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Sharp Second-Order Hankel Determinants Bounds for Alpha-Convex Functions Connected with Modified Sigmoid Functions

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Abstract: The study of the Hankel determinant generated by the Maclaurin series of holomorphic functions belonging to particular classes of normalized univalent functions is one of the most significant problems in geometric function theory. Our goal in this study is first to define a family of alpha-convex functions associated with modified sigmoid functions and then to investigate sharp bounds of initial coefficients, Fekete-Szegö inequality, and second-order Hankel determinants. Moreover, we also examine the logarithmic and inverse coefficients of functions within a defined family regarding recent issues. All of the estimations that were found are sharp.

Keywords: alpha-convex function; modified sigmoid function; logarithmic coefficients; inverse coefficients; coefficient bounds; Fekete-Szegö inequality; Hankel determinant problems

MSC: 30C45; 30C50

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1. Introduction and Definitions

For the reader's benefit, here, we introduce the notations and terminology commonly used in this research field. We denote with $\mathcal{A}$ the class of the analytic functions $g(\xi)$ defined on the open unit disk, $\mathbb{U}_d := \{\xi \in \mathbb{C} : |\xi| < 1\}$, which has the normalized form:

$$g(\xi) = \xi + \sum_{r=2}^{\infty} d_r \xi^r, \quad \xi \in \mathbb{U}_d. \tag{1}$$

With $\mathcal{S}$, we also indicate the family univalent (i.e., meromorphic and injective) functions in $\mathcal{A}$. Geometric Function Theory is the study of the geometric properties of functions belonging to $\mathcal{S}$ or to some particular subset of it. Interest in these types of problems originates with the famous Bieberbach conjecture. The conjecture stated in 1916, see [1], claims that for $g \in \mathcal{S}$, expressed through the power series expansion (1), then $|d_r| \leq r$ for all $r \geq 2$. Notice that the equality holds if and only if $g(\xi) = \xi/(1-\xi)^2$ (Koebe function) or one of its rotation. Bieberbach demonstrated this statement for $r = 2$. Löwner [2], Garabedian and Schiffer [3], Pederson and Schiffer [4], and Pederson [5], examined it for the cases $r = 3$, 4, 5, and 6, respectively. The case $r \geq 7$ remained unsolved until 1985, when de Branges [6] utilized hypergeometric functions to demonstrate Bieberbach conjecture for every $r \geq 2$. In 1960, Lawrence Zalcman conjectured that the coefficients of a function belonging to $\mathcal{S}$ satisfy the sharp inequality

$$|d_n^2 - d_{2n-1}| \leq (n-1)^2.$$

This led to the publication of several papers [7–9] regarding the generalized form of the original Zalcman inequality, namely:

$$\left|\lambda d_r^2 - d_{2r-1}\right| \leq \lambda r^2 - 2r + 1, \text{ with } \lambda \geq 0$$

for various subfamilies of $\mathcal{S}$.

The Zalcman conjecture in the case $r \leq 6$ was proven by Krushkal in [10] by utilizing the holomorphic homotopy of univalent functions presented in an unpublished manuscript [11] for $r \geq 2$. It was also proven that, for $g \in \mathcal{S}$:

$$\left|d_r^t - d_2^{t(r-1)}\right| \leq 2^{t(r-1)} - r^t, \text{ with } r,\, t \geq 2.$$

Ma, in 1999 [12], presented the following variation of Zalcman conjecture:

$$|d_s d_r - d_{s+r-1}| \leq (s-1)(r-1), \quad s,\, r \geq 2$$

for a specific subset of $\mathcal{S}$, but it remains unsolved for arbitrary elements of $\mathcal{S}$.

Another concept of great importance in geometric function theory is subordination, which we briefly recall. Given two functions $g_1,\, g_2 \in \mathcal{A}$, we say that g_1 is subordinate to g_2 and we write $g_1 \prec g_2$, if there exists a Schwarz function w, analytic but not necessarily univalent, i.e., $w(0) = 0$ and $|w(\xi)| < 1$ for any $\xi \in \mathbb{U}_d$, such that $g_1(\varsigma) = g_2(w(\xi))$ for any $\xi \in \mathbb{U}_d$. Subordination can be expressed in an equivalent form when g_2 is univalent in $\mathbb{U}_d$; in this case, $g_1 \prec g_2$ if and only if

$$g_1(0) = g_2(0) \quad \text{and} \quad g_1(\mathbb{U}_d) \subset g_2(\mathbb{U}_d).$$

Let $\mathcal{P}$ be the Carathéodory family of all analytic functions p in $\mathbb{U}_d$ having $\mathfrak{Re}(p(\xi)) > 0$ and normalized by

$$p(\xi) = 1 + \sum_{n=1}^{\infty} \varepsilon_n \xi^n, \quad \xi \in \mathbb{U}_d. \tag{2}$$

Mocanu presented and investigated the renowned class of α-convex functions in [13], which is

$$\mathcal{M}_\alpha := \left\{ g \in \mathcal{A} : \mathfrak{Re}\left[(1-\alpha)\frac{\xi g'(\xi)}{g(\xi)} + \alpha\left(1 + \frac{\xi g''(\xi)}{g'(\xi)}\right) \right] > 0, \quad \xi \in \mathbb{U}_d \right\}, \quad \alpha \geq 0.$$

Numerous authors have extensively analyzed the characteristics of this class of functions over a long period, including [14–16]. It was demonstrated in [17] that all α-convex functions are univalent and starlike, whereas the class of starlike (normalized) functions in $\mathbb{U}_d$ is represented by the subclass $\mathcal{S}^* := \mathcal{M}_0$, and the class of convex (normalized) functions in $\mathbb{U}_d$ is represented by $\mathcal{C} := \mathcal{M}_0$.

Keep in mind that a function is said to be starlike in $\mathbb{U}_d$ if mapping the open unit disk onto a star-shaped domain and is univalent in $\mathbb{U}_d$, while it is convex in $\mathbb{U}_d$ if mapping the open unit disk onto a convex domain and is univalent in $\mathbb{U}_d$. Thus, both of these two classes are extended by the α-convex functions, which also creates a “continuous”transition among these notable classes (see [13,17] for more information). We want to highlight the significance of the concept of subordination in geometric function theory due to its equivalence with the function g_2, which deals with the open unit disk $\mathbb{U}_d$. Thus, for a function $p \in \mathcal{P}$, if and $p(\xi) \prec \frac{1+\xi}{1-\xi} =: \Pi(\xi)$, and for a function $g \in \mathcal{A}$, we have the equivalences

$$g \in \mathcal{S}^* \Leftrightarrow \frac{\xi g'(\xi)}{g(\xi)} \prec \Pi(\xi), \quad g \in \mathcal{C} \Leftrightarrow 1 + \frac{\xi g''(\xi)}{g'(\xi)} \prec \Pi(\xi),$$

while
$$\mathcal{M}_\alpha := \left\{g \in \mathcal{A} : (1-\alpha)\frac{\xi g'(\xi)}{g(\xi)} + \alpha\left(1 + \frac{\xi g''(\xi)}{g'(\xi)}\right) \prec \Pi(\xi)\right\}, \quad \alpha \geq 0.$$
A sigmoid function is an example of a special function, which is a mathematical function characterized by an S shape. The function is of the form

$$g(\xi) = \frac{1}{1+e^{-\xi}} = \frac{e^{\xi}}{1+e^{\xi}} = \frac{1}{2} + \frac{1}{4}\xi - \frac{1}{48}\xi^3 + \frac{1}{480}\xi^5 + \cdots.$$

