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REGULARITY OF A $\overline{\partial}$-SOLUTION OPERATOR FOR STRONGLY C-LINEARLY CONVEX DOMAINS WITH MINIMAL SMOOTHNESS

XIANGHONG GONG† AND LOREDANA LANZANI††

ABSTRACT. We prove regularity of solutions of the $\bar{\partial}$-problem in the Hölder-Zygmund spaces of bounded, strongly $\mathbf{C}$-linearly convex domains of class $C^{1,1}$. The proofs rely on a new analytic characterization of said domains which is of independent interest, and on techniques that were recently developed by the first-named author to prove estimates for the $\bar{\partial}$-problem on strongly pseudoconvex domains of class C^2.

For Eli

1. Introduction

Let D be a bounded domain in $\mathbf{C}^n$ with a defining function r that is C^1-smooth on a neighborhood of $\overline{D}$. We say that D is *strongly* $\mathbf{C}$*-linearly convex* if

$$|r_\zeta \cdot (\zeta - z)| \geq c|\zeta - z|^2, \quad z \in \overline{D}, \quad \zeta \in \partial D \tag{1.1}$$

for a positive constant c that may depend on r.

We say that D is *strictly* $\mathbf{C}$*-linearly convex* if it satisfies the weaker condition

$$|\, r_\zeta \cdot (\zeta - z)| > 0, \quad \zeta \in \partial D, \quad z \in \overline{D} \setminus \{\zeta\}. \tag{1.2}$$

We say that D is *weakly* $\mathbf{C}$-linearly convex[1] if it satisfies the even milder condition

$$|\, r_\zeta \cdot (\zeta - z)| > 0, \quad \zeta \in \partial D, \quad z \in D. \tag{1.3}$$

The notion of strong (resp., strict; weak) $\mathbf{C}$-linear convexity is essentially intermediate between strong (resp., strict; weak) convexity and strong (resp. weak) pseudoconvexity[2]; it was first introduced by Behnke and Peschl [2] in 1935 and has since played a central role in the theory of Hardy spaces and holomorphic singular integral operators. The purpose of this paper is to extend the analysis of these domains to the $\overline{\partial}$-problem. Strongly $\mathbf{C}$-linearly convex domains of class C^2 are, in particular, strongly pseudoconvex; see [13]. Thus the $\overline{\partial}$-problem for such domains is well understood; our main goal here is to go below the C^2-category and extend the theory of $\overline{\partial}$ to the class $C^{1,1}$. As is well known, $\mathbf{C}$-linearly convex domains support a Cauchy-Fantappié kernel, the *Cauchy-Leray kernel*, that is holomorphic in the output variable $z \in D$ and is "canonical" in the sense that it is independent of

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[1] sometime simply referred to as "weakly linearly convex".

[2] while the notions of "strong" and "strict" convexity (resp., "strong" and "strict" $\mathbf{C}$-linear convexity) are distinct from one another, there is no distinction between "strong" and "strict" pseudoconvexity and the two terms are often interchanged. See [13, p. 261].

the choice of defining function r (unlike other instances of Cauchy-Fantappiè kernels). The Cauchy-Leray kernel determines a singular integral operator (the *Cauchy-Leray transform* acting on functions supported in the topological boundary ∂D) whose operator-theoretic properties depend on the boundary regularity and on the amount of $\mathbf{C}$-linear convexity enjoyed by the domain. E. M. Stein and the second-named author have shown [14] that the Cauchy-Leray transform associated to a bounded, strongly $\mathbf{C}$-linearly convex domain $D \subset \mathbf{C}^n$ with boundary of class $C^{1,1}$, initially defined for functions in $C^1(\partial D)$, extends to a bounded operator: $L^p(\partial D, \mu) \to L^p(\partial D, \mu)$ for $1 < p < \infty$ and with μ belonging to a family of boundary measures that includes induced Lebesgue measure σ. The proof relies, among other things, on the analysis of the (suitably defined) action of the complex Hessian of r (assumed to be only of class $C^{1,1}$) over the complex tangent space of ∂D. In the subsequent paper [15] examples were supplied that indicate that the two hypotheses of strong $\mathbf{C}$-linear convexity and class $C^{1,1}$ are essentially optimal.

In this paper we provide integral formulas-based solutions to the $\overline{\partial}$-problem for bounded, strongly $\mathbf{C}$-linearly convex domains of class $C^{1,1}$: we first construct homotopy formulas based on a hierarchy of Cauchy-Leray-Koppelman kernels that give rise to integral operators acting on forms of type $(0, q)$, $q = 0, \ldots, n$ with coefficients defined on $\overline{D}$, the closure of the ambient domain. From these we obtain new estimates in the Hölder and Zygmund spaces that give the expected optimal gain of 1/2 derivatives. It turns out that the classical Leray-Koppelman homotopy formulas are in fact true under milder notions of $\mathbf{C}$-linear convexity: this is the reason why we mentioned condition (1.3) above. Strong $\mathbf{C}$-linear convexity is however needed both to justify the use of the regularity estimates that were obtained by the first-named author in [6], and to prove a new homotopy formula in this paper.

Our proofs rely on the following characterization, which is of independent interest, of strong $\mathbf{C}$-linear convexity for domains whose boundary is assumed to be of class $C^{1,1}$.

Theorem 1.1. *Let D be a bounded domain of class $C^{1,1}$, and let r be any defining function for D that is of class $C^{1,1}$ in a neighborhood of $\overline{D}$. Then we have that condition* (1.1) *being satisfied by r is equivalent to the same r satisfying each of the following:*

$$|r_\zeta \cdot (\zeta - z)| \geq c_1 |\zeta - z|^2 \quad \textit{for} \quad z \in \overline{D} \quad \textit{and} \quad \zeta \in U \setminus D. \tag{1.4}$$

$$|r_\zeta \cdot (\zeta - z)| \geq c_2 (r(\zeta) - r(z) + |\zeta - z|^2) \quad \textit{for} \quad z \in \overline{D} \quad \textit{and} \quad \zeta \in U \setminus D. \tag{1.5}$$

$$|r_\zeta \cdot (\zeta - z)| \geq c_3 |\zeta - z|^2 \quad \textit{for} \quad \zeta, z \in \partial D. \tag{1.6}$$

Here U is a neighborhood of $\overline{D}$ which may not be the same for (1.4) *and* (1.5).

We obtain the following main results.

Theorem 1.2. *Let $D \subset \mathbf{C}^n$ be a bounded domain of class $C^{1,1}$, and suppose that D has a $C^{1,1}$ defining function r in a neighborhood U of $\overline{D}$ with the property that*

$$|r_\zeta \cdot (\zeta - z)| > 0, \quad z \in D, \quad \zeta \in U \setminus D\,. \tag{1.7}$$

Then there exist homotopy formulas on D

$$\varphi = \overline{\partial} H_q \varphi + H_{q+1} \overline{\partial} \varphi\,, \quad q = 1, \ldots, n-1; \tag{1.8}$$

$$\varphi = H_0 \varphi + H_1 \overline{\partial} \varphi, \quad q = 0 \tag{1.9}$$

for forms φ of type $(0, q)$ satisfying the assumption that φ and $\overline{\partial}\varphi$ have coefficients in $C^1(\overline{D})$.

Furthermore, if D is strongly $\mathbf{C}$-linearly convex, then for any $a \in (1,\infty)$ we have

$$|H_q\varphi|_{\Lambda^{a+1/2}(\overline{D})} \leq C_a|\varphi|_{\Lambda^a(\overline{D})}, \qquad q \geq 1, \tag{1.10}$$

$$|H_0\varphi|_{\Lambda^a(\overline{D})} \leq C_a|\varphi|_{\Lambda^a(\overline{D})}. \tag{1.11}$$

Here Λ^β is the standard Hölder space when $\beta \in (0,\infty) \setminus \mathbf{N}$, and is the Zygmund space when β is a positive integer. Note that condition (1.7) is trivially implied by e.g., condition (1.4) in Theorem 1.1. The operators H_q, $q = 0, 1, \ldots n$, are defined in Proposition 3.3, where the homotopy formulas (1.8) and (1.9) are obtained. In the proof of the regularity estimates (1.10) and (1.11) it will be important to work with a particular defining function for D: it is well known that condition (1.1) and hence (1.6) are independent of the choice of defining function for D in the sense that only the constants will be affected by the choice of r. In Section 2 we show that such stability is also satisfied by conditions (1.4) and (1.5) (see Lemma 2.3 for the precise statement). Condition (1.5) for a specialized choice of $r \in C^{1,1}(U) \cap C^\infty(U \setminus \overline{D})$ is then needed to justify the application to such r of the results [6, Propositions 4.4 and 4.10] which in turn give the estimates (1.10)-(1.11).

A statement analogous to Theorem 1.2 was proved by the first-named author [6] under the assumptions that the bounded domain D is strongly pseudoconvex and has boundary of class C^2. In Theorem 1.2 we essentially *increase* the amount of convexity to strong $\mathbf{C}$-linear convexity, and *reduce* the amount of boundary regularity to the class $C^{1,1}$.

