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Explicit convergence rate parameters for linear autonomous systems

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Abstract:
In this paper, we propose a change of coordinates that brings the state matrix of an autonomous linear system into a modified Jordan Block form. Such a change of coordinates allows us to obtain exact values of the scaling factor and the convergence rate of the exponential stability bound for linear systems. The analysis is then applied to a control design with requirements on the evolution of the state norm. Numerical examples are also provided to illustrate the effectiveness of the proposed approach.

Keywords: Exponential convergence parameters, Lyapunov equation, Lyapunov inequality, modified Jordan form

1. INTRODUCTION AND MOTIVATIONS

Since the origin of Lyapunov's direct method (Lyapunov, 1992), assessing the stability of an unforced system is done by considering a nonnegative scalar function of the state vector and checking that its value is monotonically decreasing in time. The approach was also described and extended in (Bertram and Kalman, 1960; Massera, 1956; Barbashin and Krasovskii, 1961; Hahn et al., 1967), see also (Martynyuk, 2002) for a review on the development of the definition of stability during the last century and (Leine, 2010) for a historical perspective on the stability concept.

For an autonomous linear system $\dot{x} = Ax$, with $x(0) = x_0 \in \mathbb{R}^n$, stability analysis is usually done by looking for a time-(in)dependent quadratic Lyapunov function, i.e., one has to look for a symmetric positive definite matrix P *solution* of a particular Linear Matrix Inequality (LMI), i.e., the Lyapunov inequality

$$PA + A^\top P \prec 0. \tag{1}$$

Such a solution P can be computed via numerical tools (i.e., optimization solvers), but, to the best of the authors' knowledge, it has not been solved in any direct or closed form. As a consequence, the standard approach to obtain P is reverted, (Rugh, 1996)[Ch.7, pg.124], i.e., rather than directly specifying the matrix P, one has to involve an additional positive definite matrix $Q = Q^\top \succ 0$, and look for a solution P of the so-called Lyapunov Equation

$$PA + A^\top P = -Q, \tag{2}$$

where Q now is a degree of freedom. For example, it can be taken as $Q = 2qI$, for some real $q > 0$.

It is well-known, that such a P , solution of (2) and thus of (1), exists and is unique if and only if A has all negative real parts eigenvalues, and such a P is given by

$$P = \int_0^\infty \exp(A^\top s) Q \exp(As) ds.$$

It is also possible to numerically compute P by exploiting the vectorization of matrices as introduced in (Bellman, 1957). Other approaches to compute the Lyapunov function for nonlinear systems can be found in (Khalil, 2002), or (Davison and Kurak, 1971), (Blanchini, 1995), see (Giesl and Hafstein, 2015) for a recent review on the topic.

The pair P, Q can be exploited to determine the positive real constants κ and α, respectively, the *scaling factor* and the *convergence rate*, in the standard definition of uniform exponential stability, (Rugh, 1996; Khalil, 2002), i.e.,

$$|x(t)| \leq \kappa \exp(-\alpha t)|x(0)| \tag{3}$$

where $|\cdot|$ denotes the standard Euclidean norm. In particular, by considering $\sigma_{\min}(Q)$ to be the minimum singular value of Q, it is easy to show, by the comparison lemma (Khalil, 2002), that

$$|x(t)| \leq \frac{\sigma_{\max}(P)}{\sigma_{\min}(P)} \exp\left(-\frac{\sigma_{\min}(Q)}{\sigma_{\max}(P)}\, t\right)|x(0)| \quad \forall\, t \geq 0, \tag{4}$$

where the κ is then given by the condition number of P and the maximum eigenvalue of P is involved in the convergence rate α, along with the minimum eigenvalue of Q. Although very general, these constants for the norm upper bound highly depend on the choice of the matrix Q. To the best of the authors' knowledge, it does not exist any (constructive) approach to select Q to obtain the (in some sense) optimal values of the exponential convergence, i.e., to estimate κ and α such that the upper bound is as close as possible to the real norm evolution. Usually, taking Q with a large $\sigma_{\min}(Q)$ (aiming at obtaining a large convergence rate α), makes P highly ill-conditioned, thus increasing the scaling factor κ in (3). Although this can be accepted by common sense, or simply by intuition, we were not able to find any result in the literature properly describing these facts.

