On the complexity of non-orientable Seifert fibre spaces

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Abstract

In this paper we deal with Seifert fibre spaces, which are compact 3-manifolds admitting a foliation by circles. We give a combinatorial description for these manifolds in all the possible cases: orientable, non-orientable, closed, with boundary. Moreover, we compute a potentially sharp upper bound for their complexity in terms of the invariants of the combinatorial description, extending to the non-orientable case results by Fominykh and Wiest for the orientable case with boundary and by Martelli and Petronio for the closed orientable case. Our upper bound is indeed sharp for all Seifert fibre spaces contained in the census of non-orientable closed 3-manifolds classified with respect to complexity.

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1 Introduction and preliminaries

The family of Seifert fibre spaces (see [Sc]) is a generalization of Seifert's original one ([Se]), since it contains also manifolds locally modeled over solid Klein bottles (which are always non-orientable). This family coincides with the class of compact 3-manifolds foliated by circles and have a central role in Thurston geometrization theory (see for example [Sc]). Indeed, in the closed case, each Seifert fibre space is geometric and each geometric 3-manifold is either a Seifert fiber space or admits hyperbolic or Sol geometry. In other words the class of Seifert fibre spaces coincides with the class of geometric manifold admitting six of the eight possible geometries, that is $\mathbb{E}^3, \mathbb{S}^3, \mathbb{S}^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}, \mathrm{Nil}, \widetilde{SL_2}(\mathbb{R})$. Moreover, Seifert fibre spaces with non-empty boundary are one of the building blocks of the relevant class of Waldhausen graph manifold (see [Wa]).

While the theory of Seifert fibre spaces is well established in the orientable case, including the construction of special spines and the estimation of complexity, in the non-orientable one this is not the case: the knowledge about construction and classification of non-orientable Seifert fibre space, their special spines and complexity is very modest. This paper is devoted to the closure of this gap, dealing with both closed and bordered case.

The notion of complexity for compact 3-dimensional manifolds has been introduced by the second author in [M1] (see also [M2]) as a way to measure how "complicated" a manifold is. Indeed for closed irreducible and $\mathbb{P}^2$-irreducible manifolds, the complexity coincides with the minimum number of tetrahedra needed to construct a manifold, with the only exceptions of S^3, $\mathbb{RP}^3$ and $L(3,1)$, all having complexity zero. Moreover, complexity is additive under connected sum, it does not increase when cutting along incompressible surfaces, and it is finite-to-one in the closed irreducible case. The last property has been used in order to construct a census of manifolds according to complexity: exact values of it are listed for the orientable case at `http://matlas.math.csu.ru/?page=search` (up to complexity 12) and for the non-orientable case at `https://regina-normal.github.io` (up to complexity 11). The main goal of the paper is to furnish a potentially sharp upper bound for the complexity of Seifert fibre spaces, extending to the non-orientable case results of [FW] for the orientable case with boundary and of [MP2] for the closed orientable case. It is worth noting that, in the non-orientable closed case, our upper-bound coincides with the exact value of the complexity for all tabulated manifolds (which are about 350).

The organization of the paper is the following. In Section 2 we recall the definition of Seifert fibre spaces and give a combinatorial description of them by a set of parameters which completely classify the spaces, up to fibre-preserving homeomorphism, proving the following result (see Theorem 2 for more details).

Theorem A. *Every Seifert fibre space is uniquely determined, up to fibre-preserving homeomorphism, by the normalized set of parameters*

$$\left\{b; (\epsilon, g, (t,k))\, ; \left(h_1, \ldots, h_{m_+} \mid k_1, \ldots, k_{m_-}\right); ((p_1, q_1). \ldots, (p_r, q_r))\right\}.$$

Section 3 is devoted to the computation of the complexity. We first deal with the case with boundary obtaining the following upper bound (see Theorem 4 for details), valid both in the orientable and in the non-orientable case.

Theorem B. *Let* $M = \left\{b; (\epsilon, g, (t,k))\, ; \left(h_1, \ldots, h_{m_+} \mid k_1, \ldots, k_{m_-}\right); ((p_1, q_1), \ldots, (p_r, q_r))\right\}$ *be a Seifert fibre space with non-empty boundary. Then*

$$c(M) \leq t + \sum_{j=1}^{r} \max\left\{S(p_j, q_j) - 3, 0\right\}.$$

The second part of Section 3 refers to the closed case: we state and prove the result in complete generality, i.e., both for the orientable and for the non-orientable case (see Theorem 5). For non-orientable manifolds, which is the relevant new case, we obtain the following result.

Theorem C. *Let* $M = \{b; (\epsilon, g, (t,k))\, ; (\ \mid\ ); ((p_1, q_1), \ldots, (p_r, q_r))\}$ *be a non-orientable closed irreducible and* $\mathbb{P}^2$*-irreducible Seifert fibre space, then*

$$c(M) \leq 6(1-\chi) + 6t + \sum_{j=1}^{r} (S(p_j, q_j) + 1)\,;$$

where $\chi = 2-2g$ if the base space of M is orientable and $\chi = 2-g$ otherwise.

We end this section by recalling some preliminary notions on spines and complexity of 3-manifolds.

Let S be a simplicial complex and let $\sigma^n, \delta^{n-1} \in S$ be two open simplices such that (i) σ^n is *principal*, i.e. σ^n is not a proper face of any simplex in S and (ii) δ^{n-1} is a *free* face of σ^n, that is δ^{n-1} is not a proper face of any simplex in S different from σ^n. The transition from S to $S \setminus (\sigma^n \cup \delta^{n-1})$ is called an *elementary simplicial collapse*. A polyhedron P *collapses* to a sub-polyhedron Q (denoted by $P \searrow Q$) if for some triangulation (S, L) of the pair (P, Q) the complex S collapses onto L by a finite sequence of elementary simplicial collapses.

A 2-dimensional polyhedron P is said to be *almost simple* if the link of each point $x \in P$ can be embedded into K_4, the complete graph with four vertices. In particular, the polyhedron is called *simple* if the link is homeomorphic to either a circle, or a circle with a diameter, or K_4. A *true vertex* of an (almost) simple polyhedron P is a point $x \in P$ whose link is homeomorphic to K_4.

A *spine* of a compact connected 3-manifold M with $\partial M \neq \emptyset$ is a polyhedron P embedded in $\mathrm{int}(M)$ such that M collapses to P. A spine of a closed connected 3-manifold M is a spine of $M \setminus \mathrm{int}(B^3)$, where B^3 is an embedded closed 3-ball. If $P \subset \mathrm{int}(M)$ is a polyhedron, then P is a spine of M if and only if $M \setminus P \cong \partial M \times [0, 1)$, if $\partial M \neq \emptyset$, and $M \setminus P \cong B^3$ otherwise. The *complexity* $c(M)$ of M is the minimum number of true vertices among all almost simple spines of M.

To construct a spine for a given manifold, we will decompose the manifold into blocks (also called bricks) by cutting it along embedded tori or Klein bottles, then providing skeletons for each block and finally assembling the pairs block-skeleton together. We adapt in this way the definition of skeleton given in [FW] in order to cover also the case of non-orientable Seifert fibre spaces (a general theory in this direction is developed [MP]). Denote by $\mathcal{H}$ the class of pairs $(M, \partial_- M)$, where M is a compact connected 3-manifolds whose (possibly empty) boundary is composed by tori and Klein bottles and $\partial_- M \subseteq \partial M$ is a (possibly empty) union of connected components of ∂M. Moreover, let $\partial_+ M = \partial M \setminus \partial_- M$. A *skeleton of* $(M, \partial_- M) \in \mathcal{H}$ is a sub-polyhedron P of M such that (i) $P \cup \partial M$ is simple, (ii) $M \searrow (P \cup \partial_- M)$ if $\partial_+ M \neq \emptyset$ or $(M \setminus \mathrm{int}(B^3)) \searrow (P \cup \partial_- M)$ if $\partial_+ M = \emptyset$, where B^3 is an embedded closed 3-ball, and (iii) for each component C of ∂M the space $P \cap C$ is either empty or a non-trivial theta curve[1] or a non-trivial simple closed curve. Note that if $P \cap \partial_+ M = \emptyset$ then a skeleton of $(M, \emptyset)$ is a simple spine for M. Given two pairs $(M_1, \partial_- M_1)$ and $(M_2, \partial_- M_2)$ in $\mathcal{H}$, let P_i be a skeleton of $(M_i, \partial_- M_i)$ for $i = 1, 2$. Take two components $C_1 \subset \partial_+ M_1$ and $C_2 \subset \partial_- M_2$ such that $P_i \cap C_i \neq \emptyset$ and $(C_1, P_1 \cap C_1)$ is homeomorphic to $(C_2, P_2 \cap C_2)$ and fix a homeomorphism $\varphi : (C_1, P_1 \cap C_1) \to (C_2, P_2 \cap C_2)$. We define a new pair $(W, \partial_- W) \in \mathcal{H}$, where $W = M_1 \cup_\varphi M_2$ and $\partial_- W = \partial_- M_1 \cup_\varphi (\partial_- M_2 \setminus C_2)$, and we say that $(W, \partial_- W)$ is obtained by *assembling* $(M_1, \partial_- M_1)$ and $(M_2, \partial_- M_2)$ and the skeleton $P = P_1 \cup_\varphi P_2$ of $(W, \partial_- W)$ is obtained by *assembling* P_1 and P_2.