Goel et al. [18] established the modified sigmoid function to achieve the normalized form of $g(\xi)$, which is defined as

$$2g(\xi) = \frac{2}{1+e^{-\xi}} = F(\xi) = 1 + \frac{1}{2}\xi - \frac{1}{24}\xi^3 + \frac{1}{240}\xi^5 - \cdots. \quad (3)$$

The modified sigmoid function maps $\mathbb{U}_d$ onto a domain $\Lambda_{sig} := \left\{w \in \mathbb{C} : \left|\log\left(\frac{w}{2-w}\right)\right| < 1\right\}$, which is symmetric about the real axis. Moreover, $F(\xi)$ is convex and, hence, starlike with respect to $F(0) = 1$. Also, $F(0) > 0$ and $F(\xi)$ have positive real parts in $\mathbb{U}_d$. Sigmoid functions have many important applications in neural networks [19]. For any neural net element, it uses a logistic function to produce the input signals. This function is often called the activation function, see [20]. Three common instances of activations functions are the logistic function, the Hyperbolic Tangent Activation (HTA) function and the Half Hyperbolic Tangent Activation function, which are given, respectively, by

$$\frac{1}{1+e^{-q}}, \quad \frac{e^{q}-e^{-q}}{e^{q}+e^{-q}}, \quad \frac{1-e^{-q}}{1+e^{-q}}, \quad q \in \mathbb{R}.$$

It is worth noting that also many physical and chemical processes have a sigmoidal dependence in nature; for instance, the pH variation in titration curves in chemistry [21]. By using (3), we can now introduce the new family of alpha convex function related to the modified sigmoid function, which is defined below.

Definition 1. *Let us introduce the new family $\mathcal{M}_\alpha(F(\xi))$ with $\alpha \geq 0$, connected with the modified sigmoid function, which is defined as follows:*

$$\mathcal{M}_\alpha(F(\xi)) = \left\{g \in \mathcal{A} : (1-\alpha)\frac{\xi g'(\xi)}{g(\xi)} + \alpha\left(1 + \frac{\xi g''(\xi)}{g'(\xi)}\right) \prec F(\xi), \quad \xi \in \mathbb{U}_d\right\}. \quad (4)$$

Remark 1. *The subfamilies examined in [22] can be acquired by choosing the values of $\alpha = 0$ and $\alpha = 1$ in (4), which are*

$$\mathcal{S}^*_{F(\xi)} := \mathcal{M}_0(F(\xi)), \quad \mathcal{C}_{F(\xi)} := \mathcal{M}_1(F(\xi)).$$

It follows that the family $\mathcal{M}_\alpha(F(\xi))$ is a subset of the family $\mathcal{M}_\alpha$, which is $\mathcal{M}_\alpha(F(\xi)) \subset \mathcal{M}_\alpha \subset \mathcal{S}^* \subset \mathcal{S}, \quad \alpha \geq 0$.

The Hankel determinant $\mathcal{H}_{\lambda,m}(g)$ with m, $\lambda \in \mathbb{N}$ is composed of the coefficients of the MacLaurin expansion of $g \in \mathcal{S}$, which is defined as follows:

$$\mathcal{H}_{\lambda,m}(g) = \begin{vmatrix} d_m & d_{m+1} & \dots & d_{m+\lambda-1} \\ d_{m+1} & d_{m+2} & \dots & d_{m+\lambda} \\ \vdots & \vdots & \dots & \vdots \\ d_{m+\lambda-1} & d_{m+\lambda} & \dots & d_{m+2\lambda-2} \end{vmatrix}.$$

Pommerenke [23,24], was the first to study the Hankel determinant for the elements of the class $\mathcal{S}$. The first and second-order Hankel determinants, respectively, are defined as

$$\begin{aligned}\mathcal{H}_{2,1}(g) &= \begin{vmatrix} 1 & d_2 \\ d_2 & d_3 \end{vmatrix} = d_3 - d_2^2, \\ \mathcal{H}_{2,2}(g) &= \begin{vmatrix} d_2 & d_3 \\ d_3 & d_4 \end{vmatrix} = d_2 d_4 - d_3^2.\end{aligned}$$

The determinant $\mathcal{H}_{2,1}(g)$ represents a particular type of Fekete–Szegö inequality $\left|d_3 - \eta d_2^2\right|$ for some complex η. The Fekete–Szegö [25] inequality is one of the oldest problems on coefficients of univalent analytic functions proven in 1933. Recent articles have focused on the importance of obtaining sharp estimates for the Hankel determinants whose elements are the coefficients of univalent functions for certain subclasses. The methodology mentioned in [26] has been widely adopted in all studies to attain the sharp bounds of $\mathcal{H}_{2,2}(g)$ for $g \in \mathcal{S}$ and $g \in \mathcal{C}$. The papers by Janteng et al. [26,27] and Lee et al. [28] examined the second-order Hankel determinant $\mathcal{H}_{2,2}(g)$ provided by numerous subclasses of $\mathcal{S}$. Janteng et al. established the best possible bounds for subclasses $\mathcal{S}^*$, $\mathcal{C}$, and $\mathcal{R}$, where $\mathcal{R}$ represents the set of bounded turning functions. The best possible estimations are

$$|\mathcal{H}_{2,2}(g)| \leq \begin{cases} 1, & \text{for} \quad g \in \mathcal{S}^*, \\ 1/8 & \text{for} \quad g \in \mathcal{C}, \\ 4/9, & \text{for} \quad g \in \mathcal{R}. \end{cases}$$

In [28], Lee et al. investigated $\mathcal{H}_{2,2}(g)$ for the general class $\mathcal{S}(\phi)$ of starlike functions with respect to the given function ϕ and particularly achieved the following estimates:

$$|\mathcal{H}_{2,2}(g)| \leq \begin{cases} 1/6, & \text{for} \quad g \in \mathcal{SL}, \\ \beta^2 & \text{for} \quad g \in \mathcal{SS}^*(\beta), \\ (1-\alpha)^2, & \text{for} \quad g \in \mathcal{S}^*(\alpha). \end{cases}$$

Eventually, Zaprawa [29] proved that if g belongs to the class of typically real functions, then $|\mathcal{H}_{2,m}(g)| \leq 1 + (m+1)^2$ and $|\mathcal{H}_{2,2}(g)| \leq 9$. Further discoveries about the Hankel determinants are provided in [30–33].

This article explores the sharp estimates of the initial coefficients, Fekete-Szegö inequality, and second-order Hankel determinant for the class $\mathcal{M}_\alpha(\digamma(\xi))$ of alpha-convex functions connected with the modified sigmoid function. Moreover, we studied the bounds of the inverse and logarithmic coefficients for the defined class.

We recall some lemmas from other contributions that will be used to demonstrate our main theorems.

Lemma 1. *See [34]. Let $p \in \mathcal{P}$ be of the form (2), then*

$$|\varepsilon_n| \leq 2, \quad \textit{for} \quad n \geq 1, \tag{5}$$

$$|\varepsilon_{n+k} - \lambda \varepsilon_n \varepsilon_k| \leq 2, \quad 0 \leq \lambda \leq 1. \tag{6}$$

Lemma 2. *See [35]. If $p \in \mathcal{P}$ and of the form (2) with $\varepsilon_1 \geq 0$, then*

$$2\varepsilon_2 = \varepsilon_1^2 + \left(4 - \varepsilon_1^2\right)\tau, \tag{7}$$

$$4\varepsilon_3 = \varepsilon_1^3 + 2\left(4 - \varepsilon_1^2\right)\tau\varepsilon_1 - \left(4 - \varepsilon_1^2\right)\tau^2\varepsilon_1 + 2\left(4 - \varepsilon_1^2\right)\left(1 - |\tau|^2\right)\delta. \tag{8}$$

for some $\tau, \delta \in \overline{\mathbb{U}_d} = \mathbb{U}_d \cup \{1\}$.

