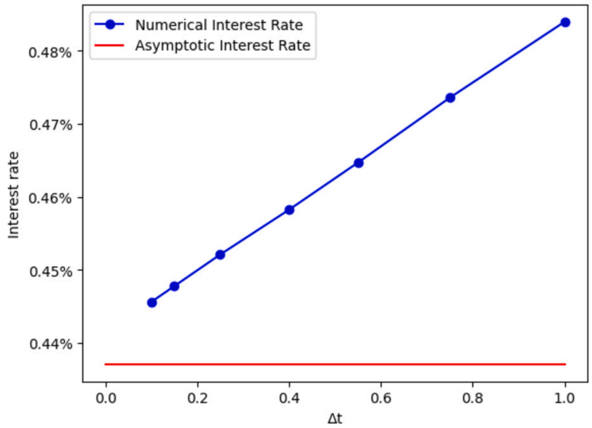

Fig. 2. Interest rate (vertical) computed numerically (marked) against step-size Δt (horizontal), compared with the closed-form expression (solid) in the second-order equilibrium.

rate at the subsequent node $c_{2\Delta t,\eta_2}(\omega_{\Delta t,\eta_1})$, and so on. Note that the consumption rate at any node depends on the path taken to reach that node, a point that is made clear in Fig. 3 in the next section.

5.2. Numerical equilibrium

In the simple economy considered, the two agents Ann and Bob have identical preferences and differ only in their initial wealth (the parameters are in Table 1). Such difference, combined with the absence of tradeable long-term bonds, renders the market incomplete. In a complete market model, asset prices would be independent of wealth distribution across agents. Additionally, since agents have constant absolute risk aversion, each agent would hold a fixed number of shares.

Fig. 1 displays the equilibrium interest rate and Ann's optimal stock holding, as functions of Ann's initial share of aggregate wealth. With unhedgeable risk, the initial wealth distribution across agents affects the market equilibrium. As discussed in Section 3, an unequal wealth distribution increases the volatility of each agent's consumption rate, leading to higher precautionary savings and a lower interest rate.

The left panel of Fig. 1 shows that the interest rate is highest when both agents hold an equal share of wealth, as predicted by the closed-form expression and closely reproduced in the numerical solution of the equilibrium.

The right panel of Fig. 1 demonstrates that – unlike the complete market model – wealthier agents hold more stocks, as agents use available assets to mimic their desired positions in long-term bonds, which are unavailable. As a result, wealthier agents hold more stocks, despite having the same constant absolute risk aversion.

The results in both panels, obtained with a time step of $\Delta t = 0.1$, are virtually indistinguishable from the closed-form second-order expansion. Note that, even in this simple economy with just two agents and one risky asset, the numerical scheme used to produce Fig. 1 is computationally very demanding, requiring the solution of 104,005 systems of equations.

Fig. 2 displays the interest rate when Ann holds 80% of aggregate initial wealth, using various time step sizes. As the number of time steps increases, the numerical solution approximates the closed-form solution obtained in a continuous-time setting.

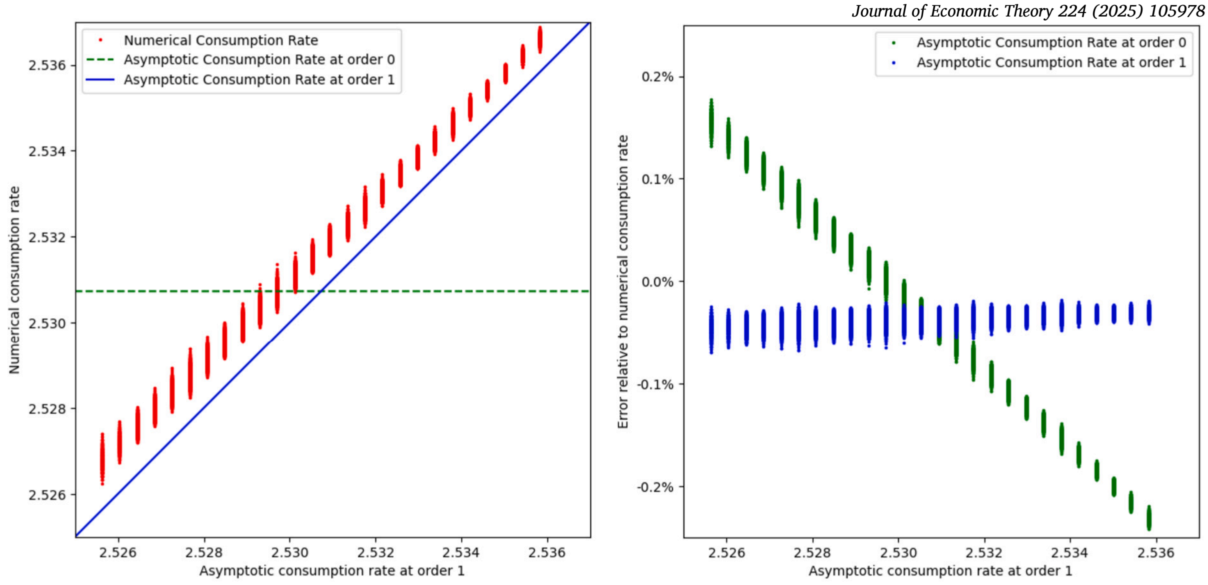

Fig. 3. Numerical consumption (vertical, left) and its relative difference (vertical, right) against first-order closed-form expansions (horizontal).

As established in Theorem 3.2, the optimal consumption rate does not depend only on the current values of the exogenous state variables D_t and z_t alone, but on their entire path. In the two-dimensional recombining binomial tree with $\Delta t = 0.1$, consider the node at time $t = 1$ that corresponds to five up-movements and five down-movements in both D_t^* and z_t^*. In other words, $D_1^* = D_0^* + \mu_D$ and $z_1^* = z_0^*$. There are $\binom{10}{5}^2 = 63,504$ possible paths leading to this node.

For each path, compute the consumption rate using both the asymptotic closed-form expression and the numerical method.[13] Each dot in the left panel of Fig. 3 represents Ann's optimal consumption rate for a different path leading to the same node (D_1^*, z_1^*), with the numerical consumption rate in the vertical axis, and the first-order closed-form expression for the same path on the horizontal axis.

The proximity of the dots to the solid identity line indicates that the first-order explicit expression for the consumption rate closely matches the numerically computed rate. The dashed line represents the consumption rate for $\varepsilon = 0$, which is constant, as it depends exclusively on the exogenous state variables at the node and not on the path taken to reach it. The right panel of Fig. 3 displays the percentage difference between the asymptotic consumption rate and the numerically computed rate. In all cases, the relative difference between the two consumption rates is below 0.1%.

6. Conclusion

This paper studies general equilibrium in an exchange economy with market incompleteness and heterogeneous agents, who trade a stock paying a stochastic dividend stream, lending to and borrowing from each other at some (locally) safe rate. Dividends' fundamentals are subject to unhedgeable shocks, rendering the market incomplete. For small fundamental noise, we find closed-form expressions for asset prices and consumption-investment decisions. Consumption policies are determined by the history of fundamental shocks (rather than current prices alone), the dispersion of agents' preferences affects asset prices and hence no representative agent exists, consumption volatility due to market incompleteness depresses the interest rate, agents dynamically trade stocks to partially hedge their exposure to fundamental shocks, and those who bear higher consumption volatility are compensated with additional expected consumption.

The novel methodology in this paper applies perturbation methods to general incomplete equilibrium by focusing on expansions around a tractable complete model, for which an explicit equilibrium is available. The present approach allows to adapt duality arguments to the incomplete setting and does not require the existence of an exact incomplete-market equilibrium – an important question beyond the scope of this paper. While the paper focuses on effects proportional to the size of the unhedgeable shocks and their variance, expansions of higher order can be readily obtained in principle, but the resulting expressions are likely to be more computationally cumbersome and less economically relevant.

Exponential utilities are instrumental in obtaining a model in which several rational agents coexist in the long run. No rational-expectations equilibrium is known to satisfy this property with power utilities, as only one agent survives in the long run (Yan, 2008; Cvitanić et al., 2012). Furthermore, exponential utilities yield all equilibrium quantities in closed form, for an arbitrary number of agents.

[13] Specifically, for each path, at every time step one computes $\Delta W^{\mu} = z_{t+\Delta t} - z_t + a z_t \Delta t$. This quantity yields W_1^{μ} for each path, whence the consumption rate from the first-order closed-form solution in Theorem 3.2.

In contrast, closed-form solutions are unavailable for heterogeneous agents with CRRA preferences, even in complete markets. Such intractability arises because, if agents have CRRA preferences with different risk aversion, the representative agent's utility is not in the CRRA family. Except in semi-explicit cases, such as two agents, one with twice the risk aversion as the other (Wang, 1996), in general the equilibrium can only be characterized through infinite sums (Bhamra and Uppal, 2014). In summary, the absence of a tractable complete-market benchmark with heterogeneous agents and CRRA preferences is the major obstacle to studying incompleteness in this setting.

CRediT authorship contribution statement

Paolo Guasoni: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Validation, Visualization, Writing – original draft, Writing – review & editing. **Marko Hans Weber:** Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing.

Declaration of competing interest

None.

Appendix A. Baseline equilibrium

Definition A.1. Let P_t be a continuous semimartingale and r_t an adapted, integrable process. The pair $\{c_t, \theta_t\}_{0\le t<\infty}$ is an admissible strategy if

(i) (c_t, θ_t) is a pair of adapted processes, adapted to the filtration generated by W^D.
(ii) There exists $\delta > 0$ such that for any T the process θ_t is $(2+\delta)$-integrable on $[0,T]\times\Omega$.
(iii) The process c_t is bounded from below in that, for some polynomials q_1, q_2,

$$c_t \ge q_1(t, D_t) + \int_0^t q_2(s, D_s)dW_s^D.$$

(iv) The budget equation

$$dX_t = \theta_t(dP + D_t dt) + r_t(X_t - \theta_t P_t)dt - c_t dt, \qquad X_0 = x_i,$$

has a strong solution X_t that is (i) $(2+\delta)$-integrable on $[0,T]\times\Omega$ for some $\delta > 0$ and any T, and (ii) bounded from below in that, for some polynomials p_1, p_2,

$$X_t \ge p_1(t, D_t) + \int_0^t p_2(s, D_s)dW_s^D.$$

$\mathcal{A}$ denotes the set of admissible strategies.

Lemma A.2. *Let P_t and r be as in Theorem 2.1, and let $M_t = \exp\{-\bar{\beta}t - \bar{\alpha}(D_t - D_0)\}$. Define $C_0 := E\left[\int_0^\infty M_t dt\right]$, $L_0 := E\left[\int_0^\infty t M_t dt\right]$. Then, for any admissible strategy (c_t, θ_t),*

$$\frac{1}{-\alpha_i} E\left[\int_0^\infty e^{-\beta_i s} U_i(c_s) ds\right] \le -\frac{C_0}{\alpha_i} \exp\left\{\frac{-E\left[\int_0^\infty M_t \log M_t dt\right] - \beta_i L_0 - x_i\alpha_i}{C_0}\right\}.$$

Proof of Lemma A.2. First, note that $E\left[\int_0^\infty M_t c_t dt\right] \le x_i$. Indeed, applying Itô's Lemma to the process $M_t X_t$, it follows that

$$x_i = M_T X_T - \frac{\sigma_D}{r}\int_0^T M_t\theta_t dW_t^D + \bar{\alpha}\sigma_D \int_0^T M_t X_t dW_t^D + \int_0^T M_t c_t dt,$$

where $X_0 = x_i$. From Hölder's inequality, $E\int_0^T |M_t\theta_t|^2 dt \le C_1 E\int_0^T |\theta_t|^{2+\delta} dt < +\infty$, where δ is chosen as in the admissibility condition for θ_t. Hence, $\int_0^T M_t\theta_t dW_t^D$ is a true (i.e., not merely local) martingale. Similarly, $\int_0^T M_t X_t dW_t^D$ is a true martingale. It follows that $x_i = \liminf_{T\to\infty} E[M_T X_T] + \liminf_{T\to\infty} E[\int_0^T M_t c_t dt]$. In view of the lower bound on X_T,

$$\liminf_{T\to\infty} E[M_T X_T] \geq \liminf_{T\to\infty} E[M_T(p_1(T, D_T) + \int_0^T p_2(s, D_s) dW_s^D)] = 0.$$

The lower bound on c_t implies that $M_t c_t$ is bounded from below by an integrable process. Hence, by Fatou's lemma, $\liminf_{T\to\infty} E[\int_0^T M_t c_t dt] \geq E[\int_0^\infty M_t c_t dt]$, whence $E\left[\int_0^\infty M_t c_t dt\right] \leq x_i$.

As $E\left[\int_0^\infty M_t c_t dt\right] \leq x_i$, for every $y > 0$

$$\begin{aligned} E\left[\int_0^\infty \frac{1}{-\alpha_i} e^{-\beta_i s - \alpha_i c_s} ds\right] &\leq E\left[\int_0^\infty \frac{1}{-\alpha_i} e^{-\beta_i s - \alpha_i c_s} ds\right] - y\left(E\left[\int_0^\infty M_t c_t dt\right] - x_i\right) \\ &\leq -\frac{1}{\alpha_i}\left((y - y\log y)C_0 - yE\left[\int_0^\infty M_t \log M_t dt\right] - y\beta_i L_0 - y x_i \alpha_i\right), \end{aligned}$$

where the second inequality is obtained by maximizing over c_t. Minimizing the last expression over y leads to the thesis. □

Proof of Theorem 2.1. With the notation from Lemma A.2, note that $C_0 = E\left[\int_0^\infty M_t dt\right] = \frac{1}{r}$ and $L_0 = E\left[\int_0^\infty t M_t dt\right] = \frac{1}{r^2}$, and therefore

$$E\left[\int_0^t M_t \log M_t dt\right] = E\left[\int_0^t M_t(-\bar{\beta}t - \bar{\alpha}(D_t - D_0))dt\right] = -\bar{\beta}L_0 - \bar{\alpha}P_0 + \bar{\alpha}D_0 C_0.$$

By Lemma A.2, the expected utility of agent i is dominated by

$$\begin{aligned} &-\frac{C_0}{\alpha_i}\exp\left\{\frac{-E\left[\int_0^\infty M_t \log M_t dt\right] - \beta_i L_0 - x_i\alpha_i}{C_0}\right\} \\ &\quad = \frac{1}{-r\alpha_i}\exp\left\{-\alpha_i\left(-\frac{\bar{\beta}-\beta_i}{r\alpha_i} + \frac{\bar{\alpha}}{\alpha_i}D_0 + r\left(x_i - \frac{\bar{\alpha}}{\alpha_i}P_0\right)\right)\right\}, \end{aligned}$$

which is the expected utility attained by the consumption plan $\hat{c}_t^i$.

To see that the consumption plan $\hat{c}_t^i$ is financed by the trading strategy $\hat{\theta}_t^i$, note that $X_t^i = x_i + \frac{\bar{\alpha}}{\alpha_i}\frac{D_t - D_0}{r} + t\left(\frac{\bar{\beta}-\beta_i}{r\alpha_i}\right)$ satisfies the budget equation $dX_t^i = \hat{\theta}_t^i(dP_t + D_t dt) + r(X_t^i - \hat{\theta}_t^i)dt - \hat{c}_t^i dt$. Finally, the market clearing conditions are satisfied, as $\sum_{i=1}^n \hat{\theta}_t^i = 1$, $\sum_{i=1}^n X_t^i - \hat{\theta}_t^i P_t = 0$, and $\sum_{i=1}^n \hat{c}_t^i = D_t$.

Recall that in a market with only the safe asset, with interest rate r, agent i's utility is $\frac{e^{-r\xi_i\alpha_i + \frac{r-\beta_i}{r}}}{-r\alpha_i}$, where ξ_i is the agent's initial endowment. The certainty equivalent in the statement follows by calculating the expected utility of $\hat{c}_t^i$ explicitly and solving the equation $E\left[\int_0^\infty e^{-\beta_i s}U_i(\hat{c}_s^i)ds\right] = \frac{e^{-r\xi_i\alpha_i + \frac{r-\beta_i}{r}}}{-r\alpha_i}$. □

Recall that a stochastic discount factor is a nonzero stochastic process M_t such that the price of both the safe and the risk assets multiplied by M_t are martingales and that in a complete model the stochastic discount factor is unique. The stochastic discount factor corresponding to the equilibrium in Theorem 2.1 is $M_t = \exp\{-\bar{\beta}t - \bar{\alpha}(D_t - D_0)\}$.

Appendix B. Deriving the incomplete-market equilibrium

To find an equilibrium in the incomplete market setting, the main idea is to identify the marginal utilities of all agents so that two conditions hold: (i) all agents must agree on the interest rate and the asset price, and (ii) the consumption plan of each agent must be replicable by saving or borrowing at the safe rate and trading in the asset. *A priori*, these are necessary conditions that any equilibrium needs to satisfy. As it is now shown, they are also sufficient to obtain a unique expansion of the equilibrium in closed form, under the assumption that marginal utilities and prices follow diffusion processes.

Let M_t^i be the rescaled marginal utility of consumption of agent i in equilibrium:

$$e^{-\beta_i t - \alpha_i \hat{c}_t^i} = y_i M_t^i, \tag{B.1}$$

where $\hat{c}_t^i$ is the agent's consumption in equilibrium and y_i is the constant such that $M_0^i = 1$.

As the unique stochastic discount factor (SDF) for $\varepsilon = 0$ is $M_t = e^{-\bar{\beta}t - \bar{\alpha}(D_t - D_0)}$, it is natural to rewrite M_t^i as a perturbation from this reference. Therefore, set $M_t^i = e^{-\bar{\beta}t - \bar{\alpha}(D_t - D_0) + Y_t^i}$, where $\lim_{\varepsilon\downarrow 0} Y_t^i = 0$ and $dY_t^i = l_t^i dt + m_t^i dW_t^\mu + n_t^i dW_t^D$, and rewrite (B.1) as

$$\frac{1}{\alpha_i}\left(-\bar{\beta}t - \bar{\alpha}(D_t - D_0) + Y_t^i\right) = -\frac{\beta_i}{\alpha_i}t - \hat{c}_t^i - \frac{1}{\alpha_i}\log y_i. \tag{B.2}$$

The first condition to identify l_t^i, m_t^i, and n_t^i is that in equilibrium all agents must agree on the interest rate and the asset price.

Lemma B.1. *Assume* $Y_t^i = \varepsilon Y_t^{i,1} + o(\varepsilon)$, *where* $dY_t^{i,1} = l_t^{i,1}dt + m_t^{i,1}dW_t^\mu + n_t^{i,1}dW_t{}^D$. *Then*

$$l_t^{i,1} = n_t^{i,1} = 0.$$

Proof. Summing (B.2) over i, and recalling that $\sum_{i=1}^n \hat{c}_t^i = D_t$, it follows that

$$\sum_{i=1}^n \frac{1}{\alpha_i} Y_t^i = -\sum_{i=1}^n \frac{1}{\alpha_i}\log y_i - D_0.$$

Hence, in particular, $\sum_{i=1}^n \frac{1}{\alpha_i} Y_t^i$ is constant, which in turn implies that

$$\sum_{i=1}^n \frac{1}{\alpha_i} l_t^i = \sum_{i=1}^n \frac{1}{\alpha_i} m_t^i = \sum_{i=1}^n \frac{1}{\alpha_i} n_t^i = 0.$$

Note that each agent's marginal utility M^i needs to imply the same interest rate and asset price through the equality $P_t = E_t\left[\int_t^\infty D_s \frac{M_s^i}{M_t^i} ds\right]$ and the condition that $M_t^i e^{\int_0^t r_s ds}$ is a local martingale, whereby the interest rate is the negative growth rate of the stochastic discount factor. Thus, for each i,

$$r_t = \bar{\beta} + \bar{\alpha}(\mu_D + \varepsilon\sigma_z z_t) - \frac{1}{2}\bar{\alpha}^2\sigma_D^2 - l_t^i - \frac{1}{2}(m_t^i)^2 - \frac{1}{2}(n_t^i)^2 + \bar{\alpha}\sigma_D n_t^i.$$

Ensuring that this expression is independent of i at the first order requires that $-l_t^{i,1} + \bar{\alpha}\sigma_D n_t^{i,1} = g_t$, for some process g_t independent of i. As $0 = \sum_{i=1}^n \frac{1}{\alpha_i}\left(-l_t^{i,1} + \bar{\alpha}\sigma_D n_t^{i,1}\right) = \frac{g_t}{\bar{\alpha}}$, it follows that

$$-l_t^{i,1} + \bar{\alpha}\sigma_D n_t^{i,1} = 0 \quad \text{for all } 1 \le i \le n.$$

Now, let the dynamics of the asset price P_t, which is the same for all agents, be

$$dP_t = pl_t dt + pm_t dW_t^\mu + pn_t dW_t{}^D,$$

for processes pl_t, pm_t, pn_t to be determined. Because $P_t = E_t\left[\int_t^\infty D_s \frac{M_s^i}{M_t^i} ds\right]$ implies that $P_t M_t^i + \int_0^t D_s M_s^i ds$ is a martingale, its drift must be null, which means that

$$-r_t P_t + pl_t - pn_t\bar{\alpha}\sigma_D + pm_t m_t^i + pn_t n_t^i + D_t = 0 \quad \text{for all } 1 \le i \le n.$$

In particular, $pm_t m_t^i + pn_t n_t^i$ is independent of i, and in fact equals zero by the same argument used for the interest rate. Note that $P_t = \frac{D_t}{\bar{r}} + \frac{\mu_D - \bar{\alpha}\sigma_D^2}{\bar{r}^2} + o(1)$, where $\bar{r} = \bar{\beta} + \bar{\alpha}\mu_D - \frac{\bar{\alpha}^2}{2}\sigma_D^2$ is the interest rate in the baseline model, hence $pm_t = o(1)$ and $pn_t = \frac{\sigma_D}{\bar{r}} + o(1)$. Therefore, $pm_t m_t^i + pn_t n_t^i = 0$ at the first order in ε if and only if $n_t^i = 0$ at first order in ε, i.e., $n_t^{i,1} = 0$. □

The second condition, which identifies $m_t^{i,1}$, requires that the optimal consumption plan for each agent is in fact replicable through a dynamic trading strategy. To achieve this goal, it suffices to verify that the wealth process of each agent, defined as the expected present value of future planned consumption under the agent's own marginal utility, indeed satisfies the budget equation, which is

$$dX_t^i = \theta_t^i(dP_t + D_t dt) + r_t(X_t^i - \theta_t^i P_t)dt - c_t^i dt. \tag{B.3}$$

The first-order condition in (B.2) yields the optimal consumption as

$$\hat{c}_t^i = \frac{\bar{\alpha}}{\alpha_i}(D_t - D_0) + \frac{\bar{\beta} - \beta_i}{\alpha_i} t - \frac{1}{\alpha_i}\log y_i - \frac{1}{\alpha_i} Y_t^i. \tag{B.4}$$

Now, multiplying equation (B.4) by M_t^i and aggregating over time and states of nature (i.e., pricing both sides), while setting $L_t^i = E_t\left[\int_t^\infty s \frac{M_s^i}{M_t^i} ds\right]$, $C_t^i = E_t\left[\int_t^\infty \frac{M_s^i}{M_t^i} ds\right]$, and $K_t^i = E_t\left[\int_t^\infty Y_s^i \frac{M_s^i}{M_t^i} ds\right]$, it follows that

$$X_t^i = \frac{\bar{\alpha}}{\alpha_i} P_t + \frac{\bar{\beta} - \beta_i}{\alpha_i} L_t^i - \frac{\bar{\alpha}}{\alpha_i} D_0 C_t^i - \frac{1}{\alpha_i}(\log y_i) C_t^i - \frac{1}{\alpha_i} K_t^i.$$

This equality for $t = 0$ determines the constants y_i, yielding $x_i = \frac{\bar{\beta}-\beta_i}{\alpha_i} L_0^i + \frac{\bar{\alpha}}{\alpha_i} P_0 - \frac{\bar{\alpha}}{\alpha_i} D_0 C_0^i - \frac{1}{\alpha_i}(\log y_i) C_0^i - \frac{1}{\alpha_i} K_0^i$, and therefore

$$X_t^i = \frac{\bar{\alpha}}{\alpha_i} P_t + \frac{\bar{\beta} - \beta_i}{\alpha_i} \left(L_t^i - \frac{L_0^i}{C_0^i} C_t^i \right) + \left(x_i - \frac{\bar{\alpha}}{\alpha_i} P_0 \right) \frac{C_t^i}{C_0^i} - \frac{1}{\alpha_i} \left(K_t^i - \frac{K_0^i}{C_0^i} C_t^i \right), \tag{B.5}$$

whence

$$dX_t^i = \frac{\bar{\alpha}}{\alpha_i} dP_t + \frac{\bar{\beta} - \beta_i}{\alpha_i} \left(dL_t^i - \frac{L_0^i}{C_0^i} dC_t^i \right) + \frac{1}{C_0^i} \left(x_i - \frac{\bar{\alpha}}{\alpha_i} P_0 \right) dC_t^i + \frac{1}{\alpha_i} \frac{K_0^i}{C_0^i} dC_t^i - \frac{1}{\alpha_i} dK_t^i. \tag{B.6}$$

The central idea is that, for a consumption plan to be replicable, the right-hand side of (B.6) must coincide with the right-hand side of (B.3), and in particular the diffusion coefficients of W^D and W^μ must be the same. To achieve this goal, it is necessary to understand the dynamics of the separate components C_t^i, L_t^i, K_t^i, and P_t.

