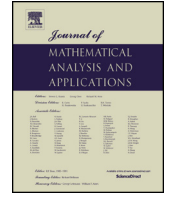




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## Regular Articles

# Approximation and interpolation in Kolmogorov-type groups

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### ABSTRACT

We prove a real interpolation characterization for some non Euclidean Hölder spaces, built on the Lie structure induced by a class of ultra-parabolic Kolmogorov-type operators satisfying the Hörmander condition. As a by-product we also obtain an approximation property for intrinsically regular functions on the whole space.

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## 1. Introduction

### 1.1. Statement of the problem

We consider the problems of approximation and interpolation for some non Euclidean Hölder spaces built on the intrinsic geometry of the following Kolmogorov operator

$$\mathcal{K} = \frac{1}{2} \sum_{i=1}^{p_0} \partial_{x_i x_i} + Y, \quad Y = \langle Bx, \nabla \rangle + \partial_t, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d, \quad (1.1)$$

where  $1 \leq p_0 \leq d$  and  $B$  is a  $d \times d$  constant matrix. If  $p_0 = d$  then  $\mathcal{K}$  is a parabolic operator while in general, for  $p_0 < d$ ,  $\mathcal{K}$  is degenerate, and not uniformly parabolic. However,  $\mathcal{K}$  can be a hypoelliptic operator, provided some structural assumptions on the matrix  $B$  are verified. This kind of PDEs appears in many linear and non-linear models in physics (see, for instance, [4], [8]) and in mathematical finance (see, for instance, [2], [7]).

In [15], Lanconelli and Polidoro first studied the non-Euclidean intrinsic Lie group structure induced by the vector fields  $\partial_{x_1}, \dots, \partial_{x_{p_0}}, Y$ , and from there on, many authors carried the study of general Kolmogorov operators in the framework of the theory of homogeneous groups. We recall, among many, [9], who proved

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the existence of a fundamental solution under optimal regularity assumptions, [10], [12] and [21], who proved Schauder type estimates for the associated Cauchy and Dirichlet problems, [13], who considered the case of integro-differential diffusions. One basic tool in the study of regularity properties of solutions to Kolmogorov operators with variable coefficients is the notion of Hölder spaces which are *intrinsic* to the aforementioned geometric structure. Over the years, different notions of Hölder classes have been proposed by several authors, based on the Carnot-Carathéodory distance associated to the group (see, among many, [10], [21], [11]). Eventually Pagliarani et al. in [22] established an optimal definition of  $C_B^{n,\alpha}$  for any  $n \in \mathbb{N}_0$ , which is given in terms of the regularity in the directions of the vector fields  $\partial_{x_1}, \dots, \partial_{x_{p_0}}$  and  $Y$  with their different formal degrees: for instance, the case  $n = 0$  consists of functions that are  $C^\alpha$  continuous in the directions  $\partial_{x_1}, \dots, \partial_{x_{p_0}}$  and  $C^{\frac{\alpha}{2}}$  in  $Y$ . With this definition they managed to prove a Taylor-type formula with related estimates of the remainder, given in terms of the intrinsic distance.

Our main result, Theorem 1.6, shows that these intrinsic spaces interpolate well with each other through the so called *real method*, namely that

$$(C_B^{n_1, \alpha_1}, C_B^{n_2, \alpha_2})_{\theta, \infty} = C_B^{n, \alpha}, \quad (1.2)$$

for any  $n_1, n_2 \in \mathbb{N}_0$ ,  $\alpha_1, \alpha_2 \in [0, 1]$ , for some expected parameters  $n \in \mathbb{N}_0$ ,  $\alpha \in (0, 1)$ . In the Euclidean case, this identification is well known (see [20], [26]), and can be proved by means of the general theory of semigroups. Similar arguments can be adapted for a different class of Hölder spaces also associated to degenerate elliptic and parabolic differential equations, the so called *anisotropic* Hölder spaces, which, unlike the intrinsic spaces, only refer to the spatial variables (see Section 1.3 for a more in-depth discussion). Here we choose to pursue a constructive approach based on explicit approximations of intrinsically regular functions (Theorem 3.1), which can possibly be of independent interest. This approach is more or less inspired by the approximation theory for Sobolev functions (see [1], Chapter 5), it does not rely on the theory of semigroups, but instead, it heavily exploits the aforementioned intrinsic Taylor formula.

Classically, characterizations of type (1.2) have remarkable applications in the study of optimal regularity in Hölder classes for linear elliptic and parabolic differential operators (see [14], [20], [18], [19]) as well as approximation theory (see [3]). In the context of Kolmogorov operators, many works already exploit some kind of interpolation inequalities to obtain different versions of Schauder estimates in different settings (see, for instance [12], [10] and [21] among others, see also [17] for the most recent global estimate for the evolution problem, as well as a complete overview of the existing literature). Let us also mention that our results, together with the embeddings of intrinsic Sobolev spaces of integer order in [24] could be applied to deduce Morrey-type embeddings for suitable notions of Sobolev-Slobodeckij and Besov intrinsic spaces as well, which, as far as our understanding, are yet to be explored in the literature in generality. Such studies will be subjects of future research. For the application of interpolation results to fractional embeddings in the Euclidean setting, see for instance, the references [1] or [25].

The paper is organized as follows: in the remaining part of this Section, we specify the precise functional framework, state the structural hypothesis on the matrix  $B$  and our main interpolation results, Proposition 1.5 and Theorem 1.6, and finally discuss the previous literature on the subject and justify our approach. In Section 2 we collect some results that are preliminary to the subsequent proofs; in Section 3 we prove the core result of this work, the approximation property for intrinsically regular functions, which eventually leads to the proof of our main results in Section 4.

## 1.2. Hölder spaces and real interpolation

We start by recalling the Lie group structure that is naturally associated to the differential operator (1.1). In [15] the authors observed that  $\mathcal{K}$  is invariant with respect to the left translations of the group  $(\mathbb{R}^{d+1}, \circ)$ , where the non-commutative group law is defined by

$$(t, x) \circ (s, \xi) = \ell_{(t,x)}(s, \xi) := (s + t, \xi + e^{sB}x), \quad (t, x), (s, \xi) \in \mathbb{R}^{d+1}. \tag{1.3}$$

Precisely,  $\ell_{(t,x)} \circ \mathcal{K} = \mathcal{K} \circ \ell_{(t,x)}$  for any  $(t, x) \in \mathbb{R}^{d+1}$ . Notice that  $(\mathbb{R}^{d+1}, \circ)$  is a group with identity  $\text{Id} = (0, 0)$  and inverse  $(t, x)^{-1} = (-t, e^{-tB}x)$ .

The authors also proved that the operator  $\mathcal{K}$  is hypoelliptic if and only if, on a certain basis of  $\mathbb{R}^{d+1}$  the matrix  $B$  takes the form

$$B = \begin{pmatrix} * & * & \cdots & * & * \\ \mathbf{B}_1 & * & \cdots & * & * \\ 0_{p_2 \times p_0} & \mathbf{B}_2 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{p_r \times p_0} & 0_{p_r \times p_1} & \cdots & \mathbf{B}_r & * \end{pmatrix}, \tag{1.4}$$

where  $\mathbf{B}_j$  is a  $p_j \times p_{j-1}$  matrix of rank  $p_j$ , with

$$p_0 \geq p_1 \geq \dots \geq p_r \geq 1, \quad \sum_{j=1}^r p_j = d,$$

$0_{p_i \times p_j}$  is a  $p_i \times p_j$  null matrix and the  $*$ -blocks are arbitrary. Throughout the paper we assume the following

**Assumption 1.1.**  $B$  is a constant  $d \times d$  matrix as in (1.4), where each  $*$ -block is null.

If (and only if) Assumption 1.1 holds, then  $\mathcal{K}$  is homogeneous of degree two with respect to the family of dilations  $(D_\lambda)_{\lambda>0}$  on  $\mathbb{R}^{d+1}$  given by

$$D_\lambda = (\lambda^2, \lambda I_{p_0}, \lambda^3 I_{p_1}, \dots, \lambda^{2r+1} I_{p_r}),$$

where  $I_{p_j}$  are  $p_j \times p_j$  identity matrices, i.e.  $(\mathcal{K}D_\lambda u)(t, x) = \lambda^2(\mathcal{K}u)(D_\lambda(t, x))$ , for any  $(t, x) \in \mathbb{R}^{d+1}$ ,  $\lambda > 0$ . In this case, the matrix  $B$  uniquely identifies the homogeneous Lie group  $\mathcal{G}_B := (\mathbb{R}^{d+1}, \circ, D(\lambda))$ . We define the  $D(\lambda)$ -homogeneous quasi-norm on  $\mathcal{G}_B$  as follows: let

$$\bar{p}_j = p_0 + \dots + p_j, \quad j = 0, \dots, r, \tag{1.5}$$

and  $\bar{p}_{-1} \equiv 0$ . Then

$$\|(t, x)\|_B := |t|^{\frac{1}{2}} + |x|_B, \quad |x|_B := \sum_{j=0}^r \sum_{i=\bar{p}_{j-1}+1}^{\bar{p}_j} |x_i|^{\frac{1}{2j+1}}. \tag{1.6}$$

Next we recall the notions of  $B$ -intrinsic regularity and  $B$ -intrinsic Hölder space introduced in [22]. We start with some useful notations: for any  $(t, x) \in \mathbb{R}^{d+1}$ ,  $i = 1, \dots, p_0$  we denote

$$e^{\delta \partial_{x_i}}(t, x) = (t, x + \delta e_i), \quad e^{\delta Y}(t, x) = (t + \delta, e^{\delta B}x), \quad \delta \in \mathbb{R}, \tag{1.7}$$

the (unique) integral curves of the vector fields  $\partial_{x_i}$  and  $Y$  respectively.