Lemma 3. *See [36]. For any real number P, Q, and R, let*

$$\chi(P,Q,R) = \max\left\{\left|P + Q\tau + R\tau^2\right| + 1 - |\tau|^2\right\}. \tag{9}$$

If $PR \geq 0$, then

$$\chi(P,Q,R) = \begin{cases} |P| + |Q| + |R|, & |Q| \geq 2(1-|R|), \\ 1 + |P| + \frac{Q^2}{4(1-|R|)} & |Q| < 2(1-|R|). \end{cases}$$

Lemma 4. *See [37]. Let $p \in \mathcal{P}$ be represented as in (2) and if $K \in [0,1]$ with $K(2K-1) \leq L \leq K$, then, we have*

$$\left|\varepsilon_3 - 2K\varepsilon_1\varepsilon_2 + L\varepsilon_1^3\right| \leq 2. \tag{10}$$

2. Coefficient Bounds

Our original contribution begins with investigating the bounds of some initial coefficients for $g \in \mathcal{M}_\alpha(F(\xi))$.

Theorem 1. *Let $g \in \mathcal{M}_\alpha(F(\xi))$ has the form (1). Then*

$$\begin{aligned} |d_2| &\leq \frac{1}{2(1+\alpha)}, \\ |d_3| &\leq \frac{1}{4(1+2\alpha)}, \\ |d_4| &\leq \frac{1}{6(1+3\alpha)}. \end{aligned}$$

These three initial outcomes are the best possible.

Proof. If $g \in \mathcal{M}_\alpha(F(\xi))$, then, from the use of subordination relationship, we have

$$(1-\alpha)\frac{\xi g'(\xi)}{g(\xi)} + \alpha\left(1 + \frac{\xi g''(\xi)}{g'(\xi)}\right) = \frac{2}{1+e^{-w(\xi)}}, \quad \xi \in \mathbb{U}_d. \tag{11}$$

Assume that $p(\xi) = \frac{1+w(\xi)}{1-w(\xi)}$ and

$$p(\xi) = 1 + \varepsilon_1\xi + \varepsilon_2\xi^2 + \varepsilon_3\xi^3 + \varepsilon_4\xi^4 + \cdots, \quad \xi \in \mathbb{U}_d.$$

Clearly, $p \in \mathcal{P}$ and

$$\begin{aligned} \frac{2}{1+e^{-w(\xi)}} &= 1 + \left(\frac{1}{4}\varepsilon_1\right)\xi + \left(-\frac{1}{8}\varepsilon_1^2 + \frac{1}{4}\varepsilon_2\right)\xi^2 + \left(-\frac{1}{4}\varepsilon_1\varepsilon_2 + \frac{11}{192}\varepsilon_1^3 + \frac{1}{4}\varepsilon_3\right)\xi^3 \\ &\quad + \left(\frac{11}{64}\varepsilon_1^2\varepsilon_2 - \frac{1}{4}\varepsilon_1\varepsilon_3 - \frac{3}{128}\varepsilon_1^4 + \frac{1}{4}\varepsilon_4 - \frac{1}{8}\varepsilon_2^2\right)\xi^4 + \cdots. \end{aligned} \tag{12}$$

Utilizing (1), we obtain

$$\begin{aligned} (1-\alpha)\frac{\xi g'(\xi)}{g(\xi)} + \alpha\left(1 + \frac{\xi g''(\xi)}{g'(\xi)}\right) &: = 1 + (1+\alpha)d_2\xi + \left[(2+4\alpha)d_3 - (1+3\alpha)d_2^2\right]\xi^2 \\ &\quad + \left[(1+7\alpha)d_2^3 + (3+9\alpha)d_4 - (3+15\alpha)d_2d_3\right]\xi^3 + \cdots. \end{aligned} \tag{13}$$

Now, by comparing (12) and (13), we obtain

$$d_2 = -\frac{1}{4(1+\alpha)}\varepsilon_1, \tag{14}$$

$$d_3 = -\frac{(2\alpha^2+\alpha+1)\varepsilon_1^2 - 4(1+\alpha)^2\varepsilon_2}{32(1+\alpha)^2(1+2\alpha)}, \tag{15}$$

$$\begin{aligned} d_4 = & -\frac{1}{1152(1+\alpha)^3(1+2\alpha)(1+3\alpha)}\Big[\Big(44\alpha^4+64\alpha^3+51\alpha^2+2\alpha+7\Big)\varepsilon_1^3 \\ & -12\Big(16\alpha^2+9\alpha+5\Big)(1+\alpha)^2\varepsilon_1\varepsilon_2 + 96(1+\alpha)^3(1+2\alpha)\varepsilon_3\Big]. \end{aligned} \tag{16}$$

From (14), applying triangle inequality and (5), we obtain

$$|d_2| \leq \frac{1}{2(1+\alpha)}.$$

Rearranging (15), we obtain

$$|d_3| = \frac{1}{8(1+2\alpha)}\left|\varepsilon_2 - \left(\frac{2\alpha^2+\alpha+1}{4(1+\alpha)^2}\right)\varepsilon_1^2\right|.$$

Using (6) and triangle inequality, we observe that $0 < \left(\frac{2\alpha^2+\alpha+1}{4(1+\alpha)^2}\right) < 1$ holds for $0 \leq \alpha \leq 1$, we obtain

$$|d_3| \leq \frac{1}{4(1+2\alpha)}.$$

Rearranging (16), we obtain

$$\begin{aligned} |d_4| = & \frac{1}{12(1+3\alpha)}\left|\varepsilon_3 - 2\left(\frac{96\alpha^2+54\alpha+30}{96(1+\alpha)(1+2\alpha)}\right)\varepsilon_1\varepsilon_2\right. \\ & \left.+\left(\frac{44\alpha^4+64\alpha^3+51\alpha^2+2\alpha+7}{96(1+\alpha)^3(1+2\alpha)}\right)\varepsilon_1^3\right|. \end{aligned}$$

From (10), let

$$K = \frac{96\alpha^2+54\alpha+30}{96(1+\alpha)(1+2\alpha)} \quad \text{and} \quad L = \frac{44\alpha^4+64\alpha^3+51\alpha^2+2\alpha+7}{96(1+\alpha)^3(1+2\alpha)}.$$

It is clear that

$$K - L = \frac{52\alpha^4+182\alpha^3+183\alpha^2+112\alpha+23}{96(1+\alpha)^3(1+2\alpha)} \geq 0,$$

and

$$L - K(2K-1) = \frac{352\alpha^5+1408\alpha^4+1933\alpha^3+1075\alpha^2+415\alpha+73}{384(1+\alpha)^3(1+2\alpha)^2} \geq 0.$$

It is true for $0 \leq \alpha \leq 1$. Hence, satisfying all of the conditions of Lemma 4, we achieve

$$|d_4| \leq \frac{1}{6(1+3\alpha)},$$

These three initial outcomes are the best possible, and the extremal functions are provided by

$$(1-\alpha)\frac{\xi g'(\xi)}{g(\xi)}+\alpha\left(1+\frac{\xi g''(\xi)}{g'(\xi)}\right) = 1+\frac{1}{2}\xi-\frac{1}{24}\xi^3+\cdots, \tag{17}$$

$$(1-\alpha)\frac{\xi g'(\xi)}{g(\xi)}+\alpha\left(1+\frac{\xi g''(\xi)}{g'(\xi)}\right) = 1+\frac{1}{2}\xi^2-\frac{1}{24}\xi^6+\cdots, \tag{18}$$

$$(1-\alpha)\frac{\xi g'(\xi)}{g(\xi)}+\alpha\left(1+\frac{\xi g''(\xi)}{g'(\xi)}\right) = 1+\frac{1}{2}\xi^3-\frac{1}{24}\xi^9+\cdots. \tag{19}$$

□

Theorem 2. *Let $g \in \mathcal{M}_\alpha(F(\xi))$, then*

$$\left|d_3-d_2^2\right| \leq \frac{1}{4(1+2\alpha)}.$$

The above outcome is the best possible. Equality is achieved for the function provided in (18).