One particular form of the Lyapunov inequality (1), that can be used to directly obtain the parameters of the exponential stability in (3), is the following

$$PA + A^\top P \preceq -2\alpha P. \tag{5}$$

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where α is precisely the convergence rate of the definition, while κ is the P conditioning number. However, solving (5) has never been done analytically, and this problem is usually addressed via numerical optimization algorithms. Moreover, the value of α cannot be chosen arbitrarily. When A is Hurwitz and diagonalizable, one can select $\alpha = |\Re\lambda_{\min}(A)|$, where $\lambda_{\min}(A)$ (resp., $\lambda_{\max}(A)$) refers to the eigenvalue of A with largest (resp., smallest) real part, but for a non-trivial geometric multiplicity of $\lambda_{\min}(A)$ one can find a positive definite solution P to (5) only with $\alpha < |\Re\lambda_{\min}(A)|$. To the best of the authors' knowledge, there is no other general approach available in the literature to **explicitly** relate the scaling factor κ and the convergence rate α for linear systems to the eigenstructure of the state matrix A in a formal way, although the link with the system eigenvalues of A is intuitive and well-understood by the community. The closer results we were able to find are (Hu and Seiler, 2016), in which the authors test the exponential convergence rate using integral quadratic constraints, and (Applelby et al., 2006) the author obtains the rate of decay of solutions of a class of convolution Volterra difference equations. However, none of the available results in the literature are suitable to explicitly obtain the parameters of exponential convergence κ and α in closed form, neither for the case of linear time-invariant systems.

It is well known that the exponential rate of convergence for a linear autonomous system can be deduced from the eigenvalues of the corresponding system's state matrix A. However, in this paper, we want to explicitly characterize the dependence between the eigenvalues (more properly, the eigenstructure) of the state matrix A and the convergence rate α, and provide a direct link of such an eigenstructure of A with the scaling factor κ. The reader can verify that this type of characterization is easy to get in two particular cases, i.e., the symmetric part of the state matrix A is negative definite (Rugh, 1996)[Ch.7, pg 114], or when the matrix A is diagonalizable. Indeed, when A is not diagonalizable, due to the structure of its Jordan blocks, the exponential behavior due to the system eigenvalues is "rescaled" by a polynomial function of time whose upper bound is not trivial to characterize, see discussion in Section 3.

Thus the aim of this paper is twofold:

(1) Provide a review of the fundamental concepts related to the stability of linear autonomous systems;
(2) Provide a characterization of the solution of the Lyapunov inequality (5), or equivalently (1), also for non-diagonalizable matrices A;

The rest of this article is structured as follows. We first collect some preliminary results in Section 2, by recalling some instrumental notions for the definition of the Jordan block form and the proposed modification of such a normal form. Based on such a transformation, we then define a Lyapunov function which allows us to obtain the explicit exponential stability parameters (the values of κ and α) in terms of the algebraic properties of the matrix eigenvalues and the (generalized) eigenvectors, as shown in Section 3. In the same section, we also discuss some properties of the proposed solution compared to a numerical solution. In Section 4, we show an application of the analysis in the design of closed loop system performances. Finally, in Section 5, we show the effectiveness of the proposed bounds on an autonomous system with large geometric multiplicity. Some conclusions and perspectives are given in Section 6.

Notation. $\mathbb{R}$, resp. $\mathbb{N}$, resp. $\mathbb{C}$, denotes the real, resp. natural, resp. complex, numbers. Given a matrix $A \in \mathbb{R}^{n\times n}$, we denote the spectrum of A as $\sigma(A)$, where $\lambda_i(A) \in \sigma(A)$ is the i-th eigenvalue of A, and $\sigma_i(A)$ is its i-th singular value. Furthermore, $\sigma_{\max}(A)$ and $\sigma_{\min}(A)$ are respectively the maximum and minimum singular value of A. The L_2-norm of matrix $M \in \mathbb{R}^{n\times n}$ is denoted by $\|M\|$, and we denote the condition number of M as

$$\mu(M) := \sqrt{\frac{\lambda_{\max}(M^\top M)}{\lambda_{\min}(M^\top M)}}$$

and $\mu(M) = \mu(M^{-1})$. For two square matrices $A, B \in \mathbb{R}^{n\times n}$ we have

$$\begin{aligned}\mu(AB) &= \|AB\|\|(AB)^{-1}\| \\ &\leq \|A\|\|B\|\|B^{-1}\|\|A^{-1}\| = \mu(A)\mu(B).\end{aligned}$$