By employing condition (1.1) and adapting the method of proof of the classical $C^{1/2}$ estimate for strongly pseudoconvex domains of class C^2 (see [10, Theorem 2.2.2]), we also obtain

Proposition 1.3. *Let $D \subset \mathbf{C}^n$ be a bounded weakly $\mathbf{C}$-linearly convex domain of class $C^{1,1}$. There exists a homotopy formula on D*

$$\varphi = \overline{\partial} T_q\varphi + T_{q+1}\overline{\partial}\varphi\,, \quad q = 1, \ldots, n-1 \tag{1.12}$$

for forms φ of type $(0,q)$ when φ and $\overline{\partial}\varphi$ have coefficients in $C^0(\overline{D})$. Furthermore, if D is strongly $\mathbf{C}$-linearly convex we have that

$$|T_q\varphi|_{C^{1/2}(\overline{D})} \leq C|\varphi|_{C^0(\overline{D})}, \quad q \geq 1. \tag{1.13}$$

Here T_q, $q = 1, \ldots, n$ are the classical Leray-Koppelman operators [3, p. 273], which must be suitably interpreted when D is merely of class $C^{1,1}$; see Section 3 for the precise statements and the proofs.

Note that Theorem 1.2 and Proposition 1.3 do not include data φ of maximal type $(0,n)$ because such data can be treated with techniques already available in the literature. Indeed it was observed by the first-named author [6] that if φ has maximal type $(0,n)$ and D is a bounded Lipschitz domain, the solutions of $\overline{\partial}u = \varphi$ can be easily obtained by extending φ to a form with compact support in $\mathbf{C}^n$. Here φ is obviously $\overline{\partial}$-closed and no convexity of D is required. Then one obtains solutions that gain one full derivative in Hölder and Zygmund spaces.

Theorem 1.2 and Proposition 1.3 effectively illustrate that from the point of view of the $\overline{\partial}$-problem with data in the Hölder-Zygmund spaces, strongly $\mathbf{C}$-linearly convex $C^{1,1}$ domains behave like strongly pseudoconvex C^2 domains. On the other hand, this analogy may fail to hold for data taken from other functional spaces. For instance, our proof of Proposition 1.3

does rely on the continuity of φ and $\bar\partial\varphi$ up to $\overline{D}$; in particular, we do not know whether $\bar\partial u = \varphi$ has an $L^\infty(D)$-solution when φ is a $\bar\partial$-closed form whose coefficients are merely in $L^\infty(D)$. The answer to such question would be positive if one knew that the closure of a strongly **C**-linearly convex domain D of class $C^{1,1}$ can be exhausted by strongly pseudoconvex subdomains $\{D_j\}_j$ whose Levi forms are positive definite on the complex tangent spaces with bounds uniform in j, but when ∂D is merely $C^{1,1}$ it is not known whether $\overline{D}$ admits such an exhaustion; see Remark 2.8 in Section 2. It would be interesting to understand the regularity of $\bar\partial$-solutions on a domain whose boundary is locally biholomorphic to a strongly **C**-linearly convex $C^{1,1}$ real hypersurface (whose definition is embedded in the statement of Proposition 2.6 below). However, it remains to be seen whether Theorem 1.2 extends to such domains.

Finally, in the last section we derive ad-hoc estimates for the relevant counter-example in [15] indicating that some regularity of the $\bar\partial$-problem in the strong **C**-linearly convex category may persist below the class $C^{1,1}$; see Section 4 for the precise statements.

The study of regularity of the solutions of the $\bar\partial$-problem via integral representations has a long and rich history. A detailed review of the existing literature may be found in [20] and, for the most recent results, [6] and [22]. Here we briefly recall that for smooth, strongly pseudoconvex domains the optimal 1/2-estimate of Proposition 1.3 was achieved by Henkin-Romanov [11] for $\bar\partial$-closed forms after Grauert-Lieb [7], Henkin [9], Kerzman [12] proved that a C^β-estimate holds for any $\beta < 1/2$. Proposition 1.3 for forms that are not necessarily $\bar\partial$-closed is due to Range-Siu [21]. The $C^{k+1/2}$ solutions for $\bar\partial u \in C^k$ were obtained by Siu [23] for $q = 1$ and by Lieb-Range [16] for all degrees, and both require $\partial D \in C^{k+2}$ and $k \in \mathbf{N}$. The results in the aforementioned [6] were recently extended to weighted L^p Sobolev spaces by Shi [22]. A survey of the extensive literature on the solutions of the $\bar\partial$-problem with methods other than integral formulas may be found in e.g., Harrington [8].

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2. More about C-linear convexity

In this section we prove Theorem 1.1. We will henceforth denote small positive constants by c, c_1, c_*, and large constants by C, C_1, C_*. All these constants may depend on the choice of defining function r for the domain D when such an r is involved. We will also deal with a neighborhood U of $\overline{D}$ in which case the constants may also depend on U.

We recall the following stability property for the notion of strong (resp. strict) **C**-linear convexity, see also [14].

Lemma 2.1. *Let $r^{(j)}$, $j = 1, 2$, be two defining functions for D that are of class C^1 in neighborhoods U_j of $\overline{D}$, $j = 1, 2$. If $r^{(1)}$ satisfies condition* (1.2) (*resp., condition* (1.1)) *with constant $c = c_1$, then $r^{(2)}$ also satisfies condition* (1.2) (*resp., condition* (1.1)) *with constant $c = c_2$.*

Proof. As is well known, see Range [20, Lemma II.2.5], there is a positive and continuous function $h : U_1 \cap U_2 \to \mathbb{R}^+$ such that

$$r^{(2)}(z) = h(z)\, r^{(1)}(z), \quad z \in U_1 \cap U_2\,, \quad \text{and}$$
$$dr^{(2)}(\zeta) = h(\zeta)\, dr^{(1)}(\zeta), \quad \zeta \in U_1 \cap U_2 \cap \partial D.$$

It follows from the second condition above, that

$$|\, r^{(2)}_\zeta \cdot (\zeta - z)| = h(\zeta)\, |r^{(1)}_\zeta \cdot (\zeta - z)| \quad \text{for all} \quad \zeta \in \partial D \;\; \text{and} \;\; \text{for all } z \in \mathbf{C}^n,$$

giving the desired conclusion. □

Thus *any* defining function of a strongly (resp. strictly) **C**-linearly convex domain will satisfy condition (1.1) (resp. (1.2)). In particular, by Whitney extension, any strongly (resp. strictly) **C**-linearly convex domain admits a defining function $r \in C^\infty(\mathbf{C}^n \setminus \partial D) \cap C^1(\mathbf{C}^n)$ that satisfies (1.1) (resp. (1.2)). Furthermore, if the regularity of D is improved to class $C^{1,1}$, then D will admit a defining function $r \in C^\infty(\mathbf{C}^n \setminus \partial D) \cap C^{1,1}(\mathbf{C}^n))$ that satisfies condition (1.1) (resp. (1.2)).

Proof of Theorem 1.1. We first verify that (1.4) is independent of the choice of the $C^{1,1}$-defining functions by proving an equivalent condition. We will use the notation

$$d(z) = \operatorname{dist}^E(z, \partial D)$$

where dist^E stands for Euclidean distance in $\mathbf{C}^n$.

Lemma 2.2. *Let D be a bounded domain with a $C^{1,1}$ defining function r defined on a neighborhood U of $\overline{D}$. Then (1.4) is equivalent to*

$$|r_\zeta \cdot (\zeta - z)| \geq c(d(\zeta) + d(z) + |\operatorname{Im}(r_\zeta \cdot (\zeta - z))| + |\zeta - z|^2), \quad z \in \overline{D}, \quad \zeta \in U \setminus D, \tag{2.1}$$

for some constant c depending on the $C^{1,1}$-norm of r.

The proof below will in fact show that (1.4) is equivalent to the (seemingly weaker) condition

$$|r_\zeta \cdot (\zeta - z)| \geq \tilde{c}\,(d(\zeta) + d(z) + |\zeta - z|^2), \quad z \in \overline{D}, \quad \zeta \in U \setminus D, \tag{2.2}$$

but here we choose to state the Lemma with condition (2.1) as it is this particular formulation that is most frequently stated in the literature.

Proof of Lemma 2.2. The implication (2.1) ⇒ (1.4) is trivial. Suppose now that r satisfies (1.4), and let $\zeta \in U \setminus D$ and $z \in \overline{D}$. Since

$$2|r_\zeta \cdot (\zeta - z)| \geq |\operatorname{Re}(r_\zeta \cdot (\zeta - z))| + |\operatorname{Im}(r_\zeta \cdot (\zeta - z))|, \tag{2.3}$$

$$|\zeta - z| \geq d(z) + d(\zeta), \tag{2.4}$$

$$c \leq \frac{-r(z)}{d(z)} \leq C, \quad \text{and} \quad c \leq \frac{r(\zeta)}{d(\zeta)} \leq C, \tag{2.5}$$

it suffices to show that (1.4) implies

$$|r_\zeta \cdot (\zeta - z)| \geq c\big(r(\zeta) - r(z) + |\zeta - z|^2\big) \tag{2.6}$$

for a (possibly different) constant $c > 0$.