An explicit computation detailed in Appendix E yields

$$C_t^i = \frac{1}{\bar{r}} - \varepsilon \frac{\bar{\alpha}}{\bar{r}(a+\bar{r})} \sigma_z z_t + o(\varepsilon), \tag{B.7}$$

$$L_t^i = \frac{1}{\bar{r}} \cdot t + \frac{1}{\bar{r}^2} - \varepsilon \left(t + \frac{1}{\bar{r}} + \frac{1}{a+\bar{r}} \right) \frac{\bar{\alpha}\sigma_z}{\bar{r}(a+\bar{r})} z_t + o(\varepsilon), \tag{B.8}$$

$$P_t = D_t \cdot C_t^i + (\mu_D - \bar{\alpha}\sigma_D^2) \left(\frac{1}{\bar{r}^2} - \varepsilon \left(\frac{1}{\bar{r}} + \frac{1}{a+\bar{r}} \right) \frac{\bar{\alpha}\sigma_z}{\bar{r}(a+\bar{r})} z_t \right) + \varepsilon \frac{1}{\bar{r}(a+\bar{r})} \sigma_z z_t + o(\varepsilon), \tag{B.9}$$

$$K_t^i = \frac{\varepsilon}{\bar{r}} Y_t^{i,1} + o(\varepsilon). \tag{B.10}$$

Hence, the wealth dynamics at the first order in ε is (the drift is omitted as it is inconsequential)

$$\begin{aligned} dX_t^i &= \frac{\bar{\alpha}}{\alpha_i} dP_t + \frac{\bar{\beta} - \beta_i}{\alpha_i} \left(dL_t^i - \frac{L_0^i}{C_0^i} dC_t^i \right) + \frac{1}{C_0^i} \left(x_i - \frac{\bar{\alpha}}{\alpha_i} P_0 \right) dC_t^i + \frac{1}{\alpha_i} \frac{K_0^i}{C_0^i} dC_t^i - \frac{1}{\alpha_i} dK_t^i \\ &= (\dots)dt + \frac{\bar{\alpha}}{\alpha_i} pm_t dW_t^\mu + \frac{\bar{\alpha}}{\alpha_i} pn_t dW_t^D - \frac{\bar{\beta} - \beta_i}{\alpha_i} \varepsilon\bar{\alpha} \left(\frac{1}{\bar{r}(a+\bar{r})} t + \frac{a+2\bar{r}}{\bar{r}^2(a+\bar{r})^2} - \frac{L_0^i}{C_0^i} \frac{1}{\bar{r}(a+\bar{r})} \right) \sigma_z dW_t^\mu \\ &\quad - \varepsilon \frac{1}{C_0^i} \left(x_i - \frac{\bar{\alpha}}{\alpha_i} P_0 \right) \frac{\bar{\alpha}}{\bar{r}(a+\bar{r})} \sigma_z dW_t^\mu - \varepsilon \frac{1}{\alpha_i \bar{r}} m_t^{i,1} dW_t^\mu + o(\varepsilon). \end{aligned}$$

Setting $\hat{\theta}_t^i = \frac{\bar{\alpha}}{\alpha_i} + \varepsilon \hat{\theta}_t^{i,1} + o(\varepsilon)$, the diffusion component of $\hat{\theta}_t^i \cdot dP_t$ equals the diffusion component of dX_t^i at the first order, if

$$\begin{aligned} \frac{\bar{\alpha}}{\alpha_i} pm_t + \varepsilon \hat{\theta}_t^{i,1} pm_t &= \frac{\bar{\alpha}}{\alpha_i} pm_t - \varepsilon \frac{\bar{\beta} - \beta_i}{\alpha_i} \bar{\alpha}\sigma_z \left(\frac{1}{\bar{r}(a+\bar{r})} t + \frac{a+2\bar{r}}{\bar{r}^2(a+\bar{r})^2} - \frac{L_0^i}{C_0^i} \frac{1}{\bar{r}(a+\bar{r})} \right) \\ &\quad - \varepsilon \frac{1}{C_0^i} \left(x_i - \frac{\bar{\alpha}}{\alpha_i} P_0 \right) \bar{\alpha}\sigma_z \frac{1}{\bar{r}(a+\bar{r})} - \varepsilon \frac{1}{\alpha_i} \frac{m_t^{i,1}}{\bar{r}}, \\ \frac{\bar{\alpha}}{\alpha_i} pn_t + \varepsilon \hat{\theta}_t^{i,1} pn_t &= \frac{\bar{\alpha}}{\alpha_i} pn_t. \end{aligned}$$

The second equation immediately implies that $\hat{\theta}_t^{i,1} = 0$, while the first one yields

$$\begin{aligned} m_t^{i,1} &= -(\bar{\beta} - \beta_i)\bar{\alpha}\sigma_z \left(\frac{1}{a+\bar{r}} t + \frac{a+2\bar{r}}{\bar{r}(a+\bar{r})^2} \right) - \left(\frac{x_i}{C_0} - \frac{\bar{\beta} - \beta_i}{\alpha_i} \frac{L_0^i}{C_0^i} - \frac{\bar{\alpha}}{\alpha_i} \frac{P_0}{C_0} \right) \alpha_i \bar{\alpha}\sigma_z \frac{1}{a+\bar{r}} \\ &= -\frac{\bar{\alpha}\alpha_i\sigma_z}{a+\bar{r}} \left(\frac{\bar{\beta} - \beta_i}{\alpha_i} \left(t + \frac{1}{a+\bar{r}} \right) + \left(x_i - \frac{\bar{\alpha}}{\alpha_i} P_0 \right) \bar{r} \right), \end{aligned}$$

thereby identifying the terms of order ε in the equilibrium. Similar calculations, detailed in Section D, yield the terms of order ε^2.

Appendix C. Incomplete-market equilibrium: ε-terms

To rigorously solve for the incomplete-market equilibrium, we first specify the class of consumption-investment policies available to the agents in the market, for a given combination of interest rate and asset prices. The next definition makes precise the notion that an admissible strategy should not require an infinite line of credit or unbounded positions, as to exclude doubling schemes.

Definition C.1. Let $P_t(\varepsilon)$ be a continuous semimartingale and $r_t(\varepsilon)$ an adapted, integrable process, for all $\varepsilon \in (0, \bar{\varepsilon})$. A family $\{c_t(\varepsilon), \theta_t(\varepsilon)\}_{t\in[0,\infty)}^{\varepsilon\in[0,\bar{\varepsilon})}$ is an admissible strategy if

(i) For all $\varepsilon \in (0, \bar{\varepsilon})$, $(c_t(\varepsilon), \theta_t(\varepsilon))$ is a pair of processes, adapted to the filtration generated by W^D and W^μ.
(ii) There exists $\delta > 0$ independent of ε such that for any $\varepsilon \in (0, \bar{\varepsilon})$ and any T the process $\theta_t(\varepsilon)$ is $(2+\delta)$-integrable on $[0, T] \times \Omega$.

(iii) The process $c_t(\varepsilon)$ is bounded from below uniformly in ε. That is, for some polynomial functions q_1, q_2, q_3 independent of ε,

$$c_t(\varepsilon) \geq q_1(t, z_t, D_t) + \int_0^t q_2(s, z_s, D_s) dW_s^D + \int_0^t q_3(s, z_s, D_s) dW_s^\mu.$$

(iv) For every $\varepsilon \in (0, \bar{\varepsilon})$, the budget equation

$$dX_t(\varepsilon) = \theta_t(\varepsilon)(dP_t(\varepsilon) + D_t dt) + r_t(\varepsilon)(X_t(\varepsilon) - \theta_t(\varepsilon)P_t(\varepsilon))dt - c_t(\varepsilon)dt, \qquad X_0(\varepsilon) = x_i,$$

has a strong solution $X_t(\varepsilon)$ that is (i) $(2+\delta)$-integrable on $[0,T]\times\Omega$ for some $\delta > 0$ independent of ε and any T, and (ii) bounded from below uniformly in ε, in that, for some polynomial functions p_1, p_2, p_3 independent of ε,

$$X_t(\varepsilon) \geq p_1(t, z_t, D_t) + \int_0^t p_2(s, z_s, D_s) dW_s^D + \int_0^t p_3(s, z_s, D_s) dW_s^\mu.$$

$\mathcal{A}$ denotes the set of admissible strategies.

Next, we introduce the notion of first-order equilibrium, i.e., we require that agents optimize their expected utility up to order ε.

Definition C.2. A first-order equilibrium is a family $\{r_t(\varepsilon), P_t(\varepsilon), (\hat{c}_t^i(\varepsilon))_{1\leq i\leq n}, (\hat{\theta}_t^i(\varepsilon))_{1\leq i\leq n}\}_{\varepsilon\in[0,\bar{\varepsilon})}$ such that (henceforth, unless ambiguity arises, the dependence on ε is omitted)

(i) $P_t(\varepsilon)_{0\leq\varepsilon\leq\bar{\varepsilon}}$ is a continuous semimartingale and $r_t(\varepsilon)_{0\leq\varepsilon\leq\bar{\varepsilon}}$ an adapted, integrable process;
(ii) $\{\hat{c}_t^i, \hat{\theta}_t^i\}_{0\leq\varepsilon\leq\bar{\varepsilon}}$ is admissible for all $1 \leq i \leq n$;
(iii) For every ε the market clearing conditions hold:

$$\sum_{i=1}^n \hat{\theta}_t^i = 1, \qquad \sum_{i=1}^n (X_t^i - \hat{\theta}_t^i P_t) = 0, \qquad \sum_{i=1}^n \hat{c}_t^i = D_t;$$

(iv) For every agent i, the family of strategies $\{\hat{c}_t^i, \hat{\theta}_t^i\}_{0\leq\varepsilon\leq\bar{\varepsilon}}$ is first-order optimal in ε, i.e.,

$$E\left[\int_0^\infty \frac{1}{-\alpha_i} e^{-\beta_i t - \alpha_i \hat{c}_t^i} dt\right] \geq \sup_{(c_t,\theta_t)\in\mathcal{A}} E\left[\int_0^\infty \frac{1}{-\alpha_i} e^{-\beta_i t - \alpha_i c_t} dt\right] + o(\varepsilon).$$

We now state a version of Theorem 3.5 restricted to the terms of order ε. The derivation of the terms of order ε^2 is the focus of Section D.

Theorem C.3. *A first-order equilibrium consists of the following interest rate r_t, price process P_t, consumption $\hat{c}_t^i$, and asset holding $\hat{\theta}_t^i$. For $t \leq T^\varepsilon := -\frac{2}{\bar{r}} \log \varepsilon$,*

$$\begin{aligned}
r_t &= \bar{r} + \varepsilon \bar{\alpha} \sigma_z z_t \\
P_t &= D_t \left(\frac{1}{\bar{r}} - \varepsilon \frac{\bar{\alpha}}{\bar{r}(a+\bar{r})} \sigma_z z_t \right) + (\mu_D - \bar{\alpha}\sigma_D^2) \left(\frac{1}{\bar{r}^2} - \varepsilon \left(\frac{1}{\bar{r}} + \frac{1}{a+\bar{r}} \right) \frac{\bar{\alpha}\sigma_z}{\bar{r}(a+\bar{r})} z_t \right) \\
&\quad + \varepsilon \frac{1}{\bar{r}(a+\bar{r})} \sigma_z z_t, \\
\hat{\theta}_t^i &= \frac{\bar{\alpha}}{\alpha_i}, \\
\hat{c}_t^i &= \frac{\bar{\beta} - \beta_i}{\alpha_i} \left(t - \frac{1}{\bar{r}} + \varepsilon \frac{\bar{\alpha}\sigma_z}{(a+\bar{r})^2} z_0 \right) + \frac{\bar{\alpha}}{\alpha_i} D_t + \left(\bar{r} + \varepsilon \frac{\bar{\alpha}\bar{r}}{a+\bar{r}} \sigma_z z_0 \right) \left(x_i - \frac{\bar{\alpha}}{\alpha_i} P_0 \right) \\
&\quad + \varepsilon \int_0^t \left(\frac{\bar{\beta} - \beta_i}{\alpha_i} \frac{1}{a+\bar{r}} \bar{\alpha} \sigma_z \left(s + \frac{1}{a+\bar{r}} \right) + \left(x_i - \frac{\bar{\alpha}}{\alpha_i} P_0 \right) \frac{\bar{\alpha}\bar{r}}{a+\bar{r}} \sigma_z \right) dW_s^\mu + \varepsilon^2 R_t,
\end{aligned}$$

where

$$R_t := \left(-\frac{1}{\bar{r}} \frac{\bar{\beta} - \beta_i}{\alpha_i} \left(z_t \cdot t + \frac{1}{(a+\bar{r})} (z_t - z_0) \right) - \left(x_i - \frac{\bar{\alpha}}{\alpha_i} P_0 \right) (z_t - z_0) \right. \tag{C.1}$$

$$+\int_0^t \left(\frac{1}{\bar r}\frac{\bar\beta-\beta_i}{\alpha_i}\left(s+\frac{1}{a+\bar r}\right)+\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right)\right)dW_s^\mu\Bigg)\frac{1}{a+\bar r}\bar\alpha^2\sigma_z^2 z_t.$$

For $t > T^\varepsilon$,

$$r_t = \bar r, \tag{C.2}$$

$$P_t = D_t\left(\frac{1}{\bar r}-\varepsilon\frac{\bar\alpha}{\bar r(a+\bar r)}\sigma_z z_t\right)+(\mu_D-\bar\alpha\sigma_D^2)\left(\frac{1}{\bar r^2}-\varepsilon\left(\frac{1}{\bar r}+\frac{1}{a+\bar r}\right)\frac{\bar\alpha\sigma_z}{\bar r(a+\bar r)}z_t\right) \tag{C.3}$$

$$+\varepsilon\frac{1}{\bar r(a+\bar r)}\sigma_z z_t, \tag{C.4}$$

$$\hat\theta_t^i = \frac{\bar\alpha}{\alpha_i}, \tag{C.5}$$

$$\hat c_t^i = \frac{\bar\beta-\beta_i}{\alpha_i}\left((t-T^\varepsilon)-\frac{1}{\bar r}\right)+\frac{\bar\alpha}{\alpha_i}D_t+\bar r\left(X^i_{T^\varepsilon}-\frac{\bar\alpha}{\alpha_i}P_{T^\varepsilon}\right). \tag{C.6}$$

Remark C.4. For $t \le T^\varepsilon$, the wealth process of agent i is

$$X_t^i = \frac{\bar\beta-\beta_i}{\alpha_i}\left(\frac{1}{\bar r}\cdot t-\varepsilon\bar\alpha\left(\frac{1}{\bar r(a+\bar r)}t+\frac{1}{\bar r(a+\bar r)^2}\right)\sigma_z z_t+\varepsilon\frac{\bar\alpha z_0}{\bar r(a+\bar r)^2}\sigma_z\right)$$
$$+\frac{\bar\alpha}{\alpha_i}P_t+\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right)\left(1-\varepsilon\bar\alpha\frac{1}{a+\bar r}\sigma_z(z_t-z_0)\right)$$
$$+\varepsilon\int_0^t\left(\frac{\bar\beta-\beta_i}{\alpha_i}\frac{1}{\bar r(a+\bar r)}\bar\alpha\sigma_z\left(s+\frac{1}{a+\bar r}\right)+\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right)\bar\alpha\sigma_z\frac{1}{a+\bar r}\right)dW_s^\mu.$$

For $t > T^\varepsilon$,

$$X_t^i = X_{T^\varepsilon}+\frac{\bar\alpha}{\alpha_i}(P_t-P_{T^\varepsilon})+(t-T^\varepsilon)\left(\frac{\bar\beta-\beta_i}{\bar r\alpha_i}\right).$$

For $t \le T^\varepsilon$, agent i's discounted marginal utility is

$$\log M_t^i = -\bar\beta t-\bar\alpha(D_t-D_0)-\varepsilon\int_0^t\left((\bar\beta-\beta_i)\frac{1}{a+\bar r}\bar\alpha\sigma_z\left(s+\frac{1}{a+\bar r}\right)+\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right)\alpha_i\bar\alpha\sigma_z\frac{\bar r}{a+\bar r}\right)dW_s^\mu.$$

For $t > T^\varepsilon$,

$$\log M_t^i = -\bar\beta t-\bar\alpha(\mu_D\cdot t+\sigma_D W_t^D).$$

Lemma C.5. *Let M_t^i be the process defined in Remark C.4. Given asset price P_t and interest rate r_t as in Theorem C.3, for every admissible family $\{c_t,\theta_t\}_{0\le\varepsilon\le\bar\varepsilon}$ with initial wealth x, we have $E\left[\int_0^\infty M_t^i c_t dt\right] \le x+o(\varepsilon)$.*

Proof. Agent i's marginal utility M_t^i has dynamics

$$dM_t^i = M_t^i\left(-r_t dt+\frac{1}{2}\varepsilon^2 f(t)^2 dt-\bar\alpha\sigma_D dW_t^D-\varepsilon f(t)dW_t^\mu\right),$$

where $f(t) = \mathbf{1}_{\{t\le T^\varepsilon\}}\cdot\left((\bar\beta-\beta_i)\frac{1}{a+\bar r}\bar\alpha\sigma_z\left(t+\frac{1}{a+\bar r}\right)+\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right)\alpha_i\bar\alpha\sigma_z\frac{\bar r}{a+\bar r}\right)$. Given stock price P_t and interest rate r_t as in Theorem C.3 and an admissible strategy $\{c_t,\theta_t\}_{0\le\varepsilon\le\bar\varepsilon}$, the corresponding wealth process X_t has dynamics

$$dX_t = \theta_t(dP_t+D_t dt)+r_t(X_t-\theta_t P_t)dt-c_t dt.$$

From Itô's lemma,

$$d(M_t^i X_t) = X_t dM_t^i+M_t^i dX_t+dM_t^i dX_t$$
$$= -M_t^i c_t dt-M_t^i X_t\bar\alpha\sigma_D dW_t^D+M_t^i\theta_t\frac{\sigma_D}{\bar r}dW_t^D+\varepsilon M_t^i\theta_t\Bigg(-\bar\alpha\frac{1}{\bar r(a+\bar r)}\sigma_z(z_t\sigma_D dW_t^D+D_t dW_t^\mu)$$
$$-\frac{(a+2\bar r)\bar\alpha(\mu_D-\bar\alpha\sigma_D^2)-\bar r(a+\bar r)}{\bar r^2(a+\bar r)^2}\sigma_z dW_t^\mu+\varepsilon\bar\alpha\sigma_z^2 z_t^2\frac{1}{\bar r(a+\bar r)}\left(\bar\alpha D_t+\frac{(a+2\bar r)\bar\alpha(\mu_D-\bar\alpha\sigma_D^2)}{\bar r(a+\bar r)}-2\right)\mathbf{1}_{\{t\le T^\varepsilon\}}dt\Bigg)$$

$$+ M_t^i X_t \left(\frac{1}{2}\varepsilon^2 f(t)^2 dt - \varepsilon f(t) dW_t^\mu\right) + \varepsilon^2 M_t^i \theta_t f(t) \frac{\sigma_z}{\bar{r}(a+\bar{r})} \left(\left(D_t + \frac{\mu_D - \bar{\alpha}\sigma_D^2}{\bar{r}}\right)\bar{\alpha} + \frac{\bar{\alpha}(\mu_D - \bar{\alpha}\sigma_D^2)}{(a+\bar{r})} - 1\right) dt.$$

Since, for any $n \geq 1$, $E\int_0^T |M_t^i|^n dt < \infty$, $E\int_0^T |M_t^i z_t|^n dt < \infty$ and $E\int_0^T |M_t^i D_t|^n dt < \infty$, and θ_t is $2+\delta$-integrable, from Hölder's inequality we get that $E\int_0^T |\theta_t M_t^i|^2 dt < \infty$, $E\int_0^T |\theta_t M_t^i z_t|^2 dt < \infty$ and $E\int_0^T |\theta_t M_t^i D_t|^2 dt < \infty$. Hence, $\int_0^T \theta_t M_t^i dW_t^D$, $\int_0^T \theta_t M_t^i z_t dW_t^D$ and $\int_0^T \theta_t M_t^i D_t dW_t^\mu$ are true martingales. Similarly, also $\int_0^T X_t M_t^i dW_t^D$ and $\int_0^T X_t M_t^i f(t) dW_t^\mu$ are true martingales. Hence,

$$E[M_T^i X_T] = x_i - E\left[\int_0^T M_t^i c_t dt\right] + \varepsilon^2 \frac{\bar{\alpha}\sigma_z^2}{\bar{r}(a+\bar{r})} E\left[\int_0^T M_t^i \theta_t \left(\bar{\alpha} D_t + \frac{(a+2\bar{r})\bar{\alpha}(\mu_D - \bar{\alpha}\sigma_D^2)}{\bar{r}(a+\bar{r})} - 2\right) z_t^2 \mathbf{1}_{\{t \leq T^\varepsilon\}} dt\right]$$
$$+ \varepsilon^2 E\left[\int_0^T \left(\frac{1}{2} M_t^i X_t f(t)^2 + M_t^i \theta_t f(t) \frac{\sigma_z}{\bar{r}(a+\bar{r})} \left(\left(D_t + \frac{\mu_D - \bar{\alpha}\sigma_D^2}{\bar{r}}\right)\bar{\alpha} + \frac{\bar{\alpha}(\mu_D - \bar{\alpha}\sigma_D^2)}{(a+\bar{r})} - 1\right)\right) dt\right].$$

From the admissibility condition, we get that

$$\lim_{T\to\infty} E[M_T^i X_T] \geq \lim_{T\to\infty} E\left[M_T^i \left(p_1(T, z_T, D_T) + \int_0^T p_2(s, z_s, D_s) dW_s^D + \int_0^T p_3(s, z_s, D_s) dW_s^\mu\right)\right] = 0.$$

From the admissibility condition on c_t we get that $M_t^i c_t$ is bounded from below by an integrable process. Fatou's lemma then implies that $\liminf_{T\to\infty} E\left[\int_0^T M_t^i c_t dt\right] \geq E\left[\int_0^\infty M_t^i c_t dt\right]$. Finally, the integrability conditions on θ_t and X_t imply

$$\varepsilon^2 \frac{\bar{\alpha}\sigma_z^2}{\bar{r}(a+\bar{r})} E\left[\int_0^{T^\varepsilon} M_t^i |\theta_t| \left|\bar{\alpha} D_t + \frac{(a+2\bar{r})\bar{\alpha}(\mu_D - \bar{\alpha}\sigma_D^2)}{\bar{r}(a+\bar{r})} - 2\right| z_t^2 dt\right]$$
$$+ \varepsilon^2 E\left[\int_0^{T^\varepsilon} M_t^i \left|\frac{1}{2} X_t f(t)^2 + \theta_t f(t) \frac{\sigma_z}{\bar{r}(a+\bar{r})} \left(\left(D_t + \frac{\mu_D - \bar{\alpha}\sigma_D^2}{\bar{r}}\right)\bar{\alpha} + \frac{\bar{\alpha}(\mu_D - \bar{\alpha}\sigma_D^2)}{(a+\bar{r})} - 1\right)\right| dt\right] = o(\varepsilon),$$

which in turn proves the thesis. □

Lemma C.6. *Let M_t^i be the process defined in Remark C.4, and define $C_0^i := E\left[\int_0^\infty M_t^i dt\right]$, $L_0^i := E\left[\int_0^\infty t M_t^i dt\right]$. Then, for every admissible consumption process c_t and initial wealth x,*

$$\frac{1}{-\alpha_i} E\left[\int_0^\infty e^{-\beta_i s} U_i(c_s) ds\right] \leq -\frac{C_0^i}{\alpha_i} \exp\left\{\frac{-E\left[\int_0^\infty M_t^i \log M_t^i dt\right] - \beta_i L_0^i - x\alpha_i + o(\varepsilon)}{C_0^i}\right\}.$$

Proof. From Lemma C.5, we get $E\left[\int_0^\infty M_t^i c_t dt\right] \leq x + o(\varepsilon)$. The remainder of the proof is identical to the proof of Lemma A.2. □

Proof of Theorem C.3. It can be verified that the consumption and stock holding strategies are admissible and satisfy the market clearing conditions.

Recall that

$$D_t = D_0 + \mu_D t + \sigma_D W_t^D + \varepsilon \frac{\sigma_z z_0}{a}(1 - e^{-at}) + \varepsilon \frac{\sigma_z}{a} \int_0^t (1 - e^{-a(t-s)}) dW_s^\mu,$$

$$z_t = z_0 e^{-at} + \int_0^t e^{-a(t-s)} dW_s^\mu,$$

and $E[XYe^Z] = e^{\frac{\sigma_Z^2}{2}}\left[\rho_{XY}\sigma_X\sigma_Y + (\rho_{XZ}\sigma_X\sigma_Z)(\rho_{YZ}\sigma_Y\sigma_Z)\right]$, where (X, Y, Z) is a centered Gaussian vector with variances σ_X^2, σ_Y^2, σ_Z^2, and correlations ρ_{XY}, ρ_{XZ}, ρ_{YZ}.

Agent i's welfare from consumption $\hat{c}_t^i$ is proportional to $E\left[\int_0^\infty e^{-\beta_i t - \alpha_i \hat{c}_t^i} dt\right] = E\left[\int_0^{T^\varepsilon} e^{-\beta_i t - \alpha_i \hat{c}_t^i} dt\right] + E\left[\int_{T^\varepsilon}^\infty e^{-\beta_i t - \alpha_i \hat{c}_t^i} dt\right]$. The first term is

$$
\begin{aligned}
E\left[\int_0^{T^\varepsilon} e^{-\beta_i t-\alpha_i \hat{c}_t^i}dt\right] &= E\Big[\int_0^{T^\varepsilon} \exp\Big\{-\beta_i t-\alpha_i\Big(\frac{\bar\beta-\beta_i}{\alpha_i}\left(t-\frac{1}{\bar r}+\varepsilon\frac{\bar\alpha\sigma_z}{(a+\bar r)^2}z_0\right)+\frac{\bar\alpha}{\alpha_i}D_t \\
&+\left(\bar r+\varepsilon\frac{\bar\alpha\bar r}{a+\bar r}\sigma_z z_0\right)\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right)+\varepsilon\int_0^t\left(\frac{\bar\beta-\beta_i}{\alpha_i}\frac{1}{a+\bar r}\bar\alpha\sigma_z\left(s+\frac{1}{a+\bar r}\right)+\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right)\frac{\bar\alpha\bar r}{a+\bar r}\sigma_z\right)dW_s^\mu \\
&+\varepsilon^2\Bigg(-\frac{1}{\bar r}\frac{\bar\beta-\beta_i}{\alpha_i}\left(z_t\cdot t+\frac{1}{(a+\bar r)}(z_t-z_0)\right)-\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right)(z_t-z_0) \\
&+\int_0^t\left(\frac{1}{\bar r}\frac{\bar\beta-\beta_i}{\alpha_i}\left(s+\frac{1}{a+\bar r}\right)+\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right)\right)dW_s^\mu\Bigg)\frac{1}{a+\bar r}\bar\alpha^2\sigma_z^2 z_t\Big)\Big\}dt\Big] \\
&= E\Big[\int_0^{T^\varepsilon}\exp\Big\{-\bar r t-(\bar\beta-\beta_i)\left(-\frac{1}{\bar r}+\varepsilon\frac{\bar\alpha\sigma_z}{(a+\bar r)^2}z_0\right)-\bar\alpha D_0-\varepsilon\bar\alpha\frac{\sigma_z z_0}{a}(1-e^{-at}) \\
&-\varepsilon\bar\alpha\frac{\sigma_z}{a}\int_0^t(1-e^{-a(t-s)})dW_s^\mu-\alpha_i\left(\bar r+\varepsilon\frac{\bar\alpha\bar r}{a+\bar r}\sigma_z z_0\right)\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right) \\
&-\varepsilon\alpha_i\int_0^t\left(\frac{\bar\beta-\beta_i}{\alpha_i}\frac{1}{a+\bar r}\bar\alpha\sigma_z\left(s+\frac{1}{a+\bar r}\right)+\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right)\frac{\bar\alpha\bar r}{a+\bar r}\sigma_z\right)dW_s^\mu \\
&-\varepsilon^2\alpha_i\Bigg(-\frac{1}{\bar r}\frac{\bar\beta-\beta_i}{\alpha_i}\left(z_t\cdot t+\frac{1}{(a+\bar r)}(z_t-z_0)\right)-\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right)(z_t-z_0) \\
&+\int_0^t\left(\frac{1}{\bar r}\frac{\bar\beta-\beta_i}{\alpha_i}\left(s+\frac{1}{a+\bar r}\right)+\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right)\right)dW_s^\mu\Bigg)\frac{1}{a+\bar r}\bar\alpha^2\sigma_z^2 z_t\Big\}dt\Big] \\
&\le \int_0^\infty\exp\Big\{-\bar r t-(\bar\beta-\beta_i)\left(-\frac{1}{\bar r}+\varepsilon\frac{\bar\alpha\sigma_z}{(a+\bar r)^2}z_0\right)-\bar\alpha D_0-\varepsilon\bar\alpha\frac{\sigma_z z_0}{a}(1-e^{-at}) \\
&-\alpha_i\left(\bar r+\varepsilon\frac{\bar\alpha\bar r}{a+\bar r}\sigma_z z_0\right)\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right)+K_1\varepsilon^2(T^\varepsilon)^3\Big\}dt \\
&\le \frac{1}{\bar r}\exp\left\{\frac{\bar\beta-\beta_i}{\bar r}-\bar\alpha D_0-\alpha_i\bar r\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right)\right\}\Big(1-\varepsilon(\bar\beta-\beta_i)\frac{\bar\alpha\sigma_z}{(a+\bar r)^2}z_0-\varepsilon\bar\alpha\sigma_z z_0\frac{1}{a+\bar r} \\
&-\varepsilon\alpha_i\frac{\bar\alpha\bar r}{a+\bar r}\sigma_z z_0\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right)+K_2\varepsilon^2(T^\varepsilon)^3\Big).
\end{aligned}
$$