We define $M \in \mathbb{R}^{n\times n}$ to be orthogonally diagonalizable if it has an orthonormal set of eigenvectors. Given a matrix $M \in \mathbb{R}^{n\times n}$, $\mathrm{sym}(M)$ refers to the symmetric part of such a matrix, i.e., $\mathrm{sym}(M) = (M + M^\top)/2$. Given a complex number $\mu \in \mathbb{C}$ and an explicit number $n \in \mathbb{N}$, we denote

$$D_n(\lambda) := \begin{cases} \mathrm{diag}(1, \lambda, \ldots, \lambda^{n-1}), & \text{if } \lambda \neq 0, \\ I_n, & \text{if } \lambda = 0, \end{cases} \tag{6}$$

where I_n is the identity matrix of dimension n. Then, the following identity holds

$$N_n D_n(\lambda) = \lambda D_n(\lambda) N_n \tag{7}$$

for any $\lambda \in \mathbb{C}/\{0\}$ and $n \in \mathbb{N}$, where $N_n \in \mathbb{R}^{n\times n}$ is the matrix of zero elements with only 1 on the first super diagonal, i.e., a shifted identity matrix of dimension $n-1$. Similarly, for a matrix $\Lambda \in \mathbb{R}^{r\times r}$ we can define

$$D_n(\Lambda) := \begin{cases} \mathrm{blckdiag}(I_r, \Lambda, \ldots, \Lambda^{n-1}), & \text{if } \Lambda \neq 0, \\ I_{rn}, & \text{if } \Lambda = 0, \end{cases} \tag{8}$$

and, in this case, the following identity can be verified for $\Lambda \neq 0$

$$D_n(\Lambda)^{-1}(N_n \otimes I_r)D_n(\Lambda) = N_n \otimes \Lambda\,. \tag{9}$$

Finally, given any $n \in \mathbb{N}$, we define the following matrix

$$\mathbb{J}_n := I_n + N_n, \quad \mathbb{J}_n \in \mathbb{R}^{n\times n}. \tag{10}$$

2. PRELIMINARIES

2.1 Some explicit eigenvalues

From (Meyer, 2000, Ex. 7.2.5), we have an explicit formula for the eigenvalues of a matrix $M \in \mathbb{R}^{n\times n}$, with $a \neq 0$ and $c \neq 0$,

$$M = \begin{bmatrix} b & a & & & \\ c & b & a & & \\ & \ddots & \ddots & \ddots & \\ & & c & b & a \\ & & & c & b \end{bmatrix} = bI_n + aN_n + cN_n^\top \tag{11}$$

are explicitly given by

$$\lambda_i(M) = b + 2a\sqrt{\frac{c}{a}}\cos\left(\pi\frac{i}{n+1}\right), \quad i \in \{1, \ldots, n\}. \tag{12}$$

For our purposes we only deal with the case $c = a$, as also exploited in (Baggio and Zampieri, 2022), yielding the explicit eigenvalues formula

$$\lambda_i(M) = b + 2a\cos\left(\pi\frac{i}{n+1}\right), \quad i \in \{1,\ldots,n\}. \tag{13}$$

2.2 The Jordan normal form

In the next subsection, we introduce a modification of the Jordan normal form that will be instrumental in showing the paper's result. We first recall and define some quantities related to the standard Jordan normal form.

Given a matrix A of dimensions $n \times n$, we suppose, without loss of generality, that its eigenvalues are ordered in decreasing order with respect to the real part, namely

$$\Re\{\lambda_1\} \geq \Re\{\lambda_2\} \geq \ldots \geq \Re\{\lambda_n\}.$$

Let $m \leq n$ be the total number of linearly independent (non-generalized) eigenvectors $T_i^1 \neq 0$ associated with an eigenvalue $\bar\lambda_i \in \sigma(A)$, $i = 1,\ldots,m$, such that

$$AT_i^1 = \bar\lambda_i T_i^1 \quad \forall i = 1,\ldots,m.$$

Definition 1. (Jordan blocks dimension). *For each $i \in \{1,\ldots,m\}$, we define the values $g_i \geq 1$ satisfying $\sum_{i=1}^m g_i = n$, such that there exist $g_i - 1$ linearly independent generalized eigenvectors $T_i^k \neq 0$, for $k = 2,\ldots,g_i$, associated to the corresponding eigenvalue $\bar\lambda_i$, i.e., satisfying*

$$(A - \bar\lambda_i I)T_i^k = T_i^{k-1} \quad \forall k = 2,\ldots,g_i.$$

The introduced notation allows us to determine in advance the number of distinct Jordan blocks m and their relative dimensions g_i, when the matrix A is transformed into its Jordan form J. Moreover, these m distinct Jordan blocks are associated with, differently from the standard nomenclature, the m *distinct eigenvalues* $\bar\lambda_i \in \sigma(A)$, $i = 1,\ldots,m$.