Proof. Utilizing (14) and (15), we achieve

$$\begin{aligned}\left|d_3-d_2^2\right| &= \left|-\frac{(2\alpha+3)\varepsilon_1^2-4(1+\alpha)\varepsilon_2}{32(1+\alpha)(1+2\alpha)}\right|. \\ &= \frac{1}{8(1+2\alpha)}\left|\varepsilon_2-\left(\frac{2\alpha+3}{4(1+\alpha)}\right)\varepsilon_1^2\right|.\end{aligned}$$

The triangle inequality and (6) illustrate that $0<\frac{2\alpha+3}{4(1+\alpha)}<1$ true for $0\leq\alpha\leq 1$, we conclude

$$\left|d_3-d_2^2\right| \leq \frac{1}{4(1+2\alpha)}.$$

Which ends the proof. □

Theorem 3. *If $g \in \mathcal{M}_\alpha(F(\xi))$ then:*

$$|d_2d_3-d_4| \leq \frac{1}{6(1+3\alpha)}.$$

The inequality is sharp, and the equality is attained for the function defined in (19).

Proof. From (14)–(16), we obtain

$$\begin{aligned}|d_2d_3-d_4| &= \frac{1}{576(1+\alpha)^2(1+2\alpha)(1+3\alpha)}\left|-\left(96\alpha^3+240\alpha^2+192\alpha+48\right)\varepsilon_3\right. \\ &\quad \left.+\left(96\alpha^3+204\alpha^2+156\alpha+48\right)\varepsilon_1\varepsilon_2-\left(22\alpha^3+37\alpha^2+11\alpha+8\right)\varepsilon_1^3\right|.\end{aligned}$$

After some simple calculations, we have

$$\begin{aligned}|d_2d_3-d_4| &= \frac{1}{12(1+3\alpha)}\left|\varepsilon_3-2\left(\frac{48\alpha^2+54\alpha+24}{48(1+\alpha)(1+2\alpha)}\right)\varepsilon_1\varepsilon_2\right. \\ &\quad \left.+\left(\frac{22\alpha^3+37\alpha^2+11\alpha+8}{48(1+\alpha)^2(1+2\alpha)}\right)\varepsilon_1^3\right|\end{aligned}$$

From (10), let

$$K = \frac{48\alpha^2 + 54\alpha + 24}{48(1+\alpha)(1+2\alpha)} \quad \text{and} \quad L = \frac{22\alpha^3 + 37\alpha^2 + 11\alpha + 8}{48(1+\alpha)^2(1+2\alpha)}.$$

It is clear that

$$K - L = \frac{26\alpha^3 + 65\alpha^2 + 67\alpha + 16}{48(1+\alpha)^2(1+2\alpha)} \geq 0,$$

and

$$L - K(2K-1) = \frac{88\alpha^4 + 264\alpha^3 + 199\alpha^2 + 90\alpha + 16}{96(1+\alpha)^2(1+2\alpha)^2} \geq 0$$

holds for $0 \leq \alpha \leq 1$. Hence, as all the hypotheses of Lemma 4 are met, we achieve

$$|d_2 d_3 - d_4| \leq \frac{1}{6(1+3\alpha)}.$$

The provided proof is the required one. □

Theorem 4. *Let $g \in \mathcal{M}_\alpha(F(\xi))$ be of the form (1), then*

$$|\mathcal{H}_{2,2}(g)| \leq \frac{1}{16(1+2\alpha)^2}.$$

This is the sharp result for the function provided by (18).

Proof. From (14)–(16), we obtain

$$\begin{aligned} |\mathcal{H}_{2,2}(g)| &= \frac{(68\alpha^4 + 132\alpha^3 + 29\alpha^2 - 18\alpha + 5)\varepsilon_1^4}{9216(1+\alpha)^3(1+2\alpha)^2(1+3\alpha)} - \frac{(14\alpha^2 + 5\alpha + 2)\varepsilon_1^2\varepsilon_2}{1152(1+\alpha)(1+3\alpha)(1+2\alpha)^2} \\ &\quad - \frac{\varepsilon_2^2}{64(1+2\alpha)^2} + \frac{\varepsilon_1\varepsilon_3}{48(1+\alpha)(1+3\alpha)}. \end{aligned}$$

Assuming that $\varepsilon_1 = \varepsilon \in [0,2]$ is possible due to the rotation invariant characteristic for the family $\mathcal{M}_\alpha(F(\xi))$ and determinant $\mathcal{H}_{2,2}(g)$. The coefficients ε_2 and ε_3 can be expressed in terms of ε_1 by utilizing Lemma 2, then

$$\begin{aligned} |\mathcal{H}_{2,2}(g)| &= \Bigg| -\frac{(16\alpha^4 + 48\alpha^3 + 91\alpha^2 + 54\alpha + 7)\varepsilon^4}{9216(1+\alpha)^3(1+2\alpha)^2(1+3\alpha)} + -\frac{(4-\varepsilon^2)\tau^2\varepsilon^2}{192(1+\alpha)(1+3\alpha)} - \frac{(4-\varepsilon^2)^2\tau^2}{256(1+2\alpha)^2} \\ &\quad + \frac{(4-\varepsilon^2)\alpha\tau\varepsilon^2}{256(1+\alpha)(1+2\alpha)^2(1+3\alpha)} + \frac{(4-\varepsilon^2)\left(1-|\tau|^2\right)\varepsilon\delta}{96(1+\alpha)(1+3\alpha)} \Bigg|. \end{aligned}$$

It is obvious that $|\mathcal{H}_{2,2}(g)| \leq \frac{1}{16(1+2\alpha)^2}$ for $\varepsilon = 0$. For $\varepsilon = 2$, then

$$|\mathcal{H}_{2,2}(g)| = \frac{16\alpha^4 + 48\alpha^3 + 91\alpha^2 + 54\alpha + 7}{576(1+\alpha)^3(1+2\alpha)^2(1+3\alpha)}.$$

Utilizing $|\delta| \leq 1$ for the case $\varepsilon \in (0,2)$, then

$$\begin{aligned} |\mathcal{H}_{2,2}(g)| &\leq \frac{(4-\varepsilon^2)\varepsilon}{96(1+\alpha)(1+3\alpha)} \Bigg(\Bigg| -\frac{(16\alpha^4 + 48\alpha^3 + 91\alpha^2 + 54\alpha + 7)\varepsilon^3}{96(1+\alpha)^2(1+2\alpha)^2(4-\varepsilon^2)} + \frac{3\alpha\varepsilon}{8(1+2\alpha)^2}\tau \\ &\quad - \frac{\varepsilon^2(7\alpha^2 + 4\alpha + 1) + 12(1+\alpha)(1+3\alpha)}{8(1+2\alpha)^2\varepsilon}\tau^2 \Bigg| + 1 - |\tau|^2 \Bigg). \end{aligned}$$

