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BOUNDED ACYCLICITY AND RELATIVE SIMPLICIAL VOLUME

KEVIN LI, CLARA LÖH, AND MARCO MORASCHINI

Abstract. We provide new vanishing and glueing results for relative simplicial volume, following up on two current themes in bounded cohomology: The passage from amenable groups to boundedly acyclic groups and the use of equivariant topology.

More precisely, we consider equivariant nerve pairs and relative classifying spaces for families of subgroups. Typically, we apply this to uniformly boundedly acyclic families of subgroups. Our methods also lead to vanishing results for ℓ^2-Betti numbers of aspherical CW-pairs with small relative amenable category and to a relative version of a result by Dranishnikov and Rudyak concerning mapping degrees and the inheritance of freeness of fundamental groups.

1. Introduction

Bounded cohomology is defined as the cohomology of the bounded dual of the singular or bar chain complex [Gro82] and it has many applications in group theory and in the study of the geometry of manifolds. A fundamental phenomenon is that bounded cohomology of amenable groups is trivial (i.e., amenable groups are boundedly acyclic). On the other hand, the bounded cohomology of negatively curved groups surjects onto ordinary cohomology. In manifold topology, the simplicial volume of an oriented compact manifold is a homotopy invariant defined as the ℓ^1-seminorm of the $\mathbb{R}$-fundamental class [Gro82]. Using a duality argument, the simplicial volume can be expressed in terms of bounded cohomology.

We provide new vanishing results for relative simplicial volume, following up on two current themes in bounded cohomology:

- The passage from amenable groups to boundedly acyclic groups;
- The use of equivariant topology, most notably of classifying spaces for families of subgroups.

A technical difficulty in the passage from amenable to boundedly acyclic groups is that the class of amenable groups possesses a large degree of uniformity when it comes to bounded cohomology. This includes the fact that the class of amenable groups is closed under subgroups and quotients and the fact that amenable groups are not only boundedly acyclic, but uniformly boundedly acyclic. Therefore, in the setting of boundedly acyclic groups, generalised vanishing results for simplicial volume come with additional uniformity and closure hypotheses.

As we aim at results for relative bounded cohomology and relative simplicial volume, we adapt tools from equivariant topology to this relative setting.

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Key words and phrases. Simplicial volume, bounded cohomology, bounded acyclicity, classifying spaces for families of subgroups.

1.1. **Uniform bounded acyclicity.** Group actions with amenable stabilisers have proved to be a valuable tool to compute bounded cohomology [Mon01, BM02, BI09]. Similarly, also *uniformly boundedly acyclic actions* allow us to compute bounded cohomology, where the uniformity refers to a uniform bound on the norms of primitives (it is not known how restrictive this uniformity hypothesis is). Recently, uniformly boundedly acyclic actions have been used to compute the bounded cohomology of geometrically relevant groups [FFLM23, MN23].

Let X be a path-connected space. We say that a set of path-connected subspaces $\mathcal{A}$ of X is *uniformly boundedly acyclic* [*of order* n] *in* X if the collection of all finite [resp. n-fold] intersections of conjugates of the subgroups

$$\big(\mathrm{im}(\pi_1(A \hookrightarrow X))\big)_{A \in \mathcal{A}}$$

in $\pi_1(X)$ is uniformly boundedly acyclic (Definition 5.11). In the special case when the above groups are amenable, we also speak of an *amenable* set of subspaces. The issue of basepoints is addressed in Section 1.5. We have two geometric situations in which uniformly boundedly acyclic sets of subspaces lead to interesting uniformly boundedly acyclic actions: Open covers and glueing loci of manifolds obtained by glueing manifolds with boundary.

1.2. **Vanishing via relative open covers.** Gromov [Gro82] and Ivanov [Iva85] established vanishing results for the comparison map (and thus for simplicial volume) in the presence of amenable open covers with small multiplicity.

Following the approach by Löh and Sauer [LS20] through equivariant nerves and classifying spaces for families, we generalise these vanishing results in two directions. First, we allow more general covers: A cover $\mathcal{U}$ of X by path-connected open subsets is *uniformly boundedly acyclic* if the underlying set of subsets of X is uniformly boundedly acyclic in X. Second, we adapt the setting to pairs of CW-complexes (X, A), where A is π_1-injective in X (Theorem 6.11). To this end, we introduce the notion of [*weakly convex*] *relative covers* (Definition 4.9). Using equivariant nerve pairs and classifying spaces of group pairs for families, we obtain:

Theorem 1.1 (Corollary 6.13)**.** *Let (X, A) be a CW-pair with path-connected ambient space X. Assume that A has only finitely many connected components, each of which is π_1-injective in X. Let $\mathcal{U}$ be a relative cover of (X, A) that is uniformly boundedly acyclic.*

(i) *If $\mathcal{U}$ is weakly convex, then the comparison map*

$$\mathrm{comp}^k \colon H_b^k(X, A; \mathbb{R}) \to H^k(X, A; \mathbb{R})$$

vanishes in all degrees $k \geq \mathrm{mult}_A(\mathcal{U})$.

(ii) *Let $\nu \colon (X, A) \to (|N(\mathcal{U})|, |N_A(\mathcal{U})|)$ be a nerve map. If $\mathcal{U}$ is convex, then the comparison map* comp^* *factors through ν:*

$$\begin{array}{ccc} H_b^*(X, A; \mathbb{R}) & \xrightarrow{\mathrm{comp}^*} & H^*(X, A; \mathbb{R}) \\ & \searrow & \uparrow{\scriptstyle H^*(\nu;\mathbb{R})} \\ & & H^*(|N(\mathcal{U})|, |N_A(\mathcal{U})|; \mathbb{R})\,. \end{array}$$

Here $\mathrm{mult}_A(\mathcal{U})$ denotes the relative multiplicity of $\mathcal{U}$ with respect to A (Definition 4.3) and the simplicial complex $N_A(\mathcal{U})$ is a suitable subcomplex of the nerve $N(\mathcal{U})$ (Definition 4.4).

In the absolute case, Ivanov proved a similar vanishing theorem for *weakly boundedly acyclic covers* using spectral sequences [Iva20]. Our notion of uniformly boundedly acyclic covers is similar, but the relation between the two is unclear (Remark 6.4).

Theorem 1.1 applies in particular to relative covers that are amenable. We introduce the *relative amenable multiplicity* $\mathrm{mult}_{\mathsf{Am}}(X, A)$ (Definition 4.13) as the minimal relative multiplicity of weakly convex relative amenable covers of (X, A) by path-connected open subsets.

Theorem 1.2 (Corollary 6.15)**.** *Let (X, A) be a CW-pair with path-connected ambient space X. Assume that A consists of finitely many connected components, each of which is π_1-injective in X. Then the comparison map*

$$\mathrm{comp}^k \colon H_b^k(X, A; \mathbb{R}) \to H^k(X, A; \mathbb{R})$$

vanishes in all degrees $k \geq \mathrm{mult}_{\mathsf{Am}}(X, A)$.

In particular, if $(M, \partial M)$ is an oriented compact connected triangulable manifold with π_1-injective boundary components and $\mathrm{mult}_{\mathsf{Am}}(M, \partial M) \leq \dim(M)$, then the relative simplicial volume $\|M, \partial M\|$ vanishes.

In the absolute case, every cover is a weakly convex relative cover and hence $\mathrm{mult}_{\mathsf{Am}}(X, \emptyset)$ is the minimal multiplicity of amenable covers of X. For a CW-complex X, this coincides with the minimal *cardinality* of amenable covers of X by not necessarily path-connected subsets [CLM22, Remark 3.13]. The latter quantity is called the *amenable category* $\mathrm{cat}_{\mathsf{Am}}(X)$ (Remark 4.15), a notion that is modelled on the classical LS-category [CLOT03].

As an application of Theorem 1.2, we give an alternative proof of a relative vanishing theorem, which is a consequence of Gromov's vanishing theorem for non-compact manifolds (Theorem 6.17).

Our methods for equivariant nerve pairs and relative classifying spaces also lead to vanishing results for ℓ^2-Betti numbers of aspherical CW-pairs with small relative amenable multiplicity (Theorem 7.3). In the absolute case (Corollary 7.4), this recovers a result by Sauer [Sau09, Theorem C].

1.3. **Glueings.** We adapt the additivity of relative simplicial volume for glueings along amenable boundaries [Gro82, BBF$^+$14, Kue15] to situations with boundedly acyclic boundaries. As we move away from amenability, we lose control on the norm, and thus only retain control on the vanishing behaviour.

Theorem 1.3 (Theorem 8.1)**.** *Let $n \geq 3$ and $(M_i, \partial M_i)_{i \in I}$ be a finite collection of oriented compact connected n-manifolds. Assume that every connected component of every boundary component ∂M_i has boundedly acyclic fundamental group. Let $\mathcal{N}$ be a set of π_1-injective boundary components of the $(M_i)_{i \in I}$ and let $(M, \partial M)$ be obtained from $(M_i, \partial M_i)_{i \in I}$ by a pairwise glueing of the boundary components in $\mathcal{N}$.*

If $\mathcal{N}$, viewed as a set of subsets of M, is uniformly boundedly acyclic of order n in M, then the following are equivalent:

(i) *We have $\|M, \partial M\| = 0$;*
(ii) *For all $i \in I$, we have $\|M_i, \partial M_i\| = 0$.*

1.4. **Mapping degrees.** One of the classical applications of simplicial volume is an a priori estimate on mapping degrees [Gro82, Thu79, FM21]. In contrast, the exact relation between mapping degrees and monotonicity of LS-category is still

wide open [Rud17]. Similarly, this problem is also open for the amenable category and other generalised LS-category invariants [CLM22, Definition 2.16, Section 6].

In the absolute case, Eilenberg and Ganea showed that the LS-category invariant for the family containing only the trivial subgroup of an aspherical space recovers the cohomological dimension of its fundamental group [EG57]. Moreover, cohomological dimension one can be characterised in terms of freeness. Thus, the monotonicity problem for (generalised) LS-category leads to inheritance properties of fundamental groups under maps of non-zero degree.

We use equivariant and group cohomological methods to establish the following relative version (and a simplified proof) of a monotonicity result by Dranishnikov and Rudyak for closed manifolds [DR09]:

Theorem 1.4 (Corollary 3.8)**.** *Let $f\colon (M,\partial M) \to (N,\partial N)$ be a map between oriented compact connected manifolds of the same dimension with π_1-injective boundary components. Let $\partial M = \coprod_{i=1}^m M_i$ and $\partial N = \coprod_{i=1}^n N_i$ be decompositions into connected components. If $\deg(f) = \pm 1$ and there exists a free group F_M such that*

$$\pi_1(M) \cong F_M * \ast_{i=1}^m \pi_1(M_i)\,,$$

then there exists a free group F_N such that $\pi_1(N) \cong F_N * \ast_{i=1}^n \pi_1(N_i)$.

For closed manifolds our approach also yields inheritance properties for virtual freeness:

Theorem 1.5 (Corollary 3.10)**.** *Let $f\colon M \to N$ be a map between oriented closed connected manifolds of the same dimension. If $\deg(f) \neq 0$ and $\pi_1(M)$ is virtually free, then also $\pi_1(N)$ is virtually free.*

1.5. **Conventions.** In this article, we adhere to the following conventions:

Instead of the usual notion of families of sets, groups, modules, ..., we will speak of *collections*; this is to avoid confusion with the term "families of subgroups". I.e., a collection $(H_i)_{i\in I}$ of groups [or sets, ...] is a map $I \to \mathrm{Group}$, $i \mapsto H_i$ from a set I to the class of all groups [or sets, ...]. In particular, collections can contain repetitions.

Families of subgroups will only be closed under conjugation but not necessarily under finite intersections or taking subgroups (Definition 2.4).

All groups will be discrete groups; in particular, we consider bounded cohomology of discrete groups and G-CW-complexes for discrete groups G. The geometric realisation of G-simplicial complexes will always be equipped with the G-CW-structure coming from the barycentric subdivision (Example 2.2).

Given a compact manifold M with non-empty boundary, we say that M has *π_1-injective boundary* if every connected component of ∂M is π_1-injective in M.

We usually refrain from spelling out basepoints for fundamental groups. Strictly speaking, fixing basepoints is necessary to make the notion of the image of "the" fundamental group of a subspace in "the" fundamental group of an ambient space precise. However, we will always deal with situations concerning conjugation-invariant properties or concerning collections of all conjugates of such subgroups. Therefore, all choices of basepoints would lead to the same outcome.

We always work with open covers consisting of path-connected sets. We explain in Remark 4.15 why this condition is not restrictive in our setting.

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2. Classifying spaces of group pairs for families of subgroups

The goal of this section is to introduce classifying spaces for families of subgroups in a relative setting. We first recall classifying spaces for families of subgroups and then explain the extension to the relative setting.

This is motivated by our geometric situations of topological pairs (X, A) (e.g., manifolds with boundary), where often two classes of subgroups of the fundamental group $G \coloneqq \pi_1(X)$ will be involved:

- A family $\mathcal{F}$ of subgroups of G, describing the allowed fundamental groups of sets in open covers of X;
- A collection $\mathcal{H}$ of subgroups of G, coming from the fundamental groups of the components of A.

2.1. G-CW-complexes.

We briefly recall the definitions of G-CW-complexes and the induction functor. For more background on G-CW-complexes we refer the reader to the literature [Lüc89, Lüc05].

Definition 2.1 (G-CW-complex)**.** A *G-CW-complex* Y is a G-space equipped with a G-invariant filtration

$$\emptyset = Y^{(-1)} \subset Y^{(0)} \subset Y^{(1)} \subset \cdots \subset Y^{(n)} \subset \cdots \subset Y \tag{2.1}$$

such that the following hold:

- $Y = \bigcup_{n \geq 0} Y^{(n)}$;
- Y carries the weak topology with respect to the filtration (2.1);
- $Y^{(n)}$ is obtained from $Y^{(n-1)}$ as a G-pushout of the form

$$\begin{array}{ccc} \coprod_{i \in I_n} G/H_i \times S^{n-1} & \longrightarrow & Y^{(n-1)} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} G/H_i \times D^n & \longrightarrow & Y^{(n)}\,. \end{array}$$

The subgroups H_i of G and their conjugates are called *isotropy groups* of Y. If all isotropy groups of Y are trivial, we also say that Y is a *free* G-CW-complex.

A morphism of G-CW-complexes is a (not necessarily cellular) G-map.

For example, the universal covering $\widetilde{X}$ of a path-connected CW-complex X is a free $\pi_1(X)$-CW-complex with respect to the CW-structure inherited from X.

Example 2.2 (Barycentric subdivision of G-simplicial complexes)**.** Let N be an (abstract) simplicial complex and let G be a group acting on N via simplicial automorphisms. Then the geometric realisation $|N'|$ of the barycentric subdivision is a G-CW-complex, while $|N|$ need not be a G-CW-complex in general. The standard homeomorphism between the geometric realisations $|N| \to |N'|$ is a G-homeomorphism. Therefore, $|N|$ admits a canonical structure as a G-CW-complex and we will always use this G-CW-structure.

Given a subgroup $H \subset G$, there is a natural way to associate to an H-CW-complex a G-CW-complex.

Definition 2.3 (Induction)**.** Let H be a subgroup of G. The *induction (along the inclusion $H \subset G$)* is the functor

$$G \times_H (-) \colon H\text{-CW-complexes} \to G\text{-CW-complexes}$$

that assigns to an H-CW-complex B the G-CW-complex $G \times_H B$, that is the quotient of $G \times B$ by the (right) H-action $(g, b) \cdot h = (gh, h^{-1}b)$. Here G acts on $G \times_H B$ by left multiplication. We denote elements of $G \times_H B$ by $[g, b]$.

For an H-map $f\colon B \to C$ between H-CW-complexes, the induced G-map

$$G \times_H f\colon G \times_H B \to G \times_H C$$

is given by $G \times_H f([g, b]) = [g, f(b)]$.

The induction functor is left-adjoint to the restriction functor, which associates to a G-CW-complex the same space viewed as an H-CW-complex.

2.2. **Classifying spaces for families of subgroups.** We use the following (non-standard) convention for families of subgroups:

Definition 2.4 (Family of subgroups)**.** Let G be a group and $\mathcal{F}$ be a non-empty set of subgroups of G. We say that $\mathcal{F}$ is a *family of subgroups* (or *conjugation-closed family of subgroups*) of G if it is closed under conjugation. We say that $\mathcal{F}$ is an *intersection-closed family of subgroups* of G if it is closed under conjugation and taking finite intersections.

Example 2.5. The following are examples of families of subgroups:

(i) The set of isotropy groups of a G-CW-complex;
(ii) The family Tr consisting only of the trivial subgroup;
(iii) The family Fin consisting of all finite subgroups;
(iv) The family Am consisting of all amenable subgroups;
(v) Let H be a subgroup of G and let $\mathcal{F}$ be a family of subgroups of G. Then the set $\mathcal{F}|_H = \{L \subset H \mid L \in \mathcal{F}\}$ is a family of subgroups of H.

Example 2.6 (Families generated by a set of subgroups)**.** Let G be a group and let $\mathcal{G}$ be a non-empty set of subgroups. The *intersection-closed family $\mathcal{F}\langle\mathcal{G}\rangle$ generated by $\mathcal{G}$* is defined to be the smallest (with respect to inclusion) intersection-closed family containing $\mathcal{G}$, that is

$$\mathcal{F}\langle\mathcal{G}\rangle = \left\{ \bigcap_{i=1}^{n} g_i H_i g_i^{-1} \;\middle|\; n \in \mathbb{N}, H_i \in \mathcal{G}, g_i \in G \right\}.$$

For $n \in \mathbb{N}$ we define the (conjugation-closed) family

$$\mathcal{F}_n\langle\mathcal{G}\rangle \coloneqq \left\{ \bigcap_{i=1}^{n} g_i H_i g_i^{-1} \;\middle|\; H_i \in \mathcal{G}, g_i \in G \right\}.$$

Recall that EG, the universal covering of an Eilenberg–MacLane space BG, is a terminal object in the G-homotopy category of free G-CW-complexes. The following is a generalisation to G-CW-complexes with not necessarily trivial isotropy groups.

Definition 2.7 (Classifying space for a family of subgroups)**.** Let G be a group and let $\mathcal{F}$ be a (conjugation-closed) family of subgroups of G. A *classifying space for G with respect to $\mathcal{F}$* is a G-CW-complex E with the following universal property:

- All isotropy groups of E lie in $\mathcal{F}$;
- For each G-CW-complex Y whose isotropy groups all lie in $\mathcal{F}$, there is up to G-homotopy exactly one G-map $Y \to E$.

We usually denote such classifying spaces by $E_{\mathcal{F}}G$ (even though they are only unique up to G-homotopy equivalence).

If $\mathcal{F}$ contains the trivial group, then the universal property of $E_{\mathcal{F}}G$ ensures that there exists a G-map $EG \to E_{\mathcal{F}}G$, which is unique up to G-homotopy.

Theorem 2.8 (Existence of classifying spaces for families of subgroups)**.** *Let G be a group and let $\mathcal{F}$ be a (conjugation-closed) family of subgroups.*

(i) *A G-CW-complex Y is a classifying space for G with respect to $\mathcal{F}$ if and only if the following conditions are satisfied:*
   - *All isotropy groups of Y lie in $\mathcal{F}$;*
   - *For all $H \in \mathcal{F}$, the fixed point set Y^H is contractible.*

(ii) *There exists a classifying space for G with respect to $\mathcal{F}$.*

Proof. (i) This follows from an equivariant version of the Whitehead theorem [Lüc05, Theorem 1.6] applied to the map $Y \to G/G$.

(ii) In view of the first part, such a classifying space can be constructed by inductively attaching cells to kill homotopy groups of the fixed point sets [Lüc89, Proposition 2.3]. □

Remark 2.9. We point out that the construction of classifying spaces in Theorem 2.8 (ii) indeed works for (conjugation-closed) families of subgroups with no additional closure properties. This level of generality is usually not considered in the literature, where classifying spaces are often defined only for families that are intersection-closed or closed under taking arbitrary subgroups.

Many interesting constructions of classifying spaces for intersection-closed families, most notably for Fin, are known [Lüc05, Section 4]. For a (conjugation-closed) family $\mathcal{G}$ of subgroups of G, we can consider the intersection-closed family $\mathcal{F}\langle\mathcal{G}\rangle$ generated by $\mathcal{G}$ (Example 2.6). Then a model for $E_{\mathcal{G}}G$ is given by the G-CW-subcomplex of $E_{\mathcal{F}\langle\mathcal{G}\rangle}G$ consisting of all cells with isotropy in $\mathcal{G}$.

Example 2.10. Let $D_\infty = \langle s, t \mid s^2 = t^2 = e\rangle$ be the infinite dihedral group. A model for $E_{\mathsf{Fin}}D_\infty$ is given by the real line $\mathbb{R}$ on which s and t act via reflection at $0, 1 \in \mathbb{R}$, respectively. Considering the (conjugation-closed) family $\mathsf{Fin} \setminus \mathsf{Tr}$, a model for $E_{\mathsf{Fin}\setminus\mathsf{Tr}}D_\infty$ is given by the subcomplex $\mathbb{Z} \subset \mathbb{R}$, that is $D_\infty/\langle s\rangle \sqcup D_\infty/\langle t\rangle$.

2.3. **$(G, \mathcal{H})$-CW-pairs.** In this section we introduce a notion of pairs of equivariant CW-complexes adapted to a collection of subgroups.

Definition 2.11 (Group pair)**.** A *group pair* is a pair $(G, \mathcal{H})$, consisting of a group G and a collection $\mathcal{H}$ of subgroups of G (see Section 1.5 for the term "collection").

For our purposes, the most important examples of group pairs will arise as pairs of fundamental groups.

Example 2.12 (Fundamental group pair)**.** Let (X, A) be a CW-pair, where X is path-connected, and let $x_0 \in X$. Moreover, we assume that each connected component of A is π_1-injective in X. A group pair $(G, \mathcal{H})$ is a *fundamental group pair* of (X, A) (at the basepoint x_0) if:

- $G = \pi_1(X, x_0)$ and

- $\mathcal{H} = (H_i)_{i\in I}$, where $A = \coprod_{i\in I} A_i$ is a decomposition of A into connected components and for each $i \in I$, there exists a basepoint $x_i \in A_i$ and a path γ_i in X from x_0 to x_i, such that H_i is the subgroup of $\pi_1(X, x_0)$ isomorphic to $\pi_1(A_i, x_i)$ via the homomorphism induced by γ_i.