The second term in the sum is $o(\varepsilon)$:

$$
\begin{aligned}
&E\left[\int_{T^\varepsilon}^\infty e^{-\beta_i t-\alpha_i\hat{c}_t^i}dt\right] \\
&= E\left[\int_{T^\varepsilon}^\infty\exp\left\{-\beta_i t-(\bar\beta-\beta_i)(t-T^\varepsilon)-\bar\alpha D_t-\alpha_i\bar r\left(X_{T^\varepsilon}^i-\frac{\bar\alpha}{\alpha_i}P_{T^\varepsilon}\right)+\frac{\bar\beta-\beta_i}{\bar r}\right\}dt\right] \\
&= E\Bigg[\int_{T^\varepsilon}^\infty\exp\Bigg\{-\bar r t-\bar\alpha D_0+\frac{\bar\beta-\beta_i}{\bar r}-\alpha_i\bar r\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right) \\
&+\varepsilon\alpha_i\bar r\Bigg(\frac{\bar\beta-\beta_i}{\alpha_i}\Big(\bar\alpha\left(\frac{1}{\bar r(a+\bar r)}T^\varepsilon+\frac{1}{\bar r(a+\bar r)^2}\right)\sigma_z z_{T^\varepsilon}-\frac{\bar\alpha z_0}{\bar r(a+\bar r)^2}\sigma_z\Big) \\
&+\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right)\frac{\bar\alpha\sigma_z}{a+\bar r}(z_{T^\varepsilon}-z_0)
\end{aligned}
$$

$$-\int_0^{T^\varepsilon}\left(\frac{\bar\beta-\beta_i}{\alpha_i}\frac{1}{\bar r(a+\bar r)}\bar\alpha\sigma_z\left(s+\frac{1}{a+\bar r}\right)+\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right)\bar\alpha\sigma_z\frac{1}{a+\bar r}\right)dW_s^\mu\bigg)\bigg\}dt\bigg]$$

$$=\varepsilon^2 e^{-\bar\alpha D_0+\frac{\bar\beta-\beta_i}{\bar r}-\alpha_i\bar r\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right)}E\left[\exp\left\{\varepsilon\alpha_i\bar r\left(\frac{\bar\beta-\beta_i}{\alpha_i}\left(\bar\alpha\left(\frac{1}{\bar r(a+\bar r)}T^\varepsilon+\frac{1}{\bar r(a+\bar r)^2}\right)\sigma_z z_{T^\varepsilon}-\frac{\bar\alpha z_0}{\bar r(a+\bar r)^2}\sigma_z\right)\right.\right.\right.$$

$$\left.\left.\left.+\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right)\frac{\bar\alpha\sigma_z}{a+\bar r}(z_{T^\varepsilon}-z_0)-\int_0^{T^\varepsilon}\left(\frac{\bar\beta-\beta_i}{\alpha_i}\frac{1}{\bar r(a+\bar r)}\bar\alpha\sigma_z\left(s+\frac{1}{a+\bar r}\right)+\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right)\bar\alpha\sigma_z\frac{1}{a+\bar r}\right)dW_s^\mu\right)\right\}\right]$$

$$=o(\varepsilon).$$

Therefore,

$$\frac{1}{-\alpha_i}E\left[\int_0^\infty e^{-\beta_i t-\alpha_i\hat c_t^i}dt\right]\geq\frac{1}{-\bar r\alpha_i}e^{\frac{\bar\beta-\beta_i}{\bar r}-\bar\alpha D_0-\alpha_i\bar r\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right)}\times$$

$$\times\left(1-\varepsilon(\bar\beta-\beta_i)\frac{\bar\alpha\sigma_z}{(a+\bar r)^2}z_0-\varepsilon\bar\alpha\sigma_z z_0\frac{1}{a+\bar r}-\varepsilon\alpha_i\frac{\bar\alpha\bar r}{a+\bar r}\sigma_z z_0\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right)\right)+o(\varepsilon).$$

Next, we compute the upper bound on welfare from Lemma C.6.

$$C_0^i=E\left[\int_0^\infty M_t^i dt\right]=E\Big[\int_0^\infty\exp\{-\bar rt-\varepsilon\bar\alpha\frac{\sigma_z z_0}{a}\mathbf{1}_{\{t\leq T^\varepsilon\}}(1-e^{-at})-\varepsilon\bar\alpha\frac{\sigma_z}{a}\mathbf{1}_{\{t\leq T^\varepsilon\}}\int_0^t(1-e^{-a(t-s)})dW_s^\mu$$

$$-\varepsilon\mathbf{1}_{\{t\leq T^\varepsilon\}}\int_0^t\left((\bar\beta-\beta_i)\frac{1}{a+\bar r}\bar\alpha\sigma_z\left(s+\frac{1}{a+\bar r}\right)+\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right)\alpha_i\bar\alpha\sigma_z\frac{\bar r}{a+\bar r}\right)dW_s^\mu\}dt\Big]$$

$$=\int_0^\infty\exp\{-\bar rt-\varepsilon\bar\alpha\frac{\sigma_z z_0}{a}\mathbf{1}_{\{t\leq T^\varepsilon\}}(1-e^{-at})+\frac{1}{2}\varepsilon^2\mathbf{1}_{\{t\leq T^\varepsilon\}}\int_0^t\left(\bar\alpha\frac{\sigma_z}{a}(1-e^{-a(t-s)})\right.$$

$$\left.+\left((\bar\beta-\beta_i)\frac{1}{a+\bar r}\bar\alpha\sigma_z\left(s+\frac{1}{a+\bar r}\right)+\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right)\alpha_i\bar\alpha\sigma_z\frac{\bar r}{a+\bar r}\right)\right)^2ds\}dt$$

$$=\int_0^\infty e^{-\bar rt}\left(1-\varepsilon\bar\alpha\frac{\sigma_z z_0}{a}\mathbf{1}_{\{t\leq T^\varepsilon\}}(1-e^{-at})+\frac{1}{2}\varepsilon^2\mathbf{1}_{\{t\leq T^\varepsilon\}}\int_0^t\left(\bar\alpha\frac{\sigma_z}{a}(1-e^{-a(t-s)})\right.\right.$$

$$\left.\left.+\left((\bar\beta-\beta_i)\frac{1}{a+\bar r}\bar\alpha\sigma_z\left(s+\frac{1}{a+\bar r}\right)+\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right)\alpha_i\bar\alpha\sigma_z\frac{\bar r}{a+\bar r}\right)\right)^2ds\right)dt+o(\varepsilon)$$

$$=\frac{1}{\bar r}-\varepsilon\bar\alpha\sigma_z z_0\frac{1}{\bar r(a+\bar r)}+o(\varepsilon),$$

and

$$L_0^i=E\left[\int_0^\infty tM_t^i dt\right]=E\Big[\int_0^\infty t\cdot\exp\{-\bar rt-\varepsilon\bar\alpha\frac{\sigma_z z_0}{a}\mathbf{1}_{\{t\leq T^\varepsilon\}}(1-e^{-at})-\varepsilon\bar\alpha\frac{\sigma_z}{a}\mathbf{1}_{\{t\leq T^\varepsilon\}}\int_0^t(1-e^{-a(t-s)})dW_s^\mu$$

$$-\varepsilon\mathbf{1}_{\{t\leq T^\varepsilon\}}\int_0^t\left((\bar\beta-\beta_i)\frac{1}{a+\bar r}\bar\alpha\sigma_z\left(s+\frac{1}{a+\bar r}\right)+\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right)\alpha_i\bar\alpha\sigma_z\frac{\bar r}{a+\bar r}\right)dW_s^\mu\}dt\Big]$$

$$=\int_0^\infty t\cdot\exp\{-\bar rt-\varepsilon\bar\alpha\frac{\sigma_z z_0}{a}\mathbf{1}_{\{t\leq T^\varepsilon\}}(1-e^{-at})+\frac{1}{2}\varepsilon^2\mathbf{1}_{\{t\leq T^\varepsilon\}}\int_0^t\left(\bar\alpha\frac{\sigma_z}{a}(1-e^{-a(t-s)})\right.$$

$$\left.+\left((\bar\beta-\beta_i)\frac{1}{a+\bar r}\bar\alpha\sigma_z\left(s+\frac{1}{a+\bar r}\right)+\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right)\alpha_i\bar\alpha\sigma_z\frac{\bar r}{a+\bar r}\right)\right)^2ds\}dt$$

$$=\int_0^\infty t\cdot e^{-\bar rt}\left(1-\varepsilon\bar\alpha\frac{\sigma_z z_0}{a}(1-e^{-at})+\frac{1}{2}\varepsilon^2\int_0^t\left(\bar\alpha\frac{\sigma_z}{a}(1-e^{-a(t-s)})\right.\right.$$

$$\left.\left.+\left((\bar\beta-\beta_i)\frac{1}{a+\bar r}\bar\alpha\sigma_z\left(s+\frac{1}{a+\bar r}\right)+\left(x_i-\frac{\bar\alpha}{\alpha_i}P_0\right)\alpha_i\bar\alpha\sigma_z\frac{\bar r}{a+\bar r}\right)\right)^2ds\right)dt+o(\varepsilon)$$

$$= \frac{1}{\bar r^2} - \varepsilon \bar\alpha \frac{\sigma_z z_0}{a}\left(\frac{1}{\bar r^2} - \frac{1}{(a+\bar r)^2}\right) + o(\varepsilon).$$

Moreover,

$$E\Big[\int_0^\infty M_t^i \log M_t^i dt\Big] = \int_0^\infty \Big(-\bar\beta t - \bar\alpha \mu_D t + \bar\alpha^2 \sigma_D^2 t - \varepsilon \bar\alpha \frac{\sigma_z z_0}{a} \mathbf{1}_{\{t\le T^\varepsilon\}}(1-e^{-at})$$
$$+ \varepsilon^2 \mathbf{1}_{\{t\le T^\varepsilon\}} \int_0^t \Big(\bar\alpha \frac{\sigma_z}{a}(1-e^{-a(t-s)})$$
$$+ \left((\bar\beta - \beta_i)\frac{1}{a+\bar r}\bar\alpha\sigma_z\left(s + \frac{1}{a+\bar r}\right) + \left(x_i - \frac{\bar\alpha}{\alpha_i}P_0\right)\alpha_i\bar\alpha\sigma_z \frac{\bar r}{a+\bar r}\right)\Big)^2 ds\Big)$$
$$\times \exp\{-\bar r t - \varepsilon\bar\alpha \frac{\sigma_z z_0}{a}\mathbf{1}_{\{t\le T^\varepsilon\}}(1-e^{-at}) + \frac{1}{2}\varepsilon^2 \mathbf{1}_{\{t\le T^\varepsilon\}}\int_0^t \Big(\bar\alpha\frac{\sigma_z}{a}(1-e^{-a(t-s)})$$
$$+ \left((\bar\beta - \beta_i)\frac{1}{a+\bar r}\bar\alpha\sigma_z\left(s + \frac{1}{a+\bar r}\right) + \left(x_i - \frac{\bar\alpha}{\alpha_i}P_0\right)\alpha_i\bar\alpha\sigma_z \frac{\bar r}{a+\bar r}\right)\Big)^2 ds\}dt$$
$$= -\frac{1}{\bar r^2}\left(\bar\beta + \bar\alpha\mu_D - \bar\alpha^2\sigma_D^2\right) - \varepsilon\bar\alpha\sigma_z z_0 \frac{1}{\bar r(a+\bar r)} + \varepsilon\left(\bar\beta + \bar\alpha\mu_D - \bar\alpha^2\sigma_D^2\right)\bar\alpha\sigma_z z_0 \frac{a+2\bar r}{\bar r^2(a+\bar r)^2} + o(\varepsilon).$$

Hence,

$$-E\left[\int_0^\infty M_t^i \log M_t^i dt\right] - \beta_i L_0^i - x_i\alpha_i + o(\varepsilon) = \frac{1}{\bar r^2}\left(\bar\beta - \beta_i + \bar\alpha\mu_D - \bar\alpha^2\sigma_D^2\right) - x_i\alpha_i + \varepsilon\bar\alpha\sigma_z z_0 \frac{1}{\bar r(a+\bar r)}$$
$$- \varepsilon\left(\bar\beta + \bar\alpha\mu_D - \bar\alpha^2\sigma_D^2\right)\bar\alpha\sigma_z z_0 \frac{a+2\bar r}{\bar r^2(a+\bar r)^2} + \varepsilon\beta_i\bar\alpha\frac{\sigma_z z_0}{a}\left(\frac{1}{\bar r^2} - \frac{1}{(a+\bar r)^2}\right) + o(\varepsilon)$$

and

$$\exp\left\{\frac{-E\left[\int_0^\infty M_t^i \log M_t^i dt\right] - \beta_i L_0^i - x_i\alpha_i + o(\varepsilon)}{C_0^i}\right\}$$
$$= \frac{\bar\beta - \beta_i}{\bar r} + \bar\alpha\frac{\mu_D - \bar\alpha\sigma_D^2}{\bar r} - \bar r x_i\alpha_i + \varepsilon\bar\alpha\sigma_z z_0 \frac{1}{a+\bar r} - \varepsilon\left(\bar\beta - \beta_i + \bar\alpha\mu_D - \bar\alpha^2\sigma_D^2\right)\frac{\bar\alpha\sigma_z z_0}{(a+\bar r)^2} - \varepsilon\bar\alpha\sigma_z z_0 \frac{x_i\alpha_i\bar r}{a+\bar r} + o(\varepsilon)$$
$$= \frac{\bar\beta - \beta_i}{\bar r} - \bar\alpha D_0 - \alpha_i\bar r\left(x_i - \frac{\bar\alpha}{\alpha_i}P_0\right) - \varepsilon(\bar\beta - \beta_i)\frac{\bar\alpha\sigma_z z_0}{(a+\bar r)^2} - \varepsilon\alpha_i\sigma_z z_0 \frac{\bar\alpha\bar r}{a+\bar r}\left(x_i - \frac{\bar\alpha}{\alpha_i}P_0\right) + o(\varepsilon).$$

Finally,

$$-\frac{C_0^i}{\alpha_i}\exp\left\{\frac{-E\left[\int_0^\infty M_t^i \log M_t^i dt\right] - \beta_i L_0^i - x_i\alpha_i + o(\varepsilon)}{C_0^i}\right\}$$
$$= \frac{1}{-\bar r\alpha_i}\exp\left\{\frac{\bar\beta - \beta_i}{\bar r} - \bar\alpha D_0 - \alpha_i\bar r\left(x_i - \frac{\bar\alpha}{\alpha_i}P_0\right)\right\}$$
$$\times\left(1 - \varepsilon(\bar\beta - \beta_i)\bar\alpha\sigma_z z_0 \frac{1}{(a+\bar r)^2} - \varepsilon\alpha_i\sigma_z z_0 \frac{\bar\alpha\bar r}{a+\bar r}\left(x_i - \frac{\bar\alpha}{\alpha_i}P_0\right) - \varepsilon\bar\alpha\sigma_z z_0 \frac{1}{a+\bar r}\right) + o(\varepsilon) \quad \square$$

Appendix D. Second-order equilibrium

Next, we state a complete version of Theorem 3.5.

Theorem D.1. *Assume $\bar r > 0$. The interest rate r_t, price P_t, consumption $\hat c_t^i$ and investment $\hat\theta_t^i$ strategies yield a second-order equilibrium. For $t \le -\frac{3}{\bar r}\log\varepsilon$, set*

$$r_t = \bar r + \varepsilon\bar\alpha\sigma_z z_t - \frac{\varepsilon^2}{2}\sum_{j=1}^n \frac{\bar\alpha}{\alpha_j}(m_t^{j,1})^2, \tag{D.1}$$

$$P_t = D_t\tilde C_t + (\mu_D - \bar\alpha\sigma_D^2)\tilde L_t + \varepsilon\frac{1}{\bar r(a+\bar r)}\sigma_z z_t - \varepsilon^2\frac{2\bar\alpha\sigma_z^2}{\bar r(a+\bar r)(2a+\bar r)}\left(z_t^2 + \frac{1}{\bar r}\right), \tag{D.2}$$

$$\hat{\theta}_t^i = \frac{\bar{\alpha}}{\alpha_i} + \varepsilon^2 \frac{\bar{r}}{\sigma_D^2} pm_t^1 \frac{m_t^{i,1}}{\alpha_i}, \tag{D.3}$$

$$\hat{c}_t^i = \frac{\bar{\alpha}}{\alpha_i} D_t + \frac{1}{C_0^i}\left(x_i - \frac{\bar{\alpha}}{\alpha_i} P_0\right) + \frac{\bar{\beta} - \beta_i}{\alpha_i}\left(t - \frac{L_0^i}{C_0^i}\right) \tag{D.4}$$

$$+ \frac{1}{\alpha_i}\frac{K_0^i}{C_0^i} - \frac{\varepsilon}{\alpha_i}\int_0^t m_s^{i,1} dW_s^\mu - \frac{\varepsilon^2}{\alpha_i}\left(\int_0^t l_s^{i,2} ds + \int_0^t m_s^{i,2} dW_s^\mu + \int_0^t n_s^{i,2} dW_s^D\right), \tag{D.5}$$

$$\log M_t^i = -\bar{\beta} t - \bar{\alpha}(D_t - D_0) + \varepsilon \int_0^t m_s^{i,1} dW_s^\mu + \varepsilon^2\left(\int_0^t l_s^{i,2} ds + \int_0^t m_s^{i,2} dW_s^\mu + \int_0^t n_s^{i,2} dW_s^D\right), \tag{D.6}$$

where

$$C_t^i = \frac{1}{\bar{r}} - \varepsilon \frac{\bar{\alpha}}{\bar{r}(a+\bar{r})}\sigma_z z_t + \varepsilon^2\Big[\frac{\bar{\alpha}^2\sigma_z^2}{\bar{r}(a+\bar{r})(2a+\bar{r})}\left(z_t^2 + \frac{1}{\bar{r}}\right) + \frac{1}{2}\sum_{j=1}^N \frac{\bar{\alpha}}{\alpha_j} M_t^{j,C} - \bar{\alpha}\frac{\sigma_z}{a}\int_t^\infty \frac{a e^{-r(u-t)}}{r(a+r)} m_u^{i,1} du\Big], \tag{D.7}$$

$$L_t^i = t \cdot C_t^i + \frac{1}{\bar{r}^2} - \varepsilon\left(\frac{1}{\bar{r}} + \frac{1}{a+\bar{r}}\right)\frac{\bar{\alpha}\sigma_z}{\bar{r}(a+\bar{r})} z_t + \varepsilon^2\Big[\frac{\bar{\alpha}^2\sigma_z^2}{\bar{r}^2(a+\bar{r})(2a+\bar{r})}\left(\frac{1}{\bar{r}} + \frac{2a^2+6ar+3r^2}{(a+\bar{r})(2a+\bar{r})}\left(z_t^2 + \frac{1}{\bar{r}}\right)\right) \tag{D.8}$$

$$+ \frac{1}{2}\sum_{j=1}^N \frac{\bar{\alpha}}{\alpha_j} M_t^{j,L} - \bar{\alpha}\frac{\sigma_z}{a}\int_t^\infty \frac{a(r(t-u)(a+r)+a+2r)}{r^2(a+r)^2} e^{-r(t-u)} m_u^{i,1} du\Big],$$

$$M_t^{i,C} = \int_t^\infty \frac{e^{-\bar{r}(u-t)}}{\bar{r}}(m_u^{i,1})^2 du, \qquad M_t^{i,L} = \int_t^\infty e^{-\bar{r}(u-t)}\frac{1+\bar{r}(u-t)}{\bar{r}^2}(m_u^{i,1})^2 du, \tag{D.9}$$

$$\tilde{C}_t = \sum_{j=1}^N \frac{\bar{\alpha}}{\alpha_j} C_t^j, \qquad \tilde{L}_t = \sum_{j=1}^N \frac{\bar{\alpha}}{\alpha_j}(L_t^j - t \cdot C_t^j), \tag{D.10}$$

$$K_t^i = Y_t^i C_t^i + \frac{\varepsilon^2}{2} M_t^{i,C} + \frac{\varepsilon^2}{2}\sum_{j=1}^N \frac{\bar{\alpha}}{\alpha_j} M_t^{j,C} - \varepsilon^2\bar{\alpha}\frac{\sigma_z}{a}\int_t^\infty e^{-\bar{r}(s-t)}\left(\int_t^s m_u^{i,1}(1 - e^{-a(s-u)}) du\right) ds, \tag{D.11}$$

$$Y_t^i = \varepsilon \int_0^t m_s^{i,1} dW_s^\mu + \varepsilon^2\left(\int_0^t l_s^{i,2} ds + \int_0^t m_s^{i,2} dW_s^\mu + \int_0^t n_s^{i,2} dW_s^D\right), \tag{D.12}$$

and

$$\hat{\theta}_t^{i,2} = \hat{\theta}_t^i - \frac{\bar{\alpha}}{\alpha_i}, \tag{D.13}$$

$$m_t^{i,1} = -\alpha_i \frac{\bar{\alpha}\sigma_z}{a+\bar{r}}\left(\frac{\bar{\beta}-\beta_i}{\alpha_i}\left(t + \frac{1}{a+\bar{r}}\right) + \left(x_i - \frac{\bar{\alpha}}{\alpha_i} P_0\right)\bar{r}\right), \tag{D.14}$$

$$Q_t^i = \frac{1}{C_0^i}\left(x_i - \frac{\bar{\alpha}}{\alpha_i} P_0\right) C_t^i + \frac{\bar{\beta}-\beta_i}{\alpha_i}\left(L_t^i - \frac{L_0^i}{C_0^i} C_t^i\right), \tag{D.15}$$

$$qm_t^{i,1} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\frac{d\langle Q^i, W^\mu\rangle_t}{dt}, \qquad qm_t^{i,2} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\left(\frac{d\langle Q^i, W^\mu\rangle_t}{dt} - \varepsilon qm_t^{i,1}\right) \tag{D.16}$$

$$pm_t^1 = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\frac{d\langle P, W^\mu\rangle_t}{dt} = -\frac{\sigma_z}{\bar{r}(a+\bar{r})}\left(D_t\bar{\alpha} + \frac{(a+2\bar{r})\bar{\alpha}(\mu_D - \bar{\alpha}\sigma_D^2) - \bar{r}(a+\bar{r})}{\bar{r}(a+\bar{r})}\right), \tag{D.17}$$

$$l_t^{i,2} = -\alpha_i\bar{\alpha}\sigma_D^2\hat{\theta}_t^{i,2} - \frac{1}{2}(m_t^{i,1})^2 + \frac{1}{2}\sum_{j=1}^N \frac{\bar{\alpha}}{\alpha_j}(m_t^{j,1})^2, \tag{D.18}$$

$$m_t^{i,2} = \alpha_i\bar{r} qm_t^{i,2} - \bar{r}\cdot\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\frac{d\langle YC^i, W^\mu\rangle_t}{dt} \tag{D.19}$$

$$n_t^{i,2} = -\frac{\bar{r}}{\sigma_D} pm_t^1 m_t^{i,1} = -\sigma_D\alpha_i\hat{\theta}_t^{i,2}. \tag{D.20}$$

The rigorous proof of Theorem D.1 follows the same lines as the proof of Theorem C.3. In this section we present the formal derivation of the second-order equilibrium.

Similarly to the heuristic derivation of the first-order equilibrium, we obtain the second-order equilibrium by (i) imposing that the interest rate is the same across agents, (ii) imposing that the stock price is the same across agents, and (iii) imposing that each agent's wealth satisfies the budget equation.

Recall that

$$r_t = \bar{\beta} + \bar{\alpha}(\mu_D + \varepsilon\sigma_z z_t) - \frac{1}{2}\bar{\alpha}^2\sigma_D^2 - l_t^i - \frac{1}{2}(m_t^i)^2 - \frac{1}{2}(n_t^i)^2 + \bar{\alpha}\sigma_D n_t^i,$$

and $l_t^i = l_t^{i,2}\varepsilon^2 + o(\varepsilon^2)$, $m_t^i = m_t^{i,1}\varepsilon + m_t^{i,2}\varepsilon^2 + o(\varepsilon^2)$, and $n_t^i = n_t^{i,2}\varepsilon^2 + o(\varepsilon^2)$.

Because in equilibrium the interest rate r_t cannot depend on i, the second-order term in the expansion of r_t, i.e., $g_t = -l_t^{i,2} - \frac{1}{2}(m_t^{i,1})^2 + \bar{\alpha}\sigma_D n_t^{i,2}$, is independent of i. Therefore, from $0 = \sum_i -\frac{1}{\alpha_i} l_t^{i,2} + \frac{1}{\alpha_i}\bar{\alpha}\sigma_D n_t^{i,2} = \frac{1}{\bar{\alpha}} g_t + \frac{1}{2}\sum_i \frac{1}{\alpha_i}(m_t^{i,1})^2$ it follows that $g_t = -\frac{1}{2}\sum_i \frac{\bar{\alpha}}{\alpha_i}(m_t^{i,1})^2$ and the equilibrium interest rate is

$$r_t = \bar{r} + \varepsilon\bar{\alpha}\sigma_z z_t - \frac{1}{2}\varepsilon^2 \sum_i \frac{\bar{\alpha}}{\alpha_i}(m_t^{i,1})^2 + o(\varepsilon^2).$$

Furthermore, $l_t^{i,2}$ and $n_t^{i,2}$ satisfy the relation

$$-l_t^{i,2} + \bar{\alpha}\sigma_D n_t^{i,2} = \frac{1}{2}(m_t^{i,1})^2 - \frac{1}{2}\sum_{j=1}^N \frac{\bar{\alpha}}{\alpha_j}(m_t^{j,1})^2.$$

To compute $l_t^{i,2}$ and $n_t^{i,2}$, we impose that in equilibrium the stock price P_t is independent of i. Recall that

$$-r_t P_t + pl_t - pn_t\bar{\alpha}\sigma_D + pm_t m_t^i + pn_t n_t^i + D_t = 0 \quad \text{for all } 1 \le i \le n,$$

where $dP_t = pl_t dt + pm_t dW_t^\mu + pn_t dW_t^D$ and $pm_t = \varepsilon pm_t^1 + o(\varepsilon)$, $pn_t = \frac{\sigma_D}{\bar{r}} + \varepsilon pn_t^1 + o(\varepsilon)$. The second-order term of $pm_t m_t^i + pn_t n_t^i$, i.e., $g_t = pm_t^1 m_t^{i,1} + \frac{\sigma_D}{\bar{r}} n_t^{i,2}$, is independent of i. Dividing by α_i and summing up of i yields $g_t = 0$. Hence, $n_t^{i,2} = -\frac{\bar{r}}{\sigma_D} pm_t^1 m_t^{i,1}$.

The equilibrium conditions on r_t and P_t determine $l_t^{i,2}$ and n_t^2, as summarized in the following Remark.