**Definition 1.2.** Let  $X \in \{\partial_{x_1}, \dots, \partial_{x_{p_0}}, Y\}$  and  $\alpha \in ]0, 1]$ . We say that  $u \in C_X^\alpha$  if

$$[u]_{C_X^\alpha} := \sup_{\substack{z \in \mathbb{R}^{d+1} \\ \delta \in \mathbb{R} \setminus \{0\}}} \frac{|u(e^{\delta X}z) - u(z)|}{|\delta|^\alpha} < \infty. \tag{1.8}$$

We say that  $u$  is  $X$ -differentiable in  $z$  if the function  $\delta \mapsto u(e^{\delta X} z)$  is differentiable in 0 and refer to the function  $z \mapsto Xu(z) := \frac{d}{d\delta} u(e^{\delta X} z) |_{\delta=0}$  as the  $X$ -Lie derivative of  $u$ .

**Definition 1.3.** Let  $\alpha \in [0, 1]$ , then:

- i) If  $\alpha = 0$ ,  $C_B^{0,0} \equiv C(\mathbb{R}^{d+1})$  is the space of bounded and continuous function on  $\mathbb{R}^{d+1}$ , endowed with the sup-norm

$$\|u\|_{C_B^{0,0}} := |u|_\infty.$$

If  $\alpha \in ]0, 1]$  we say that  $u \in C_B^{0,\alpha}$  if  $u \in C_Y^{\frac{\alpha}{2}}$  and  $u \in C_{\partial x_i}^\alpha$  for any  $i = 1, \dots, p_0$ , with norm

$$\|u\|_{C_B^{0,\alpha}} := |u|_\infty + [u]_{C_Y^{\frac{\alpha}{2}}} + \sum_{i=1}^{p_0} [u]_{C_{\partial x_i}^\alpha} < \infty, \quad \alpha \in ]0, 1].$$

- ii)  $u \in C_B^{1,\alpha}$  if  $u \in C_Y^{\frac{1+\alpha}{2}}$  and  $\partial_{x_i} u \in C_B^\alpha$  for any  $i = 1, \dots, p_0$ , with norm

$$\|u\|_{C_B^{1,\alpha}} := |u|_\infty + [u]_{C_Y^{\frac{1+\alpha}{2}}} + \sum_{i=1}^{p_0} \|\partial_{x_i} u\|_{C_B^{0,\alpha}} < \infty.$$

- iii) for  $n \in \mathbb{N}$  with  $n \geq 2$ ,  $u \in C_B^{n,\alpha}$  if  $Yu \in C_B^{n-2,\alpha}$  and  $\partial_{x_i} u \in C_B^{n-1,\alpha}$  for any  $i = 1, \dots, p_0$ , with norm

$$\|u\|_{C_B^n} := |u|_\infty + \|Yu\|_{C_B^{n-2,\alpha}} + \sum_{i=1}^{p_0} \|\partial_{x_i} u\|_{C_B^{n-1,\alpha}} < \infty.$$

**Remark 1.4.**  $C_B^{n,\alpha}$  is a Banach space. Notice also that  $C_B^{n,\alpha} \subseteq C_B^{n',\alpha'}$  for  $0 \leq n \leq n'$  and  $0 \leq \alpha' \leq \alpha \leq 1$ .

For any two real Banach spaces  $Z_1, Z_2$ , by  $Z_1 = Z_2$  we mean that  $Z_1$  and  $Z_2$  have the same elements with equivalent norms and by  $Z_1 \subseteq Z_2$  we mean that  $Z_1$  is continuously embedded in  $Z_2$ . We recall that the couple  $\{Z_1, Z_2\}$  is called *interpolation couple* if both  $Z_1$  and  $Z_2$  are continuously embedded in some Hausdorff topological vector space  $\mathcal{Z}$ ; in this case, the intersection  $Z_1 \cap Z_2$  and the sum  $Z_1 + Z_2 = \{u_1 + u_2, u_1 \in Z_1, u_2 \in Z_2\}$ , endowed with the norms

$$\|u\|_{Z_1 \cap Z_2} := \max\{\|u\|_{Z_1}, \|u\|_{Z_2}\}, \quad \|u\|_{Z_1 + Z_2} := \inf\{\|u_1\|_{Z_1} + \|u_2\|_{Z_2}, u_1 \in Z_1, u_2 \in Z_2\},$$

are Banach spaces. Moreover, any Banach space  $E$  such that

$$Z_1 \cap Z_2 \subseteq E \subseteq Z_1 + Z_2, \tag{1.9}$$

is called an *intermediate space*. Among these, we have the *real interpolation space*  $(Z_1, Z_2)_{\theta, \infty}$ , which can be defined, for any  $\theta \in (0, 1)$  (see, for instance [20]), as

$$(Z_1, Z_2)_{\theta, \infty} := \{u \in Z_1 + Z_2, \|u\|_{(Z_1, Z_2)_{\theta, \infty}} := \sup_{\lambda \geq 0} \lambda^{-\theta} K(\lambda, u; Z_1, Z_2) < \infty\} \tag{1.10}$$

where

$$K(\lambda, u; Z_1, Z_2) := \inf \{\|u_1\|_{Z_1} + \lambda \|u_2\|_{Z_2}, u = u_1 + u_2, u_1 \in Z_1, u_2 \in Z_2\}.$$

Our first results states that any space  $C_B^{n,0}$  belongs to a special class of *intermediate spaces* between the larger space  $C_B^{n_1,0}$ ,  $n_1 < n$  and a smaller space  $C_B^{n_2,0}$ ,  $n < n_2$ , with  $n, n_1, n_2 \in \mathbb{N}_0$ .

**Proposition 1.5.** *Let  $n, n_1, n_2 \in \mathbb{N}_0$ ,  $n_1 < n < n_2$ .*

i) *There exists a constant  $c_1$  such that, for any  $u \in C_B^{n_2,0}$ , we have*

$$\|u\|_{C_B^{n,0}} \leq c_1 \|u\|_{C_B^{n_1,0}}^{\frac{n_2-n}{n_2-n_1}} \|u\|_{C_B^{n_2,0}}^{\frac{n-n_1}{n_2-n_1}}; \tag{1.11}$$

i) *There exists a constant  $c_2$  such that, for any  $u \in C_B^{n,0}$ , we have*

$$K(\lambda, u; C_B^{n_1,0}, C_B^{n_2,0}) \leq c_2 \lambda^{\frac{n-n_1}{n_2-n_1}} \|u\|_{C_B^{n,0}}, \quad \lambda > 0. \tag{1.12}$$

*Equivalently (see, for instance [20], Proposition 1.20) it holds that*

$$\left(C_B^{n_1,0}, C_B^{n_2,0}\right)_{\frac{n-n_1}{n_2-n_1}, 1} \subseteq C_B^{n,0} \subseteq \left(C_B^{n_1,0}, C_B^{n_2,0}\right)_{\frac{n-n_1}{n_2-n_1}, \infty}. \tag{1.13}$$

Next, our main result, gives a characterization of  $B$ -Hölder spaces of any order as interpolation spaces.

**Theorem 1.6.** *Let  $n_1, n_2 \in \mathbb{N}_0$  and  $\alpha_1, \alpha_2 \in [0, 1]$ , with  $n_1 + \alpha_1 \leq n_2 + \alpha_2$ , and let  $\theta \in (0, 1)$ . Let  $n \in \mathbb{N}_0$ ,  $n_1 \leq n \leq n_2$  such that*

$$\frac{n - (n_1 + \alpha_1)}{(n_2 + \alpha_2) - (n_1 + \alpha_1)} < \theta < \frac{n + 1 - (n_1 + \alpha_1)}{(n_2 + \alpha_2) - (n_1 + \alpha_1)}. \tag{1.14}$$

*If  $\alpha := (n_1 + \alpha_1) + \theta[n_2 + \alpha_2 - (n_1 + \alpha_1)] - n \neq 0, 1$  then it holds that*

$$\left(C_B^{n_1, \alpha_1}, C_B^{n_2, \alpha_2}\right)_{\theta, \infty} = C_B^{n, \alpha} \tag{1.15}$$

**Remark 1.7.** If  $\Omega$  is an open set of  $\mathbb{R}^{d+1}$ , one can define the Hölder spaces  $C_B^{n, \alpha}(\Omega)$ ,  $n \in \mathbb{N}_0$ ,  $\alpha \in (0, 1)$  as in the case  $\Omega = \mathbb{R}^{d+1}$ , where in particular, for  $X \in \{\partial_{x_1}, \dots, \partial_{x_{p_0}}, Y\}$ ,

$$[u]_{C_B^\alpha(\Omega)} := \sup \left\{ \frac{|u(e^{\delta X} z) - u(z)|}{|\delta|^\alpha}, \quad z \in \Omega, \delta \in \mathbb{R} \setminus \{0\} \text{ s.t. } e^{\delta' X} z \in \Omega, |\delta'| < |\delta| \right\}.$$

Then it is likely that our results also hold for the spaces  $C_B^{n, \alpha}(\Omega)$ , provided that the domain  $\Omega$  is smooth enough (see [26], Theorem 1.2.4 and Section 4.5). The construction of the appropriate extension operators under optimal regularity assumptions for  $\Omega$  is outside the scope of the present paper and is postponed to future research.