In this situation, we will also abuse notation and just write $\pi_1(A_i)$ for this subgroup H_i in G. It should be noted that there is always an implicit (fixed) choice of basepoints and paths involved. In many cases, these choices will be of no consequence; when these choices would matter, we will be in situations, where we include all G-conjugates of the $(H_i)_{i\in I}$ in the collection of subgroups in question, and thus avoid ambiguities.

In the sequel, when we speak of a CW-pair (X, A) with fundamental group pair $(G, \mathcal{H})$, the assumption of π_1-injectivity is left implicit.

Definition 2.13 ($(G, \mathcal{H})$-CW-pair)**.** Let $(G, \mathcal{H})$ be a group pair with $\mathcal{H} = (H_i)_{i\in I}$. A $(G, \mathcal{H})$-*CW-pair* is a G-CW-pair (Y, B) together with a decomposition

$$B = \coprod_{i\in I} G \times_{H_i} B_i \,,$$

where B_i is an H_i-CW-complex.

Let $\mathcal{F}$ be a family of subgroups of G. We say that (Y, B) *has isotropy in* $\mathcal{F}$ if all isotropy groups of Y lie in $\mathcal{F}$.

The *relative dimension* $\dim(Y, B) \in \mathbb{N} \cup \{\infty\}$ is the dimension of the relative G-CW-complex (Y, B), that is the largest dimension of a cell of Y which is not contained in B.

A map of $(G, \mathcal{H})$-CW-pairs $f\colon (Y, B) \to (Z, C)$ is a G-map of pairs such that the restriction $f|_B$ is of the form $\coprod_{i\in I} G \times_{H_i} f_i$, where $f_i\colon B_i \to C_i$ is an H_i-map.

If a $(G, \mathcal{H})$-CW-pair (Y, B) as above has isotropy in $\mathcal{F}$, then the isotropy groups of the H_i-CW-complex B_i lie in $\mathcal{F}|_{H_i}$.

In the situation of Definition 2.13, replacing a subgroup H_i by a conjugate gH_ig^{-1} with $g \in G$ corresponds to changing the reference point in the description of the induced space $G \times_{H_i} B_i$. Thus, the collection $\mathcal{H}$ can also be viewed as a collection of representatives of conjugacy classes of subgroups of G.

Example 2.14 (Universal covering pair)**.** Let (X, A) be a CW-pair with fundamental group pair $(G, \mathcal{H})$, where $A = \coprod_{i\in I} A_i$ and $\mathcal{H} = (H_i)_{i\in I} = (\pi_1(A_i))_{i\in I}$ are as in Example 2.12. Denote by $p\colon \widetilde{X} \to X$ the universal covering map. Then there is a G-homeomorphism $p^{-1}(A) \cong \coprod_{i\in I} G \times_{H_i} \widetilde{A_i}$, where $\widetilde{A_i}$ is the universal covering of A_i. This shows that $(\widetilde{X}, p^{-1}(A))$ is a $(G, \mathcal{H})$-CW-pair with isotropy in the trivial family Tr.

2.4. **Classifying spaces of group pairs with respect to a family.** We now let families of subgroups and an additional collection of subgroups interact:

Definition 2.15 (Classifying space for group pairs)**.** Let G be a group, $\mathcal{H}$ be a collection of subgroups of G, and $\mathcal{F}$ be a family of subgroups of G. A *classifying space for the group pair* $(G, \mathcal{H})$ *with respect to* $\mathcal{F}$ is a $(G, \mathcal{H})$-CW-pair (E, D) with the following universal property:

- The pair (E, D) has isotropy in $\mathcal{F}$;
- For each $(G, \mathcal{H})$-CW-pair (Y, B) with isotropy in $\mathcal{F}$, there is up to G-homotopy exactly one G-map $(Y, B) \to (E, D)$ of $(G, \mathcal{H})$-CW-pairs.

We usually denote such classifying spaces by $E_{\mathcal{F}}(G,\mathcal{H})$ (even though they are only unique up to G-homotopy equivalence of pairs).

Models for $E_{\mathcal{F}}(G,\mathcal{H})$ can be constructed as mapping cylinders:

Lemma 2.16 (Existence of classifying spaces for group pairs)**.** *Let $(G,\mathcal{H})$ be a group pair and let $\mathcal{F}$ be family of subgroups of G. Then there exists a classifying space for the group pair $(G,\mathcal{H})$ with respect to $\mathcal{F}$.*

Proof. We write the collection $\mathcal{H}$ as $(H_i)_{i\in I}$. For every $i \in I$, let $E_{\mathcal{F}|_{H_i}} H_i$ be a classifying space for H_i with respect to the family $\mathcal{F}|_{H_i}$. The induced G-CW-complex $G \times_{H_i} (E_{\mathcal{F}|_{H_i}} H_i)$ has isotropy in the family $\mathcal{F}$ and hence there exists a G-map $G \times_{H_i} (E_{\mathcal{F}|_{H_i}} H_i) \to E_{\mathcal{F}}G$ (that is unique up to G-homotopy). Then the mapping cylinder of the G-map

$$\coprod_{i\in I} G \times_{H_i} (E_{\mathcal{F}|_{H_i}} H_i) \to E_{\mathcal{F}}G$$

is a model for $E_{\mathcal{F}}(G,\mathcal{H})$. This follows from the universal properties of the classifying spaces $E_{\mathcal{F}|_{H_i}} H_i$ and $E_{\mathcal{F}}G$. □

3. Relative cohomological dimension and mapping degrees

We discuss an application of classifying spaces for group pairs and relative group cohomology to maps between manifolds. We obtain inheritance results for the freeness of fundamental groups of manifolds in terms of mapping degrees. The results of this section are independent from the rest of the paper.

3.1. Relative group cohomology.

We recall the definition of relative cohomology of a group pair and a characterisation of groups pairs with relative cohomological dimension one.

Definition 3.1 (Relative cohomology of group pairs)**.** Let $(G,\mathcal{H})$ be a group pair and R be a commutative ring. We define the *relative cohomology* $H^*(G,\mathcal{H};V)$ with coefficients in an RG-module V to be the R-module

$$H^*(G,\mathcal{H};V) \coloneqq H^*_G(E_{\mathsf{Tr}}(G,\mathcal{H});V)\,,$$

where $E_{\mathsf{Tr}}(G,\mathcal{H})$ is a classifying space for $(G,\mathcal{H})$ with respect to the trivial family. Here the equivariant cohomology $H^*_G(E_{\mathsf{Tr}}(G,\mathcal{H});V)$ is by definition the cohomology $H^*(BG, \coprod_{H\in\mathcal{H}} BH; V)$ with twisted coefficients.

Definition 3.2 (Relative cohomological dimension)**.** The *relative cohomological dimension* $\mathrm{cd}_R(G,\mathcal{H})$ of the group pair $(G,\mathcal{H})$ over the ring R is defined as follows:

$$\mathrm{cd}_R(G,\mathcal{H}) \coloneqq \sup\{n \in \mathbb{N} \mid H^n(G,\mathcal{H};V) \ncong 0 \text{ for some } RG\text{-module } V\}.$$

For simplicity we denote $\mathrm{cd}_{\mathbb{Z}}(G,\mathcal{H})$ by $\mathrm{cd}(G,\mathcal{H})$.

To illustrate these definitions, we mention that for $\mathcal{H} = (H_i)_{i\in I}$ the long exact sequence for the pair $E_{\mathsf{Tr}}(G,\mathcal{H})$ takes the following form:

$$\cdots \to \prod_{i\in I} H^{n-1}(H_i;V) \to H^n(G,\mathcal{H};V) \to H^n(G;V) \to \prod_{i\in I} H^n(H_i;V) \to \cdots\,.$$

This shows that $\mathrm{cd}_R(G,\mathcal{H}) \le n$ if and only if for all RG-modules V the restriction map $H^k(G;V) \to \prod_{i\in I} H^k(H_i;V)$ is an isomorphism for $k > n$ and an epimorphism for $k = n$.

Remark 3.3. Let $(G,\mathcal{H})$ be a group pair with $\mathcal{H}=(H_i)_{i\in I}$. While our definition of $H^*(G,\mathcal{H};V)$ is purely topological, one may also define it algebraically via derived functors. More precisely, consider the augmentation RG-map $R[\coprod_{i\in I} G/H_i]\to R$ and let Δ denote its kernel. Then there exists a natural isomorphism

$$H^*(G,\mathcal{H};V)\cong \operatorname{Ext}^{*-1}_{RG}(\Delta,V)\,.$$

In this situation we have that $\operatorname{cd}_R(G,\mathcal{H})=\operatorname{pd}_{RG}(\Delta)+1$, where pd_{RG} denotes the projective dimension.

Many of the usual cohomological tools for group cohomology have been developed for the relative case as well [Tak59, BE78, Alo91].

In the case of a single subgroup H, the relation between $H^*(G,H;V)$ and the Bredon cohomology $H^*_G(E_{\mathcal{F}\langle H\rangle}G;V)$ has recently been investigated [ANCM17, ANCMSS21].

By the work of Stallings [Sta68] and Swan [Swa69] groups of cohomological dimension one are precisely the free groups. The following is a generalisation to the relative setting.

Theorem 3.4 (Group pairs of relative cohomological dimension one [Dic80, Alo91])**.** *Let $(G,\mathcal{H})$ be a group pair with $\mathcal{H}=(H_i)_{i\in I}$. Then the following are equivalent:*

(i) $\operatorname{cd}(G,\mathcal{H})=1$*;*
(ii) *There exists a free group F such that $G\cong F**_{i\in I}H_i$.*

3.2. **Mapping degrees and monotonicity.** For maps between manifolds with π_1-injective boundary components, we prove monotonicity results on the cohomological dimension of the fundamental group pairs.

First, we introduce some notation. Let (X,A) be a CW-pair with X path-connected. Let $A=\coprod_{i\in I}A_i$ be a decomposition into connected components and assume that each A_i is π_1-injective in X. We denote by $(\pi_1(X),\pi_1(A))$ a fundamental group pair of (X,A) (Example 2.12) and by $p\colon \widetilde{X}\to X$ the universal covering map. Let $H^*(X,A;V)$ be the cohomology with twisted coefficients in a $\pi_1(X)$-module V. Then the classifying map $\varphi_{(X,A)}\colon (\widetilde{X},p^{-1}(A))\to E_{\mathsf{Tr}}(\pi_1(X),\pi_1(A))$ induces a map on cohomology:

$$H^*(\varphi_{(X,A)})\colon H^*(\pi_1(X),\pi_1(A);V)\to H^*(X,A;V)\,.$$

Lemma 3.5. *Let (X,A) be a CW-pair with fundamental group pair $(\pi_1(X),\pi_1(A))$ as above. The map $H^2(\varphi_{(X,A)})\colon H^2(\pi_1(X),\pi_1(A);V)\to H^2(X,A;V)$ is injective for every $\pi_1(X)$-module V.*

Proof. We argue via the four lemma for monomorphisms. We consider the following commutative diagram, where the coefficient module is omitted:

$$\begin{array}{ccccccc}
H^1(\pi_1(X)) & \longrightarrow & \prod_{i\in I} H^1(\pi_1(A_i)) & \longrightarrow & H^2(\pi_1(X),\pi_1(A)) & \longrightarrow & H^2(\pi_1(X)) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle H^2(\varphi_{(X,A)})} & & \downarrow \\
H^1(X) & \longrightarrow & H^1(A) & \longrightarrow & H^2(X,A) & \longrightarrow & H^2(X)\,.
\end{array}$$

Here all vertical maps are induced by the respective classifying maps. Since a model for $E_{\mathsf{Tr}}(\pi_1(X))$ [resp. for $E_{\mathsf{Tr}}(\pi_1(A_i))$] can be built from $\widetilde{X}$ [resp. from $\widetilde{A_i}$] by attaching cells of dimension greater than or equal to 3, the first and second vertical arrows are isomorphisms, while the last vertical arrow is injective. Applying the four lemma for monomorphisms, we conclude that $H^2(\varphi_{(X,A)})$ is injective. □

We use the convention (Section 1.5) to say that a manifold M has *π_1-injective boundary* ∂M if every component of ∂M is π_1-injective in M.

Theorem 3.6 (Mapping degree and relative cohomological dimension one)**.** *Let $f\colon (M,\partial M) \to (N,\partial N)$ be a map between oriented compact connected manifolds of the same dimension with π_1-injective boundary. Then the following hold:*

(i) *If $\deg(f) = \pm 1$ and $\mathrm{cd}(\pi_1(M),\pi_1(\partial M)) \leq 1$, then $\mathrm{cd}(\pi_1(N),\pi_1(\partial N)) \leq 1$;*
(ii) *If $\deg(f) \neq 0$ and $\mathrm{cd}_{\mathbb{Q}}(\pi_1(M),\pi_1(\partial M)) \leq 1$, then $\mathrm{cd}_{\mathbb{Q}}(\pi_1(N),\pi_1(\partial N)) \leq 1$.*

Proof. We proceed by contraposition. Let $R = \mathbb{Z}$ [resp. $R = \mathbb{Q}$] and suppose that $\mathrm{cd}_R(\pi_1(N),\pi_1(\partial N)) > 1$. Then by a dimension shifting argument, there exists an $R\pi_1(N)$-module V such that $H^2(\pi_1(N),\pi_1(\partial N);V)$ is non-trivial (this works as in the absolute case [DKR08, Lemma 2.6][Bro94, Chapter III.7]). We denote by $f^{-1}V$ the $R\pi_1(M)$-module that is obtained from V by restriction along $\pi_1(f)$. Consider the following commutative diagram:

$$\begin{array}{ccc} H^2(N,\partial N;V) & \xrightarrow{\quad H^2(f)\quad} & H^2(M,\partial M;f^{-1}V) \\ {\scriptstyle H^2(\varphi_{(N,\partial N)})}\uparrow & & \uparrow{\scriptstyle H^2(\varphi_{(M,\partial M)})} \\ H^2(\pi_1(N),\pi_1(\partial N);V) & \xrightarrow{H^2(\pi_1(f))} & H^2(\pi_1(M),\pi_1(\partial M);f^{-1}V)\,. \end{array}$$

Here the vertical maps, which are induced by the respective classifying maps, are injective in degree 2 (Lemma 3.5). By Poincaré–Lefschetz duality with twisted coefficients, there exists an Umkehr map

$$f_!\colon H^2(M,\partial M;f^{-1}V) \to H^2(N,\partial N;V)$$

such that the composition $f_! \circ H^2(f)\colon H^2(N,\partial N;V) \to H^2(N,\partial N;V)$ is given by multiplication with $\deg(f)$ [Bre97, Definition VI.11.2, Proposition VI.14.1]. Hence the map $H^2(f)$ is injective in each of the following cases:

(i) If $\deg(f) = \pm 1$ and $R = \mathbb{Z}$;
(ii) If $\deg(f) \neq 0$ and $R = \mathbb{Q}$.

This shows that in the situations (i) and (ii) the composition

$$H^2(f) \circ H^2(\varphi_{(N,\partial N)}) = H^2(\varphi_{(M,\partial M)}) \circ H^2(\pi_1(f))$$

is injective, whence $H^2(\pi_1(f))$ is injective. Therefore the relative cohomology group $H^2(\pi_1(M),\pi_1(\partial M);f^{-1}V)$ is non-trivial and we have $\mathrm{cd}_R(\pi_1(M),\pi_1(\partial M)) > 1$. □

Remark 3.7. If the universal coverings of N and ∂N are k-connected, then the map $H^{k+1}(\varphi_{(N,\partial N)})$ is injective and hence similar monotonicity results hold for cohomological dimension at most k.

Theorem 3.6 readily implies the following:

Corollary 3.8. *Let $f\colon (M,\partial M) \to (N,\partial N)$ be a map between oriented compact connected manifolds of the same dimension with π_1-injective boundary. Suppose that $\partial M = \coprod_{i=1}^m M_i$ and $\partial N = \coprod_{i=1}^n N_i$ are decompositions into connected components. If $\deg(f) = \pm 1$ and there exists a free group F_M such that*

$$\pi_1(M) \cong F_M * \ast_{i=1}^m \pi_1(M_i)\,,$$

then there exists a free group F_N such that $\pi_1(N) \cong F_N * \ast_{i=1}^n \pi_1(N_i)$.

Proof. This follows from Theorem 3.6 (i) and the group-theoretic characterisation of relative cohomological dimension one (Theorem 3.4). □

Examples of manifolds satisfying the assumptions of Corollary 3.8 are the following:

Example 3.9. Let F_k be a free group of rank k, and let $\mathcal{H} = (H_1, \dots, H_m)$ be a finite collection of finitely presented groups. Then for every $n \geq 7$, there exists a compact connected n-dimensional manifold $(M, \partial M)$ with fundamental group pair $(G, \mathcal{H})$ such that $G \cong F_k * *_{i=1}^m H_i$. Indeed, let L_i be an oriented closed connected 4-manifold with $\pi_1(L_i) \cong H_i$ and consider

$$(M, \partial M) := (\#_{i=1}^k S^1 \times S^{n-1}) \# (\#_{i=1}^m L_i \times D^{n-4}).$$

Then $\partial M \cong \coprod_{i=1}^m L_i \times S^{n-5}$ and $\pi_1(M) \cong F_k * *_{i=1}^m \pi_1(L_i)$.

In the case of closed manifolds, Corollary 3.8 provides a simplified proof of a result by Dranishnikov and Rudyak [DR09, Theorem 5.2], without making use of the Berstein class. We also obtain the following analogue for maps of non-zero degree:

Corollary 3.10. *Let $f\colon M \to N$ be a map between oriented closed connected manifolds of the same dimension with $\deg(f) \neq 0$. If $\pi_1(M)$ is the fundamental group of a graph of finite groups, then so is $\pi_1(N)$. In particular, if $\pi_1(M)$ is virtually free, then $\pi_1(N)$ is virtually free.*

Proof. This follows from Theorem 3.6 (ii) and Dunwoody's characterisation [Dun79] of groups of cohomological dimension one over arbitrary rings. As fundamental groups of closed manifolds are finitely generated, this characterisation can also be expressed in terms of virtual freeness [Dun79, Corollary 1.2]. □

We are not aware of a relative version of Dunwoody's result that would characterise group pairs of relative cohomological dimension one over arbitrary rings. The results in this section motivate the following:

Question 3.11. For which other classes $\mathcal{C}$ of groups does the following hold? Whenever $f\colon M \to N$ is a map between oriented closed connected manifolds of the same dimension with $\deg(f) \neq 0$ [or $\deg(f) = 1$] and $\pi_1(M)$ is the fundamental group of a graph of groups from $\mathcal{C}$, also $\pi_1(N)$ must be the fundamental group of a graph of groups from $\mathcal{C}$.

Positive answers to Question 3.11 lead to corresponding monotonicity results for the generalised LS-category ≤ 2 associated with the class $\mathcal{C}$, provided that $\mathcal{C}$ is closed under isomorphisms, subgroups, and quotients [CLM22, Corollary 5.4 and the following paragraph].

4. Relative open covers and equivariant nerve pairs

In this section, we study equivariant nerves of open covers in a relative setting. Given an open cover of a space, the nerve of the cover gives an approximation of the space. Considering actions on spaces and compatible covers leads to equivariant nerves, studied by Löh and Sauer [LS20] for universal covering spaces with the action by the fundamental group. We adapt this approach to pairs of spaces.

4.1. **Open covers and nerve pairs.** We fix some notation and terminology on open covers and their nerves.

Let Y be a space and $\mathcal{V}$ be a cover of Y by path-connected open subsets. We regard $\mathcal{V}$ as a set of subsets of Y (and not as a collection of subsets).

Definition 4.1 ($\mathcal{F}$-Cover)**.** Let $\mathcal{F}$ be a family of subgroups of $\pi_1(Y)$. We say that $\mathcal{V}$ is an *$\mathcal{F}$-cover* of Y if $\operatorname{im}(\pi_1(V,x) \to \pi_1(Y,x)) \in \mathcal{F}$ for all $V \in \mathcal{V}$ and all $x \in V$.

Definition 4.2 (Convex cover)**.** The cover $\mathcal{V}$ of Y is said to be *convex* if every intersection of finitely many elements of $\mathcal{V}$ is path-connected or empty.

The *multiplicity* $\operatorname{mult}(\mathcal{V})$ of $\mathcal{V}$ is defined as follows:

$$\operatorname{mult}(\mathcal{V}) \coloneqq \sup\Big\{ n \in \mathbb{N} \;\Big|\; \bigcap_{i=1}^{n} V_i \neq \emptyset \text{ for some pairwise different } V_1, \dots, V_n \in \mathcal{V} \Big\}.$$

The *nerve* $N(\mathcal{V})$ of $\mathcal{V}$ is the (abstract) simplicial complex with vertex set $\mathcal{V}$; pairwise different $V_0, \dots, V_n \in \mathcal{V}$ span an n-simplex in $N(\mathcal{V})$ if $V_0 \cap \dots \cap V_n \neq \emptyset$. By definition, $\dim(N(\mathcal{V})) = \operatorname{mult}(\mathcal{V}) - 1$.

Let $|N(\mathcal{V})|$ be the geometric realisation of the nerve $N(\mathcal{V})$. Given a partition of unity $(\psi_V)_{V \in \mathcal{V}}$ on Y subordinate to $\mathcal{V}$, there is an associated *nerve map*

$$\mu \colon Y \to |N(\mathcal{V})|\,, \quad y \mapsto \mu(y) \coloneqq \sum_{V \in \mathcal{V}} \psi_V(y) \cdot V\,. \tag{4.1}$$

The nerve map is unique up to homotopy, since different choices of partitions of unity lead to homotopic nerve maps.

