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On the motive of the Quot scheme of finite quotients of a locally free sheaf

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# ON THE MOTIVE OF THE QUOT SCHEME OF FINITE QUOTIENTS OF A LOCALLY FREE SHEAF

ANDREA T. RICOLFI

ABSTRACT. Let  $X$  be a smooth variety,  $E$  a locally free sheaf on  $X$ . We express the generating function of the motives  $[\mathrm{Quot}_X(E, n)]$  in terms of the power structure on the Grothendieck ring of varieties. This extends a recent result of Bagnarol, Fantechi and Perroni for curves, and a result of Gusein-Zade, Luengo and Melle-Hernández for Hilbert schemes. We compute this generating function for curves and we express the relative motive  $[\mathrm{Quot}_{\mathbb{A}^d}(\mathcal{O}^{\oplus r}) \rightarrow \mathrm{Sym} \mathbb{A}^d]$  as a plethystic exponential.

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## 0. INTRODUCTION

Let  $X$  be a smooth quasi-projective variety over  $\mathbb{C}$ , and let  $E$  be a locally free sheaf of rank  $r$  on  $X$ . The Quot scheme  $\mathrm{Quot}_X(E, n)$  parameterises quotients  $E \rightarrow Q$  such that  $Q$  is a zero-dimensional sheaf of length  $n$ . Recently Bagnarol, Fantechi and Perroni [1] have shown that if  $C$  is a smooth proper curve, the class

$$[\mathrm{Quot}_C(E, n)] \in K_0(\mathrm{Var}_{\mathbb{C}})$$

in the Grothendieck ring of varieties does not depend on  $E$ . We use the theory of *power structures* [9] to extend their result to arbitrary dimension. Roughly speaking, a power structure on a ring  $R$  is a way of making sense of expressions  $A(t)^m$ , where  $A(t) = 1 + A_1 t + A_2 t^2 + \dots$  is a power series with coefficients in  $R$  and  $m \in R$ .

For  $(X, E)$  as above, we form the generating function

$$Z_E(t) = \sum_{n \geq 0} [\mathrm{Quot}_X(E, n)] t^n,$$

and we denote by  $P_{r,n} \in K_0(\mathrm{Var}_{\mathbb{C}})$  the motive of the *punctual Quot scheme*, namely the closed subscheme  $P_{r,n} \subset \mathrm{Quot}_X(E, n)$  parameterising quotients that are entirely supported at a single (fixed) point in  $X$ .

Our first main result (proved in Theorem 2.3) is the following.

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*Key words and phrases.* Quot schemes, Moduli spaces of sheaves, Grothendieck ring of varieties.

**Theorem A.** *There is an identity*

$$Z_E(t) = \left( \sum_{n \geq 0} P_{r,n} t^n \right)^{[X]}.$$

Since the punctual Quot scheme only depends on  $r$ ,  $n$  and  $\dim X$ , it follows that the same holds true for the motive of  $\text{Quot}_X(E, n)$ . Note that this was proved for  $r = 1$  (the Hilbert scheme case) by Gusein-Zade, Luengo and Melle-Hernández [10].

Our second main result is of *relative* nature and concerns  $X = \mathbb{A}^d$ . The Quot-to-Chow morphism

$$\text{Quot}_X(E, n) \rightarrow \text{Sym}^n X$$

sends a quotient  $E \rightarrow Q$  to the support of  $Q$ , viewed as a zero-cycle with multiplicities. We consider the relative motive

$$Z^{\text{rel}}(\mathbb{A}^d, r) = \sum_{n \geq 0} [\text{Quot}_{\mathbb{A}^d}(\mathcal{O}^{\oplus r}, n) \rightarrow \text{Sym}^n \mathbb{A}^d] \in K_0(\text{Var}_{\text{Sym} \mathbb{A}^d})$$

over the symmetric product of  $\mathbb{A}^d$ . We define classes  $\Omega_{r,n} \in K_0(\text{Var}_{\mathbb{C}})$  by

$$\sum_{n \geq 0} P_{r,n} t^n = \text{Exp} \left( \sum_{n > 0} \Omega_{r,n} t^n \right)$$

where  $\text{Exp}$  is the *motivic exponential* (see Section 1.5) induced by the lambda ring structure on  $K_0(\text{Var}_{\mathbb{C}})$ . For  $\mathbb{A}^d$ , we refine Theorem A by showing (see Theorem 2.9) that  $Z^{\text{rel}}(\mathbb{A}^d, r)$  is generated on the small diagonal by the absolute motives  $\Omega_{r,n}$ .

**Theorem B.** *There is an identity*

$$Z^{\text{rel}}(\mathbb{A}^d, r) = \text{Exp}_{\cup} \left( \sum_{n > 0} \Omega_{r,n} \boxtimes [\mathbb{A}^d \xrightarrow{\Delta_n} \text{Sym}^n \mathbb{A}^d] \right).$$

See [6, 14] for analogues of this result in the context of motivic Donaldson–Thomas theory and [3] for the calculation of the (absolute) *virtual motive* of  $\text{Hilb}^n(\mathbb{A}^3)$ .

Finally, our last result (see Section 3.1) is the full “solution” of the motivic theory of the Quot scheme of a smooth curve, which can be summarised by the identities

$$\Omega_{r,n} = \begin{cases} [\mathbb{P}^{r-1}] & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$$

**Theorem C.** *If  $E$  is a locally free sheaf on a smooth curve  $C$ , there is an identity*

$$Z_E(t) = \text{Exp}([C \times \mathbb{P}^{r-1}]t).$$

Moreover, in  $K_0(\text{Var}_{\text{Sym} \mathbb{A}^1})$  there is an identity

$$Z^{\text{rel}}(\mathbb{A}^1, r) = \text{Exp}_{\cup}([[\mathbb{P}^{r-1}] \boxtimes [\mathbb{A}^1 \xrightarrow{\text{id}} \mathbb{A}^1]]).$$

We use the first relation to compute the Hodge–Deligne polynomial of the smooth space  $\text{Quot}_C(E, n)$  for a proper curve  $C$  (Proposition 3.5). We stress that the formula for  $Z_E$  in the proper case was already implicit in the calculation of [1, Prop. 4.5].

In Section 3.3 we discuss the case  $r = 1$  on a surface, where we find  $\Omega_{1,n} = \mathbb{L}^{n-1}$  according to Göttsche’s formula [7]. Finally, we conclude by proposing a geometric open problem related to punctual Quot schemes on curves.

We work over the field of complex numbers throughout.

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## 1. MOTIVIC PRELIMINARIES

In this section we recall a few motivic constructions that will be needed later. Most of this material is a simplified version of [6, Section 1], adapted to suit the purposes of this paper.