Under condition (1.4) it is enough to verify (2.6) when $|\zeta - z|$ is small. By Taylor formula, which is applicable because $r \in C^{1,1}$, we have

$$2\operatorname{Re}(r_\zeta \cdot (\zeta - z)) \geq r(\zeta) - r(z) - C_0|\zeta - z|^2. \tag{2.7}$$

Note that $r(\zeta) - r(z) > 0$. If

$$2C_0|\zeta - z|^2 < r(\zeta) - r(z) \tag{2.8}$$

we obtain

$$\operatorname{Re} r_\zeta \cdot (\zeta - z) \geq (r(\zeta) - r(z))/2 \geq (r(\zeta) - r(z) + c'|\zeta - z|^2)/4.$$

Thus $|r_\zeta \cdot (\zeta - z)| \geq (r(\zeta) - r(z))/2 \geq (r(\zeta) - r(z) + c'|\zeta - z|^2)/4$, which is equivalent to (2.6) for a possibly different constant c. If

$$2C_0|\zeta - z|^2 \geq r(\zeta) - r(z), \tag{2.9}$$

we obtain by (1.4)

$$|r_\zeta \cdot (\zeta - z)| \geq c|\zeta - z|^2 \geq c(|\zeta - z|^2 + \frac{c_0}{4}(r(\zeta) - r(z)))/2,$$

which also gives us (2.6) for a possibly different c. $\square$.

Lemma 2.3. *Let D be a bounded domain of class $C^{1,1}$. Then condition* (2.1) *is independent of the choice of the $C^{1,1}$-defining function for D.*

Here and in what follows we may shrink the neighborhoods U of D in (2.1) and (1.4).

Proof. Let $r \in C^{1,1}(U)$ and $\tilde{r} \in C^{1,1}(\tilde{U})$ be two defining functions for D, and suppose that r satisfies (2.1). We need to show that $\tilde{r}$ also satisfies (2.1), or equivalently that

$$|\tilde{r}_\zeta \cdot (\zeta - z)| \geq \tilde{c}(d(\zeta) + d(z) + |\zeta - z|^2), \quad z \in D, \quad \zeta \in \tilde{U} \setminus D, \tag{2.10}$$

where $\tilde{U}$ is some neighborhood of $\overline{D}$; see (2.2). But Lemma 2.2 shows that (2.1) and (1.4) are equivalent, thus r satisfies (1.4) as well, and we will use the latter to show that $\tilde{r}$ satisfies (2.10) (and hence (2.1)).

We start with $\tilde{r} = hr$ with $h \in Lip(U')$ and $h \in C^{1,1}_{loc}(U' \setminus \partial D)$, where U' is an open neighborhood of ∂D. Suppose that $\zeta \in U' \setminus D$. We have

$$\tilde{r}_\zeta = hr_\zeta + rh_\zeta. \tag{2.11}$$

We may assume that $c_0 < h < C_0$ and (by Rademacher Theorem) that $|h_\zeta| < C_1$ for some C_0 and C_1. Then

$$\tilde{r}_\zeta \cdot (\zeta - z) = h(\zeta)r_\zeta \cdot (\zeta - z) + r(\zeta)h_\zeta \cdot (\zeta - z).$$

Combining this with condition (1.4) applied to r, we have

$$|\tilde{r}_\zeta \cdot (\zeta - z)| \geq c_1|\zeta - z|^2 - C_1 r(\zeta)|\zeta - z|. \tag{2.12}$$

(To be precise, we obtain the above inequality first at those points $\zeta \in U'$ where the Lipschitz function h is differentiable, and then we extend to any $\zeta \in U'$ by the continuity of $\tilde{r}_\zeta$.) From this inequality and (2.4)-(2.5), we see that by possibly shrinking U, it suffices to verify (2.10) when $|\zeta - z|$ and hence $r(\zeta)$ are sufficiently small. Invoking the elementary inequality:

$$2ab \leq \delta^2 a^2 + \delta^{-2} b^2 \quad \text{for any} \quad \delta > 0,$$

we see that

$$r(\zeta)|\zeta - z| \leq \epsilon|\zeta - z|^2 + \epsilon^{-1}|r(\zeta)|^2.$$

Without loss of generality, we may choose to make U' so small that $r(\zeta) < c_*\epsilon$ for $\zeta \in \tilde{U}$ for a suitable small $c_* > 0$, which we reserve to choose later.

Plugging the above in (2.12) we obtain

$$\begin{aligned}|\tilde{r}_\zeta \cdot (\zeta - z)| &\geq (c_1 - C_1\epsilon)|\zeta - z|^2 - C_1 c_* r(\zeta) \geq c_2|\zeta - z|^2 - C_1' c_* \tilde{r}(\zeta) \\ &\geq c_2|\zeta - z|^2 - C_1' c_* (\tilde{r}(\zeta) - \tilde{r}(z))\end{aligned} \tag{2.13}$$

where the last inequality is due to the fact that $z \in D$, that is $\tilde{r}(z) < 0$.

We also have

$$2|\tilde{r}_\zeta \cdot (\zeta - z)| \geq 2\operatorname{Re}(\tilde{r}_\zeta \cdot (\zeta - z)) \geq \tilde{r}(\zeta) - \tilde{r}(z) - C_2|\zeta - z|^2 \tag{2.14}$$

where the second inequality is obtained by applying Taylor's theorem to $\tilde{r}$ (see the proof of Lemma 2.2 for a similar argument), and we may assume without loss of generality that $C_2 > 1$. We now choose c_* (and thus U') so that $c_2/(c_* C_1') > 8C_2$. This gives us

$$|\tilde{r}_\zeta \cdot (\zeta - z)| \geq \frac{c_2}{4}|\zeta - z|^2 + \frac{C_1' c_*}{2}(\tilde{r}(\zeta) - \tilde{r}(z)).$$

Here the inequality is obtained by invoking (2.13) if $c_2|\zeta - z|^2 > 2C_1' c_*(\tilde{r}(\zeta) - \tilde{r}(z))$ or if not, by invoking (2.14) to obtain

$$\begin{aligned}2|\tilde{r}_\zeta \cdot (\zeta - z)| &\geq \frac{1}{2}(\tilde{r}(\zeta) - \tilde{r}(z)) + \frac{1}{2}(\tilde{r}(\zeta) - \tilde{r}(z)) - C_2|\zeta - z|^2 \\ &\geq \frac{1}{2}(\tilde{r}(\zeta) - \tilde{r}(z)) + \frac{c_2}{4C_1' c_*}|\zeta - z|^2 - C_2|\zeta - z|^2 \\ &\geq \frac{1}{2}(\tilde{r}(\zeta) - \tilde{r}(z)) + \frac{C_2}{2}|\zeta - z|^2,\end{aligned}$$

which proves (2.10). □

Corollary 2.4. *Condition* (1.4) *is independent of the choice of $C^{1,1}$ defining function.*

Proof. Let r and $\tilde{r}$ be two defining functions for D, and suppose that r satisfies condition (1.4). We proceed by contradiction and suppose that $\tilde{r}$ does not satisfy condition (1.4). Then $\tilde{r}$ does not satisfy condition (2.1) (Lemma 2.2); but this in turn implies that r does not satisfy condition (2.1) (Lemma 2.3). It follows by Lemma 2.2 that r does not satisfy condition (1.4), giving a contradiction. □

The proof of Theorem 1.1 now continues with the following

Proposition 2.5. *If D is a bounded, strongly* **C***-linearly convex domain of class $C^{1,1}$, then condition* (1.1) *is equivalent to* (1.4) *for $r \in C^{1,1}$.*

Proof. It's clear that (1.4) $\Rightarrow$ (1.1); we need to show the implication: (1.1) $\Rightarrow$ (1.4) i.e., that

$$|r_\zeta \cdot (\zeta - z)| \geq c_0|\zeta - z|^2 \tag{2.15}$$

holds for some positive c_0, $z \in \overline{D}$ and all $\zeta \in U \setminus D$ for some neighborhood U of $\overline{D}$. By the assumed strong **C**-linear convexity of D, i.e. (1.1), we have that the above holds for $z \in D$ and $\zeta \in \partial D$ for a possibly different constant c_0.

By the continuity of r_ζ, it is clear that (2.15) holds when $U \setminus D$ is sufficiently "narrow", which we assume, and when $\zeta \in U \setminus D, z \in D$ and $|\zeta - z| > c$, where c is any positive constant and U depends on c. Without loss of generality, we assume that $|\zeta - z| < c_*$ for a small c_* to be determined.