Definition 4.3 (Relative multiplicity)**.** For a subspace B of Y, we define the *relative multiplicity* $\operatorname{mult}_B(\mathcal{V})$ of $\mathcal{V}$ (with respect to B) as follows:

$$\operatorname{mult}_B(\mathcal{V}) \coloneqq \sup\Big\{ n \in \mathbb{N} \;\Big|\; \bigcap_{i=1}^{n} V_i \neq \emptyset \text{ and } B \cap \Big(\bigcap_{i=1}^{n} V_i\Big) = \emptyset \text{ for some pairwise different } V_1, \dots, V_n \in \mathcal{V} \Big\}.$$

Definition 4.4 (Nerve pair)**.** For a subspace B of Y, we denote by $N_B(\mathcal{V})$ the simplicial subcomplex of $N(\mathcal{V})$ with vertex set $\mathcal{V}_B \coloneqq \{V \in \mathcal{V} \mid V \cap B \neq \emptyset\}$, and pairwise different $V_0, \dots, V_n \in \mathcal{V}_B$ span a simplex in $N_B(\mathcal{V})$ if $V_0 \cap \dots \cap V_n \cap B \neq \emptyset$. By construction we have $N(\mathcal{V}) = N_Y(\mathcal{V})$.

The nerve map $\mu \colon Y \to |N(\mathcal{V})|$ induces a map of pairs

$$\mu \colon (Y, B) \to (|N(\mathcal{V})|, |N_B(\mathcal{V})|)\,.$$

The *relative dimension* $\dim(N(\mathcal{V}), N_B(\mathcal{V}))$ is the dimension of the relative simplicial complex $(N(\mathcal{V}), N_B(\mathcal{V}))$. By definition, $\dim(N(\mathcal{V}), N_B(\mathcal{V})) = \operatorname{mult}_B(\mathcal{V}) - 1$.

4.2. **Equivariant nerve pairs.** We now consider open covers that are compatible with a group action, giving rise to an action on their nerve.

Definition 4.5 (Invariant cover and partition of unity)**.** Let Y be a G-CW-complex and $\mathcal{V}$ be a cover of Y by path-connected open subsets. We say that $\mathcal{V}$ is *G-invariant* if for all $g \in G$ and $V \in \mathcal{V}$, we have $g \cdot V \in \mathcal{V}$. For $V \in \mathcal{V}$, we write

$$\operatorname{Stab}_G(V) \coloneqq \{g \in G \mid g \cdot V = V\}\,.$$

For a family of subgroups $\mathcal{F}$ of G, we say that the cover $\mathcal{V}$ *has isotropy in* $\mathcal{F}$ if for all $V \in \mathcal{V}$, we have $\operatorname{Stab}_G(V) \in \mathcal{F}$.

A partition of unity $(\psi_V)_{V\in\mathcal{V}}$ on Y subordinate to a G-invariant cover $\mathcal{V}$ is said to be *G-invariant* if for all $g \in G$ and $y \in Y$, we have

$$\psi_V(y) = \psi_{g\cdot V}(g \cdot y)\,.$$

The key examples will come from covers of a space X giving rise to $\pi_1(X)$-invariant covers of the universal covering space $\widetilde{X}$ (Example 4.8).

We recall basic properties of equivariant nerves [LS20, Lemmas 4.8 and 4.11] and their proofs for completeness.

Lemma 4.6 (Equivariant nerve)**.** *Let Y be a G-CW-complex and $\mathcal{V}$ be a G-invariant cover of Y. Then the following hold:*

(i) *Let B be an H-invariant subcomplex of Y for a subgroup H of G. Then $N_B(\mathcal{V})$ is an H-simplicial complex and its geometric realisation $|N_B(\mathcal{V})|$ is an H-CW-complex. In particular, $N(\mathcal{V})$ is a G-simplicial complex;*

(ii) *Suppose that $g\cdot V \cap V \neq \emptyset$ implies $g\cdot V = V$ for all $g \in G, V \in \mathcal{V}$. Let $\mathcal{F}$ be an intersection-closed family of subgroups of G. If the cover $\mathcal{V}$ has isotropy in $\mathcal{F}$, then the G-CW-complex $|N(\mathcal{V})|$ has isotropy in $\mathcal{F}$;*

(iii) *Let $(\psi_V)_{V\in\mathcal{V}}$ be a G-invariant partition of unity on Y subordinate to $\mathcal{V}$. Then the induced nerve map $\mu\colon Y \to |N(\mathcal{V})|$ is G-equivariant.*

Proof. (i) To show that $N_B(\mathcal{V})$ is an H-simplicial complex, it suffices to prove that the H-action sends simplices of $N_B(\mathcal{V})$ to simplices of $N_B(\mathcal{V})$. Let v be a vertex of $N_B(\mathcal{V})$ corresponding to $V \in \mathcal{V}$ with $V \cap B \neq \emptyset$. Then for every $h \in H$, we have $\emptyset \neq h \cdot (V \cap B) = (h \cdot V) \cap (h \cdot B) \subset (h \cdot V) \cap B$. This shows that the vertex $h \cdot v$ of $N(\mathcal{V})$ corresponding to $h \cdot V \in \mathcal{V}$ lies in $N_B(\mathcal{V})$. The same argument also extends to higher-dimensional simplices, which proves the claim. Then the geometric realisation $|N_B(\mathcal{V})|$ is an H-CW-complex (Example 2.2).

(ii) We show that the isotropy groups of the vertices of the barycentric subdivision of $N(\mathcal{V})$ lie in $\mathcal{F}$. This is indeed sufficient; since the action is simplicial the stabiliser of every interior point of a k-simplex in the barycentric subdivision of $N(\mathcal{V})$ is the intersection of the stabilisers of its $k+1$ vertices. Then the fact that $\mathcal{F}$ is closed under finite intersections yields the thesis.

Let v be a vertex in the barycentric subdivision of $N(\mathcal{V})$, associated to a k-simplex corresponding to $V_0, \dots, V_k \in \mathcal{V}$ with $V_0 \cap \dots \cap V_k \neq \emptyset$. It remains to show that the subgroup

$$G_v = \big\{ g \in G \;\big|\; \{g\cdot V_0, \dots, g\cdot V_k\} = \{V_0, \dots, V_k\} \big\}$$

of G lies in the family $\mathcal{F}$. For $k = 0$, we know that $G_v = \mathrm{Stab}_G(V_0) \in \mathcal{F}$ by the assumption that $\mathcal{V}$ has isotropy in $\mathcal{F}$. On the other hand, for $k > 0$, by the assumption that $g \cdot V \cap V \neq \emptyset$ implies $g \cdot V = V$, we have that $g \cdot V_i = V_j$ for $i, j \in \{0, \dots, k\}$ implies $i = j$. Hence $G_v = \mathrm{Stab}_G(V_0) \cap \dots \cap \mathrm{Stab}_G(V_k)$. This group lies in $\mathcal{F}$ since $\mathcal{F}$ is closed under finite intersections.

(iii) The G-invariance of the partition of unity implies the G-equivariance of the nerve map (4.1) as follows: For all $g \in G$ and all $y \in Y$, we have

$$\begin{aligned}\mu(g \cdot y) &= \sum_{V\in\mathcal{V}} \psi_V(g \cdot y) \cdot V = \sum_{V\in\mathcal{V}} \psi_{g^{-1}\cdot V}(y) \cdot V \\ &= \sum_{V\in\mathcal{V}} \psi_V(y) \cdot (g \cdot V) = g \cdot \mu(y)\,.\end{aligned}$$

This finishes the proof. □

We extend the previous results to the relative situation:

Lemma 4.7 (Equivariant nerve pair)**.** *Let $(G,\mathcal{H})$ be a group pair and let (Y,B) be a $(G,\mathcal{H})$-CW-pair. Let $\mathcal{F}$ be an intersection-closed family of subgroups of G and $\mathcal{V}$ be a G-invariant cover of Y with isotropy in $\mathcal{F}$. Suppose that the following hold:*

(i) *For all $V \in \mathcal{V}, g \in G$ with $g \cdot V \cap V \neq \emptyset$, we have $g \cdot V = V$;*
(ii) *There exists a G-invariant partition of unity on Y subordinate to $\mathcal{V}$;*
(iii) *For all $V \in \mathcal{V}$ with $V \cap B \neq \emptyset$, the intersection $V \cap B$ is connected.*

Then $(|N(\mathcal{V})|, |N_B(\mathcal{V})|)$ is a $(G,\mathcal{H})$-CW-pair with isotropy in $\mathcal{F}$. Moreover, the nerve map $\mu\colon Y \to |N(\mathcal{V})|$ induces a map of $(G,\mathcal{H})$-CW-pairs:

$$\mu\colon (Y,B) \to \big(|N(\mathcal{V})|, |N_B(\mathcal{V})|\big)\,.$$

Proof. By Lemma 4.6, assumption (i) implies that $|N(\mathcal{V})|$ is a G-CW-complex with isotropy in $\mathcal{F}$, and assumption (ii) implies that the nerve map $\mu\colon Y \to |N(\mathcal{V})|$ is G-equivariant.

We write the collection $\mathcal{H}$ as $(H_i)_{i\in I}$. Since (Y,B) is a $(G,\mathcal{H})$-CW-pair, we have a decomposition $B = \coprod_{i\in I} G \times_{H_i} B_i$, where B_i is an H_i-CW-complex. We identify B_i with the subset $[e,B_i] \subset B \subset Y$, where $e \in G$ denotes the neutral element. We also identify the set of vertices of $|N(\mathcal{V})|$ with $\mathcal{V}$. There is a G-map

$$\Phi\colon \coprod_{i\in I} G \times_{H_i} |N_{B_i}(\mathcal{V})| \to |N_B(\mathcal{V})|\,,$$

mapping a vertex $[g,V]$ of $G \times_{H_i} |N_{B_i}(\mathcal{V})|$ to the vertex $g\cdot V$ of $|N_B(\mathcal{V})|$, and that is defined by affine extension. The affine extension is well-defined, since the images of vertices spanning a simplex in $|N_{B_i}(\mathcal{V})|$ also span a simplex in $|N_B(\mathcal{V})|$. We claim that assumption (iii) implies that Φ is a G-homeomorphism.

Indeed, the inverse map Φ^{-1} is given as follows: For a vertex V of $|N_B(\mathcal{V})|$, we have $V \cap B \neq \emptyset$ and hence the intersection $V \cap B$ is connected by assumption (iii). Thus, there exists a unique element $i \in I$ and a unique coset $gH_i \in G/H_i$ such that $V \cap [g,B_i] \neq \emptyset$. Since $V \cap [g,B_i] = g(g^{-1}\cdot V \cap B_i)$, we may define Φ^{-1} to map the vertex V to the vertex $[g, g^{-1}\cdot V]$. This assignment is G-equivariant: Indeed, for every $g' \in G$, we have $\emptyset \neq g'(V \cap [g,B_i]) = g'g(g^{-1}\cdot V \cap B_i)$. Hence under Φ^{-1} the vertex $g'\cdot V$ is mapped to the vertex $[g'g, g^{-1}\cdot V]$.

Then Φ^{-1} is determined by affine extension. This is well-defined because the images of vertices spanning a simplex in $|N_B(\mathcal{V})|$ also span a simplex in the corresponding $|N_{B_i}(\mathcal{V})|$. Thus Φ is a G-homeomorphism, showing that $(|N(\mathcal{V})|, |N_B(\mathcal{V})|)$ is a $(G,\mathcal{H})$-CW-pair.

The G-map $\mu\colon (Y,B) \to (|N(\mathcal{V})|, |N_B(\mathcal{V})|)$ is a map of $(G,\mathcal{H})$-CW-pairs, since we have $\mu(B_i) \subset |N_{B_i}(\mathcal{V})|$. □

4.3. **Relative open covers.** We study our main example of equivariant nerve pairs coming from lifted covers of CW-pairs.

Example 4.8 (Lifted cover)**.** Let X be a connected CW-complex, let $G \coloneqq \pi_1(X)$, and let $p\colon \widetilde{X} \to X$ denote the universal covering. For a cover $\mathcal{U}$ of X by path-connected open subsets, we consider the *lifted cover* $\widetilde{\mathcal{U}}$ of $\widetilde{X}$:

$$\widetilde{\mathcal{U}} \coloneqq \big\{V \subset \widetilde{X} \;\big|\; V \text{ is a path-connected component of } p^{-1}(U) \text{ for some } U \in \mathcal{U}\big\}\,.$$

Clearly, $\widetilde{\mathcal{U}}$ is a G-invariant cover of $\widetilde{X}$. Note that for every $g \in G$, $V \in \widetilde{\mathcal{U}}$, the condition $g \cdot V \cap V \neq \emptyset$ implies $g \cdot V = V$. Moreover, for every $V \in \widetilde{\mathcal{U}}$ we have

that $\mathrm{Stab}_G(V)$ is conjugate to $\mathrm{im}(\pi_1(p(V)) \to \pi_1(X))$. This shows that if $\mathcal{U}$ is an $\mathcal{F}$-cover of X, then $\widetilde{\mathcal{U}}$ has isotropy in $\mathcal{F}$.

Every given partition of unity $(\varphi_U)_{U\in\mathcal{U}}$ on X subordinate to $\mathcal{U}$ lifts to a G-invariant partition of unity $(\widetilde{\varphi}_V)_{V\in\widetilde{\mathcal{U}}}$ on $\widetilde{X}$ subordinate to $\widetilde{\mathcal{U}}$ as follows: For $V \in \widetilde{\mathcal{U}}$, define

$$\widetilde{\varphi}_V \coloneqq \chi_V \cdot (\varphi_{p(V)} \circ p)\colon \widetilde{X} \to [0,1]\,,$$

where $\chi_V\colon \widetilde{X} \to [0,1]$ denotes the characteristic function on $V \subset \widetilde{X}$. Let ν and $\widetilde{\nu}$ be the nerve maps associated to $(\varphi_U)_{U\in\mathcal{U}}$ and $(\widetilde{\varphi}_V)_{V\in\widetilde{\mathcal{U}}}$, respectively.

The simplicial map $N(p)\colon N(\widetilde{\mathcal{U}}) \to N(\mathcal{U})$, under which a simplex of $N(\widetilde{\mathcal{U}})$ corresponding to $V_0,\dots,V_k \in \widetilde{\mathcal{U}}$ is mapped to the simplex of $N(\mathcal{U})$ corresponding to $p(V_0),\dots,p(V_k) \in \mathcal{U}$, makes the following diagram commute:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\ \widetilde{\nu}\ } & |N(\widetilde{\mathcal{U}})| \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle |N(p)|} \\ X & \xrightarrow{\ \nu\ } & |N(\mathcal{U})|\,. \end{array}$$

Given a pair of spaces (X, A), we introduce conditions on covers of X requiring a certain regularity near the subspace A. These conditions will give rise to desirable properties of the lifted covers.

Definition 4.9 (Relative cover)**.** Let (X, A) be a pair of spaces with path-connected ambient space X. A *relative cover* of (X, A) is a cover $\mathcal{U}$ of X by path-connected open subsets such that for all $U \in \mathcal{U}$ the following hold:

(RC1) If $U \cap A \neq \emptyset$, then $U \cap A$ is path-connected;
(RC2) If $U \cap A \neq \emptyset$, then the inclusion

$$\mathrm{im}\big(\pi_1(U \cap A, x) \to \pi_1(X, x)\big) \hookrightarrow \mathrm{im}\big(\pi_1(U, x) \to \pi_1(X, x)\big)$$

is an isomorphism for some (whence every) $x \in U \cap A$.

A relative open cover $\mathcal{U}$ is *weakly convex* if for every $k \in \mathbb{N}$ and all $U_1,\dots,U_k \in \mathcal{U}$ with $U_1 \cap \dots \cap U_k \cap A \neq \emptyset$, each path-connected component of $U_1 \cap \dots \cap U_k$ intersects A.

We also say that a relative open cover $\mathcal{U}$ of (X, A) is *convex* if the underlying cover $\mathcal{U}$ of X is convex. Clearly, every convex relative cover is in particular weakly convex.

Given a family $\mathcal{F}$ of subgroups of $\pi_1(X)$, a relative cover $\mathcal{U}$ of (X, A) is a *relative $\mathcal{F}$-cover* if the cover $\mathcal{U}$ of X is an $\mathcal{F}$-cover.

Keeping the same notation as in Example 4.8, we have the following:

Proposition 4.10 (Equivariant nerve pair of lifted covers)**.** *Let (X, A) be a CW-pair with fundamental group pair $(G, \mathcal{H})$. Let $\mathcal{F}$ be an intersection-closed family of subgroups of G and $\mathcal{U}$ be a relative $\mathcal{F}$-cover of X. Let ν and $\widetilde{\nu}$ be the above nerve maps of $\mathcal{U}$ and $\widetilde{\mathcal{U}}$, respectively.*

Then $(|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|)$ is a $(G, \mathcal{H})$-CW-pair with isotropy in $\mathcal{F}$ and the map $\widetilde{\nu}$ induces a map of $(G, \mathcal{H})$-CW-pairs

$$\widetilde{\nu}\colon \big(\widetilde{X}, p^{-1}(A)\big) \to \big(|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|\big)$$

that makes the following diagram commute:

$$\begin{array}{ccc} \big(\widetilde{X}, p^{-1}(A)\big) & \xrightarrow{\widetilde{\nu}} & \big(|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|\big) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle |N(p)|} \\ (X, A) & \xrightarrow{\nu} & \big(|N(\mathcal{U})|, |N_A(\mathcal{U})|\big)\,. \end{array}$$

Moreover, we have the following:

(i) *If $\mathcal{U}$ is weakly convex, then*

$$\dim\big(N(\widetilde{\mathcal{U}}), N_{p^{-1}(A)}(\widetilde{\mathcal{U}})\big) = \operatorname{mult}_A(\mathcal{U}) - 1\,;$$

(ii) *If $\mathcal{U}$ is convex, then the map $N(p)$ induces isomorphisms of simplicial complexes:*

$$G\backslash N(\widetilde{\mathcal{U}}) \cong N(\mathcal{U})\,;$$
$$G\backslash N_{p^{-1}(A)}(\widetilde{\mathcal{U}}) \cong N_A(\mathcal{U})\,.$$

Proof. To show that $(|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|)$ is a $(G, \mathcal{H})$-CW-pair with isotropy in $\mathcal{F}$, we verify that the lifted G-invariant cover $\widetilde{\mathcal{U}}$ of $\widetilde{X}$ satisfies all assumptions of Lemma 4.7. By Example 4.8, we know that $\widetilde{\mathcal{U}}$ has isotropy in $\mathcal{F}$, that there exists a G-invariant partition of unity on $\widetilde{X}$ subordinate to $\widetilde{\mathcal{U}}$, and that $g \cdot V \cap V \neq \emptyset$ implies $g \cdot V = V$ for all $g \in G, V \in \widetilde{\mathcal{U}}$. Hence, we are left to show that if $V \in \widetilde{\mathcal{U}}$ with $V \cap p^{-1}(A) \neq \emptyset$, then $V \cap p^{-1}(A)$ is connected.

Assume for a contradiction that $V \cap p^{-1}(A)$ is disconnected. Let us set $U \coloneqq p(V)$. By condition (RC1) we know that $U \cap A$ is connected. This shows that there exists a point $a \in U \cap A$ with two lifts $\widetilde{a}_1, \widetilde{a}_2$ contained in different components of $V \cap p^{-1}(A)$. Since V is path-connected, there exists a path γ in V connecting $\widetilde{a}_1$ to $\widetilde{a}_2$. By construction, the image of γ under p is a loop $p_*\gamma$ in U based at a. Then, by condition (RC2), the homotopy class $[p_*\gamma] \in \pi_1(X, a)$ admits a representative whose support is contained in $U \cap A$. Thus, there exists a lifted homotopy in $\widetilde{X}$ relative to the endpoints from γ to a path in $V \cap p^{-1}(A)$. This contradicts the fact that $\widetilde{a}_1$ and $\widetilde{a}_2$ lie in different components of $V \cap p^{-1}(A)$. Hence Lemma 4.7 applies and yields the claim.

(i) We show that $\operatorname{mult}_{p^{-1}(A)}(\widetilde{\mathcal{U}}) = \operatorname{mult}_A(\mathcal{U})$, which immediately implies the claim. The inequality $\operatorname{mult}_{p^{-1}(A)}(\widetilde{\mathcal{U}}) \geq \operatorname{mult}_A(\mathcal{U})$ is clear. To show the opposite inequality, let $V_1, \dots, V_k \in \widetilde{\mathcal{U}}$ with $\bigcap_{i=1}^k V_i \neq \emptyset$ and $\bigcap_{i=1}^k V_i \cap p^{-1}(A) = \emptyset$. We claim that $\bigcap_{i=1}^k p(V_i) \cap A = \emptyset$, whence $k \leq \operatorname{mult}_A(\mathcal{U})$ because the $(p(V_i))_i$ are pairwise different. Indeed, assume for a contradiction that $\bigcap_{i=1}^k p(V_i) \cap A \neq \emptyset$. Take a point $\widetilde{x} \in \bigcap_{i=1}^k V_i$ and consider $p(\widetilde{x}) \in \bigcap_{i=1}^k p(V_i)$. Then the component of $\bigcap_{i=1}^k p(V_i)$ containing $p(\widetilde{x})$ intersects A by weak convexity of $\mathcal{U}$. Hence, we can choose a path τ in $\bigcap_{i=1}^k p(V_i)$ connecting $p(\widetilde{x})$ to some point in A. Then the lifted path $\widetilde{\tau}$ of τ in $\widetilde{X}$ with starting point $\widetilde{x}$ has endpoint in $p^{-1}(A)$ and is supported in $\bigcap_{i=1}^k V_i$. This shows that $\bigcap_{i=1}^k V_i \cap p^{-1}(A) \neq \emptyset$, which is a contradiction.