Remark D.2. Define $Y_t^i = \varepsilon Y_t^{i,1} + \varepsilon^2 Y_t^{i,2} + o(\varepsilon^2)$, where $dY_t^{i,2} = l_t^{i,2}dt + m_t^{i,2}dW_t^\mu + n_t^{i,2}dW_t^D$. Then

$$n_t^{i,2} = -\frac{\bar{r}}{\sigma_D} pm_t^1 m_t^{i,1},$$

$$l_t^{i,2} = \bar{\alpha}\sigma_D n_t^{i,2} - \frac{1}{2}(m_t^{i,1})^2 + \frac{1}{2}\sum_{j=1}^N \frac{\bar{\alpha}}{\alpha_j}(m_t^{j,1})^2 = -\bar{\alpha}\cdot r \cdot pm_t^1 m_t^{i,1} - \frac{1}{2}(m_t^{i,1})^2 + \frac{1}{2}\sum_{j=1}^N \frac{\bar{\alpha}}{\alpha_j}(m_t^{j,1})^2,$$

where

$$m_t^{i,1} = -\alpha_i \frac{\bar{\alpha}\sigma_z}{a+\bar{r}}\left[\frac{\bar{\beta}-\beta_i}{\alpha_i}\left(t + \frac{1}{a+\bar{r}}\right) + \bar{r}\left(x_i - \frac{\bar{\alpha}}{\alpha_i}P_0\right)\right],$$

$$pm_t^1 = \frac{1}{\varepsilon}\frac{d\langle P^1, W^\mu\rangle_t}{dt}.$$

To obtain $m_t^{i,2}$ and $\hat{\theta}_t^i$ we match the dynamics of wealth derived from the first-order condition with those from the budget equation. Recall that the dynamics of the wealth process determined by the first-order condition is

$$\begin{aligned} dX_t^i &= \frac{\bar{\alpha}}{\alpha_i}dP_t + \frac{\bar{\beta}-\beta_i}{\alpha_i}\left(dL_t^i - \frac{L_0^i}{C_0^i}dC_t^i\right) + \frac{1}{C_0^i}\left(x_i - \frac{\bar{\alpha}}{\alpha_i}P_0\right)dC_t^i + \frac{1}{\alpha_i}\frac{K_0^i}{C_0^i}dC_t^i - \frac{1}{\alpha_i}dK_t^i \\ &= \frac{\bar{\alpha}}{\alpha_i}dP_t + dQ_t^i + \frac{1}{\alpha_i}\frac{K_0^i}{C_0^i}dC_t^i - \frac{1}{\alpha_i}dK_t^i, \end{aligned}$$

where $Q_t^i = \frac{\bar{\beta}-\beta_i}{\alpha_i}\left(L_t^i - \frac{L_0^i}{C_0^i}C_t^i\right) + \frac{1}{C_0^i}\left(x_i - \frac{\bar{\alpha}}{\alpha_i}P_0\right)C_t^i$. Define $qm_t^i = \frac{d\langle Q^i, W^\mu\rangle_t}{dt}$ and $qn_t^i = \frac{d\langle Q^i, W^D\rangle_t}{dt}$.

From the budget equation, $dX_t^i = \hat{\theta}_t^i dP_t + (\ldots)dt$. By matching the diffusion terms, we get

$$\hat{\theta}_t^i pm_t = \frac{\bar{\alpha}}{\alpha_i}pm_t + qm_t^i + \frac{1}{\alpha_i}\frac{K_0^i}{C_0^i}cm_t^i - \frac{1}{\alpha_i}km_t^i,$$

$$\hat{\theta}_t^i pn_t = \frac{\bar{\alpha}}{\alpha_i}pn_t + qn_t^i + \frac{1}{\alpha_i}\frac{K_0^i}{C_0^i}cn_t^i - \frac{1}{\alpha_i}kn_t^i.$$

At second order,

$$
\begin{aligned}
0 &= qm_t^{i,2} + \frac{\bar r}{\alpha_i} K_0^{i,1} cm_t^{i,1} - \frac{1}{\alpha_i} km_t^{i,2}, \\
\hat\theta_t^{i,2}\frac{\sigma_D}{\bar r} &= qn_t^{i,2} + \frac{\bar r}{\alpha_i} K_0^{i,1} cn_t^{i,1} - \frac{1}{\alpha_i} kn_t^{i,2},
\end{aligned} \tag{D.21}
$$

where $qm_t^i = \varepsilon qm_t^{i,1} + \varepsilon^2 qm_t^{i,2} + o(\varepsilon^2)$ and $qn_t^i = \varepsilon qn_t^{i,1} + \varepsilon^2 qn_t^{i,2} + o(\varepsilon^2)$.

To solve equations (D.21), we need to compute the second-order expansion of C_t^i, L_t^i, and K_t^i. The following Lemma will be used in the computations.

Lemma D.3. *We have*

$$
E_t\left[e^{-\bar\alpha\sigma_D(W_s - W_t)} \int_t^s (a_0 u + b_0)(W_u - W_t) du \right] = -\frac{1}{6}\bar\alpha\sigma_D e^{\frac12 \bar\alpha^2\sigma_D^2(s-t)}(3b_0 + 2a_0 s + a_0 t)(s-t)^2,
$$

$$
E_t\left[e^{-\bar\alpha\sigma_D(W_s - W_t)} \int_t^s (a_0 u + b_0)(W_u - W_t) dW_u \right] = \frac{1}{6}\bar\alpha^2\sigma_D^2 e^{\frac12 \bar\alpha^2\sigma_D^2(s-t)}(3b_0 + 2a_0 s + a_0 t)(s-t)^2.
$$

Proof. Because $\int_t^s u(W_u - W_t)du = \frac12 \int_t^s (s^2 - u^2) dW_u$ and $\int_t^s (W_u - W_t) du = \int_t^s (s-u) dW_u$, we get $\int_t^s (a_0 u + b_0)(W_u - W_t)du = \int_t^s \left(\frac12 a_0(s^2 - u^2) + b_0(s-u)\right) dW_u$. The first equation follows.

Define $Y_s := \int_t^s (a_0 u + b_0)(W_u - W_t) dW_u \exp\{-\bar\alpha\sigma_D(W_s - W_t)\}$, calculate its dynamics via Itô's lemma, and solve the corresponding ODE for $E_t[Y_s]$. □

Lemma D.4. *Assume that $m_t^{i,2}$ is independent of W_t^D. The second-order expansions of C_t^i, L_t^i and K_t^i are*

$$
\begin{aligned}
C_t^i &= \frac{1}{\bar r} - \varepsilon\bar\alpha\sigma_z \frac{1}{\bar r(a+\bar r)} z_t + \varepsilon^2\Bigg(\frac{\bar\alpha^2\sigma_z^2}{\bar r(a+\bar r)(2a+\bar r)}\left(z_t^2 + \frac{1}{\bar r}\right) + \frac12 \sum_{j=1}^N \frac{\bar\alpha}{\alpha_j} M_t^{j,C} \\
&\quad - \bar\alpha\frac{\sigma_z}{a}\int_t^\infty e^{-\bar r(s-t)} \int_t^s m_u^{i,1}\left(1 - e^{-a(s-u)}\right) du ds \Bigg) + o(\varepsilon^2), \\
L_t^i &= t\cdot C_t^i + \frac{1}{\bar r^2} - \varepsilon\bar\alpha \frac{\sigma_z z_t}{a}\left(\frac{1}{\bar r^2} - \frac{1}{(a+\bar r)^2}\right) + \varepsilon^2\Bigg(\frac{\bar\alpha^2\sigma_z^2}{\bar r^3(a+\bar r)(2a+\bar r)} \\
&\quad + \bar\alpha^2\sigma_z^2 \frac{(a+\bar r)(2a+\bar r) + \bar r(2a+\bar r) + \bar r(a+\bar r)}{\bar r^2(a+\bar r)^2(2a+\bar r)^2}\left(z_t^2 + \frac{1}{\bar r}\right) + \frac12\sum_{j=1}^N \frac{\bar\alpha}{\alpha_j} M_t^{j,L} \\
&\quad - \bar\alpha\frac{\sigma_z}{a}\int_t^\infty (s-t) e^{-\bar r(s-t)} \int_t^s m_u^{i,1}(1 - e^{-a(s-u)}) du ds \Bigg) + o(\varepsilon^2), \\
K_t^i &= Y_t^i C_t^i + \frac{\varepsilon^2}{2} M_t^{i,C} + \frac{\varepsilon^2}{2}\sum_{j=1}^N \frac{\bar\alpha}{\alpha_j} M_t^{j,C} - \varepsilon^2\bar\alpha\frac{\sigma_z}{a}\int_t^\infty e^{-\bar r(s-t)}\left(\int_t^s m_u^{i,1}(1 - e^{-a(s-u)}) du \right) ds + o(\varepsilon^2),
\end{aligned}
$$

where $M_t^{j,C} := \int_t^\infty e^{-\bar r(s-t)} \int_t^s (m_u^{j,1})^2 du ds$ and $M_t^{j,L} := \int_t^\infty e^{-\bar r(s-t)}(s-t)\int_t^s (m_u^{j,1})^2 du ds$.

Proof. Define a_m^i, b_m^i such that $m_t^{i,1} = a_m^i \cdot t + b_m^i$.

$$
\begin{aligned}
C_t^i &\approx E_t \int_t^\infty \exp\{-\bar\beta(s-t) - \bar\alpha(D_s - D_t) + \varepsilon(Y_s^{i,1} - Y_t^{i,1}) + \varepsilon^2(Y_s^{i,2} - Y_t^{i,2})\} ds \\
&\approx E_t \int_t^\infty \Bigg(1 - \varepsilon\bar\alpha\frac{\sigma_z z_t}{a}(1 - e^{-a(s-t)}) - \varepsilon\bar\alpha\frac{\sigma_z}{a}\int_t^s (1 - e^{-a(s-u)}) dW_u^\mu + \varepsilon\int_t^s m_u^{i,1} dW_u^\mu + \\
&\quad \frac{\varepsilon^2}{2}\left(-\bar\alpha\frac{\sigma_z z_t}{a}(1 - e^{-a(s-t)}) - \bar\alpha\frac{\sigma_z}{a}\int_t^s (1 - e^{-a(s-u)}) dW_u^\mu + \int_t^s m_u^{i,1} dW_u^\mu \right)^2 + \varepsilon^2(Y_s^{i,2} - Y_t^{i,2}) \Bigg).
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \exp\{-\bar\beta(s-t)-\bar\alpha\mu_D(s-t)-\bar\alpha\sigma_D(W_s^D-W_t^D)\}ds \\
&=\frac{1}{\bar r}-\varepsilon\bar\alpha\sigma_z\frac{1}{\bar r(a+\bar r)}z_t \\
&+\varepsilon^2 E_t\int_t^\infty\left(\frac{1}{2}\left(-\bar\alpha\frac{\sigma_z z_t}{a}(1-e^{-a(s-t)})-\bar\alpha\frac{\sigma_z}{a}\int_t^s(1-e^{-a(s-u)})dW_u^\mu+\int_t^s m_u^{i,1}dW_u^\mu\right)^2+(Y_s^{i,2}-Y_t^{i,2})\right) \\
&\exp\{-\bar\beta(s-t)-\bar\alpha\mu_D(s-t)-\bar\alpha\sigma_D(W_s^D-W_t^D)\}ds.
\end{aligned}
$$

The first term of order ε^2 is

$$
\begin{aligned}
&\frac{1}{2}E_t\int_t^\infty\left(\bar\alpha\frac{\sigma_z z_t}{a}(1-e^{-a(s-t)})+\int_t^s\left(\bar\alpha\frac{\sigma_z}{a}(1-e^{-a(s-u)})-m_u^{i,1}\right)dW_u^\mu\right)^2\times \\
&\quad\times\exp\{-\bar\beta(s-t)-\bar\alpha\mu_D(s-t)-\bar\alpha\sigma_D(W_s^D-W_t^D)\}ds \\
&\quad=\frac{1}{2}\frac{\bar\alpha^2\sigma_z^2}{a^2}\int_t^\infty(1-e^{-a(s-t)})^2e^{-\bar r(s-t)}ds\,z_t^2+\frac{1}{2}\int_t^\infty\int_t^s\left(\bar\alpha\frac{\sigma_z}{a}(1-e^{-a(s-u)})-m_u^{i,1}\right)^2du\,e^{-\bar r(s-t)}ds \\
&\quad=\frac{1}{2}\frac{\bar\alpha^2\sigma_z^2}{a^2}\left(\frac{1}{\bar r}-2\frac{1}{a+\bar r}+\frac{1}{2a+\bar r}\right)z_t^2+\frac{1}{2}\int_t^\infty\int_t^s\left(\bar\alpha\frac{\sigma_z}{a}(1-e^{-a(s-u)})\right)^2du\,e^{-\bar r(s-t)}ds \\
&\quad-\bar\alpha\frac{\sigma_z}{a}\int_t^\infty e^{-\bar r(s-t)}\int_t^s m_u^{i,1}(1-e^{-a(s-u)})du\,ds+\frac{1}{2}M_t^{i,C} \\
&\quad=\frac{\bar\alpha^2\sigma_z^2}{\bar r(a+\bar r)(2a+\bar r)}(z_t^2+\frac{1}{\bar r})+\frac{1}{2}M_t^{i,C}-\bar\alpha\frac{\sigma_z}{a}\int_t^\infty e^{-\bar r(s-t)}\int_t^s m_u^{i,1}(1-e^{-a(s-u)})du\,ds.
\end{aligned}
$$

Next, we consider the second term of order ε^2, i.e., $E_t\int_t^\infty(Y_s^{i,2}-Y_t^{i,2})\exp\{-\bar\beta(s-t)-\bar\alpha\mu_D(s-t)-\bar\alpha\sigma_D(W_s^D-W_t^D)\}ds$. Recall that $n_u^{i,2}=(GD_t+G\mu_D(u-t)+G\sigma_D(W_u^D-W_t^D)+F)(a_m^it+b_m^i)$ for some constants G, F. It follows from Lemma D.3 that

$$
E_t\left[\left(\int_t^s\bar\alpha\sigma_D n_u^{i,2}du+\int_t^s n_u^{i,2}dW_u^D\right)\exp\{-\bar\alpha\sigma_D(W_s^D-W_t^D)\}\right]=0.
$$

Furthermore, assuming that $m_t^{i,2}$ is independent of the Wiener process W_t^D, we also get that $E_t\left[\left(\int_t^s m_u^{i,2}dW_u^\mu\right)\exp\{-\bar\alpha\sigma_D(W_s^D-W_t^D)\}\right]=0$.

Therefore, from the expression for $l_t^{i,2}$, it follows that

$$
\begin{aligned}
&E_t\int_t^\infty(Y_s^{i,2}-Y_t^{i,2})\exp\{-\bar\beta(s-t)-\bar\alpha\mu_D(s-t)-\bar\alpha\sigma_D(W_s^D-W_t^D)\}= \\
&\quad=E_t\int_t^\infty\int_t^s\left(\frac{1}{2}\sum_{j=1}^N\frac{\bar\alpha}{\alpha_j}(m_u^{j,1})^2-\frac{1}{2}(m_u^{i,1})^2\right)du\exp\{-\bar\beta(s-t)-\bar\alpha\mu_D(s-t)-\bar\alpha\sigma_D(W_s^D-W_t^D)\}ds \\
&\quad=\frac{1}{2}\sum_{j=1}^N\frac{\bar\alpha}{\alpha_j}M_t^{j,C}-\frac{1}{2}M_t^{i,C}.
\end{aligned}
$$

Next, we compute the price of the linear consol bond L_t^i:

$$
\begin{aligned}
&L_t^i\approx t\cdot C_t^i+E_t\int_t^\infty(s-t)\cdot\exp\{-\bar\beta(s-t)-\bar\alpha(D_s-D_t)+\varepsilon(Y_s^{i,1}-Y_t^{i,1})+\varepsilon^2(Y_s^{i,2}-Y_t^{i,2})\}ds \\
&\approx t\cdot C_t^i+E_t\int_t^\infty(s-t)\cdot\left(1-\varepsilon\bar\alpha\frac{\sigma_z z_t}{a}(1-e^{-a(s-t)})-\varepsilon\bar\alpha\frac{\sigma_z}{a}\int_t^s(1-e^{-a(s-u)})dW_u^\mu+\varepsilon\int_t^s m_u^i dW_u^\mu+\right. \\
&\left.\frac{\varepsilon^2}{2}\left(-\bar\alpha\frac{\sigma_z z_t}{a}(1-e^{-a(s-t)})-\bar\alpha\frac{\sigma_z}{a}\int_t^s(1-e^{-a(s-u)})dW_u^\mu+\int_t^s m_u^i dW_u^\mu\right)^2+\varepsilon^2(Y_s^{i,2}-Y_t^{i,2})\right)\times
\end{aligned}
$$

$$\times \exp\{-\bar\beta(s-t) - \bar\alpha\mu_D(s-t) - \bar\alpha\sigma_D(W_s^D - W_t^D)\}ds$$
$$= t\cdot C_t^i + \frac{1}{\bar r^2} - \varepsilon\bar\alpha\frac{\sigma_z z_t}{a}\left(\frac{1}{\bar r^2} - \frac{1}{(a+\bar r)^2}\right)$$
$$+\varepsilon^2 E_t\int_t^\infty (s-t)\Big(\frac{1}{2}\Big(-\bar\alpha\frac{\sigma_z z_t}{a}(1-e^{-a(s-t)}) - \bar\alpha\frac{\sigma_z}{a}\int_t^s(1-e^{-a(s-u)})dW_u^\mu + \int_t^s m_u^i dW_u^\mu\Big)^2 + (Y_s^{i,2} - Y_t^{i,2})\Big)$$
$$\exp\{-\bar\beta(s-t) - \bar\alpha\mu_D(s-t) - \bar\alpha\sigma_D(W_s^D - W_t^D)\}ds.$$

The first term of order ε^2 is

$$\frac{1}{2}E_t\int_t^\infty (s-t)\left(\bar\alpha\frac{\sigma_z z_t}{a}(1-e^{-a(s-t)}) + \bar\alpha\frac{\sigma_z}{a}\int_t^s(1-e^{-a(s-u)})dW_u^\mu - \int_t^s m_u^{i,1}dW_u^\mu\right)^2\times$$
$$\times\exp\{-\bar\beta(s-t) - \bar\alpha\mu_D(s-t) - \bar\alpha\sigma_D(W_s^D - W_t^D)\}ds$$
$$= \frac{1}{2}\frac{\bar\alpha^2\sigma_z^2}{a^2}\int_t^\infty (s-t)(1-e^{-a(s-t)})^2 e^{-\bar r(s-t)}ds\, z_t^2$$
$$+\frac{1}{2}\int_t^\infty (s-t)e^{-\bar r(s-t)}\int_t^s\left(\bar\alpha\frac{\sigma_z}{a}(1-e^{-a(s-u)}) - m_u^{i,1}\right)^2 du ds$$
$$= \frac{1}{2}\frac{\bar\alpha^2\sigma_z^2}{a^2}\left(\frac{1}{\bar r^2} - 2\frac{1}{(a+\bar r)^2} + \frac{1}{(2a+\bar r)^2}\right)z_t^2 - \int_t^\infty (s-t)e^{-\bar r(s-t)}\int_t^s \bar\alpha\frac{\sigma_z}{a}(1-e^{-a(s-u)})m_u^{i,1}du ds$$
$$+\frac{1}{2}\int_t^\infty (s-t)e^{-\bar r(s-t)}\int_t^s\left(\bar\alpha\frac{\sigma_z}{a}(1-e^{-a(s-u)})\right)^2 du ds + \frac{1}{2}M_t^{i,L}$$
$$= \bar\alpha^2\sigma_z^2\frac{2a^2+6a\bar r+3\bar r^2}{\bar r^2(a+\bar r)^2(2a+\bar r)^2}(z_t^2 + \frac{1}{\bar r}) + \frac{\bar\alpha^2\sigma_z^2}{\bar r^3(a+\bar r)(2a+\bar r)} + \frac{1}{2}M_t^{i,L}$$
$$-\bar\alpha\frac{\sigma_z}{a}\int_t^\infty (s-t)e^{-\bar r(s-t)}\int_t^s m_u^{i,1}(1-e^{-a(s-u)})du ds.$$

Recall that from Lemma D.3 we have $E_t\left[\left(\int_t^s \bar\alpha\sigma_D n_u^{i,2}du + \int_t^s n_u^{i,2}dW_u^D\right)\exp\{-\bar\alpha\sigma_D(W_s^D - W_t^D)\}\right] = 0$. Assuming that $m_t^{i,2}$ is independent of W^D and using the expression for $l_t^{i,2}$, we get

$$E_t\int_t^\infty (s-t)\Big(\int_t^s l_u^{i,2}du + \int_t^s m_u^{i,2}dW_u^\mu + \int_t^s n_u^{i,2}dW_u^D\Big)\exp\{-\bar\beta(s-t) - \bar\alpha\mu_D(s-t) - \bar\alpha\sigma_D(W_s^D - W_t^D)\}ds$$
$$= E_t\int_t^\infty (s-t)\int_t^s\Big(\frac{1}{2}\sum_{j=1}^N\frac{\bar\alpha}{\alpha_j}(m_u^{j,1})^2 - \frac{1}{2}(m_u^{i,1})^2\Big)du\exp\{-\bar\beta(s-t) - \bar\alpha\mu_D(s-t) - \bar\alpha\sigma_D(W_s^D - W_t^D)\}ds$$
$$= \frac{1}{2}\sum_{j=1}^N\frac{\bar\alpha}{\alpha_j}M_t^{j,L} - \frac{1}{2}M_t^{i,L}.$$

Next, we compute K_t^i:

$$K_t^i \approx Y_t^i C_t^i + \varepsilon E_t\int_t^\infty\left(\int_t^s m_u^{i,1}dW_u^\mu\right)\exp\left\{-\bar\beta(s-t) - \bar\alpha(D_s - D_t) + \varepsilon\int_t^s m_u^{i,1}dW_u^\mu\right\}ds$$
$$+\varepsilon^2 E_t\int_t^\infty (Y_s^{i,2} - Y_t^{i,2})\exp\{-\bar\beta(s-t) - \bar\alpha\mu_D(s-t) - \bar\alpha\sigma_D(W_s^D - W_t^D)\}ds$$
$$= Y_t^i C_t^i + \varepsilon^2\int_t^\infty e^{-\bar r(s-t)}\left(-\bar\alpha\frac{\sigma_z}{a}\int_t^s(1-e^{-a(s-u)})m_u^{i,1}du + \int_t^s (m_u^{i,1})^2 du\right)ds$$

$$+\varepsilon^2\int_t^\infty e^{-\bar{r}(s-t)}\int_t^s\left(-\frac{1}{2}(m_u^{i,1})^2+\frac{1}{2}\sum_{j=1}^N\frac{\bar{\alpha}}{\alpha_j}(m_u^{j,1})^2\right)du\,ds$$

$$=Y_t^iC_t^i+\frac{\varepsilon^2}{2}M_t^{i,C}+\frac{\varepsilon^2}{2}\sum_{j=1}^N\frac{\bar{\alpha}}{\alpha_j}M_t^{j,C}-\varepsilon^2\bar{\alpha}\frac{\sigma_z}{a}\int_t^\infty e^{-\bar{r}(s-t)}\left(\int_t^s m_u^{i,1}(1-e^{-a(s-u)})du\right)ds.\quad\square$$

From Lemma D.4, it follows

$$km_t^{i,2}=\frac{m_t^{i,2}}{\bar{r}}+\frac{d\langle Y^1C^{i,1},W^\mu\rangle_t}{dt},\qquad kn_t^{i,2}=\frac{n_t^{i,2}}{\bar{r}},$$

and $K_0^{i,1}=0$, $qn_t^{i,2}=0$. Hence, equation (D.21) reduces to

$$0=qm_t^{i,2}-\frac{1}{\alpha_i}\frac{m_t^{i,2}}{\bar{r}}-\frac{1}{\alpha_i}\frac{d\langle Y^1C^{i,1},W^\mu\rangle_t}{dt},$$

$$\hat{\theta}_t^{i,2}\frac{\sigma_D}{\bar{r}}=-\frac{1}{\alpha_i}\frac{n_t^{i,2}}{\bar{r}}.$$

From the first equation we get

$$m_t^{i,2}=\alpha_i\bar{r}qm_t^{i,2}-\bar{r}\frac{d\langle Y^1C^{i,1},W^\mu\rangle_t}{dt},$$

where all quantities can be computed explicitly from Lemma D.4. Notice that $m_t^{i,2}$ is independent of W_t^D, consistently with the assumption made in the derivation.

The second equation yields the optimal trading strategy

$$\hat{\theta}_t^{i,2}=\frac{1}{\alpha_i}\frac{\bar{r}}{\sigma_D^2}pm_t^1m_t^{i,1}.$$

The optimal consumption process follows from the first-order condition:

$$\hat{c}_t^i=-\frac{1}{\alpha_i}\log y_i+\frac{\bar{\beta}-\beta_i}{\alpha_i}t+\frac{\bar{\alpha}}{\alpha_i}(D_t-D_0)-\frac{1}{\alpha_i}Y_t^i$$

$$=\frac{\bar{\alpha}}{\alpha_i}D_t+\frac{1}{C_0^i}\left(x_i-\frac{\bar{\alpha}}{\alpha_i}P_0\right)+\frac{\bar{\beta}-\beta_i}{\alpha_i}\left(t-\frac{L_0^i}{C_0^i}\right)-\frac{1}{\alpha_i}\left(Y_t^i-\frac{K_0^i}{C_0^i}\right).$$

Finally, we compute the equilibrium stock price P_t at second order. The following Lemma will be needed in the computations.