**Remark 1.8.** Notice that the assumption  $n_1 + \alpha_1 \leq n_2 + \alpha_2$  is not restrictive. Indeed we can always write

$$\left(C_B^{n_1, \alpha_1}, C_B^{n_2, \alpha_2}\right)_{\theta, \infty} = \left(C_B^{n_2, \alpha_2}, C_B^{n_1, \alpha_1}\right)_{1-\theta, \infty}.$$

### 1.3. Related works and strategy of the proof

As already mentioned, in the Euclidean case, the interpolation identity (1.2) can be proved by means of the theory of semigroups ([20], [26]): let  $C, C^1$  denote the spaces of bounded continuous functions and

bounded continuously differentiable functions with bounded derivatives respectively, and let  $C^\theta$  denote the space of bounded and uniformly  $\theta$ -Hölder continuous functions, all endowed with the usual norms. Roughly speaking, denoting with  $T_i$  the translation semigroup of the  $i$ -th variable, and  $\mathcal{D}(\partial_{x_i}) = \{f \in C, \exists \partial_{x_i} f \in C\}$ , we have

$$(C, C^1)_{\theta,p} = \left( C, \bigcap_{i=1}^d \mathcal{D}(\partial_{x_i}) \right)_{\theta,p} = \bigcap_{i=1}^d \left( C, \mathcal{D}(\partial_{x_i}) \right)_{\theta,p}, \quad (1.16)$$

for any  $p \geq 1$ , where

$$(C, \mathcal{D}(\partial_{x_i}))_{\theta,\infty} = \{f \in C, t \mapsto \phi(t) := t^{-\theta} \|T_i(t)f - f\|_\infty \in L^\infty(0, \infty)\}, \quad (1.17)$$

that is the space of  $\theta$ -Hölder continuous functions in the  $i$ -th variable. Therefore  $(C, C^1)_{\theta,\infty} = C^\theta$ . In particular (1.17) holds more generally for  $p \geq 1$ , and substituting  $T_i$  and  $\partial_{x_i}$  with any bounded semigroup  $T$  and its infinitesimal generator on a Banach space, while the second equality in (1.16) precisely holds because the semigroups  $T_i$ ,  $i = 1, \dots, d$  commute.

In [19], Lunardi adapted these arguments to derive an interpolation result for anisotropic Hölder spaces associated with operators of the type (1.1), which are spaces of functions on  $\mathbb{R}^d$  (only the spatial variables are considered) defined with respect to the metric induced by  $|\cdot|_B$  (cf. (1.6)). For instance, let us consider the degenerate prototype of (1.1), that is the Langevin operator

$$\mathcal{K} = \frac{1}{2} \partial_{x_1 x_1} + Y, \quad Y = x_1 \partial_{x_2} + \partial_t,$$

which corresponds to the matrix

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

In this case the anisotropic Hölder regularity in the space variables reads:

$$|u(x) - u(\xi)| \leq C |x - \xi|_B^\alpha \sim |x_1 - \xi_1|^\alpha + |x_2 - \xi_2|^{\frac{\alpha}{3}}, \quad x, \xi \in \mathbb{R}^2. \quad (1.18)$$

A regularity theory for equations with variable coefficients, built on this functional setting, is useful when one needs to consider coefficients or data that are irregular in time (see also [5], [23], [6], [16] for recent developments). This approach leads to mild or weak/distributional solutions also because it cannot benefit from any regularizing effect of the semigroup in time.

On the other hand, for  $B$  as above, a function  $u \in C_B^{0,\alpha}$  satisfies (see Theorem 2.4 in the next Section), for any  $(t, x), (s, \xi) \in \mathbb{R}^3$ ,

$$|u(t, x) - u(s, \xi)| \leq C \|(s, \xi)^{-1} \circ (t, x)\|_B^\alpha \sim |t - s|^{\frac{\alpha}{2}} + |x_1 - \xi_1|^\alpha + |x_2 - \xi_2 - (t - s)\xi_1|^{\frac{\alpha}{3}}.$$

Notice that for points that are on the same time level, the increments are controlled as for (1.18); otherwise, as opposed to the standard parabolic case, the regularity in the euclidean directions is somehow entangled due to the group law associated to  $B$ . This functional setting, which exploits some regularity in the full set of variables, is more suited to obtain classical solutions (see [10], [21] and the references therein). To the best of our knowledge this is the first attempt at obtaining a general interpolation characterization for the spaces  $C_B^{m,\alpha}$ . For this purpose, notice that while it is true that

$$(C, \mathcal{D}(Y))_{\theta,\infty} = \{f \in C, t \mapsto \phi(t) := t^{-\theta} \|e^{tY} f - f\|_\infty \in L^\infty(0, \infty)\} = C_Y^\theta,$$

since the vector fields  $\partial_{x_1}$  and  $Y$  don't commute, it's not at all clear whether

$$(C, \mathcal{D}(\partial_{x_1}^2) \cap \mathcal{D}(Y))_{\frac{\theta}{2}, \infty} = (C, \mathcal{D}(\partial_{x_1}^2))_{\frac{\theta}{2}, \infty} \cap (C, \mathcal{D}(Y))_{\frac{\theta}{2}, \infty}. \tag{1.19}$$

This makes it difficult to pursue an approach based on semigroup theory. Instead of trying to prove (1.19) we follow a constructive approach based on approximation: for instance, for any  $u \in C_B^{0,\alpha}$  we can define

$$u_\varepsilon(z) = \int_{\mathbb{R}^{d+1}} u(\zeta) \phi(D_{\varepsilon^{-1}}(\zeta^{-1} \circ z)) \frac{d\zeta}{\varepsilon^{Q+2}}, \quad z \in \mathbb{R}^{d+1},$$

where  $\phi$  is a test function with unit integral, supported on  $\|y\|_B \leq 1$ . This approximation complies with the Lie structure induced by  $B$ , and indeed we prove the following expected controls

$$|u - u_\varepsilon|_\infty \leq c\varepsilon^\alpha \|u\|_{C_B^{0,\alpha}}, \quad \|u_\varepsilon\|_{C_B^{1,0}} \leq c\varepsilon^{\alpha-1} \|u\|_{C_B^{0,\alpha}}, \tag{1.20}$$

so that

$$\varepsilon^{-\alpha} K(\varepsilon, u; C, C_B^{1,0}) \leq \varepsilon^{-\alpha} (|u - u_\varepsilon|_\infty + \varepsilon \|u_\varepsilon\|_{C_B^{1,0}}) \leq 2c \|u\|_{C_B^{0,\alpha}}.$$

This way of reasoning is fairly standard (see, for instance, Chapter 1 in [20]), though the estimates (1.20) are new for the intrinsic spaces and we present a generalization for any order  $n \in \mathbb{N}_0$  in Theorem 3.1.

## 2. Preliminaries

Here we collect some remarks and known preliminary results which will play a central role for the next Sections.

**Remark 2.1.** Under Assumption 1.1 the matrix  $B$  is a nilpotent matrix of degree  $r + 1$ . In particular

$$e^{\delta B} = I_d + \sum_{j=1}^r \frac{B^j}{j!} \delta^j,$$

is a lower triangular matrix with diagonal  $(1, \dots, 1)$  and therefore it has determinant equal to 1. Indeed, by Assumption 1.1, a direct computation shows that, for any  $n \leq r$  (recalling the notation (1.5) for  $\bar{p}_j$ ) we have

$$B^n = \begin{pmatrix} 0_{\bar{p}_{n-1} \times p_0} & 0_{\bar{p}_{n-1} \times p_1} & \cdots & 0_{\bar{p}_{n-1} \times p_{r-n}} & 0_{\bar{p}_{n-1} \times (\bar{p}_r - \bar{p}_{r-n})} \\ \prod_{j=1}^n \mathbf{B}_j & 0_{p_n \times p_1} & \cdots & 0_{p_n \times p_{r-n}} & 0_{p_n \times (\bar{p}_r - \bar{p}_{r-n})} \\ 0_{p_{n+1} \times p_0} & \prod_{j=2}^{n+1} \mathbf{B}_j & \cdots & 0_{p_{n+1} \times p_{r-n}} & 0_{p_{n+1} \times (\bar{p}_r - \bar{p}_{r-n})} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{p_r \times p_0} & 0_{p_r \times p_1} & \cdots & \prod_{j=r-n+1}^r \mathbf{B}_j & 0_{p_r \times (\bar{p}_r - \bar{p}_{r-n})} \end{pmatrix}, \quad \prod_{j=1}^n \mathbf{B}_j = \mathbf{B}_n \mathbf{B}_{n-1} \cdots \mathbf{B}_1, \tag{2.1}$$

and  $B^n = 0$  for  $n > r$ .

**Remark 2.2.** A simple computation shows that, for any  $z, y \in \mathbb{R}^{d+1}$ ,

$$y^{-1} \circ e^{\delta Y} z = e^{\delta Y} (y^{-1} \circ z), \quad y^{-1} \circ e^{\delta \partial_{x_i}} z = e^{\delta \partial_{x_i}} (y^{-1} \circ z), \quad i = 1, \dots, d, \tag{2.2}$$

which is a way to restate the invariance of operator (1.1) w.r.t. the group law.

**Lemma 2.3.** *The following identities hold:*

$$D_\lambda e^{\delta Y}(z) = e^{\delta \lambda^2 Y}(D_\lambda z), \quad D_\lambda e^{\delta \partial_{x_i}}(z) = e^{\delta \lambda^{2j+1} \partial_{x_i}} D_\lambda z, \quad i = \bar{p}_{j-1} + 1, \dots, \bar{p}_j, \quad (2.3)$$

for any  $z \in \mathbb{R}^{d+1}$  and  $\delta, \lambda \in \mathbb{R}$ .