2 Seifert fibre spaces

In this section we first recall the definition of Seifert fibre spaces given in [Sc], then we give a combinatorial description of these spaces as well as a classification up to fibre-preserving homeomorphism, extending the results of [Fi] to the case with boundary.

2.1 Definitions and examples

Denote by $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$ the closed unit disk and by $I = [0, 1]$ the real unit interval. Moreover, let $S^1 = \partial D$ and $D^+ = \{z \in D \mid \mathrm{Re}(z) \geq 0\}$.

[1] A *non-trivial* theta curve θ on a torus or a Klein bottle C is a subset of C homeomorphic to the theta-graph (i.e., the graph with 2 vertices and 3 edges joining them), such that $C \setminus \theta$ is an open disk.

Finally, let N, K and T be the Möbius strip, the Klein bottle and the torus, respectively.

A *fibred solid torus* $T(p,r)$ *of type* (p,r) with $p, r \in \mathbb{Z}$, $p > 0$ and $\gcd(p,r) = 1$, is the 3-manifold obtained from $D \times I$ by identifying $D \times \{0\}$ with $D \times \{1\}$ by the homeomorphism $\varphi_{p,r}$ defined by

$$\varphi_{p,r} : \ (z, 0) \longmapsto (ze^{2i\pi \frac{r}{p}}, 1).$$

The fibred solid torus $T(p,r)$ is the union of the disjoint circles, called *fibres*, $\bigcup_{k=0}^{p-1} \left\{ ze^{2i\pi k \frac{r}{p}} \right\} \times I$ under the identification, for each $z \in D$. The fibre corresponding to $z = 0$ is called *the axis* of $T(p,r)$. The map obtained by collapsing each fibre to a point is a (regular) S^1-fibre bundle if $p = 1$, while it has a singularity corresponding to the axis if $p > 1$. Moreover, we call $T(1,0)$ *the trivial solid torus* which p-fold covers $T(p,r)$. It is well known that two fibred solid tori $T(p,r)$ and $T(p',r')$ are fibre-preserving homeomorphic if and only if $p = p'$ and $r \equiv \pm r' \mod p$.

Analogously, we can define the *(fibred) solid Klein bottle* SK as the manifold which can be obtained from $D \times I$ by identifying $D \times \{0\}$ with $D \times \{1\}$ by the (orientation reversing) homeomorphism φ defined by [2]

$$\varphi : \ (z, 0) \longmapsto (\overline{z}, 1).$$

The fibred solid Klein bottle is the union of the disjoint circles, called *fibres*, $(\{z\} \times I) \cup (\{\overline{z}\} \times I)$ under the identification, for each $z \in D$. Note that $SK \cong N \times I$ and it is double covered by a trivial fibred solid torus.

Moreover, we call *half solid torus* (resp. *half solid Klein bottle*) the fibred manifold obtained from $D^+ \times I$ by gluing $D^+ \times \{0\}$ with $D^+ \times \{1\}$ by the restriction of $\varphi_{1,0}$ (resp. φ) to $D^+ \times \{0\}$.

A *Seifert fibre space* M is a compact connected 3-manifold admitting a decomposition into disjoint circles, called *fibres*, such that each fibre has a neighborhood in M which is a union of fibres and it is fibre-preserving homeomorphic to

[2] Observe that the replacing of φ with another reflection on D does not affect the fibre-preserving homeomorphism type of the resulting space.

- either a fibred solid torus or Klein bottle, if the fibre is contained in $\mathrm{int}(M)$;
- either a half solid torus or a half solid Klein bottle, if the fibre is contained in ∂M.

Note that the original definition of Seifert manifolds given in [Se] excludes the case of fibred solid Klein bottles. One of the interesting features of this more general definition is that the class of Seifert fibre spaces coincides with that of compact connected 3-manifolds foliated by circles (see [E]).

We say that a fibre of M is *regular* if it has a neighborhood fibre-preserving homeomorphic to a trivial fibred solid torus or to a half solid torus, and *exceptional* otherwise. Hence exceptional fibres are either isolated, corresponding to the axis of $T(p,r)$ with $p>1$, or form properly embedded compact surfaces, corresponding to the points $(\{z\}\times I)/\sim_{\varphi}\subset SK$ with $\mathrm{Im}(z)=0$. Moreover, each connected exceptional surface is either a properly embedded annulus or it is a closed surface obtained by gluing together the two boundaries of an annulus, so it is either a torus or a Klein bottle. We denote by $E(M)$ (resp. $SE(M)$) the union of all isolated (resp. non-isolated) exceptional fibres of M and call *E-fibre* (resp. *SE-fibre*) any fibre contained in $E(M)$ (resp. $SE(M)$). Finally, we set $SE(M)=CE(M)\cup AE(M)$, where $CE(M)$ contains the closed components of $SE(M)$, while $AE(M)$ contains the non-closed ones. Note that if M is orientable then $SE(M)=\emptyset$.

The components of ∂M are either tori or Klein bottles: the toric components are regularly fibred, while a Klein bottle component is either regularly fibred (see the left part of Figure 1) or it contains two exceptional fibres of $AE(M)$ (see the right part of Figure 1).

Given a Seifert fibre space M, denote by B the space obtained by collapsing each fibre to a point and by $f:M\to B$ the projection map. If $P\in B$ is the projection of a regular fibre ϕ, then a tubular neighborhood of P is either a disk (if $\phi\subset\mathrm{int}(M)$) or a half-disk (if $\phi\subset\partial M$). The possible models around points which are projections of an exceptional fibre are depicted in

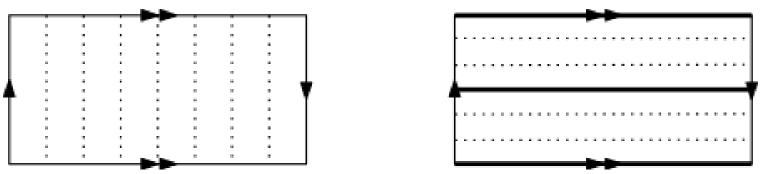

Figure 1: The two different fibre structures of the Klein bottle boundary components of a Seifert fibre space.

Figure 2. As a consequence, B is a compact 2-dimensional orbifold, called *base space*, whose singular locus $\mathcal{S}$ coincides with the projection of all the exceptional fibres $E(M) \cup SE(M)$ of M (thick lines and points in the figures represent singularities of the orbifold). More precisely, the singularities of the orbifold are (see also [Sc]):

- cone points of cone angle $2\pi/p$ for $p > 1$, corresponding to E-fibres having a neighborhood fibre-preserving homeomorphic to $T(p,r)$;
- reflector arcs, corresponding to components of $AE(M)$ (i.e., annulus exceptional surfaces);
- reflector circles, corresponding to components of $CE(M)$ (i.e., tori or Klein bottles exceptional surfaces).

Note that the orbifold B has no corner points in its singular locus and that the restriction of f to the counter-image of the complement of an open tubular neighborhood $N(\mathcal{S})$ of $\mathcal{S} \subset B$ is an S^1-bundle over the compact surface $B \setminus N(\mathcal{S})$.