**1.1. The Grothendieck ring of varieties.** Fix a complex scheme  $S$  locally of finite type over  $\mathbb{C}$ . The Grothendieck ring of  $S$ -varieties

$$K_0(\text{Var}_S)$$

is the free abelian group generated by isomorphism classes  $[X \rightarrow S]$  of  $S$ -varieties modulo the *scissor relations*, namely the identities

$$[X \xrightarrow{f} S] = [Y \xrightarrow{f|_Y} S] + [X \setminus Y \xrightarrow{f|_{X \setminus Y}} S]$$

imposed whenever  $Y \subset X$  is a closed  $S$ -subvariety of  $X$ . The ring structure is given on generators by fibre product over  $S$ ,

$$(1.1) \quad [X \rightarrow S] \cdot [Y \rightarrow S] = [X \times_S Y \rightarrow S].$$

The element

$$\mathbb{L} = [\mathbb{A}^1 \times_{\mathbb{C}} S \rightarrow S] \in K_0(\text{Var}_S)$$

is called the *Lefschetz motive* (over  $S$ ). If  $S'$  is another complex scheme, there is an external product

$$(1.2) \quad K_0(\text{Var}_S) \times K_0(\text{Var}_{S'}) \xrightarrow{\boxtimes} K_0(\text{Var}_{S \times S'})$$

defined on generators by sending  $([f: X \rightarrow S], [g: X' \rightarrow S']) \mapsto [f \times g: X \times X' \rightarrow S \times S']$ .

A morphism  $f: S \rightarrow T$  induces a ring homomorphism  $f^*: K_0(\text{Var}_T) \rightarrow K_0(\text{Var}_S)$  by base change and a  $K_0(\text{Var}_T)$ -linear map  $f_!: K_0(\text{Var}_S) \rightarrow K_0(\text{Var}_T)$  defined on generators by composition with  $f$ .

**Definition 1.1.** We denote by  $S_0(\text{Var}_S)$  the semigroup of *effective* motives, i.e. the semigroup generated by isomorphism classes  $[X \rightarrow S]$  of complex quasi-projective  $S$ -varieties modulo the scissor relations. The product (1.1) turns  $S_0(\text{Var}_S)$  into a semiring. There is a natural semiring map  $S_0(\text{Var}_S) \rightarrow K_0(\text{Var}_S)$ , and we say that  $\alpha \in K_0(\text{Var}_S)$  is effective if it lies in the image of this map.

**1.2. Equivariant motives and the quotient map.** Recall that if  $S$  is a scheme with a *good* action by a finite group  $G$  (i.e. an action such that every point of  $S$  has an affine  $G$ -invariant open neighborhood), the quotient  $S/G$  exists as a scheme. For instance, finite group actions on quasi-projective varieties are good.

**Definition 1.2.** Let  $G$  be a finite group,  $S$  a scheme with good  $G$ -action. We denote by  $\tilde{K}_0^G(\text{Var}_S)$  the free abelian group generated by isomorphism classes  $[X \rightarrow S]$  of  $G$ -equivariant  $S$ -varieties with good action, modulo the  $G$ -equivariant scissor relations. We denote by  $K_0^G(\text{Var}_S)$  the quotient of  $\tilde{K}_0^G(\text{Var}_S)$  by the relations

$$[V \rightarrow X \rightarrow S] = [\mathbb{A}_X^r \rightarrow S],$$

where  $V \rightarrow X$  is a  $G$ -equivariant vector bundle of rank  $r$  over a  $G$ -equivariant  $S$ -variety  $X$ .

There is a natural ring structure on  $\tilde{K}_0^G(\text{Var}_S)$ , where the product of two classes  $[X \rightarrow S]$  and  $[Y \rightarrow S]$  is given by taking the diagonal action on  $X \times_S Y$ . The structures  $f^*$ ,  $f_!$  and  $\boxtimes$  naturally extend to the equivariant setting, along with their basic compatibilities. For instance, if  $f: S \rightarrow T$  (resp.  $g: S' \rightarrow T'$ ) is a  $G$ -equivariant (resp.  $G'$ -equivariant) map, and  $u, v$  are equivariant motives over  $S, S'$ , then

$$(1.3) \quad (f \times g)_!(u \boxtimes v) = f_! u \boxtimes g_! v$$

in the  $(G \times G')$ -equivariant  $K$ -group over  $T \times T'$ .

One can define a  $K_0(\text{Var}_{S/G})$ -linear map (cf. [6, Lemma 1.5])

$$(1.4) \quad \pi_G: \tilde{K}_0^G(\text{Var}_S) \rightarrow K_0(\text{Var}_{S/G})$$

given on generators by taking the orbit space,

$$\pi_G[X \rightarrow S] = [X/G \rightarrow S/G].$$

This map does not always extend to  $K_0^G(\text{Var}_S)$ . It does when  $G$  acts freely on  $S$ , by [5, Lemma 3.2].

**1.3. Lambda ring structures.** Let  $n > 0$  be an integer, and let  $\mathfrak{S}_n$  be the symmetric group of  $n$  elements. By [6, Lemma 1.6], namely the relative version of [3, Lemma 2.4], there exist “ $n$ -th power” maps

$$(1.5) \quad (\cdot)^{\otimes n}: K_0(\text{Var}_S) \rightarrow \tilde{K}_0^{\mathfrak{S}_n}(\text{Var}_{S^n})$$

where  $S^n = S \times \cdots \times S$  is endowed with the natural  $\mathfrak{S}_n$ -action. The power map takes  $[f: X \rightarrow S]$  to the class of the equivariant function  $f^n: X^n \rightarrow S^n$ . For  $A \in K_0(\text{Var}_S)$ , consider the classes

$$\pi_{\mathfrak{S}_n}(A^{\otimes n}) \in K_0(\text{Var}_{S^n/\mathfrak{S}_n}).$$

The *lambda ring* operations on  $K_0(\text{Var}_{\mathbb{C}})$  are defined by

$$A \mapsto \sigma^n(A) = \pi_{\mathfrak{S}_n}(A^{\otimes n}) \in K_0(\text{Var}_{\mathbb{C}})$$

for effective classes  $A \in K_0(\text{Var}_{\mathbb{C}})$ , and then taking the unique extension to a lambda ring structure on  $K_0(\text{Var}_{\mathbb{C}})$ , determined by the relation

$$(1.6) \quad \sum_{i=0}^n \sigma^i([X] - [Y]) \sigma^{n-i}[Y] = \sigma^n[X].$$

If  $S$  comes with a commutative associative map  $\nu: S \times S \rightarrow S$ , we likewise define

$$\sigma_\nu^n(A) = \bar{\nu}_! \pi_{\mathfrak{S}_n}(A^{\otimes n}) \in K_0(\text{Var}_S)$$

on effective classes  $A = [X \rightarrow S]$ , where  $\bar{\nu}$  is the map  $S^n/\mathfrak{S}_n \rightarrow S$ . One then uses the analogue of the relation (1.6) to find a unique set of lambda ring operators  $\sigma_\nu^n$  restricting to the previous identity on effective motives.

As a special case, one can consider  $(S, \nu) = (\mathbb{N}, +)$ , viewed as a symmetric monoid in the category of schemes. We obtain lambda operations  $\sigma^n = \sigma_+^n$  on  $K_0(\text{Var}_{\mathbb{C}})[[t]]$  via the isomorphism

$$(1.7) \quad K_0(\text{Var}_{\mathbb{C}})[[t]] \xrightarrow{\sim} K_0(\text{Var}_{\mathbb{N}})$$

defined by sending  $\sum_{n \geq 0} [Y_n] t^n \mapsto [\coprod_{n \in \mathbb{N}} Y_n \rightarrow \{n\}]$ .