**Proof.** Let  $z = (t, x)$ , and let  $D_\lambda^0$  be the spatial part of the dilation operator  $D_\lambda$ , namely  $D_\lambda^0 = (\lambda I_{p_0}, \lambda^3 I_{p_1}, \dots, \lambda^{2r+1} I_{p_r})$ . Denoting with  $x_{\{i\}}$ ,  $i = 0, \dots, r$ , the projection of  $x$  on  $\mathbb{R}^{p_i}$  defined by

$$x_{\{i\}} := (x_{\bar{p}_{i-1}+1}, \dots, x_{\bar{p}_i}),$$

by (2.1) we can write

$$(e^{\delta B} x)_{\{i\}} = x_{\{i\}} + \sum_{j=0}^{i-1} \delta^{i-j} \left( \prod_{k=j+1}^i \mathbf{B}_k \right) x_{\{j\}}, \quad i = 0, \dots, r. \quad (2.4)$$

Then we have

$$\begin{aligned} (D_\lambda^0 e^{\delta B} x)_{\{i\}} &= \lambda^{2i-1} x_{\{i\}} + \lambda^{2i-1} \sum_{j=0}^{i-1} \delta^{i-j} \left( \prod_{k=j+1}^i \mathbf{B}_k \right) x_{\{j\}} \\ &= \lambda^{2i-1} x_{\{i\}} + \sum_{j=0}^{i-1} (\lambda^2 \delta)^{i-j} \left( \prod_{k=j+1}^i \mathbf{B}_k \right) \lambda^{2j-1} x_{\{j\}} = \left( e^{\delta \lambda^2 B} (D_\lambda^0 x) \right)_{\{i\}}. \end{aligned}$$

where in the last equality we used that  $\lambda^{2i-1} x_{\{i\}} = (D_\lambda x)_{\{i\}}$  and  $\lambda^{2j-1} x_{\{j\}} = (D_\lambda x)_{\{j\}}$ . Thus we get

$$D_\lambda e^{\delta Y} z = \left( \lambda^2(t + \delta), e^{\delta \lambda^2 B} (D_\lambda^0 x) \right) = e^{\delta \lambda^2 Y} (D(\lambda) z).$$

The second equality is clear from the definitions.  $\square$

We conclude this Section by recalling the crucial intrinsic Taylor expansion proved in [22]. Here  $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}_0^d$  denotes a multi-index, and as usual  $\beta! := \prod_{j=1}^d (\beta_j!)$ . Moreover, for any  $x \in \mathbb{R}^d$ ,  $x^\beta = x_1^{\beta_1} \dots, x_d^{\beta_d}$  and  $\partial^\beta = \partial_x^\beta = \partial_{x_1}^{\beta_1} \dots \partial_{x_d}^{\beta_d}$ . The  $B$ -length of  $\beta$  is defined as

$$|\beta|_B := \sum_{i=0}^r (2i + 1) \sum_{j=\bar{p}_{i-1}+1}^{\bar{p}_i} \beta_j.$$

**Theorem 2.4** (Intrinsic Taylor formula). *If  $u \in C_B^{n,\alpha}$  then we have*

$$Y^k \partial_x^\beta u \in C_B^{n-2k-|\beta|_B, \alpha}, \quad 0 \leq 2k + |\beta|_B \leq n, \quad (2.5)$$

and

$$|u(z) - T_n u(\zeta, z)| \leq c_B \|u\|_{C_B^{n,\alpha}} \|\zeta^{-1} \circ z\|_B^{n+\alpha}, \quad z, \zeta \in \mathbb{R}^{d+1}, \quad (2.6)$$

where  $c_B$  is a positive constant which only depend on  $B$  and  $T_n u(\zeta, \cdot)$  is the  $n$ -th order  $B$ -Taylor polynomial of  $u$  around  $\zeta = (s, \xi)$  defined as

$$T_n u(\zeta, z) = \sum_{0 \leq 2k + |\beta|_B \leq n} \frac{1}{k! \beta!} \left( Y^k \partial_\xi^\beta u(s, \xi) \right) (t - s)^k (x - e^{(t-s)B} \xi)^\beta, \quad z = (t, x) \in \mathbb{R}^{d+1}. \quad (2.7)$$



**Notation:** Eventually, we will frequently use the notation  $\lesssim$ . For two quantities  $Q_1$  and  $Q_2$ , we mean by  $Q_1 \lesssim Q_2$  that there exists a constant  $c_B$ , only dependent on the matrix  $B$ , such that  $Q_1 \leq c_B Q_2$ .

### 3. Approximation of $C_B^{n,\alpha}$ functions

This Section is mainly devoted to the proof of the following Theorem 3.1, which establishes an approximation property for our  $B$ -Hölder continuous functions.

**Theorem 3.1** (Approximation property). *Let  $n, l \in \mathbb{N}_0$  and  $\alpha \in [0, 1]$ . There exists a constant  $c_B$ , only dependent on  $B$  and  $d$ , such that for any  $u \in C_B^{n,\alpha}$  and  $0 < \varepsilon \leq 1$ , there exists  $u_\varepsilon \in C^\infty$  such that*

$$\|u - u_\varepsilon\|_{C_B^{l,0}} \leq c_B \varepsilon^{n+\alpha-l} \|u\|_{C_B^{n,\alpha}}, \quad l \leq n, \tag{3.1}$$

$$\|u_\varepsilon\|_{C_B^{l,0}} \leq c_B \varepsilon^{n+\alpha-l} \|u\|_{C_B^{n,\alpha}} \quad l > n. \tag{3.2}$$

The proof is based on the intrinsic Hölder expansion of Theorem 2.4 and the invariance properties of Lemma 2.3 and Remark 2.2. We also need the following technical Lemma, which remarkably says that we can exchange the differentiation of the intrinsic Taylor polynomial with the differentiation of the inner function, taking into account the intrinsic degree of the derivatives  $\partial_i, i = 1, \dots, p_0$  and  $Y$ .

To avoid ambiguity, when necessary, we will write  $Y_z, Y_\zeta$  as well as  $\partial_{x_i}, \partial_{\xi_i}$  to clearly distinguish the variables with respect to which the derivation is applied.

**Lemma 3.2.** *For any  $u \in C_B^{n,\alpha}$  and  $z = (t, x), \zeta = (s, \xi) \in \mathbb{R}^{d+1}$  we have*

$$\partial_{x_i} T_n u(\zeta, z) = T_{n-1}(\partial_i u)(\zeta, z), \quad n \geq 1, \quad i = 1, \dots, p_0, \tag{3.3}$$

$$Y_z T_n u(\zeta, z) = T_{n-2}(Y u)(\zeta, z), \quad n \geq 2. \tag{3.4}$$

**Proof.** Equality (3.3) is analogous to the Euclidean case and follows by a direct computation. Indeed it suffices to notice that, for  $i = 1, \dots, p_0$ ,  $T_n u(\zeta, z)$  depends on  $x_i$  only in the terms  $(x - e^{(t-s)B}\zeta)_i$  and  $\partial_{x_j}(x - e^{(t-s)B}\zeta)_i = \delta_{ij}$  ( $\delta_{ij}$  being the Kronecker Delta) for  $i, j = 1, \dots, p_0$ . Equality (3.4) on the other hand, needs a more careful analysis: the directional derivative  $Y_z$  is not null when applied both to the terms  $(t-s)^k$  and  $(x - e^{(t-s)B}\zeta)_i$  for any  $i > p_0$  and, most importantly, some commutators need to be introduced in order to exchange the order of derivation between  $\partial_{\xi_i}$  and  $Y_\zeta$ , and make  $Y_\zeta u$  appear in any term of the sum.

First we introduce some notations: let  $v := x - e^{(t-s)B}\zeta \in \mathbb{R}^d$  and let us denote with  $v^{[i]}, i = 0, \dots, r$ , the projection of  $v$  on  $\{0\}^{\bar{p}_{i-1}} \times \mathbb{R}^{p_i} \times \{0\}^{d-\bar{p}_i}$ ; then we denote with  $X_i$  the vector field  $X_i := \langle v^{[i]}, \nabla \rangle$ , were  $\nabla$  represent the usual Nabla operator. Clearly we have

$$Y_z X_0 = 0, \tag{3.5}$$

Moreover we have

$$\begin{aligned} Y_z X_i u(\zeta) &= \sum_{j=\bar{p}_{i-1}+1}^{\bar{p}_i} (Bv)_j \partial_{\xi_j} u(\zeta) = \langle (Bv)^{[i]}, \nabla u(\zeta) \rangle \\ &= \langle Bv^{[i-1]}, \nabla u(\zeta) \rangle = [X_{i-1}, Y_\zeta] u(\zeta) =: X_{i-1}^{(1)} u(\zeta). \end{aligned} \tag{3.6}$$

To prove (3.4) we proceed by induction. For  $n = 2$  we directly get

$$Y_z T_2 u(\zeta, z) = Y_z \left( u(\zeta) + X_0 u(\zeta) + \frac{1}{2!} X_0^2 u(\zeta) + Y_\zeta u(\zeta)(t-s) \right) = Y_z Y_\zeta u(\zeta)(t-s) = Y_\zeta u(\zeta).$$

Then we proceed in two steps, namely, the induction from  $2n+1$  to  $2n+2$  and the induction from  $2n$  to  $2n+1$ .