Lemma 1. *Let A be a $n \times n$ matrix, let m be the number of distinct eigenvalues $\bar\lambda_i$ with associated Jordan block dimension g_i, so that $\sum_{i=1}^m g_i = n$. Then, there exists a $T \in \mathbb{R}^{n\times n}$ such that*

$$T^{-1}AT = J, \quad \begin{cases} J := \text{blckdiag}\left(J_{\bar\lambda_1},\ldots,J_{\bar\lambda_m}\right), \\ J_{\bar\lambda_i} := \bar\lambda_i I_{g_i} + N_{g_i}, \quad i = 1,\ldots,m. \end{cases} \tag{14}$$

For complex eigenvalues, one can always consider a real Jordan form in which a Jordan block associated with a complex pair of eigenvalues $\bar\lambda_{i/i+1} = a_i \pm \mathtt{i}b_i$, with $g_i - 1$ pairs of associated generalized eigenvectors, given by

$$J_{\bar\Lambda_i} := I_{g_i} \otimes \bar\Lambda_i + N_{g_i} \otimes I_2 = \begin{bmatrix} \bar\Lambda_i & I_2 & 0 & \ldots & 0 \\ 0 & \bar\Lambda_i & I_2 & \ldots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ & & & \bar\Lambda_i & I_2 \\ 0 & \ldots & & 0 & \bar\Lambda_i \end{bmatrix}_{2g_i \times 2g_i} \tag{15}$$

where $\bar\Lambda_i$ is defined as

$$\bar\Lambda_i = \begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix}$$

for which one can verify that $\sigma(\bar\Lambda_i) = \{a + \mathtt{i}b, a - \mathtt{i}b\}$. Note also that

$$\bar\Lambda_i + \bar\Lambda_i^\top = 2a_i I_2 = 2\Re\{\bar\lambda_{i/i+1}\}I_2.$$

2.3 The modified Jordan Form

In order to construct the modified Jordan normal form, for any non-zero distinct eigenvalue $\bar\lambda \neq 0$ with associate Jordan block dimension g, by implying the matrix $D_g(\bar\lambda)$ defined in (6) we obtain

$$D_g^{-1}(\bar\lambda) J_{\bar\lambda} D_g(\bar\lambda) = \bar\lambda \mathbb{J}_g \tag{16}$$

with the matrix $\mathbb{J}_g$ defined according to (10). Hence, by defining the matrix

$$\mathbb{D} := \text{blckdiag}\left(D_{g_1}(\bar\lambda_1),\,\ldots,\,D_{g_m}(\bar\lambda_m)\right) \tag{17}$$

the *modified Jordan form* $\mathbb{J}$ is defined as

$$\mathbb{J} := \mathbb{T}^{-1}A\mathbb{T}, \qquad \mathbb{T} := T\mathbb{D}, \tag{18}$$

where T satisfies (14). One can verify, using the property (16), that $\mathbb{J}$ as the following form

$$\mathbb{J} = \text{blckdiag}\left(\bar\lambda_1\mathbb{J}_{g_1},\,\ldots\,,\,\bar\lambda_m\mathbb{J}_{g_m}\right) = \Lambda(I + N), \tag{19}$$

with the matrix $\mathbb{J}_{g_i}$ defined as in (10), for each $i = 1,\ldots,m$, where

$$\Lambda = \text{diag}\left(\lambda_1,\ldots,\lambda_n\right)$$

and

$$N = \text{blckdiag}\left(N_{g_1},\ldots,N_{g_m}\right).$$

As we shall see in the next section, the matrix $\mathbb{J}$ is now in a more convenient form to study Lyapunov matrix inequalities. Note that in the case of complex eigenvalues, one has to define a slightly different change of coordinates to deal with the real representation of the generalized Jordan block (15). In particular, by using the definition (8) one can verify the following identity

$$\mathbb{J}_{g_i} \otimes \bar\Lambda_i = (I_{g_i} + N_{g_i}) \otimes \bar\Lambda_i = D_{g_i}^{-1}(\bar\Lambda_i) J_{\bar\Lambda_i} D_{g_i}(\bar\Lambda_i)$$

which gives

$$\mathbb{J}_{g_i} \otimes \bar\Lambda_i = \begin{bmatrix} \bar\Lambda_i & \bar\Lambda_i & 0 & \ldots & 0 \\ 0 & \bar\Lambda_i & \bar\Lambda_i & \ldots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ & & & \bar\Lambda_i & \bar\Lambda_i \\ 0 & \ldots & & 0 & \bar\Lambda_i \end{bmatrix}. \tag{20}$$

In the following, we give two relevant properties of the matrix $\mathbb{J}_n$.