Let $\zeta \in U \setminus \overline{D}$, $z \in D$, where $|\zeta - z|$ is sufficiently small, and let $\zeta_* \in \partial D$ be such that

$$\operatorname{dist}^E(\zeta, \partial D) = |\zeta - \zeta_*|\,.$$

Since D is, in particular, of class C^1, we have that the line through ζ and ζ_* is perpendicular to $T^{\mathbb{R}}_{\zeta_*}(\partial D)$, the tangent space to ∂D at ζ^*. By a translation and a unitary change of coordinates, we may assume that

$$\zeta_* = 0, \quad \text{and} \quad \zeta = (0', -i\lambda), \quad \lambda > 0.$$

Thus the real tangent space $T^{\mathbb{R}}_0(\partial D)$ is defined by $y_n = 0$ and near the origin D is defined by

$$\hat{r} = -y_n + R(z', x_n) < 0, \quad R(0', 0) = 0,\ \nabla R(0', 0) = \vec{0}, \tag{2.16}$$

where $z' \in \mathbf{C}^{n-1}$. With the above choice of coordinates, we can easily relate $\hat{r}_\zeta \cdot (\zeta - z)$ to its value at $\zeta = \zeta_*$. Set $z = (z', x_n + iy_n)$; we have

$$\hat{r}_\zeta \cdot (\zeta - z) = \frac{1}{2i}(x_n + i(\lambda + y_n)) = \hat{r}_\eta \cdot (\eta - z)|_{\eta=0} + \frac{1}{2}\lambda. \tag{2.17}$$

Thus

$$|\hat{r}_\zeta \cdot (\zeta - z)| = \frac{1}{2}|x_n + i(y_n + \lambda)|,$$
$$|\zeta - z|^2 = |z'|^2 + |x_n|^2 + |y_n + \lambda|^2.$$

By the hypothesis and Lemma 2.1, we have that

$$\frac{1}{2}|x_n + iy_n| = \left|\hat{r}_\eta \cdot (\eta - z)|_{\eta=0}\right| \geq c_1|z|^2.$$

This shows that when x_n, y_n are sufficiently small, we have

$$|x_n| + |y_n| \geq c_1|z'|^2. \tag{2.18}$$

We claim that it suffices to show

$$|x_n| + |y_n + \lambda| \geq c_*|z'|^2. \tag{2.19}$$

Indeed, assuming the truth of (2.19), we see that it and (2.17) give us

$$\begin{aligned}|\hat{r}_\zeta \cdot (\zeta - z)| &= \frac{1}{2}|x_n + i(\lambda + y_n)| \\ &\geq \frac{1}{4}(|x_n + i(\lambda + y_n)| + c_*|z'|^2/2) \geq c|\zeta - z|^2.\end{aligned}$$

This gives us the required estimate in terms of $\hat{r}$. By Lemma 2.3, we obtain (1.4) for a possibly different U.

We are left to prove (2.19). We first give a simple argument in the case when D is strongly convex. Indeed, for such D we have $y_n > 0$. We immediately obtain

$$|x_n| + |y_n + \lambda| = |x_n| + |y_n| + \lambda \geq |x_n| + |y_n| \geq c_1|z'|^2,$$

where the last inequality follows from (2.18).

We now consider the general case, and again use the original assumption (1.1) in the local coordinate system. For any z' as above, we momentarily consider an auxiliary point $\tilde{z} := (z', \tilde{x}_n + i\tilde{y}_n)$ by setting $\tilde{x}_n := 0$ and $\tilde{y}_n := R(z', 0)$. Then $\tilde{z}$ is in ∂D. Thus by (1.1), we get

$$\frac{1}{2}|\tilde{y}_n| = |\hat{r}_\zeta \cdot (\zeta - \tilde{z})|_{\zeta=0}| \geq c_1|\tilde{z}|^2 \geq c_1|z'|^2.$$

This shows that

$$|R(z',0)| \geq 2c_1|z'|^2, \quad \forall z'. \tag{2.20}$$

We next make the stronger claim that

$$R(z',0) \geq 2c_1|z'|^2, \quad \forall z'. \tag{2.21}$$

On account of (2.20) just proved, it suffices to show that

$$R(z',0) \geq 0, \quad \forall z'.$$

To this end, take any point $\zeta_* \in \partial D$. Let $\hat{z} = L_{\zeta_*}(z) = U(\zeta_*)(z-\zeta_*)$ be the composition of a translation and unitary transformation such that $L_{\zeta_*}(\zeta_*) = 0$ and $L_{\zeta*}D$ is defined by

$$-\hat{y}_n + R(\hat{z}', \hat{x}_n; \zeta_*) < 0,$$

where $R(0;\zeta_*) = 0$ and $\nabla R(0;\zeta_*) = 0$. At the origin the complex tangent space of $L_{\zeta_*}D$ is defined by $\hat{z}_n = 0$, which is uniquely determined by our construction. If $\tilde{U}(\zeta_*)$ is another choice that yields a different $\tilde{R}$, we have

$$-\hat{y}_n + \tilde{R}(\hat{z}', \hat{x}_n; \zeta_*) = (-\hat{y}_n + R(V\hat{z}', \hat{x}_n; \zeta_*))h,$$

where V is a unitary matrix and h is a positive function. Therefore, the sign function

$$S(\zeta_*) = \operatorname{sign} R(\hat{z}', 0; \zeta_*) \in \{-1, 1\}$$

is independent of the choices of L_{ζ_*} and z' when $z' \neq 0$ is sufficiently small.

Locally, we can choose L_{ζ_*} depending on $\zeta_* \in \partial D$ continuously. This shows that S is well-defined and continuous on ∂D. Next, we find a boundary point $\hat{\zeta}$ such that D is located on one side of the real tangent space of ∂D at $\hat{\zeta}$. For such $\hat{\zeta}$, it is clear that $R(z',0;\hat{\zeta}) < 0$ cannot occur for small z'. Thus S is positive on the component of ∂D containing $\hat{\zeta}$. But ∂D (the boundary of the bounded domain of holomorphy $D \subset \mathbf{C}^n$ with $n \geq 2$) must be connected by Hartogs's extension. It follows that $S > 0$, thus proving (2.21).

Now by the assumed $C^{1,1}$ regularity of R and the Taylor remainder theorem, we obtain

$$|R(z',x_n) - R(z',0) - (\partial_{x^n} R(z',x_n))|_{x_n=0} x_n| \leq C x_n^2, \quad \forall z', x_n.$$

We have $\nabla R(0) = 0$. When (z', x_n) is sufficiently small (recall that $|\zeta - z|$ is small), we obtain

$$|R(z',x_n) - R(z',0)| \leq \epsilon |x_n| + C x_n^2. \tag{2.22}$$

Here $\epsilon > 0$ can be made small by assuming that ζ, z', x_n are sufficiently small. Recall that

$$y_n > R(z', x_n),$$

by (2.16). From the latter together with (2.21) and (2.22), we see that

$$\begin{aligned} y_n > R(z',x_n) &= R(z',0) + (R(z',x_n) - R(z',0)) \\ &\geq c_1|z'|^2 - C|x_n|^2 - \epsilon|x_n| \geq c_1|z'|^2 - \epsilon'|x_n|, \end{aligned} \tag{2.23}$$

where ϵ' can be arbitrarily small by assuming ζ, z to be sufficiently close. Since $\lambda > 0$, then

$$|x_n| + |y_n + \lambda| \geq |x_n| + y_n + \lambda \geq |x_n| + c_1|z'|^2 - \epsilon'|x_n| \geq c_1|z'|^2. \tag{2.24}$$

Thus (2.19) holds.

The above gives us the estimate for $\hat{r}_\zeta \cdot (\zeta - z)$; invoking Lemma 2.3 one more time, we conclude that $|r_\zeta \cdot (\zeta - z)| \geq c_2|\zeta - z|^2$ for a possibly different U, thus concluding the proof of Proposition 2.5. □

The following result allows us to introduce a notion of strongly **C**-linearly convex real *hypersurface.*

Proposition 2.6. *Let D be a bounded domain with a $C^{1,1}$ defining function r. Then* (1.1) *and hence* (1.4) *are equivalent to* (1.6)*:*

$$|r_\zeta \cdot (\zeta - z)| \geq c|\zeta - z|^2, \quad \forall \zeta, z \in \partial D.$$

Of course, condition (1.6) alone cannot tell which side of ∂D is the domain D: it is the additional assumption of boundedness of D that determines D.

Proof. Note that Proposition 2.6 is meaningful also in the 1-dimensional setting, that is for $D \Subset \mathbf{C}$, but in this case its conclusion is obvious and so in what follows we assume that $D \Subset \mathbf{C}^n$ with $n \geq 2$. The direction (1.4) $\Rightarrow$ (1.6) is trivial. We prove the opposite direction: suppose that (1.6) holds. We first show that D is strictly **C**-linearly convex, i.e. that $r_\zeta \cdot (\zeta - z) \neq 0$ for $\zeta \in \partial D$, $z \in \overline{D} \setminus \{\zeta\}$, by (1.2). Indeed, suppose for the sake of contradiction that $r_\zeta \cdot (\zeta - z) = 0$ for some $\zeta \in \partial D$ and $z \in D$ (note that case $z \in \partial D$ is ruled out by (1.6)).

Then $H(\zeta) := \{z \in D \colon r_\zeta \cdot (\zeta - z) = 0\}$ is non empty. Since $H(\zeta)$ is a bounded domain in the complex hyperplane $\{z \in \mathbf{C}^n \colon r_\zeta \cdot (\zeta - z) = 0\}$ and $n \geq 2$, the boundary of $H(\zeta)$ contains more than one point and is a subset of ∂D. But $z \in \partial H(\zeta) \setminus \{\zeta\}$ gives $r_\zeta \cdot (\zeta - z) = 0$, which contradicts the assumption (1.6). Thus D is strictly **C**-linearly convex.