(ii) If $\mathcal{U}$ is convex, then the map $N(p)\colon N(\widetilde{\mathcal{U}}) \to N(\mathcal{U})$ induces the first isomorphism $G\backslash N(\widetilde{\mathcal{U}}) \cong N(\mathcal{U})$ by [LS20, Lemma 4.5 (3)]. Hence, to deduce the second isomorphism, it suffices to show that $N_{p^{-1}(A)}(\widetilde{\mathcal{U}}) = N(p)^{-1}(N_A(\mathcal{U}))$. The

inclusion $N_{p^{-1}(A)}(\widetilde{\mathcal{U}}) \subset N(p)^{-1}(N_A(\mathcal{U}))$ is clear. To prove the opposite inclusion, let $V_1, \dots, V_k \in \widetilde{\mathcal{U}}$ span a simplex in $N(p)^{-1}(N_A(\mathcal{U}))$. This means that $\bigcap_{i=1}^k V_i \neq \emptyset$ and $\bigcap_{i=1}^k p(V_i) \cap A \neq \emptyset$, and we need to show that $\bigcap_{i=1}^k V_i \cap p^{-1}(A) \neq \emptyset$. Take a point $\widetilde{x} \in \bigcap_{i=1}^k V_i$ and consider $p(\widetilde{x}) \in \bigcap_{i=1}^k p(V_i)$. Since $\mathcal{U}$ is convex and $\bigcap_{i=1}^k p(V_i) \cap A \neq \emptyset$, we can choose a path τ in $\bigcap_{i=1}^k p(V_i)$ with starting point $p(\widetilde{x})$ and endpoint in A. As before, the lifted path $\widetilde{\tau}$ of τ in $\widetilde{X}$ starting at $\widetilde{x}$ shows that $\bigcap_{i=1}^k V_i \cap p^{-1}(A) \neq \emptyset$. □

4.4. **Relative generalised LS-category.** We introduce a relative version of the generalised Lusternik–Schnirelmann category for families of subgroups [CLM22, Definition 2.16].

Definition 4.11 (Relative $\mathcal{F}$-category)**.** Let (X, A) be a CW-pair with fundamental group pair $(G, \mathcal{H})$ and let $p\colon \widetilde{X} \to X$ denote the universal covering. Let $\mathcal{F}$ be a family of subgroups of G that contains the trivial subgroup. The *relative $\mathcal{F}$-category* of (X, A), denoted by $\mathrm{cat}_{\mathcal{F}}(X, A)$, is the minimal $n \in \mathbb{N}$ such that there exists a $(G, \mathcal{H})$-CW-pair (Y, B) with isotropy in $\mathcal{F}$ of relative dimension $n-1$ and a map of $(G, \mathcal{H})$-CW-pairs $(\widetilde{X}, p^{-1}(A)) \to (Y, B)$. If no such integer n exists, we set $\mathrm{cat}_{\mathcal{F}}(X, A) \coloneqq +\infty$.

We will refer to $\mathrm{cat}_{\mathsf{Am}}(X, A)$ also as the *relative amenable category* of (X, A).

Remark 4.12. In the situation of Definition 4.11, let $E_{\mathcal{F}}(G, \mathcal{H})$ be a model for the classifying space of the group pair $(G, \mathcal{H})$ with respect to the family $\mathcal{F}$. Consider the (up to G-homotopy unique) map of $(G, \mathcal{H})$-CW-pairs

$$f\colon \big(\widetilde{X}, p^{-1}(A)\big) \to E_{\mathcal{F}}(G, \mathcal{H})\,.$$

Let $n \in \mathbb{N}$. Then the following are equivalent:

(i) We have $\mathrm{cat}_{\mathcal{F}}(X, A) \leq n$;
(ii) The map f factors (up to G-homotpy) through a $(G, \mathcal{H})$-CW-pair (Y, B) with isotropy in $\mathcal{F}$ of relative dimension $n-1$;
(iii) The map f is G-homotopic to a map of $(G, \mathcal{H})$-CW-pairs with values in the relative $(n-1)$-skeleton of $E_{\mathcal{F}}(G, \mathcal{H})$.

Indeed, the equivalence of these conditions follows from the universal property of $E_{\mathcal{F}}(G, \mathcal{H})$ and the equivariant cellular approximation theorem [Lüc89, Theorem 2.1].

By definition, the relative $\mathcal{F}$-category satisfies $\mathrm{cat}_{\mathcal{F}}(X, A) \leq \dim(X, A) + 1$. A more efficient upper bound for the relative category is provided by the existence of weakly convex relative covers:

Definition 4.13 (Relative $\mathcal{F}$-multiplicity)**.** Let (X, A) be a pair of spaces and let $\mathcal{F}$ be a family of subgroups of $\pi_1(X)$. The *relative $\mathcal{F}$-multiplicity* of (X, A), denoted by $\mathrm{mult}_{\mathcal{F}}(X, A)$, is the minimal $n \in \mathbb{N}$ such that there exists a weakly convex relative $\mathcal{F}$-cover $\mathcal{U}$ of (X, A) with $\mathrm{mult}_A(\mathcal{U}) = n$. If no such integer n exists, we set $\mathrm{mult}_{\mathcal{F}}(X, A) \coloneqq +\infty$.

We will refer to $\mathrm{mult}_{\mathsf{Am}}(X, A)$ also as the *relative amenable multiplicity* of (X, A).

Lemma 4.14. *Let (X, A) be a CW-pair with fundamental group pair $(G, \mathcal{H})$. Let $\mathcal{F}$ be a family of subgroups of G that contains the trivial subgroup. Then we have*

$$\mathrm{cat}_{\mathcal{F}}(X, A) \leq \mathrm{mult}_{\mathcal{F}}(X, A)\,.$$

Proof. We may assume that $n \coloneqq \mathrm{mult}_{\mathcal{F}}(X, A)$ is finite. Let $\mathcal{U}$ be a weakly convex relative $\mathcal{F}$-cover of (X, A) with $\mathrm{mult}_A(\mathcal{U}) = n$. By Proposition 4.10, the equivariant nerve pair $(|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|)$ of the lifted cover $\widetilde{\mathcal{U}}$ of $\widetilde{X}$ is a $(G, \mathcal{H})$-CW-pair with isotropy in $\mathcal{F}$ and of relative dimension $n-1$. Hence the nerve map

$$\widetilde{\nu}\colon \big(\widetilde{X}, p^{-1}(A)\big) \to \big(|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|\big)$$

exhibits the desired inequality. □

We do not know whether the inequality in Lemma 4.14 can be strict. Any such example would have to be of a relative nature, since in the absolute case equality holds:

Remark 4.15 (Category and multiplicity, absolute case)*.* Let X be a path-connected CW-complex with fundamental group G and let $\mathcal{F}$ be a family of subgroups of G. In the absolute case, the generalised Lusternik–Schnirelmann category $\mathrm{cat}_{\mathcal{F}}(X)$ is defined as the minimal n for which there exists an open $\mathcal{F}$-cover of X by n many *not necessarily path-connected* subsets. If the family $\mathcal{F}$ is closed under taking subgroups, this is compatible with Definition 4.11 in the sense that $\mathrm{cat}_{\mathcal{F}}(X, \emptyset) = \mathrm{cat}_{\mathcal{F}}(X)$ by [CLM22, Lemma 7.6].

In particular, also the converse estimate of Lemma 4.14 holds in the absolute case: Indeed, taking path-connected components of open $\mathcal{F}$-covers with n not necessarily path-connected members produces an $\mathcal{F}$-cover of multiplicity at most n. Therefore, we obtain $\mathrm{cat}_{\mathcal{F}}(X, \emptyset) = \mathrm{mult}_{\mathcal{F}}(X, \emptyset)$.

If $f\colon (Z, C) \to (X, A)$ is a homotopy equivalence of CW-pairs, then pulling back fundamental group pairs and families of subgroups along $\pi_1(f)$ shows that $\mathrm{cat}_{\pi_1(f)^*\mathcal{F}}(Z, C) = \mathrm{cat}_{\mathcal{F}}(X, A)$. In contrast, it is not clear whether $\mathrm{mult}_{\mathcal{F}}$ is also a relative homotopy invariant.

5. Simplicial volume, bounded cohomology, and acyclicity

In this section we recall the notions of simplicial volume and bounded cohomology. We also recall bounded acyclicity and we introduce a uniform version of bounded acyclicity. In particular, we explain how uniformly boundedly acyclic actions lead to computations of bounded cohomology. This is an adaptation of standard techniques in bounded cohomology [Mon01, MR23]; similar results are also discussed in recent computations of bounded cohomology groups [FFLM23, MN23].

5.1. **Simplicial volume.** We recall the definition of simplicial volume [Gro82]. Let (X, A) be a pair of spaces. For every singular n-chain $c = \sum_{i=1}^{k} a_i \sigma_i \in C_n(X, A; \mathbb{R})$, written in reduced form, we define the ℓ^1-*norm* as follows:

$$|c|_1 \coloneqq \sum_{i=1}^{k} |a_i|\,.$$

The restriction of the ℓ^1-norm to the subspace of relative cycles induces a quotient ℓ^1-seminorm (denoted by $\|\cdot\|_1$) on the homology group $H_n(X, A; \mathbb{R})$.

Definition 5.1 (Relative simplicial volume)**.** Let M be an oriented connected compact n-manifold with (possibly non-empty) boundary. Then the *relative simplicial volume* of M is

$$\|M, \partial M\| \coloneqq \big\|[M, \partial M]\big\|_1\,,$$

where $[M, \partial M] \in H_n(M, \partial M; \mathbb{R}) \cong \mathbb{R}$ denotes the relative fundamental class of M.

Example 5.2. Let M be an oriented compact connected n-manifold.

(i) If M° admits a complete finite-volume hyperbolic metric, then we have $\|M, \partial M\| = \mathrm{vol}(M)/v_n$ [Gro82, FM11];
(ii) If M is a handlebody of genus $g \geq 2$, then $\|M, \partial M\| = 3 \cdot (g-1)$ [BFP15];
(iii) If $M = \Sigma_g \times I$, where Σ_g is a surface of genus $g \geq 2$, then we have $\|M, \partial M\| = \frac{5}{4} \cdot \|\partial M\|$ [BFP15];
(iv) The simplicial volume of graph manifolds is zero [Som81, Gro82].
(v) If M admits a self-map f with $|\deg(f)| \geq 2$, then $\|M, \partial M\| = 0$ [Gro82].
(vi) If M is closed and admits an open cover by amenable subsets of multiplicity at most $\dim(M)$, then $\|M\| = 0$ [Gro82]. Many examples are known to satisfy this condition [LMS22, Section 1.1].

Further computations of simplicial volume are surveyed in the literature [LMR22].

We also recall the locally finite version of simplicial volume for non-compact manifolds [Gro82, Löh08, FM23]. Given a topological space X, a (possibly infinite) real singular n-chain $c = \sum_{\sigma \in \mathrm{map}(\Delta^n, X)} a_\sigma \sigma$ is *locally finite* if every compact subset of X intersects only finitely many simplices with non-trivial coefficient. We define $C_*^{\mathrm{lf}}(X;\mathbb{R})$ as the $\mathbb{R}$-module of locally finite chains on X. The usual boundary operator for finite chains admits a canonical extension to $C_*^{\mathrm{lf}}(X;\mathbb{R})$. The *locally finite homology* $H_*^{\mathrm{lf}}(X;\mathbb{R})$ of X is the homology of the complex $C_*^{\mathrm{lf}}(X;\mathbb{R})$.

As in the finite case, the ℓ^1-norm of a locally finite chain $c = \sum_{\sigma \in \mathrm{map}(\Delta^n, X)} a_\sigma \sigma$ in $C_n^{\mathrm{lf}}(X;\mathbb{R})$ is given by

$$|c|_1 \coloneqq \sum_{\sigma \in \mathrm{map}(\Delta^n, X)} |a_\sigma| \in [0, +\infty] .$$

As before, this norm induces an ℓ^1-seminorm $\|\cdot\|_1$ on $H_n^{\mathrm{lf}}(X;\mathbb{R})$.

Definition 5.3 (Locally finite simplicial volume)**.** Let M be an oriented (possibly non-compact) connected n-manifold without boundary. The *locally finite simplicial volume* of M is defined by

$$\|M\|_{\mathrm{lf}} \coloneqq \big\|[M]_{\mathrm{lf}}\big\|_1 ,$$

where $[M]_{\mathrm{lf}} \in H_n^{\mathrm{lf}}(M;\mathbb{R}) \cong \mathbb{R}$ denotes the locally finite fundamental class.

The [locally finite] simplicial volume can be defined for every normed ring R. In this case, we will consider the ℓ^1-seminorm on $H_*^{[\mathrm{lf}]}(-;R)$ and we will talk about [*locally finite*] *R-simplicial volume* $\|\cdot\|_{R,[\mathrm{lf}]}$.

5.2. **Bounded cohomology.** We recall the definition of bounded cohomology of groups and spaces [Gro82, Iva85, Mon01, Fri17] as well as its equivariant version [LS20, Li23].

For a group G and a normed $\mathbb{R}G$-module V, we write

$$C_b^*(G;V) \coloneqq \ell^\infty(G^{*+1}, V)^G$$

(equipped with the simplicial coboundary operator) for the *bounded cochain complex of G with coefficients in V*.

Definition 5.4 (Bounded cohomology of groups)**.** The *bounded cohomology of G with coefficients in V* is defined by

$$H_b^*(G;V) \coloneqq H^*\big(C_b^*(G;V)\big) .$$

Similarly, if (X, A) is a topological pair, we can consider the singular cochain complex

$$C^*(X, A; \mathbb{R}) \coloneqq \big\{ f \in C^*(X; \mathbb{R}) \bigm| f(\sigma) = 0 \text{ for all } \sigma \text{ supported in } A \big\},$$

where a singular n-simplex σ is *supported in* A if $\sigma(\Delta^n) \subset A$. We can restrict to the subcomplex of bounded cochains:

$$C_b^*(X, A; \mathbb{R}) \coloneqq \big\{ f \in C^*(X, A; \mathbb{R}) \bigm| \sup\nolimits_{\sigma \in \mathrm{map}(\Delta^*, X)} |f(\sigma)| < \infty \big\}.$$

Definition 5.5 (Bounded cohomology of spaces)**.** Let (X, A) be a pair of spaces. The *bounded cohomology of* (X, A) (with real coefficients) is defined by

$$H_b^*(X, A; \mathbb{R}) \coloneqq H^*\big(C_b^*(X, A; \mathbb{R})\big).$$

The inclusion of complexes $C_b^*(X, A; \mathbb{R}) \hookrightarrow C^*(X, A; \mathbb{R})$ induces a natural map from bounded cohomology to ordinary cohomology, the *comparison map*:

$$\mathrm{comp}^*_{(X,A)} \colon H_b^*(X, A; \mathbb{R}) \to H^*(X, A; \mathbb{R}).$$

The connection between bounded cohomology and simplicial volume is encoded in the following classical result:

Proposition 5.6 (Duality principle, qualitative version [Gro82, Fri17])**.** *Let M be an oriented connected compact n-manifold with (possibly empty) boundary. Then the following are equivalent:*

(i) $\|M, \partial M\| > 0$*;*
(ii) *The comparison map* $\mathrm{comp}^n_{(M,\partial M)}$ *is surjective.*

We also recall the equivariant version of bounded cohomology [LS20, Definition 5.1]:

Definition 5.7 (Equivariant [bounded] cohomology)**.** Let Y be a G-space and let $C_*(Y; \mathbb{R})$ denote the singular chain complex. For coefficients in a [normed] $\mathbb{R}G$-module V, we define the cochain complex

$$C_G^*(Y; V) \coloneqq \mathrm{Hom}_{\mathbb{R}G}\big(C_*(Y; \mathbb{R}), V\big)$$

and the subcomplex $C_{G,b}^*(Y; V) \subset C_G^*(Y; V)$ consisting of [bounded] $\mathbb{R}G$-homomorphisms. Then we set

$$H_G^n(Y; V) \coloneqq H^n\big(C_G^*(Y; V)\big);$$
$$H_{G,b}^n(Y; V) \coloneqq H^n\big(C_{G,b}^*(Y; V)\big).$$

For a pair of G-spaces (Y, B) one similarly defines $H_G^n(Y, B; V)$ and $H_{G,b}^n(Y, B; V)$. As in the absolute case, there is a *comparison map*

$$\mathrm{comp}^n_{G,(Y,B)} \colon H_{G,b}^n(Y, B; V) \to H_G^n(Y, B; V).$$

We have the following induction isomorphisms:

Lemma 5.8. *Let H be a subgroup of G, let B be an H-space, and let $n \in \mathbb{N}$. Then there are natural isomorphisms of $\mathbb{R}$-vector spaces:*

$$H_G^n(G \times_H B; \mathbb{R}) \xrightarrow{\cong} H_H^n(B; \mathbb{R});$$
$$H_{G,b}^n(G \times_H B; \mathbb{R}) \xrightarrow{\cong} H_{H,b}^n(B; \mathbb{R}).$$

Proof. Let $C_*(-)$ denote the singular chain complexes (with real coefficients). Since the induced G-space $G \times_H B$ consists of disjoint copies of B, the image of a singular simplex in $G \times_H B$ is contained in a single copy. Hence we have a natural isomorphism $C_*(G \times_H B) \cong \mathbb{R}G \otimes_{\mathbb{R}H} C_*(B)$ of $\mathbb{R}G$-chain complexes and thus an adjunction isomorphism

$$\Phi\colon C_G^*(G \times_H B; \mathbb{R}) \xrightarrow{\cong} C_H^*(B; \mathbb{R})\,.$$

This yields the isomorphism on equivariant cohomology. The claim on equivariant bounded cohomology follows by observing that Φ restricts to an isomorphism $C_{G,b}^*(G \times_H B; \mathbb{R}) \to C_{H,b}^*(B; \mathbb{R})$ on the subcomplexes of bounded cochains. □

5.3. **Bounded acyclicity.** We recall the definition of bounded acyclicity for modules and groups:

Definition 5.9 (Boundedly acyclic group)**.** Let G be a group and let $n \in \mathbb{N}$. A normed $\mathbb{R}G$-module V is *boundedly n-acyclic* if $H_b^k(G; V) \cong 0$ for all $k \in \{1, \dots, n\}$. A normed $\mathbb{R}G$-module V is *boundedly acyclic* if $H_b^k(G; V) \cong 0$ for all $k \in \mathbb{N}_{\geq 1}$.

The group G is *boundedly n-acyclic* [resp. *boundedly acyclic*] if the trivial $\mathbb{R}G$-module $\mathbb{R}$ is boundedly n-acyclic [resp. boundedly acyclic].

Amenable groups are boundedly acyclic; by now, there is a wide range of known examples of non-amenable boundedly acyclic groups, including finitely presented examples [MM85, Löh17, FFLM24, FFLM23, MN23, Mon22].

Resolutions by boundedly acyclic modules can be used to compute bounded cohomology [MR23, Proposition 2.5.4 and Remark 2.5.5]:

Theorem 5.10 (Fundamental lemma for boundedly acyclic resolutions [MR23])**.** *Let G be a group and let $n \in \mathbb{N}$. Let $0 \to V \to V^*$ be a resolution of normed $\mathbb{R}G$-modules such that V^j is a dual normed $\mathbb{R}G$-module and boundedly $(n-j)$-acyclic for every $j \in \{0, \dots, n-1\}$. Then there is a canonical isomorphism (of $\mathbb{R}$-vector spaces)*

$$H^k(V^{*G}) \xrightarrow{\cong} H_b^k(G; V)$$

for all $k \in \{0, \dots, n\}$ and a canonical injective map

$$H^{n+1}(V^{*G}) \hookrightarrow H_b^{n+1}(G; V)\,.$$

Moreover, if the given resolution is strong, then these maps are the ones induced by the canonical G-cochain homotopy class $V^* \to \ell^\infty(G^{*+1}, V)$.

5.4. **Uniform bounded acyclicity.** To formulate uniform bounded acyclicity for collections of groups, we need additional control on the norms of primitives. This can be expressed in terms of the uniform boundary condition (see Appendix A for a definition and some properties). More precisely, we use the *uniform uniform boundary condition* denoted by UUBC (Definition A.9).

Definition 5.11 (Uniformly boundedly acyclic collection of groups)**.** A collection $\mathcal{G}$ of groups is *uniformly boundedly acyclic* if

- all members of $\mathcal{G}$ are boundedly acyclic and
- the collection $(C_b^*(H; \mathbb{R}))_{H \in \mathcal{G}}$ satisfies UUBC^k for all $k \in \mathbb{N}$.

Similarly, for $n \in \mathbb{N}$, we define *uniformly boundedly n-acyclic* collections of groups if the previous conditions are satisfied up to degree n. Moreover, we extend these definitions to sets of groups.

For example, all collections consisting of amenable groups are uniformly boundedly acyclic (Example A.10). Also, all *finite* collections of boundedly acyclic groups are uniformly boundedly acyclic (Example A.11).

Proposition 5.12. *Let $n \in \mathbb{N}$, let G be a group, let $(H_i)_{i\in I}$ be a uniformly boundedly n-acyclic collection of subgroups of G, and let $k \in \{1,\dots,n\}$. Then*

$$H_b^k\big(G; \ell^\infty(G/H_i,\mathbb{R})\big) \cong 0$$

for all $i \in I$ and the collection $(C_b^*(G;\ell^\infty(G/H_i,\mathbb{R})))_{i\in I}$ satisfies UUBC^k.