**1.4. Power structures.** The main references for power structures are [9, 10].

**Definition 1.3** ([9]). A *power structure* on a (semi)ring  $R$  is a map

$$\begin{aligned} (1 + tR[[t]]) \times R &\rightarrow 1 + tR[[t]] \\ (A(t), m) &\mapsto A(t)^m \end{aligned}$$

satisfying the following conditions:

- (1)  $A(t)^0 = 1$ ,
- (2)  $A(t)^1 = A(t)$ ,
- (3)  $(A(t) \cdot B(t))^m = A(t)^m \cdot B(t)^m$ ,
- (4)  $A(t)^{m+m'} = A(t)^m \cdot A(t)^{m'}$ ,
- (5)  $A(t)^{mm'} = (A(t)^m)^{m'}$ ,
- (6)  $(1+t)^m = 1 + mt + O(t^2)$ ,
- (7)  $A(t)^m|_{t \rightarrow t^e} = A(t^e)^m$ .

Throughout we use the following:

**Notation 1.4.** Partitions  $\alpha \vdash n$  are written as  $\alpha = (1^{\alpha_1} \dots i^{\alpha_i} \dots s^{\alpha_s})$ , meaning that there are  $\alpha_i$  parts of size  $i$ . In particular we recover  $n = \sum_i i\alpha_i$ . The *automorphism group* of  $\alpha$  is the product of symmetric groups  $G_\alpha = \prod_i \mathfrak{S}_{\alpha_i}$ .

**Example 1.5.** If  $R = \mathbb{Z}$ ,  $A(t) = 1 + \sum_{n>0} A_n t^n \in \mathbb{Z}[[t]]$  and  $m \in \mathbb{N}$ , the known formula [16, p. 40]

$$A(t)^m = 1 + \sum_{n \geq 0} \sum_{\alpha \vdash n} \left( \prod_{i=0}^{|\alpha|-1} (m-i) \cdot \frac{\prod_i A_i^{\alpha_i}}{\prod_i \alpha_i!} \right) t^n$$

defines a power structure on  $\mathbb{Z}$ , where we have set  $|\alpha| = \sum_i \alpha_i$ .

Gusein-Zade, Luengo and Melle-Hernández have proved [9, Thm. 2] that there is a unique power structure

$$(A(t), m) \mapsto A(t)^m$$

on  $K_0(\text{Var}_{\mathbb{C}})$  extending the one defined in *loc. cit.* on the semiring  $S_0(\text{Var}_{\mathbb{C}})$  of effective motives. The latter is given by the formula

$$(1.8) \quad A(t)^{[X]} = 1 + \sum_{n \geq 0} \sum_{\alpha \vdash n} \pi_{G_\alpha} \left( \left[ \prod_i X^{\alpha_i} \setminus \Delta \right] \cdot \prod_i A_i^{\otimes \alpha_i} \right) t^n.$$

Here,  $\Delta \subset \prod_i X^{\alpha_i}$  is the “big diagonal” (the locus in the product where at least two entries are equal), and the product in big round brackets is a  $G_\alpha$ -equivariant motive in  $\tilde{K}_0^{G_\alpha}(\text{Var}_{\mathbb{C}})$ , thanks to the power map (1.5).

**Remark 1.6.** We will not encounter non-effective coefficients in this paper, so we will have direct access to Formula (1.8).

**1.5. Motivic exponential.** The *motivic exponential* is a group isomorphism

$$\text{Exp}: t K_0(\text{Var}_{\mathbb{C}})[[t]] \xrightarrow{\sim} 1 + t K_0(\text{Var}_{\mathbb{C}})[[t]],$$

converting sums into products and preserving effectiveness. If  $A = \sum_{n>0} A_n t^n$  is an effective power series, one has by definition

$$\text{Exp}\left(\sum_{n>0} A_n t^n\right) = \prod_{n>0} (1 - t^n)^{-A_n},$$

and if  $A$  and  $B$  are effective, one sets

$$(1.9) \quad \text{Exp}(A - B) = \prod_{n>0} (1 - t^n)^{-A_n} \cdot \left( \prod_{n>0} (1 - t^n)^{-B_n} \right)^{-1}.$$

More generally, if  $(S, \nu: S \times S \rightarrow S)$  is a commutative monoid in the category of schemes, with a submonoid  $S_+ \subset S$  such that the induced map  $\coprod_{n \geq 1} S_+^{\times n} \rightarrow S$  is of finite type, we similarly define

$$\text{Exp}_{\nu}(A) = \sum_{n \geq 0} \sigma_{\nu}^n(A)$$

on effective classes, and for  $A$  and  $B$  two effective classes, we define  $\text{Exp}_{\nu}(A - B)$  by the analogue of (1.9), i.e. by  $\text{Exp}_{\nu}(A) \cdot \text{Exp}_{\nu}(B)^{-1}$ .

**1.6. Motives over symmetric products.** The machinery described so far will be applied to the following situation. For a variety  $X$ , we will consider  $(\text{Sym}(X), \cup)$ , where

$$\text{Sym}(X) = \coprod_{n \geq 0} \text{Sym}^n(X)$$

can be viewed as a monoid via the morphism

$$\text{Sym}(X) \times \text{Sym}(X) \xrightarrow{\cup} \text{Sym}(X)$$

sending two zero-cycles (with multiplicities) on  $X$  to their union. The submonoid  $\text{Sym}(X)_+ = \coprod_{n>0} \text{Sym}^n(X)$  allows one to construct the map  $\text{Exp}_{\cup}$  as in Section 1.5.

In order to recover a formal power series in  $K_0(\text{Var}_{\mathbb{C}})[[t]]$  from a relative motive over  $\text{Sym}(X)$ , we consider the operation

$$(1.10) \quad \#_! \left( \sum_{n \geq 0} [Y_n \rightarrow \text{Sym}^n X] \right) = \sum_{n \geq 0} [Y_n] t^n.$$

In other words we take the direct image along the “tautological” map  $\#: \text{Sym}(X) \rightarrow \mathbb{N}$  which collapses  $\text{Sym}^n(X)$  onto the point  $n$ . In the right hand side of (1.10), we use the isomorphism (1.7) to identify relative motives over  $\mathbb{N}$  and formal power series with coefficients in  $K_0(\text{Var}_{\mathbb{C}})$ .

The following result, a special case of [6, Prop. 1.12], will be needed in the proof of Theorem 2.9.