Since $r_\zeta \cdot (\zeta - z) \neq 0$ for $\zeta \in \partial D$ and $z \in \overline{D} \setminus \{\zeta\}$, by (1.2), then for any $\delta > 0$

$$\inf \left\{ |\zeta - z|^{-2} |r_\zeta \cdot (\zeta - z)| \colon \zeta \in \partial D, z \in \overline{D}, |\zeta - z| \geq \delta \right\} > 0.$$

We now proceed to prove (1.4): we may assume without loss of generality that z is in a small neighborhood of $\zeta \in \partial D$. As in the proof of Proposition 2.5, by a unitary transformation and translation, we may assume that $\zeta = 0$ and that near the origin D is defined by

$$y_n > R(z', x_n), \quad R(0', 0) = 0, \quad \nabla R(0', 0) = \vec{0}.$$

Let $z = (z', x_n + iy_n)$ with z', x_n and y_n as above, and again consider the auxiliary point $\tilde{z} := (z', i\tilde{y}_n)$ where $\tilde{y}_n := R(z', 0)$. Then z satisfies (2.22) and (2.23). Proceeding as in the proof of (2.24) we find

$$|x_n| + |y_n| \geq |x_n| + y_n \geq c|z'|^2.$$

For $\hat{r} = -y_n + R(z', x_n)$, we obtain

$$|\hat{r}_\zeta \cdot (\zeta - z)| = \frac{1}{2}|x_n + iy_n| \geq \frac{1}{4}(|x_n + iy_n| + \frac{1}{2}(|x_n| + |y_n|)) \geq \frac{1}{4}(|x_n + iy_n| + \frac{c}{2}|z'|^2).$$

We have verified (1.1) and hence (1.4). □

In the sequel we will need the following version of the Whitney extension theorem.

Proposition 2.7. *Let $k \geq 0$ be an integer. Let D be a bounded domain with Lipschitz boundary. If $f \in C^{k,\beta}$ with $0 \leq \beta \leq 1$, then there exists an extension $E_k f \in C^{k,\beta}(\mathbf{R}^N)$ such that $E_k f = f$ on $\overline{D}$ and*

$$|\partial_x^\ell E_k f| \leq C_{k,\ell}(1 + \operatorname{dist}(x, \partial D)^{k+\beta-\ell}), \quad x \in \mathbf{C}^n \setminus \overline{D}, \quad \ell = 0, 1, \ldots. \tag{2.25}$$

For a proof, see [5, 6, 24]. The proof in [6] assumes that $f \in C^{k+\alpha}(\overline{D}) := C^{k,\alpha}(\overline{D})$ with $0 \le \alpha < 1$, but the same argument works for $f \in C^{k,1}(\overline{D})$.

Remark 2.8. While the work carried out in this section here will be sufficient to prove the main result Theorem 1.2, one would like to know whether the closure of a strongly **C**-linearly convex domain D whose boundary is merely of class $C^{1,1}$ has a basis of pseudo-convex neighborhoods; and whether such D can be exhausted by strongly **C**-linearly convex $C^{1,1}$ domains that are relatively compact in D: at present we do not know the answers to such questions.

3. Homotopy formulas

In this section, we first derive a homotopy formulas for $C^{1,1}$ domains D that are *weakly* **C**-*linearly convex*, see (1.3), in terms of the classical Leray-Koppelman operators T_q which however need to be properly interpreted here, since the classical construction of T_q requires two continuous derivatives of the (any) defining function of D and these, in our context, are not available. These are the operators that occur in Proposition 1.3. We mention in passing that the notion of **C**-linear convexity is also meaningful for domains below the C^1-category (and in this context such notion is often referred to simply as "linear convexity") but here it is of no import. The interested reader is referred to [1] for a detailed discussion of this more general setting.

We next derive a homotopy formula for bounded $C^{1,1}$ domains admitting a defining function r that obeys the stronger condition (1.7) in a neighborhood U of $\overline{D}$ which, on account of Theorem 1.1, is satisfied e.g, by strongly **C**-linearly convex D. Here the homotopy formulas are given in terms of operators H_q that are constructed in Proposition 3.3 below.

3.1. **The Leray-Koppelman homotopy operators and homotopy formulas for weakly C-linearly convex domains of class $C^{1,1}$.** Let D be a bounded domain defined by a $C^{1,1}$ function r defined on a bounded, open neighborhood U of $\overline{D}$. We first assume that r satisfies (1.3), which we recall here:

$$|r_\zeta \cdot (\zeta - z)| > 0, \quad \zeta \in \partial D, \quad z \in D\,.$$

Set

$$g^0(z,\zeta) := \overline{\zeta} - \overline{z}\,; \quad g^1(z,\zeta) := r_\zeta \quad \text{and} \quad w = \zeta - z. \tag{3.1}$$

Note that while $g^1(\zeta, z)$ does not depend on z, we maintain this notation to conform with the literature for Cauchy-Fantappiè forms. In particular g^1 is (trivially) holomorphic in z.

Also let

$$V := D \times (U \setminus D)\,, \quad \text{and} \quad S := \{(z,\zeta) \in V : r_\zeta \cdot (\zeta - z) = 0\}.$$

Note that S is a (possibly empty) closed subset of V, and that

$$g^0 \cdot w \neq 0, \quad \zeta \neq z; \qquad g^1 \cdot w \neq 0 \quad (z,\zeta) \in V \setminus S. \tag{3.2}$$

We formally define

$$\omega^i \;:=\; \frac{1}{2\pi i}\frac{g^i\cdot dw}{g^i\cdot w}\,;\quad \Omega^i(z,\zeta)\;:=\;\omega^i\wedge(\overline{\partial}\omega^i)^{n-1},\quad i=0,1,\quad\text{and} \tag{3.3}$$

$$\Omega^{01}(z,\zeta)\;:=\omega^0\wedge\omega^1\wedge\sum_{\alpha+\beta=n-2}(\overline{\partial}\omega^0)^\alpha\wedge(\overline{\partial}\omega^1)^\beta,\quad (\zeta,z)\in V\setminus S. \tag{3.4}$$

Here the implied variable (z,ζ) is taken in V, and the differentials d and $\overline{\partial}$ are taken with respect to both z and ζ.

Note that since the components of ∇r, and hence of g^1, are only Lipschitz, the formal definitions of $\Omega^1(z,\zeta)$ and of $\Omega^{01}(z,\zeta)$ given above must be suitably interpreted. More precisely, let g^1_k be a smooth approximation of g^1, see (3.1), where $k\in\mathbb{N}$ is sufficiently large so that (3.2) is true with g^1_k in place of g^1. Hence (3.3) and (3.4) with g^1_k in place of g^1 give meaningful notions of $\Omega^1_k(z,\zeta)$ and $\Omega^{01}_k(z,\zeta)$. The Rademacher Theorem now ensures that the limits as $k\to\infty$ of $\Omega^1_k(z,\zeta)$ and $\Omega^{01}_k(z,\zeta)$ exist for every $z\in D$ and a.e. $\zeta\in U_K\setminus D$ and are in $L^\infty(K\times(U_K\setminus D))$ for any compact subset $K\subset D$, and we take $\Omega^1(z,\zeta)$ and $\Omega^{01}(z,\zeta)$ to be such limits. Here U_K is an open set containing $\overline{D}$. We have the following representations:

$$\Omega^i(z,\zeta)\;=\;\frac{1}{(2\pi\sqrt{-1})^n}\frac{g^i\cdot d\zeta}{g^i\cdot(\zeta-z)}\wedge\left(\frac{\overline{\partial}g^i\dot{\wedge}\,d\zeta}{g^i\cdot(\zeta-z)}\right)^{n-1}, \tag{3.5}$$

where we have adopted the shorthand: $\overline{\partial}g^i\dot{\wedge}\,d\zeta:=\overline{\partial}g^i_1\wedge d\zeta_1+\cdots+\overline{\partial}g^i_n\wedge d\zeta_n$, and

$$\Omega^{01}(z,\zeta)\;=\;\frac{1}{(2\pi\sqrt{-1})^n}\sum_{\alpha+\beta=n-2}\frac{g^0\cdot d\zeta\wedge(\overline{\partial}g^0\dot{\wedge}\,d\zeta)^\alpha}{(g^0\cdot(\zeta-z))^{\alpha+1}}\wedge\frac{g^1\cdot d\zeta\wedge(\overline{\partial}g^1\dot{\wedge}\,d\zeta)^\beta}{(g^1\cdot(\zeta-z))^{\beta+1}}. \tag{3.6}$$

Furthermore, we decompose

$$\Omega^i(z,\zeta)=\sum_{q=0}^{n-1}\Omega^i_{0,q}(z,\zeta),\quad\text{and}\quad\Omega^{01}(z,\zeta)=\sum_{q=0}^{n-2}\Omega^{01}_{0,q}(z,\zeta). \tag{3.7}$$

Here both $\Omega^i_{0,q}(z,\zeta)$ and $\Omega^{01}_{0,q}(z,\zeta)$ have type $(0,q)$ in the variable z; on the other hand, in the variable ζ the type of $\Omega^i_{0,q}(z,\zeta)$ is $(n,n-1-q)$, while $\Omega^{01}_{0,q}(z,\zeta)$ has type $(n,n-2-q)$. And we have set $\Omega^1_{0,-1}(z,\zeta):=0$ and $\Omega^{01}_{0,-1}(z,\zeta):=0$. The previous argument gives that each term $\Omega^i_{0,q}(z,\zeta)$ and $\Omega^{01}_{0,q}(z,\zeta)$ in the decompositions (3.7) is in $L^\infty(K\times(U_K\setminus D))$ for any compact set $K\subset D$.