Example 1. The solid torus $D^2 \times S^1$ and the solid Klein bottle SK are both examples of Seifert fibre spaces with non-empty boundary: the first one admits infinitely many fibre space structures $T(p,r)$ with one isolated exceptional fibre when $p > 1$, while SK admits a unique Seifert fibre space structure, having an annulus as exceptional set (see Figure 2 (a) and (b) for a representation of the base orbifold). Other interesting examples of Seifert fibre spaces are the two N-bundles over S^1, namely $N \times S^1$ and $N \widetilde{\times} S^1$. We

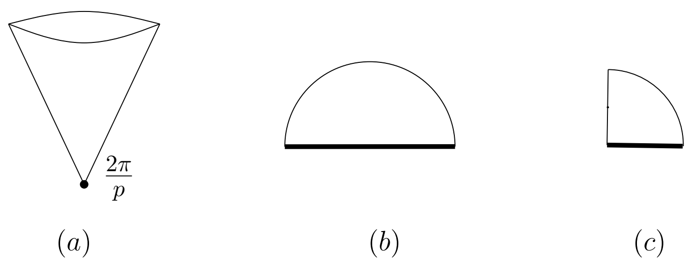

Figure 2: Local models for singular points of the base orbifold B of a Seifert fibre space: (a) cone points, (b) reflector points corresponding to internal fibres and (c) reflector points corresponding to boundary fibres.

recall that $N \tilde{\times} S^1$ is the manifold obtained from $N \times I$ by gluing $(x, 0)$ with $(g(x), 1)$, where, referring to Figure 3, the map g is the composition of a reflection along the exceptional fiber of N (the thick line) with a reflection along the ℓ axis. In this case $\partial(N \tilde{\times} S^1) = K$. The manifold $N \times S^1$ admits the trivial product fibration (without exceptional fibres) and a Seifert fibration with a toric exceptional surface; while $N \tilde{\times} S^1$ admits a Seifert fibration having an isolated exceptional fibre of type $(2, 1)$ and an exceptional annulus, and another one with a Klein bottle exceptional surface. The pictures in the first two rows of Figure 6 represent the base orbifold of all such fibrations (the meaning of the labels in the figure will be explained in the next subsection).

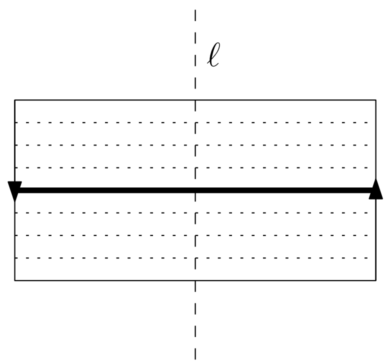

Figure 3: The Möbius strip foliated by circles.

2.2 Combinatorial description and fibre-preserving classification

A combinatorial description for closed Seifert fibre spaces is given in [Fi] as well as the classification of these spaces up to fibre-preserving homeomorphisms. In this section we extend that description to the case with boundary.

Let M be a Seifert fibre space with non-empty boundary and without exceptional fibres, then $f : M \to B$ is an S^1-bundle over B. Denote by $\omega : H_1(B) \to \{1, -1\}$ the group homomorphism such that $\omega(\alpha) = 1$ if and only if the orientation of a fibre in M is preserved when a representative loop of α in B is traversed. If B has genus $g \geq 0$ and $n > 0$ boundary components then, referring to Figure 4,

$$H_1(B) = \langle a_i, b_i, s_j \mid s_1 + \cdots + s_n = 0 \rangle_{i=1,\ldots,g,\ j=1,\ldots,n}$$

if B is orientable, and

$$H_1(B) = \langle v_i, s_j \mid s_1 + \cdots + s_n + 2v_1 + \cdots + 2v_g = 0 \rangle_{i=1,\ldots,g,\ j=1,\ldots,n} \qquad (g \geq 1)$$

if B is non-orientable. We say that the S^1-bundle $f : M \to B$ is of type:

- o_1 if $\omega(a_i) = \omega(b_i) = 1$ for all $i = 1, \ldots, g$;
- o_2 if $\omega(a_i) = \omega(b_i) = -1$ for all $i = 1, \ldots, g$ $(g \geq 1)$;
- n_1 if $\omega(v_i) = 1$ for all $i = 1, \ldots, g$ $(g \geq 1)$;
- n_2 if $\omega(v_i) = -1$ for all $i = 1, \ldots, g$ $(g \geq 1)$;
- n_3 if $\omega(v_1) = 1$ and $\omega(v_i) = -1$ for all $i = 2, \ldots, g$ $(g \geq 2)$;
- n_4 if $\omega(v_1) = \omega(v_2) = 1$ and $\omega(v_i) = -1$ for all $i = 3, \ldots, g$ $(g \geq 3)$.

The following theorem describes the classification of S^1-bundles over a fixed surface, up to fibre-preserving homeomorphisms.

Theorem 1 ([Fi]). *Let B be a compact connected surface with non-empty boundary. The fibre-preserving homeomorphism classes of S^1-bundles over B are in 1-1 correspondence with the pairs (k, ϵ), where k is an even non-negative number which counts the number of s_j such that $\omega(s_j) = -1$ and*

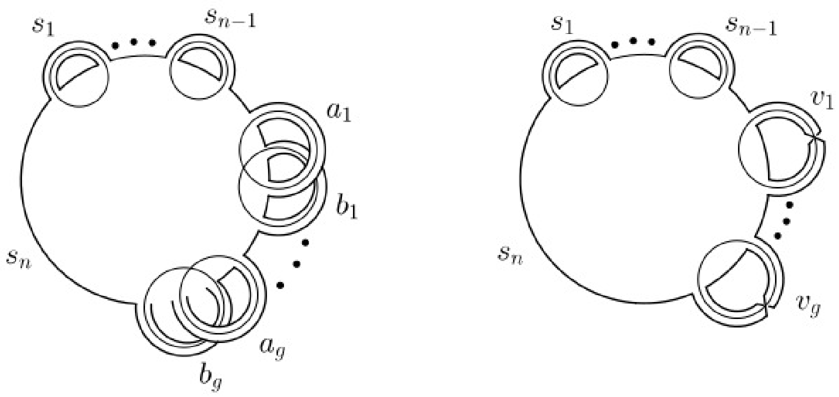

Figure 4: Generators of $H_1(B)$.

(i) $\epsilon = o_1, o_2$ *when* B *is orientable and* $\epsilon = n_1, n_2, n_3, n_4$ *when* B *is non-orientable, if* $k = 0$ *or (ii)* $\epsilon = o$ *with* $o := o_1 = o_2$ *when* B *is orientable and* $\epsilon = n$ *with* $n := n_1 = n_2 = n_3 = n_4$ *when* B *is non-orientable, if* $k > 0$.

Now we are ready to introduce the combinatorial description for Seifert fibre spaces. Let

- g, t, k, m_+, m_-, r be non-negative integers such that $k + m_-$ is even and $k \leq t$;
- ϵ be a symbol belonging to the set $\mathcal{E} = \{o, o_1, o_2, n, n_1, n_2, n_3, n_4\}$ such that (i) $\epsilon = o, n$ if and only if $k + m_- > 0$, (ii) if $\epsilon = n_4$ then $g \geq 3$, (iii) if $\epsilon = n_3$ then $g \geq 2$ and (iv) if $\epsilon = o_2, n, n_1, n_2$ then $g \geq 1$;
- $h_1, \ldots, h_{m_+}$ and $k_1, \ldots, k_{m_-}$ be non-negative integers such that $h_1 \leq \cdots \leq h_{m_+}$ and $k_1 \leq \cdots \leq k_{m_-}$;
- (p_j, q_j) be lexicographically ordered pairs of coprime integers such that $0 < q_j < p_j$ if $\epsilon = o_1, n_2$ and $0 < q_j \leq p_j/2$ otherwise, for $j = 1, \ldots, r$;
- b be an arbitrary integer if $t = m_+ = m_- = 0$ and $\epsilon = o_1, n_2$; $b = 0$ or 1 if $t = m_+ = m_- = 0$ and $\epsilon = o_2, n_1, n_3, n_4$ and no $p_j = 2$; $b = 0$

otherwise.

The previous parameters with the given conditions are called *normalized*, and we denote by

$$\left\{b; (\epsilon, g, (t, k))\, ; \left(h_1, \ldots, h_{m_+} \mid k_1, \ldots, k_{m_-}\right) ; ((p_1, q_1), \ldots, (p_r, q_r))\right\}$$

the Seifert fibre space constructed as follows.

If $b = 0$, denote by B^* a compact connected genus g surface having $s = r+t+m_+ +m_-$ boundary components and being orientable if $\epsilon = o, o_1, o_2$ and non-orientable otherwise. By Theorem 1 there is a unique S^1-bundle over B^* associated to the pair $(k+m_-, \epsilon)$, up to fibre-preserving homeomorphism: call it M^* (see Remark 3 for the details of this construction). Note that M^* has $k+m_-$ boundary components which are Klein bottles and the remaining $r+t-k+m_+$ ones are tori. Denote by $c_1, \ldots, c_s$ the boundary components of B^*, numbering them such that the last $k+m_-$ correspond to Klein bottles in M^*. Let $\partial_1 B^* = c_1 \cup \ldots \cup c_{r+t-k+m_+}$ and $\partial_2 B^* = \partial B^* \setminus \partial_1 B^*$. Finally, denote by $s^* : B^* \to M^*$ a section of $f^* : M^* \to B^*$.