Now, to use Lemma 3, we can rewrite the above inequality in terms of P, Q, and R, as given in (9) by

$$|\mathcal{H}_{2,2}(g)| \leq \frac{(4-\varepsilon^2)\varepsilon}{96(1+\alpha)(1+3\alpha)}\chi(P,Q,R),$$

where

$$\chi(P,Q,R) = \left|P + Q\tau + R\tau^2\right| + 1 - |\tau|^2,$$

with

$$\begin{aligned}
P &= -\frac{(16\alpha^4+48\alpha^3+91\alpha^2+54\alpha+7)\varepsilon^3}{96(1+\alpha)^2(1+2\alpha)^2(4-\varepsilon^2)}, \\
Q &= \frac{3\alpha\varepsilon}{8(1+2\alpha)^2}, \\
R &= -\frac{\varepsilon^2(7\alpha^2+4\alpha+1)+12(1+\alpha)(1+3\alpha)}{8(1+2\alpha)^2\varepsilon}.
\end{aligned}$$

Obviously, $PR \geq 0$ and the maxima of χ can be identified by employing Lemma 3. Notice that $|Q| \geq 2(1-|R|)$ is equal to

$$\sigma(\varepsilon,\alpha) = \frac{(14\alpha^2+11\alpha+2)\varepsilon^2 - 16(1+2\alpha)^2\varepsilon + 24(1+\alpha)(1+3\alpha)}{8(1+2\alpha)^2\varepsilon} \geq 0.$$

In order to demonstrate that $\sigma(\varepsilon,\alpha) > 2$, we have to prove that the minima of $\sigma(\varepsilon,\alpha)$ is positive for all $\varepsilon \in [0,2]$ and $0 \leq \alpha \leq 1$. Using basic math, we easily determine that

$$\min \sigma(\varepsilon,\alpha) = \sigma(2,\alpha) = \frac{3\alpha}{4(1+2\alpha)^2} > 0.$$

Using Lemma 3, we obtain

$$\chi(P,Q,R) \leq (|P|+|Q|+|R|),$$

and thus

$$\begin{aligned}
|\mathcal{H}_{2,2}(g)| &\leq \frac{(4-\varepsilon^2)\varepsilon}{96(1+\alpha)(1+3\alpha)}(|P|+|Q|+|R|) \\
&= -\frac{(68\alpha^4+204\alpha^3+173\alpha^2+54\alpha+5)}{9216(1+\alpha)^3(1+2\alpha)^2(1+3\alpha)}\varepsilon^4 \\
&\quad -\frac{(6\alpha^2+15\alpha+6)}{576(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\varepsilon^2 + \frac{1}{16(1+2\alpha)^2} \\
&= \phi_1(\varepsilon).
\end{aligned}$$

It is quite easy to determine that ϕ_1 acquires its maxima of $\frac{1}{16(1+2\alpha)^2}$ at $\varepsilon = 0$, then

$$|\mathcal{H}_{2,2}(g)| \leq \frac{1}{16(1+2\alpha)^2}.$$

This completes the proof. □

3. Logarithmic Coefficients

The logarithmic coefficient ν_n of $g \in \mathcal{S}$ is defined as

$$\frac{1}{2}\log\left(\frac{g(\xi)}{\xi}\right) = \sum_{n=1}^{\infty} \nu_n \xi^n. \tag{20}$$

The logarithmic coefficients of g are represented by ν_n, which are essential for studying univalent functions. The Hankel determinant with logarithmic coefficient entries appear to be a natural consideration. Kowalczyk et al. initially presented the Hankel determinant utilizing logarithmic coefficients in [38,39], and obtaining

$$\mathcal{H}_{q,n}(G_g/2) := \begin{vmatrix} \nu_n & \nu_{n+1} & \cdots & \nu_{n+q-1} \\ \nu_{n+1} & \nu_{n+2} & \cdots & \nu_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ \nu_{n+q-1} & \nu_{n+q} & \cdots & \nu_{n+2q-2} \end{vmatrix}. \tag{21}$$

It is specifically stated that

$$\mathcal{H}_{2,1}(G_g/2) = \begin{vmatrix} \nu_1 & \nu_2 \\ \nu_2 & \nu_3 \end{vmatrix} = \left|\nu_1\nu_3 - \nu_2^2\right|.$$

For more results about the logarithmic coefficients, we refer to [40–45].

Note that the logarithmic coefficients of g are provided by (1), and are given as follows

$$\nu_1 = \frac{1}{2}d_2 \tag{22}$$

$$\nu_2 = \frac{1}{2}\left(d_3 - \frac{1}{2}d_2^2\right) \tag{23}$$

$$\nu_3 = \frac{1}{2}\left(d_4 - d_2d_3 + \frac{1}{3}d_2^3\right). \tag{24}$$

Theorem 5. *Let $g \in \mathcal{M}_\alpha(\digamma(\xi))$ have the form (1). Then*

$$|\nu_1| \leq \frac{1}{4(1+\alpha)},$$

$$|\nu_2| \leq \frac{1}{8(1+2\alpha)},$$

$$|\nu_3| \leq \frac{1}{12(1+3\alpha)}.$$

These logarithmic coefficients are sharp.

Proof. Applying (14)–(16) in (22)–(24), we obtain

$$\nu_1 = \frac{1}{8(1+\alpha)}\varepsilon_1, \tag{25}$$

$$\nu_2 = -\frac{(2\alpha^2+3\alpha+2)\varepsilon_1^2 - 4(1+\alpha)^2\varepsilon_2}{64(1+\alpha)^2(1+2\alpha)}, \tag{26}$$

$$\nu_3 = \frac{1}{1152(1+\alpha)^3(1+2\alpha)(1+3\alpha)}\Big[\Big(22\alpha^4+59\alpha^3+66\alpha^2+34\alpha+11\Big)\varepsilon_1^3 - 12\Big(8\alpha^2+9\alpha+4\Big)(1+\alpha)^2\varepsilon_1\varepsilon_2 + 48(1+\alpha)^3(1+2\alpha)\varepsilon_3\Big]. \tag{27}$$

From (25) applying (5) and triangle inequality, we obtain

$$|\nu_1| \leq \frac{1}{4(1+\alpha)}.$$

By rearranging (26), we have

$$|\nu_2| = \frac{1}{16(1+2\alpha)}\left|\varepsilon_2 - \left(\frac{2\alpha^2+3\alpha+2}{4(1+\alpha)^2}\right)\varepsilon_1^2\right|.$$

By using (6) and triangle inequality, we observe that $0 < \left(\frac{2\alpha^2+3\alpha+2}{4(1+\alpha)^2}\right) < 1$ holds for $0 \leq \alpha \leq 1$, we obtain

$$|\nu_2| \leq \frac{1}{8(1+2\alpha)}.$$

Reshuffling (27), we have

$$\begin{aligned} |\nu_3| &= \frac{1}{24(1+3\alpha)}\left|\varepsilon_3 - 2\left(\frac{48\alpha^2+54\alpha+24}{48(1+\alpha)(1+2\alpha)}\right)\varepsilon_1\varepsilon_2 \right. \\ &\quad \left. + \left(\frac{22\alpha^4+59\alpha^3+66\alpha^2+34\alpha+11}{48(1+\alpha)^3(1+2\alpha)}\right)\varepsilon_1^3\right|. \end{aligned} \tag{28}$$

From (10), let

$$K = \frac{48\alpha^2+54\alpha+24}{48(1+\alpha)(1+2\alpha)} \quad \text{and} \quad L = \frac{22\alpha^4+59\alpha^3+66\alpha^2+34\alpha+11}{48(1+\alpha)^3(1+2\alpha)}.$$

It is clear that

$$K - L = \frac{26\alpha^4+91\alpha^3+114\alpha^2+68\alpha+13}{48(1+\alpha)^3(1+2\alpha)} \geq 0,$$

and

$$L - K(2K-1) = \frac{88\alpha^5+352\alpha^4+535\alpha^3+385\alpha^2+148\alpha+22}{96(1+\alpha)^3(1+2\alpha)^2} \geq 0.$$

It is true for $0 \leq \alpha \leq 1$. Hence, satisfying all the conditions of Lemma 4, we achieve

$$|\nu_3| \leq \frac{1}{12(1+3\alpha)},$$

The equalities holds for the function given by using (22)–(24) and (17)–(19). □

Theorem 6. *Let $g \in \mathcal{M}_\alpha(F(\xi))$. Then*

$$\left|\nu_2 - \nu_1^2\right| \leq \frac{1}{8(1+2\alpha)}.$$

The outcome is sharp.