Lemma D.5. *We have*

$$E_t\left[(W_s-W_t)\int_t^s(a_0u+b_0)(W_u-W_t)du\exp\{-\bar{\alpha}\sigma_D(W_s-W_t)\}\right]=$$

$$=\frac{1}{6}(s-t)^2(3b_0+2a_0s+a_0t)(1+\bar{\alpha}^2\sigma_D^2(s-t))\exp\left\{\frac{1}{2}\bar{\alpha}^2\sigma_D^2(s-t)\right\},$$

$$E_t\left[(W_s-W_t)\int_t^s(a_0u+b_0)(W_u-W_t)dW_u\exp\{-\bar{\alpha}\sigma_D(W_s-W_t)\}\right]$$

$$=-\frac{1}{6}\bar{\alpha}\sigma_D(s-t)^2(3b_0+2a_0s+a_0t)(2+\bar{\alpha}^2\sigma_D^2(s-t))\exp\left\{\frac{1}{2}\bar{\alpha}^2\sigma_D^2(s-t)\right\}.$$

Proof. Define $Y_s:=(W_s-W_t)\int_t^s(a_0u+b_0)(W_u-W_t)du\exp\{-\bar{\alpha}\sigma_D(W_s-W_t)\}$. Applying Itô's Lemma yields

$$E_t[Y_s]=E_t\int_t^s(W_u-W_t)(a_0u+b_0)(W_u-W_t)\exp\{-\bar{\alpha}\sigma_D(W_u-W_t)\}du+\frac{1}{2}\bar{\alpha}^2\sigma_D^2E_t\int_t^sY_udu$$

$$-E_t\bar{\alpha}\sigma_D\int_t^s\int_t^u(a_0w+b_0)(W_w-W_t)dw\exp\{-\bar{\alpha}\sigma_D(W_u-W_t)\}du$$

$$= \int_t^s (a_0 u + b_0)(u-t)(1+\bar\alpha^2\sigma_D^2(u-t))\exp\left\{\frac{1}{2}\bar\alpha^2\sigma_D^2(u-t)\right\}du + \frac{1}{2}\bar\alpha^2\sigma_D^2 E_t\int_t^s Y_u du$$

$$- E_t\bar\alpha\sigma_D \int_t^s\int_t^u \left(\frac{1}{2}a_0(u^2-w^2)+b_0(u-w)\right)dW_w \exp\{-\bar\alpha\sigma_D(W_u-W_t)\}du$$

$$= \int_t^s (a_0 u + b_0)(u-t)(1+\bar\alpha^2\sigma_D^2(u-t))\exp\left\{\frac{1}{2}\bar\alpha^2\sigma_D^2(u-t)\right\}du + \frac{1}{2}\bar\alpha^2\sigma_D^2 E_t\int_t^s Y_u du$$

$$+ \bar\alpha^2\sigma_D^2\int_t^s \frac{1}{6}(t-u)^2\left(3b_0 + a_0(t+2u)\right)\exp\left\{\frac{1}{2}\bar\alpha^2\sigma_D^2(u-t)\right\}du.$$

Hence, $g(s) = E_t[Y_s]$ solves the ODE

$$g'(s) = (s-t)\left((a_0 s + b_0) + \bar\alpha^2\sigma_D^2(s-t)\left(\frac{4}{3}a_0 s + \frac{1}{6}a_0 t + \frac{3}{2}b_0\right)\right)\exp\left\{\frac{1}{2}\bar\alpha^2\sigma_D^2(s-t)\right\} + \frac{1}{2}\bar\alpha^2\sigma_D^2 g(s).$$

The solution of this ODE with initial condition $g(t) = 0$ is

$$g(s) = \frac{1}{6}(s-t)^2(3b_0 + 2a_0 s + a_0 t)(1+\bar\alpha^2\sigma_D^2(s-t))\exp\left\{\frac{1}{2}\bar\alpha^2\sigma_D^2(s-t)\right\}.$$

Next, define $Y_s := (W_s - W_t)\int_t^s (a_0 u + b_0)(W_u - W_t)dW_u \exp\{-\bar\alpha\sigma_D(W_s - W_t)\}$. Applying Itô's Lemma yields

$$E_t[Y_s] = \frac{1}{2}\bar\alpha^2\sigma_D^2 E_t\int_t^s Y_u du + E_t\int_t^s (a_0 u + b_0)(W_u - W_t)\exp\{-\bar\alpha\sigma_D(W_u-W_t)\}du$$

$$- E_t\bar\alpha\sigma_D\int_t^s\int_t^u (a_0 w + b_0)(W_w - W_t)dW_w\exp\{-\bar\alpha\sigma_D(W_u - W_t)\}du$$

$$- E_t\bar\alpha\sigma_D\int_t^s (a_0 u + b_0)(W_u - W_t)^2\exp\{-\bar\alpha\sigma_D(W_u - W_t)\}du$$

$$= \frac{1}{2}\bar\alpha^2\sigma_D^2 E_t\int_t^s Y_u du - \bar\alpha\sigma_D\int_t^s (a_0 u + b_0)(u-t)\exp\left\{\frac{1}{2}\bar\alpha^2\sigma_D^2(u-t)\right\}du$$

$$- \bar\alpha\sigma_D\int_t^s \frac{1}{6}\bar\alpha^2\sigma_D^2(3b_0 + 2a_0 u + a_0 t)(u-t)^2\exp\left\{\frac{1}{2}\bar\alpha^2\sigma_D^2(u-t)\right\}du$$

$$- \bar\alpha\sigma_D\int_t^s (a_0 u + b_0)(u-t)(1+\bar\alpha^2\sigma_D^2(u-t))\exp\left\{\frac{1}{2}\bar\alpha^2\sigma_D^2(u-t)\right\}du.$$

Hence, $g(s) = E_t[Y_s]$ solves the ODE

$$g'(s) = -\bar\alpha\sigma_D\left(2(a_0 s + b_0)(s-t) + \bar\alpha^2\sigma_D^2\left(\frac{3}{2}b_0 + \frac{4}{3}a_0 s + \frac{1}{6}a_0 t\right)(s-t)^2\right)\times$$
$$\exp\left\{\frac{1}{2}\bar\alpha^2\sigma_D^2(s-t)\right\} + \frac{1}{2}\bar\alpha^2\sigma_D^2 g(s).$$

The solution of this ODE with initial condition $g(t) = 0$ is

$$g(s) = -\frac{1}{6}\bar\alpha\sigma_D(s-t)^2(3b_0 + 2a_0 s + a_0 t)(2+\bar\alpha^2\sigma_D^2(s-t))\exp\left\{\frac{1}{2}\bar\alpha^2\sigma_D^2(s-t)\right\}. \quad \square$$

The price of the asset is $P_t = E_t\int_t^\infty D_s\frac{M_s^i}{M_t^i}ds = D_t\cdot C_t^i + E_t\int_t^\infty (D_s - D_t)\frac{M_s^i}{M_t^i}ds$.
As derived earlier, the term of order ε^2 of C_t^i is

$$E_t\int_t^\infty\left(\frac{1}{2}\left(-\bar\alpha\frac{\sigma_z z_t}{a}(1-e^{-a(s-t)}) - \bar\alpha\frac{\sigma_z}{a}\int_t^s (1-e^{-a(s-u)})dW_u^\mu + \int_t^s m_u^{i,1}dW_u^\mu\right)^2 + (Y_s^{i,2} - Y_t^{i,2})\right)\times$$
$$\times\exp\{-\bar\beta(s-t) - \bar\alpha\mu_D(s-t) - \bar\alpha\sigma_D(W_s^D - W_t^D)\}ds$$

$$= \frac{\bar{\alpha}^2\sigma_z^2}{\bar{r}(a+\bar{r})(2a+\bar{r})} z_t^2 + \frac{1}{2}\int_t^\infty e^{-\bar{r}(s-t)} \int_t^s \left(\bar{\alpha}\frac{\sigma_z}{a}(1-e^{-a(s-u)}) - m_u^{i,1}\right)^2 du\, ds$$

$$+ E_t \int_t^\infty (Y_s^{i,2} - Y_t^{i,2}) \exp\{-\bar{\beta}(s-t) - \bar{\alpha}\mu_D(s-t) - \bar{\alpha}\sigma_D(W_s^D - W_t^D)\} ds$$

$$= \frac{\bar{\alpha}^2\sigma_z^2}{\bar{r}(a+\bar{r})(2a+\bar{r})} z_t^2 + \int_t^\infty e^{-\bar{r}(s-t)} \int_t^s \left[\frac{1}{2}\left(\bar{\alpha}\frac{\sigma_z}{a}(1-e^{-a(s-u)}) - m_u^{i,1}\right)^2 - \frac{1}{2}(m_u^{i,1})^2 + \frac{1}{2}\sum_{j=1}^N \frac{\bar{\alpha}}{\alpha_j}(m_u^{j,1})^2\right] du\, ds$$

$$= \frac{\bar{\alpha}^2\sigma_z^2}{\bar{r}(a+\bar{r})(2a+\bar{r})} z_t^2 + \int_t^\infty e^{-\bar{r}(s-t)} \int_t^s \left[\frac{\bar{\alpha}^2\sigma_z^2}{2a^2}(1-e^{-a(s-u)})^2 - \frac{\bar{\alpha}\sigma_z}{a}(1-e^{-a(s-u)})m_u^{i,1} + \frac{1}{2}\sum_{j=1}^N \frac{\bar{\alpha}}{\alpha_j}(m_u^{j,1})^2\right] du\, ds.$$

The second term of the asset's price is

$$E_t \int_t^\infty (D_s - D_t) \exp\{-\bar{\beta}(s-t) - \bar{\alpha}(D_s - D_t) + \varepsilon(Y_s^{i,1} - Y_t^{i,1}) + \varepsilon^2(Y_s^{i,2} - Y_t^{i,2})\} ds$$

$$= E_t \int_t^\infty \left(\mu_D(s-t) + \sigma_D(W_s^D - W_t^D) + \varepsilon\frac{\sigma_z z_t}{a}(1-e^{-a(s-t)}) + \varepsilon\frac{\sigma_z}{a}\int_t^s (1-e^{-a(s-u)}) dW_u^\mu\right) \times$$

$$\times \exp\Big\{ - \bar{\beta}(s-t) - \bar{\alpha}\mu_D(s-t) - \bar{\alpha}\sigma_D(W_s^D - W_t^D) - \varepsilon\frac{\bar{\alpha}\sigma_z z_t}{a}(1-e^{-a(s-t)}) - \varepsilon\frac{\bar{\alpha}\sigma_z}{a}\int_t^s (1-e^{-a(s-u)}) dW_u^\mu$$

$$+ \varepsilon\int_t^s m_u^{i,1} dW_u^\mu + \varepsilon^2(Y_s^{i,2} - Y_t^{i,2})\Big\} ds$$

$$= E_t \int_t^\infty \left(\mu_D(s-t) + \sigma_D(W_s^D - W_t^D) + \varepsilon\frac{\sigma_z z_t}{a}(1-e^{-a(s-t)}) + \varepsilon\frac{\sigma_z}{a}\int_t^s (1-e^{-a(s-u)}) dW_u^\mu\right) \times$$

$$\times \exp\Big\{ - \bar{\beta}(s-t) - \bar{\alpha}\mu_D(s-t) - \bar{\alpha}\sigma_D(W_s^D - W_t^D)\Big\} \times \Big(1 - \varepsilon\frac{\bar{\alpha}\sigma_z z_t}{a}(1-e^{-a(s-t)})$$

$$- \varepsilon\frac{\bar{\alpha}\sigma_z}{a}\int_t^s (1-e^{-a(s-u)}) dW_u^\mu + \varepsilon\int_t^s m_u^{i,1} dW_u^\mu$$

$$+ \frac{1}{2}\varepsilon^2\Big(\frac{\bar{\alpha}\sigma_z z_t}{a}(1-e^{-a(s-t)}) + \int_t^s \left(\frac{\bar{\alpha}\sigma_z}{a}(1-e^{-a(s-u)}) - m_u^{i,1}\right) dW_u^\mu\Big)^2 + \varepsilon^2(Y_s^{i,2} - Y_t^{i,2})\Big) ds$$

$$= (\dots) + \varepsilon(\dots) + \varepsilon^2 E_t \int_t^\infty \exp\Big\{ - \bar{\beta}(s-t) - \bar{\alpha}\mu_D(s-t) - \bar{\alpha}\sigma_D(W_s^D - W_t^D)\Big\} \times$$

$$\Big[\big(\mu_D(s-t) + \sigma_D(W_s^D - W_t^D)\big) \times$$

$$\times \Big(\frac{1}{2}\Big(\frac{\bar{\alpha}\sigma_z z_t}{a}(1-e^{-a(s-t)}) + \int_t^s \left(\frac{\bar{\alpha}\sigma_z}{a}(1-e^{-a(s-u)}) - m_u^{i,1}\right) dW_u^\mu\Big)^2 + (Y_s^{i,2} - Y_t^{i,2})\Big)$$

$$+ \left(\frac{\sigma_z z_t}{a}(1-e^{-a(s-t)}) + \frac{\sigma_z}{a}\int_t^s (1-e^{-a(s-u)}) dW_u^\mu\right) \times$$

$$\times \Big(- \frac{\bar{\alpha}\sigma_z z_t}{a}(1-e^{-a(s-t)}) - \int_t^s \left(\frac{\bar{\alpha}\sigma_z}{a}(1-e^{-a(s-u)}) - m_u^{i,1}\right) dW_u^\mu\Big)\Big] ds$$

$$= (\dots) + \varepsilon(\dots) + \varepsilon^2 E_t \int_t^\infty \exp\Big\{ - \bar{\beta}(s-t) - \bar{\alpha}\mu_D(s-t) - \bar{\alpha}\sigma_D(W_s^D - W_t^D)\Big\} \times$$

$$\Big[\big(\mu_D(s-t) + \sigma_D(W_s^D - W_t^D)\big) \times$$

$$\times\Big(\frac{\bar\alpha^2\sigma_z^2 z_t^2}{2a^2}(1-e^{-a(s-t)})^2+\frac{1}{2}\int_t^s\Big(\frac{\bar\alpha\sigma_z}{a}(1-e^{-a(s-u)})-m_u^{i,1}\Big)^2du+\int_t^s l_u^{i,2}du+\int_t^s n_u^{i,1}dW_u^D\Big)$$
$$-\left(\frac{\bar\alpha\sigma_z^2 z_t^2}{a^2}(1-e^{-a(s-t)})^2+\int_t^s\frac{\sigma_z}{a}(1-e^{-a(s-u)})\left(\frac{\bar\alpha\sigma_z}{a}(1-e^{-a(s-u)})-m_u^{i,1}\right)du\right)\Bigg]ds.$$

Recall that $n_t^{i,2}=-\frac{\bar r}{\sigma_D}pm_t^1 m_t^{i,1}$ and $l_t^{i,2}=-\bar\alpha\bar r\cdot pm_t^1m_t^{i,1}-\frac{1}{2}(m_t^{i,1})^2+\frac{1}{2}\sum_{j=1}^N\frac{\bar\alpha}{\alpha_j}(m_t^{j,1})^2$, where $m_t^{i,1}=a_m^it+b_m^i$ and $pm_t^1=GD_t+F$ for some constants a_m^i,b_m^i,F,G.

The term of order ε^2 in $E_t\int_t^\infty(D_s-D_t)M_s^i ds$ can be rewritten as

$$E_t\int_t^\infty\exp\Big\{-\bar\beta(s-t)-\bar\alpha\mu_D(s-t)-\bar\alpha\sigma_D(W_s^D-W_t^D)\Big\}\times$$
$$\Big[\big(\mu_D(s-t)+\sigma_D(W_s^D-W_t^D)\big)\times$$
$$\times\Big(\frac{\bar\alpha^2\sigma_z^2z_t^2}{2a^2}(1-e^{-a(s-t)})^2+\frac{1}{2}\int_t^s\frac{\bar\alpha^2\sigma_z^2}{a^2}(1-e^{-a(s-u)})^2du-\int_t^s\frac{\bar\alpha\sigma_z}{a}(1-e^{-a(s-u)})m_u^{i,1}du$$
$$-\bar\alpha\bar r\int_t^s pm_u^1m_u^{i,1}du+\frac{1}{2}\sum_{j=1}^N\frac{\bar\alpha}{\alpha_j}\int_t^s(m_u^{j,1})^2du-\frac{\bar r}{\sigma_D}\int_t^s pm_u^1m_u^{i,1}dW_u^D\Big)$$
$$-\left(\frac{\bar\alpha\sigma_z^2 z_t^2}{a^2}(1-e^{-a(s-t)})^2+\int_t^s\frac{\sigma_z}{a}(1-e^{-a(s-u)})\left(\frac{\bar\alpha\sigma_z}{a}(1-e^{-a(s-u)})-m_u^{i,1}\right)du\right)\Bigg]ds$$
$$=E_t\int_t^\infty\exp\Big\{-\bar\beta(s-t)-\bar\alpha\mu_D(s-t)-\bar\alpha\sigma_D(W_s^D-W_t^D)\Big\}\times$$
$$\Big[\big(\mu_D(s-t)+\sigma_D(W_s^D-W_t^D)\big)\times$$
$$\times\Big(\frac{\bar\alpha^2\sigma_z^2z_t^2}{2a^2}(1-e^{-a(s-t)})^2+\frac{1}{2}\int_t^s\frac{\bar\alpha^2\sigma_z^2}{a^2}(1-e^{-a(s-u)})^2du-\int_t^s\frac{\bar\alpha\sigma_z}{a}(1-e^{-a(s-u)})m_u^{i,1}du$$
$$-\bar\alpha\bar r\int_t^s(G(D_u-D_t)+F+GD_t)m_u^{i,1}du+\frac{1}{2}\sum_{j=1}^N\frac{\bar\alpha}{\alpha_j}\int_t^s(m_u^{j,1})^2du-\frac{\bar r}{\sigma_D}\int_t^s(G(D_u-D_t)+F+GD_t)m_u^{i,1}dW_u^D\Big)$$
$$-\left(\frac{\bar\alpha\sigma_z^2 z_t^2}{a^2}(1-e^{-a(s-t)})^2+\int_t^s\frac{\sigma_z}{a}(1-e^{-a(s-u)})\left(\frac{\bar\alpha\sigma_z}{a}(1-e^{-a(s-u)})-m_u^{i,1}\right)du\right)\Bigg]ds$$
$$=E_t\int_t^\infty\exp\Big\{-\bar\beta(s-t)-\bar\alpha\mu_D(s-t)-\bar\alpha\sigma_D(W_s^D-W_t^D)\Big\}\times$$
$$\Big[\big(\mu_D(s-t)+\sigma_D(W_s^D-W_t^D)\big)\times$$
$$\times\Big(\frac{\bar\alpha^2\sigma_z^2z_t^2}{2a^2}(1-e^{-a(s-t)})^2+\frac{1}{2}\int_t^s\frac{\bar\alpha^2\sigma_z^2}{a^2}(1-e^{-a(s-u)})^2du-\int_t^s\frac{\bar\alpha\sigma_z}{a}(1-e^{-a(s-u)})m_u^{i,1}du$$
$$-\bar\alpha\bar r\int_t^s(G\mu_D(u-t)+G\sigma_D(W_u^D-W_t^D)+F+GD_t)m_u^{i,1}du$$
$$+\frac{1}{2}\sum_{j=1}^N\frac{\bar\alpha}{\alpha_j}\int_t^s(m_u^{j,1})^2du-\frac{\bar r}{\sigma_D}\int_t^s(G\mu_D(u-t)+G\sigma_D(W_u^D-W_t^D)+F+GD_t)m_u^{i,1}dW_u^D\Big)$$
$$-\left(\frac{\bar\alpha\sigma_z^2 z_t^2}{a^2}(1-e^{-a(s-t)})^2+\int_t^s\frac{\sigma_z}{a}(1-e^{-a(s-u)})\left(\frac{\bar\alpha\sigma_z}{a}(1-e^{-a(s-u)})-m_u^{i,1}\right)du\right)\Bigg]ds.$$

Using Lemma D.3, this integral becomes

$$E_t \int_t^\infty \exp\Big\{ -\bar\beta(s-t) - \bar\alpha\mu_D(s-t) - \bar\alpha\sigma_D(W_s^D - W_t^D)\Big\}\times$$

$$\mu_D(s-t)\times$$

$$\times\Big(\frac{\bar\alpha^2\sigma_z^2 z_t^2}{2a^2}(1-e^{-a(s-t)})^2 + \frac{1}{2}\int_t^s \frac{\bar\alpha^2\sigma_z^2}{a^2}(1-e^{-a(s-u)})^2 du - \int_t^s \frac{\bar\alpha\sigma_z}{a}(1-e^{-a(s-u)})m_u^{i,1} du$$

$$+ \frac{1}{2}\sum_{j=1}^N \frac{\bar\alpha}{\alpha_j}\int_t^s (m_u^{j,1})^2 du - \bar\alpha\bar r \int_t^s (G\mu_D(u-t) + G\sigma_D(W_u^D - W_t^D) + F + GD_t)m_u^{i,1} du$$

$$- \frac{\bar r}{\sigma_D}\int_t^s (G\mu_D(u-t) + G\sigma_D(W_u^D - W_t^D) + F + GD_t)m_u^{i,1} dW_u^D\Big)ds$$

$$+ E_t \int_t^\infty \exp\Big\{ -\bar\beta(s-t) - \bar\alpha\mu_D(s-t) - \bar\alpha\sigma_D(W_s^D - W_t^D)\Big\}\times$$

$$\sigma_D(W_s^D - W_t^D)\times$$

$$\times\Big(\frac{\bar\alpha^2\sigma_z^2 z_t^2}{2a^2}(1-e^{-a(s-t)})^2 + \frac{1}{2}\int_t^s \frac{\bar\alpha^2\sigma_z^2}{a^2}(1-e^{-a(s-u)})^2 du - \int_t^s \frac{\bar\alpha\sigma_z}{a}(1-e^{-a(s-u)})m_u^{i,1} du$$

$$+ \frac{1}{2}\sum_{j=1}^N \frac{\bar\alpha}{\alpha_j}\int_t^s (m_u^{j,1})^2 du - \bar\alpha\bar r \int_t^s (G\mu_D(u-t) + G\sigma_D(W_u^D - W_t^D) + F + GD_t)m_u^{i,1} du$$

$$- \frac{\bar r}{\sigma_D}\int_t^s (G\mu_D(u-t) + G\sigma_D(W_u^D - W_t^D) + F + GD_t)m_u^{i,1} dW_u^D\Big)ds$$

$$- E_t \int_t^\infty \exp\Big\{ -\bar\beta(s-t) - \bar\alpha\mu_D(s-t) - \bar\alpha\sigma_D(W_s^D - W_t^D)\Big\}\times$$

$$\left(\frac{\bar\alpha\sigma_z^2 z_t^2}{a^2}(1-e^{-a(s-t)})^2 + \int_t^s \frac{\sigma_z}{a}(1-e^{-a(s-u)})\left(\frac{\bar\alpha\sigma_z}{a}(1-e^{-a(s-u)}) - m_u^{i,1}\right)du\right)ds$$

$$= \mu_D \int_t^\infty e^{-\bar r(s-t)}(s-t)\Big(\frac{\bar\alpha^2\sigma_z^2 z_t^2}{2a^2}(1-e^{-a(s-t)})^2 + \frac{1}{2}\int_t^s \frac{\bar\alpha^2\sigma_z^2}{a^2}(1-e^{-a(s-u)})^2 du$$

$$- \int_t^s \frac{\bar\alpha\sigma_z}{a}(1-e^{-a(s-u)})m_u^{i,1} du + \frac{1}{2}\sum_{j=1}^N \frac{\bar\alpha}{\alpha_j}\int_t^s (m_u^{j,1})^2 du\Big)ds$$

$$- \int_t^\infty e^{-\bar r(s-t)}\left(\frac{\bar\alpha\sigma_z^2 z_t^2}{a^2}(1-e^{-a(s-t)})^2 + \int_t^s \frac{\sigma_z}{a}(1-e^{-a(s-u)})\left(\frac{\bar\alpha\sigma_z}{a}(1-e^{-a(s-u)}) - m_u^{i,1}\right)du\right)ds$$

$$- \bar\alpha\sigma_D^2 \int_t^\infty e^{-\bar r(s-t)}(s-t)\Big[\frac{\bar\alpha^2\sigma_z^2 z_t^2}{2a^2}(1-e^{-a(s-t)})^2 + \frac{1}{2}\int_t^s \frac{\bar\alpha^2\sigma_z^2}{a^2}(1-e^{-a(s-u)})^2 du$$

$$- \int_t^s \frac{\bar\alpha\sigma_z}{a}(1-e^{-a(s-u)})m_u^{i,1} du + \frac{1}{2}\sum_{j=1}^N \frac{\bar\alpha}{\alpha_j}\int_t^s (m_u^{j,1})^2 du - \bar\alpha\bar r\int_t^s (G\mu_D(u-t) + F + GD_t)m_u^{i,1} du\Big]ds$$

$$- \bar r \int_t^\infty e^{-\bar r(s-t)}(1 + \bar\alpha^2\sigma_D^2(s-t))\int_t^s (G\mu_D(u-t) + F + GD_t)m_u^{i,1} du ds$$

$$- \sigma_D \bar r G E_t \int_t^\infty \exp\Big\{ -\bar\beta(s-t) - \bar\alpha\mu_D(s-t) - \bar\alpha\sigma_D(W_s^D - W_t^D)\Big\}(W_s^D - W_t^D)\times$$

$$\times\Big[\bar{\alpha}\sigma_D\int_t^s (W_u^D - W_t^D)m_u^{i,1}du + \int_t^s (W_u^D - W_t^D)m_u^{i,1}dW_u^D\Big]ds.$$

We can compute the last expectation using Lemma D.5:

$$-\sigma_D\bar{r}GE_t\int_t^\infty \exp\Big\{-\bar{\beta}(s-t)-\bar{\alpha}\mu_D(s-t)-\bar{\alpha}\sigma_D(W_s^D-W_t^D)\Big\}(W_s^D-W_t^D)\times$$

$$\times\Big[\bar{\alpha}\sigma_D\int_t^s (W_u^D - W_t^D)m_u^{i,1}du + \int_t^s (W_u^D - W_t^D)m_u^{i,1}dW_u^D\Big]ds$$

$$=\frac{1}{6}\bar{\alpha}\sigma_D^2\bar{r}G\int_t^\infty e^{-\bar{r}(s-t)}(s-t)^2(3b_m^i+2a_m^i s+a_m^i t)ds.$$

Summing up all terms, we get that the term of order ε^2 of the price of the asset is

$$D_t\Bigg(\frac{\bar{\alpha}^2\sigma_z^2}{\bar{r}(a+\bar{r})(2a+\bar{r})}z_t^2+\int_t^\infty e^{-\bar{r}(s-t)}\int_t^s\Big[\frac{\bar{\alpha}^2\sigma_z^2}{2a^2}(1-e^{-a(s-u)})^2-\bar{\alpha}\frac{\sigma_z}{a}(1-e^{-a(s-u)})m_u^{i,1}+\frac{1}{2}\sum_{j=1}^N\frac{\bar{\alpha}}{\alpha_j}(m_u^{j,1})^2\Big]duds\Bigg)$$

$$+(\mu_D-\bar{\alpha}\sigma_D^2)\int_t^\infty e^{-\bar{r}(s-t)}(s-t)\Big(\frac{\bar{\alpha}^2\sigma_z^2z_t^2}{2a^2}(1-e^{-a(s-t)})^2+\frac{1}{2}\int_t^s\frac{\bar{\alpha}^2\sigma_z^2}{a^2}(1-e^{-a(s-u)})^2du$$

$$-\int_t^s\frac{\bar{\alpha}\sigma_z}{a}(1-e^{-a(s-u)})m_u^{i,1}du+\frac{1}{2}\sum_{j=1}^N\frac{\bar{\alpha}}{\alpha_j}\int_t^s(m_u^{j,1})^2du\Big)ds$$

$$-\int_t^\infty e^{-\bar{r}(s-t)}\Bigg(\frac{\bar{\alpha}\sigma_z^2z_t^2}{a^2}(1-e^{-a(s-t)})^2+\int_t^s\frac{\sigma_z}{a}(1-e^{-a(s-u)})\left(\frac{\bar{\alpha}\sigma_z}{a}(1-e^{-a(s-u)})-m_u^{i,1}\right)du\Bigg)ds$$

$$-\bar{r}\int_t^\infty e^{-\bar{r}(s-t)}\int_t^s(G\mu_D(u-t)+F+GD_t)m_u^{i,1}duds$$

$$+\frac{1}{6}\bar{\alpha}\sigma_D^2\bar{r}G\int_t^\infty e^{-\bar{r}(s-t)}(s-t)^2(3b_m^i+2a_m^i s+a_m^i t)ds$$

$$=D_t\Bigg(\frac{\bar{\alpha}^2\sigma_z^2}{\bar{r}(a+\bar{r})(2a+\bar{r})}z_t^2+\int_t^\infty e^{-\bar{r}(s-t)}\int_t^s\Big[\frac{\bar{\alpha}^2\sigma_z^2}{2a^2}(1-e^{-a(s-u)})^2+\frac{1}{2}\sum_{j=1}^N\frac{\bar{\alpha}}{\alpha_j}(m_u^{j,1})^2\Big]duds\Bigg)$$

$$+(\mu_D-\bar{\alpha}\sigma_D^2)\int_t^\infty e^{-\bar{r}(s-t)}(s-t)\Big(\frac{\bar{\alpha}^2\sigma_z^2z_t^2}{2a^2}(1-e^{-a(s-t)})^2+\frac{1}{2}\int_t^s\frac{\bar{\alpha}^2\sigma_z^2}{a^2}(1-e^{-a(s-u)})^2du$$

$$-\int_t^s\frac{\bar{\alpha}\sigma_z}{a}(1-e^{-a(s-u)})m_u^{i,1}du+\frac{1}{2}\sum_{j=1}^N\frac{\bar{\alpha}}{\alpha_j}\int_t^s(m_u^{j,1})^2du\Big)ds$$

$$-\int_t^\infty e^{-\bar{r}(s-t)}\Bigg(\frac{\bar{\alpha}\sigma_z^2z_t^2}{a^2}(1-e^{-a(s-t)})^2+\int_t^s\frac{\sigma_z}{a}(1-e^{-a(s-u)})\left(\frac{\bar{\alpha}\sigma_z}{a}(1-e^{-a(s-u)})-m_u^{i,1}\right)du\Bigg)ds$$

$$-\bar{r}\int_t^\infty e^{-\bar{r}(s-t)}\int_t^s(G\mu_D(u-t)+F)m_u^{i,1}duds$$

$$+\frac{1}{6}\bar{\alpha}\sigma_D^2\bar{r}G\int_t^\infty e^{-\bar{r}(s-t)}(s-t)^2(3b_m^i+2a_m^i s+a_m^i t)ds$$