Proof. This is a boundedly controlled version of the Shapiro lemma: By the Shapiro lemma in bounded cohomology [Mon01, Proposition 10.1.3], we have

$$H_b^k\big(G; \ell^\infty(G/H_i,\mathbb{R})\big) \cong H_b^k(H_i;\mathbb{R}) \cong 0$$

for all $i \in I$ and all $k \in \{1,\dots,n\}$. In order to conclude that $(C_b^*(G;\ell^\infty(G/H_i,\mathbb{R})))_{i\in I}$ satisfies UUBC^k, we make the proof of the Shapiro lemma more explicit:

Let $H \subset G$ be a subgroup of G. Then there is a non-empty set J such that G, as an H-space, is isomorphic to $J \times H$ (with the translation action on the H-factor). Therefore, on the one hand, we obtain isometric isomorphisms

$$\begin{aligned} C_b^*\big(G;\ell^\infty(G/H,\mathbb{R})\big) &\cong \ell^\infty\big(G^{*+1}, \ell^\infty(G/H,\mathbb{R})\big)^G \cong \ell^\infty\big(\mathrm{res}_H^G\, G^{*+1},\mathbb{R}\big)^H \\ &\cong \ell^\infty\big((J\times H)^{*+1},\mathbb{R}\big)^H \end{aligned}$$

of cochain complexes (each equipped with the simplicial coboundary operator). On the other hand, $C_b^*(H;\mathbb{R}) = \ell^\infty(H^{*+1},\mathbb{R})^H$. Both sides are connected through mutually homotopy inverse cochain homotopy equivalences

$$\ell^\infty\big(H^{*+1},\mathbb{R}\big)^H \leftrightarrow \ell^\infty\big((J\times H)^{*+1},\mathbb{R}\big)^H$$

given by (where $0 \in J$ is a chosen basepoint)

$$\begin{aligned} \varphi^*\colon \ell^\infty\big(H^{*+1},\mathbb{R}\big)^H &\to \ell^\infty\big((J\times H)^{*+1},\mathbb{R}\big)^H \\ f &\mapsto \big(((i_0,h_0),\dots,(i_k,h_k)) \mapsto f(h_0,\dots,h_k)\big) \\ \psi^*\colon \ell^\infty\big((J\times H)^{*+1},\mathbb{R}\big)^H &\to \ell^\infty\big(H^{*+1},\mathbb{R}\big)^H \\ f &\mapsto \big((h_0,\dots,h_k) \mapsto f\big((0,h_0),\dots,(0,h_k)\big)\big)\,; \end{aligned}$$

these cochain maps have norm 1 in each degree. Indeed, $\psi^* \circ \varphi^*$ is the identity on $\ell^\infty(H^{*+1},\mathbb{R})^H$ and the standard map

$$\begin{aligned} \ell^\infty\big((J\times H)^{*+1},\mathbb{R}\big)^H &\to \ell^\infty\big((J\times H)^{*},\mathbb{R}\big)^H \\ f &\mapsto \Bigg(((i_0,h_0),\dots,(i_{k-1},h_{k-1})) \mapsto \\ &\qquad \sum_{j=0}^{k-1} (-1)^j \cdot f\big((i_0,h_0),\dots,(i_j,h_j),(0,h_j),\dots,(0,h_{k-1})\big)\Bigg) \end{aligned}$$

is a cochain homotopy between $\varphi^* \circ \psi^*$ and the identity on $\ell^\infty((J\times H)^{*+1},\mathbb{R})^H$, with norm k in degree k. In particular, all these norms are independent of the subgroup $H \subset G$.

Hence, the claim follows by applying these considerations and homotopy inheritance of UBC (Proposition A.3) to the subgroups $(H_i)_{i\in I}$ of G. □

5.5. **Uniformly boundedly acyclic actions.** Group actions with amenable stabilisers, so-called amenable actions, have proved to be a valuable tool to compute bounded cohomology in specific cases [Mon01, BM02, BI09]. Similarly, also uniformly boundedly acyclic actions allow us to compute bounded cohomology. This is an easy application of the fact that bounded cohomology can be computed via acyclic resolutions (Theorem 5.10). However, usually, in this approach we cannot compute the seminorm on bounded cohomology.

Definition 5.13 (Uniformly boundedly acyclic action)**.** A group action on a set is *uniformly boundedly acyclic* if the collection of all stabilisers forms a uniformly boundedly acyclic collection of groups. Similarly, for $n \in \mathbb{N}$, we introduce the notion of *uniformly boundedly n-acyclic actions*.

Uniformly boundedly acyclic actions lead to boundedly acyclic modules:

Proposition 5.14. *Let G be a group, let $n \in \mathbb{N}$, and let $G \curvearrowright S$ be a uniformly boundedly n-acyclic action on a set S. Then, for all $k \in \{1, \dots, n\}$, we have*

$$H_b^k\big(G; \ell^\infty(S, \mathbb{R})\big) \cong 0\,.$$

Proof. Without loss of generality, we may assume that $S = \coprod_{i\in I} G/H_i$ with the left translation action on each summand. Using the uniform version of the Shapiro lemma (Proposition 5.12) and the compatibility with bounded products (Theorem A.15), we obtain for every $k \in \{1, \dots, n\}$

$$\begin{aligned}
0 &\cong \prod_{i\in I}{}^b H_b^k\big(G; \ell^\infty(G/H_i, \mathbb{R})\big) && \text{(Proposition 5.12)}\\
&\cong H^k\Big(\prod_{i\in I}{}^b C_b^*\big(G; \ell^\infty(G/H_i, \mathbb{R})\big)\Big) && \text{(Theorem A.15)}\\
&\cong H^k\Big(C_b^*\Big(G; \prod_{i\in I}{}^b \ell^\infty(G/H_i, \mathbb{R})\Big)\Big) && \text{(direct computation)}\\
&\cong H^k\Big(C_b^*\Big(G; \ell^\infty\Big(\coprod_{i\in I} G/H_i, \mathbb{R}\Big)\Big)\Big) && \text{(Example A.13)}\\
&= H_b^k\big(G; \ell^\infty(S, \mathbb{R})\big)\,,
\end{aligned}$$

as claimed. □

Corollary 5.15 (Bounded cohomology via uniformly boundedly acyclic actions)**.** *Let G be a group, let $G \curvearrowright S$ be an action on a non-empty set S. Let $n \in \mathbb{N}_{>0}$ and suppose that the diagonal action $G \curvearrowright S^n$ is uniformly boundedly n-acyclic. Then the cohomology of the simplicial cochain complex $\ell^\infty(S^{*+1}, \mathbb{R})^G$ is canonically isomorphic to $H_b^*(G; \mathbb{R})$ in all degrees $\leq n$ and there exists a canonical injective map*

$$H^{n+1}\big(\ell^\infty(S^{*+1}, \mathbb{R})^G\big) \hookrightarrow H^{n+1}(G; \mathbb{R})\,.$$

More precisely, every G-cochain map $\ell^\infty(S^{*+1}, \mathbb{R}) \to \ell^\infty(G^{*+1}, \mathbb{R})$ that is degreewise bounded and extends $\mathrm{id}_{\mathbb{R}} \colon \mathbb{R} \to \mathbb{R}$ induces an isomorphism [resp. injection] $H^k(\ell^\infty(S^{*+1}, \mathbb{R})^G) \to H_b^k(G; \mathbb{R})$ in the corresponding range for k.

Proof. Bounded cohomology can be computed through boundedly acyclic resolutions (Theorem 5.10). As S is non-empty, $\ell^\infty(S^{*+1}, \mathbb{R})$ is a strong resolution of $\mathbb{R}$

by normed $\mathbb{R}G$-modules. Therefore, it suffices to notice that in the present situation the $\mathbb{R}G$-modules $\ell^\infty(S^{k+1},\mathbb{R})$ are boundedly acyclic (Definition 5.9) for every $k \in \{0,\dots,n-1\}$ by Proposition 5.14. □

Remark 5.16 (Amenable actions). Proposition 5.14 and Corollary 5.15 are analogous to the corresponding results for amenable actions [Mon01][Fri17, Section 4.9]: If the action of G on a set S is amenable, then the normed $\mathbb{R}G$-module $\ell^\infty(S,\mathbb{R})$ is relatively injective and hence the cochain complex $\ell^\infty(S^{*+1},\mathbb{R})^G$ computes $H_b^*(G;\mathbb{R})$.

Remark 5.17 (Bounded cohomology via alternating cochains). Let G be a group and let $G \curvearrowright S$ be an action on a non-empty set S. A bounded function $f \in \ell^\infty(S^k,\mathbb{R})$ is *alternating* if

$$f\big(s_{\sigma(1)},\dots,s_{\sigma(k)}\big) = \operatorname{sign}(\sigma)\cdot f(s_1,\dots,s_k)$$

holds for every permutation $\sigma \in \Sigma_k$ and all $(s_1,\dots,s_k) \in S^k$. We write

$$\ell^\infty_{\mathrm{alt}}(S^{*+1},\mathbb{R}) \subset \ell^\infty(S^{*+1},\mathbb{R})$$

for the subcomplex of alternating functions, which is well-defined since the coboundary operator preserves being alternating.

Let $n \in \mathbb{N}_{>0}$ and suppose that the diagonal action $G \curvearrowright S^n$ is uniformly boundedly n-acyclic. Then also the cohomology of the simplicial cochain complex

$$\ell^\infty_{\mathrm{alt}}(S,\mathbb{R})^G \to \ell^\infty_{\mathrm{alt}}(S^2,\mathbb{R})^G \to \ell^\infty_{\mathrm{alt}}(S^3,\mathbb{R})^G \to \cdots$$

is canonically isomorphic to $H_b^*(G;\mathbb{R})$ in all degrees $\leq n$ and the canonical map $H^{n+1}(\ell^\infty_{\mathrm{alt}}(S^{*+1},\mathbb{R})^G) \to H^{n+1}(G;\mathbb{R})$ is injective. Indeed, by Corollary 5.15, we already know that the previous result holds for the non-alternating complex. Moreover, the inclusion $\ell^\infty_{\mathrm{alt}}(S^{*+1},\mathbb{R}) \hookrightarrow \ell^\infty(S^{*+1},\mathbb{R})$ induces an isomorphism on cohomology; this can be seen from the same computation as in the case of the complex $\ell^\infty(G^{*+1},\mathbb{R})$ [Fri17, Proposition 4.26].

We conclude this section by showing that the computation of bounded cohomology via alternating cochains of boundedly acyclic actions is natural in the following sense. This is analogous to the case of amenable actions [BBF+14, Lemma 2.2].

Lemma 5.18. *Let $i\colon H \to G$ be a group homomorphism. Let $H \curvearrowright S_H$ and $G \curvearrowright S_G$ be actions on non-empty sets S_H and S_G, respectively. Let $\varphi\colon S_H \to S_G$ be an i-equivariant map.*

(i) *Then the following diagram commutes, where the horizontal arrows are the canonical maps (induced by restriction to a single orbit):*

$$\begin{array}{ccc} H^*\big(\ell^\infty_{\mathrm{alt}}(S_G^{*+1},\mathbb{R})^G\big) & \longrightarrow & H_b^*(G;\mathbb{R}) \\ \big\downarrow{\scriptstyle H^*(\varphi^{*+1})} & & \big\downarrow{\scriptstyle H_b^*(i)} \\ H^*\big(\ell^\infty_{\mathrm{alt}}(S_H^{*+1},\mathbb{R})^H\big) & \longrightarrow & H_b^*(H;\mathbb{R})\,. \end{array}$$

(ii) *Let $n \in \mathbb{N}_{>0}$ and suppose that the diagonal actions $H \curvearrowright S_H^n$ and $G \curvearrowright S_G^n$ are uniformly boundedly n-acyclic. Then the horizontal arrows are isomorphisms in all degrees $\leq n$ and injective in degree $n+1$.*

Proof. Part (ii) is shown in Remark 5.17.

Part (i) is a straightforward computation: As S_H and S_G are non-empty, we can choose a point $x_H \in S_H$ and set $x_G := \varphi(x_H)$. The orbit maps $\psi_H\colon H \to S_H$

for x_H and $\psi_G\colon G \to S_G$ for x_G induce cochain maps extending $\mathrm{id}_{\mathbb{R}}\colon \mathbb{R} \to \mathbb{R}$ and therefore induce the canonical maps $H^*(\ell^\infty_{\mathrm{alt}}(S_H^{*+1},\mathbb{R})^H) \to H_b^*(H;\mathbb{R})$ and $H^*(\ell^\infty_{\mathrm{alt}}(S_G^{*+1},\mathbb{R})^G) \to H_b^*(G;\mathbb{R})$, respectively. Because φ is i-equivariant and because $\varphi(x_H) = x_G$, the diagram

$$\begin{array}{ccc} \ell^\infty_{\mathrm{alt}}(S_G^{*+1},\mathbb{R})^G & \xrightarrow{\psi_G^{*+1}} & C_b^*(G;\mathbb{R}) \\ \downarrow{\scriptstyle \varphi^{*+1}} & & \downarrow{\scriptstyle C_b^*(i;\mathbb{R})} \\ \ell^\infty_{\mathrm{alt}}(S_H^{*+1},\mathbb{R})^H & \xrightarrow[\psi_H^{*+1}]{} & C_b^*(H;\mathbb{R})\,, \end{array}$$

commutes. Taking cohomology proves part (i). □

6. A vanishing theorem for relative simplicial volume

We prove vanishing theorems for the comparison map and for relative simplicial volume in the presence of uniformly boundedly acyclic open covers with small multiplicity, as outlined in Section 1.2. The proof uses equivariant nerve pairs and equivariant bounded cohomology with respect to families of boundedly acyclic subgroups.

6.1. Uniformly boundedly acyclic open covers.

We introduce the notion of uniformly boundedly acyclic open covers.

Definition 6.1 (Families associated to a set of subspaces)**.** Let X be a path-connected space with $\pi_1(X) = G$ and $\mathcal{A}$ be a set of path-connected subspaces of X. Consider the set

$$\mathcal{G} \coloneqq \big\{ \mathrm{im}\big(\pi_1(A \hookrightarrow X)\big) \;\big|\; A \in \mathcal{A} \big\}$$

of subgroups of G; again, we implicitly use the convention on basepoints (Section 1.5). Then the *intersection-closed family of subgroups of G associated to $\mathcal{A}$* is defined as

$$\mathcal{F}\langle\mathcal{A}\rangle \coloneqq \mathcal{F}\langle\mathcal{G}\rangle = \left\{ \bigcap_{i=1}^n g_i H_i g_i^{-1} \;\middle|\; n \in \mathbb{N}, H_i \in \mathcal{G}, g_i \in G \right\}.$$

For fixed $n \in \mathbb{N}$, we define the (conjugation-closed) family of subgroups of G associated to $\mathcal{A}$ as

$$\mathcal{F}_n\langle\mathcal{A}\rangle \coloneqq \mathcal{F}_n\langle\mathcal{G}\rangle = \left\{ \bigcap_{i=1}^n g_i H_i g_i^{-1} \;\middle|\; H_i \in \mathcal{G}, g_i \in G \right\}.$$

Using the notion of uniformly boundedly acyclic collections of groups (Definition 5.11), we define the following:

Definition 6.2 (Uniformly boundedly acyclic set of subspaces)**.** Let X be a path-connected space and $\mathcal{A}$ be a set of path-connected subspaces of X. We say that $\mathcal{A}$ is *uniformly boundedly acylic* [*of order n*] *in X* if the associated family $\mathcal{F}\langle\mathcal{A}\rangle$ [resp. $\mathcal{F}_n\langle\mathcal{A}\rangle$] is uniformly boundedly acyclic.

Definition 6.3 (Uniformly boundedly acyclic open cover)**.** Let X be a path-connected space and $\mathcal{U}$ be a cover of X by path-connected open subsets. We say that $\mathcal{U}$ is *uniformly boundedly acyclic* if it is uniformly boundedly acyclic in X when viewed as a set of subspaces of X.

By Example A.10, every amenable cover is uniformly boundedly acyclic.

Remark 6.4. The above notion of uniformly boundedly acyclic open covers is similar to Ivanov's notion of weakly boundedly acyclic open covers [Iva20, Section 4]. The difference is that we consider intersections of subgroups of the fundamental group whereas Ivanov considers fundamental groups of intersections of subspaces. The key steps of our arguments happen on the level of bounded cohomology. In contrast, Ivanov's arguments via spectral sequences target the comparison map more directly. It is not clear to us whether one of the concepts contains the other.

6.2. **Strong H_b^*-admissibility.** We show that classifying spaces with respect to uniformly boundedly acyclic families can be used to compute bounded cohomology. This is phrased in terms of equivariant bounded cohomology (Definition 5.7).

Definition 6.5 (Strongly H_b^*-admissible family)**.** Let G be a group and let $\mathcal{F}$ be a (conjugation-closed) family of subgroups of G that contains the trivial subgroup 1. Since $1 \in \mathcal{F}$, there is a canonical (up to G-homotopy) G-map $f\colon EG \to E_{\mathcal{F}}G$. The family $\mathcal{F}$ is *strongly H_b^*-admissible* if the induced map

$$H_{G,b}^*(f;\mathbb{R})\colon H_{G,b}^*(E_{\mathcal{F}}G;\mathbb{R}) \to H_{G,b}^*(EG;\mathbb{R}) \cong H_b^*(G;\mathbb{R})$$

is bijective.

Definition 6.5 is a slight generalisation of the original definition [LS20, Definition 5.1], where only families are considered that are closed under taking subgroups.

Proposition 6.6. *Let G be a group and let $\mathcal{F}$ be an intersection-closed family of subgroups of G that is uniformly boundedly acyclic and contains the trivial subgroup. Then $\mathcal{F}$ is strongly H_b^*-admissible.*

Proof. We use Proposition 5.14 and the fact that bounded cohomology can be computed using acyclic resolutions (Theorem 5.10). Let $E_{\mathcal{F}}G$ be a model for the classifying space of G with respect to the family $\mathcal{F}$. We show that the chain complex $C_b^*(E_{\mathcal{F}}G;\mathbb{R})$ together with the canonical augmentation $\mathbb{R} \to C_b^0(E_{\mathcal{F}}G;\mathbb{R})$ is a boundedly acylic resolution of $\mathbb{R}$ over G: As $1 \in \mathcal{F}$, the space $E_{\mathcal{F}}G$ is contractible (Theorem 2.8). Therefore, $C_b^*(E_{\mathcal{F}}G;\mathbb{R})$ is a resolution of $\mathbb{R}$.

Moreover, for each $n \in \mathbb{N}$, the Banach G-module $C_b^n(E_{\mathcal{F}}G;\mathbb{R})$ is boundedly acyclic: By definition, $C_b^n(E_{\mathcal{F}}G;\mathbb{R}) = \ell^\infty(S,\mathbb{R})$ with $S := \operatorname{map}(\Delta^n, E_{\mathcal{F}}G)$. In view of Proposition 5.14, it thus suffices to show that the stabilisers of the G-action on S lie in the uniformly boundedly acyclic collection $\mathcal{F}$. If $\sigma\colon \Delta^n \to E_{\mathcal{F}}G$ is a singular simplex, then $\sigma(\Delta^n)$ meets only finitely many cells of $E_{\mathcal{F}}G$. Therefore, the stabiliser of σ is an intersection of finitely many elements of $\mathcal{F}$ and thus lies in $\mathcal{F}$.

Let $f\colon EG \to E_{\mathcal{F}}G$ be the canonical (up to G-homotopy) G-map. Then the fundamental lemma for boundedly acyclic resolutions (Theorem 5.10) shows that the cochain map

$$C_b^*(f;\mathbb{R})^G\colon C_b^*(E_{\mathcal{F}}G;\mathbb{R})^G \to C_b^*(EG;\mathbb{R})^G$$

induces an isomorphism in bounded cohomology. □

Corollary 6.7 ([LS20, Proposition 5.2])**.** *Every intersection-closed family $\mathcal{F}$ that consists of amenable groups and contains the trivial subgroup is strongly H_b^*-admissible.* □

6.3. **A relative vanishing theorem.** In this section, we prove the relative vanishing theorem by making use of the relative equivariant setting developed in the previous sections. This result extends the classical vanishing theorem by Ivanov [Iva85, Iva20] to the relative setting. By now there are several alternative proofs of Ivanov's result [FM23, LS20, Rap24]; moreover, in the case of aspherical manifolds, the vanishing of simplicial volume can also be obtained directly through the amenable reduction lemma [AK16, FM23, LMS22]. We will follow the approach via classifying spaces by Löh and Sauer [LS20].

First, we prove a vanishing result for the comparison map in terms of the relative generalised LS-category (Definition 4.11).

Setup 6.8 (CW-pair with a family of subgroups)**.** Let (X, A) be a CW-pair with X path-connected and let A have only finitely many connected components.

- We suppose that the inclusion of every component of A into X is π_1-injective and we let $(G, \mathcal{H} = (H_i)_{i\in I})$ be a fundamental group pair for (X, A) (Example 2.12);
- Let $\mathcal{F}$ be a family of subgroups of G that contains the trivial subgroup;
- For every $i \in I$, let $\mathcal{F}|_{H_i}$ be the restricted family of subgroups of H_i (Example 2.5 (v));
- Let $p\colon \widetilde{X} \to X$ be the universal covering of X.

Lemma 6.9. *In the situation of Setup 6.8, suppose that the family $\mathcal{F}$ of subgroups of G is strongly H_b^*-admissible and that for every $i \in I$ the family $\mathcal{F}|_{H_i}$ of subgroups of H_i is strongly H_b^*-admissible.*

Then the canonical map $f\colon (\widetilde{X}, p^{-1}(A)) \to E_{\mathcal{F}}(G, \mathcal{H})$ of $(G, \mathcal{H})$-CW-pairs induces an isomorphism in equivariant bounded cohomology:

$$H^*_{G,b}(f)\colon H^*_{G,b}\big(E_{\mathcal{F}}(G, \mathcal{H}); \mathbb{R}\big) \xrightarrow{\cong} H^*_{G,b}\big(\widetilde{X}, p^{-1}(A); \mathbb{R}\big)\,.$$

In particular, the comparison map $\operatorname{comp}^k_{(X,A)}\colon H^k_b(X, A; \mathbb{R}) \to H^k(X, A; \mathbb{R})$ vanishes in all degrees $k \geq \operatorname{cat}_{\mathcal{F}}(X, A)$.

Proof. By Example 2.14, $(\widetilde{X}, p^{-1}(A))$ is a $(G, \mathcal{H})$-CW-pair with isotropy in the trivial family Tr. Thus, since $\mathcal{F}$ contains the trivial subgroup, the G-map f can be factored (up to G-homotopy) as

$$\big(\widetilde{X}, p^{-1}(A)\big) \xrightarrow{f_1} E_{\mathsf{Tr}}(G, \mathcal{H}) \xrightarrow{f_2} E_{\mathcal{F}}(G, \mathcal{H})\,.$$

Using the long exact sequence for pairs in equivariant bounded cohomology together with the induction isomorphism (Lemma 5.8), we have the following: The induced map $H^*_{G,b}(f_1)$ is a (not necessarily isometric) isomorphism by the mapping theorem [Gro82] and the five lemma. The map $H^*_{G,b}(f_2)$ is an isomorphism by the five lemma and the assumption that the families $\mathcal{F}$ and $\mathcal{F}|_{H_i}$ are strongly H_b^*-admissible. Together, this yields the desired isomorphism.