Lemma 2. *For any $n \in \mathbb{N}$, the matrix $\mathbb{J}_n$ satisfies the following properties.*

(1) The eigenvalues of $\mathbb{J}_n + \mathbb{J}_n^\top$ satisfies

$$\sigma_i(\mathbb{J}_n + \mathbb{J}_n^\top) = 2\left[1 + \cos\left(\pi\frac{i}{n+1}\right)\right],\ i = \{1,\ldots,n\}.$$

(2) The matrix $\mathbb{J}_n + \mathbb{J}_n^\top$ satisfies

$$2\left[1 + \cos\left(\pi\tfrac{n}{n+1}\right)\right] I_n \preceq \mathbb{J}_n + \mathbb{J}_n^\top \preceq 2\left[1 + \cos\left(\pi\tfrac{1}{n+1}\right)\right] I_n.$$

Proof. The explicit formula of the eigenvalues of $\mathbb{J} + \mathbb{J}^\top$ follows from (13), showing items 1 and 2. □

Remark 1. From the first item of the Lemma right above, one can easily check that $\mathbb{J}_n + \mathbb{J}_n^\top$ is a positive definite matrix for any $n \in \mathbb{N}$.

Hence, as a consequence, by defining

$$\mathbb{J} = \text{blckdiag}\left(\mathbb{J}_{g_1},\ldots,\mathbb{J}_{g_m}\right) = I + N$$

we can also say that $\text{sym}(\mathbb{J})$ is a positive definite matrix.

Remark 2. In some sense, we construct the modified Jordan block form by defining the $\mathbb{D}$ change of coordinates. This kind of change of coordinates is also used in the high gain (observer/stabilizer) framework, see e.g. (Bernard et al., 2022, Section 6) or (Isidori, 1995, Chapter 4.7). In our case, by exploiting the transformation matrix $D_{g_i}(\bar{\lambda}_i)$, $i = 1, \ldots, m$, we are able to place the value of the eigenvalue on the super diagonal of the related Jordan block, as shown in (16).

3. EXPONENTIAL PARAMETERS FROM A EXPLICIT LYAPUNOV FUNCTION

Given a Hurwitz matrix A, via the Jordan block normal form, we can explicitly write the solution of the system dynamics

$$\dot{x} = Ax, \quad x(0) = x_0 \in \mathbb{R}^n, \tag{21}$$

as

$$x(t) = T \exp(\Lambda t) \sum_{k=0}^{g_{\max}} \frac{N^k t^k}{k!} T^{-1} x(0)$$

where $g_{\max} = \max_{i \in \{1,\ldots,m\}} \{g_i\}$. The solution proposed in this paper allows us to upper bound the norm of $x(t)$ via an exponential convergence function, i.e.,

$$\begin{aligned} |x(t)| =& |T \exp(\Lambda t) \sum_{k=0}^{g_{\max}} \frac{N^k t^k}{k!} T^{-1} x(0)| \\ \le& \kappa \exp(-\alpha t)|x(0)| \end{aligned}$$

such that α dependence explicitly on the eigenstructure and not only on the system's eigenvalues. This upper bound cannot be obtained by exploiting the normal form coordinates $z = T^{-1}x$. Indeed, considering $P = T^{-\top}T^{-1}$ as a solution of the Lyapunov matrix inequality (5), thus obtaining $J + J^\top \preceq -2\alpha I$, may not satisfy the inequality for any $\alpha > 0$. In other words, we can easily see, when computing the eigenvalues of $J + J^\top$, that from (13) each Jordan block form $J_{\bar{\lambda}_i}$ satisfies

$$\sigma_{\max}(J_{\bar{\lambda}_i} + J_{\bar{\lambda}_i}^\top) = 2\Re\{\bar{\lambda}_i\} + 2\cos\left(\frac{\pi}{g_i + 1}\right)$$

which is negative only for $\Re\{\lambda_i\} \le -\cos(\pi/(g_i+1)) \le -1$.

This problem does not arise if we take into account the modified Jordan form coordinates $z = \mathbb{T}^{-1}x$, with $\mathbb{T}$ defined in (18), and we define a Lyapunov function candidate

$$V = x^\top \mathbb{T}^{-\top}\mathbb{T}^{-1} x, \tag{22}$$

that we call the *explicit* Lyapunov functions [2]. One can easily notice that $\mathbb{T}^{-\top}\mathbb{T}^{-1}$ is a positive definite matrix, because $\mathbb{T}$ is full rank since it has all linearly independent columns, hence $V(x) > 0$ for $x \neq 0$, with $V(0) = 0$, and it is radially unbounded with

$$\sigma_{\min}(\mathbb{T}^{-1})|x|^2 \le V \le \sigma_{\max}(\mathbb{T}^{-1})|x|^2.$$

Theorem 1. *Consider an asymptotically stable system* (21) *and the transformation matrix* $\mathbb{T}$ *defined in* (18). *Then, the norm of the system state is upper bounded by* (3) *with*

$$\kappa = \mu(\mathbb{T}), \ \alpha = -\Re\{\lambda_{\max}(\mathrm{sym}(\mathbb{J}))\}. \tag{23}$$

[2] We denote as explicit Lyapunov function any Lyapunov function that can be written directly in terms of the algebraic property of a Hurwitz A without considering the solution of a Lyapunov equation (2), or inequality (1), and thus without the choice of positive definite matrix Q.