1: Induction from  $2n+1$  to  $2n+2$

Let  $\beta = (\beta_0, \dots, \beta_r) \in \mathbb{N}_0^{r+1}$  and let us denote just in the context of this proof

$$|\beta|_B = \sum_{i=0}^r (2i+1)\beta_i, \quad \beta! = \beta_1! \cdots \beta_r!$$

Then, with the notation introduced above we can conveniently write

$$T_{2n+2} u(\zeta, z) = T_{2n+1} u(\zeta, z) + \sum_{k=0}^{n+1} \sum_{|\beta|_B=2(n+1)-2k} \frac{1}{k! \beta!} Y_\zeta^k X_0^{\beta_0} \cdots X_{n-k}^{\beta_{n-k}} u(\zeta)(t-s)^k \quad (3.7)$$

$$=: T_{2n+1} u(\zeta, z) + \tilde{T}_{2n+1} u(\zeta, z). \quad (3.8)$$

By (3.5) and (3.6) we have

$$Y_z Y_\zeta^k (t-s)^k = Y_\zeta^k (t-s)^{k-1}, \quad (3.9)$$

$$Y_z X_i^{\beta_i} = \beta_i X_i^{\beta_i-1} X_{i-1}^{(1)} \quad (3.10)$$

$$Y_\zeta X_i^{\beta_i} = X_i^{\beta_i} Y_\zeta - \beta_i X_i^{\beta_i-1} X_i^{(1)}. \quad (3.11)$$

Therefore, applying  $Y_z$  to  $\tilde{T}_{2n+1} u(\zeta, z)$  we get  $Y_z \tilde{T}_{2n+1} u(\zeta, z) = \sum_{k=0}^{n+1} S_k$  with

$$\begin{aligned} S_0 &= \sum_{|\beta|_B=2n+2} \frac{1}{\beta!} \sum_{i=0}^n \beta_i X_0^{\beta_0} \cdots X_i^{\beta_i-1} \cdots X_n^{\beta_n} X_{i-1}^{(1)} u(\zeta) \\ &= \sum_{|\beta|_B=2n+2} \sum_{i=0}^{n-1} \frac{1}{\beta_0! \cdots (\beta_{i+1}-1)! \cdots \beta_n!} X_0^{\beta_0} \cdots X_{i+1}^{\beta_{i+1}-1} \cdots X_n^{\beta_n} X_i^{(1)} u(\zeta) \\ &= \sum_{i=0}^{n-1} \sum_{|\beta|_B=2(n-i)-1} \frac{1}{\beta!} X_0^{\beta_0} \cdots X_{n-i-1}^{\beta_{n-i-1}} X_i^{(1)} u(\zeta), \end{aligned}$$

and, for  $k = 1, \dots, n+1$ ,

$$\begin{aligned} S_k &= \sum_{|\beta|_B=2(n+1)-2k} \frac{(t-s)^{k-1}}{(k-1)! \beta!} Y_\zeta^{k-1} \left( X_0^{\beta_0} \cdots X_{n-k}^{\beta_{n-k}} Y_\zeta - \sum_{i=0}^{n-k} \beta_i X_0^{\beta_0} \cdots X_i^{\beta_i-1} \cdots X_{n-k}^{\beta_{n-k}} X_i^{(1)} \right) u(\zeta) \\ &\quad + \sum_{|\beta|_B=2(n+1)-2k} \frac{(t-s)^k}{k! \beta!} Y_\zeta^k \sum_{i=0}^{n-k} \beta_i X_0^{\beta_0} \cdots X_i^{\beta_i-1} \cdots X_{n-k}^{\beta_{n-k}} X_{i-1}^{(1)} u(\zeta) \\ &= S_k^{(1)} - \sum_{i=0}^{n-k} \sum_{|\beta|_B=2(n-k-i)+1} \frac{1}{(k-1)! \beta!} Y_\zeta^{k-1} X_0^{\beta_0} \cdots X_{n-i-k}^{\beta_{n-i-k}} X_i^{(1)} u(\zeta)(t-s)^{k-1} \\ &\quad + \sum_{i=0}^{n-k-1} \sum_{|\beta|_B=2(n-k-i)-1} \frac{1}{k! \beta!} Y_\zeta^k X_0^{\beta_0} \cdots X_{n-i-k-1}^{\beta_{n-i-k-1}} X_i^{(1)} u(\zeta)(t-s)^k \end{aligned}$$

$$= S_k^{(1)} + S_k^{(2)} + S_k^{(3)}.$$

Now notice that

$$S_0 + S_1^{(2)} = 0 \quad S_k^{(3)} + S_{k+1}^{(2)} = 0, \quad k = 1, \dots, n. \tag{3.12}$$

Moreover  $S_n^{(3)} = S_{n+1}^{(2)} = S_{n+1}^{(3)} = 0$  by (3.9). Thus we finally get

$$Y_z \tilde{T}_{2n+2} u(\zeta, z) = \sum_{k=0}^n S_{k+1}^{(1)} = \sum_{k=0}^n \sum_{|\beta|_B=2n-2k} \frac{1}{k! \beta!} Y_\zeta^k X_0^{\beta_0} \dots X_{n-k-1}^{\beta_{n-k-1}} Y_\zeta u(\zeta) (t-s)^k = \tilde{T}_{2n} Y u(\zeta, z),$$

and therefore, using the induction hypotheses

$$Y_z T_{2n+2} u(\zeta, z) = T_{2n-1} Y u(\zeta, z) + \tilde{T}_{2n} Y u(\zeta, z) = T_{2n} Y u(\zeta, z).$$

2: Induction from  $2n$  to  $2n + 1$

This Step only differs from the previous one by the tracking of the indexes throughout the computations and the fact that the polynomial  $\tilde{T}_{2n+1}$  always contains at least one field of type  $X_i$ . We write  $Y_z \tilde{T}_{2n+1} u(\zeta, z) = \sum_{k=0}^n S_k$  and we derive, with similar computations than in the previous step

$$S_0 = \sum_{i=0}^{n-1} \sum_{|\beta|_B=2(n-i)} \frac{1}{\beta!} X_0^{\beta_0} \dots X_{n-i-1}^{\beta_{n-i-1}} X_i^{(1)} u(\zeta),$$

and, for any  $k = 1, \dots, n - 1$ ,

$$\begin{aligned} S_k &= \sum_{|\beta|_B=2(n-k)+1} \frac{1}{(k-1)! \beta!} Y_\zeta^{k-1} X_0^{\beta_0} \dots X_{n-k}^{\beta_{n-k}} Y_\zeta u(\zeta) (t-s)^{k-1} \\ &\quad - \sum_{i=0}^{n-k} \sum_{|\beta|_B=2(n-k-i)} \frac{1}{(k-1)! \beta!} Y_\zeta^{k-1} X_0^{\beta_0} \dots X_{n-i-k}^{\beta_{n-i-k}} X_i^{(1)} u(\zeta) (t-s)^{k-1} \\ &\quad + \sum_{i=0}^{n-k-1} \sum_{|\beta|_B=2(n-k-i-1)} \frac{1}{k! \beta!} Y_\zeta^k X_0^{\beta_0} \dots X_{n-i-k-1}^{\beta_{n-i-k-1}} X_i^{(1)} u(\zeta) (t-s)^k \\ &= S_k^{(1)} + S_k^{(2)} + S_k^{(3)}. \end{aligned}$$

In particular,  $S_0 + S_1^{(2)} = 0$ ,  $S_k^{(3)} + S_{k+1}^{(2)} = 0$  for any  $k = 1, \dots, n - 1$  and  $S_n^{(3)} = 0$  by (3.5). Therefore we can eventually derive (3.4).  $\square$

We are now ready to prove Theorem 3.1.

**Proof of Theorem 3.1.** Let  $u \in C_B^{n,\alpha}$  and let  $\phi$  be a test function supported on  $\|z\|_B \leq 1$  with unit integral. We define our candidate approximation by

$$u_\varepsilon^{(n)}(z) = T_n u(\cdot, z) \star_B \phi_\varepsilon(z) := \int_{\mathbb{R}^{d+1}} T_n u(\zeta, z) \phi(D_{\varepsilon^{-1}}(\zeta^{-1} \circ z)) \frac{d\zeta}{\varepsilon^{Q+2}},$$

where  $T_n u(\zeta, z)$  is the Taylor polynomial in (2.7) and  $Q$  is the spatial homogeneous dimension of  $\mathbb{R}^{d+1}$  w.r.t.  $(D_\lambda)_{\lambda>0}$ , that is the positive integer

$$Q = p_0 + 3p_1 + \dots + (2r + 1)p_r.$$

Notice that, by the change of variables  $\bar{\zeta} = D_{\varepsilon^{-1}}(\zeta^{-1} \circ z)$ , recalling Remark 2.1, it is easy to verify that

$$\int_{\mathbb{R}^{d+1}} \phi(D_{\varepsilon^{-1}}(\zeta^{-1} \circ z)) \frac{d\zeta}{\varepsilon^{Q+2}} = \int_{\|\zeta\|_B \leq 1} \phi(\zeta) d\zeta = 1. \tag{3.13}$$

In particular we have

$$u(z) - u_{\varepsilon}^{(n)}(z) = \int_{\mathbb{R}^{d+1}} (u(z) - T_n u(\zeta, z)) \phi(D_{\varepsilon^{-1}}(\zeta^{-1} \circ z)) \frac{d\zeta}{\varepsilon^{Q+2}}.$$