Moreover, by naturality of the comparison map we have a commutative diagram, where we omit the trivial $\mathbb{R}$-coefficients:

$$\begin{array}{ccccc}
H^k_{G,b}\big(E_{\mathcal{F}}(G, \mathcal{H})\big) & \xrightarrow[\cong]{H^k_{G,b}(f)} & H^k_{G,b}\big(\widetilde{X}, p^{-1}(A)\big) & \xleftarrow[\cong]{H^*_b(p)} & H^k_b(X, A) \\
\downarrow{\scriptstyle \operatorname{comp}^k_{G,E_{\mathcal{F}}(G,\mathcal{H})}} & & \downarrow{\scriptstyle \operatorname{comp}^k_{G,(\widetilde{X},p^{-1}(A))}} & & \downarrow{\scriptstyle \operatorname{comp}^k_{(X,A)}} \\
H^k_G\big(E_{\mathcal{F}}(G, \mathcal{H})\big) & \xrightarrow[H^k_G(f)]{} & H^k_G\big(\widetilde{X}, p^{-1}(A)\big) & \xleftarrow[H^*(p)]{\cong} & H^k(X, A)\,.
\end{array}$$

The claim follows, since the map $H_G^k(f)$ is trivial in all degrees $k \geq \operatorname{cat}_{\mathcal{F}}(X, A)$ (Remark 4.12). □

Second, we use the relation between relative category and relative multiplicity to derive a vanishing theorem via relative open covers:

Setup 6.10 (Relative $\mathcal{F}$-cover and its nerves)**.** In the situation of Setup 6.8, we additionally consider a relative $\mathcal{F}$-cover $\mathcal{U}$ of (X, A) (Definition 4.9). We will use the notions of relative multiplicity (Definition 4.3) and weak convexity (Definition 4.9) of relative covers. Moreover, we fix the following notation:

- Let $\widetilde{\mathcal{U}}$ be the lifted G-invariant cover of $\widetilde{X}$ (Example 4.8);
- Let $\nu\colon \big(X, A\big) \to \big(|N(\mathcal{U})|, |N_A(\mathcal{U})|\big)$ be a nerve map;
- Let $\widetilde{\nu}\colon \big(\widetilde{X}, p^{-1}(A)\big) \to \big(|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|\big)$ be a corresponding nerve map of $(G, \mathcal{H})$-CW-pairs (Proposition 4.10).

The following theorem is a relative version of the vanishing theorem for strongly H_b^*-admissible families [LS20, Theorem 5.3].

Theorem 6.11 (Relative vanishing theorem)**.** *In the situation of Setup 6.8 and Setup 6.10, suppose that the family $\mathcal{F}$ of subgroups of G is strongly H_b^*-admissible and that for all $i \in I$ the family $\mathcal{F}|_{H_i}$ of subgroups of H_i is strongly H_b^*-admissible.*

Then the comparison map comp_G^* *factors through the equivariant nerve map $\widetilde{\nu}$:*

$$\begin{array}{ccc} H_{G,b}^*(\widetilde{X}, p^{-1}(A); \mathbb{R}) & \xrightarrow{\operatorname{comp}_G^*} & H_G^*(\widetilde{X}, p^{-1}(A); \mathbb{R}) \\ & \searrow & \uparrow{\scriptstyle H_G^*(\widetilde{\nu};\mathbb{R})} \\ & & H_G^*(|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|; \mathbb{R})\,. \end{array}$$

In particular, the following hold:

(i) *If $\mathcal{U}$ is weakly convex, then the comparison map*

$$\operatorname{comp}^k\colon H_b^k(X, A; \mathbb{R}) \to H^k(X, A; \mathbb{R})$$

vanishes in all degrees $k \geq \operatorname{mult}_A(\mathcal{U})$;

(ii) *If $\mathcal{U}$ is convex, then the comparison map* comp^* *factors through the nerve map ν:*

$$\begin{array}{ccc} H_b^*(X, A; \mathbb{R}) & \xrightarrow{\operatorname{comp}^*} & H^*(X, A; \mathbb{R}) \\ & \searrow & \uparrow{\scriptstyle H^*(\nu;\mathbb{R})} \\ & & H^*(|N(\mathcal{U})|, |N_A(\mathcal{U})|; \mathbb{R})\,. \end{array}$$

Proof. Let $E_{\mathcal{F}}(G, \mathcal{H})$ be a model for the classifying space of $(G, \mathcal{H})$ with respect to the family $\mathcal{F}$ (Lemma 2.16). By Proposition 4.10, $(|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|)$ is a $(G, \mathcal{H})$-CW-pair with isotropy in $\mathcal{F}$. By Example 2.14, $(\widetilde{X}, p^{-1}(A))$ is a $(G, \mathcal{H})$-CW-pair with isotropy in the trivial family Tr. The universal property of $E_{\mathcal{F}}(G, \mathcal{H})$ yields that the canonical map $f\colon (\widetilde{X}, p^{-1}(A)) \to E_{\mathcal{F}}(G, \mathcal{H})$ of $(G, \mathcal{H})$-CW-pairs factors (up to G-homotopy) through the equivariant nerve map $\widetilde{\nu}$:

$$\big(\widetilde{X}, p^{-1}(A)\big) \xrightarrow{\widetilde{\nu}} \big(|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|\big) \xrightarrow{\varphi} E_{\mathcal{F}}(G, \mathcal{H})\,.$$

Hence, we have the following commutative diagram, where the trivial coefficient module $\mathbb{R}$ is omitted from the notation.

$$\begin{array}{ccccc}
H_b^*(X,A) & \xrightarrow{\mathrm{comp}^*} & H^*(X,A) & \xleftarrow{H^*(\nu)} & H^*\big(|N(\mathcal{U})|, |N_A(\mathcal{U})|\big) \\
{\scriptstyle H_b^*(p)}\downarrow{\scriptstyle\cong} & & {\scriptstyle H^*(p)}\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle H^*(|N(p)|)} \\
H_{G,b}^*\big(\widetilde{X}, p^{-1}(A)\big) & \xrightarrow{\mathrm{comp}_G^*} & H_G^*\big(\widetilde{X}, p^{-1}(A)\big) & \xleftarrow{H_G^*(\widetilde{\nu})} & H_G^*\big(|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|\big) \\
{\scriptstyle H_{G,b}^*(f)}\uparrow{\scriptstyle\cong} & & {\scriptstyle H_G^*(f)}\uparrow & \nearrow_{H_G^*(\varphi)} & \\
H_{G,b}^*\big(E_{\mathcal{F}}(G,\mathcal{H})\big) & \xrightarrow{\mathrm{comp}^*_{G,E_{\mathcal{F}}(G,\mathcal{H})}} & H_G^*\big(E_{\mathcal{F}}(G,\mathcal{H})\big) & &
\end{array}$$

The map $H_{G,b}^*(f)$ is an isomorphism by Lemma 6.9. Therefore, the desired factorisation of the comparison map comp_G^* is given by

$$\mathrm{comp}_G^* = H_G^*(\widetilde{\nu}) \circ H_G^*(\varphi) \circ \mathrm{comp}^*_{G,E_{\mathcal{F}}(G,\mathcal{H})} \circ H_{G,b}^*(f)^{-1}\,.$$

(i) If $\mathcal{U}$ is weakly convex, then we have $\mathrm{cat}_{\mathcal{F}}(X,A) \leq \mathrm{mult}_A(\mathcal{U})$ (Lemma 4.14) and we conclude by Lemma 6.9. More explicitly, let $n = \mathrm{mult}_A(\mathcal{U})$. Then, by Proposition 4.10 (i), we have $\dim(|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|) = n-1$ and hence $H_G^k(|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|) = 0$ for $k \geq n$. This shows that comp^k vanishes in every degree $k \geq n$.

(ii) If $\mathcal{U}$ is convex, then the map $H^*(|N(p)|)$ is an isomorphism by Proposition 4.10 (ii). Hence the desired factorisation of the comparison map comp^* is given by

$$\mathrm{comp}^* = H^*(\nu) \circ H^*(|N(p)|)^{-1} \circ H_G^*(\varphi) \circ \mathrm{comp}^*_{G,E_{\mathcal{F}}(G,\mathcal{H})} \circ H_{G,b}^*(f)^{-1} \circ H_b^*(p)\,.$$

This proves the statement. □

Remark 6.12 (H_b^*-admissible families). A family $\mathcal{F}$ of subgroups of G containing the trivial subgroup is called (not necessarily strongly) *H_b^*-admissible* if the canonical G-map $EG \to E_{\mathcal{F}}G$ induces a surjective map in equivariant bounded cohomology in all degrees [LS20, Definition 5.1].

The conclusions of Theorem 6.11 hold more generally, if in the situation of Setup 6.10 the family $\mathcal{F}$ of subgroups of G is only assumed to be (not necessarily strongly) H_b^*-admissible, while the families $\mathcal{F}|_{H_i}$ of subgroups of H_i are still assumed to be strongly H_b^*-admissible. Indeed, similarly to Lemma 6.9, in this case it follows from the four lemma for epimorphisms that the canonical G-map $(\widetilde{X}, p^{-1}(A)) \to E_{\mathcal{F}}(G,\mathcal{H})$ induces a surjective map in equivariant bounded cohomology in all degrees. This suffices to carry out the above proof of Theorem 6.11.

As an application of the previous theorem, we can deduce the vanishing theorem for uniformly boundedly acyclic open covers (Definition 6.3), which complements a recent result by Ivanov [Iva20].

Corollary 6.13 (Relative vanishing theorem for uniformly boundedly acyclic covers)**.** *In the situation of Setup 6.8 and Setup 6.10, suppose that the relative cover $\mathcal{U}$ of (X,A) is uniformly boundedly acyclic, viewed as an open cover of X.*

Then all statements in Theorem 6.11 hold.

Proof. We take $\mathcal{F}$ to be the family $\mathcal{F}\langle\mathcal{U}\rangle \cup \{1\}$ of subgroups of G. Since the cover $\mathcal{U}$ is uniformly boundedly acyclic, the family $\mathcal{F}$ is uniformly boundedly acyclic and

hence strongly H_b^*-admissible (Proposition 6.6). As subsets of uniformly boundedly acyclic sets of groups are again uniformly boundedly acyclic, the restricted families $\mathcal{F}|_{H_i}$ of subgroups of H_i are strongly H_b^*-admissible for every $i \in I$. Thus, Theorem 6.11 applies and yields the thesis. □

Remark 6.14 (ℓ^1-Homology). In view of *strong* H_b^*-admissibility, the analogues of Theorem 6.11 and Corollary 6.13 for ℓ^1-homology also hold. One can simply argue through duality as in the absolute case [LS20, Section 6].

In particular, Theorem 6.11 gives a relative vanishing theorem in the presence of "small" relative amenable multiplicity (Definition 4.13).

Corollary 6.15 (Relative vanishing theorem for amenable covers)**.** *Let (X, A) be a CW-pair with path-connected ambient space X. Assume that A consists of finitely many connected components, each of which is π_1-injective in X. Then the comparison map*

$$\mathrm{comp}^k_{(X,A)} \colon H_b^k(X, A; \mathbb{R}) \to H^k(X, A; \mathbb{R})$$

vanishes in all degrees $k \geq \mathrm{mult}_{\mathsf{Am}}(X, A)$.

In particular, if $(M, \partial M)$ is an oriented compact connected triangulable manifold with (possibly empty) π_1-injective boundary and $\mathrm{mult}_{\mathsf{Am}}(M, \partial M) \leq \dim(M)$, then the relative simplicial volume $\|M, \partial M\|$ vanishes.

Proof. The claim on the comparison map follows from the strong H_b^*-admissibility of the family Am (Corollary 6.7).

Because $(M, \partial M)$ is triangulable, both M and ∂M admit compatible triangulations; in particular, we can view $(M, \partial M)$ as a CW-pair. Therefore, the statement on relative simplicial volume follows from the duality principle (Proposition 5.6) □

Remark 6.16 (Optimality of assumptions). Let Σ denote the surface of genus 1 with one boundary component. Since the interior of Σ admits a complete hyperbolic metric with finite volume, we know that $\|\Sigma, \partial\Sigma\| > 0$ (Example 5.2.1). This shows that $\mathrm{mult}_{\mathsf{Am}}(\Sigma, \partial\Sigma) = 3$. However, when either one of the conditions (RC1), (RC2), or weak convexity on the open cover is dropped, then it is not difficult to construct amenable covers of $(\Sigma, \partial\Sigma)$ with relative multiplicity at most 2. In this sense, our set of assumptions in Corollary 6.15 is optimal.

6.4. **Amenable covers with small multiplicity on the boundary.** We compare Corollary 6.15 with existing results in the literature [LMR22, Section 3.3]. The main available relative vanishing result is the following, which is based on Gromov's vanishing theorem for non-compact manifolds [Gro82][FM23, Corollary 11]:

Theorem 6.17 ([LMR22, Theorem 3.13])**.** *Let $(M, \partial M)$ be an oriented compact connected n-manifold with non-empty boundary. Let $\mathcal{U}$ be an open cover of M and let $\mathcal{U}|_{\partial M}$ denote the restriction of $\mathcal{U}$ to ∂M, i.e., $\mathcal{U}|_{\partial M} := \{U \cap \partial M \mid U \in \mathcal{U}\}$. Suppose that the following hold:*

(i) $\mathrm{mult}(\mathcal{U}) \leq n$;
(ii) $\mathrm{mult}(\mathcal{U}|_{\partial M}) \leq n - 1$;
(iii) *The open covers $\mathcal{U}$ of M and $\mathcal{U}|_{\partial M}$ of ∂M are amenable.*

Then $\|M, \partial M\| = 0$.

We emphasise that (contrary to our convention) here the restricted cover $\mathcal{U}|_{\partial M}$ of ∂M may consist of disconnected subsets.

Since this result neither assumes that the manifold M is triangulable, nor that the boundary inclusion is π_1-injective (in the sense of Section 1.5), while our Corollary 6.15 does so, we cannot recover Theorem 6.17 in full generality. However, in the special situation of triangulable manifolds with π_1-injective boundary, we can provide a simplified proof that does not make use of Gromov's theory of diffusion of chains:

Proof of Theorem 6.17 for triangulable manifolds with π_1-injective boundary. We show that there exists a weakly convex relative Am-cover $\mathcal{V}$ of $(M, \partial M)$ such that $\mathrm{mult}_{\partial M}(\mathcal{V}) \leq n = \dim(M)$. This implies that $\mathrm{mult}_{\mathsf{Am}}(M, \partial M) \leq n$ and thus by Corollary 6.15 the thesis.

Let $m \coloneqq \{\mathrm{mult}(\mathcal{U}), \mathrm{mult}(\mathcal{U}|_{\partial M}) + 1\}$. By conditions (i) and (ii), we have $m \leq n$. From $\mathcal{U}$ we will construct a new cover $\mathcal{V}$ of $(M, \partial M)$ that is a weakly convex relative Am-cover with $\mathrm{mult}_{\partial M}(\mathcal{V}) \leq m$. To this end, we follow a classical strategy for modifying open covers (while controlling the multiplicity) in the case of compact manifolds with boundary [FM23, proof of Theorem 11.2.3][LS09a, proof of Theorem 5.3].

By compactness, we may assume that $\mathcal{U}$ is finite, say $\mathcal{U} = \{U_1, \dots, U_k\}$. Since ∂M is collared in M, we have the following identification:

$$(M, \partial M) \cong (M', \partial M') \coloneqq \left(M \cup_{\partial M \cong (\partial M \times \{0\})} (\partial M \times [0,1]), \partial M \times \{1\}\right).$$

Let $\varepsilon \coloneqq 1/3(k+1)$ and $t_i \coloneqq i/(k+1)$. Moreover, for $i \in \{1, \dots, k\}$, let $\mathcal{U}(i)$ denote the set of connected components of $U_i \cap \partial M$. We set for every $i \in \{1, \dots, k\}$ such that $U_i \cap \partial M \neq \emptyset$

$$U_i' \coloneqq U_i \cup \left((U_i \cap \partial M) \times [0, t_i + \varepsilon)\right) \subset M'$$

and

$$\mathcal{U}'(i) \coloneqq \left\{U \times (t_i - \varepsilon, 1] \;\middle|\; U \in \mathcal{U}(i)\right\}.$$

When $U_i \cap \partial M = \emptyset$, we just set $U_i' \coloneqq U_i$ and $\mathcal{U}'(i) \coloneqq \emptyset$. This produces a new open amenable cover

$$\mathcal{U}' \coloneqq \left\{U_1', \dots, U_k'\right\} \cup \mathcal{U}'(1) \cup \dots \cup \mathcal{U}'(k)$$

of M'.

Finally, we obtain $\mathcal{V}$ from $\mathcal{U}'$ by discarding all sets of the form $U \times (t_i - \varepsilon, 1]$ with $U \in \mathcal{U}(j)$ for some $j > i$. Then $\mathcal{V}$ is an amenable open cover of M' and a straightforward case analysis shows that $\mathrm{mult}(\mathcal{V}) \leq m$. In particular, this implies that $\mathrm{mult}_{\partial M'}(\mathcal{V}) \leq \mathrm{mult}(\mathcal{V}) \leq m \leq n$ by assumptions (i) and (ii).

We are left to show that $\mathcal{V}$ satisfies all the conditions in Definition 4.9. The key observation is the following: Only sets $V \in \mathcal{V}$ of the form $V = U \times (t_i - \varepsilon, 1]$ with $U \in \mathcal{U}(i)$ intersect $\partial M'$. In particular, if $V \in \mathcal{V}$ satisfies $V \cap \partial M' \neq \emptyset$, then $V \cap \partial M'$ is connected. Thus, (RC1) is satisfied. Moreover, for the same reason every $V \in \mathcal{V}$ with $V \cap \partial M' \neq \emptyset$ deformation retracts onto $V \cap \partial M'$, whence (RC2) holds. It remains to show that the relative cover $\mathcal{V}$ is weakly convex. If $V_1, \dots, V_j \in \mathcal{V}$ satisfy $V_1 \cap \dots \cap V_j \cap \partial M' \neq \emptyset$, then $V_1 \cap \dots \cap V_j$ is of the form $(V_1 \cap \dots \cap V_j \cap \partial M') \times (r, 1]$ for some $r \in (0,1)$. In particular, each component of $V_1 \cap \dots \cap V_j$ intersects $\partial M'$. Hence, $\mathcal{V}$ is weakly convex.

We conclude that $\mathcal{V}$ is a weakly convex relative Am-cover of $(M', \partial M')$ with $\mathrm{mult}_{\partial M'}(\mathcal{V}) \leq n$. Using the identification of $(M', \partial M')$ with $(M, \partial M)$, we get the thesis. □

We conclude this section by showing that for general CW-pairs (X, A) such that the inclusion of A into X is π_1-injective the hypotheses of Corollary 6.15 are weaker than the ones of Theorem 6.17:

Example 6.18. Let $n \geq 3$. Let M be an oriented closed connected hyperbolic $(n-1)$-manifold and denote $I = [0,1]$. We consider the CW-pair $(M \times I, M \times \{1\})$. Since $M \times \{1\} \cong M$ is hyperbolic, we have $\|M \times \{1\}\| > 0$ [Thu79, Gro82]. Hence, $M \times \{1\}$ cannot admit an open amenable cover with multiplicity at most $n-1$. This shows that there is no open amenable cover of M whose restriction to $M \times \{1\}$ is both amenable and with multiplicity at most $n-1$.

On the other hand, it is easy to construct a weakly convex relative Am-cover $\mathcal{V}$ of $(M \times I, M \times \{1\})$ with $\mathrm{mult}_{M \times \{1\}}(\mathcal{V}) \leq n$. Let $\mathcal{U}$ be the open star cover of $M \times \{1\}$ and let $\mathcal{V}$ be the cover of $M \times I$ defined as follows:

$$\mathcal{V} \coloneqq \{U \times I \mid U \in \mathcal{U}\}.$$

Since by construction $\mathcal{V}$ consists of contractible sets, $\mathcal{V}$ is an amenable open cover. Moreover, each member of $\mathcal{V}$ intersects $M \times \{1\}$ in a contractible, whence connected, set. The same argument also applies to multiple intersections. Finally, since each element V in $\mathcal{V}$ is a product $U \times I$ with $U \in \mathcal{U}$, it follows that V retracts by deformation onto $V \cap (M \times \{1\})$. This shows that $\mathcal{V}$ is a weakly convex relative Am-cover of $(M \times I, M \times \{1\})$.

Since by construction the relative multiplicity of $\mathcal{V}$ is zero, we have obtained our desired cover.

We point out that while the manifold $M \times I$ has π_1-injective boundary, we consider $M \times I$ relative to only one of its two boundary components.

We do not know whether the previous example can be improved to the situation of compact manifolds with π_1-injective boundary relative to all boundary components.

7. A vanishing theorem for relative ℓ^2-Betti numbers

We prove a vanishing theorem for the relative ℓ^2-Betti numbers of aspherical CW-pairs with small relative amenable multiplicity using equivariant nerve pairs. In the absolute case for ℓ^2-Betti numbers, more sophisticated arguments involving nerves have previously been used by Sauer [Sau09]. For further background on ℓ^2-Betti numbers we refer to the literature [Lüc02, Kam19]. The results of this section are not used in the rest of the article.

We use the following (non-standard) notation:

Definition 7.1 (ℓ^2-Homology and ℓ^2-Betti numbers)**.** Let Y be a G-space. The *ℓ^2-homology* $H_*^{(2)}(G \curvearrowright Y)$ is defined as the singular homology of Y with twisted coefficients in the group von Neumann algebra $\mathcal{N}G$, that is the $\mathcal{N}G$-module

$$H_*^{(2)}(G \curvearrowright Y) \coloneqq H_*^G(Y; \mathcal{N}G).$$

The *n-th ℓ^2-Betti number* $b_n^{(2)}(G \curvearrowright Y) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ is

$$b_n^{(2)}(G \curvearrowright Y) \coloneqq \dim_{\mathcal{N}G}\big(H_n^{(2)}(G \curvearrowright Y)\big),$$

where $\dim_{\mathcal{N}G}$ is the von Neumann dimension function. For a pair (Y, B) of G-spaces, one similarly defines $H_*^{(2)}(G \curvearrowright (Y, B))$ and $b_*^{(2)}(G \curvearrowright (Y, B))$.