**Lemma 1.7.** *Let  $U$  be a variety and let  $\Delta_n: U \rightarrow \text{Sym}^n U$  be the small diagonal. Let  $A = \sum_{n>0} A_n$  be an effective motive over  $\mathbb{N}_{>0}$  and set  $B = \text{Exp}(A) = 1 + \sum_{n>0} B_n$ . Define*

$$Z = \sum_{n \geq 0} \sum_{\alpha \vdash n} \cup_! \pi_{G_{\alpha}} j_{\alpha}^* \left( \boxtimes_{i|\alpha_i \neq 0} \Delta_{i!} ([U \xrightarrow{\text{id}} U] \boxtimes B_i)^{\otimes \alpha_i} \right) \in K_0(\text{Var}_{\text{Sym} U}),$$



where  $j_\alpha$  is the  $G_\alpha$ -equivariant open immersion  $\prod_i \mathrm{Sym}^i(U)^{\alpha_i} \setminus \Delta \hookrightarrow \prod_i \mathrm{Sym}^i(U)^{\alpha_i}$ . Then there is an identity

$$Z = \mathrm{Exp}_U \left( \sum_{n>0} A_n \boxtimes [U \xrightarrow{\Delta_n} \mathrm{Sym}^n U] \right)$$

Moreover,

$$\#_! \mathrm{Exp}_U \left( \sum_{n>0} A_n \boxtimes [U \xrightarrow{\Delta_n} \mathrm{Sym}^n U] \right) = B^{[U]} \in K_0(\mathrm{Var}_{\mathbb{C}})[[t]].$$

We briefly explain how to read the right hand side of the first equation of the lemma. First of all, we view  $U$  as a map  $\mathrm{Sym}(U)^b \rightarrow \mathrm{Sym}(U)$  for any  $b > 0$ . The map  $\pi_{G_\alpha}$  appearing in the definition of  $Z$  sends a  $G_\alpha$ -equivariant relative motive over  $\prod_i \mathrm{Sym}^i(U)^{\alpha_i} \setminus \Delta$  to a relative motive over  $\prod_i \mathrm{Sym}^{i\alpha_i}(U) \setminus \Delta$ , therefore we can apply the direct image  $U_!$  to get a relative motive over  $\mathrm{Sym}^n U$ , where  $n = \sum_i i\alpha_i$ .

## 2. THE MOTIVE OF THE QUOT SCHEME

**2.1. Main characters.** Let  $X$  be a smooth quasi-projective variety of dimension  $d$ . Let  $E$  be a rank  $r$  locally free sheaf on  $X$ . For a given integer  $n \geq 0$ , the Quot scheme

$$\mathrm{Quot}_X(E, n)$$

parameterises quotients  $E \rightarrow Q$  such that

$$\dim(\mathrm{Supp} Q) = 0, \quad \chi(Q) = n.$$

The Quot-to-Chow map

$$\sigma_n: \mathrm{Quot}_X(E, n) \rightarrow \mathrm{Sym}^n X$$

constructed in [8, Section 6] (see also [15, Cor. 7.15] for a modern treatment) takes a quotient  $E \rightarrow Q$  to the zero-cycle (with multiplicities) determined by the set-theoretic support of  $Q$ . We define the *punctual Quot scheme* to be the preimage

$$\mathrm{Quot}_X(E, n)_p = \sigma_n^{-1}(n \cdot p)$$

of the cycle  $n \cdot p \in \mathrm{Sym}^n X$ , where  $p \in X$  is a point. This is easily seen to only depend on a formal neighborhood of  $p \in X$  (but not on  $p, X$  or  $E$ ). In particular, one has isomorphisms

$$(2.1) \quad \mathrm{Quot}_X(E, n)_p \cong \mathrm{Quot}_X(\mathcal{O}_X^{\oplus r}, n)_p \cong \mathrm{Quot}_{\mathbb{A}^d}(\mathcal{O}_{\mathbb{A}^d}^{\oplus r}, n)_0$$

where 0 is the origin in  $\mathbb{A}^d$ . This scheme will be denoted  $P_{r,n}$  from now on, and

$$P_{r,n} = [P_{r,n}] \in K_0(\mathrm{Var}_{\mathbb{C}})$$

will denote its motive.

We pause for a second to explain how to prove the second isomorphism in (2.1). Using smoothness of  $X$ , we can fix étale coordinates around  $p \in X$ . This means we can find a pair  $(U, \varphi)$  where  $p \in U \subset X$  is an open neighborhood and  $\varphi: U \rightarrow \mathbb{A}^d$  is an étale map such that  $\varphi(p) = 0 \in \mathbb{A}^d$ . As in the proof of [2, Lemma A.1], we can further shrink  $U$  until  $U \cap \varphi^{-1}(0)$  is the single point  $p$ . Then, we consider the open subscheme  $W \subset \mathrm{Quot}_U(\mathcal{O}_U^{\oplus r}, n) \subset \mathrm{Quot}_X(\mathcal{O}_X^{\oplus r}, n)$  consisting of quotients  $\mathcal{O}_U^{\oplus r} \twoheadrightarrow Q$  such that  $\varphi|_{\mathrm{Supp} Q}$  is injective. Note that  $W$  contains  $\mathrm{Quot}_U(\mathcal{O}_U^{\oplus r}, n)_p = \mathrm{Quot}_X(\mathcal{O}_X^{\oplus r}, n)_p$  as a closed subscheme. By [2, Proposition A.3], sending

$$(\mathcal{O}_U^{\oplus r} \twoheadrightarrow Q) \mapsto (\mathcal{O}_{\mathbb{A}^d}^{\oplus r} \rightarrow \varphi_* \varphi^* \mathcal{O}_U^{\oplus r} = \varphi_* \mathcal{O}_U^{\oplus r} \twoheadrightarrow \varphi_* Q)$$

defines an étale morphism  $\Phi: W \rightarrow \text{Quot}_{\mathbb{A}^d}(\mathcal{O}_{\mathbb{A}^d}^{\oplus r}, n)$ . Its restriction

$$(2.2) \quad \Phi^{-1}(\text{Quot}_{\mathbb{A}^d}(\mathcal{O}_{\mathbb{A}^d}^{\oplus r}, n)_0) \rightarrow \text{Quot}_{\mathbb{A}^d}(\mathcal{O}_{\mathbb{A}^d}^{\oplus r}, n)_0$$

to the punctual Quot scheme of  $\mathbb{A}^d$  is étale and bijective, hence an isomorphism. For surjectivity, use that  $p$  is the only point in  $U \cap \varphi^{-1}(0)$ , and for injectivity use that  $\varphi|_U$  is an immersion around  $p$ , so that  $\varphi^* \varphi_* Q \xrightarrow{\sim} Q$  is an isomorphism for all  $Q$  supported entirely at  $p$ . Finally, again by our choice of  $U$ , the source of the morphism (2.2) is naturally identified with  $\text{Quot}_U(\mathcal{O}_U^{\oplus r}, n)_p$ .

**Remark 2.1.** The *punctual motives*  $P_{r,n}$  clearly depend on the dimension  $d = \dim X$ , but we omit  $d$  from the notation.

**Example 2.2.** On a curve (i.e. if  $d = 1$ ), by [1, Prop. 2.6] we have

$$(2.3) \quad P_{r,1} = [\mathbb{P}^{r-1}].$$

**2.2. Absolute motives.** Let  $X$  and  $E$  be as in the previous section. Define the generating functions

$$\begin{aligned} P_r(t) &= \sum_{n \geq 0} P_{r,n} t^n, \\ Z_E(t) &= \sum_{n \geq 0} [\text{Quot}_X(E, n)] t^n \end{aligned}$$

in the power series ring  $K_0(\text{Var}_{\mathbb{C}})[[t]]$ . The following result (namely Theorem A from the Introduction) is the higher rank analogue of the corresponding statement for the Hilbert scheme of points [10, Thm. 1], obtained by setting  $r = 1$ .