Next we formally define the *Leray-Koppelman operators*:

$$T_q\varphi(z):=-\int\limits_{\zeta\in\partial D}\Omega^{01}_{0,q-1}(z,\zeta)\wedge\varphi(\zeta)+\int\limits_{\zeta\in D}\Omega^0_{0,q-1}(z,\zeta)\wedge\varphi(\zeta),\quad q=1,\dots,n \tag{3.8}$$

where φ is a form of type $(0,q)$ whose coefficients are continuous on $\overline{D}$. In giving this definition we face a new conceptual difficulty again stemming from the hypothesis that r is only of class $C^{1,1}(U)$: the Rademacher theorem grants that the second-order derivatives of r are in $L^\infty(U)$, hence $\partial^2 r|_{\partial D}$ may be undefined on ∂D as the latter has Lebesgue measure

0 in $\mathbf{C}^n$. Thus the boundary integral in (3.8) is, in principle, problematic. However one can show that

$$\int_{\zeta\in\partial D}\Omega^{01}_{0,q-1}(z,\zeta)\wedge\varphi(\zeta) \tag{3.9}$$

is nonetheless meaningful. This can be verified e.g., by expressing ∂D as the graph of a function $\psi\in C^{1,1}(\mathbf{R}^{2n-1})$ and employing the argument in [14] which we briefly recall here. Using a partition of unity, we may assume that the coefficients of φ have compact support in a small neighborhood V of $\zeta_0\in\partial D$. On V, we assume that ∂D is given by

$$y_n=\psi(z',x_n),$$

where $\psi\in C^{1,1}(\mathbb{R}^{2n-1})$. We take $r:=-y_n+\psi(z',x_n)$ and invoke the $C^{1,1}$-regularity of ψ to get a sequence of smooth functions $\{\psi^{(k)}\}_k$ that converges to ψ in the C^1 norm, while $\partial^2\psi^{(k)}$ are uniformly bounded and, furthermore, $\partial^2\psi^{(k)}(\zeta)\to\partial^2\psi(\zeta)$ as $k\to\infty$, whenever ζ is a Lebesgue point of $\partial^2\psi$.

We now work with the following defining function for D:

$$r:=-y_n+\psi(z',x_n) \tag{3.10}$$

and with its smooth approximates

$$r^{(k)}:=-y_n+\psi^{(k)}(z',x_n). \tag{3.11}$$

We define the form g^1 and smooth approximations $\{g^1_k\}_k$ using this choice of r. This gives an approximation of $\Omega^{01}(z,\zeta)$ by smooth forms $\Omega^{01}_k(z,\zeta)$ which admit a decomposition analogous to (3.7) as a sum of $(0,q)$ forms $\Omega^{01}_{(0,q)\,k}(z,\zeta)$ with smooth coefficients. Proceeding as in [14] (Rademacher theorem in the variables $(z',x_n)\in\mathbb{R}^{2n-1}$) one can show that each coefficient of $\Omega^{01}_{(0,q)\,k}(z,\zeta)$ converges a.e. $\zeta\in\partial D$ to a limit which is in $L^\infty(K\times\partial D)$ for any compact subset $K\subset D$, which must agree with (the corresponding coefficient of) the limit $\Omega^{01}_{0,q}(z,\zeta)$ that was previously determined. In short, we have that

$$\Omega^{01}_{0,q}(z,\zeta)\in L^\infty(K\times(U_K\setminus D))\cap L^\infty(K\times\partial D) \tag{3.12}$$

for any compact subset $K\subset D$. It follows that

$$\int\limits_{\zeta\in\partial D}\Omega^{01}_{(0,q-1)\,k}(z,\zeta)\wedge\varphi(\zeta)\;-\int\limits_{\zeta\in\partial D}\Omega^{01}_{0,q-1}(z,\zeta)\wedge\varphi(\zeta)\to 0$$

uniformly on the compact subsets of D as $k\to\infty$. This shows that (3.9), and hence T_q is indeed well-defined for r as in (3.10).

The above arguments also show that the conclusions of *Koppelman Lemma*:

$$\overline{\partial}_\zeta\Omega^1_{0,q}(z,\zeta)+\overline{\partial}_z\Omega^1_{0,q-1}(z,\zeta)=0,\quad q=0,\dots,n-1 \tag{3.13}$$

$$\overline{\partial}_\zeta\Omega^{01}_{0,q}(z,\zeta)+\overline{\partial}_z\Omega^{01}_{0,q-1}(z,\zeta)=\Omega^0_{0,q}(z,\zeta)-\Omega^1_{0,q}(z,\zeta),\quad q=0,\dots,n-1 \tag{3.14}$$

are valid for our choice of $\Omega^i_{0,q},\ i=0,1$ and $\Omega^{01}_{0,q}$, $q=0,\dots,n-1$, for

$$(z,\zeta)\in V\setminus S=D\times(U\setminus D)\setminus\{(z,\zeta)\colon r_\zeta\cdot(\zeta-z)=0\}. \tag{3.15}$$

Indeed, by the classical Koppelman lemma [3, p. 263], identities (3.13) and (3.14) are valid for $\Omega^i_{(0,q)\,k}(z,\zeta), i=0,1$ and $\Omega^{01}_{(0,q)\,k}(z,\zeta)$ for any $k\in\mathbb{N}$ and

$$(z,\zeta)\in V\setminus S_k:=D\times(U\setminus D)\setminus\{(z,\zeta)\colon r^{(k)}_\zeta\cdot(\zeta-z)=0\}. \tag{3.16}$$

Taking the limit as $k \to \infty$ we obtain (3.13) and (3.14).

Lemma 3.1. *Let $D \subset \mathbf{C}^n$ be a bounded, weakly $\mathbf{C}$-linearly convex domain of class $C^{1,1}$. The definition of the Leray-Koppelman homotopy operators T_q, i.e.* (3.8)*, is independent of the choice of $C^{1,1}$ defining function for D.*

Proof. Recall that $T_q\varphi$ was defined via a partition of unity for $\overline{D}$ and local graph defining function of ∂D. To verify that $T_q\varphi$ is independent of the choice of $r \in C^{1,1}$, we again use a partition of unity and the above local defining function. For any defining function $\tilde{r} \in C^{1,1}$ of D, using (3.10) we have

$$\tilde{r} = hr, \quad \tilde{r}_\zeta = hr_\zeta, \quad \zeta \in \partial D.$$

Here $h \in Lip$. We now compute Ω^{01} defined by (3.6) with $g^1 = r_\zeta$ being replaced by $\tilde{g}^1 = \tilde{r}_\zeta$. We first note that since $\tilde{r} \in C^{1,1}$ then in particular $\tilde{r}|_{\partial D} \in C^{1,1}$ and hence $d(\tilde{g}^1 \cdot d\zeta) \in L^\infty(\partial D)$; see [14, Section 2]. Thus $T_q\varphi$ is well-defined when g^1 is replaced by $\tilde{g}^1$. With $h \in Lip$ additionally, we conclude that on ∂D,

$$\tilde{g}^1 \cdot d\zeta \wedge (d_\zeta(\tilde{g}^1 \cdot d\zeta))^\alpha = hg^1 \cdot d\zeta \wedge (d_\zeta(hg^1 \cdot d\zeta))^\alpha = h^{\alpha+1} g^1 \cdot d\zeta \wedge (d_\zeta(g^1 \cdot d\zeta))^\alpha.$$

We can also verify that the above expressions have L^∞ coefficients, and the identities hold in the sense of distributions. Approximating $\varphi \in C^0$ by C^1 functions, we conclude that $T_q\varphi$ is independent of the $\tilde{r}$. ☐

Proposition 3.2. *Let D be a bounded weakly $\mathbf{C}$-linearly convex domain with a defining function $r \in C^{1,1}$. Then on D*

$$\varphi(z) = \overline{\partial}_z T_q \varphi + T_{q+1} \overline{\partial}_z \varphi, \quad 1 \le q \le n, \tag{3.17}$$

$$\varphi(z) = \int_{\partial D} \Omega^1_{0,0} \varphi + T_1 \overline{\partial} \varphi, \quad q = 0, \tag{3.18}$$

hold for for any $(0,q)$-form $\varphi \in C^0(\overline{D})$ whose distributional derivatives $\overline{\partial}\varphi$ on D extend to a form whose coefficients are in $C^0(\overline{D})$.

Proof. By the Whitney extension theorem, we may assume that $r \in C^\infty(U \setminus \partial D)$. By Lemma 2.1, we have that such an r satisfies condition (1.2):

$$|r_\zeta \cdot (\zeta - z)| > 0, \quad \zeta \in \partial D, \quad z \in D.$$

Let D' be any relatively compact subdomain of D. We get from the above that

$$|r_\zeta \cdot (\zeta - z)| > c, \quad \zeta \in \partial D, \quad z \in D'. \tag{3.19}$$

We now define

$$D_j := \{r < -c'/j\} \quad \text{where } \ c' \ \text{ is a small positive number.}$$

Note that D_j is relatively compact in D and D_j increase to D as $j \to \infty$. Thus we may henceforth assume that D' is relatively compact in each D_j (for $j > j_0$), and that (3.19) holds for $z \in D'$ and $\zeta \in \partial D_j$.