Proof. By taking the time derivative of V in 22, we have

$$\begin{aligned} \dot{V} &= 2x^\top \mathbb{T}^{-\top}\mathbb{T}^{-1}\dot{x} = 2x^\top \left(\mathbb{T}^{-\top}\mathbb{T}^{-1}A\right)x \\ &= 2x^\top \mathbb{T}^{-\top}\left(\mathbb{T}^{-1}A\mathbb{T}\right)\mathbb{T}^{-1}x = x^\top \mathbb{T}^{-\top}(\mathbb{J}^\top + \mathbb{J})\mathbb{T}^{-1}x \\ &\le \left(\Re\{\lambda_{\max}(\mathbb{J}^\top + \mathbb{J})\}\right) V. \end{aligned}$$

Hence, $\dot{V}$ is strictly less than zero for nonzero x and, moreover, V is exponentially decreasing. Indeed, by the comparison Lemma (Khalil, 2002)[Lemma 3.4], we have

$$V(t) \le \exp(\Re\{\lambda_{\max}(\mathbb{J}^\top + \mathbb{J})t)V(0)$$

from which we get

$$|x|^2 \le \frac{\sigma_{\max}(\mathbb{T}^{-1})}{\sigma_{\min}(\mathbb{T}^{-1})} \exp(\Re\{\lambda_{\max}(\mathbb{J}^\top + \mathbb{J})t)|x(0)|^2.$$

Finally, taking the square root of both sides of the inequality proves the theorem. □ The eigenvalues of $\mathbb{J}^\top + \mathbb{J}$ can be explicitly written as a function of the eigenvalues of A and of the related geometric multiplicity since each block of $\mathbb{J}^\top + \mathbb{J}$ has the form of (11). We can thus obtain an explicit expression for α as the following one

$$\alpha = -\max_{i \in \{1,\ldots,m\}} \left\{ \Re\{\bar{\lambda}_i\} \left[1 - \cos\left(\frac{\pi}{g_i + 1}\right)\right] \right\}. \tag{24}$$

Remark 3. The values of α and κ given in Theorem 1 are related to properties of the eigenvalues and the (generalized) eigenvectors of the matrix A, rather than to the properties of P and Q satisfying a Lyapunov equation (2).

Remark 4. Due to the presence of the generalized eigenvectors, i.e., given by the geometric multiplicity of the associated eigenvalue $\bar{\lambda}_i$, the columns of $\mathbb{T}$, i.e., $\mathbb{T}_i$, $i = 1, \ldots, n$, are not all orthogonal one each other. Thus, they yield a bad condition number of matrix $\mathbb{T}$. Only when A is *orthogonally diagonalizable*, we have $\mathbb{D} = I$ and we can always normalize the columns of T so to get an orthonormal matrix, i.e., $T^\top T = I$. In this case, the x state evolution norm upper bond simplifies to the one one with standard *dominant pole* convergence rate, i.e. $\alpha = |\lambda_1|$,

$$|x| \le \exp\left(-|\lambda_1|t\right)|x(0)|.$$

3.1 Comparison with a standard solution of Eq. (5)

When we solve for (5), with a fixed convergence rate (24), the obtained solution P can be always rewritten as $P = \mathbb{T}^{-\top}P'\mathbb{T}^{-1}$, for some positive definite P'. We can thus write, from the dynamics of $V' = x^\top P x = x^\top \mathbb{T}^{-\top}P'\mathbb{T}^{-1}x$, that

$$|x| \le \mu(\mathbb{T}^{-\top}P'\mathbb{T}^{-1})\exp(-\alpha t)|x(0)|.$$

However, we cannot determine a priori if $\mu(\mathbb{T}^{-\top}\mathbb{T}^{-1}) \le \mu(\mathbb{T}^{-\top}P'\mathbb{T}^{-1})$, we can only provide a conservative upper bound of the form $\mu(\mathbb{T}^{-\top}P'\mathbb{T}^{-1}) \le \mu(\mathbb{T}^{-\top}\mathbb{T}^{-1})\mu(P')$. Thus, the condition number of P could be in principle better than the one provided by the choice $\mathbb{T}^{-\top}\mathbb{T}^{-1}$. However, we cannot mathematically prove whether or not other solutions of the LMI (5) can be either better or worse than the particular choice $P = \mathbb{T}^{-\top}\mathbb{T}^{-1}$. We thus report, in Sec. 5.2, the results from a set of numerical tests showing that most of the time $P = \mathbb{T}^{-\top}\mathbb{T}^{-1}$ is better conditioned than the numerical solution P of the LMI.