**Theorem 2.3.** *Let  $X$  be a smooth quasi-projective variety. Let  $E$  be a rank  $r$  locally free sheaf on  $X$ . There is an identity*

$$(2.4) \quad Z_E(t) = P_r(t)^{[X]}.$$

*Proof.* For  $\alpha$  a partition of  $n$ , let  $\text{Sym}^\alpha X \subset \text{Sym}^n X$  be the locally closed subvariety parameterising zero-cycles whose support is distributed according to  $\alpha$ . We get a motivic decomposition

$$(2.5) \quad [\text{Quot}_X(E, n)] = \sum_{\alpha \vdash n} [\text{Quot}_X(E, n)_\alpha],$$

where we have set  $\text{Quot}_X(E, n)_\alpha = \sigma_n^{-1}(\text{Sym}^\alpha X)$ . By standard arguments (see e.g. [4, Sec. 4] and [13, Sec. 3]), one sees that the deepest stratum of the Quot-to-Chow map

$$\sigma_{(n)}: \text{Quot}_X(E, n)_{(n)} \rightarrow X$$

is a Zariski locally trivial fibration with fibre  $P_{r,n}$ . This relies on the local case  $X = \mathbb{A}^d$ , where one has a global decomposition

$$\text{Quot}_{\mathbb{A}^d}(\mathcal{O}^{\oplus r}, n)_{(n)} \cong \mathbb{A}^d \times P_{r,n}$$

under which  $\sigma_{(n)}$  is identified with the first projection.

For a fixed partition  $\alpha \vdash n$ , let

$$V_\alpha \hookrightarrow \prod_i \text{Quot}_X(E, i)^{\alpha_i}$$

be the open subscheme parameterising finite quotients with disjoint supports. By [2, Prop. A.3] (but see also [4, Lemma 4.10] for the Hilbert scheme version), taking the union of points gives an étale map

$$u_\alpha: V_\alpha \rightarrow \text{Quot}_X(E, n)$$

and we let  $U_\alpha$  denote its image. The stratum  $\text{Quot}_X(E, n)_\alpha$  sits inside  $U_\alpha$  as a closed subscheme. We let the cartesian diagram

$$(2.6) \quad \begin{array}{ccc} Z_\alpha & \hookrightarrow & V_\alpha \\ \tilde{u}_\alpha \downarrow & \square & \downarrow u_\alpha \\ \text{Quot}_X(E, n)_\alpha & \hookrightarrow & U_\alpha \end{array}$$

define the scheme  $Z_\alpha$ . The map  $\tilde{u}_\alpha$  is a finite étale cover with Galois group  $G_\alpha$ , in particular we have

$$(2.7) \quad \text{Quot}_X(E, n)_\alpha = Z_\alpha / G_\alpha.$$

In fact,  $Z_\alpha$  can also be realised as the fibre product

$$(2.8) \quad \begin{array}{ccc} Z_\alpha & \hookrightarrow & \prod_i \text{Quot}_X(E, i)_{(i)}^{\alpha_i} \\ f_\alpha \downarrow & \square & \downarrow \\ \prod_i X^{\alpha_i} \setminus \Delta & \hookrightarrow & \prod_i X^{\alpha_i} \end{array}$$

where the bottom open immersion is the complement of the big diagonal and the map  $f_\alpha$  is a  $G_\alpha$ -equivariant piecewise trivial fibration with fibre  $\prod_i P_{r,i}^{\alpha_i}$ . This implies the identity

$$[Z_\alpha] = \left[ \prod_i X^{\alpha_i} \setminus \Delta \right] \cdot \prod_i P_{r,i}^{\otimes \alpha_i}$$

in  $K_0^{G_\alpha}(\text{Var}_\mathbb{C})$ . Using (2.7), it follows that

$$\begin{aligned} [\text{Quot}_X(E, n)_\alpha] &= \pi_{G_\alpha} [Z_\alpha] \\ &= \pi_{G_\alpha} \left( \left[ \prod_i X^{\alpha_i} \setminus \Delta \right] \cdot \prod_i P_{r,i}^{\otimes \alpha_i} \right), \end{aligned}$$

where  $\pi_{G_\alpha}: K_0^{G_\alpha}(\text{Var}_\mathbb{C}) \rightarrow K_0(\text{Var}_\mathbb{C})$  is the quotient map extending (1.4). Since the classes  $P_{r,i}$  are effective, combining the decomposition (2.5) with the power structure formula (1.8) and summing over  $n$  proves the result.  $\square$

The following is a generalisation of [1, Thm. 4.1] to arbitrary varieties.

**Corollary 2.4.** *The series  $Z_E(t)$  does not depend on  $E$ . In particular, the identity*

$$[\text{Quot}_X(E, n)] = [\text{Quot}_X(\mathcal{O}_X^{\oplus r}, n)]$$

*holds in  $K_0(\text{Var}_\mathbb{C})$  for all locally free sheaves  $E$  of rank  $r$  on  $X$ .*

**Definition 2.5.** Define absolute classes  $\Omega_{r,n} \in K_0(\text{Var}_\mathbb{C})$  via

$$(2.9) \quad \text{Exp} \left( \sum_{n>0} \Omega_{r,n} t^n \right) = P_r(t).$$

**Remark 2.6.** In terms of the motivic exponential, we can rephrase Equation (2.4) as

$$(2.10) \quad Z_E(t) = \text{Exp} \left( [X] \sum_{n>0} \Omega_{r,n} t^n \right).$$

It is then clear that to determine the series  $Z_E$  one has to compute the fully punctual classes  $\Omega_{r,n}$ . We will do this in the case of curves (for arbitrary  $r$ ) in Section 3.1, and for surfaces (only for  $r = 1$ ) in Section 3.3.

**2.3. Relative motives.** Let  $(X, E)$  be as in the previous sections. Consider the relative motive

$$Z_E^{\text{rel}} = \sum_{n \geq 0} \left[ \text{Quot}_X(E, n) \xrightarrow{\sigma_n} \text{Sym}^n X \right] \in K_0(\text{Var}_{\text{Sym} X}).$$

In other words,  $Z_E^{\text{rel}} = [\text{Quot}_X(E) \rightarrow \text{Sym} X]$ , the class of  $\text{Quot}_X(E) = \coprod_n \text{Quot}_X(E, n)$  over  $\text{Sym} X$ . Note that  $Z_E^{\text{rel}}$  is a refinement of  $Z_E$ , in the sense that

$$\#_! Z_E^{\text{rel}} = Z_E(t),$$

where  $\#_!$  is the operation introduced in (1.10).

We simply write

$$Z^{\text{rel}}(X, r) = Z_{\mathcal{O}^{\oplus r}}^{\text{rel}}$$

when  $E = \mathcal{O}^{\oplus r}$  is the trivial bundle over  $X$ . We will show below (Theorem 2.9) that the relative motive  $Z^{\text{rel}}(\mathbb{A}^d, r) \in K_0(\text{Var}_{\text{Sym} \mathbb{A}^d})$  is generated under  $\text{Exp}_{\cup}$  by the motives  $\Omega_{r,n}$  defined in (2.9), extended on the small diagonal

$$\mathbb{A}^d \xrightarrow{\Delta_n} \text{Sym}^n \mathbb{A}^d.$$

**Example 2.7.** Set  $r = 1, d = 1$  (i.e. we consider line bundles on curves). Then  $\text{Quot}_X(L, n) = \text{Hilb}^n X = \text{Sym}^n X$  for all line bundles  $L$  on  $X$ , and

$$Z^{\text{rel}}(X, 1) = Z_{\mathcal{O}_X}^{\text{rel}} = [\text{Sym} X \xrightarrow{\text{id}} \text{Sym} X] = \mathbb{1} \in K_0(\text{Var}_{\text{Sym} X}).$$

Pushing this forward via  $\#$  yields

$$Z_{\mathcal{O}_X}(t) = \sum_{n \geq 0} [\text{Sym}^n X] t^n = \zeta_X(t),$$

the Kapranov *motivic zeta function* of the curve  $X$ .