Observe now that the differentiation of  $u_{\varepsilon}^{(n)}$  falls back both on the test function and the polynomial  $T_n u$ , the latter being handled by Lemma 3.2; for the former case, by Lemma 2.3 and Remark 2.2 we deduce

$$\begin{aligned} Y_z \phi(D_{\varepsilon^{-1}}(\zeta^{-1} \circ z)) &= \varepsilon^{-2} Y \phi(\bar{\zeta}) \Big|_{\bar{\zeta}=D_{\varepsilon^{-1}}(\zeta^{-1} \circ z)}, \\ \partial_{z_j} \phi(D_{\varepsilon^{-1}}(\zeta^{-1} \circ z)) &= \varepsilon^{-(2i+1)} \partial_j \phi(\bar{\zeta}) \Big|_{\bar{\zeta}=D_{\varepsilon^{-1}}(\zeta^{-1} \circ z)}, \quad \bar{p}_{i-1} < j \leq \bar{p}_i, \end{aligned}$$

and, more generally, for any  $\beta \in \mathbb{N}_0^d$ ,

$$Y_z^k \partial_z^\beta \phi(D_{\varepsilon^{-1}}(\zeta^{-1} \circ z)) = \varepsilon^{-2k-|\beta|_B} Y^k \partial^\beta \phi(\bar{z}) \Big|_{\bar{z}=D_{\varepsilon^{-1}}(\zeta^{-1} \circ z)}. \tag{3.14}$$

1: Preliminary controls

Let us denote a general intrinsic derivative of order  $m$ ,  $\mathfrak{D}^m = \sum_{2k+|\beta|_B=m} Y^k \partial^\beta$ , and let

$$I_{\varepsilon}^{(n,m)} u(z) := \int_{\mathbb{R}^{d+1}} (u(z) - T_n u(\zeta, z)) \mathfrak{D}_z^m \phi(D_{\varepsilon^{-1}}(\zeta^{-1} \circ z)) \frac{d\zeta}{\varepsilon^{Q+2}}.$$

We want to prove that

$$|I_{\varepsilon}^{(n,m)} u|_{\infty} \lesssim \varepsilon^{n+\alpha-m} \|u\|_{C_B^{n,\alpha}}, \tag{3.15}$$

$$|I_{\varepsilon}^{(n,m)} u|_{C^{\frac{1}{2}}} \lesssim \varepsilon^{n+\alpha-(m+1)} \|u\|_{C_B^{n,\alpha}}, \quad n \geq 1. \tag{3.16}$$

By (3.14), the change of variable  $\bar{\zeta} = D_{\varepsilon^{-1}}(\zeta^{-1} \circ z)$ , and Theorem 2.4, we have

$$\begin{aligned} |I_{\varepsilon}^{(n,m)} u|_{\infty} &\leq \int_{\mathbb{R}^{d+1}} |(u(z) - T_n u(\zeta, z)) \mathfrak{D}_z^m \phi(D_{\varepsilon^{-1}}(\zeta^{-1} \circ z))| \frac{d\zeta}{\varepsilon^{Q+2}} \\ &\lesssim \|u\|_{C_B^{n,\alpha}} \int_{\mathbb{R}^{d+1}} \|\zeta^{-1} \circ z\|_B^{n+\alpha} \varepsilon^{-m} \mathfrak{D}^m \phi(\bar{\zeta}) \Big|_{\bar{\zeta}=D_{\varepsilon^{-1}}(\zeta^{-1} \circ z)} \frac{d\zeta}{\varepsilon^{Q+2}} \\ &\lesssim \|u\|_{C_B^{n,\alpha}} \int_{\|\bar{\zeta}\|_B \leq 1} \|D_{\varepsilon} \bar{\zeta}\|_B^{n+\alpha} \varepsilon^{-m} \mathfrak{D}^m \phi(\bar{\zeta}) d\bar{\zeta} \lesssim \varepsilon^{n+\alpha-m} \|u\|_{C_B^{n,\alpha}}, \end{aligned}$$

were we used that  $\|D_{\varepsilon} \bar{\zeta}\|_B = \varepsilon \|\bar{\zeta}\|_B$  in the last inequality, and this proves (3.15). On the other hand we have

$$\begin{aligned}
 & |I_\varepsilon^{(n,m)}u(z) - I_\varepsilon^{(n,m)}u(e^{\delta Y}z)| \\
 & \leq \int_{\mathbb{R}^{d+1}} |u(z) - T_nu(\zeta, z)| |\mathfrak{D}_z^m\phi(D_{\varepsilon^{-1}}(\zeta^{-1} \circ z')) - \mathfrak{D}_z^m\phi(D_{\varepsilon^{-1}}(\zeta^{-1} \circ z))|_{z'=e^{\delta Y}z} \frac{d\zeta}{\varepsilon^{Q+2}} \\
 & \quad + \int_{\mathbb{R}^{d+1}} |u(e^{\delta Y}z) - T_nu(\zeta, e^{\delta Y}z) - (u(z) - T_nu(\zeta, z))| |\mathfrak{D}_z^m\phi(D_{\varepsilon^{-1}}(\zeta^{-1} \circ z))| \frac{d\zeta}{\varepsilon^{Q+2}} \\
 & =: \Delta I_{\varepsilon,1}^{(n,m)} + \Delta I_{\varepsilon,2}^{(n,m)}.
 \end{aligned}$$

By (3.14), Theorem 3.3, Lemma 2.3, and the usual change of variables  $\bar{\zeta} = D_{\varepsilon^{-1}}(\zeta^{-1} \circ z)$ , we have

$$\begin{aligned}
 \Delta I_{\varepsilon,1}^{(n,m)} & \lesssim \|u\|_{C_B^{n,\alpha}} \int_{\mathbb{R}^{d+1}} \|\zeta^{-1} \circ z\|_B^{n+\alpha} \varepsilon^{-m} |\mathfrak{D}^m\phi(\bar{\zeta}') - \mathfrak{D}^m\phi(\bar{\zeta})|_{\bar{\zeta}'=e^{\frac{\delta}{\varepsilon^2}Y}\bar{\zeta}} \frac{d\zeta}{\varepsilon^{Q+2}} \\
 & \lesssim \|u\|_{C_B^{n,\alpha}} \int_{\|\bar{\zeta}\|_B \leq 1} \varepsilon^{n+\alpha-m} \left| \frac{\delta}{\varepsilon^2} \right|^{\frac{1}{2}} \|\phi\|_{C_B^{m+1,0}} d\bar{\zeta} \lesssim |\delta|^{\frac{1}{2}} \varepsilon^{n+\alpha-(m+1)} \|u\|_{C_B^{n,\alpha}}.
 \end{aligned}$$

To control  $\Delta I_{\varepsilon,2}^{(n,m)}$  we write

$$\Delta I_{\varepsilon,2}^{(n,m)} = \Delta I_{\varepsilon,2}^{(n,m)} \mathbb{I}_{\delta > \varepsilon^2} + \Delta I_{\varepsilon,2}^{(n,m)} \mathbb{I}_{\delta \leq \varepsilon^2}.$$

We have

$$\Delta I_{\varepsilon,2}^{(n,m)} \mathbb{I}_{\delta > \varepsilon^2} \leq \int_{\mathbb{R}^{d+1}} (|u(e^{\delta Y}z) - T_nu(\zeta, e^{\delta Y}z)| + |u(z) - T_nu(\zeta, z)|) \varepsilon^{-m} |\mathfrak{D}^m\phi(\bar{\zeta})| \frac{d\zeta}{\varepsilon^{Q+2}} \mathbb{I}_{\delta > \varepsilon^2}$$

(using that  $\zeta^{-1} \circ e^{\delta Y}z = D_\varepsilon e^{\delta \varepsilon^2 Y} \bar{\zeta}$  by Lemma 2.3 and Remark 2.2)

$$\begin{aligned}
 & \lesssim \|u\|_{C_B^{n,\alpha}} \int_{\|\bar{\zeta}\|_B \leq 1} \varepsilon^{n+\alpha-m} (\|e^{\delta \varepsilon^2 Y} \bar{\zeta}\|_B^{n+\alpha} + \|\bar{\zeta}\|_B^{n+\alpha}) |\mathfrak{D}^m\phi(\bar{\zeta})| d\bar{\zeta} \mathbb{I}_{\delta > \varepsilon^2} \\
 & \lesssim \varepsilon^{n+\alpha-m} \|u\|_{C_B^{n,\alpha}} \mathbb{I}_{\delta > \varepsilon^2} \lesssim |\delta|^{\frac{1}{2}} \varepsilon^{n+\alpha-(m+1)} \|u\|_{C_B^{n,\alpha}}.
 \end{aligned}$$

Finally, assume for a moment  $n \geq 2$ . By the standard mean value theorem and Lemma 3.2, there exists  $\lambda \in [0, 1]$  such that

$$\begin{aligned}
 \Delta I_{\varepsilon,2}^{(n,m)} \mathbb{I}_{\delta \leq \varepsilon^2} & \leq |\delta| \int_{\mathbb{R}^{d+1}} |Yu(z') - T_{n-2}Yu(\zeta, z')|_{\zeta'=e^{\lambda \delta Y}z} \varepsilon^{-m} |\mathfrak{D}^m\phi(\bar{\zeta})| \frac{d\zeta}{\varepsilon^{Q+2}} \mathbb{I}_{\delta \leq \varepsilon^2} \\
 & \lesssim |\delta| \|Yu\|_{C_B^{n-2,\alpha}} \int_{\mathbb{R}^{d+1}} \|\zeta^{-1} \circ e^{\lambda \delta Y}z\|_B^{n-2+\alpha} \varepsilon^{-m} \|\phi\|_{C_B^{m,0}} \frac{d\zeta}{\varepsilon^{Q+2}} \mathbb{I}_{\delta \leq \varepsilon^2} \\
 & \lesssim |\delta| \|u\|_{C_B^{n,\alpha}} \int_{\|\zeta\|_B \leq 1} \varepsilon^{n-2+\alpha-m} \|e^{\lambda \delta \varepsilon^2 Y} \bar{\zeta}\|_B^{n-2+\alpha} d\bar{\zeta} \mathbb{I}_{\delta \leq \varepsilon^2} \\
 & \lesssim |\delta| \varepsilon^{n-2+\alpha-m} \|u\|_{C_B^{n,\alpha}} \mathbb{I}_{\delta \leq \varepsilon^2} \lesssim |\delta|^{\frac{1}{2}} \varepsilon^{n+\alpha-(m+1)} \|u\|_{C_B^{n,\alpha}}.
 \end{aligned}$$

If  $n = 0$  or  $n = 1$  we have  $T_1u(\zeta, e^{\delta Y}z) = T_1u(\zeta, z)$  and  $T_0u(\zeta, z) = T_0u(\zeta, e^{\delta Y}z) = u(\zeta)$ , so that the expression simplifies and it suffices to exploit the regularity of  $u$  w.r.t. the field  $Y$ : this proves (3.16).