4. APPLICATION IN CONTROLLER DESIGN

Consider the controllable linear system

$$\dot{x} = Ax + Bu, \quad x(0) = x_0 \tag{25}$$

with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, and consider designing a stabilizing state feedback gain K such that $A - BK$ is Hurwitz with the states' norm guaranteeing some desired performance for *all initial conditions* x_0 in a ball of radius ρ_0, i.e., $\forall x_0 \in \mathcal{B}_{\rho_0}$. These performances on $|x(t)|$ can read as

- limiting the 'overshoot' with respect to the initial condition, i.e., the evolution of the state norm must be constrained to remain inside a certain ball of fixed radius $\overline{\rho} \geq \rho_0$;
- finding a 'good' estimate of the time $t^\star$ after which the state norm reaches a certain ball of radius $\underline{\rho}$, i.e., find $t^\star \in \mathbb{R}$ such that $|x(t)| \leq \underline{\rho}$ for all $t \geq t^\star$, when the system is initialized in $\mathcal{B}_{\rho_0}$;
- forcing the state norm to reach a certain ball $|x(t)| \leq \underline{\rho}$ within a prescribed fixed time $t^\star$.

In order to illustrate the potential of the application on an example, let us consider, for the sake of exposition, the case in which

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and take as static feedback gain $K = [\alpha_1\alpha_2, \ \alpha_1 + \alpha_2]$, with $\alpha_1 = \gamma$ and $\alpha_2 = 2\gamma$, as in the standard high-gain formalism, where $\gamma > 0$ plays the role of the high gain and imposes the closed-loop eigenvalues to be $\lambda_1 = -\gamma$ and $\lambda_2 = -2\gamma$. Since the closed-loop matrix is in companion form, it can be diagonalized (due to the presence of simple eigenvalues) via the Vandermonde matrix

$$T = \begin{bmatrix} 1 & 1 \\ -\alpha_1 & -\alpha_2 \end{bmatrix}, \quad T^{-1} = \frac{1}{\alpha_1 - \alpha_2} \begin{bmatrix} -\alpha_2 & -1 \\ \alpha_1 & 1 \end{bmatrix},$$

whose conditioning number $\mu(T)$, for $\alpha_1 = \gamma$ and $\alpha_2 = 2\gamma$, reads as

$$\mu(T) = \sqrt{\frac{5\gamma^2 + 2 + \sqrt{25\gamma^4 + 16\gamma^2 + 4}}{5\gamma^2 + 2 - \sqrt{25\gamma^4 + 16\gamma^2 + 4}}}$$

and thus have $|x(t)| \leq \mu(T)\exp(-\gamma t)|x(0)|$. For this choice of the closed loop characteristic polynomial, we have a minimum of $\mu(T)$ at $\gamma = \sqrt{\frac{2}{5}} \approx 0.6325$, to which it corresponds a condition number $\mu(T) \approx 6.1623$.
We cannot prove that this solution is the best for this set of parameters, although, to the best of the authors' knowledge, this is the only constructive approach available in the literature to achieve such a result and that does not rely on testing all initial conditions in a compact set $\mathcal{B}_{\rho_0}$.

The proposed approach has other possible applications, such as the design of the observer dynamics with prescribed convergence performances and their employment in output dynamical feedback controller design (motivating this work at an early stage). However, due to a lack of space, we postpone a deeper analysis to a journal version of this paper. Another application can be found in defining the Hamiltonian structural matrices of a stable linear system as shown in Spirito et al. (2024).

5. SOME NUMERICAL RESULTS

5.1 An autonomous system example

We consider the evolution of the norm of $z(t)$ for an eigenvalue, $\lambda = -1$, with associated geometric multiplicity $g = n = 10$, i.e., we consider an autonomous system dynamics with state matrix $A = -1(I + N) \in \mathbb{R}^{10\times 10}$, whose corresponding convergence rate, given (24), is $\alpha \approx 0.0405$. The evolution of the state norm $|z(t)|$, from initial conditions

$$\begin{aligned} z_0 = [&0.01001979, \ 0.02185996, \ -0.01413963 \\ &0.08315555, \ -0.14693675, \ 0.31947075 \\ &-0.4683713, \ 0.59627956, \ -0.4827674 \\ &0.24631203]^\top. \end{aligned} \tag{26}$$