An explicit calculation shows that the terms that depend on the choice of the agent i sum up to zero:

$$
\begin{aligned}
&-(\mu_D-\bar\alpha\sigma_D^2)\int_t^\infty e^{-\bar r(s-t)}(s-t)\int_t^s \frac{\bar\alpha\sigma_z}{a}(1-e^{-a(s-u)})m_u^{i,1}duds\\
&+\frac{\sigma_z}{a}\int_t^\infty e^{-\bar r(s-t)}\int_t^s(1-e^{-a(s-u)})m_u^{i,1}duds-\bar r\int_t^\infty e^{-\bar r(s-t)}\int_t^s(G\mu_D(u-t)+F)m_u^{i,1}duds\\
&+\frac{1}{6}\bar\alpha\sigma_D^2\bar r G\int_t^\infty e^{-\bar r(s-t)}(s-t)^2(3b_m^i+2a_m^i s+a_m^i t)ds=0.
\end{aligned}
$$

Hence, the term of order ε^2 of the price of the asset P_t is

$$
\begin{aligned}
&D_t\left(\frac{\bar\alpha^2\sigma_z^2}{\bar r(a+\bar r)(2a+\bar r)}z_t^2+\int_t^\infty e^{-\bar r(s-t)}\int_t^s\left[\frac{\bar\alpha^2\sigma_z^2}{2a^2}(1-e^{-a(s-u)})^2+\frac{1}{2}\sum_{j=1}^N\frac{\bar\alpha}{\alpha_j}(m_u^{j,1})^2\right]duds\right)\\
&+(\mu_D-\bar\alpha\sigma_D^2)\frac{\bar\alpha^2\sigma_z^2}{2a^2}\int_t^\infty e^{-\bar r(s-t)}(s-t)\Big(z_t^2(1-e^{-a(s-t)})^2+\int_t^s(1-e^{-a(s-u)})^2du\Big)ds\\
&+(\mu_D-\bar\alpha\sigma_D^2)\frac{1}{2}\int_t^\infty e^{-\bar r(s-t)}(s-t)\Big(\sum_{j=1}^N\frac{\bar\alpha}{\alpha_j}\int_t^s(m_u^{j,1})^2du\Big)ds\\
&-\frac{\bar\alpha\sigma_z^2}{a^2}\int_t^\infty e^{-\bar r(s-t)}\left(z_t^2(1-e^{-a(s-t)})^2+\int_t^s(1-e^{-a(s-u)})^2du\right)ds\\
&=D_t\Big(\frac{\bar\alpha^2\sigma_z^2}{\bar r(a+\bar r)(2a+\bar r)}z_t^2+\frac{\bar\alpha^2\sigma_z^2}{2a^2}\left(\frac{1}{\bar r^2}-\frac{3}{2a\bar r}+\frac{2}{a(a+\bar r)}-\frac{1}{2a(2a+\bar r)}\right)+\frac{1}{2}\sum_{j=1}^N\frac{\bar\alpha}{\alpha_j}M_t^{j,C}\Big)\\
&-\frac{\bar\alpha\sigma_z^2}{a^2}\left(z_t^2\left(\frac{1}{\bar r}-\frac{2}{a+\bar r}+\frac{1}{2a+\bar r}\right)+\frac{1}{\bar r^2}-\frac{3}{2a\bar r}+\frac{2}{a(a+\bar r)}-\frac{1}{2a(2a+\bar r)}\right)\\
&+(\mu_D-\bar\alpha\sigma_D^2)\frac{\bar\alpha^2\sigma_z^2}{2a^2}\left(z_t^2\left(\frac{1}{\bar r^2}-\frac{2}{(a+\bar r)^2}+\frac{1}{(2a+\bar r)^2}\right)+\frac{2}{\bar r^3}-\frac{3}{2a\bar r^2}+\frac{2}{a(a+\bar r)^2}-\frac{1}{2a(2a+\bar r)^2}\right)\\
&+(\mu_D-\bar\alpha\sigma_D^2)\frac{1}{2}\sum_{j=1}^N\frac{\bar\alpha}{\alpha_j}M_t^{j,L}\\
&=\frac{\bar\alpha^2\sigma_z^2}{\bar r(a+\bar r)(2a+\bar r)}\left(z_t^2+\frac{1}{\bar r}\right)D_t+\frac{1}{2}\left(\sum_{j=1}^N\frac{\bar\alpha}{\alpha_j}M_t^{j,C}\right)D_t-\frac{2\bar\alpha\sigma_z^2}{\bar r(a+\bar r)(2a+\bar r)}\left(z_t^2+\frac{1}{\bar r}\right)\\
&+(\mu_D-\bar\alpha\sigma_D^2)\frac{\bar\alpha^2\sigma_z^2}{2a^2}\left(z_t^2\left(\frac{1}{\bar r^2}-\frac{2}{(a+\bar r)^2}+\frac{1}{(2a+\bar r)^2}\right)+\frac{2}{\bar r^3}-\frac{3}{2a\bar r^2}+\frac{2}{a(a+\bar r)^2}-\frac{1}{2a(2a+\bar r)^2}\right)\\
&+(\mu_D-\bar\alpha\sigma_D^2)\frac{1}{2}\sum_{j=1}^N\frac{\bar\alpha}{\alpha_j}M_t^{j,L}.
\end{aligned}
$$

Appendix E. Deriving (B.7), (B.8), (B.9), (B.10) and other proofs

Recall that the dividend process equals

$$
D_s=D_t+\mu_D(s-t)+\sigma_D(W_s^D-W_t^D)+\varepsilon\frac{\sigma_z z_t}{a}(1-e^{-a(s-t)})+\varepsilon\frac{\sigma_z}{a}\int_t^s(1-e^{-a(s-u)})dW_u^\mu.
$$

Therefore,

$$
C_t^i=E_t\int_t^\infty \exp\{-\bar\beta(s-t)-\bar\alpha(D_s-D_t)+(Y_s^i-Y_t^i)\}ds
$$

$$
\begin{aligned}
&= E_t \int_t^\infty \left(1 - \varepsilon\bar{\alpha}\frac{\sigma_z z_t}{a}(1-e^{-a(s-t)}) - \varepsilon\bar{\alpha}\frac{\sigma_z}{a}\int_t^s (1-e^{-a(s-u)})dW_u^\mu + \varepsilon\int_t^s m_u^{i,1}dW_u^\mu \right)\exp\{-\bar{\beta}(s-t) \\
&- \bar{\alpha}\mu_D(s-t) - \bar{\alpha}\sigma_D(W_s^D - W_t^D)\}ds + o(\varepsilon) \\
&= \frac{1}{\bar{r}} - \varepsilon\bar{\alpha}\frac{\sigma_z}{a}z_t E_t\int_t^\infty \left(1-e^{-a(s-t)}\right)\exp\{-\bar{\beta}(s-t) - \bar{\alpha}\mu_D(s-t) - \bar{\alpha}\sigma_D(W_s^D - W_t^D)\}ds + o(\varepsilon) \\
&= \frac{1}{\bar{r}} - \varepsilon\bar{\alpha}\frac{\sigma_z}{a}z_t E_t\int_t^\infty (1-e^{-a(s-t)})\exp\{-\bar{r}(s-t)\}ds + o(\varepsilon) = \frac{1}{\bar{r}} - \varepsilon\bar{\alpha}\frac{1}{\bar{r}(a+\bar{r})}\sigma_z z_t + o(\varepsilon).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
L_t^i &= E_t\int_t^\infty s\cdot\exp\{-\bar{\beta}(s-t) - \bar{\alpha}(D_s - D_t) + (Y_s^i - Y_t^i)\}ds \\
&= E_t\int_t^\infty s\cdot\left(1 - \varepsilon\bar{\alpha}\frac{\sigma_z z_t}{a}(1-e^{-a(s-t)}) - \varepsilon\bar{\alpha}\frac{\sigma_z}{a}\int_t^s (1-e^{-a(s-u)})dW_u^\mu + \varepsilon\int_t^s m_u^i dW_u^\mu \right)\exp\{-\bar{\beta}(s-t) \\
&- \bar{\alpha}\mu_D(s-t) - \bar{\alpha}\sigma_D(W_s^D - W_t^D)\}ds + o(\varepsilon) \\
&= \frac{1}{\bar{r}}\cdot t + \frac{1}{\bar{r}^2} - \varepsilon\bar{\alpha}\frac{\sigma_z}{a}z_t E_t\int_t^\infty s\cdot\left(1-e^{-a(s-t)}\right)\exp\{-\bar{\beta}(s-t) - \bar{\alpha}\mu_D(s-t) - \bar{\alpha}\sigma_D(W_s^D - W_t^D)\}ds + o(\varepsilon) \\
&= \frac{1}{\bar{r}}\cdot t + \frac{1}{\bar{r}^2} - \varepsilon\bar{\alpha}\frac{\sigma_z}{a}z_t E_t\int_t^\infty s\cdot(1-e^{-a(s-t)})\exp\{-\bar{r}(s-t)\}ds + o(\varepsilon) \\
&= \frac{1}{\bar{r}}\cdot t + \frac{1}{\bar{r}^2} - \varepsilon\bar{\alpha}\left(\frac{1}{\bar{r}(a+\bar{r})}t + \frac{a+2\bar{r}}{\bar{r}^2(a+\bar{r})^2}\right)\sigma_z z_t + o(\varepsilon)
\end{aligned}
$$

and

$$
\begin{aligned}
P_t &= E_t\int_t^\infty D_s\cdot\exp\{-\bar{\beta}(s-t) - \bar{\alpha}(D_s - D_t) + (Y_s^i - Y_t^i)\}ds \\
&= D_t\cdot C_t^i + E_t\int_t^\infty (D_s - D_t)\cdot\exp\{-\bar{\beta}(s-t) - \bar{\alpha}(D_s - D_t) + (Y_s^i - Y_t^i)\}ds \\
&= D_t\cdot C_t^i + E_t\int_t^\infty \Big[\mu_D(s-t) + \sigma_D(W_s^D - W_t^D) + \varepsilon\frac{\sigma_z z_t}{a}(1-e^{-a(s-t)}) + \varepsilon\frac{\sigma_z}{a}\int_t^s (1-e^{-a(s-u)})dW_u^\mu \\
&- \varepsilon\bar{\alpha}\left(\mu_D(s-t) + \sigma_D(W_s^D - W_t^D)\right)\left(\frac{\sigma_z z_t}{a}(1-e^{-a(s-t)}) + \frac{\sigma_z}{a}\int_t^s (1-e^{-a(s-u)})dW_u^\mu + \int_t^s m_u^i dW_u^\mu\right)\Big]\cdot \\
&\cdot\exp\{-\bar{\beta}(s-t) - \bar{\alpha}(\mu_D(s-t) + \sigma_D(W_s^D - W_t^D))\}ds + o(\varepsilon) \\
&= D_t\cdot C_t^i + \int_t^\infty \left[(\mu_D - \bar{\alpha}\sigma_D^2)(s-t) + \varepsilon\frac{\sigma_z z_t}{a}\left(1 - \bar{\alpha}(\mu_D - \bar{\alpha}\sigma_D^2)(s-t)\right)(1-e^{-a(s-t)})\right]e^{-\bar{r}(s-t)}ds + o(\varepsilon) \\
&= D_t\cdot C_t^i + \frac{\mu_D - \bar{\alpha}\sigma_D^2}{\bar{r}^2} + \varepsilon\frac{\sigma_z z_t}{a}\left(\frac{1}{\bar{r}} - \frac{1}{a+\bar{r}} - \bar{\alpha}(\mu_D - \bar{\alpha}\sigma_D^2)\left(\frac{1}{\bar{r}^2} - \frac{1}{(a+\bar{r})^2}\right)\right) + o(\varepsilon).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
K_t^i &= E_t\int_t^\infty (Y_t^i + (Y_s^i - Y_t^i))\exp\{-\bar{\beta}(s-t) - \bar{\alpha}(D_s - D_t) + (Y_s^i - Y_t^i)\}ds \\
&= Y_t^i C_t^i + E_t\int_t^\infty (Y_s^i - Y_t^i)\exp\{-\bar{\beta}(s-t) - \bar{\alpha}(D_s - D_t) + (Y_s^i - Y_t^i)\}ds
\end{aligned}
$$

$$= \varepsilon \frac{1}{\bar{r}} Y_t^{i,1} + \varepsilon E_t \int_t^\infty \left(\int_t^s m_u^i dW_u^\mu \right) \exp\{-\bar{\beta}(s-t) - \bar{\alpha}\mu_D(s-t) - \bar{\alpha}\sigma_D(W_s^D - W_t^D)\} ds + o(\varepsilon)$$

$$= \varepsilon \frac{1}{\bar{r}} Y_t^{i,1} + o(\varepsilon).$$

Proof of Proposition 3.3. Follows from a direct calculation from Theorem 3.2. □

Proof of Proposition 3.4. Recall that the interest rate is the negative drift of each agent's marginal utility of optimal consumption, i.e.,

$$r_t = \bar{\beta} + \bar{\alpha}(\mu_D + \varepsilon\sigma_z z_t) - \frac{1}{2}\bar{\alpha}^2\sigma_D^2 - l_t^i - \frac{1}{2}(m_t^i)^2 - \frac{1}{2}(n_t^i)^2 + \bar{\alpha}\sigma_D n_t^i,$$

where $l_t^i = l_t^{i,2}\varepsilon^2 + l_t^{i,3}\varepsilon^3 + o(\varepsilon^3)$, $m_t^i = m_t^{i,1}\varepsilon + m_t^{i,2}\varepsilon^2 + m_t^{i,3}\varepsilon^3 + o(\varepsilon^3)$ and $n_t^i = n_t^{i,2}\varepsilon^2 + n_t^{i,3}\varepsilon^3 + o(\varepsilon^3)$.
All agents must agree on the equilibrium interest rate r_t. Therefore, $-l_t^i - \frac{1}{2}(m_t^i)^2 - \frac{1}{2}(n_t^i)^2 + \bar{\alpha}\sigma_D n_t^i = g_t$, for some $g_t = g_t^0 + g_t^1\varepsilon + g_t^2\varepsilon^2 + g_t^3\varepsilon^3 + o(\varepsilon^3)$ independent of i. By equating the terms of order ε^3 and using $n_t^{i,1} = 0$, it follows that

$$-l_t^{i,3} - m_t^{i,1}m_t^{i,2} + \bar{\alpha}\sigma_D n_t^{i,3} = g_t^3.$$

Recall that $\sum_{i=1}^n \frac{1}{\alpha_i} l_t^{i,3} = \sum_{i=1}^n \frac{1}{\alpha_i} m_t^{i,3} = \sum_{i=1}^n \frac{1}{\alpha_i} n_t^{i,3} = 0$. Hence,

$$\frac{1}{\bar{\alpha}} g_t^3 = \sum_{i=1}^n \frac{1}{\alpha_i} g_t^3 = -\sum_{i=1}^n \frac{1}{\alpha_i} l_t^{i,3} - \sum_{i=1}^n \frac{1}{\alpha_i} m_t^{i,1}m_t^{i,2} + \bar{\alpha}\sigma_D \sum_{i=1}^n \frac{1}{\alpha_i} n_t^{i,3} = -\sum_{i=1}^n \frac{1}{\alpha_i} m_t^{i,1}m_t^{i,2}.$$

Therefore, at the third order, the equilibrium interest rate is

$$r_t = \bar{r} + \varepsilon\bar{\alpha}\sigma_z z_t - \frac{1}{2}\varepsilon^2 \sum_{i=1}^n \frac{\bar{\alpha}}{\alpha_i}(m_t^{i,1})^2 - \varepsilon^3 \sum_{i=1}^n \frac{1}{\alpha_i} m_t^{i,1}m_t^{i,2} + o(\varepsilon^3).$$

Recall that $m_t^{i,2} = \alpha_i \bar{r} q m_t^{i,2} - \bar{r}\frac{d\langle Y^{i,1}C^{i,1}, W^\mu\rangle_t}{dt}$, where $C_t^{i,1} = -\bar{\alpha}\sigma_z \frac{1}{\bar{r}(a+\bar{r})} z_t$ and $Y_t^{i,1} = \int_0^t m_s^{i,1} dW_s^\mu$. Hence,

$$\frac{d\langle Y^{i,1}C^{i,1}, W^\mu\rangle_t}{dt} = -\bar{\alpha}\sigma_z \frac{1}{\bar{r}(a+\bar{r})}(Y_t^{i,1} + m_t^{i,1} z_t).$$

Homogeneous time-preferences ($\beta_i = \bar{\beta}$ for all i) yield

$$q m_t^{i,2} = \left(x_i - \frac{\bar{\alpha}}{\alpha_i}P_0\right)\left[2\bar{r}\frac{\bar{\alpha}^2\sigma_z^2}{\bar{r}(a+\bar{r})(2a+\bar{r})} z_t - \bar{r}^2 \frac{\bar{\alpha}^2}{\bar{r}^2(a+\bar{r})^2}\sigma_z^2 z_0\right]$$

and $Y_t^{i,1} = -\alpha_i \frac{\bar{\alpha}\sigma_z}{a+\bar{r}}\left(x_i - \frac{\bar{\alpha}}{\alpha_i}P_0\right)\bar{r}W_t^\mu$. Therefore, the term of order ε^3 in the equilibrium rate is

$$-\sum_{i=1}^n \frac{1}{\alpha_i} m_t^{i,1}m_t^{i,2} = \sum_{i=1}^n \frac{\bar{\alpha}\sigma_z}{a+\bar{r}}\left(x_i - \frac{\bar{\alpha}}{\alpha_i}P_0\right)\bar{r}\left(\alpha_i \bar{r} q m_t^{i,2} - \bar{r}\frac{d\langle Y^{i,1}C^{i,1}, W^\mu\rangle_t}{dt}\right)$$
$$= \frac{\bar{\alpha}^3\sigma_z^3}{(a+\bar{r})^3}\bar{r}^2\left(\frac{\bar{r}}{2a+\bar{r}} z_t - z_0 - W_t^\mu\right)\sum_{i=1}^n \alpha_i \left(x_i - \frac{\bar{\alpha}}{\alpha_i}P_0\right)^2,$$

which proves the claim. □

Proof of Proposition 4.1. The result is a particular case of Proposition 4.2. □

Proof of Proposition 4.2. Assuming correlation ρ between W^D and W^μ, i.e., $W_t^\mu = \rho W_t^D + \sqrt{1-\rho^2}\tilde{W}_t^\mu$ with W^D and $\tilde{W}^\mu$ independent, the same arguments as in Theorem C.3 lead to the stock process

$$P_t = D_t\left(\frac{1}{\bar{r}} + \bar{\alpha}\sigma_D\varepsilon\bar{\alpha}\sigma_z\frac{1}{\bar{r}^2(a+\bar{r})}\rho - \varepsilon\bar{\alpha}\sigma_z\frac{1}{\bar{r}(a+\bar{r})}z_t\right) - \varepsilon\bar{\alpha}\frac{2}{\bar{r}^2(a+\bar{r})}\sigma_D\rho\sigma_z + \varepsilon\frac{1}{\bar{r}(a+\bar{r})}\sigma_z z_t$$
$$+ \left[\frac{1}{\bar{r}^2} - \bar{\alpha}\sigma_z\varepsilon\frac{z_t}{a}\left(\frac{1}{\bar{r}^2} - \frac{1}{(a+\bar{r})^2}\right) + \frac{\bar{\alpha}^2}{a}\varepsilon\sigma_z\left(\frac{2}{\bar{r}^3} - \frac{1}{a\bar{r}^2} + \frac{1}{a(a+\bar{r})^2}\right)\rho\sigma_D\right]\cdot(\mu_D - \bar{\alpha}\sigma_D^2) + o(\varepsilon)$$

and interest rate $r_t = \bar{r} + \varepsilon\bar{\alpha}\sigma_z z_t + o(\varepsilon)$. Recall that

$$D_t = D_s + \mu_D(t-s) + \sigma_D(W_t^D - W_s^D) + \varepsilon \frac{\sigma_z z_s}{a}(1-e^{-a(t-s)}) + \varepsilon \frac{\sigma_z}{a} \int_s^t (1-e^{-a(t-u)})[\rho dW_u^D + \sqrt{1-\rho^2} d\tilde{W}_u^\mu],$$

$$z_t = z_s e^{-a(t-s)} + \int_s^t e^{-a(t-u)}[\rho dW_u^D + \sqrt{1-\rho^2} d\tilde{W}_u^\mu]$$

Therefore,

$$\begin{aligned} E[P_t e^{-\int_s^t r_u du}|\mathcal{F}_s] &= e^{-\bar{r}(t-s)}[D_s + \mu_D(t-s)]\left(\frac{1}{\bar{r}} + \bar{\alpha}\sigma_D \varepsilon \bar{\alpha}\sigma_z \frac{1}{\bar{r}^2(a+\bar{r})}\rho - \varepsilon\bar{\alpha}\sigma_z \frac{1}{\bar{r}(a+\bar{r})} z_s e^{-a(t-s)}\right) \\ &- \varepsilon\bar{\alpha}\sigma_z \frac{1}{\bar{r}(a+\bar{r})}\sigma_D e^{-\bar{r}(t-s)}\rho \frac{1-e^{-a(t-s)}}{a} + \varepsilon e^{-\bar{r}(t-s)} \frac{\sigma_z z_s}{a}(1-e^{-a(t-s)})\frac{1}{\bar{r}} \\ &+ e^{-\bar{r}(t-s)}\Bigg(-\varepsilon\bar{\alpha}\frac{2}{\bar{r}^2(a+\bar{r})}\sigma_D\rho\sigma_z + \varepsilon \frac{1}{\bar{r}(a+\bar{r})}\sigma_z z_s e^{-a(t-s)} \\ &+ \left[\frac{1}{\bar{r}^2} - z_s e^{-a(t-s)}\bar{\alpha}\sigma_z \varepsilon \frac{1}{a}\left(\frac{1}{\bar{r}^2} - \frac{1}{(a+\bar{r})^2}\right) + \frac{\bar{\alpha}^2}{a}\varepsilon\sigma_z\left(\frac{2}{\bar{r}^3} - \frac{1}{a\bar{r}^2} + \frac{1}{a(a+\bar{r})^2}\right)\rho\sigma_D\right] \times (\mu_D - \bar{\alpha}\sigma_D^2)\Bigg) \\ &- \varepsilon\bar{\alpha}\sigma_z\left(\frac{1}{\bar{r}}(D_s + \mu_D(t-s)) + \frac{1}{\bar{r}^2}(\mu_D - \bar{\alpha}\sigma_D^2)\right)e^{-\bar{r}(t-s)}\frac{z_s}{a}\left(1-e^{-a(t-s)}\right) \\ &- \varepsilon\bar{\alpha}\rho\sigma_z \frac{1}{\bar{r}}\sigma_D e^{-\bar{r}(t-s)}\frac{1}{a}\left(t-s-\frac{1-e^{-a(t-s)}}{a}\right) \end{aligned}$$

and

$$\begin{aligned} E\left[\int_s^t D_u e^{-\int_s^u r_l dl} du|\mathcal{F}_s\right] &= -\varepsilon\rho\bar{\alpha}\sigma_z \frac{\sigma_D}{a}\left[-e^{-\bar{r}(t-s)}\frac{1}{\bar{r}}(t-s) - e^{-(a+\bar{r})(t-s)}\frac{1}{a(a+\bar{r})} + \frac{a}{\bar{r}^2(a+\bar{r})} + \frac{\bar{r}-a}{a\bar{r}^2}e^{-\bar{r}(t-s)}\right] \\ &+ D_s \frac{1-e^{-\bar{r}(t-s)}}{\bar{r}} + \mu_D \frac{1-e^{-\bar{r}(t-s)} - \bar{r}e^{-\bar{r}(t-s)}(t-s)}{\bar{r}^2} + \varepsilon \frac{\sigma_z z_s}{a}\left(\frac{1-e^{-\bar{r}(t-s)}}{\bar{r}} - \frac{1-e^{-(a+\bar{r})(t-s)}}{a+\bar{r}}\right) \\ &- \varepsilon\left(D_s \frac{1-e^{-\bar{r}(t-s)}}{\bar{r}} + \mu_D \frac{1-e^{-\bar{r}(t-s)} - \bar{r}e^{-\bar{r}(t-s)}(t-s)}{\bar{r}^2}\right)\frac{\bar{\alpha}}{a}\sigma_z z_s \\ &+ \varepsilon\left(D_s \frac{1-e^{-(a+\bar{r})(t-s)}}{a+\bar{r}} + \mu_D \frac{1-e^{-(a+\bar{r})(t-s)} - (a+\bar{r})e^{-(a+\bar{r})(t-s)}(t-s)}{(a+\bar{r})^2}\right)\frac{\bar{\alpha}}{a}\sigma_z z_s + o(\varepsilon). \end{aligned}$$

Combining the expressions in $E[P_t e^{-\int_s^t r_u du} + \int_s^t D_u e^{-\int_s^u r_l dl} du|\mathcal{F}_s] - P_s$ yields the result.
To find the expected total return of the consol bond, notice that $C_t = \frac{1}{\bar{r}} - \varepsilon\bar{\alpha}\sigma_z \frac{1}{\bar{r}(a+\bar{r})} z_t + \varepsilon\bar{\alpha}^2\sigma_D\sigma_z \frac{1}{\bar{r}^2(a+\bar{r})}\rho$. Then

$$E_{s,t}C = -\varepsilon\bar{\alpha}^2\sigma_D\sigma_z \frac{1}{\bar{r}^2(a+\bar{r})}\rho(1-e^{-\bar{r}(t-s)}) + o(\varepsilon) \quad \square$$

Sketch of proof of Theorem 4.3. The rigorous proof follows the same lines as the proof of Theorem 2.1. Here we present the formal derivation of the equilibrium.