**Remark 2.8.** By definition of the power structure and of the motivic exponential, one has

$$\zeta_Y(t) = (1 - t)^{-[Y]} = \text{Exp}([Y]t),$$

for every variety  $Y$ . Moreover, the identities

$$(2.11) \quad \zeta_Y(\mathbb{L}^s t) = \zeta_{\mathbb{A}^s \times Y}(t) = \text{Exp}(\mathbb{L}^s [Y]t)$$

hold in  $K_0(\text{Var}_{\mathbb{C}})[[t]]$  for every  $s \in \mathbb{N}$ .

We now prove Theorem B from the Introduction.

Before we begin, let us observe that for a morphism of varieties  $f: S \rightarrow T$  and an integer  $n > 0$ , there is a commutative diagram

$$(2.12) \quad \begin{array}{ccc} K_0(\text{Var}_S) & \xrightarrow{f_!} & K_0(\text{Var}_T) \\ (\cdot)^{\otimes n} \downarrow & \circ & \downarrow (\cdot)^{\otimes n} \\ \tilde{K}_0^{\mathfrak{S}_n}(\text{Var}_{S^n}) & \xrightarrow{f_!^n} & \tilde{K}_0^{\mathfrak{S}_n}(\text{Var}_{T^n}) \end{array}$$

where  $(\cdot)^{\otimes n}$  is the power map (1.5).

**Theorem 2.9.** *There is an identity*

$$Z^{\text{rel}}(\mathbb{A}^d, r) = \text{Exp}_{\cup} \left( \sum_{n>0} \Omega_{r,n} \boxtimes [\mathbb{A}^d \xrightarrow{\Delta_n} \text{Sym}^n \mathbb{A}^d] \right) \in K_0(\text{Var}_{\text{Sym} \mathbb{A}^d}).$$

*Proof.* For a partition  $\alpha \vdash n$ , set  $Q_{\alpha}^n = \text{Quot}_{\mathbb{A}^d}(\mathcal{O}^{\oplus r}, n)_{\alpha}$ . One has a decomposition

$$Z^{\text{rel}}(\mathbb{A}^d, r) = \sum_{n \geq 0} \sum_{\alpha \vdash n} [Q_{\alpha}^n \rightarrow \text{Sym}^n \mathbb{A}^d].$$

Let us consider the  $G_{\alpha}$ -equivariant cartesian diagram

$$(2.13) \quad \begin{array}{ccc} Z_{\alpha} & \hookrightarrow & \prod_i Q_{(i)}^{\alpha_i} \\ \downarrow & \square & \downarrow \\ \prod_i (\mathbb{A}^d)^{\alpha_i} \setminus \Delta & \xrightarrow{\iota_{\alpha}} & \prod_i (\mathbb{A}^d)^{\alpha_i} \\ \downarrow \Delta & \square & \downarrow \Delta \\ \prod_i \text{Sym}^i(\mathbb{A}^d)^{\alpha_i} \setminus \Delta & \xrightarrow{j_{\alpha}} & \prod_i \text{Sym}^i(\mathbb{A}^d)^{\alpha_i} \end{array}$$

where the top square is Diagram (2.8), the horizontal maps are open immersions (the complements of the big diagonals) and the vertical inclusions are products of small diagonals. We have a base change identity

$$(2.14) \quad j_{\alpha}^* \Delta_i = \Delta_i \iota_{\alpha}^*.$$

On the deepest stratum, we have a commutative diagram

$$\begin{array}{ccc} Q_{(n)}^n & \xrightarrow{\sim} & \mathbb{A}^d \times P_{r,n} \\ \sigma_{(n)} \downarrow & \swarrow \text{pr}_1 & \\ \mathbb{A}^d & & \end{array}$$

inducing an identity

$$(2.15) \quad [Q_{(n)}^n \rightarrow \mathbb{A}^d] = [\mathbb{A}^d \xrightarrow{\text{id}} \mathbb{A}^d] \boxtimes P_{r,n} \in K_0(\text{Var}_{\mathbb{A}^d}).$$

For a general partition  $\alpha$  of  $n$ , consider the equivariant motives

$$[Q_{(i)}^i \rightarrow \mathbb{A}^d]^{\otimes \alpha_i} \in \tilde{K}_0^{\mathfrak{S}_{\alpha_i}}(\text{Var}_{(\mathbb{A}^d)^{\alpha_i}}).$$

If  $\iota_{\alpha}$  is as in Diagram (2.13), one has

$$\Delta_i \left[ Z_{\alpha} \rightarrow \prod_i (\mathbb{A}^d)^{\alpha_i} \setminus \Delta \right] = \Delta_i \iota_{\alpha}^* \left( \boxtimes_{i|\alpha_i \neq 0} [Q_{(i)}^i \rightarrow \mathbb{A}^d]^{\otimes \alpha_i} \right) \in \tilde{K}_0^{G_{\alpha}}(\text{Var}_{\prod_i \text{Sym}^i(\mathbb{A}^d)^{\alpha_i} \setminus \Delta}).$$

Applying the quotient map  $\pi_{G_\alpha}$  to the last identity, followed by the pushforward along the union of points map, we obtain

$$\begin{aligned}
[Q_\alpha^n \rightarrow \text{Sym}^n \mathbb{A}^d] &= \cup_! \pi_{G_\alpha} \Delta_! t_\alpha^* \left( \boxtimes_{i|\alpha_i \neq 0} [Q_{(i)}^i \rightarrow \mathbb{A}^d]^{\otimes \alpha_i} \right) \\
&= \cup_! \pi_{G_\alpha} J_\alpha^* \Delta_! \left( \boxtimes_{i|\alpha_i \neq 0} [Q_{(i)}^i \rightarrow \mathbb{A}^d]^{\otimes \alpha_i} \right) && \text{by (2.14)} \\
&= \cup_! \pi_{G_\alpha} J_\alpha^* \left( \boxtimes_{i|\alpha_i \neq 0} (\Delta_i^{\alpha_i})_! [Q_{(i)}^i \rightarrow \mathbb{A}^d]^{\otimes \alpha_i} \right) && \text{by (1.3)} \\
&= \cup_! \pi_{G_\alpha} J_\alpha^* \left( \boxtimes_{i|\alpha_i \neq 0} (\Delta_i)_! [Q_{(i)}^i \rightarrow \mathbb{A}^d]^{\otimes \alpha_i} \right) && \text{by (2.12)} \\
&= \cup_! \pi_{G_\alpha} J_\alpha^* \left( \boxtimes_{i|\alpha_i \neq 0} \Delta_i! \left( [\mathbb{A}^d \xrightarrow{\text{id}} \mathbb{A}^d] \boxtimes P_{r,i} \right)^{\otimes \alpha_i} \right) && \text{by (2.15)}
\end{aligned}$$

so that summing these classes over all partitions of integers and noting that  $\Omega_{r,n}$  are effective (because  $P_{r,n}$  are effective) yields precisely

$$\text{Exp}_\cup \left( \sum_{n>0} \Omega_{r,n} \boxtimes [\mathbb{A}^d \xrightarrow{\Delta_n} \text{Sym}^n \mathbb{A}^d] \right)$$

by an application of Lemma 1.7. □

**Remark 2.10.** By the last part of Lemma 1.7, the theorem implies the formula

$$Z_{\mathcal{O}^{\text{or}}}(t) = P_r(t)^{\mathbb{L}^d}$$

of Theorem 2.3 for  $X = \mathbb{A}^d$ .