Proof. From (25) and (26), we obtain

$$\begin{aligned} \left|\nu_2 - \nu_1^2\right| &= \left|-\frac{(2\alpha+3)\varepsilon_1^2 - 4(1+\alpha)\varepsilon_2}{64(1+\alpha)(1+2\alpha)}\right| \\ &= \frac{1}{16(1+2\alpha)}\left|\varepsilon_2 - \left(\frac{2\alpha+3}{4(1+\alpha)}\right)\varepsilon_1^2\right|. \end{aligned}$$

The triangle inequality and (6) illustrate that $0 < \frac{2\alpha+3}{4(1+\alpha)} < 1$ true for $0 \leq \alpha \leq 1$, we conclude

$$\left|\nu_2 - \nu_1^2\right| \leq \frac{1}{8(1+2\alpha)}.$$

Equality is achieved by utilizing (22), (23) and (18). □

Theorem 7. *If $g \in \mathcal{M}_\alpha(\digamma(\xi))$, then*

$$|\nu_3 - \nu_1\nu_2| \leq \frac{1}{12(1+3\alpha)}.$$

This is the sharp result for the function provided in (22)–(24) and (19).

Proof. From (25)–(27), we obtain

$$|\nu_3 - \nu_1\nu_2| = \frac{1}{4608(1+\alpha)^3(1+2\alpha)(1+3\alpha)}\Big|192(1+2\alpha)(1+\alpha)^3\varepsilon_3 - 12\Big(32\alpha^2+45\alpha+19\Big)$$
$$(1+\alpha)^2\varepsilon_1\varepsilon_2 + \Big(88\alpha^4+290\alpha^3+363\alpha^2+217\alpha+62\Big)\varepsilon_1^3\Big|.$$

After some simple calculations, we have

$$|\nu_3 - \nu_1\nu_2| = \frac{1}{24(1+3\alpha)}\left|\varepsilon_3 - 2\left(\frac{192\alpha^2+270\alpha+114}{192(1+\alpha)(1+2\alpha)}\right)\varepsilon_1\varepsilon_2 + \left(\frac{88\alpha^4+290\alpha^3+363\alpha^2+217\alpha+62}{192(1+\alpha)^3(1+2\alpha)}\right)\varepsilon_1^3\right|.$$

From (10), let

$$K = \frac{192\alpha^2+270\alpha+114}{192(1+\alpha)(1+2\alpha)} \quad \text{and} \quad L = \frac{88\alpha^4+290\alpha^3+363\alpha^2+217\alpha+62}{192(1+\alpha)^3(1+2\alpha)}.$$

It is clear that

$$K - L = \frac{104\alpha^4+364\alpha^3+483\alpha^2+281\alpha+52}{192(1+\alpha)^3(1+2\alpha)} \geq 0,$$

and

$$L - K(2K-1) = \frac{1408\alpha^5+5632\alpha^4+8533\alpha^3+6259\alpha^2+2323\alpha+325}{1536(1+\alpha)^3(1+2\alpha)^2} \geq 0.$$

It is true for $0 \leq \alpha \leq 1$. Hence, satisfy all the conditions of Lemma 4, we achieve

$$|\nu_3 - \nu_1\nu_2| \leq \frac{1}{12(1+3\alpha)}.$$

Which completes the proof. □

Theorem 8. *Let $g \in \mathcal{M}_\alpha(\digamma(\xi))$. Then*

$$\left|\mathcal{H}_{2,1}(G_g/2)\right| \leq \frac{1}{64(1+2\alpha)^2}.$$

This result is sharp. Equality is determined by using (22)–(24) and (18).

Proof. From (25)–(27), we obtain

$$\left|\nu_1\nu_3-\nu_2^2\right| = \left|\frac{(68\alpha^5+200\alpha^4+197\alpha^3+59\alpha^2+8\alpha+8)\varepsilon_1^4}{36864(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)}-\frac{\varepsilon_2^2}{256(1+2\alpha)^2}\right.$$
$$\left.-\frac{(14\alpha^2+5\alpha+2)\varepsilon_1^2\varepsilon_2}{1536(1+\alpha)(1+2\alpha)^2(1+3\alpha)}+\frac{\varepsilon_1\varepsilon_3}{192(1+\alpha)(1+3\alpha)}\right|.$$

Assuming that $\varepsilon_1=\varepsilon\in[0,2]$ is possible due to the rotation invariant characteristic for the family $\mathcal{M}_\alpha(F(\xi))$ and determinant $\mathcal{H}_{2,1}(G_g/2)$. The coefficients ε_2 and ε_3 can be expressed in terms of ε_1 by utilizing Lemma 2, then

$$\left|\mathcal{H}_{2,1}(G_g/2)\right| = \left|-\frac{(16\alpha^5+64\alpha^4+103\alpha^3+97\alpha^2+40\alpha+4)\varepsilon^4}{36864(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)}+\frac{(4-\varepsilon^2)\alpha\tau\varepsilon^2}{1024(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\right.$$
$$\left.-\frac{(4-\varepsilon^2)\tau^2\varepsilon^2}{768(1+\alpha)(1+3\alpha)}+\frac{(4-\varepsilon^2)\left(1-|\tau|^2\right)\varepsilon\delta}{384(1+\alpha)(1+3\alpha)}-\frac{(4-\varepsilon^2)^2\tau^2}{1024(1+2\alpha)^2}\right|,$$

It is obvious that $\left|\mathcal{H}_{2,1}(G_g/2)\right|\le\frac{1}{64(1+2\alpha)^2}$ for $\varepsilon=0$. For $\varepsilon=2$, then

$$\left|\mathcal{H}_{2,1}(G_g/2)\right|=\frac{16\alpha^5+64\alpha^4+103\alpha^3+97\alpha^2+40\alpha+4}{2304(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)}.$$

For the case of $\varepsilon\in(0,2)$, using $|\delta|\le1$, it is seen that

$$\begin{aligned}\left|\mathcal{H}_{2,1}(G_g/2)\right| &\le \frac{(4-\varepsilon^2)\varepsilon}{384(1+\alpha)(1+3\alpha)}\left(\left|-\frac{(16\alpha^5+64\alpha^4+103\alpha^3+97\alpha^2+40\alpha+4)\varepsilon^3}{96(1+\alpha)^3(1+2\alpha)^2(4-\varepsilon^2)}\right.\right.\\ &\left.\left.+\frac{3\alpha\varepsilon}{8(1+2\alpha)^2}\tau-\frac{\varepsilon^2(7\alpha^2+4\alpha+1)+12(1+\alpha)(1+3\alpha)}{8(1+2\alpha)^2\varepsilon}\tau^2\right|+1-|\tau|^2\right).\\ &= \frac{(4-\varepsilon^2)\varepsilon}{384(1+\alpha)(1+3\alpha)}\chi(P,Q,R),\end{aligned}$$

where we have achieved the last inequality by using (9) with

$$\chi(P,Q,R)=\left|P+Q\tau+R\tau^2\right|+1-|\tau|^2,$$

and

$$\begin{aligned}P &= -\frac{(16\alpha^5+64\alpha^4+103\alpha^3+97\alpha^2+40\alpha+4)\varepsilon^3}{96(1+\alpha)^3(1+2\alpha)^2(4-\varepsilon^2)},\\ Q &= \frac{3\alpha\varepsilon}{8(1+2\alpha)^2},\\ R &= -\frac{\varepsilon^2(7\alpha^2+4\alpha+1)+12(1+\alpha)(1+3\alpha)}{8(1+2\alpha)^2\varepsilon}.\end{aligned}$$