The ℓ^2-Betti numbers $b_*^{(2)}(G)$ of a group G are defined as

$$b_*^{(2)}(G) \coloneqq b_*^{(2)}(G \curvearrowright EG)\,.$$

We say that G is *ℓ^2-acyclic* if $b_k^{(2)}(G) = 0$ for all $k > 0$.

For example, amenable groups are ℓ^2-acyclic [Lüc02, Corollary 6.75]. The following shows that ℓ^2-Betti numbers can be computed using classifying spaces for families consisting of ℓ^2-acyclic subgroups.

Proposition 7.2 ([Kam19, Theorem 4.14])**.** *Let G be a group and $\mathcal{F}$ be a (conjugation-closed) family of subgroups of G that consists of ℓ^2-acyclic groups and contains the trivial subgroup. Then the canonical G-map $EG \to E_{\mathcal{F}}G$ induces a dimension-isomorphism in ℓ^2-homology*

$$H_*^{(2)}(G \curvearrowright EG) \xrightarrow{\cong_{\dim}} H_*^{(2)}(G \curvearrowright E_{\mathcal{F}}G)\,.$$

In particular, $b_*^{(2)}(G) = b_*^{(2)}(G \curvearrowright E_{\mathcal{F}}G)$.

Proof. Since $\mathcal{F}$ contains the trivial subgroup, $EG \times E_{\mathcal{F}}G$ equipped with the diagonal G-action is a model for EG. The projection $EG \times E_{\mathcal{F}}G \to E_{\mathcal{F}}G$ onto the second factor induces a dimension-isomorphism

$$H_*^{(2)}(G \curvearrowright EG) \cong H_*^{(2)}(G \curvearrowright EG \times E_{\mathcal{F}}G) \xrightarrow{\cong_{\dim}} H_*^{(2)}(G \curvearrowright E_{\mathcal{F}}G)\,,$$

because all members of $\mathcal{F}$ are ℓ^2-acyclic [Lüc02, proof of Theorem 6.54 (2)]. □

For the vanishing theorem, we consider the following situation: Let (X, A) be a CW-pair with X path-connected. Suppose that A has only finitely many connected components and let $A = \coprod_{i\in I} A_i$ be a decomposition into connected components. Assume that each A_i is π_1-injective in X and let $(G, \mathcal{H})$ be a fundamental group pair for (X, A) (Example 2.12). Let $\mathcal{F}$ be a family of subgroups of G that contains the trivial subgroup. Denote by $p\colon \widetilde{X} \to X$ the universal covering.

Moreover, let $\mathcal{U}$ be a relative $\mathcal{F}$-cover of (X, A) (Definition 4.9) with relative multiplicity $\mathrm{mult}_A(\mathcal{U})$ (Definition 4.3). Let $\widetilde{\mathcal{U}}$ be the lifted G-invariant cover of $\widetilde{X}$ (Example 4.8) and $(|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|)$ be its equivariant nerve pair (Proposition 4.10). We will also use the notion of weakly convex relative covers (Definition 4.9).

Theorem 7.3 (Relative vanishing theorem for ℓ^2-Betti numbers)**.** *In the above situation, if X and all the $(A_i)_{i\in I}$ are aspherical and $\mathcal{F}$ consists of ℓ^2-acyclic groups, then we have*

$$b_*^{(2)}\big(G \curvearrowright (\widetilde{X}, p^{-1}(A))\big) \leq b_*^{(2)}\big(G \curvearrowright (|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|)\big)\,.$$

In particular, if $\mathcal{U}$ is weakly convex, then

$$b_k^{(2)}\big(G \curvearrowright (\widetilde{X}, p^{-1}(A))\big) = 0$$

for all $k \geq \mathrm{mult}_A(\mathcal{U})$.

Proof. By Proposition 4.10, the equivariant nerve pair $(|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|)$ is a $(G, \mathcal{H})$-CW-pair with isotropy in $\mathcal{F}$. By the universal property of the classifying space $E_{\mathcal{F}}(G, \mathcal{H})$, the canonical G-map $f\colon (\widetilde{X}, p^{-1}(A)) \to E_{\mathcal{F}}(G, \mathcal{H})$ factors (up to G-homotopy) through the equivariant nerve map $\widetilde{\nu}$:

$$\big(\widetilde{X}, p^{-1}(A)\big) \xrightarrow{\widetilde{\nu}} \big(|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|\big) \to E_{\mathcal{F}}(G, \mathcal{H})\,.$$

Since X and all the $(A_i)_{i\in I}$ are aspherical by assumption, Proposition 7.2 and a five lemma for dimension-isomorphisms [Sau05, Section 2] imply that f induces a dimension-isomorphism

$$H_*^{(2)}\big(G \curvearrowright (\widetilde{X}, p^{-1}(A))\big) \xrightarrow{\cong_{\dim}} H_*^{(2)}\big(G \curvearrowright E_{\mathcal{F}}(G,\mathcal{H})\big)\,.$$

Thus, the above factorisation of the G-map f shows

$$b_*^{(2)}\big(G \curvearrowright (\widetilde{X}, p^{-1}(A))\big) \leq b_*^{(2)}\big(G \curvearrowright (|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|)\big)\,,$$

as claimed.

To conclude the vanishing result, suppose that the relative cover $\mathcal{U}$ is weakly convex with $\operatorname{mult}_A(\mathcal{U}) = n$. Then we have $\dim(|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|) = n-1$ by Proposition 4.10 (i) and hence $b_k^{(2)}(G \curvearrowright (|N(\widetilde{\mathcal{U}})|, |N_{p^{-1}(A)}(\widetilde{\mathcal{U}})|)) = 0$ for all degrees $k \geq n$. □

In the absolute case for the family Am, we recover a result of Sauer [Sau09, Theorem C]:

Corollary 7.4. *Let X be a path-connected aspherical CW-complex. Then we have $b_k^{(2)}(\pi_1(X) \curvearrowright \widetilde{X}) = 0$ for all $k \geq \operatorname{cat}_{\mathsf{Am}}(X)$.* □

8. Glueing estimates for relative simplicial volume

The classical glueing estimates for simplicial volume [Gro82, BBF$^+$14, Kue15] require that the boundary components used in the glueing have amenable fundamental groups. Replacing amenability with bounded acyclicity and the uniform boundary condition also leads to (albeit weaker) glueing estimates for simplicial volume (Theorem 8.1). The uniform boundary condition also has been used for versions of simplicial volume where the strong glueing formulae are unknown and no suitable dual theory is available [FL21, FFL19].

Theorem 8.1 (Vanishing inheritance for boundedly acyclic glueings)**.** *Let $n \geq 3$ and $(M_i, \partial M_i)_{i\in I}$ be a finite collection of oriented compact connected n-manifolds. Assume that every connected component of every boundary component ∂M_i has boundedly acyclic fundamental group. Let $\mathcal{N}$ be a set of π_1-injective boundary components of the $(M_i)_{i\in I}$ and let $(M, \partial M)$ be obtained from $(M_i, \partial M_i)_{i\in I}$ by a pairwise glueing of the boundary components in $\mathcal{N}$ (along orientation-reversing homeomorphisms).*

If $\mathcal{N}$, viewed as a set of subsets of M, is uniformly boundedly acyclic of order n in M (Definition 6.2) then the following are equivalent:

(i) *We have $\|M, \partial M\| = 0$;*
(ii) *For all $i \in I$, we have $\|M_i, \partial M_i\| = 0$.*

The implication (ii) ⇒ (i) is proved in Section 8.1. The implication (i) ⇒ (ii) is established in Section 8.2.

Remark 8.2. More generally, the conclusion of Theorem 8.1 holds for *compatible* glueings [BBF$^+$14], where the boundary components in $\mathcal{N}$ need not be π_1-injective. On the other hand, it remains unclear whether the assumption of bounded acyclicity on the boundary components that are not in $\mathcal{N}$ can be dropped.

Remark 8.3. In the situation of Theorem 8.1, if the collection of fundamental groups of all members of $\mathcal{N}$ is *malnormal* in $\pi_1(M)$, then the uniform bounded acyclicity condition is automatically satisfied as soon as all members of $\mathcal{N}$ have boundedly acyclic fundamental group.

8.1. **Upper glueing estimates via the uniform boundary condition.** We begin with the, easier, upper glueing estimate; this estimate works over all normed rings and also gives rough estimates in the non-vanishing case:

Proposition 8.4. *Let R be a normed ring, let $K \in \mathbb{R}_{>0}$, let I be a finite set, and let $(M_i, \partial M_i)_{i\in I}$ be a finite collection of oriented compact connected manifolds of the same dimension n. Moreover, let $(M, \partial M)$ be obtained from $(M_i, \partial M_i)_{i\in I}$ by a pairwise glueing (along orientation-reversing homeomorphisms) of a set $\mathcal{N}$ of boundary components such that K is a* UBC_{n-1}*-constant of $C_*(\bigcup \mathcal{N}; R)$. Then*

$$\|M, \partial M\|_R \leq \big(1 + K \cdot (n+1)\big) \cdot \sum_{i\in I} \|M_i, \partial M_i\|_R \,.$$

In particular, if $\|M_i, \partial M_i\|_R = 0$ for all $i \in I$, then $\|M, \partial M\|_R = 0$.

Proof. This is the standard filling argument [Löh07, Example 6.18][BBF+14, Remark 6.2], adapted to the general UBC-setting; for the sake of completeness, we give the argument:

For notational simplicity, we view the $(M_i)_{i\in I}$ as subspaces of the glued manifold M. Moreover, we write $N \subset M$ for the (disjoint) union of the glueing loci. Hence, K is a UBC_{n-1}-constant for N. Let $(z_i \in C_n(M_i; R))_{i\in I}$ be a collection of relative fundamental cycles of $(M_i, \partial M_i)_{i\in I}$. As we glue along orientation-reversing homeomorphisms, the chain

$$b \coloneqq \sum_{i\in I} \partial z_i|_N \in C_{n-1}(N; R)$$

is null-homologous. By UBC_{n-1} for $C_*(N; R)$, there exists a chain $c \in C_n(N; R)$ with

$$\partial c = b \quad \text{and} \quad |c|_1 \leq K \cdot |b|_1 \leq K \cdot \sum_{i\in I} |\partial z_i|_1 \leq K \cdot (n+1) \cdot \sum_{i\in I} |z_i|_1 \,.$$

Then $z \coloneqq \sum_{i\in I} z_i - c \in C_n(M; R)$ is a relative cycle on $(M, \partial M)$; checking the local contributions on the components $(M_i, \partial M_i)$ shows that z is a relative R-fundamental cycle of $(M, \partial M)$. Therefore, we obtain

$$\begin{aligned} \|M, \partial M\|_R \leq |z|_1 &\leq \sum_{i\in I} |z_i|_1 + |c|_1 \\ &\leq \sum_{i\in I} |z_i|_1 + K \cdot (n+1) \cdot \sum_{i\in I} |z_i|_1 \,. \end{aligned}$$

Taking the infimum over all relative fundamental cycles $(z_i)_{i\in I}$ proves the claim. □

Proof of Theorem 8.1, (ii) ⇒ (i). All boundedly acyclic groups satisfy the uniform boundary condition in all degrees (Theorem A.6). As only finitely many components are involved, we also find a joint UBC_{n-1}-constant for $C_*(\bigcup \mathcal{N}; \mathbb{R})$. Applying Proposition 8.4 thus proves the implication (ii) ⇒ (i). □

In the same way, we also obtain the following estimate for the locally finite simplicial volume [Gro82, Löh07] for interiors of compact manifolds with UBC-boundary; similar results have been obtained previously for amenable boundaries (or other more restrictive conditions on the boundary) via the uniform boundary condition [LS09b, Löh07] or bounded cohomology [KK15].

Proposition 8.5. *Let R be a normed ring and let M be an oriented connected compact n-manifold with boundary satisfying the following properties:*

- *We have $\|\partial M\|_R = 0$;*
- *The boundary ∂M satisfies UBC_{n-1} over R; let K be a UBC_{n-1}-constant for $C_*(\partial M; R)$.*

Then

$$\|M^\circ\|_{R,\mathrm{lf}} \leq \big(K \cdot (n+1) + 1\big) \cdot \|M, \partial M\|_R \,.$$

Proof. We will follow the previously known UBC-arguments: Let $c \in C_n(M; R)$ be a relative fundamental cycle of $(M, \partial M)$ and let $\varepsilon \in \mathbb{R}_{>0}$.

Because of $\|\partial M\|_R = 0$, there exists a sequence $(z_k)_{k\in\mathbb{N}}$ in $C_{n-1}(\partial M; R)$ of fundamental cycles of ∂M with $|z_k|_1 \leq \varepsilon \cdot 1/2^k$ for all $k \in \mathbb{N}$. From this sequence, we can construct a locally finite relative fundamental cycle $z \in C_n^{\mathrm{lf}}(\partial M \times \mathbb{R}_{\geq 0}; R)$ of the half-open cylinder $\partial M \times \mathbb{R}_{\geq 0}$ with $\partial z = -\partial c$ and

$$\begin{aligned}
|z|_1 &\leq K \cdot \big(|\partial c|_1 + |z_0|_1\big) + n \cdot |z_0|_1 + \sum_{k=0}^{\infty} \big(K \cdot \big(|z_k|_1 + |z_{k+1}|_1\big) + n \cdot |z_{k+1}|_1\big) \\
&\leq K \cdot |\partial c|_1 + n \cdot \varepsilon \cdot 2 + K \cdot 2 \cdot \varepsilon \cdot 2 \\
&\leq K \cdot (n+1) \cdot |c|_1 + 2 \cdot \varepsilon \cdot (n + 2 \cdot K)
\end{aligned}$$

by UBC-filling the differences between subsequent z_k with "small" chains and then spreading out the result over the half-open cylinder [Löh07, proof of Theorem 6.1]. Here we filled the "cylinders" by using the canonical triangulation of $\Delta^{n-1} \times [0,1]$ into n simplices of dimension n.

Moreover, $c + z$ is a locally finite fundamental cycle of $M \cup_{\partial M} (\partial M \times \mathbb{R}_{\geq 0}) \cong M^\circ$ (via the topological collar theorem) and so

$$\|M^\circ\|_{R,\mathrm{lf}} \leq |c+z|_1 \leq |c|_1 + K \cdot (n+1) \cdot |c|_1 + 2 \cdot \varepsilon \cdot (n + 2 \cdot K) \,.$$

Taking the infimum over $\varepsilon \to 0$ and then over all relative fundamental cycles c thus shows that

$$\|M^\circ\|_{R,\mathrm{lf}} \leq \|M, \partial M\|_R + K \cdot (n+1) \cdot \|M, \partial M\|_R \,,$$

as claimed. □

Corollary 8.6. *Let M be an oriented connected compact n-manifold with boundary satisfying the following properties:*

- *We have $\|M, \partial M\| = 0$;*
- *The boundary ∂M satisfies UBC_{n-1}.*

Then $\|M^\circ\|_{\mathrm{lf}} = 0$.

Proof. Since $\|\partial M\| \leq (n+1) \cdot \|M, \partial M\| = 0$, we can apply Proposition 8.5. □

In particular, the conditions on the boundary in Proposition 8.5 (over $\mathbb{R}$) and Corollary 8.6 are satisfied if the boundary has vanishing bounded cohomology in positive degrees.

Remark 8.7 (Group-theoretic Dehn fillings). A classical application of upper glueing estimates are (generalised) Dehn fillings of manifolds [FM11, BBF$^+$14]. The simplicial volume of *group-theoretic* Dehn fillings was recently investigated [PS24]. In particular, the simplicial volume does not increase when performing a group-theoretic Dehn filling with resulting peripheral subgroups that are amenable and of small cohomological dimension [PS24, Theorem 6.3]. One obtains an analogous result for the vanishing behaviour of simplicial volume if amenability is replaced by bounded acyclicity.

8.2. **Lower glueing estimates via bounded acyclicity.** We prove the lower glueing estimate (i) $\Rightarrow$ (ii) of Theorem 8.1, adapting the argument in the amenable case by Bucher, Burger, Frigerio, Iozzi, Pagliantini, and Pozzetti [BBF$^+$14].

In this section, all [bounded] cohomology groups are taken with trivial coefficients in $\mathbb{R}$.

Proof of Theorem 8.1, (i) $\Rightarrow$ (ii). We proceed by contraposition, i.e., we assume that one of the building blocks satisfies $\|M_i, \partial M_i\| > 0$ and show that $\|M, \partial M\| > 0$. By the duality principle (Proposition 5.6), it suffices to show that the comparison map $H_b^n(M, \partial M) \to H^n(M, \partial M)$ is non-zero.

Let $N \subset M$ be the union of the glueing loci. We consider the following diagram. Here, all unlabelled arrows are induced by inclusions (and direct sums) and the maps labelled by ev are given by evaluation on the fundamental class.

$$\begin{array}{ccccccc}
H_b^n(M) & \xleftarrow{\cong} & H_b^n(M, \partial M) & \xrightarrow[\mathrm{comp}^n_{(M,\partial M)}]{} & H^n(M, \partial M) & \xrightarrow[\mathrm{ev}]{\cong} & \mathbb{R} \\
\downarrow{\scriptstyle (**)} & & \uparrow{\scriptstyle \cong} & & \uparrow & & \uparrow{\scriptstyle \mathrm{sum}} \\
 & & H_b^n(M, N \cup \partial M) & \xrightarrow[\mathrm{comp}^n_{(M,N\cup\partial M)}]{} & H^n(M, N \cup \partial M) & & \\
 & & \downarrow{\scriptstyle (***)} & & \downarrow & & \\
\bigoplus_{i\in I} H_b^n(M_i) & \xleftarrow{\cong} & \bigoplus_{i\in I} H_b^n(M_i, \partial M_i) & \xrightarrow[\bigoplus_{i\in I} \mathrm{comp}^n_{(M_i,\partial M_i)}]{(*)} & \bigoplus_{i\in I} H^n(M_i, \partial M_i) & \xrightarrow[\mathrm{ev}]{\cong} & \bigoplus_I \mathbb{R}
\end{array}$$

This diagram is commutative: The leftmost and middle squares are commutative by functoriality of bounded cohomology and naturality of the comparison map. The rightmost square is commutative because one can construct a relative fundamental cycle of $(M, \partial M)$ out of relative fundamental cycles of the $(M_i, \partial M_i)$ plus a chain on N (see proof of Proposition 8.4).

By the duality principle and the hypothesis that one of the $\|M_i, \partial M_i\|$ is non-zero, the arrow $(*)$ is non-zero; isolating the corresponding index shows that also the composition $\mathrm{sum} \circ \mathrm{ev} \circ (*)$ is non-zero.

In the leftmost square, both horizontal arrows and the upper right vertical arrow are isomorphisms by bounded acyclicity of all boundary components (and the long exact sequence of pairs in bounded cohomology). In Section 8.3, using graphs of groups (Theorem 8.11 and Example 8.12), we will show that the left vertical arrow $(**)$ induced by the inclusions $M_i \hookrightarrow M$ is surjective.

Therefore, the leftmost square shows that $(***)$ is surjective. Together with the non-triviality of the composition $\mathrm{sum} \circ \mathrm{ev} \circ (*)$, we thus obtain that the comparison map $H_b^n(M, \partial M) \to H^n(M, \partial M)$ must be non-zero, as desired. □

8.3. **Graphs of groups with boundedly acyclic edge groups.** We consider the bounded cohomology of finite graphs of groups with boundedly acyclic edge groups in relation to the bounded cohomology of the vertex groups. We adapt the proof in the amenable case by Bucher et al. [BBF$^+$14] to the boundedly acyclic situation by using uniformly boundedly acyclic actions instead of amenable actions.

We first fix basic notation.

Definition 8.8 (Graph)**.** A *graph* is a tuple $\Gamma = (V, E, o, t, \overline{\cdot})$, consisting of a set V, a set E, a map $(o,t)\colon E \to V^2$, and a fixed point-free involution $\overline{\cdot}\colon E \to E$ with

$$o(e) = t(\overline{e})$$

for all $e \in E$. The elements of V are called *vertices*, the elements of E are called *edges*. The set of *geometric edges* is defined by

$$\overline{E} \coloneqq \left\{ (e, \overline{e}) \;\middle|\; e \in E \right\}.$$

Definition 8.9 (Graph of groups)**.** Let $\Gamma = (V, E, o, t, \overline{\cdot})$ be a finite graph (i.e., V and E are finite). A *graph of groups* $\mathbb{G}$ *over* Γ is a Γ-shaped diagram in the category of groups and injective group homomorphisms, i.e., $\mathbb{G}$ consists of the following data:

- A map that associates a group G_v to each $v \in V$;
- A map that associates a group G_e to each $e \in E$ such that $G_e = G_{\overline{e}}$;
- A map that associates to each edge $e \in E$ an injective group homomorphism $h_e\colon G_e \to G_{t(e)}$.

If G is the fundamental group of a graph of groups, then the vertex and edge groups admit canonical inclusions into G [Ser03, Chapter 5] and we will identify these groups with their image inside of G.

We consider finite graphs of groups with boundedly acyclic edge groups, in analogy with the amenable case [BBF$^+$14, Theorem 1.1]; more precisely:

Setup 8.10.

- Let $n \geq 1$;
- Let $\Gamma = (V, E, o, t, \overline{\cdot})$ be a finite graph;
- Let $\mathbb{G}$ be a graph of groups over Γ;
- Let G be the fundamental group of $\mathbb{G}$; for $v \in V$, we denote the corresponding inclusion by $i_v\colon G_v \hookrightarrow G$;
- The edge groups $(G_e)_{e \in E}$ are *uniformly boundedly acyclic of order n in G*, i.e., the set

$$\left\{ \bigcap_{i=1}^{n} g_i G_{e_i} g_i^{-1} \;\middle|\; g_1, \dots, g_n \in G,\ e_1, \dots, e_n \in E \right\}$$

  of subgroups of G is uniformly boundedly acyclic.

Theorem 8.11. *In the situation of Setup 8.10, let $n \geq 3$ and $k \in \{3, \dots, n\}$. Then the map*

$$\bigoplus_{v \in V} H_b^k(i_v)\colon H_b^k(G) \to \bigoplus_{v \in V} H_b^k(G_v)$$

induced by the inclusions is surjective.