2: Conclusions

Let

$$J_\varepsilon^{(n,m)}u(z) := \int_{\mathbb{R}^{d+1}} T_n u(\zeta, z) \mathfrak{D}_z^m \phi(D_{\varepsilon^{-1}}(\zeta^{-1} \circ z)) \frac{d\zeta}{\varepsilon^{Q+2}}.$$

By iterative applications of the generalized Leibniz formula and Lemma 3.2, for  $l \leq n$ :

$$\|u - u_\varepsilon^{(n)}\|_{C_\alpha^{l,0}} \lesssim \sum_{i=0}^l \left| I_\varepsilon^{(n-i,l-i)} \mathfrak{D}^i u \right|_\infty + \sum_{i=0}^{l-1} \left[ I_\varepsilon^{(n-i,l-i-1)} \mathfrak{D}^i u \right]_{C_Y^{\frac{1}{2}}}$$

(using that  $\mathfrak{D}^i u \in C_B^{n-i,\alpha}$  by Theorem 2.4, and (3.15)-(3.16) with  $n - i$  instead of  $n$  and  $m = l - i$ )

$$\lesssim \sum_{i=0}^l \varepsilon^{n+\alpha-l} \|\mathfrak{D}^i u\|_{C_B^{n-i,\alpha}} + \sum_{i=0}^{l-1} \varepsilon^{n+\alpha-l} \|\mathfrak{D}^i u\|_{C_B^{n-i,\alpha}} \lesssim \varepsilon^{n+\alpha-l} \|u\|_{C_B^{n,\alpha}}. \tag{3.17}$$

Similarly, for  $l > n$ , we have

$$\|u_\varepsilon^{(n)}\|_{C_\alpha^{l,0}} \lesssim \sum_{i=0}^n \left( \left| J_\varepsilon^{(n-i,l-i)} \mathfrak{D}^i u \right|_\infty + \left[ J_\varepsilon^{(n-i,l-i-1)} \mathfrak{D}^i u \right]_{C_Y^{\frac{1}{2}}} \right)$$

Since  $l > n$  any term of the first sum sports, at least, one derivative applied on  $\phi$ ; since the integral of the mollifier (3.13) is constant in  $z$ , the cancellation property  $J_\varepsilon^{(n-i,l-i)} \mathfrak{D}^i u = I_\varepsilon^{(n-i,l-i)} \mathfrak{D}^i u$  holds for any  $i = 0, \dots, n$ . Similarly for the second sum. Therefore we finally get

$$\begin{aligned} \|u_\varepsilon^{(n)}\|_{C_\alpha^{l,0}} &\lesssim \sum_{i=0}^n \left( \left| I_\varepsilon^{(n-i,l-i)} \mathfrak{D}^i u \right|_\infty + \left[ I_\varepsilon^{(n-i,l-i-1)} \mathfrak{D}^i u \right]_{C_Y^{\frac{1}{2}}} \right) \\ &\lesssim \sum_{i=0}^n \varepsilon^{n+\alpha-l} \|\mathfrak{D}^i u\|_{C_B^{n-i,\alpha}} \lesssim \varepsilon^{n+\alpha-l} \|u\|_{C_B^{n,\alpha}}. \end{aligned} \tag{3.18}$$

The proof is completed.  $\square$

4. Proofs of Proposition 1.5 and Theorem 1.6

**Proof of Proposition 1.5.** Let  $u \in C_B^{n,0}$ . By Proposition 3.1 we have

$$K(\lambda, u; C_B^{n_1,0}, C_B^{n_2,0}) \lesssim \|u - u_\varepsilon\|_{C_B^{n_1,0}} + \lambda \|u_\varepsilon\|_{C_B^{n_2,0}} \lesssim (\varepsilon^{n-n_1} + \lambda \varepsilon^{n-n_2}) \|u\|_{C_B^{n,0}}.$$

Then, taking  $\varepsilon = \lambda^{\frac{1}{n_2-n_1}}$  we directly get  $K(\lambda, u; C_B^{n_1,0}, C_B^{n_2,0}) \lesssim \lambda^{\frac{n-n_1}{n_2-n_1}} \|u\|_{C_B^{n,0}}$ , which gives the set inclusion on the right. It remains to prove that, for any  $u \in C_B^{n,0}$

$$\|u\|_{C_B^{n,0}} \lesssim \|u\|_{C_B^{n_1,0}}^{1-\frac{n-n_1}{n_2-n_1}} \|u\|_{C_B^{n_2,0}}^{\frac{n-n_1}{n_2-n_1}}. \tag{4.1}$$

We check that (4.1) holds for  $(n_1, n_2) = (n - 1, n + 1)$ ,  $n \in \mathbb{N}$ , in which case, the inequality reads

$$\|u\|_{C_B^{n,0}} \lesssim \|u\|_{C_B^{n-1,0}}^{\frac{1}{2}} \|u\|_{C_B^{n+1,0}}^{\frac{1}{2}}. \tag{4.2}$$

Then it is straightforward to see that (4.1) follows by repeated applications of (4.2) with itself. We proceed by induction on  $n$ . By the standard Mean Value Theorem we have

$$\begin{aligned} |u(e^{\delta\partial_{x_i}} z) - u(z) - \partial_{x_i} u(z)\delta| &\leq \frac{1}{2} |\partial_{x_i}^2 u|_\infty \delta^2, \quad i = 1, \dots, p_0 \\ |u(e^{\delta^2 Y} z) - u(z)| &\leq |Y u|_\infty \delta^2, \end{aligned}$$

and therefore

$$\sum_{i=1}^{p_0} |\partial_{x_i} u|_\infty + [u]_{C_Y^1} \lesssim \delta^{-1} |u|_\infty + \delta (|\partial_{x_i}^2 u|_\infty + |Y u|_\infty).$$

Taking the optimal  $\delta > 0$  we get

$$\|u\|_{C_B^{1,0}} \lesssim |u|_\infty + |u|_\infty^{\frac{1}{2}} (|\partial_{x_i}^2 u|_\infty + |Y u|_\infty)^{\frac{1}{2}} \lesssim |u|_\infty^{\frac{1}{2}} \|u\|_{C_B^{2,0}}^{\frac{1}{2}}. \tag{4.3}$$

This proves (4.2) for  $n = 1$ . Next, by the mean value Theorem along the vector field  $Y$  we have that, for every  $z \in R^{d+1}$  and  $\delta \neq 0$ , there exists  $\bar{\delta}$ ,  $|\bar{\delta}| \leq |\delta|$  such that

$$u(e^{\delta Y} z) - u(z) - \delta Y u(z) = \delta \left( Y u(e^{\bar{\delta} Y} z) - Y u(z) \right).$$

Then, dividing by  $|\delta|^{\frac{1}{2}}$  we easily derive

$$|\delta|^{\frac{1}{2}} |Y u|_\infty \lesssim [u]_{C_Y^{\frac{1}{2}}} + |\delta| [Y u]_{C_Y^{\frac{1}{2}}} \tag{4.4}$$

By (4.3) and (4.4), taking the optimal  $\delta$ , we eventually get

$$\begin{aligned} \|u\|_{C_B^{2,0}} &= |u|_\infty + |Y u|_\infty + \sum_{i=0}^{p_0} \|\partial_{x_i} u\|_{C_B^{1,0}} \\ &\lesssim |u|_\infty + [u]_{C_Y^{\frac{1}{2}}}^{\frac{1}{2}} [Y u]_{C_Y^{\frac{1}{2}}}^{\frac{1}{2}} + \sum_{i=0}^{p_0} |\partial_{x_i} u|_\infty^{\frac{1}{2}} \|\partial_{x_i} u\|_{C_B^{2,0}}^{\frac{1}{2}} \lesssim \|u\|_{C^{1,0}}^{\frac{1}{2}} \|u\|_{C^{3,0}}^{\frac{1}{2}}. \end{aligned}$$

This proves (4.2) for  $n = 2$ . The general case simply follows by the iterative definition of the spaces and the induction hypothesis.  $\square$

Before we proceed with the proof of Theorem 1.6, we recall another important tool in the theory of interpolation, that is the well known Reiteration Theorem (see [20], Theorem 1.23 or [26], Section 1.10.2).