Obviously $PR\ge0$ and the maxima of χ can be identified by employing Lemma 3. Notice that $|Q|\ge2(1-|R|)$ is equal to

$$\sigma(\varepsilon,\alpha)=\frac{(14\alpha^2+11\alpha+2)\varepsilon^2-16(1+2\alpha)^2\varepsilon+24(1+\alpha)(1+3\alpha)}{8(1+2\alpha)^2\varepsilon}\ge0,$$

In order to demonstrate that $\sigma(\varepsilon,\alpha) > 2$, we have to prove that the minima of $\sigma(\varepsilon,\alpha)$ is positive for all $\varepsilon \in [0,2]$ and $0 \le \alpha \le 1$. Using basic math, we easily determine that

$$\min \sigma(\varepsilon,\alpha) = \sigma(2,\alpha) = \frac{3\alpha}{4(1+2\alpha)^2} > 0.$$

Using Lemma 3, we obtain

$$\chi(P,Q,R) \le (|P| + |Q| + |R|),$$

and thus

$$\begin{aligned} \left|\mathcal{H}_{2,1}(G_g/2)\right| &\le \frac{(4-\varepsilon^2)\varepsilon}{384(1+\alpha)(1+3\alpha)}(|P|+|Q|+|R|) \\ &= -\frac{(68\alpha^5 + 272\alpha^4 + 413\alpha^3 + 275\alpha^2 + 80\alpha + 8)}{36864(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)}\varepsilon^4 \\ &\quad -\frac{(8\alpha^2 + 20\alpha + 8)}{3072(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\varepsilon^2 + \frac{1}{64(1+2\alpha)^2} \\ &= \phi_1(\varepsilon). \end{aligned}$$

It is quite easy to determine that ϕ_1 acquires its maxima of $\frac{1}{16(1+2\alpha)^2}$ at $\varepsilon = 0$, then

$$\left|\mathcal{H}_{2,1}(G_g/2)\right| \le \frac{1}{16(1+2\alpha)^2}.$$

Hence, the proof is completed. □

4. Inverse Coefficients

The renowned Köebe 1/4-theorem ensures that, for each univalent function g defined in $\mathbb{U}_d$, its inverse g^{-1} exists at least on a disc of radius 1/4 with Taylor's series of the form representation

$$g^{-1}(w) = w + \sum_{n=2}^{\infty} \mu_n w^n, \quad \left(|w| < \frac{1}{4}\right). \tag{29}$$

From $g(g^{-1}(w)) = w$, we obtain

$$\mu_2 = -d_2, \tag{30}$$

$$\mu_3 = -d_3 + 2d_2^2, \tag{31}$$

$$\mu_4 = -d_4 + 5d_2d_3 - 5d_2^3, \tag{32}$$

Many authors studied Hankel determinants for the inverse functions, see [46–49].

Theorem 9. *Let $g \in \mathcal{M}_\alpha(F(\xi))$ has the form (1), then*

$$\left|\mu_3 - \mu_2^2\right| \le \frac{1}{4(1+2\alpha)},$$

This result is sharp. Equality is determined by using (30), (31) and (18).

Proof. Applying (14)–(16) in (30)–(32), we obtain

$$\mu_2 = -\frac{1}{4(1+\alpha)}\varepsilon_1, \tag{33}$$

$$\mu_3 = \frac{(2\alpha^2+9\alpha+5)\varepsilon_1^2-4(1+\alpha)^2\varepsilon_2}{32(1+\alpha)^2(1+2\alpha)}, \tag{34}$$

$$\mu_4 = -\frac{1}{576(1+\alpha)^3(1+2\alpha)(1+3\alpha)}\Big[\Big(22\alpha^4+167\alpha^3+408\alpha^2+316\alpha+71\Big)\varepsilon_1^3 -12\Big(8\alpha^2+27\alpha+10\Big)(1+\alpha)^2\varepsilon_1\varepsilon_2+48(1+\alpha)^3(1+2\alpha)\varepsilon_3\Big]. \tag{35}$$

From (33) and (34), we obtain

$$\begin{aligned}\left|\mu_3-\mu_2^2\right| &= \left|\frac{(2\alpha+3)\varepsilon_1^2-4(1+\alpha)\varepsilon_2}{32(1+\alpha)(1+2\alpha)}\right| \\ &= \frac{1}{8(1+2\alpha)}\left|\varepsilon_2-\left(\frac{2\alpha+3}{4(1+\alpha)}\right)\varepsilon_1^2\right|.\end{aligned}$$

By using (6) and triangle inequality, we observe that $0<\left(\frac{2\alpha+3}{4(1+\alpha)}\right)<1$ holds for $0\le\alpha\le 1$, we obtain

$$\left|\mu_3-\mu_2^2\right|\le\frac{1}{4(1+2\alpha)}.$$

Hence, the proof is completed. □

Theorem 10. *If $g\in\mathcal{M}_\alpha(\digamma(\xi))$. Then*

$$\left|\mathcal{H}_{2,2}\left(g^{-1}\right)\right|\le\frac{1}{64(1+2\alpha)^2}.$$

This result is sharp. Equality is determined by using (30)–(32) and (18).

Proof. From (33)–(35), we obtain

$$\begin{aligned}\left|\mathcal{H}_{2,2}\left(g^{-1}\right)\right| &= \left|\frac{(68\alpha^4+384\alpha^3+533\alpha^2+288\alpha+59)\varepsilon_1^4}{9216(1+\alpha)^3(1+2\alpha)^2(1+3\alpha)}-\frac{\varepsilon_2^2}{64(1+2\alpha)^2}\right. \\ &\quad\left.-\frac{(14\alpha^3+37\alpha^2+22\alpha+5)\varepsilon_1^2\varepsilon_2}{384(1+\alpha)^2(1+2\alpha)^2(1+3\alpha)}+\frac{\varepsilon_1\varepsilon_3}{48(1+\alpha)(1+3\alpha)}\right|.\end{aligned}$$

Assuming that $\varepsilon_1=\varepsilon\in[0,2]$ is possible due to the rotation invariant characteristic for the family $\mathcal{M}_\alpha(\digamma(\xi))$ and determinant $\mathcal{H}_{2,2}(g^{-1})$. The coefficients ε_2 and ε_3 can be expressed in terms of ε_1 by utilizing Lemma 2, then

$$\begin{aligned}\left|\mathcal{H}_{2,2}\left(g^{-1}\right)\right| &= \left|-\frac{(16\alpha^4+48\alpha^3-17\alpha^2-36\alpha-11)\varepsilon^4}{9216(1+\alpha)^3(1+2\alpha)^2(1+3\alpha)}-\frac{(4-\varepsilon^2)(5\alpha^2+4\alpha+1)\tau\varepsilon^2}{256(1+\alpha)^2(1+2\alpha)^2(1+3\alpha)}\right. \\ &\quad\left.-\frac{(4-\varepsilon^2)\tau^2\varepsilon^2}{192(1+\alpha)(1+3\alpha)}+\frac{(4-\varepsilon^2)\left(1-|\tau|^2\right)\varepsilon\delta}{96(1+\alpha)(1+3\alpha)}-\frac{(4-\varepsilon^2)^2\tau^2}{256(1+2\alpha)^2}\right|,\end{aligned}$$