(i) For $j = 1, \ldots, r$ fill the toric boundary component $(f^*)^{-1}(c_j)$ of M^* with a solid torus by sending the boundary of a meridian disk of the solid torus into the curve $p_j d_j + q_j f_j$, where f_j is a fibre and $d_j = s^*(c_j)$;

(ii) for $i = 1, \ldots, m_+$ (resp. $j = 1, \ldots, m_-$) consider h_i (resp. k_j) disjoint closed arcs inside the boundary component c_{i+r} of $\partial_1 B^*$ (resp. $c_{j+r+t-k+m_+}$ of $\partial_2 B^*$) and, for each arc and each point x of the arc, attach a Möbius strip along the boundary to the fibre $(f^*)^{-1}(x)$, where the Möbius strip is foliated by circles as depicted in Figure 3. On the whole, we attach h_i (resp. k_j) disjoint copies of $N \times I$ to the boundary of M^* corresponding to the counter-image of c_{i+r} (resp. $c_{j+r+t-k+m_+}$). So the boundary component remains unchanged if $h_i = 0$ (resp. $k_j = 0$) and it is partially filled otherwise. In the latter case instead of the initial boundary component we have h_i (resp. k_j) Klein bottle boundary components;

(iii) for $i = 1, \ldots, t-k$ (resp. $j = 1, \ldots, k$) glue a copy of $N \times S^1$ (resp. $N \widetilde{\times} S^1$) to each toric (resp. Klein bottle) boundary component of M^* along the boundary via a homeomorphism which is fibre-preserving with respect to the fibration depicted in Figure 6 (a') (resp. (b')). Namely, as in the previous step, for each point $x \in c_{i+r+m_+}$ (resp. $x \in c_{j+r+t-k+m_++m_-}$) we attach a Möbius strip along the boundary to the fibre $(f^*)^{-1}(x)$.

If $b \neq 0$ (and therefore $t = m_+ = m_- = 0$) we modify the above construction as follows: we take a surface B^* with $r+1$ boundary components and fill the first r-ones boundary components of M^* as described in (i) and the last one by sending the boundary of a meridian disk of the solid torus into $d_{r+1} + bf_{r+1}$ (i.e., with $(p_{r+1}, q_{r+1}) = (1, b)$).

The resulting manifold is the Seifert fibre space

$$M = \left\{b; (\epsilon, g, (t,k))\,; \left(h_1, \ldots, h_{m_+} \mid k_1, \ldots, k_{m_-}\right); ((p_1, q_1), \ldots, (p_r, q_r))\right\}.$$

Note that when $t = m_+ = m_- = 0$, the above construction gives the classical closed Seifert fibre space $(b; \epsilon, g; (p_1, q_1), \ldots, (p_r, q_r))$ of [Se].

From the above construction it follows that the exceptional set of M is composed by: (i) an E-fibre of type[3] (p_j, q_j) for $j = 1, \ldots, r$, (ii) t closed exceptional surfaces, k of which are Klein bottles while the remaining $t-k$ are tori and (iii) $t' = h_1 + \cdots + h_{m_+} + k_1 + \cdots + k_{m_-}$ exceptional surfaces homeomorphic to annuli. Moreover, the boundary of M has t' components which are Klein bottles with two exceptional fibres (contained in $AE(M)$) and, for each $h_i = 0$ (resp. $k_j = 0$), a toric (resp. Klein bottle) boundary component without exceptional fibres.

The singular locus $\mathcal{S}$ of the base orbifold B (that will be depicted using thick lines and points in figures) consists of: (i) r cone points of cone angles $2\pi/p_1, \ldots, 2\pi/p_r$ (in figures each cone point will be decorated with the corresponding pair (p_j, q_j)), (ii) t reflector circles and (iii) t' reflector arcs. The

[3] Note that a fibred tubular neighborhood of an E-fibre of type (p_j, q_j) is fibre-preserving equivalent to $T(p_j, r_j)$ with $r_j q_j \equiv 1 \mod p_j$.

underlying surface of the orbifold has genus g and it is orientable if and only if $\epsilon = o, o_1, o_2$. Moreover, it has $m_+ + m_- + t$ boundary components: one boundary component containing h_i (resp. k_j) disjoint reflector arcs for each $i = 1, \ldots, m_+$ (resp. $j = 1, \ldots, m_-$), and one boundary components for each reflector circle. We decorate by the symbol "$-$" each boundary component of the underlying surface having a Klein bottle as counterimage in M.

Remark 1. Conditions on the invariants ensuring the orientability and the closeness of a Seifert fibre space are the following:

(i) M is orientable if and only if $t = m_- = 0$, $h_i = 0$ for all $i = 1, \ldots, m_+$ and $\epsilon = o_1, n_2$;

(ii) M is closed if and only if $m_+ = m_- = 0$. In this case the combinatorial description coincide with the one of [Fi].

Example 2. The Seifert fibre space $\{0; (o, 4, (1,1))\,; (1 \mid 0)\,; ((3,1), (5,2))\}$ is depicted in Figure 5. It has two E-fibres of type $(3,1)$ and $(5,2)$, one Klein bottle exceptional surface and one annulus exceptional surface. The boundary consists of two Klein bottles, one with two exceptional fibres and another without exceptional fibres.

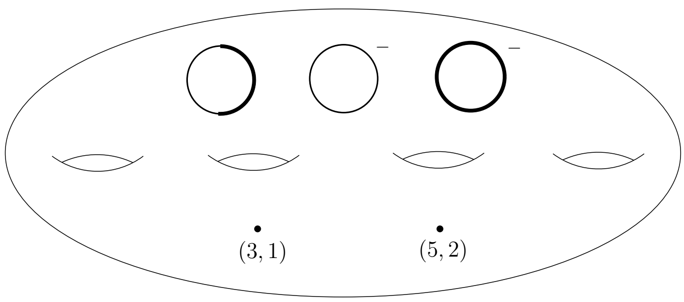

Figure 5: The Seifert fibre space $\{0; (o, 4, (1,1))\,; (1 \mid 0)\,; ((3,1), (5,2))\}$.

Remark 2. Let M be a Seifert fibre space such that $M \setminus SE(M)$ is orientable, and therefore $M = \left\{b; (\epsilon, g, (t,0))\,; \left(h_1, \ldots, h_{m_+} \mid \right)\,; ((p_1,q_1), \ldots, (p_r,q_r))\right\}$

with $\epsilon \in \{o_1, n_2\}$. If M is closed and orientable (i.e., $t = m_+ = 0$), by reversing a fixed orientation on M we obtain

$$-M = \{-b - r; (\epsilon, g, (0,0)) ; (\mid) ; ((p_1, p_1 - q_1), \ldots, (p_r, p_r - q_r))\}$$

(see [O, p.18]). So, if we do not take care of the orientation, we can suppose $b \geq -r/2$. Moreover, if $b = -r/2$ we can assume $0 < q_l < p_l/2$, where l is the minimum j, if any, such that $p_j > 2$. In all the other cases (i.e., $\partial M \neq \emptyset$ or M non orientable) $b = 0$, and, reversing the orientation on $M \setminus SE$ we get the equivalent space $\left\{0; (\epsilon, g, (t, 0)) ; \left(h_1, \ldots, h_{m_+} \mid \right) ; ((p_1, p_1 - q_1), \ldots, (p_r, p_r - q_r))\right\}$. So, we can suppose $0 < q_l < p_l/2$, where l is as above.

Theorem 2. *Every Seifert fibre space is uniquely determined, up to fibre-preserving homeomorphism, by the normalized set of parameters*

$$\left\{b; (\epsilon, g, (t, k)) ; \left(h_1, \ldots, h_{m_+} \mid k_1, \ldots, k_{m_-}\right) ; ((p_1, q_1), \ldots, (p_r, q_r))\right\},$$

and, when $M \setminus SE(M)$ is orientable (i.e., $\epsilon \in \{o_1, n_2\}$), by the following additional conditions: (i) if M is closed and orientable, then $b \geq -r/2$ and, if $b = -r/2$, $0 < q_l < p_l/2$, (ii) if M is non-closed or non-orientable then $0 < q_l < p_l/2$; where l is the minimum j, if any, such that $p_j > 2$.