is depicted in blue in Fig.1. When compared to the numerical solution of the LMI (5), by fixing α as in (24), we get a matrix P_{LMI} with condition number of $\mu(P_{LMI}) = 5.0957$, which is worse than the unitary condition number of $\mathbb{T} = I$. Whereas, solving (2), with $Q = I$ or with $Q = \alpha I$, where α is given in (24), gives solutions P_I and P_α whose maximum eigenvalues are respectively $\lambda_{\max}(P_I) \approx 5.2605$ and $\lambda_{\max}(P_\alpha) \approx 0.2131$. Furthermore, they share the same condition number $\mu(P_I) = \mu(P_\alpha) \approx 20.5959$, and provide the same convergence rate $\alpha_Q = \lambda_{\min}(I)/\lambda_{\max}(P_I) \approx 0.1901$ which is approximately 5 times larger than the one obtained via (24).

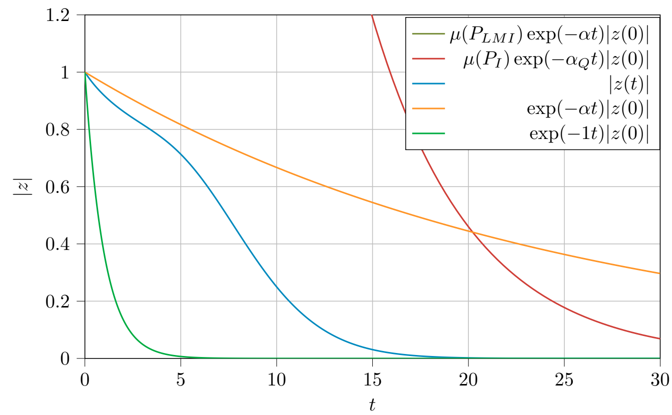

Fig. 1. Evolution of the norm of $z(t)$ compared to (3) with different parameters, i.e., the one obtained from (5) in dark green, from (2) in red, and the one with parameters in (23) in orange.

In Fig.1, we compare the proposed parameters (24) with the two sets of parameters obtained from the solutions of (5) and (2). In particular, we can notice that during a first transient, the proposed approach provides a better approximation of norm evolution, while the one obtained from (2), depicted in red, performs better on a larger time interval. Due to the fact the solution of (5) can only be obtained numerically, depicted in dark green, it provides by far the most conservative result [3].

5.2 Numerical tests

We consider as numerical test a matrix A with dimension 9 with 3 distinct eigenvalues $\bar{\lambda}$, i.e., $\{-0.5, -2, -4\}$ with associated Jordan block dimensions $(g_1, g_2, g_3) = (3, 5, 1)$. That is, according to the value convergence rate formula (24) we have

[3] Note that solving (5) by fixing $\alpha = \lambda_{\min}(I)/\lambda_{\max}(P_I) \approx 0.1901$, provides a numerical solution with conditioning number $\mu(P) \approx 729.4461$.

$$\alpha = 0.5\left(1+\cos\left(\frac{3}{4}\pi\right)\right) \approx 0.1464$$

with $P = \mathbb{T}^{-\top}\mathbb{T}^{-1}$, thus the condition number of P is equivalent to the condition number of $\mathbb{T}$. By comparing the solution P_{LMI} of Linear Matrix Inequality solver (getlmis()) implemented in the Robust Control Toolbox™ of Matlab version R2022a with the given α, we get, over 5000 comparisons, 99% of the times a better condition number of $\mathbb{T}$ than that of the numerical solution P_{LMI}. The numerical testing details are summarized in Table 1. Other testing results are also available in (Spirito and Astolfi, 2024), in which we show that with a free convergence rate α in (5), the solution $\mathbb{T}^{-\top}\mathbb{T}^{-1}$ performance better than the numerical solution obtain by the Matlab solver.

Table 1. Numerical results

# tests	success %	average $\mu(\mathbb{T}^\top\mathbb{T})$	average $\mu(P_{LMI})$
5000	99%	$1.4839e+03$	$1.0417e+06$

6. CONCLUSIONS

In this work, we present a modification of the Jordan block normal form of the state matrix of an autonomous linear system. We use this analysis to obtain, in closed form, the constant values for the scaling factor κ and the converge rate α in the definition of global exponential convergence for stable linear autonomous systems. This approach thus allows to obtain a constructive characterization of the convergence property of a linear system. Moreover, we discuss how this approach can be used to enhance/define the closed-loop performances of a controlled system. Numerical examples are correlated to show the effectiveness of the analysis.

The presented result is a preliminary work that has been extended to the case of non-Hurwitz matrices typical of k-contraction scenarios, see e.g. (Zoboli et al., 2023), in (Spirito and Astolfi, 2024).

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