For the proof, we describe the bounded cohomology of G and the vertex groups via suitable uniformly boundedly acyclic actions. In the situation of Setup 8.10, we consider the set

$$S \coloneqq (G \times V) \sqcup \coprod_{e \in \overline{E}} G/G_e$$

with the G-action

- given on $G \times V$ by left translation on the first factor;
- given on each G/G_e by left translation of cosets.

In particular, for $k \in \{0, \dots, n-1\}$, the diagonal action of G on S^{k+1} is uniformly boundedly acyclic, since we assumed uniform bounded acyclicity of order n for the collection of edge groups. By Remark 5.17, the bounded cohomology of G is canonically isomorphic to the cohomology of the complex $\ell^\infty_{\mathrm{alt}}(S^{*+1}, \mathbb{R})^G$ in degrees $\leq n$.

Similarly, for each vertex $v \in V$, we consider the G_v-set

$$S_v \coloneqq G_v \sqcup \coprod_{e \in E, t(e) = v} G_v/G_e\,.$$

with the left translation action. In the situation of Setup 8.10, the diagonal action on S_v^{k+1} is uniformly boundedly acyclic for $k \in \{0, \dots, n-1\}$ and thus the bounded cohomology of G_v is canonically isomorphic to the cohomology of the complex $\ell^\infty_{\mathrm{alt}}(S_v^{*+1}, \mathbb{R})^{G_v}$ in degrees $\leq n$ (Remark 5.17).

For each vertex $v \in V$, there is a canonical inclusion $\varphi_v \colon S_v \to S$; on the first summand, this is given by $\varphi_v(g) \coloneqq (g, v)$ for all $g \in G_v$, on the other summands, we use the canonical maps induced by the canonical inclusions $i_v \colon G_v \hookrightarrow G$. By construction, φ_v is G_v-equivariant with respect to the inclusion i_v.

With this preparation, we give the proof of Theorem 8.11:

Proof of Theorem 8.11. We write

$$\varphi^* \coloneqq \bigoplus_{v \in V} \varphi_v^* \colon \ell^\infty_{\mathrm{alt}}(S^{*+1}, \mathbb{R})^G \to \bigoplus_{v \in V} \ell^\infty_{\mathrm{alt}}(S_v^{*+1}, \mathbb{R})^{G_v}$$

for the combination of the φ_v^*. Bucher et al. [BBF+14, Theorem 4.1] provide a construction of a cochain map $\psi^* \colon \bigoplus_{v \in V} \ell^\infty_{\mathrm{alt}}(S_v^{*+1}, \mathbb{R})^{G_v} \to \ell^\infty_{\mathrm{alt}}(S^{*+1}, \mathbb{R})^G$ in degrees ≥ 2 that is right-inverse to φ^*. Then also $H^k(\varphi^*)$ has a right inverse in degrees ≥ 3.

Let $k \in \{3, \dots, n\}$. By Lemma 5.18 (applied to i_v and φ_v for each vertex $v \in V$) and using that V is finite, we obtain the following commutative diagram:

$$\begin{array}{ccccc}
H^k\big(\ell^\infty_{\mathrm{alt}}(S^{*+1}, \mathbb{R})^G\big) & & \longrightarrow & & H_b^k(G; \mathbb{R}) \\
\downarrow{\scriptstyle H^k(\varphi^*)} & & & & \downarrow{\scriptstyle \bigoplus_{v \in V} H_b^k(i_v)} \\
H^k\big(\bigoplus_{v \in V} \ell^\infty_{\mathrm{alt}}(S_v^{*+1}, \mathbb{R})^{G_v}\big) & \xleftarrow{\cong} & \bigoplus_{v \in V} H^k\big(\ell^\infty_{\mathrm{alt}}(S_v^{*+1}, \mathbb{R})^{G_v}\big) & \longrightarrow & \bigoplus_{v \in V} H_b^k(G_v; \mathbb{R})\,.
\end{array}$$

Here the horizontal maps are the canonical maps. As $k \in \{3, \dots, n\}$, these horizontal maps are isomorphism and the left vertical arrow admits a right inverse (given by ψ^*). Therefore, also the right vertical arrow has a right inverse. In particular, the right vertical arrow $\bigoplus_{v \in V} H_b^k(i_v)$ is surjective. □

In particular, Theorem 8.11 applies to the glueing situation of Theorem 8.1:

Example 8.12. In the situation of Theorem 8.11, the fundamental group $\pi_1(M)$ is isomorphic to the fundamental group of a finite graph of groups that satisfies the conditions of Setup 8.10. More specifically, the vertex groups are isomorphic to the fundamental groups of the $(M_i)_{i\in I}$ and the edge groups are isomorphic to the fundamental groups of the boundary components in $\mathcal{N}$ along which we glue.

This concludes the proof of Theorem 8.1.

Appendix A. The uniform boundary condition

We recall the uniform boundary condition and its basic properties and consequences in the context of bounded cohomology. Moreover, we introduce the uniform uniform boundary condition and use it to compute the bounded cohomology of bounded products.

A.1. **Normed chain complexes.** We begin with basic terminology for normed chain complexes. A *normed chain complex* over a normed ring R is a chain complex in the category of normed R-modules with bounded linear maps; i.e., the boundary operators in normed chain complexes are degree-wise bounded linear operators. A *Banach chain complex* is a normed chain complex over $\mathbb{R}$ consisting of Banach spaces. Similarly, one has *normed* [*resp. Banach*] *cochain complexes*.

Let C_* be a normed chain complex over a normed ring R with boundary operator $\partial_*\colon C_* \to C_{*-1}$.

- We write $B(C_*, R)$ for the normed cochain complex whose cochain modules are the bounded duals of the chain modules of C_* and whose coboundary operators are the duals of the boundary operators ∂_*.
- We write $\overline{C}_*$ for the degree-wise completion of C_* with the degree-wise continuous extension $\overline{\partial}_*$ of the boundary operator ∂_*.

Over $\mathbb{R}$, both $B(C_*, \mathbb{R})$ and $\overline{C}_*$ are Banach [co]chain complexes.

If C_* is a normed chain complex, then we obtain an induced seminorm on $H_*(C_*)$ via

$$\|\alpha\| \coloneqq \inf\bigl\{|c| \bigm| c \in C_* \text{ is a cycle representing } \alpha\bigr\}$$

for all $\alpha \in H_*(C_*)$. Similarly, this also works for normed cochain complexes.

A.2. **The uniform boundary condition.** The uniform boundary condition asks for uniform control on fillings of null-homologous [co]cycles in normed [co]chain complexes. In some contexts, similar properties are encoded in the language of isoperimetric inequalities. In the following, we will stick to the terminology of Matsumoto and Morita [MM85], see also Frigerio's book [Fri17, Chapter 6].

Definition A.1 (Uniform boundary condition)**.** Let C_* be a normed chain complex over a normed ring and let $k \in \mathbb{N}$. Then C_* satisfies the *uniform boundary condition in degree k* if there exists a constant $K \in \mathbb{R}_{>0}$ with

$$\forall_{b\in \operatorname{im}\partial_{k+1}\subset C_k} \quad \exists_{c\in C_{k+1}} \quad \partial_{k+1}(c) = b \quad \text{and} \quad |c| \leq K \cdot |b|\,.$$

We abbreviate the uniform boundary condition in degree k by UBC_k.

Remark A.2. Through re-indexing, we can translate the notion of uniform boundary condition also to normed cochain complexes. In this case, we use the notation UBC^k for the uniform boundary condition in degree k. Moreover, all of the results below apply both to chain and cochain complexes (with appropriate re-indexing).

We recall basic inheritance properties of UBC:

Proposition A.3 (Homotopy inheritance of UBC)**.** *Let $k \in \mathbb{N}$ and let C_*, D_* be normed chain complexes over a normed ring R that are chain homotopic in the category of normed R-chain complexes. Then C_* satisfies* UBC_k *if and only if D_* satisfies* UBC_k*.*

More precisely, let $f_*\colon C_* \to D_*$ and $g_*\colon D_* \to C_*$ be chain maps that are bounded in each degree and let $h_*\colon D_* \to D_{*+1}$ be a corresponding degree-wise bounded chain homotopy between $f_* \circ g_*$ and id_{D_*}. If $K \in \mathbb{R}_{>0}$ is a UBC_k*-constant for C_*, then*

$$\|f_{k+1}\| \cdot \|g_k\| \cdot K + \|h_k\|$$

is a UBC_k*-constant for D_*.*

Proof. Let $x \in \operatorname{im} \partial^D_{k+1}$. Then $g_k(x) \in \operatorname{im} \partial^C_{k+1}$. As K is a UBC_k-constant for C_*, there exists $\widetilde{y} \in C_{k+1}$ with $\partial^C_{k+1}(\widetilde{y}) = g_k(x)$ and $|\widetilde{y}| \leq K \cdot \big|g_k(x)\big|$. Then

$$y \coloneqq f_{k+1}(\widetilde{y}) - h_k(x)$$

satisfies $\partial^D_{k+1}(y) = x$ and $|y| \leq \big(\|f_{k+1}\| \cdot \|g_k\| \cdot K + \|h_k\|\big) \cdot |x|$, as desired. □

Proposition A.4 (Dense subcomplexes and UBC)**.** *Let R be a normed ring, let D_* be a normed chain complex over R, and let $C_* \subset D_*$ be a dense subcomplex. Let $k \in \mathbb{N}$. Then the following are equivalent:*

(i) *C_* satisfies* UBC_k*;*
(ii) *$\overline{D}_*$ satisfies* UBC_k *and* $\ker \partial^C_{k+1}$ *is dense in* $\ker \partial^{\overline{D}}_{k+1}$*;*
(iii) *D_* satisfies* UBC_k *and* $\ker \partial^C_{k+1}$ *is dense in* $\ker \partial^D_{k+1}$*.*

Proof. Because C_* is dense in D_*, the completion $\overline{D}_*$ of D_* is also a completion of C_*. The argument by Matsumoto and Morita [MM85, Theorem 2.8] applies to all normed chain complexes and their completions; this shows the equivalence of (i) and (ii).

We may apply this to D_* and its completion $\overline{D}_*$, since with $\ker \partial^C_{k+1}$ also $\ker \partial^D_{k+1}$ is dense in $\ker \partial^{\overline{D}}_{k+1}$. Hence (ii) implies that D_* satisfies UBC_k; whence (ii) implies (iii), because $\ker \partial^C_{k+1}$ being dense in $\ker \partial^{\overline{D}}_{k+1}$ also implies density of $\ker \partial^C_{k+1}$ in $\ker \partial^D_{k+1}$.

Also, the argument by Matsumoto and Morita shows that (iii) implies (i), as this implication does not rely on completeness of the ambient complex. □

The following characterisations of UBC apply to normed [resp. Banach] chain complexes over $\mathbb{R}$.

Theorem A.5 ([MM85, Theorem 2.8])**.** *Let C_* be a normed chain complex over $\mathbb{R}$ and let $k \in \mathbb{N}$. Then the following are equivalent:*

(i) *C_* satisfies* UBC_k*;*
(ii) *$\overline{C}_*$ satisfies* UBC_k *and* $\ker \partial_{k+1}$ *is dense in* $\ker \overline{\partial}_{k+1}$*;*
(iii) *The comparison map $H^{k+1}(B(C_*, \mathbb{R})) \to H^{k+1}(\mathrm{Hom}_{\mathbb{R}}(C_*, \mathbb{R}))$ is injective.*

Proof. The equivalence of the first two items is contained in Proposition A.4. The argument by Matsumoto and Morita [MM85, Theorem 2.8] for the remaining implications applies to all normed chain complexes over $\mathbb{R}$. □

Theorem A.6 ([MM85, Theorem 2.3])**.** *Let C_* be a Banach chain complex over $\mathbb{R}$ and let $k \in \mathbb{N}$. Then the following are equivalent:*

(i) C_* *satisfies* UBC_k*;*
(ii) $\operatorname{im}\partial_{k+1}$ *is closed in* C_k*;*
(iii) $H_k(C_*)$ *is a Banach space with respect to the induced seminorm;*
(iv) $H^{k+1}(B(C_*,\mathbb{R}))$ *is a Banach space with respect to the induced seminorm.*

Proof. The argument by Matsumoto and Morita [MM85, Theorem 2.3] applies to all Banach chain complexes. □

Example A.7. Let X be a space or a group with $H_b^*(X;\mathbb{R}) \cong H_b^*(1;\mathbb{R})$. In particular, this bounded cohomology is Banach in all degrees. Then the cochain complex version of Theorem A.6 (Remark A.2) shows that $C_b^*(X;\mathbb{R})$ satisfies UBC^k for all $k \in \mathbb{N}$. Moreover, Theorem A.5 shows that $C_*(X;\mathbb{R})$ satisfies UBC_k for all $k \in \mathbb{N}$.

This applies to all path-connected spaces with amenable fundamental group, to all amenable groups, and to the known boundedly acyclic groups [MM85, Löh17, FFLM24, FFLM23, MN23, Mon22]. In particular, there exist finitely presented non-amenable groups G such that $C_b^*(G;\mathbb{R})$ satisfies UBC^k for all $k \in \mathbb{N}$ [FFLM24, Corollary 5.2][Mon22].

Remark A.8. For the free group F_2 of rank 2 it is well known that $H_b^2(F_2;\mathbb{R})$ and $H_b^3(F_2;\mathbb{R})$ are infinite-dimensional. But it is unknown whether the higher bounded cohomology of F_2 is trivial or not. We outline an approach through the uniform boundary condition: Let $k \in \mathbb{N}_{\geq 4}$. Then the following are equivalent:

(i) $H_b^k(F_2;\mathbb{R}) \cong 0$;
(ii) $C_*(F_2;\mathbb{R})$ satisfies UBC_{k-1};
(iii) $C_*(F_2;\mathbb{Q})$ satisfies UBC_{k-1};

Indeed, the first two items are equivalent by Theorem A.5 and the fact that $H^k(F_2;\mathbb{R}) \cong 0$. The equivalence of the last two items follows from Proposition A.4, the fact that $C_*(F_2;\mathbb{Q}) \hookrightarrow C_*(F_2;\mathbb{R})$ induces an isometric embedding on the level of homology [Löh07, Proposition 1.7], and a small computation using the universal coefficient theorem.

Moreover, the last condition can be reformulated as follows:

$$\exists_{K\in\mathbb{R}_{>0}} \quad \forall_{c\in C_k(F_2;\mathbb{Z})} \quad \exists_{c'\in C_k(F_2;\mathbb{Q})} \quad \partial_k(c') = \partial_k(c) \quad \text{and} \quad |c'|_1 \leq K \cdot |\partial_k(c)|_1 \,.$$

In principle, this allows for experimental testing whether $H_b^k(F_2;\mathbb{R})$ is trivial or not [Lan19]. Of course, the main challenge is to efficiently generate large amounts of "interesting" chains in $C_k(F_2;\mathbb{Z})$.

The uniform boundary condition is useful, e.g., in glueing estimates for simplicial volume (Section 8.1), in vanishing results for bounded cohomology of certain groups [MM85, Löh17, FFLM24, FFLM23], and in vanishing results for ℓ^1-homology [MM85, FL21].

A.3. **The uniform uniform boundary condition.** We introduce a uniform version of the uniform boundary condition for collections of normed cochain complexes.

Definition A.9 (Uniform uniform boundary condition)**.** Let R be a normed ring and let $k \in \mathbb{N}$. A collection $(C_i^*)_{i\in I}$ of normed cochain complexes over R satisfies the *uniform uniform boundary condition in degree* k (UUBC^k for short) if there exists $K \in \mathbb{R}_{>0}$ that is a UBC^k-constant for all C_i^* with $i \in I$.

Example A.10 ([FFLM23, Example 4.11])**.** We consider a collection $(H_i)_{i\in I}$ of amenable groups. Then $(C_b^*(H_i;\mathbb{R}))_{i\in I}$ satisfies UUBC^k for all $k \in \mathbb{N}$. Indeed, if H is amenable, then 1 is a UBC^k-constant for $C_b^*(H;\mathbb{R})$: There exists a contracting cochain homotopy s for $C_b^*(H;\mathbb{R})$ with $\|s\| \leq 1$ [Fri17, Theorem 3.6]; thus, for every cocycle b, the cochain $c \coloneqq s(b)$ satisfies

$$b = \delta \circ s(b) + s \circ \delta(b) = \delta(c) \quad \text{and} \quad |c|_\infty \leq \|s\| \cdot |b|_\infty \leq |b|_\infty .$$

Example A.11. Every finite collection of cochain complexes whose members all satisfy UBC^k satisfies UUBC^k.

In particular, from Example A.7, we obtain: If $(H_i)_{i\in I}$ is a *finite* collection of boundedly acyclic groups, then $(C_b^*(H_i;\mathbb{R}))_{i\in I}$ satisfies UUBC^k for all $k \in \mathbb{N}$.

It is unknown whether all collections of boundedly acyclic groups satisfy UUBC in all degrees (as the open mapping theorem used in the proof of Theorem A.6 does not give a priori estimates on the norms of the partial inverses). Every collection of boundedly acyclic groups satisfies UUBC in degree 2 [FFLM23, Proposition 4.15].

A.4. **Bounded products.** Finite degree-wise products are compatible with taking cohomology of bounded cochain complexes. For infinite degree-wise products, in general, one needs to impose boundedness conditions in a uniform way. To this end, we introduce bounded products and prove a compatibility statement for cohomology of certain degree-wise bounded products.

Definition A.12 (Bounded product)**.** Let R be a normed ring. Let $(V_i)_{i\in I}$ be a collection of normed modules over R. The *bounded product* of $(V_i)_{i\in I}$ is the normed R-module

$$\prod_{i\in I}^b V_i \coloneqq \Big\{ x \in \prod_{i\in I} V_i \;\Big|\; \sup_{i\in I} |x_i| < \infty \Big\} \subset \prod_{i\in I} V_i$$

with respect to the supremum norm $|\cdot|_\infty$.

Example A.13. Let $(S_i)_{i\in I}$ be a collection of sets and let R be a normed ring. Then the canonical inclusions $(S_j \hookrightarrow \coprod_{i\in I} S_i)_{j\in I}$ induce a natural isometry

$$\ell^\infty\Big(\coprod_{i\in I} S_i, R\Big) \to \prod_{i\in I}^b \ell^\infty(S_i, R)$$

of normed R-modules.

Remark A.14 (Bounded product of normed cochain complexes)*.* Let R be a normed ring. A collection $(C_i^*)_{i\in I}$ of normed cochain complexes over R is *uniform* if for each $k \in \mathbb{N}$, the supremum $\sup_{i\in I} \|\delta_i^k\|$ is finite. For example, all collections of normed cochain complexes built using simplicial coboundary operators are uniform (such as bounded cochain complexes of groups or spaces).

If $(C_i^*)_{i\in I}$ is a uniform collection of normed cochain complexes over R, then the degree-wise bounded product $(\prod_{i\in I}^b C_i^k)_{k\in\mathbb{N}}$ is a normed cochain complex over R with respect to the supremum norm and the degree-wise product coboundary operator

$$\begin{aligned} \prod_{i\in I}^b C_i^* &\to \prod_{i\in I}^b C_i^{*+1} \\ (x_i)_{i\in I} &\mapsto (\delta_i^*(x_i))_{i\in I} . \end{aligned}$$

Theorem A.15 (Cohomology of bounded products)**.** *Let $k \in \mathbb{N}$. Let $(C_i^*)_{i\in I}$ be a uniform collection of normed cochain complexes over a normed ring R that satisfies* UUBC^k*. Then the map*

$$\Phi\colon H^k\Big(\prod_{i\in I}^b C_i^*\Big) \to \prod_{i\in I}^b H^k(C_i^*)$$

induced by the canonical projections is a continuous isomorphism of R-modules with continuous inverse. Here, we equip $H^k(C_i^*)$ with the seminorm induced by the given norm on C_i^*.

Proof. Clearly, the map Φ is well-defined and continuous.

We construct an explicit inverse: Let $\varepsilon \in \mathbb{R}_{>0}$. Let $(\varphi_i)_{i\in I} \in \prod_{i\in I}^b H^k(C_i^*)$; for each $i \in I$, there exists a cocycle $f_i \in C_i^k$ representing φ_i in $H^k(C_i^*)$ with

$$|f_i| \le \|\varphi_i\| + \varepsilon\,.$$

Because $(\varphi_i)_{i\in I}$ lies in the bounded product, the norms $(|f_i|)_{i\in I}$ are a bounded collection, and so $f \coloneqq (f_i)_{i\in I} \in \prod_{i\in I}^b C_i^k$; moreover, $\delta(f) = (\delta_i(f_i))_{i\in I} = 0$. Therefore, we obtain a cohomology class

$$\varphi \coloneqq [f] \in H^k\Big(\prod_{i\in I}^b C_i^*\Big)\,.$$

By construction, $\Phi(\varphi) = (\varphi_i)_{i\in I}$.

If this construction is independent of the chosen collection $(f_i)_{i\in I}$, then it provides an R-linear inverse of Φ; moreover, as we can take $\varepsilon \to 0$, we also see that this inverse is bounded.

Thus, it remains to show that φ is independent of the choice of the collection $(f_i)_{i\in I}$. In order to show this, we use the uniform uniform boundary condition: Let $(f_i')_{i\in I} \in \prod_{i\in I} C_i^k$ be a collection of cocycles with

$$[f_i'] = \varphi_i \in H^k(C_i^*) \quad \text{and} \quad |f_i'| \le \|\varphi_i\| + 1$$

for all $i \in I$. Let K be a UUBC^k-constant for $(C_i^*)_{i\in I}$. Then, for each $i \in I$, there is a cochain $c_i \in C_i^{k-1}$ with

$$\delta_i(c_i) = f_i - f_i' \quad \text{and} \quad |c_i| \le K \cdot |f_i - f_i'| \le K \cdot 2 \cdot \big(\|\varphi_i\| + 1\big)\,.$$

Thus, $c \coloneqq (c_i)_{i\in I}$ lies in $\prod_{i\in I}^b C_i^{k-1}$ and $\delta(c) = f - f'$. □

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Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany
E-mail address: kevin.li@ur.de

Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany
E-mail address: clara.loeh@ur.de

Dipartimento di Matematica, Università di Bologna, 40126 Bologna, Italy
E-mail address: marco.moraschini2@unibo.it