Solving the first-order condition

$$e^{-\beta_i t - \alpha_i \hat{c}_t^i} = y_i M_t,$$

for the consumption rate c_t^i, and aggregating across agents, it follows that

$$\log M_t = -\bar{\beta}t - \bar{\alpha}\mathbf{1}^\top D_t - \bar{\alpha}\sum_{i=1}^n \frac{\log y_i}{\alpha_i},$$

whence, up to a multiplicative constant, $M_t = \exp\{-\bar{\beta}t - \bar{\alpha}\mathbf{1}^\top(D_t - D_0)\}$ and hence

$$\hat{c}_t^i = \frac{\bar{\beta}-\beta_i}{\alpha_i}t + \frac{\bar{\alpha}}{\alpha_i}\mathbf{1}^\top(D_t - D_0) - \frac{\log y_i}{\alpha_i}.$$

The dynamics of M_t implies the equilibrium interest rate

$$r = \bar{\beta} + \bar{\alpha}\mathbf{1}^\top \boldsymbol{\mu_D} - \frac{1}{2}\bar{\alpha}^2\mathbf{1}^\top \boldsymbol{\sigma_D}\boldsymbol{\sigma_D}^\top \mathbf{1}.$$

The price of asset j is

$$P_t^j = E_t\left[\int_t^\infty D_s^j \frac{M_s}{M_t} ds\right]$$

$$= E_t\left[\int_t^\infty (D_t^j + \mu_D^j(s-t) + \sigma_D^j(W_s^D - W_t^D))\exp\{-\bar\beta(s-t) - \bar\alpha \mathbf{1}^\top \boldsymbol{\mu_D}(s-t) - \bar\alpha\mathbf{1}^\top\boldsymbol{\sigma_D}(W_s^D - W_t^D)\}ds\right]$$

$$= \int_t^\infty (D_t^j + \mu_D^j(s-t) - \bar\alpha\mathbf{1}^\top\boldsymbol{\sigma_D}\sigma_D^j(s-t))\exp\{-r(s-t)\}ds = \frac{D_t^j}{r} + \frac{1}{r^2}(\mu_D^j - \bar\alpha\mathbf{1}^\top\boldsymbol{\sigma_D}\sigma_D^j),$$

where σ_D^j is the j-th row of $\boldsymbol{\sigma_D}$.

Note that $E_t\left[\int_t^\infty \frac{M_s}{M_t}ds\right] = \frac{1}{r}$ and $E_t\left[\int_t^\infty s\frac{M_s}{M_t}ds\right] = \frac{t}{r} + \frac{1}{r^2}$. Then, from the first-order condition,

$$X_t^i = \frac{\bar\beta - \beta_i}{\alpha_i}\left(\frac{t}{r} + \frac{1}{r^2}\right) + \frac{\bar\alpha}{\alpha_i}\mathbf{1}^\top\left(P_t - \frac{D_0}{r}\right) - \frac{1}{r}\frac{\log y_i}{\alpha_i}$$

and $\frac{1}{r}\frac{\log y_i}{\alpha_i} = \frac{\bar\beta-\beta_i}{\alpha_i}\frac{1}{r^2} + \frac{\bar\alpha}{\alpha_i}\mathbf{1}^\top\left(P_0 - \frac{D_0}{r}\right) - x_i$. Hence,

$$X_t^i = x_i + \frac{\bar\alpha}{\alpha_i}\mathbf{1}^\top\left(P_t - P_0\right) + t\frac{\bar\beta - \beta_i}{r\alpha_i},$$

$$\hat c_t^i = \frac{\bar\beta - \beta_i}{\alpha_i}\left(t - \frac{1}{r}\right) + \frac{\bar\alpha}{\alpha_i}\mathbf{1}^\top D_t + r\left(x_i - \frac{\bar\alpha}{\alpha_i}\mathbf{1}^\top P_0\right),$$

whence $\theta_t^i = \frac{\bar\alpha}{\alpha_i}\mathbf{1}$ by inspection of the first equation above. □

Proof of Corollary 4.4. It follows immediately from Theorem 4.3:

$$R_t^i := \frac{\bar\alpha}{r^2}(\sigma_D^i)^\top\boldsymbol{\sigma_D}^\top\mathbf{1} = Beta_i\frac{\bar\alpha}{r^2}\mathbf{1}^\top\boldsymbol{\sigma_D}\boldsymbol{\sigma_D}^\top\mathbf{1} = Beta_i R_t^m. \quad \square \tag{E.1}$$

Proof of Corollary 4.5. From the proof of Theorem 4.3 we have that $X_t^i = x_i + \frac{\bar\alpha}{\alpha_i}\mathbf{1}^\top\left(P_t - P_0\right) + t\frac{\bar\beta-\beta_i}{r\alpha_i}$. Hence, for large t, the proportion of personal wealth in the j-th risky asset is

$$\frac{\theta_t^{i,j}P_t^j}{X_t^i} = \frac{\frac{\bar\alpha}{\alpha_i}\left(\frac{D_t^j}{r} + \frac{\mu_D^j - \bar\alpha\sigma_D^2}{r^2}\right)}{x_i + \frac{\bar\alpha}{\alpha_i}\mathbf{1}^\top\left(P_t - P_0\right) + t\frac{\bar\beta-\beta_i}{r\alpha_i}} \sim \frac{\frac{\bar\alpha}{\alpha_i}\frac{D_t^j}{r}}{\frac{\bar\alpha}{\alpha_i}\frac{\mathbf{1}^\top D_t}{r} + t\left(\frac{\bar\beta-\beta_i}{r\alpha_i}\right)} \sim \frac{\mu_D^j}{\mathbf{1}^\top\boldsymbol{\mu_D} + \frac{\bar\beta-\beta_i}{\bar\alpha}},$$

because in the long-run $D_t \approx \boldsymbol{\mu_D}t$. Similarly, the share of aggregate wealth owned by agent i is

$$\frac{X_t^i}{\mathbf{1}^\top P_t} = \frac{x_i + \frac{\bar\alpha}{\alpha_i}\mathbf{1}^\top\left(P_t - P_0\right) + t\frac{\bar\beta-\beta_i}{r\alpha_i}}{\frac{\mathbf{1}^\top D_t}{r} + \frac{\mathbf{1}^\top\boldsymbol{\mu_D} - \bar\alpha\mathbf{1}^\top\boldsymbol{\sigma_D}\boldsymbol{\sigma_D}^\top\mathbf{1}}{r^2}} \sim \frac{\frac{\bar\alpha}{\alpha_i}\frac{\mathbf{1}^\top D_t}{r} + t\left(\frac{\bar\beta-\beta_i}{r\alpha_i}\right)}{\frac{\mathbf{1}^\top D_t}{r}} \sim \frac{\bar\alpha}{\alpha_i} + \frac{\bar\beta - \beta_i}{\alpha_i\mathbf{1}^\top\boldsymbol{\mu_D}}. \quad \square$$

Sketch of proof of Theorem 4.6. Following the same arguments as in the one-dimensional model, it follows that $r_t = \bar r + \varepsilon\bar\alpha\mathbf{1}^T\boldsymbol{\sigma_z}\boldsymbol{z}_t$. The present value of the i-th stock is

$$S_t^i = E_t\left[\int_t^\infty e^{-\int_t^s r_u du}D_s^i ds\right]$$

$$= E_t\Big[\int_t^\infty e^{-\bar r(s-t) - \varepsilon\bar\alpha\mathbf{1}^T\boldsymbol{\sigma_z}\int_t^s \boldsymbol{z}_u du}(D_t^i + \mu_D^i(s-t) + \varepsilon e^{i^T}\boldsymbol{\sigma_z}(I - e^{-A(s-t)})A^{-1}\boldsymbol{z}_t$$

$$+ \varepsilon e^{i^T}\boldsymbol{\sigma_z}\int_t^s (I - e^{-A(s-u)})A^{-1}dW_u^\mu)ds\Big]$$

$$= E_t\Big[\int_t^\infty e^{-\bar r(s-t) - \varepsilon\bar\alpha\mathbf{1}^T\boldsymbol{\sigma_z}(I - e^{-A(s-t)})A^{-1}z_t - \varepsilon\bar\alpha\mathbf{1}^T\boldsymbol{\sigma_z}\int_t^s(I - e^{-A(s-u)})A^{-1}dW_u}(D_t^i + \mu_D^i(s-t)$$

$$+\varepsilon e^{i^T}\boldsymbol{\sigma_z}(I-e^{-A(s-t)})A^{-1}\mathbf{z}_t+\varepsilon e^{i^T}\boldsymbol{\sigma_z}\int_t^s(I-e^{-A(s-u)})A^{-1}dW_u^{\mu})ds\Big]$$

$$=\int_t^\infty e^{-\bar r(s-t)-\varepsilon\bar\alpha\mathbf{1}^T\boldsymbol{\sigma_z}(I-e^{-A(s-t)})A^{-1}z_t+\frac12\varepsilon^2\bar\alpha^2\mathbf{1}^T\boldsymbol{\sigma_z}A^{-1}\int_t^s(I-e^{-A(s-u)})(I-e^{-A(s-u)})^Tdu(A^T)^{-1}\boldsymbol{\sigma_z}^T1}\Big(D_t^i+\mu_D^i(s-t)$$

$$+\varepsilon e^{i^T}\boldsymbol{\sigma_z}(I-e^{-A(s-t)})A^{-1}\mathbf{z}_t-\varepsilon^2\bar\alpha\mathbf{1}^T\boldsymbol{\sigma_z}A^{-1}\int_t^s(I-e^{-A(s-u)})(I-e^{-A(s-u)})^Tdu(A^T)^{-1}\boldsymbol{\sigma_z}^Te^i\Big)ds$$

$$\approx\int_t^\infty e^{-\bar r(s-t)}\Big(D_t^i+\mu_D^i(s-t)\Big)$$

$$+\varepsilon e^{-\bar r(s-t)}\Big(e^{i^T}\boldsymbol{\sigma_z}(I-e^{-A(s-t)})A^{-1}\mathbf{z}_t-(D_t^i+\mu_D^i(s-t))\bar\alpha\mathbf{1}^T\boldsymbol{\sigma_z}(I-e^{-A(s-t)})A^{-1}z_t\Big)ds$$

$$=\frac{D_t^i}{\bar r}+\frac{\mu_D^i}{\bar r^2}+\varepsilon\Big(e^{i^T}\boldsymbol{\sigma_z}\left(\frac1{\bar r}I-(\bar rI+A)^{-1}\right)A^{-1}\mathbf{z}_t-D_t^i\bar\alpha\mathbf{1}^T\boldsymbol{\sigma_z}\left(\frac1{\bar r}I-(\bar rI+A)^{-1}\right)A^{-1}z_t$$

$$-\varepsilon\mu_D^i\bar\alpha\mathbf{1}^T\boldsymbol{\sigma_z}\left(\frac1{\bar r^2}I-\int_0^\infty se^{-(\bar rI+A)s}ds\right)A^{-1}z_t\Big).$$

Similar computations show that the price of the consol bond is

$$C_t^i\approx\frac1{\bar r}-\varepsilon\bar\alpha\mathbf{1}^T\boldsymbol{\sigma_z}\left(\frac1{\bar r}I-(\bar rI+A)^{-1}\right)A^{-1}z_t.$$

Also,

$$E_t\left[\int_t^\infty\frac{M_s}{M_t}(D_s^i-D_t^i)ds\right]=E_t\Big[\int_t^\infty e^{-\bar\beta(s-t)-\bar\alpha\mathbf{1}^T(D_s-D_t)+(Y_s^\varepsilon-Y_t^\varepsilon)}(D_s^i-D_t^i)ds\Big]$$

$$=E_t\Big[\int_t^\infty e^{-\bar\beta(s-t)-\bar\alpha\mathbf{1}^T\mu_D(s-t)-\varepsilon\bar\alpha\mathbf{1}^T\boldsymbol{\sigma_z}(I-e^{-A(s-t)})A^{-1}z_t-\bar\alpha\mathbf{1}^T\boldsymbol{\sigma_D}(W_s^D-W_t^D)+\varepsilon\int_t^s(\dots)dW_u^\mu}\times$$

$$\times(\mu_D^i(s-t)+\varepsilon e^{i^T}\boldsymbol{\sigma_z}(I-e^{-A(s-t)})A^{-1}\mathbf{z}_t+e^{i^T}\boldsymbol{\sigma_D}(W_s^D-W_t^D)+\varepsilon\int_t^s(\dots)dW_u^\mu)ds\Big]$$

$$\approx\int_t^\infty e^{-\bar r(s-t)-\varepsilon\bar\alpha\mathbf{1}^T\boldsymbol{\sigma_z}(I-e^{-A(s-t)})A^{-1}z_t}\Big(\mu_D^i(s-t)+\varepsilon e^{i^T}\boldsymbol{\sigma_z}(I-e^{-A(s-t)})A^{-1}\mathbf{z}_t-\bar\alpha e^{i^T}\boldsymbol{\sigma_D}\boldsymbol{\sigma_D}^T1(s-t)\Big)ds$$

$$\approx\int_t^\infty e^{-\bar r(s-t)}\Big(\Big(\mu_D^i(s-t)-\bar\alpha e^{i^T}\boldsymbol{\sigma_D}\boldsymbol{\sigma_D}^T\mathbf{1}(s-t)\Big)(1-\varepsilon\bar\alpha\mathbf{1}^T\boldsymbol{\sigma_z}(I-e^{-A(s-t)})A^{-1}\mathbf{z}_t)$$

$$+\varepsilon e^{i^T}\boldsymbol{\sigma_z}(I-e^{-A(s-t)})A^{-1}\mathbf{z}_t\Big)ds$$

$$=\frac1{\bar r^2}\Big(\mu_D^i-\bar\alpha e^{i^T}\boldsymbol{\sigma_D}\boldsymbol{\sigma_D}^T\mathbf{1}\Big)+\varepsilon e^{i^T}\boldsymbol{\sigma_z}\left(\frac1{\bar r}I-(\bar rI+A)^{-1}\right)A^{-1}\mathbf{z}_t$$

$$-\varepsilon\Big(\mu_D^i-\bar\alpha e^{i^T}\boldsymbol{\sigma_D}\boldsymbol{\sigma_D}^T\mathbf{1}\Big)\bar\alpha\mathbf{1}^T\boldsymbol{\sigma_z}\left(\frac1{\bar r^2}I-\int_0^\infty se^{-(\bar rI+A)s}ds\right)A^{-1}\mathbf{z}_t$$

Hence,

$$R_t^i=S_t^i-P_t^i=\left(\frac{\bar\alpha}{\bar r^2}-\varepsilon\bar\alpha^2\mathbf{1}^T\boldsymbol{\sigma_z}\left(\frac1{\bar r^2}I-\int_0^\infty se^{-(\bar rI+A)s}ds\right)A^{-1}\mathbf{z}_t\right)e^{i^T}\boldsymbol{\sigma_D}\boldsymbol{\sigma_D}^T\mathbf{1}$$

$$R_t^m=\mathbf{1}^T(S_t-P_t)=\left(\frac{\bar\alpha}{\bar r^2}-\varepsilon\bar\alpha^2\mathbf{1}^T\boldsymbol{\sigma_z}\left(\frac1{\bar r^2}I-\int_0^\infty se^{-(\bar rI+A)s}ds\right)A^{-1}\mathbf{z}_t\right)\mathbf{1}^T\boldsymbol{\sigma_D}\boldsymbol{\sigma_D}^T\mathbf{1}.$$

Therefore,

$$R_t^i = \frac{(\sigma_D^i)^T \boldsymbol{\sigma_D}^T \mathbf{1}}{\mathbf{1}^T \boldsymbol{\sigma_D}\boldsymbol{\sigma_D}^T \mathbf{1}} R_t^m.$$

To compute the market beta of each asset, recall that W^D and W^μ are independent Brownian motions, hence $d\langle D^i, z^j\rangle_t = 0$ for any i, j and

$$d\langle P^i, P^j\rangle_t = \left(\frac{1}{\bar r^2} - 2\varepsilon \frac{1}{\bar r}\bar\alpha \mathbf{1}^T \boldsymbol{\sigma_z}\left(\frac{1}{\bar r}I - (\bar r I + A)^{-1}\right)A^{-1}\boldsymbol{z}\right) e^{iT} \boldsymbol{\sigma_D}\boldsymbol{\sigma_D}^T e^j\, dt + o(\varepsilon).$$

As $P^m = \mathbf{1}^T P$, it follows that

$$Beta_i := \frac{d\langle P^i, P^m\rangle_t}{d\langle P^m\rangle_t} = \frac{(\sigma_D^i)^\top \boldsymbol{\sigma_D}^\top \mathbf{1}}{\mathbf{1}^\top \boldsymbol{\sigma_D}\boldsymbol{\sigma_D}^\top \mathbf{1}}. \quad \square$$

Proof of Theorem 4.7. The result follows from similar computations as in the sketch of the proof of Theorem 4.6. $\square$

Proof of Proposition 4.8. Similar computations as in the sketch of the proof of Theorem 4.6 lead to the following expression for the vector of stock prices at first order:

$$\begin{aligned}
P_t = D_t & \left(\frac{1}{\bar r} + \bar\alpha 1^T \boldsymbol{\sigma_D}\, \varepsilon\bar\alpha 1^T \boldsymbol{\sigma_z} \frac{1}{\bar r^2(a+\bar r)}\boldsymbol{\rho} - \varepsilon\bar\alpha 1^T \boldsymbol{\sigma_z}\frac{1}{\bar r(a+\bar r)} z_t\right) \\
& - \varepsilon\bar\alpha \frac{1}{\bar r^2(a+\bar r)}\left[\boldsymbol{\sigma_D}\boldsymbol{\rho}\boldsymbol{\sigma_z}^T 1 + \boldsymbol{\sigma_z}\boldsymbol{\rho}^T\boldsymbol{\sigma_D}^T 1\right] + \varepsilon \frac{1}{\bar r(a+\bar r)}\boldsymbol{\sigma_z} z_t \\
& + \left[\frac{1}{\bar r^2} - \bar\alpha 1^T \boldsymbol{\sigma_z}\,\varepsilon \frac{z_t}{a}\left(\frac{1}{\bar r^2} - \frac{1}{(a+\bar r)^2}\right) + \frac{\bar\alpha^2}{a}\varepsilon 1^T \boldsymbol{\sigma_z}\left(\frac{2}{\bar r^3} - \frac{1}{a\bar r^2} + \frac{1}{a(a+\bar r)^2}\right)\boldsymbol{\rho}^T\boldsymbol{\sigma_D}^T 1\right] \times \\
& \times (\bar{\boldsymbol{\mu}}_{\boldsymbol{D}} - \bar\alpha \boldsymbol{\sigma_D}\boldsymbol{\sigma_D}^T 1).
\end{aligned}$$

Differentiating this expression yields the dynamics of P_t. $\square$

Proof of Corollary 4.9. The correlation follows explicitly from the dynamics of the stock price in Proposition 4.8. $\square$

Proof of Proposition 4.10. The vector of stock prices is of the form

$$P_t = D_t \cdot (f + g z_t + h z_t^2) + (\mathbf{a} + \mathbf{b}\, z_t + \mathbf{c}\, z_t^2),$$

for some scalars f, g, h and vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, where g, $\mathbf{b}$ are of order ε and h, $\mathbf{c}$ of order ε^2. Hence,

$$dP_t = (\dots)dt + (f + g z_t + h z_t^2) dD_t + [D_t \cdot (g + 2h z_t) + (\mathbf{b} + 2\mathbf{c}\, z_t)] dz_t,$$

and

$$\begin{aligned}
\frac{d\langle P, P\rangle_t}{dt} = & (f + g z_t + h z_t^2)^2 \Sigma_D + (f + g z_t + h z_t^2)\boldsymbol{\sigma_D}\boldsymbol{\rho}[D_t \cdot (g + 2h z_t) + (\mathbf{b} + 2\mathbf{c}\, z_t)]^T \\
& + (f + g z_t + h z_t^2)[D_t \cdot (g + 2h z_t) + (\mathbf{b} + 2\mathbf{c}\, z_t)]\boldsymbol{\rho}^T \boldsymbol{\sigma_D}^T \\
& + [D_t \cdot (g + 2h z_t) + (\mathbf{b} + 2\mathbf{c}\, z_t)][D_t \cdot (g + 2h z_t) + (\mathbf{b} + 2\mathbf{c}\, z_t)]^T,
\end{aligned}$$

where $\Sigma_D = \boldsymbol{\sigma_D}\boldsymbol{\sigma_D}^T$. Notice that $\lim_{\varepsilon\downarrow 0} f = \frac{1}{\bar r}$. From

$$\begin{aligned}
E_t\left[\int_t^\infty \frac{M_s^i}{M_t^i} ds\right] & \approx E_t\Big[\int_t^\infty e^{-\bar\beta(s-t) - \bar\alpha 1^T(D_s - D_t) + \varepsilon(Y_s^{i,1} - Y_t^{i,1}) + \varepsilon^2(Y_s^{i,2} - Y_t^{i,2})} ds\Big] \\
& = E_t\Big[\int_t^\infty e^{-\bar\beta(s-t) - \bar\alpha 1^T \bar{\boldsymbol{\mu}}_{\boldsymbol{D}}(s-t) - \bar\alpha 1^T \boldsymbol{\sigma_D}(W_s^D - W_t^D) - \varepsilon\bar\alpha 1^T\left(\frac{\boldsymbol{\sigma_z} z_t}{a}(1 - e^{-a(s-t)}) + \frac{\boldsymbol{\sigma_z}}{a}\int_t^s (1 - e^{-a(s-u)}) dW_u^\mu\right) + \varepsilon(Y_s^{i,1} - Y_t^{i,1}) + \varepsilon^2(Y_s^{i,2} - Y_t^{i,2})} ds\Big] \\
& = (\dots) - \varepsilon\bar\alpha 1^T \boldsymbol{\sigma_z}\frac{z_t}{a} E_t\Big[\int_t^\infty e^{-\bar\beta(s-t) - \bar\alpha 1^T \bar{\boldsymbol{\mu}}_{\boldsymbol{D}}(s-t) - \bar\alpha 1^T \boldsymbol{\sigma_D}(W_s^D - W_t^D)}(1 - e^{-a(s-t)}) ds\Big] \\
& + \frac{1}{2}\varepsilon^2\bar\alpha^2 (1^T \boldsymbol{\sigma_z})^2 \frac{z_t^2}{a^2} E_t\Big[\int_t^\infty e^{-\bar\beta(s-t) - \bar\alpha 1^T \bar{\boldsymbol{\mu}}_{\boldsymbol{D}}(s-t) - \bar\alpha 1^T \boldsymbol{\sigma_D}(W_s^D - W_t^D)}(1 - e^{-a(s-t)})^2 ds\Big] \\
& = (\dots) - \varepsilon\bar\alpha 1^T \boldsymbol{\sigma_z}\frac{z_t}{a}\left(\frac{1}{\bar r} - \frac{1}{a + \bar r}\right) + \frac{1}{2}\varepsilon^2\bar\alpha^2 (1^T \boldsymbol{\sigma_z})^2 \frac{z_t^2}{a^2}\left(\frac{1}{\bar r} - 2\frac{1}{a + \bar r} + \frac{1}{2a + \bar r}\right)
\end{aligned}$$

$$= (\dots) - \varepsilon\bar{\alpha}1^T \boldsymbol{\sigma_z} z_t \frac{1}{\bar{r}(a+\bar{r})} + \varepsilon^2\bar{\alpha}^2(1^T \boldsymbol{\sigma_z})^2 z_t^2 \frac{1}{\bar{r}(a+\bar{r})(2a+\bar{r})},$$

it follows that $g = -\varepsilon\bar{\alpha}1^T \boldsymbol{\sigma_z} \frac{1}{\bar{r}(a+\bar{r})} + o(\varepsilon^2)$ and $h = \varepsilon^2\bar{\alpha}^2(1^T \boldsymbol{\sigma_z})^2 \frac{1}{\bar{r}(a+\bar{r})(2a+\bar{r})} + o(\varepsilon^2)$. Similarly, from

$$E_t\left[\int_t^\infty (D_s - D_t)\frac{M_s^i}{M_t^i} ds\right] \approx E_t\Big[\int_t^\infty (D_s - D_t) e^{-\bar{\beta}(s-t)-\bar{\alpha}1^T(D_s-D_t)+\varepsilon(Y_s^{i,1}-Y_t^{i,1})+\varepsilon^2(Y_s^{i,2}-Y_t^{i,2})} ds\Big]$$

$$= E_t\Big[\int_t^\infty (D_s - D_t) e^{-\bar{\beta}(s-t)-\bar{\alpha}1^T \bar{\boldsymbol{\mu}}_{\boldsymbol{D}}(s-t)-\bar{\alpha}1^T \boldsymbol{\sigma_D}(W_s^D-W_t^D)}.$$

$$\cdot e^{-\varepsilon\bar{\alpha}1^T(\frac{\boldsymbol{\sigma_z} z_t}{a}(1-e^{-a(s-t)})+\frac{\boldsymbol{\sigma_z}}{a}\int_t^s(1-e^{-a(s-u)})dW_u^\mu)+\varepsilon(Y_s^{i,1}-Y_t^{i,1})+\varepsilon^2(Y_s^{i,2}-Y_t^{i,2})} ds\Big]$$

$$= E_t\Big[\int_t^\infty (\bar{\boldsymbol{\mu}}_{\boldsymbol{D}}(s-t) + \boldsymbol{\sigma_D}(W_s^D - W_t^D) + \varepsilon\frac{\boldsymbol{\sigma_z} z_t}{a}(1-e^{-a(s-t)}) + \varepsilon\frac{\boldsymbol{\sigma_z}}{a}\int_t^s (1-e^{-a(s-u)})[\boldsymbol{\rho}^T dW_u^D + \sqrt{1-|\boldsymbol{\rho}|^2} d\tilde{W}_u^\mu])$$

$$e^{-\bar{\beta}(s-t)-\bar{\alpha}1^T \bar{\boldsymbol{\mu}}_{\boldsymbol{D}}(s-t)-\bar{\alpha}1^T \boldsymbol{\sigma_D}(W_s^D-W_t^D)} e^{-\varepsilon\bar{\alpha}1^T(\frac{\boldsymbol{\sigma_z} z_t}{a}(1-e^{-a(s-t)})+\frac{\boldsymbol{\sigma_z}}{a}\int_t^s(1-e^{-a(s-u)})dW_u^\mu)+\varepsilon(Y_s^{i,1}-Y_t^{i,1})+\varepsilon^2(Y_s^{i,2}-Y_t^{i,2})} ds\Big]$$

$$= (\dots) + E_t\Big[\int_t^\infty\int_t^s (\boldsymbol{\sigma_D} + \varepsilon\frac{\boldsymbol{\sigma_z}}{a}(1-e^{-a(s-u)})\boldsymbol{\rho}^T) dW_u^D e^{-\bar{\beta}(s-t)-\bar{\alpha}1^T \bar{\boldsymbol{\mu}}_{\boldsymbol{D}}(s-t)-\bar{\alpha}1^T \boldsymbol{\sigma_D}(W_s^D-W_t^D)} e^{-\varepsilon\bar{\alpha}1^T \frac{\boldsymbol{\sigma_z} z_t}{a}(1-e^{-a(s-t)})} ds\Big]$$

$$+ E_t\Big[\int_t^\infty (\bar{\boldsymbol{\mu}}_{\boldsymbol{D}}(s-t) + \varepsilon\frac{\boldsymbol{\sigma_z} z_t}{a}(1-e^{-a(s-t)})) e^{-\bar{\beta}(s-t)-\bar{\alpha}1^T \bar{\boldsymbol{\mu}}_{\boldsymbol{D}}(s-t)-\bar{\alpha}1^T \boldsymbol{\sigma_D}(W_s^D-W_t^D)} e^{-\varepsilon\bar{\alpha}1^T \frac{\boldsymbol{\sigma_z} z_t}{a}(1-e^{-a(s-t)})} ds\Big]$$