**2.4. Related work on more general Quot schemes.** The theory developed so far relies crucially on the locally free assumption on  $E$ . Indeed, the isomorphisms (2.1) fail even if  $E$  is, say, reflexive but not locally free. However, the geometry of the Quot scheme can be interesting also in the non-locally free case. For instance, the Quot scheme of finite quotients of the ideal sheaf  $\mathcal{I}_C \subset \mathcal{O}_Y$  of a smooth curve in a 3-fold  $Y$  has been studied in [13], where essential local triviality statements on the Quot-to-Chow morphism were proved (see e.g. Corollary 3.2 in *loc. cit.*). Moreover, in [12, Thm. 2.1] it is proven that  $\text{Quot}_Y(\mathcal{I}_C, n)$  appears as the typical (scheme-theoretic) fibre of the Hilbert–Chow morphism  $\text{Hilb}(Y) \rightarrow \text{Chow}(Y)$  in a neighborhood of the cycle of the smooth curve  $C$ . (This holds in all dimensions, not just 3-folds.) This was used to prove the  $C$ -local DT/PT correspondence for Calabi–Yau 3-folds [12, Thm. 1.1]. The (virtual) motivic theory of  $\text{Quot}_Y(\mathcal{I}_C, n)$  was developed in [6].

The enumerative geometry of  $\text{Quot}_X(E, n)$ , for  $E$  a sheaf of homological dimension at most one on a 3-fold, was studied in [2] and related to the local Pandharipande–Thomas theory of  $X$ . The Appendix in *loc. cit.* develops the abstract theory comparing various Quot schemes of smooth quasi-projective varieties, and implicitly shows that the singularities of  $\text{Quot}_X(E, n)$  only depend on  $n$  and  $\dim X$ . The (virtual) motivic theory in the locally free case for 3-folds was developed in [14], along with a construction of a virtual fundamental class on the Quot scheme.

### 3. CALCULATIONS: CURVES AND SURFACES

In this section we compute the fully punctual motives

$$\Omega_{r,n}$$

in the case of curves, for all  $r > 0$  and  $n > 0$ , and in the case of surfaces for  $r = 1$  and all  $n > 0$ .

**3.1. The class of the Quot scheme on a curve.** We fix a locally free sheaf  $E$  of rank  $r$  on a smooth quasi-projective curve  $C$ .

**Lemma 3.1.** *On a curve, we have*

$$\Omega_{r,1} = [\mathbb{P}^{r-1}].$$

*Proof.* By the properties of the power structure, one has

$$\begin{aligned} P_r(t) &= \prod_{n \geq 1} (1 - t^n)^{-\Omega_{r,n}} \\ &= \prod_{n \geq 1} (1 - t)^{-\Omega_{r,n}} \Big|_{t \rightarrow t^n} \\ &= \prod_{n \geq 1} (1 + \Omega_{r,n} t + \cdots) \Big|_{t \rightarrow t^n} \\ &= \prod_{n \geq 1} (1 + \Omega_{r,n} t^n + \cdots), \end{aligned}$$

which immediately implies

$$\Omega_{r,1} = P_{r,1}.$$

On the other hand, the equality  $P_{r,1} = [\mathbb{P}^{r-1}]$  holds by (2.3).  $\square$

We now reformulate (and generalise to the quasi-projective case) the main formula proved in [1, Prop. 4.5]. The following is Theorem C from the Introduction.

**Theorem 3.2.** *There is an identity*

$$(3.1) \quad Z_E(t) = \text{Exp}([C \times \mathbb{P}^{r-1}] t)$$

in  $K_0(\text{Var}_{\mathbb{C}})[[t]]$ . Moreover, in  $K_0(\text{Var}_{\text{Sym} \mathbb{A}^1})$  there is an identity

$$Z^{\text{rel}}(\mathbb{A}^1, r) = \text{Exp}_{\cup}([\mathbb{P}^{r-1}] \boxtimes [\mathbb{A}^1 \xrightarrow{\text{id}} \mathbb{A}^1]).$$

*Proof.* By [1, Prop. 4.5], for projective  $C$  one has

$$(3.2) \quad [\text{Quot}_C(E, n)] = \sum_{n_1 + \cdots + n_r = n} [\text{Sym}^{n_1} C] \cdots [\text{Sym}^{n_r} C] \cdot \mathbb{L}^{\sum_{i=0}^{r-1} (i-1)n_i},$$

and it is clear that the generating function  $Z_E(t)$  of these motives can be expanded as a product of shifted motivic zeta functions. More precisely, one has

$$\begin{aligned} \sum_{n \geq 0} [\text{Quot}_C(E, n)] t^n &= \prod_{i=1}^r \zeta_C(\mathbb{L}^{i-1} t) \\ &= \prod_{i=1}^r \text{Exp}([C] \mathbb{L}^{i-i} \cdot t) \\ &= \text{Exp} \left( [C] \sum_{i=1}^r \mathbb{L}^{i-i} \right) \\ &= \text{Exp}([C \times \mathbb{P}^{r-1}] t), \end{aligned}$$

where the second equality follows by (2.11). So the statement is true when  $C$  is projective. In this case, comparing (3.1) with Equation (2.10) and using the injectivity of  $\text{Exp}$ , we obtain the identities

$$(3.3) \quad [C] \cdot \Omega_{r,n} = \begin{cases} [C] \cdot [\mathbb{P}^{r-1}] & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$$

By Equation (2.10), to prove the statement on an arbitrary  $C$  it is enough to show that

$$(3.4) \quad \Omega_{r,n} = \begin{cases} [\mathbb{P}^{r-1}] & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$$

By Lemma 3.1, we already know that  $\Omega_{r,1} = [\mathbb{P}^{r-1}]$ . Finally, the equation  $\Omega_{r,n} = 0$  holds for  $n > 1$  because  $\Omega_{r,n}$  is effective. Indeed, write  $\Omega_{r,n} = [Y]$  for a variety  $Y$ , so that  $0 = [C] \cdot \Omega_{r,n} = [C \times Y]$ . But the class of a variety vanishes if and only if the variety is empty, and this happens if and only if  $Y = \emptyset$ .