It is obvious that $\left|\mathcal{H}_{2,2}(g^{-1})\right|\le\frac{1}{64(1+2\alpha)^2}$ for $\varepsilon=0$. For $\varepsilon=2$, then

$$\left|\mathcal{H}_{2,2}\left(g^{-1}\right)\right|=\frac{16\alpha^4+48\alpha^3-17\alpha^2-36\alpha-11}{576(1+\alpha)^3(1+2\alpha)^2(1+3\alpha)}.$$

For the case of $\varepsilon \in (0,2)$, using $|\delta| \leq 1$, it is seen that

$$\begin{aligned}\left|\mathcal{H}_{2,2}\left(g^{-1}\right)\right| &\leq \frac{(4-\varepsilon^2)\varepsilon}{96(1+\alpha)(1+3\alpha)}\left(\left|-\frac{(16\alpha^4+48\alpha^3-17\alpha^2-36\alpha-11)\varepsilon^3}{96(1+\alpha)^2(1+2\alpha)^2(4-\varepsilon^2)}\right.\right.\\ &\left.\left.-\frac{(15\alpha^2+12\alpha+3)\varepsilon}{8(1+2\alpha)^2(1+\alpha)}\tau-\frac{\varepsilon^2(7\alpha^2+4\alpha+1)+12(1+\alpha)(1+3\alpha)}{8(1+2\alpha)^2\varepsilon}\tau^2\right|+1-|\tau|^2\right).\\ &= \frac{(4-\varepsilon^2)\varepsilon}{96(1+\alpha)(1+3\alpha)}\chi(P,Q,R),\end{aligned}$$

where we easily obtain the last inequality by using (9) with

$$\chi(P,Q,R) = \left|P+Q\tau+R\tau^2\right|+1-|\tau|^2,$$

and

$$\begin{aligned}P &= -\frac{(16\alpha^4+48\alpha^3-17\alpha^2-36\alpha-11)\varepsilon^3}{96(1+\alpha)^2(1+2\alpha)^2(4-\varepsilon^2)},\\ Q &= -\frac{(15\alpha^2+12\alpha+3)\varepsilon}{8(1+2\alpha)^2(1+\alpha)},\\ R &= -\frac{\varepsilon^2(7\alpha^2+4\alpha+1)+12(1+\alpha)(1+3\alpha)}{8(1+2\alpha)^2\varepsilon}.\end{aligned}$$

Obviously, $PR \geq 0$ and the maxima of χ can be identified by employing Lemma 3. Notice that $|Q| \geq 2(1-|R|)$ is equal to

$$\sigma(\varepsilon,\alpha) = \frac{(14\alpha^3+37\alpha^2+22\alpha+5)\varepsilon^2-16(1+2\alpha)^2(1+\alpha)\varepsilon+24(1+\alpha)^2(1+3\alpha)}{8(1+2\alpha)^2(1+\alpha)\varepsilon} \geq 0,$$

In order to demonstrate that $\sigma(\varepsilon,\alpha) > 2$, we have to prove that the minima of $\sigma(\varepsilon,\alpha)$ is positive for all $\varepsilon \in [0,2]$ and $0 \leq \alpha \leq 1$. Using basic math, we easily determine that

$$\min\sigma(\varepsilon,\alpha) = \sigma(2,\alpha) = \frac{60\alpha^2+48\alpha+12}{16(1+2\alpha)^2(1+\alpha)} > 0.$$

Using Lemma 3, we obtain

$$\chi(P,Q,R) \leq (|P|+|Q|+|R|),$$

and thus

$$\begin{aligned}\left|\mathcal{H}_{2,2}\left(g^{-1}\right)\right| &\leq \frac{(4-\varepsilon^2)\varepsilon}{96(1+\alpha)(1+3\alpha)}(|P|+|Q|+|R|)\\ &= -\frac{(68\alpha^4+348\alpha^3+533\alpha^2+288\alpha+59)}{9216(1+\alpha)^3(1+2\alpha)^2(1+3\alpha)}\varepsilon^4\\ &\quad+\frac{(-24\alpha^3+60\alpha^2+24\alpha+12)}{2304(1+\alpha)^2(1+2\alpha)^2(1+3\alpha)}\varepsilon^2+\frac{1}{16(1+2\alpha)^2}\\ &= \phi_1(\varepsilon).\end{aligned}$$

It is an elementary matter to infer that ϕ_1 attains its maximum value $\frac{1}{16(1+2\alpha)^2}$ at $\varepsilon = 0$.

$$\left|\mathcal{H}_{2,2}\left(g^{-1}\right)\right| \leq \frac{1}{16(1+2\alpha)^2}.$$

Hence, the proof is completed. □

5. Discussion

This article introduces a new class of alpha convex functions related to sigmoid functions, which generalized the scope of previous research on sigmoid convex and sigmoid starlike functions. It emphasizes the importance of considering symmetry and unique geometric features of sigmoid functions, as most previous studies (see [22,50]) have focused on classical definitions without considering them. This new approach offers a fresh perspective on geometric functions theory. Our research establishes a formal framework for studying sigmoid alpha-convex functions within a newly defined class, addressing the gap in existing literature. We explicitly represent these functions and rigorously demonstrate their geometric properties, advancing the field beyond existing literature by addressing the influence of symmetric points and distinct geometric features.

The investigation of a Fekete–Szegö sharp inequality and the derivation of sharp limits for the first four initial coefficients are this work's main contributions, which constitute significant advancements in geometric function theory. Furthermore, our findings about the sharp bounds for the inverse and logarithmic coefficients and the second Hankel determinant provide fresh perspectives that extend the bounds of current studies in this field. Our findings are consistent with the recent works, demonstrating the continued importance of investigating varied function classes in geometric function theory. However, by focusing on the geometric implications of the sigmoid function, our study takes these findings in a new direction.

By examining this previously unstudied problem, we not only fill a major gap in the literature, but also open the way for future research on symmetric behaviors across diverse geometric forms and function classes. This research will promote further investigation of symmetry in geometric function theory, resulting in breakthroughs and applications in mathematical analysis.

6. Conclusions

In the current article, we study the Hankel determinant by utilizing the coefficients of logarithmic and inverse functions for the family of holomorphic functions. This generalizes the classical definition of the Hankel determinant and could provide more knowledge into the characteristics of the logarithmic and inverse functions. We have investigated the coefficient-related problems for the logarithmic and inverse functions that belong to the family of alpha-convex functions associated with sigmoid functions. The discussed coefficient-related problems include the sharp bounds of some initial coefficients, the Fekete–Szegö inequality, and the second Hankel determinant for the defined class by using the concept of the Carathéodory function. The same problems are also studied for the logarithmic and inverse coefficients. Our research presents a novel framework for examining the Hankel determinant, considering the significance of the holomorphic function's logarithmic and inverse coefficients. This research may encourage more attention to the coefficient-related problems concerning the logarithmic and inverse functions for certain families of holomorphic functions. However, there are still so many directions [51–53] in which researchers can demonstrate their skills, such as Hankel determinants of a higher order for functions of this class and its convolution properties, partial sum inequalities, and Majorization findings.

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