Proof. If $M = \left\{b; (\epsilon, g, (t, k)) ; \left(h_1, \ldots, h_{m_+} \mid k_1, \ldots, k_{m_-}\right) ; ((p_1, q_1), \ldots, (p_r, q_r))\right\}$ and $\bar{M} = \left\{\bar{b}; \left(\bar{\epsilon}, \bar{g}, (\bar{t}, \bar{k})\right) ; \left(\bar{h}_1, \ldots, \bar{h}_{\bar{m}_+} \mid \bar{k}_1, \ldots, \bar{k}_{\bar{m}_-}\right) ; ((\bar{p}_1, \bar{q}_1), \ldots, (\bar{p}_{\bar{r}}, \bar{q}_{\bar{r}}))\right\}$ are fibre-preserving homeomorphic then, by looking at the boundaries of the base orbifolds, it is clear that $m_+ = \bar{m}_+$, $m_- = \bar{m}_-$, $h_i = \bar{h}_i$, $k_j = \bar{k}_j$ for $i = 1, \ldots, m_+$ and $j = 1, \ldots, m_-$. If we fill, respecting the fibration, each boundary component with two exceptional fibres with a solid Klein bottle, each toric boundary component with $N \times S^1$, and each Klein bottle boundary component without exceptional fibres with $N \widetilde{\times} S^1$, we obtain the two closed Seifert fibre spaces $\{b; (\epsilon, g, (t, k)) ; (\mid) ; ((p_1, q_1), \ldots, (p_r, q_r))\}$ and $\left\{\bar{b}; \left(\bar{\epsilon}, \bar{g}, (\bar{t}, \bar{k})\right) ; (\mid) ; ((\bar{p}_1, \bar{q}_1), \ldots, (\bar{p}_{\bar{r}}, \bar{q}_{\bar{r}}))\right\}$. So the result follows directly from [Fi, Theorem 2] and Remark 2. $\square$

From now on we will always suppose the parameters of Seifert fibre spaces to be normalized.

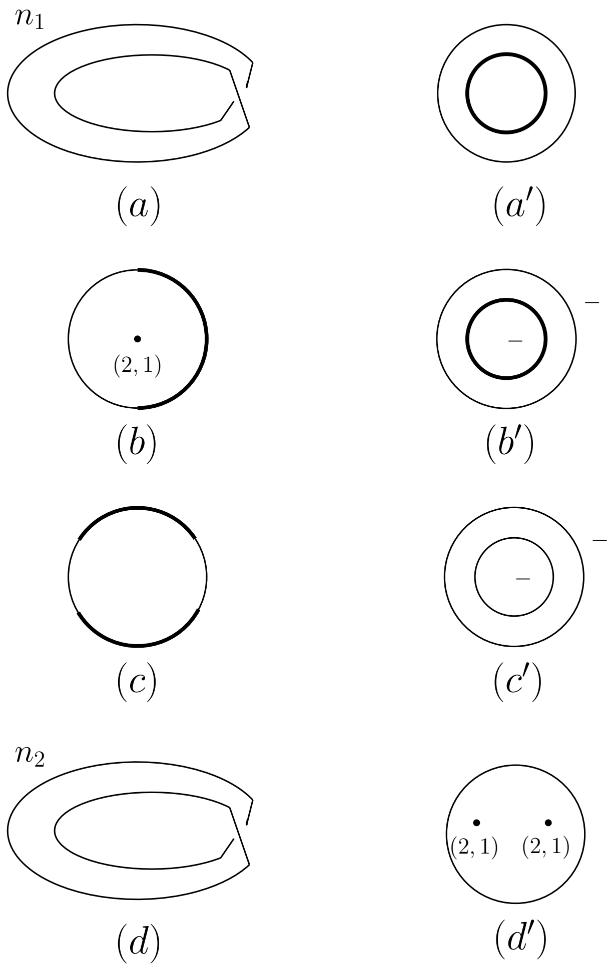

Figure 6: The Seifert fibre structures over: $(a,\ a')$ $N \times S^1$, $(b,\ b')$ $N \widetilde{\times} S^1$, $(c,\ c')$ $K \times I$, $(d,\ d')$ $K \widetilde{\times} I$.

Example 3. The solid torus $D^2 \times S^1$ admits the combinatorial descriptions $\{0; (o_1, 0, (0,0))\,; (0 \mid\ )\,; (p,q)\}$ if $p > 1$, and $\{0; (o_1, 0, (0,0))\,; (0 \mid\ )\,;\ \}$ if $p = 1$, while the solid Klein bottle admits the description $\{0; (o_1, 0, (0,0))\,; (1 \mid\ )\,;\ \}$ (see also Example 1). Other important examples are depicted in Figure 6: the manifold $N \times S^1$ has two different Seifert space structures, up to fibre-preserving homeomorphism, namely (a) the trivial S^1-bundle over N, whose description is $\{0; (n_1, 1, (0,0))\,; (0 \mid\ )\,;\ \}$, and (a') that is $\{0; (o_1, 0, (1,0))\,; (0 \mid\ )\,;\ \}$. Also $N\tilde{\times}S^1$ can be fibred both as $\{0; (o_1, 0, (0,0))\,; (1 \mid\ )\,; (2,1)\}$, depicted in (b), and $\{0; (o, 0, (1,1))\,; (\ \mid 0)\,;\ \}$, depicted in (b'). Pictures (c) and (c') represent the two possible Seifert structures over $K \times I$ (i.e., $\{0; (o_1, 0, (0,0))\,; (2 \mid\ )\,;\ \}$ and $\{0; (o, 0, (0,0))\,; (\ \mid 0, 0)\,;\ \}$, respectively). Finally, in (d) and (d') there are two different Seifert structures of $K\tilde{\times}I$ a twisted I-bundle over K, that are $\{0; (n_2, 1, (0,0))\,; (0 \mid\ )\,;\ \}$ and $\{0; (o_1, 0, (0,0))\,; (0 \mid\ )\,; ((2,1),(2,1))\}$, respectively. As proved in [AM, Proposition A.1], the previous four manifolds and $T \times I = \{0; (o_1, 0, (0,0))\,; (0, 0 \mid\ )\,;\ \}$ are all the possible I-bundles over the torus ($T \times I$ and $T\tilde{\times}I = N \times S^1$) and the Klein bottle ($K \times I$, $K\tilde{\times}I$ and $K\tilde{\tilde{\times}}I = N\tilde{\times}S^1$).

Remark 3. We recall how to construct an S^1-bundle of type (k, ϵ) over the compact connected surface B^* with $\partial B^* \neq \emptyset$. The surface B^* is homeomorphic to a disk with r attached bands $\beta_1, \ldots, \beta_r$, where $r = \text{rank}(H_1(B^*))$. Let $y_1, \ldots, y_r$ be the generators of $H_1(B^*)$ depicted in Figure 4 (i.e., $y_l = a_i, b_i, s_j$ if B^* is orientable and $y_l = v_i, s_j$ otherwise). For $l = 1, \ldots, r$ denote by d_l the oriented cocore of β_l; cutting B^* along $A = d_1 \cup \cdots \cup d_l$ we obtain a disk Δ. Let d_l' and d_l'' be the two oriented copies of d_l in Δ and, for each $x \in d_l$, denote by x' and x'' the two points in d_l' and d_l'' corresponding to x, respectively. Finally, let $\omega : H_1(B^*) \to \{-1, 1\}$ be the homomorphism associated to the pair (k, ϵ). Since Δ is contractible, $\Delta \times S^1$ is the unique S^1-bundle over Δ. Attach $d_l' \times S^1$ to $d_l'' \times S^1$ via $(x', e^{i\theta}) \mapsto (x'', e^{i\theta})$ if $\omega(x_l) = 1$ and via $(x', e^{i\theta}) \mapsto (x'', e^{-i\theta})$ otherwise. The resulting manifold M^* is the required S^1-bundle over B^*.

3 Complexity of Seifert fibre spaces

3.1 The case with non-empty boundary

In this subsection we analyze the case $\partial M \neq \emptyset$ describing a (almost) simple spine for M and using it to give an upper bound for the complexity of the manifold.

In [FW] the authors construct a (almost) simple spine for orientable Seifert fibre spaces, and therefore with $SE(M) = \emptyset$, obtaining an upper bound for the complexity. Let us recall their result. For two coprime integers p, q with $0 < q < p$ denote by $S(p, q)$ the sum of the coefficients of the expansion of p/q as a continued fraction:

$$\text{if} \quad \frac{p}{q} = a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_{k-1} + \cfrac{1}{a_k}}}, \qquad \text{then } S(p,q) = a_1 + \cdots + a_k.$$

Theorem 3 ([FW]). *Let M be an orientable Seifert fibre space with non-empty boundary, having E-fibres of types $(p_1, q_1), \ldots, (p_r, q_r)$ with $p_j > q_j > 0$ for all $j = 1, \ldots, r$. Then*

$$c(M) \leq \sum_{j=1}^{r} \max\{S(p_j, q_j) - 3, 0\}.$$

This theorem is proved by finding a spine of M: such a spine is obtained decomposing M into blocks and then assembling the skeletons of the different blocks together. In order to generalize the result to the case $SE(M) \neq \emptyset$ we adapt one of the blocks, the "main" one, in order to include $AE(M)$, and describe a new type of block for the components of $CE(M)$.