**Theorem 4.1 (Reiteration Theorem).** *Let  $\{Z_1, Z_2\}$  be an interpolation couple, and let  $E_1, E_2$  be some intermediate spaces between  $Z_1$  and  $Z_2$ . If*

$$(Z_1, Z_2)_{\theta_i, 1} \subseteq E_i \subseteq (Z_1, Z_2)_{\theta_i, \infty}, \quad i = 1, 2$$

for some  $\theta_i$  such that  $0 \leq \theta_1 < \theta_2 \leq \infty$ , then

$$(E_1, E_2)_{\alpha, p} = (Z_1, Z_2)_{(1-\alpha)\theta_1 + \alpha\theta_2, p}, \quad \alpha \in (0, 1), p \geq 1.$$

We are finally ready to prove Theorem 1.6.

**Proof of Theorem 1.6.** The proof is a fairly standard application of Theorem 3.1, Proposition 1.5 and Theorem 4.1. We proceed in two steps:

*Step 1:* We prove (1.15) for  $\alpha_1 = \alpha_2 = 0$  and  $n_1 = n, n_2 = n + 1$ . Precisely, for any  $n \in \mathbb{N}_0$ , and  $0 < \alpha < 1$ ,

$$\left(C_B^{n,0}, C_B^{n+1,0}\right)_{\alpha,\infty} = \left(C_B^{n,0}, C_B^{n,1}\right)_{\alpha,\infty} = C_B^{n,\alpha}.$$

( $\supseteq$ ): This inclusion is a direct consequence of the approximation result of Theorem 3.1. Indeed, for any  $u \in C_B^{n,\alpha}$  and  $\lambda \in [0, 1)$  we have

$$K(\lambda, u; C_B^{n,0}, C_B^{n+1,0}) \lesssim \|u - u_\epsilon\|_{C_B^{n,0}} + \lambda \|u_\epsilon\|_{C_B^{n+1,0}} \lesssim (\epsilon^\alpha + \lambda \epsilon^{\alpha-1}) \|u\|_{C_B^{n,\alpha}}.$$

Therefore, taking  $\epsilon = \lambda$  we get

$$K(\lambda, u; C_B^{n,0}, C_B^{n+1,0}) \lesssim \lambda^\alpha \|u\|_{C_B^{n,\alpha}}.$$

On the other hand, for  $\lambda \geq 1$  it suffices to take  $u_\epsilon \equiv 0$ .

( $\subseteq$ ): We proceed by induction on  $n$ . Let  $u \in C_B^{0,\alpha}$ , then, for any choice of  $a \in C, b \in C_B^{1,0}$  such that  $u = a + b$  we have  $|u|_\infty \leq |a|_\infty + |b|_\infty$  and therefore  $|u|_\infty \leq K(1, u; C, C_B^{1,0}) \leq \|u\|_{(C, C_B^{1,0})_{\alpha,\infty}}$ . Similarly, for any  $i = 1, \dots, p_0$

$$|u(e^{\delta \partial_{x_i} z}) - u(z)| \leq 2|a|_\infty + [b]_{C_B^{1,0}} |\delta|, \quad |u(e^{\delta Y z}) - u(z)| \leq 2|a|_\infty + [b]_{C_Y^{1,0}} |\delta|^{\frac{1}{2}}, \quad z \in \mathbb{R}^{d+1},$$

and thus, for any  $i = 1, \dots, p_0$

$$|u(e^{\delta \partial_{x_i} z}) - u(z)| \leq 2K(|\delta|, u; C, C_B^{1,0}) \leq 2|\delta|^\alpha \|u\|_{(C, C_B^{1,0})_{\alpha,\infty}}, \tag{4.5}$$

$$|u(e^{\delta Y z}) - u(z)| \leq 2K(|\delta|^{\frac{1}{2}}, u; C, C_B^{1,0}) \leq 2|\delta|^{\frac{\alpha}{2}} \|u\|_{(C, C_B^{1,0})_{\alpha,\infty}}. \tag{4.6}$$

Gathering together (4.5)-(4.6), we get the inclusion in the case  $n = 0$ . Next, let  $u \in C_B^{1,\alpha}$ . Clearly  $|u|_\infty \leq K(1, u; C_B^{1,0}, C_B^{2,0})$ . Moreover, for any choice of  $a \in C_B^{1,0}, b \in C_B^{2,0}$  such that  $u = a + b$  we have, for any  $i = 1, \dots, p_0$

$$|u(e^{\delta \partial_{x_i} z}) - u(z) - \delta \partial_{x_i} u(z)| \leq 2|\partial_{x_i} a|_\infty |\delta| + \delta^2 [\partial_{x_i} b]_{C_{\partial_{x_i}}^1} |\delta^2|,$$

$$|u(e^{\delta Y z}) - u(z)| \leq [a]_{C_Y^{1,0}} |\delta|^{\frac{1}{2}} + |Yb|_\infty |\delta|,$$

$$|\partial_{x_i} u(z') - \partial_{x_i} u(z)|_{z'=e^{\delta Y} z} \leq 2|\partial_{x_i} a|_\infty + [\partial_{x_i} b]_{C_Y^{1,0}} |\delta|^{\frac{1}{2}},$$

and therefore, for any  $i = 1, \dots, p_0$

$$|u(e^{\delta \partial_{x_i} z}) - u(z) - \delta \partial_{x_i} u(z)| \leq 2|\delta| K(|\delta|, u; C_B^{1,0}, C_B^{2,0}) \leq 2|\delta|^{1+\alpha} \|u\|_{(C_B^{1,0}, C_B^{2,0})_{\alpha,\infty}}, \tag{4.7}$$

$$|u(e^{\delta Y z}) - u(z)| \leq |\delta|^{\frac{1}{2}} K(|\delta|^{\frac{1}{2}}, u; C_B^{1,0}, C_B^{2,0}) \leq |\delta|^{\frac{1+\alpha}{2}} \|u\|_{(C_B^{1,0}, C_B^{2,0})_{\alpha,\infty}}, \tag{4.8}$$

$$|\partial_{x_i} u(z') - \partial_{x_i} u(z)|_{z'=e^{\delta Y} z} \leq 2K(|\delta|^{\frac{1}{2}}, u; C_B^{1,0}, C_B^{2,0}) \leq 2|\delta|^{\frac{\alpha}{2}} \|u\|_{(C_B^{1,0}, C_B^{2,0})_{\alpha,\infty}}. \tag{4.9}$$

Gathering together (4.7)-(4.9), we get the inclusion in the case  $n = 1$ . Finally, for  $u \in C_B^{n,\alpha}, n \geq 2$ , we have



$$\begin{aligned} \|u\|_{C_B^{n,\alpha}} &= |u|_\infty + \|Yu\|_{C_B^{n-2,\alpha}} + \sum_{i=1}^{p_0} \|\partial_{x_i} u\|_{C_B^{n-1,\alpha}} \\ &\lesssim |u|_\infty + \|Yu\|_{(C_B^{n-2,0}, C_B^{n-1,0})_{\alpha,\infty}} + \sum_{i=1}^{p_0} \|\partial_{x_i} u\|_{(C_B^{n-1,0}, C_B^{n,0})_{\alpha,\infty}} \lesssim \|u\|_{(C_B^{n,0}, C_B^{n+1,0})_{\alpha,\infty}}, \end{aligned}$$

where we used that  $K(\lambda, \partial_{x_i} u; C_B^{n-1,0}, C_B^{n,0}) \leq K(\lambda, u; C_B^{n,0}, C_B^{n+1,0})$  as well as  $K(\lambda, Yu; C_B^{n-2,0}, C_B^{n-1,0}) \leq K(\lambda, u; C_B^{n,0}, C_B^{n+1,0})$ , for any  $\lambda \geq 0, i = 1, \dots, p_0$ .

*Step 2:* Let now  $n_1, n_2 \in \mathbb{N}_0, n_1 < n_2$ , and let  $\alpha_1, \alpha_2 \in [0, 1]$ . By the interpolation result of *Step 1*, and the Reiteration Theorem paired with Proposition 1.5, we get

$$\begin{aligned} (C_B^{n_1,\alpha_1}, C_B^{n_2,\alpha_2})_{\theta,\infty} &= \left( (C_B^{n_1,0}, C_B^{n_1+1,0})_{\alpha_1,\infty}, (C_B^{n_2,0}, C_B^{n_2+1,0})_{\alpha_2,\infty} \right)_{\theta,\infty} \\ &= \left( (C_B^{n_1,0}, C_B^{n_2+1,0})_{\frac{\alpha_1}{n_2+1-n_1},\infty}, (C_B^{n_1,0}, C_B^{n_2+1,0})_{\frac{n_2+\alpha_2-n_1}{n_2+1-n_1},\infty} \right)_{\theta,\infty} \\ &= (C_B^{n_1,0}, C_B^{n_2+1,0})_{(1-\theta)\frac{\alpha_1}{n_2+1-n_1} + \theta\frac{n_2+\alpha_2-n_1}{n_2+1-n_1},\infty} \\ &=: (C_B^{n_1,0}, C_B^{n_2+1,0})_{\theta',\infty} \end{aligned}$$

Taking  $n$  as in (1.14) we finally obtain, again by the Reiteration Theorem,

$$(C_B^{n_1,\alpha_1}, C_B^{n_2,\alpha_2})_{\theta,\infty} = (C_B^{n,0}, C_B^{n+1,0})_{\theta'(n_2+1-n_1)-(n-n_1),\infty} = C_B^{n,\alpha_1+\theta[(n_2+\alpha_2)-(n_1+\alpha_1)]-(n-n_1)}.$$

The proof is completed.  $\square$

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