As before, we take a sequence of smooth functions $\{r^{(k)}\}_k$ that tend to r in the $C^1(\overline{D})$-norm, while $\nabla^2 r^{(k)}$ are uniformly bounded, and $\partial^2 r^{(k)} \to \partial^2 r$ pointwise a.e. on a neighborhood of $\overline{D}$. Thus, replacing r in (3.19) with $r^{(k)}$, we have

$$|r^{(k)}_\zeta \cdot (\zeta - z)| > 0, \quad \zeta \in \partial D_j, \quad z \in D'. \tag{3.20}$$

We next let $\varphi_\epsilon \in C^\infty(\overline{D}_j)$ be such that $\overline{\partial}\varphi_\epsilon - \overline{\partial}\varphi$ tends to 0 in the sup norm on $\overline{D}_j$ as $\epsilon \to 0$, on account of the assumption that $\varphi \in C^0(\overline{D})$ and $\overline{\partial}\varphi$ on D extends to a continuous form on $\overline{D}$.

Let us consider the case of $q \geq 1$. Assume that φ and $\overline{\partial}\varphi$ are in $C^0(\overline{D})$. We first recall the Bochner-Martinelli-Koppelman formula for C^1 domains [3, Theorem 11.1.2]:

$$\varphi(z) = \overline{\partial}_z \int_D \Omega^0_{0,q-1}(z,\zeta) \wedge \varphi(\zeta) + \int_D \Omega^0_{0,q}(z,\zeta) \wedge \overline{\partial}\varphi + \int_{\partial D} \Omega^0_{0,q}(z,\zeta) \wedge \varphi(\zeta). \tag{3.21}$$

Fix $z \in D'$. Applying (3.20) to D_j with (z,ζ) as in (3.16), we obtain via an (implicit) analog of (3.14)

$$\begin{aligned}\int_{\partial D_j} \Omega^1_k(z,\zeta) \wedge \varphi_\epsilon(\zeta) &= \int_{\partial D_j} \left\{\overline{\partial}_\zeta \Omega^{01}_{(0,q),k}\right\} \wedge \varphi_\epsilon + \int_{\partial D_j} \left\{\overline{\partial}_z \Omega^{01}_{(0,q-1),k}\right\} \wedge \varphi_\epsilon \\ &= -\int_{\partial D_j} \Omega^{01}_{(0,q),k} \wedge \overline{\partial}_\zeta \varphi_\epsilon - \overline{\partial}_z \int_{\partial D_j} \Omega^{01}_{(0,q-1),k} \wedge \varphi_\epsilon.\end{aligned}$$

Note that $\overline{\partial}(\varphi_\epsilon) = (\overline{\partial}\varphi)_\epsilon$ on D_j when $0 < \epsilon < \epsilon_j$ for a sufficiently small positive ϵ_j. Thus we have

$$\varphi_\epsilon = \overline{\partial} T^{j,k}_q \varphi_\epsilon + T^{j,k}_q (\overline{\partial}\varphi)_\epsilon, \quad \text{on } D' \text{ and for } \epsilon < \epsilon_j$$

where $T^{j,k}_q$ is the Leray-Koppelman operator (3.8) associated with D_j and $r^{(k)}$, given by

$$T^{j,k}_q \varphi(z) = -\int_{\zeta \in \partial D_j} \Omega^{01}_{(0,q-1),k}(z,\zeta) \wedge \varphi(\zeta) + \int_{\zeta \in D_j} \Omega^0_{(0,q-1),k}(z,\zeta) \wedge \varphi(\zeta), \quad q = 1, \dots, n.$$

We first let ϵ tend to 0 and then we let j tend to ∞. Then on D' we have

$$\varphi = \overline{\partial} T^k_q \varphi + T^k_q \overline{\partial}\varphi$$

where

$$T^k_q \varphi(z) := -\int_{\zeta \in \partial D} \Omega^{01}_{(0,q-1),k}(z,\zeta) \wedge \varphi(\zeta) + \int_{\zeta \in D} \Omega^0_{(0,q-1),k}(z,\zeta) \wedge \varphi(\zeta), \quad q = 1, \dots, n.$$

Now the argument that was used to define the quantity (3.9) shows that $T_q\varphi$ is well-defined by taking $k \to \infty$ in the expression above. We have obtained (3.17). The proof for (3.18) follows a similar strategy. □

3.2. A new homotopy formula for bounded strongly C-linearly convex $C^{1,1}$ domains.

We now prove the main result of this section by deriving a homotopy formula for a domain D admitting a $C^{1,1}$ defining function r satisfying (1.7), which we recall says

$$r_\zeta \cdot (\zeta - z) \neq 0, \quad \forall \zeta \in U \setminus D, z \in D.$$

Note that by Proposition 2.5, this condition is weaker than (1.4) and follows from the strong **C**-linear convexity of $D \in C^{1,1}$.

Proposition 3.3. *Let $D \subset \mathbf{C}^n$ be a domain with a defining function r which is of class $C^{1,1}$ in a neighborhood U of $\overline{D}$. Let $g^0(z,\zeta) = \overline{\zeta} - \overline{z}$. Let $g^1(z,\zeta) = r_\zeta$ satisfy condition* (1.7). *Let φ be a $(0,q)$-form in $\overline{D}$. Suppose that φ and $\overline{\partial}\varphi$ are in $C^1(\overline{D})$. Then in D*

$$\varphi = \overline{\partial} H_q \varphi + H_{q+1} \overline{\partial}\varphi, \quad 1 \leq q \leq n, \tag{3.22}$$

$$\varphi = H_0 \varphi + H_1 \overline{\partial}\varphi, \quad q = 0, \tag{3.23}$$

with

$$H_q\varphi := \int_U \Omega^0_{0,q-1}\wedge E\varphi + \int_{U\setminus D}\Omega^{01}_{0,q-1}\wedge[\overline{\partial},E]\varphi, \quad q>0, \tag{3.24}$$

$$H_0\varphi := \int_{\partial D}\Omega^1_{0,0}\varphi - \int_{U\setminus D}\Omega^1_{0,0}\wedge E\overline{\partial}\varphi = \int_{U\setminus D}\Omega^1_{0,0}\wedge[\overline{\partial},E]\varphi. \tag{3.25}$$

Here, E is an operator that extends φ and $\overline{\partial}\varphi$ to $C^1(U)$.

Proof. The extension operator E was constructed in [24], where it was defined for functions. Here we define $E\varphi$ by applying the original operator E component-wise to the coefficients of φ, which results in a form of the same type. We may assume without loss of generality that $E\varphi$ has compact support in U, by using cut-off functions. Thus $E\varphi, E\overline{\partial}\varphi$ are in $C^1(U)$ and $[\overline{\partial},E]\varphi, [\overline{\partial},E]\overline{\partial}\varphi = \overline{\partial}E\overline{\partial}\varphi$ are in $C^0(U)$.

Let us consider the case: $q\geq 1$. We recall again the Bochner-Martinelli-Koppelman formula (3.21):

$$\varphi(z) = \overline{\partial}_z\int_D \Omega^0_{0,q-1}(z,\zeta)\wedge\varphi(\zeta) + \int_D\Omega^0_{0,q}(z,\zeta)\wedge\overline{\partial}\varphi + \int_{\partial D}\Omega^0_{0,q}(z,\zeta)\wedge\varphi(\zeta). \tag{3.26}$$

Assume that φ and $\overline{\partial}\varphi$ are in $C^1(\overline{D})$. We want to rewrite the boundary integral in (3.26). Fix a relatively compact subdomain D' of D and let z vary in D'. Then we have by (3.13)-(3.14) applied to $g^1(\zeta,z) = r^{(k)}_\zeta$,

$$\begin{aligned}\int_{\partial D}\Omega^0_{0,q}(z,\zeta)\wedge\varphi(\zeta) &= \int_{\partial D}\overline{\partial}_z\Omega^{01}_{(0,q-1),k}(z,\zeta)\wedge\varphi(\zeta) + \int_{\partial D}\overline{\partial}_\zeta\Omega^{01}_{(0,q),k}(z,\zeta)\wedge\varphi(\zeta)\\ &= \overline{\partial}_z\int_{\partial D}\Omega^{01}_{(0,q-1),k}(z,\zeta)\wedge\varphi(\zeta) + \int_{\partial D}\Omega^{01}_{(0,q),k}(z,\zeta)\wedge\overline{\partial}_\zeta\varphi(\zeta).\end{aligned}$$

By [6, (2.12)], we have for $z\in D$ and $\varphi\in C^1(\overline{D})$,

$$\begin{aligned}&-\overline{\partial}\int_{\zeta\in\partial D}\Omega^{01}_{(0,q-1),k}(z,\zeta)\wedge\varphi(\zeta) + \overline{\partial}\int_{\zeta\in D}\Omega^0_{0,q-1}(z,\zeta)\wedge\varphi(\zeta)\\ &\qquad = \overline{\partial}\int_{U\setminus D}\Omega^{01}_{(0,q-1),k}(z,\zeta)\wedge\overline{\partial}E\varphi(\zeta) + \overline{\partial}\int_U\Omega^0_{0,q-1}(z,\zeta)\wedge E\varphi(\zeta).\end{aligned} \tag{3.27}$$