Main block. Let M_0 be a Seifert fibre space such that $\partial M_0 \neq \emptyset$ and $E(M_0) = CE(M_0) = \emptyset$ and let $f_0 : M_0 \to B_0$ be the projection map. Moreover, suppose that if B_0 is a disk, then the boundary of M_0 has at least two components (so there are at least two reflector arcs in the boundary of the disk). We take a decomposition of ∂M_0 into $\partial_+ M_0 \cup \partial_- M_0$ such that $\partial_+ M_0 \neq \emptyset$ and contains all the boundary components with exceptional fibres. Such a decomposition of ∂M_0 determines a decomposition $\partial B_0 = \partial_+ B_0 \cup \partial_- B_0$, where ∂B_0 denotes the boundary of the surface and not of the orbifold (so including the reflector arcs). Note that $\partial_+ B_0 \neq \emptyset$. Referring to Figure 7, let Γ_0 be a graph embedded in B_0 such that (i) each vertex has at most valence three and (ii) $B_0 \setminus (\Gamma_0 \cup \partial_- B_0) \cong (\partial_+ B_0 \cap \partial_{\mathcal{O}} B_0) \times [0,1)$, where $\partial_{\mathcal{O}} B_0$ denotes the boundary of the orbifold[4] B_0. By the previous assumptions it is easy to see that Γ_0 does not reduce to a single point.

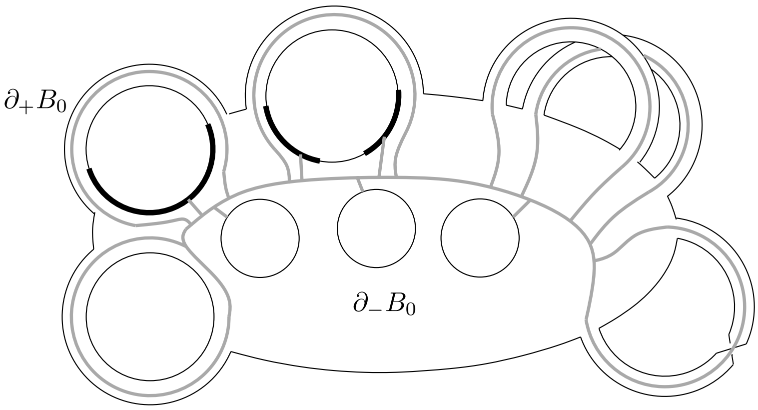

Figure 7: The surface B_0 with the graph Γ_0 depicted in gray.

Let $P_0 = f_0^{-1}(\Gamma_0)$. Since $\mathcal{S} \setminus \Gamma_0 \cong \partial\mathcal{S} \times [0,1)$, it follows that $M_0 \setminus (P_0 \cup \partial_- M_0) \cong \partial_+ M_0 \times [0,1)$, and therefore P_0 is a skeleton for the *main block* $(M_0, \partial_- M_0)$ without true vertices. Note that, since Γ_0 intersects each reflector arc in a single point, P_0 intersects each component of $AE(M)$ in an exceptional fibre ϕ and $\overline{N}(\phi) \cap P_0$ is a Möbius strip, where $\overline{N}(\phi)$ denotes a closed regular neighborhood of ϕ composed by fibres. Furthermore, $P_0 \cap \partial_+ M_0 = \emptyset$ and P_0 intersects each component of $\partial_- M_0$ in a regular fibre.

Exceptional block. Let M_E be either $N \times S^1$ or $N \widetilde{\times} S^1$, considered with the Seifert fibre space structures depicted in Figure 6 (a') and (b'), respectively. Denote by $f : M_E \to B_E$ the projection map. We represent N as in Figure 3 and M_E as $(N \times I)/\sim$, where $\sim$ is an identification between $N \times \{0\}$ and $N \times \{1\}$ via the identity on N if $M = N \times S^1$ and the composition

[4]Note that $\partial_{\mathcal{O}} B_0$ does not contain singular points except for the endpoints of the reflector arcs.

of the reflection along the thick line with that along the axis ℓ if $N \widetilde{\times} S^1$. Now let $N' \subset N$ be a closed regular neighborhood of the core of N composed by fibres and disjoint from ∂N. Of course, N' is a Möbius strip and $N \setminus \mathrm{int}(N') \cong S^1 \times I$. Referring to Figure 8, let $P_E \subset M_E$ be the polyhedron (depicted in gray) which is the union of:

- the annulus $\alpha = (N \setminus \mathrm{int}(N')) \times \left\{\frac{1}{4}\right\}$;
- the Möbius strip $N' \times \{\frac{1}{2}\}$;
- a band β obtained by taking $(L \times I)/\sim$, where $L \subset N'$ is the arc of the fixed points of the reflection along ℓ, cutting it along $L \times \left\{\frac{1}{2}\right\}$ and pushing up (resp. pushing down) the part $L \times \{x\}$ with $x \geq \frac{1}{2}$ (resp. with $x \leq \frac{1}{2}$) leaving fixed $L \times \{0\} \sim L \times \{1\}$, as shown in Figure 8. Observe that β intersects transversally in an arc each $N' \times \{x\}$, with $x \neq \frac{1}{2}$, and intersect $N' \times \left\{\frac{1}{2}\right\}$ in two disjoint arcs;
- the surface $\partial((N' \times I)/\sim) \setminus R$, either a punctured torus or a punctured Klein bottle, where R is the open dashed 2-cell depicted in Figure 9.

Note that $M_E \searrow ((N' \times I)/\sim) \cup \alpha$ and $((N' \times I)/\sim) \setminus P_E$ is a 3-ball. So, the polyhedron P_E is a skeleton for the *exceptional block* $(M_E, \emptyset)$ with only one true vertex (the thick point represented in both Figures 8 and 9). Moreover, $\partial_+ M_E = \partial M_E$ and $P_E \cap \partial M_E$ is a regular fibre (i.e., $\alpha \cap \partial M_E$).

We are ready to state our result on the complexity of bordered Seifert fibre spaces.

Theorem 4. *Let $M = \big\{b; (\epsilon, g, (t, k))\,; \big(h_1, \ldots, h_{m_+} \mid k_1, \ldots, k_{m_-}\big)\,; ((p_1, q_1), \ldots, (p_r, q_r))\big\}$ be a Seifert fibre space such that $\partial M \neq \emptyset$ (i.e., $m_+ + m_- > 0$). Then*

$$c(M) \leq t + \sum_{j=1}^{r} \max\{S(p_j, q_j) - 3, 0\}, \qquad (1)$$

where $S(p_j, q_j)$ denotes the sum of the coefficients of the expansion of p_j/q_j as a continued fraction.

Moreover, if $M = N \times S^1, N \widetilde{\times} S^1, D^2 \times S^1, SK$ then $c(M) = 0$.

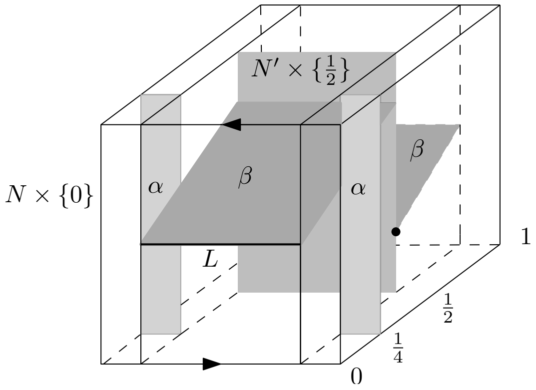

Figure 8: The exceptional block $(M_E, \emptyset)$ and his skeleton P_E (in gray).

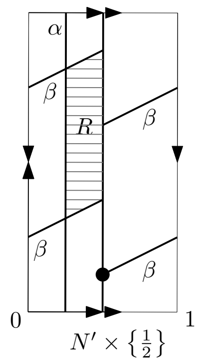

Figure 9: The surface $\partial((N' \times I)/\sim) \setminus R$.

Proof. We start by proving the last part of the statement. Referring to Example 3, we have $N \times S^1 = \{0; (o_1, 0, (1, 0)); (0 \mid); \ \}$, $N \widetilde{\times} S^1 = \{0; (o, 0, (1, 1)); (\mid 0); \ \}$, $D^2 \times S^1 = \{0; (o_1, 0, (0, 0); (0 \mid); (p, q)\} = \{0; (o_1, 0, (0, 0); (0 \mid); \}$, $SK = \{0; (o_1, 0, (0, 0); (1 \mid); \ \}$. A spine S of the first two manifolds is the exceptional set (respectively, a torus and a Klein bottle), while a spine of the last ones is a circle (i.e., the axis of the solid torus for $D^2 \times S^1$ or any exceptional fibre for SK). Indeed, in all cases $M \setminus S \cong \partial M \times [0, 1)$ and so, since S has no true vertices, all these manifolds have complexity zero.