To prove the last assertion, it is enough to combine Theorem 2.9 with the relations (3.4).  $\square$

**Remark 3.3.** The formula (3.2) is proved in [1] over a field  $k$  of arbitrary characteristic.

**Remark 3.4.** By Equation (3.4), the generating function of the punctual motives can be computed as

$$(3.5) \quad P_r(t) = \text{Exp}([\mathbb{P}^{r-1}])t = \zeta_{\mathbb{P}^{r-1}}(t) = \prod_{i=0}^{r-1} \frac{1}{1 - \mathbb{L}^i t}.$$

**3.2. The Hodge numbers of the Quot scheme on a curve.** The Hodge–Deligne polynomial (also called the E-polynomial) of a smooth complex projective variety  $Y$  is given by

$$E(Y; u, v) = \sum_{p,q} (-1)^{p+q} h^{p,q}(Y) u^p v^q$$

where  $h^{p,q}(Y) = \dim_{\mathbb{C}} H^q(Y, \Omega_Y^p)$  are the Hodge numbers of  $Y$ . For instance, one has

$$(3.6) \quad \begin{aligned} E(\mathbb{P}^{r-1}; u, v) &= \sum_{i=0}^{r-1} u^i v^i, \\ E(C; u, v) &= 1 - gu - gv + uv, \end{aligned}$$

where  $C$  is a smooth projective curve of genus  $g$ . Sending  $[Y] \mapsto E(Y; u, v)$  defines a motivic measure

$$K_0(\text{Var}_{\mathbb{C}}) \xrightarrow{E} \mathbb{Z}[u, v]$$

which is in fact a homomorphism of rings with power structure. The power structure on the polynomial ring  $\mathbb{Z}[u, v]$  is determined by the formula

$$(1-t)^{-f(u,v)} = \prod_{i,j} (1 - u^i v^j t)^{-p_{ij}},$$

where we have written  $f(u, v) = \sum_{i,j} p_{ij} u^i v^j$  for integers  $p_{ij}$ . This implies (cf. [10, Prop. 4]) the basic relation

$$(3.7) \quad E((1-t)^{-[Y]}) = (1-t)^{-E(Y;u,v)}.$$



Let  $C$  be a smooth projective curve of genus  $g$ , and let  $E$  be a rank  $r$  locally free sheaf on  $C$ . We compute the generating function

$$E_r(C, t) = \sum_{n \geq 0} E(\text{Quot}_C(E, n); u, v) t^n.$$

We already know this series does not depend on  $E$ .

**Proposition 3.5.** *There is an identity*

$$(3.8) \quad E_r(C, t) = \prod_{i=0}^{r-1} \frac{(1 - u^i v^{i+1} t)^g (1 - u^{i+1} v^i t)^g}{(1 - u^i v^i t)(1 - u^{i+1} v^{i+1} t)}$$

in the ring  $\mathbb{Z}[u, v][[t]]$ .

*Proof.* We have

$$\begin{aligned} E_r(C, t) &= E(\text{Exp}([C \times \mathbb{P}^{r-1}]t)) && \text{by (3.1)} \\ &= E((1-t)^{-[C \times \mathbb{P}^{r-1}]}) && \text{by definition of Exp} \\ &= (1-t)^{-E([C \times \mathbb{P}^{r-1}])} && \text{by (3.7)} \\ &= (1-t)^{-(1-gu-gv+uv)\sum_{i=0}^{r-1} u^i v^i}. \end{aligned}$$

We have used that  $E$  is a ring homomorphism and the identities (3.6) in the last step. The result now follows from direct computation and by definition of the power structure on  $\mathbb{Z}[u, v]$ .  $\square$

**Remark 3.6.** Setting  $u = v$  in Formula (3.8) one recovers the generating function of (signed) Poincaré polynomials computed in [1, Remark 4.6], namely

$$\sum_{n \geq 0} P(\text{Quot}_C(F, n), -u) t^n = \prod_{i=0}^{r-1} \frac{(1 - u^{2i+1} t)^{2g}}{(1 - u^{2i} t)(1 - u^{2i+2} t)}.$$

**3.3. The Hilbert scheme of points on a surface.** Let  $S$  be a smooth quasi-projective surface, and set  $r = 1$ , so that  $\text{Quot}_S(L, n) = \text{Hilb}^n S$  for every line bundle  $L$ . We know by Formula (2.10) that

$$Z_{\mathcal{O}_S}(t) = \text{Exp} \left( [S] \sum_{n > 0} \Omega_{1,n} t^n \right).$$

On the other hand, by Göttsche's formula [7],

$$Z_{\mathcal{O}_S}(t) = \text{Exp} \left( \frac{[S]t}{1 - \mathbb{L}t} \right) = \text{Exp} \left( [S] \sum_{n > 0} \mathbb{L}^{n-1} t^n \right).$$

By the injectivity of  $\text{Exp}$ , we conclude that on a surface  $S$  one has

$$(\Omega_{1,n} - \mathbb{L}^{n-1})[S] = 0.$$

However, this relation holds universally for *every* quasi-projective surface, in particular for  $S = \mathbb{P}^2$  and  $S = \mathbb{A}^1 \times \mathbb{P}^1$ . Therefore

$$(\Omega_{1,n} - \mathbb{L}^{n-1})(1 + \mathbb{L} + \mathbb{L}^2) = 0 = (\Omega_{1,n} - \mathbb{L}^{n-1})(\mathbb{L} + \mathbb{L}^2),$$

showing that

$$(3.9) \quad \Omega_{1,n} = \mathbb{L}^{n-1}, \quad n > 0.$$

In particular, we recover the known generating function of the motives of punctual Hilbert schemes, given by the formula

$$(3.10) \quad P_1(t) = \sum_{n \geq 0} [\text{Hilb}^n(\mathbb{A}^2)_0] t^n = \prod_{n \geq 1} (1 - \mathbb{L}^{n-1} t^n)^{-1}.$$

Finally, we obtain the following relative statement.

**Theorem 3.7.** *There is an identity*

$$\sum_{n \geq 0} [\text{Hilb}^n \mathbb{A}^2 \xrightarrow{\sigma_n} \text{Sym}^n \mathbb{A}^2] = \text{Exp}_{\cup} \left( \sum_{n > 0} \mathbb{L}^{n-1} \boxtimes [\mathbb{A}^2 \xrightarrow{\Delta_n} \text{Sym}^n \mathbb{A}^2] \right)$$

in  $K_0(\text{Var}_{\text{Sym} \mathbb{A}^2})$ .

*Proof.* Combine Theorem 2.9 with Equation (3.9). □

**Remark 3.8.** The relation (3.10) was already proved in [9], and it was exploited in [11] to provide a motivic check of the classification of modules of length 3 and 4 over the polynomial ring  $k[x, y]$ .

#### 4. A MOTIVIC-TO-GEOMETRIC OPEN PROBLEM

Let  $C$  be a smooth quasi-projective curve. The punctual Quot scheme

$$P_{r,n} \subset \text{Quot}_C(\mathcal{O}^{\oplus r}, n)$$

parameterises quotients  $\mathcal{O}^{\oplus r} \rightarrow Q$  entirely supported at a single (fixed) point  $p \in C$ . As proved in [1, Prop. 2.6], one has

$$P_{r,1} = \mathbb{P}^{r-1}.$$

How can one describe  $P_{r,n}$  for  $n > 1$ ? The relation

$$P_r(t) = \text{Exp}([\mathbb{P}^{r-1}]t) = \zeta_{\mathbb{P}^{r-1}}(t)$$

established in Equation (3.5) translates into the motivic identity

$$(4.1) \quad [P_{r,n}] = [\text{Sym}^n \mathbb{P}^{r-1}] = [\text{Sym}^n P_{r,1}].$$

It thus makes sense to ask the following:

**Question 4.1.** What is the geometric meaning of the relation (4.1)? Can one geometrically compare the schemes  $P_{r,n}$  and  $\text{Sym}^n P_{r,1}$ ?

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