When $\overline{\partial}\varphi\in C^1(\overline{D})$, proceeding as in the proof of [6, (2.13)] we employ [6, (2.11)] to obtain

$$\begin{aligned}-\int_{\partial D}\Omega^{01}_{(0,q),k}\wedge\overline{\partial}\varphi + \int_D\Omega^0_{0,q}\wedge\overline{\partial}\varphi &= \int_{U\setminus D}\Omega^{01}_{(0,q),k}\wedge\overline{\partial}E\overline{\partial}\varphi\\ &\quad -\overline{\partial}\int_{U\setminus D}\Omega^{01}_{(0,q-1),k}\wedge E\overline{\partial}\varphi - \int_{U\setminus D}\Omega^1_{(0,q),k}\wedge E\overline{\partial}\varphi\\ &\quad + \int_{U\setminus D}\Omega^0_{0,q}\wedge E\overline{\partial}\varphi + \int_D\Omega^0_{0,q}\wedge\overline{\partial}\varphi.\end{aligned} \tag{3.28}$$

On the right-hand side, the first term can be written via the commutator as $\overline{\partial}E\overline{\partial}\varphi = (\overline{\partial}E - E\overline{\partial})\overline{\partial}\varphi$. Since $q\geq 1$, the third term is zero. The second term, when combined with the first term on the right-hand side of (3.27), gives us the desired commutator for φ. Adding (3.27)-(3.28) yields (3.22) in which g^1 is $r^{(k)}_\zeta$. Letting $k\to\infty$, we obtain (3.22) on D.

This completes the proof of (3.22). The proof of (3.23) (that is the case: $q = 0$) follows a similar strategy. □

We mention that the commutator $[\overline{\partial}, E]$ was first employed by Peters [19], where E is the Seeley extension when ∂D is sufficiently smooth. The commutator has been useful in the construction of homotopy formulas in other settings; for instance, see Michel [17] and Michel-Shaw [18].

When $\partial D \in C^2$ is strongly pseudoconvex, the statement analogous to Proposition 3.3 has been proved recently by Shi [22] when $\varphi, \overline{\partial}\varphi$ are in the Sobolev space $W^{1,1}(D)$.

Proofs of Theorem 1.2 *and Proposition* 1.3. The inequality (2.25), where f is replaced by $r|_{\overline{D}}$, ensures that $W(\zeta, z) := r_\zeta(\zeta)$ with $r := Ef$ is a regularized Henkin-Ramirez function in the sense of [6, Definition 4.1] where the requirement that $W \in C^1$ (see condition (i.) in [6, Definition 4.1]) is relaxed to W being Lipschitz. Note that conditions (2.1) and (2.25) give that $\Phi(\zeta, z) := r_\zeta \cdot (\zeta - z)$ satisfies the hypotheses of [6, Propositions 4.4 and 4.7]; see [6, (4.4) and (4.5)]. Thus conclusions (1.10) and (1.11) follow from [6, Propositions 4.4 and 4.7], respectively. With the same reasoning, the proof of the well-known 1/2-estimate of Henkin-Romanov [10, Theorem 2.2.2] for strictly pseudoconvex C^2 domains is valid for (1.13) for our $C^{1,1}$ domain. □

4. Below the $C^{1,1}$ category: one example

Let us give an example of $C^{1,\alpha}$ domain which satisfies (1.4).

Example 4.1. Consider for $x \in \mathbf{R}^n$

$$f(x) = |x|^m, \quad 1 < m < \infty.$$

We claim that for a given $C_0 > 0$

$$f(y) - f(x) - \nabla f(x) \cdot (y - x) \geq c|y - x|^{\max(m,2)} \quad \text{whenever} \quad |x| + |y| < C_0. \tag{4.29}$$

Indeed, note that the estimate (4.29) is trivial when x or y is the origin, so in the sequel we assume that neither of x, y is the origin. We have

$$\begin{aligned} f(y) - f(x) - \nabla f(x) \cdot (y - x) &= |y|^m - |x|^m - m|x|^{m-2}x \cdot (y - x) \\ &= |y|^m + (m-1)|x|^m - m|x|^{m-2}x \cdot y. \end{aligned}$$

Thus the estimate with $f(y) - f(x) - \nabla f(x) \cdot (y - x) \geq c|y - x|^m$ holds trivially when $|x| < c_m|y|$, or $|y| < c_m|x|$, or $x \cdot y \leq 0$. Hence by dilation, we may assume that $|x| = 1$, $c < |y| < C$ and $x \cdot y \geq 0$. By rotation, we know that the real Hessian of $|x|^m$ is positive definite at any point $x \neq 0$. Consequently the restriction of $|x|^m$ to any real line segment $\theta \in [0, 1] \mapsto (1 - \theta)x + \theta y$ with positive distance to the origin is a strictly convex smooth function of θ.

Moreover, for $c_m < |y| < C_m$; $|x| = 1$, and $x \cdot y \geq 0$, we have

$$|(1-\theta)x + \theta y|^2 = (1-\theta)^2|x|^2 + \theta^2|y|^2 + 2\theta(1-\theta)x \cdot y \geq (1-\theta)^2|x|^2 + \theta^2|y|^2 \geq \frac{|x|^2|y|^2}{|x|^2 + |y|^2}.$$

This shows that $g(t) := |(1-t)x + ty|^m$ has $g''(t) \geq c_m$, and thus $g(1) - g(0) - g'(0) \geq c_m/2$. This gives us that

$$f(y) - f(x) - \nabla f(x) \cdot (y - x) \geq \tilde{c}_m|y - x|^2/2$$

and concludes the construction of Example 4.1.

As a consequence of the above, given any collection $\{m_1, \dots, m_k\}$ with $m_j > 1, j = 1, \dots, k$, and setting

$$f(x) := \sum_{j=1}^{k} a_j |x^j|^{m_j}, \qquad x^j \in \mathbf{R}^{n_j}, \quad a_j > 0, \ |x^j| < C_0,$$

where $x = (x^1, \dots, x^k) \in \mathbf{R}^{n_1+\cdots+n_k}$, we have

$$f(y) - f(x) - \nabla f(x) \cdot (y - x) \geq c_m |y - x|^{\max(m_1,\dots,m_k,2)}.$$

Here c_m depends only on $a_1, \dots, a_m$ and C_0. In particular, if

$$r(z) := |x_1|^{m_1} + |y_1|^{m_2} + \cdots + |x_n|^{m_{2n-1}} + |y_n|^{m_{2n}}, \tag{4.30}$$

then

$$r(z) - r(\zeta) - 2\,\mathrm{Re}(r_\zeta \cdot (z - \zeta)) \geq c|\zeta - z|^{\max(m_1,\dots,m_{2n},2)}, \quad |\zeta| + |z| < C,$$

where c depends on $m_1, \dots, m_{2n}, C$. Assume further that $r(z) \leq r(\zeta)$. We obtain

$$2\,\mathrm{Re}(r_\zeta \cdot (\zeta - z)) \geq c'|\zeta - z|^{\max(m_1,\dots m_{2n},2)}.$$

If $1 < m_j \leq 2$ for all j, then the domains $\{r < C\}$ are strongly **C**-linearly convex and of class $C^{1,\alpha}$ with $\alpha := \min\{m_1, \dots, m_{2n}\} - 1$, when $C > 0$. In fact condition (1.4) holds for such domains.

The above shows that the function r in (4.30) is a convex function. Thus a theorem of Dufresnoy [4] gives the following.

Proposition 4.2. *Let $q > 0$ and let r be given by* (4.30)*. Suppose that $f \in C^\infty(\overline{D})$ is a $\overline{\partial}$-closed $(0, q)$ form on $D := \{r < c\}$. Then there is a solution $u \in C^\infty(\overline{D})$ to $\overline{\partial} u = f$ on D.*

On the other hand, for the above examples and, more generally for any $C^{1,\alpha}$ strongly **C**-linearly convex domain with $0 < \alpha < 1$, our integral formulas-based method for the solution of the $\overline{\partial}$-problem in the Hölder or Zygmund space category (in fact, *any* function space!) is conceptually problematic. This is because the Leray-Koppelman homotopy operators T_q cannot be made meaningful, as the components of $\nabla^2 r$ are not Lipschitz and there is neither a global, nor a tangential analog of the Rademacher Theorem that can be applied in this context.

Remark 4.3. One can check that condition (4.29) is also valid for

$$f(x) := |x|^m g(x), \quad g \in C^{1,1}(\mathbf{R}), \quad c < g(x) < C.$$

The function above may be used to construct concrete examples of strongly **C**-linearly convex domains of class $C^{1,1}$ and no better.

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Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706, U.S.A.
Email address: `gong@math.wisc.edu`

Department of Mathematics, Syracuse University, Syracuse, NY 13244-1150, U.S.A.
Email address: `llanzani@syr.edu`