From now on, let M be a Seifert fibre space different from the previous ones. Let $f : M \to B$ be the projection map and let $\partial_+ B$ be the union of the boundary components of the underlying surface not corresponding to reflector circles. Denote by B_0 the surface obtained from B by removing disjoint open disks around each cone point and disjoint open collars around each reflector circle, clearly $B_0 \subset B$ and $\partial_+ B \subset B_0$. Let $\partial_+ B_0 = \partial_+ B$, $\partial_- B_0 = \partial B_0 \setminus \partial_+ B_0$ and $(M_0, \partial_- M_0) = (f^{-1}(B_0), f^{-1}(\partial_- B_0))$, therefore $\partial_+ M_0 = \partial M_0 \setminus \partial_- M_0 = \partial M$. Since $\partial M \neq \emptyset$ and M is neither $D^2 \times S^1$ nor SK, then $(M_0, \partial_- M_0)$ is a main block. Moreover, $M \setminus \mathrm{int}(M_0)$ is the disjoint union of t exceptional blocks M_{E_i} (one for each component of $CE(M)$) and r fibred solid tori $T_1, \ldots, T_r$. Take the skeleton P_0 (without true vertices) for M_0 and the skeleton P_{E_i} (with one true vertex) for each exceptional block. For T_j we take the skeleton described in [FW], having $\max\{S(p_j, q_j) - 3, 0\}$ true vertices, for $j = 1, \ldots, r$. By assembling P_{E_i} with P_0 via the identity and the skeleton of T_j with P_0 as described in [FW], we obtain the required result. □

Next corollary characterizes a wide class of Seifert fibre spaces having complexity zero.

Corollary 1. *Let M be a Seifert fibre space with $\partial M \neq \emptyset$, and such that*

i) $SE(M) = AE(M)$ *(i.e., $t = 0$),*

ii) *the E-fibres of M, if any, are of type $(2,1)$, $(3,1)$ and $(3,2)$,*

then $c(M) = 0$.

Proof. From the above conditions we have $S(p_j, q_j) \leq 3$. So the statement follows directly from (1). □

3.2 The closed case

Now we deal with the case $\partial M = \emptyset$. In the orientable case many results are already known: the complexity of S^3 is zero and in [M2, p.77] the following upper bound for lens space complexity is given

$$c(L(p,q)) \leq \max\{S(p,q) - 3, 0\}, \tag{2}$$

which has been proved to be sharp in many cases (see [JRT, JRT2]). Efficient upper bounds for all closed orientable Seifert fibre spaces have been obtained in [MP2], and in the following we will extend the main result of that paper to the non-orientable case.

As in the bordered case we construct a spine of M by assembling together the skeletons of the different blocks in which M is decomposed. Since the manifold is closed, we need to construct a skeleton for the space $M_0 = M \setminus N(E(M) \cup SE(M))$[5] whose complement is a 3-ball, so we will need to add a section of $f_{|M_0} : M_0 \to f(M_0)$ to the skeleton of the main block described in the case with non-empty boundary.

Main block. Let $M_0 = \{0; (\epsilon, g, (0,0))\,; (0, \ldots, 0 \mid 0, \ldots, 0)\,; \}$ be a Seifert fibre space without exceptional fibres and let $f_0 : M_0 \to B_0$ be the projection map. Denote by $s = m_+ + m_-$ the number of boundary components of both B_0 and M_0. Set $\partial_- M_0 = \partial M_0$ and $\partial_+ M_0 = \emptyset$. Suppose that B_0 is neither a sphere nor a disk and denote by χ the Euler characteristic of the closed surface B obtained from B_0 by capping off by disks all its boundary components (i.e., $\chi = 2 - 2g$ if $\epsilon = o, o_1, o_2$ and $\chi = 2 - g$ if $\epsilon = n, n_1, n_2, n_3, n_4$). As a consequence, if $\chi = 2$ then $s \geq 2$.

Let D be a closed disk embedded in $\mathrm{int}(B_0)$ and let A be the union of the disjoint arcs properly embedded in $B_0 \setminus \mathrm{int}(D)$ described in Remark 3 (depicted by thick lines in Figure 10). Then A is non-empty and it is composed by $2-\chi$ edges with both endpoints in ∂D and s edges with an endpoint in ∂D and the other in a component of ∂B_0. By construction $B_0 \setminus (A \cup \partial B_0 \cup D)$ is homeomorphic to an open disk, and therefore $B_0 \setminus (A \cup \partial B_0 \cup \partial D)$ is the disjoint union of two open disks. Note that the number of points of A belonging to ∂D is at least two.

If $s_0 : B_0 \to M_0$ is a section of f_0, then $P_0'' = s_0(B_0) \cup f_0^{-1}(A) \cup f_0^{-1}(\partial D)$ is a non-simple polyhedron since $\mathrm{int}(s_0(A))$ is a collection of quadruple lines

[5] The regular neighborhood of $E(M) \cup SE(M)$ is supposed to be a union of fibres of M.

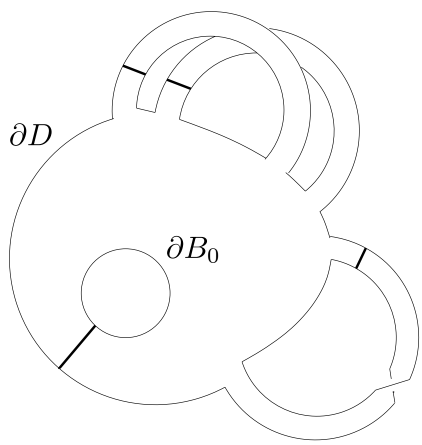

Figure 10: The set $A \subset B_0 \setminus \mathrm{int}(D)$.

in the polyhedron (the link of each point is homeomorphic to a graph with two vertices and four edges connecting them), and a similar phenomenon occurs for $s_0(\partial D \setminus A)$. In order to make the polyhedron P_0'' simple we perform "small" shifts by moving in parallel the disk $s_0(D)$ along the fibration and the components of $f_0^{-1}(A)$ as depicted in Figure 11. As shown by the pictures, the shift of any component of $f_0^{-1}(A)$ may be performed in two different ways that, as we will see, are not usually equivalent in term of complexity of the final spine. On the contrary, the two possible parallel shifts for $s_0(D)$ are equivalent as it is evident from Figure 12, which represents the torus $T = f_0^{-1}(\partial D)$. It is convenient to think the shifts of $f_0^{-1}(A)$ as performed on the components of A.

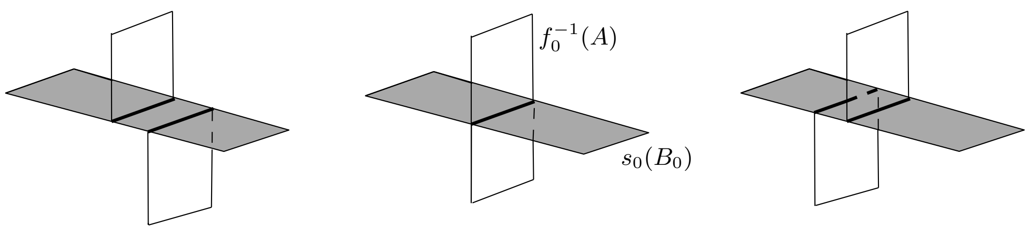

Figure 11: The two possible shifts on a component of $f_0^{-1}(A)$.

Let $P_0' = s_0(B_0 \setminus \mathrm{int}(D)) \cup D' \cup W \cup f_0^{-1}(\partial D)$ be the polyhedron obtained from P_0'' after the shifts, where D' and W are the results of the shifts of $s_0(D)$ and $f_0^{-1}(A)$, respectively. It is easy to see that $P_0' \cup \partial M_0$ is simple and P_0' intersects each component of ∂M_0 in a non-trivial theta-curve. Moreover, P_0' has 3 true vertices for each point of $A \cap \partial D$, so it has exactly $12-6\chi+3s$ true vertices. Since $M_0 \setminus (P_0' \cup \partial_- M_0)$ is the disjoint union of two open balls, in order to obtain a skeleton P_0 for the *main block* $(M_0, \partial_- M_0)$ it is enough to remove a suitable open 2-cell from the torus $T = f_0^{-1}(\partial D) \subset P_0'$, connecting in this way the two balls. Since $\Gamma = T \cap (s_0(B_0 \setminus \mathrm{int}(D)) \cup D' \cup A)$ is a graph cellularly embedded in T whose vertices are true vertices of P_0', we will remove the region R of $T \setminus \Gamma$ having in the boundary the highest number of vertices of Γ.