$$= (\dots) - \bar{\alpha}\int_t^\infty\int_t^s (\boldsymbol{\sigma_D}\boldsymbol{\sigma_D}^T 1 + \varepsilon\frac{\boldsymbol{\sigma_z}}{a}(1-e^{-a(s-u)})\boldsymbol{\rho}^T \boldsymbol{\sigma_D}^T 1) du e^{-\bar{r}(s-t)} e^{-\varepsilon\bar{\alpha}1^T \frac{\boldsymbol{\sigma_z} z_t}{a}(1-e^{-a(s-t)})} ds$$

$$+ \int_t^\infty (\bar{\boldsymbol{\mu}}_{\boldsymbol{D}}(s-t) + \varepsilon\frac{\boldsymbol{\sigma_z} z_t}{a}(1-e^{-a(s-t)})) e^{-\bar{r}(s-t)} e^{-\varepsilon\bar{\alpha}1^T \frac{\boldsymbol{\sigma_z} z_t}{a}(1-e^{-a(s-t)})} ds$$

$$= (\dots) + (\bar{\boldsymbol{\mu}}_{\boldsymbol{D}} - \bar{\alpha}\boldsymbol{\sigma_D}\boldsymbol{\sigma_D}^T 1)\int_t^\infty (s-t) e^{-\bar{r}(s-t)} e^{-\varepsilon\bar{\alpha}1^T \boldsymbol{\sigma_z} \frac{z_t}{a}(1-e^{-a(s-t)})} ds$$

$$+ \varepsilon\frac{\boldsymbol{\sigma_z} z_t}{a}\int_t^\infty (1-e^{-a(s-t)}) e^{-\bar{r}(s-t)} ds - \varepsilon^2\frac{z_t^2}{a^2}\bar{\alpha}(1^T \boldsymbol{\sigma_z})\boldsymbol{\sigma_z}\int_t^\infty e^{-\bar{r}(s-t)}(1-e^{-a(s-t)})^2 ds$$

$$+ \varepsilon^2\bar{\alpha}^2\frac{1}{a^2}(1^T \boldsymbol{\sigma_z}) z_t \boldsymbol{\rho}^T \boldsymbol{\sigma_D}^T 1 \boldsymbol{\sigma_z}\int_t^\infty\int_t^s (1-e^{-a(s-u)}) du e^{-\bar{r}(s-t)}(1-e^{-a(s-t)})) ds$$

$$= (\dots) + (\bar{\boldsymbol{\mu}}_{\boldsymbol{D}} - \bar{\alpha}\boldsymbol{\sigma_D}\boldsymbol{\sigma_D}^T 1)\int_t^\infty (s-t) e^{-\bar{r}(s-t)} e^{-\varepsilon\bar{\alpha}1^T \boldsymbol{\sigma_z} \frac{z_t}{a}(1-e^{-a(s-t)})} ds$$

$$+ \varepsilon\frac{\boldsymbol{\sigma_z} z_t}{a}\left(\frac{1}{\bar{r}} - \frac{1}{a+\bar{r}}\right) - \varepsilon^2\frac{z_t^2}{a^2}\bar{\alpha}(1^T \boldsymbol{\sigma_z})\boldsymbol{\sigma_z}\int_t^\infty e^{-\bar{r}(s-t)}(1-e^{-a(s-t)})^2 ds$$

$$+ \varepsilon^2\bar{\alpha}^2\frac{1}{a^2}(1^T \boldsymbol{\sigma_z}) z_t \boldsymbol{\rho}^T \boldsymbol{\sigma_D}^T 1 \boldsymbol{\sigma_z} \frac{a^2(2a+3\bar{r})}{\bar{r}^2(a+\bar{r})^2(2a+\bar{r})}$$

$$= (\dots) + (\bar{\boldsymbol{\mu}}_{\boldsymbol{D}} - \bar{\alpha}\boldsymbol{\sigma_D}\boldsymbol{\sigma_D}^T 1)\Big[-\varepsilon\bar{\alpha}1^T \boldsymbol{\sigma_z}\frac{z_t}{a}\left(\frac{1}{\bar{r}^2} - \frac{1}{(a+\bar{r})^2}\right) + \varepsilon^2\bar{\alpha}^2(1^T \boldsymbol{\sigma_z})^2 z_t^2 \frac{2a^2+6a\bar{r}+3\bar{r}^2}{\bar{r}^2(a+\bar{r})^2(2a+\bar{r})^2}\Big]$$

$$+ \varepsilon\frac{\boldsymbol{\sigma_z} z_t}{a}\left(\frac{1}{\bar{r}} - \frac{1}{a+\bar{r}}\right) - \varepsilon^2\frac{z_t^2}{a^2}\bar{\alpha}(1^T \boldsymbol{\sigma_z})\boldsymbol{\sigma_z}\left(\frac{1}{\bar{r}} - \frac{2}{a+\bar{r}} + \frac{1}{2a+\bar{r}}\right)$$

$$+ \varepsilon^2\bar{\alpha}^2(1^T \boldsymbol{\sigma_z}) z_t \boldsymbol{\rho}^T \boldsymbol{\sigma_D}^T 1 \boldsymbol{\sigma_z} \frac{2a+3\bar{r}}{\bar{r}^2(a+\bar{r})^2(2a+\bar{r})}$$

$$= (\dots) + (\bar{\boldsymbol{\mu}}_{\boldsymbol{D}} - \bar{\alpha}\boldsymbol{\sigma_D}\boldsymbol{\sigma_D}^T 1)\Big[-\varepsilon\bar{\alpha}1^T \boldsymbol{\sigma_z} z_t \frac{1}{\bar{r}(a+\bar{r})}\left(\frac{1}{\bar{r}} + \frac{1}{a+\bar{r}}\right) + \varepsilon^2\bar{\alpha}^2(1^T \boldsymbol{\sigma_z})^2 z_t^2 \frac{2a^2+6a\bar{r}+3\bar{r}^2}{\bar{r}^2(a+\bar{r})^2(2a+\bar{r})^2}\Big]$$

$$+\varepsilon\boldsymbol{\sigma_z} z_t \frac{1}{\bar r(a+\bar r)} - \varepsilon^2 z_t^2 \bar\alpha(1^T\boldsymbol{\sigma_z})\boldsymbol{\sigma_z}\frac{2}{\bar r(a+\bar r)(2a+\bar r)} + \varepsilon^2\bar\alpha^2(1^T\boldsymbol{\sigma_z})z_t\boldsymbol{\rho}^T\boldsymbol{\sigma_D}^T 1\boldsymbol{\sigma_z}\frac{2a+3\bar r}{\bar r^2(a+\bar r)^2(2a+\bar r)},$$

one obtains

$$\begin{aligned}\mathbf{b} &= -\varepsilon(\bar{\boldsymbol{\mu}}_{\boldsymbol D} - \bar\alpha\boldsymbol{\sigma_D}\boldsymbol{\sigma_D}^T 1)\bar\alpha 1^T\boldsymbol{\sigma_z}\frac{1}{\bar r(a+\bar r)}\left(\frac{1}{\bar r}+\frac{1}{a+\bar r}\right) + \varepsilon\boldsymbol{\sigma_z}\frac{1}{\bar r(a+\bar r)}\\ &\quad + \varepsilon^2\bar\alpha^2(1^T\boldsymbol{\sigma_z})\boldsymbol{\rho}^T\boldsymbol{\sigma_D}^T 1\boldsymbol{\sigma_z}\frac{2a+3\bar r}{\bar r^2(a+\bar r)^2(2a+\bar r)} + o(\varepsilon^2),\\ \mathbf{c} &= \varepsilon^2(\bar{\boldsymbol{\mu}}_{\boldsymbol D} - \bar\alpha\boldsymbol{\sigma_D}\boldsymbol{\sigma_D}^T 1)\bar\alpha^2(1^T\boldsymbol{\sigma_z})^2\frac{2a^2+6a\bar r+3\bar r^2}{\bar r^2(a+\bar r)^2(2a+\bar r)^2} - \varepsilon^2\bar\alpha(1^T\boldsymbol{\sigma_z})\boldsymbol{\sigma_z}\frac{2}{\bar r(a+\bar r)(2a+\bar r)} + o(\varepsilon^2).\end{aligned}$$

Define

$$\mathbf{v}_t := \frac{\bar\alpha(1^T\boldsymbol{\sigma_z})}{\bar r(a+\bar r)^2(2a+\bar r)}\left(\bar\alpha(1^T\boldsymbol{\sigma_z})D_t + (\bar{\boldsymbol{\mu}}_{\boldsymbol D} - \bar\alpha\boldsymbol{\sigma_D}\boldsymbol{\sigma_D}^T 1)\bar\alpha(1^T\boldsymbol{\sigma_z})\left(\frac{2}{\bar r}+\frac{1}{a+\bar r}+\frac{1}{2a+\bar r}\right) - \frac{2a+3\bar r}{\bar r}\boldsymbol{\sigma_z}\right)$$

and

$$B := \boldsymbol{\sigma_D}\boldsymbol{\rho}\mathbf{v}_t^T + \mathbf{v}_t\boldsymbol{\rho}^T\boldsymbol{\sigma_D}^T.$$

Let $\rho_{i,j} = \frac{d\langle P^i,P^j\rangle_t/dt}{\sqrt{d\langle P^i,P^i\rangle_t/dt}\sqrt{d\langle P^j,P^j\rangle_t/dt}}$ and $\rho^{(2)}_{i,j}$ its asymptotic expansion in ε at second order. An explicit computation shows that $\rho^{(2)}_{i,j}$ is linear in z_t with coefficient equal to

$$\frac{\bar r^2}{\sqrt{\Sigma_D^{ii}\Sigma_D^{jj}}}\left(B^{ij} - \frac{\Sigma_D^{ij}}{\Sigma_D^{ii}}\frac{B^{ii}}{2} - \frac{\Sigma_D^{ij}}{\Sigma_D^{jj}}\frac{B^{jj}}{2}\right). \quad \square$$

Appendix F. Complete equilibrium

In this section describes the complete-market equilibrium (for $\varepsilon > 0$) in which a consol bond C_t that pays a constant coupon flow equal to 1 is available to trade.

The proof relies on stochastic control arguments and is omitted.

Theorem F.1. *Assume $\bar r > 0$. A second-order equilibrium for the complete market where the consol bond is tradeable consists of the following interest rate r_t, stock price P_t, consol bond price C_t, optimal consumption $\hat c_t^i$, optimal stock holding $\hat\theta_t^{P,i}$ and optimal consol bond holding $\hat\theta_t^{C,i}$:*

$$r_t = \bar r + \varepsilon\bar\alpha\sigma_z z_t, \tag{F.1}$$

$$P_t = D_t\cdot C_t + (\mu_D - \bar\alpha\sigma_D^2)\left(L_t - t\cdot C_t\right) + \varepsilon\frac{1}{\bar r(a+\bar r)}\sigma_z z_t - \varepsilon^2\frac{2\bar\alpha\sigma_z^2}{\bar r(a+\bar r)(2a+\bar r)}\left(z_t^2+\frac{1}{\bar r}\right), \tag{F.2}$$

$$C_t = \frac{1}{\bar r} - \varepsilon\frac{\bar\alpha}{\bar r(a+\bar r)}\sigma_z z_t + \varepsilon^2\frac{\bar\alpha^2\sigma_z^2}{\bar r(a+\bar r)(2a+\bar r)}\left(z_t^2+\frac{1}{\bar r}\right), \tag{F.3}$$

$$\hat\theta_t^{P,i} = \frac{\bar\alpha}{\alpha_i}, \tag{F.4}$$

$$\begin{aligned}\hat\theta_t^{C,i} &= \left(x_i - \frac{\bar\alpha}{\alpha_i}P_0\right)\frac{1}{C_0} + \frac{\bar\beta-\beta_i}{\alpha_i}\Bigg(t + \frac{1}{a+\bar r} + \varepsilon\frac{\bar\alpha\sigma_z(2z_t - z_0)}{(a+\bar r)^2} + \varepsilon^2\frac{\bar\alpha^2\sigma_z^2(4z_t^2 - z_0^2)}{(a+\bar r)^3}\\ &\quad - \varepsilon^2\frac{\bar\alpha^2\sigma_z^2\left(2a^3+8a^2\bar r+9a\bar r^2+3\bar r^3 - a\bar r^2(3\bar r+5a)z_0^2\right)}{\bar r^2(a+\bar r)^3(2a+\bar r)^2}\Bigg),\end{aligned} \tag{F.5}$$

$$\hat c_t^i = \frac{\bar\alpha}{\alpha_i}D_t + \frac{1}{C_0}\left(x_i - \frac{\bar\alpha}{\alpha_i}P_0\right) + \frac{\bar\beta-\beta_i}{\alpha_i}\left(t - \frac{L_0}{C_0}\right), \tag{F.6}$$

where

$$L_t = t\cdot C_t + \frac{1}{\bar r^2} - \varepsilon\left(\frac{1}{\bar r}+\frac{1}{a+\bar r}\right)\frac{\bar\alpha\sigma_z}{\bar r(a+\bar r)}z_t + \varepsilon^2\frac{\bar\alpha^2\sigma_z^2}{\bar r^2(a+\bar r)(2a+\bar r)}\left(\frac{1}{\bar r} + \frac{2a^2+6ar+3r^2}{(a+\bar r)(2a+\bar r)}\left(z_t^2+\frac{1}{\bar r}\right)\right).$$

Remark F.2. Note that the interest rate in (F.1) is *exact* – not merely asymptotic. Since z_t follows an Ornstein-Uhlenbeck process, the resulting interest rate follows a Vasicek model, in analogy to the equilibrium interest rate in the single-agent, complete model by Goldstein and Zapatero (1996), where the dividend process also exhibits mean-reverting drift. Moreover, the equilibrium stock holdings are also exact, with each agent holding precisely $\bar\alpha/\alpha_i$ shares, whereas the remaining equilibrium quantities in Theorem F.1 are asymptotic.

Appendix G. Numerical method

This section adapts the numerical method in Dumas and Lyasoff (2012) to compute the market equilibrium in the incomplete model in this paper. The model assumes two agents with identical risk-aversion α and time-preference rate β, but different initial wealth allocations x_1 and x_2.

First, set a finite time horizon T and an equally-spaced time grid $t_0 = 0 < t_1 < \cdots < t_n = T$, where $t_i := i \cdot \Delta t$ and $\Delta t = \frac{T}{n}$. Applying the Euler scheme to the exogenous processes D_t and z_t yields the discrete dynamics

$$D_{t_{i+1}} = D_{t_i} + \left(\mu_D + \varepsilon\sigma_z z_{t_i}\right)\Delta t + \sigma_D \Delta W^D,$$
$$z_{t_{i+1}} = z_{t_i} - a z_{t_i}\Delta t + \Delta W^\mu,$$

where $i = 0, \dots, n-1$. These processes are approximated with a two-dimensional recombining binomial tree: define D_t^* and z_t^* as

$$\begin{bmatrix} D^*_{t_{i+1}} \\ z^*_{t_{i+1}} \end{bmatrix} = \begin{bmatrix} D^*_{t_i} + \mu_D \Delta t \\ z^*_{t_i} \end{bmatrix} + \xi_i \cdot \sqrt{\Delta t},$$

where $\{\xi_i\}_{i=0}^{n-1}$ are independent random variables such that ξ_i takes values $\begin{bmatrix} \sigma_D \\ 1 \end{bmatrix}, \begin{bmatrix} \sigma_D \\ -1 \end{bmatrix}, \begin{bmatrix} -\sigma_D \\ 1 \end{bmatrix}, \begin{bmatrix} -\sigma_D \\ -1 \end{bmatrix}$ with probabilities, respectively, p_i^{++}, p_i^{+-}, p_i^{-+}, p_i^{--}. These probabilities are chosen such that the first two moments of the increments of D_t^* and z_t^* match those of the increments of D_t and z_t, i.e.,

$$p_i^{++}\begin{bmatrix} \sigma_D \\ 1 \end{bmatrix}\sqrt{\Delta t} + p_i^{+-}\begin{bmatrix} \sigma_D \\ -1 \end{bmatrix}\sqrt{\Delta t} + p_i^{-+}\begin{bmatrix} -\sigma_D \\ 1 \end{bmatrix}\sqrt{\Delta t} + p_i^{--}\begin{bmatrix} -\sigma_D \\ -1 \end{bmatrix}\sqrt{\Delta t} = \begin{bmatrix} \varepsilon\sigma_z z^*_{t_i}\Delta t \\ -a z^*_{t_i}\Delta t \end{bmatrix},$$
$$\begin{bmatrix} \sigma_D^2 \Delta t & (p_i^{++} - p_i^{+-} - p_i^{-+} + p_i^{--})\sigma_D \Delta t \\ (p_i^{++} - p_i^{+-} - p_i^{-+} + p_i^{--})\sigma_D \Delta t & \Delta t \end{bmatrix} = \begin{bmatrix} \sigma_D^2 \Delta t & 0 \\ 0 & \Delta t \end{bmatrix}.$$

Furthermore, probabilities need to sum up to 1, i.e., $p_i^{++} + p_i^{+-} + p_i^{-+} + p_i^{--} = 1$. Therefore,

$$p_i^{++} = \frac{1}{4} + \frac{1}{4}\left(\frac{\varepsilon\sigma_z}{\sigma_D} z^*_{t_i} - a z^*_{t_i}\right)\sqrt{\Delta t}, \quad p_i^{+-} = \frac{1}{4} + \frac{1}{4}\left(\frac{\varepsilon\sigma_z}{\sigma_D} z^*_{t_i} + a z^*_{t_i}\right)\sqrt{\Delta t},$$
$$p_i^{-+} = \frac{1}{4} - \frac{1}{4}\left(\frac{\varepsilon\sigma_z}{\sigma_D} z^*_{t_i} + a z^*_{t_i}\right)\sqrt{\Delta t}, \quad p_i^{--} = \frac{1}{4} - \frac{1}{4}\left(\frac{\varepsilon\sigma_z}{\sigma_D} z^*_{t_i} - a z^*_{t_i}\right)\sqrt{\Delta t}.$$

The recursive algorithm to compute the market equilibrium is adapted from Dumas and Lyasoff (2012) and proceeds as follows. Fix a grid of values Ξ covering the interval $(0,1)$ representing Ann's consumption share at some time t, i.e., $\frac{c_t^1}{D_t}$. (In the continuous-time model, dividends may become negative. The numerical simulations in this paper are based on parameters values for which dividends always remain positive.) Note that in the current setting $\bar{\alpha} = \frac{\alpha}{2}$ and $\bar{\beta} = \beta$.

1. For each terminal node (D_T^*, z_T^*) and $\omega \in (0,1)$, set $P_T(\omega) = P^{comp}(z_T^*, D_T^*)$ and $X_T^1(\omega) = \omega \cdot D_T^* \cdot C^{comp}(z_T^*) + \frac{1}{2}(P^{comp}(z_T^*, D_T^*) - D_T^* \cdot C^{comp}(z_T^*))$, where P^{comp} and C^{comp} are the prices of the stock and the consol bond in the complete equilibrium defined in Theorem F.1. In other words, the stock price and Ann's wealth at time T are set at the corresponding values in the complete model where also consol bonds are traded. The value of ω represents $\frac{c_T^1}{D_T^*}$, the current optimal consumption share of Ann, which is yet to be determined.
2. Fix a node $(D^*_{t_{n-1}}, z^*_{t_{n-1}})$ at time t_{n-1} and a value $\omega \in \Xi$, which represents $\frac{c^1_{t_{n-1}}}{D^*_{t_{n-1}}}$. Let $\eta \in \{++, +-, -+, --\}$ index the four subsequent nodes for each of the four possible values of the random variable ξ_{n-1}, e.g., $D^*_{t_n,++}$ is the value of the dividend for $\xi_{n-1} = \begin{bmatrix} \sigma_D \\ 1 \end{bmatrix}$.
Thus, solve a system of six equations in the six variables $c^1_{t_n,\eta}$ for $\eta \in \{++, +-, -+, --\}$, $\theta^1_{t_{n-1}}$ and $\theta^{rf,1}_{t_{n-1}}$. The variables $\{c^1_{t_n,\eta}\}_\eta$ are Ann's optimal consumption rates at time t_n in each of the four subsequent nodes, $\theta^1_{t_{n-1}}$ is Ann's optimal holdings in the stock at time t_{n-1}, and $\theta^{rf,1}_{t_{n-1}}$ is the dollar value (at the next time step) of Ann's optimal position in the safe asset at time t_{n-1}. Note that Bob's consumption rate follows from the market clearing condition on consumption, i.e., $c^2_{t_n,\eta} = D^*_{t_n,\eta} - c^1_{t_n,\eta}$. The agents' marginal utilities are, respectively, $\frac{M^1_{t_n,\eta}}{M^1_{t_{n-1}}} = e^{-\beta\Delta t - \alpha(c^1_{t_n,\eta} - \omega \cdot D^*_{t_{n-1}})}$ and $\frac{M^2_{t_n,\eta}}{M^2_{t_{n-1}}} = e^{-\beta\Delta t - \alpha(c^2_{t_n,\eta} - (1-\omega)\cdot D^*_{t_{n-1}})}$.
The system of six equations is therefore:

$$X^1_{t_n,\eta}\left(\frac{c^1_{t_n,\eta}}{D^*_{t_n,\eta}}\right) = \theta^1_{t_{n-1}}\left(P_{t_n,\eta}\left(\frac{c^1_{t_n,\eta}}{D^*_{t_n,\eta}}\right) + D^*_{t_n,\eta}\Delta t\right) + \theta^{rf,1}_{t_{n-1}} - c^1_{t_n,\eta}\Delta t, \tag{G.1}$$

for $\eta \in \{++, +-, -+, --\}$,

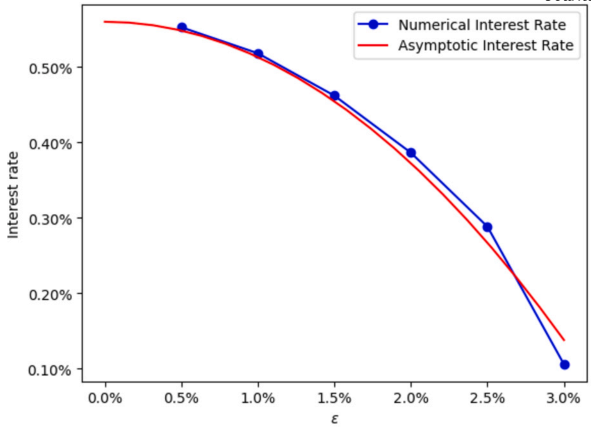

Fig. 4. Interest rate (vertical) computed numerically (marked) and from the closed-form expressions (solid) in the second-order equilibrium against ε (horizontal).

$$\frac{E_{t_{n-1}}[M^1_{t_n}(P_{t_n}(c^1_{t_n}/D^*_{t_n}) + D^*_{t_n}\Delta t)]}{M^1_{t_{n-1}}} = \frac{E_{t_{n-1}}[M^2_{t_n}(P_{t_n}(c^1_{t_n}/D^*_{t_n}) + D^*_{t_n}\Delta t)]}{M^2_{t_{n-1}}}, \tag{G.2}$$

$$-\frac{E_{t_{n-1}}[M^1_{t_n}] - M^1_{t_{n-1}}}{M^1_{t_{n-1}}\Delta t} = -\frac{E_{t_{n-1}}[M^2_{t_n}] - M^2_{t_{n-1}}}{M^2_{t_{n-1}}\Delta t}, \tag{G.3}$$

where the functions $X^1_{t_n,\eta}(\cdot)$ and $P_{t_n,\eta}(\cdot)$ have been computed in the previous step of the algorithm and the conditional expectation $E_{t_{n-1}}[\cdot]$ is taken over the four subsequent states $\eta \in \{++,+-,-+,--\}$ with the corresponding probabilities $p^{++}_{n-1}, p^{+-}_{n-1}, p^{-+}_{n-1}, p^{--}_{n-1}$. Equations (G.1) are Ann's budget equations for each of the four subsequent nodes, and equations (G.2) and (G.3) state that agents agree on stock price and interest rate. In particular, for the node $(D^*_{t_{n-1}}, z^*_{t_{n-1}})$ and value $\omega \in \Xi$ the stock price and Ann's wealth are

$$P_{t_{n-1}}(\omega) := \frac{E_{t_{n-1}}[M^1_{t_n}(P_{t_n}(c^1_{t_n}/D^*_{t_n}) + D^*_{t_n}\Delta t)]}{M^1_{t_{n-1}}},$$

$$X^1_{t_{n-1}}(\omega) := \frac{E_{t_{n-1}}[M^1_{t_n}(X^1_{t_n}(c^1_{t_n}/D^*_{t_n}) + c^1_{t_n}\Delta t)]}{M^1_{t_{n-1}}},$$

where the dependence on ω is made explicit.
For each node $(D^*_{t_{n-1}}, z^*_{t_{n-1}})$ at time t_{n-1} and for each value ω in Ξ, solve the system of equations (G.1), (G.2), (G.3) for the variables $\{c^1_{t_n,\eta}\}_\eta$, $\theta^1_{t_{n-1}}$ and $\theta^{rf,1}_{t_{n-1}}$. For each node, then compute the values $P_{t_{n-1}}(\omega)$ and $X^1_{t_{n-1}}(\omega)$, where $\omega \in \Xi$. These values are interpolated to obtain the functions $P_{t_{n-1}}(\cdot)$ and $X^1_{t_{n-1}}(\cdot)$ on the interval $(0, 1)$.
Iteratively, solve the system of equations for each node $(D^*_{t_i}, z^*_{t_i})$ at time t_i and for each value $\omega \in \Xi$, where i moves backwards from $n-1$ to 0.

3. The iteration in the previous step leads to the function $X^1_{t_0}(\cdot)$. Let $\bar{\omega}$ be the solution to the equation $X^1_{t_0}(\omega) = x_i$, where x_i is Ann's initial wealth. Ann's optimal consumption rate at time 0 is then $\hat{c}^1_0 = \bar{\omega} \cdot D_0$.

   The equilibrium interest rate at time 0 is obtained by interpolating the values

$$r_{t_0}(\omega) = -\frac{E_{t_0}[M^1_{t_1}] - M^1_{t_0}}{M^1_{t_0}\Delta t}$$

for $\omega \in \Xi$, and evaluating the resulting function at $\bar{\omega}$.

G.1. Additional numerical comparison

In the present model, the parameter ε controls the standard deviation of the interest rate, which is $\frac{\bar{\alpha}}{\sqrt{2a}}\sigma_z\varepsilon$.[14] Thus, ε is calibrated to match the historical standard deviation of the safe rate. Fig. 4 displays the equilibrium interest rate when Ann holds 80% of initial wealth for values of ε from zero to twice the calibrated value, with all other parameters as in Table 1. The discrepancy between our

[14] The parameter a determines the mean-reversion speed of the interest rate, while σ_z is observationally indistinguishable from ε and set equal to 1.

asymptotic expression and the numerical value remains small but increases with ε. Extensive numerical testing suggests that most of this small discrepancy actually stems from imprecisions in the numerical solution rather than approximation error of the asymptotic expansion: for a fixed time step Δt and large ε, the transition probabilities computed above may fall outside $[0,1]$. Truncating $p_i^{\pm\pm}$ to remain in $[0,1]$ causes the numerical imprecision.

Data availability

No data was used for the research described in the article.

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