Referring to Figure 12, the graph Γ is composed by two horizontal parallel loops $d = \partial(s_0(D))$ and $d' = \partial D'$, and an arc with both endpoints on d for each boundary point of A belonging to ∂D. Reversing the shift of a component of A performs a symmetry along d of the correspondent arc(s). Clearly, if the shift is performed on a component of A which is the cocore of a handle, both arcs corresponding to the endpoints change as just described. Moreover, if the core of an orientable (resp. non-orientable) handle is sent by ω to 1 then the corresponding two arcs (which are not necessarily consecutive in Γ, as suggested by the dots in the pictures) are as in picture (a) (resp. (b)), or in the mirrored ones with respect to d. On the contrary, if the core is sent to -1 then the rightmost arc in each picture should be symmetrized with respect to the loop d.

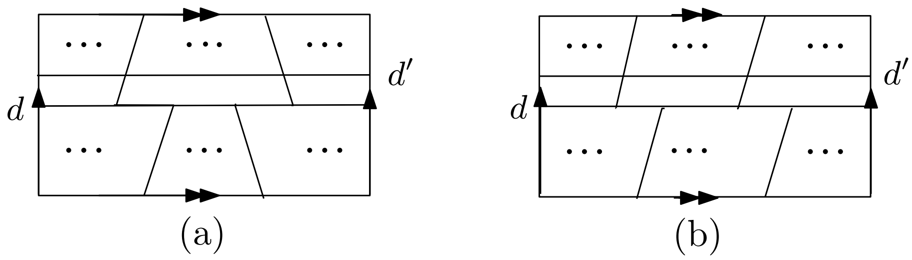

Figure 12: A fragment of the graph Γ embedded in $f_0^{-1}(\partial D)$.

A region of $T \setminus \Gamma$ has 5 vertices when the arcs belonging to its boundary are parallel and either 4 or 6 vertices otherwise. Since all regions has 5 vertices if and only if all arcs are parallel, the shifts of the elements of A can be chosen in such a way that there exists a region with 6 vertices in all cases except when $\chi = 1$, $\epsilon = n_1$ and $s = 0$. This exceptional case is the one depicted in Figure 12 (b) without dots: in that case all regions have 5 vertices. By removing such a region from P_0' we obtain a polyhedron P_0 for the main block $(M_0, \partial_- M_0)$ with $6(1-\chi) + 3s$ true vertices, while in the special case $\chi = 1$, $\epsilon = n_1$ and $s = 0$, the polyhedron has exactly one true vertex. We remark that changing the shift of a component of A intersecting ∂B_0 changes the intersection between the corresponding element of $f^{-1}(A)$ and $\partial_-(M_0)$ (which is a non-trivial theta curve) by a flip move (see bottom and top face of the block of Figure 13).

It is important to point out that when $s = 0$ the polyhedron P_0 is a simple spine for M_0.

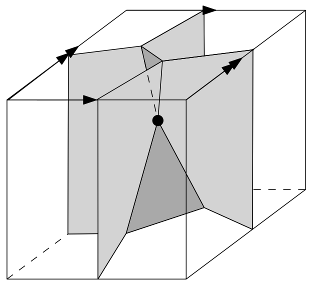

Figure 13: A flip block connecting two theta graphs.

Exceptional block. Let M_E be either $N \times S^1$ or $N \widetilde{\times} S^1$ considered with the Seifert fibre space structures depicted in Figure 6 (a') and (b'), respectively, and denote by $f : M_E \to B_E$ the projection map. Consider the skeleton P_E of the exceptional block $(M_E, \emptyset)$ constructed in the bordered case (see Figure 8). In that case $P_E \cap \partial M_E$ is a regular fibre, while in order to make the assembling with the main block the intersection should be a theta graph. Therefore, referring to Figure 14, we modify P_E as follows:

- add an annulus γ which is a section of the restriction of f to the space $M_E \setminus \operatorname{int}((N' \times I) / \sim)$;
- modify the annulus α by[6] pushing (i) the bottom part of the right strip toward $N \times \{0\}$ and the upper part of the left strip toward $N \times \{1\}$;
- take the surface $\partial((N' \times I)/\sim) \setminus R$, where the 2-cell R is the dashed region of the left picture of Figure 15.

If P_E still denote the resulting skeleton, then $M_E \searrow ((N' \times I)/\sim) \cup \alpha \cup \gamma$ and $((N' \times I)/\sim) \setminus P_E$ is a 3-ball. Therefore the polyhedron P_E is a skeleton with 3 true vertices (the thick points represented both in Figures 14 and 15) for the *exceptional block* $(M_E, \emptyset)$. Note that if we modify α in the opposite way, namely pushing (i) the upper part of the right strip toward $N \times \{0\}$ and the bottom part of the left strip toward $N \times \{1\}$ the theta graph on $P_E \cap \partial M_E$ changes by a flip. Anyway, we can still find a skeleton with 3 true vertices (see the right part of Figure 15).

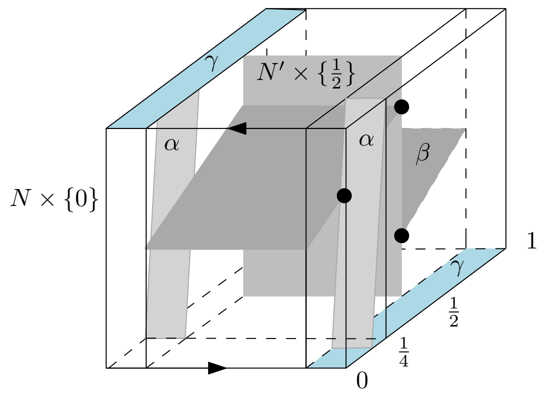

Figure 14: The exceptional skeleton P_E (in gray) for the block $(M_E, \emptyset)$.

Now we are ready to state our result on the complexity of closed Seifert fibre spaces.

[6] We perform this change in order to remove the quadruple line $\alpha \cap \gamma$.

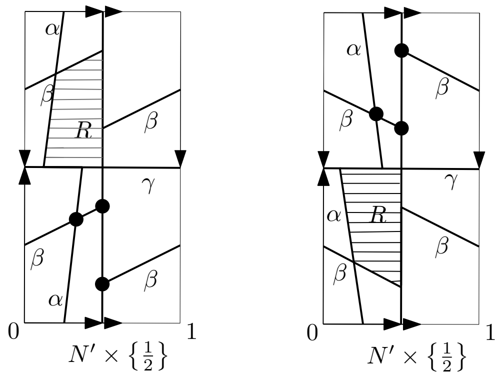

Figure 15: The surface $\partial((N' \times I)/\sim) \setminus R$, corresponding to the two different choices for P_E.

Theorem 5. *Let* $M = \{b; (\epsilon, g, (t,k)); (\mid); ((p_1, q_1), \ldots, (p_r, q_r))\}$ *be a closed Seifert fibre space with* $b \geq -r/2$, *and let* $\chi = 2 - 2g$ *if* $\epsilon = o, o_1, o_2$ *and* $\chi = 2 - g$ *if* $\epsilon = n, n_1, n_2, n_3, n_4$.

1. *If* $\chi = 2$ *and* $r = t = 0$, *then* $c(M) \leq \max\{b - 3, 0\}$;

2. *if* $\chi = 2$, $t = 0$, $r = 1$ *and* $b > 0$, *then* $c(M) \leq \max\{b + S(p_1, q_1) - 3, 0\}$;

3. *if* $\chi = 2$, $t = 0$, $r = 1$ *and* $b = 0$, *then* $c(M) \leq \max\{S(p_1, q_1) - 3 - \lfloor p_1/q_1 \rfloor, 0\}$, *where* $\lfloor \cdot \rfloor$ *denotes the integer part function;*

4. *if* $\chi = 1$, $\epsilon = n_1$, $r = t = 0$ *and* $b = 0$, *then* $c(M) \leq 1$;

5. *if* $\chi = 1$, $\epsilon = n_1$, $r = t = 0$ *and* $b = 1$, *then* $c(M) = 0$;

6. *in all other cases:*

$$c(M) \leq \max\{b - 1 + \chi, 0\} + 6(1 - \chi) + \sum_{j=1}^{r} (S(p_j, q_j) + 1)\,, \tag{3}$$

if M *is orientable*[7];

$$c(M) \leq 6(1 - \chi) + 6t + \sum_{j=1}^{r} (S(p_j, q_j) + 1)\,, \tag{4}$$

[7] Formula (3) has been introduced in [MP2]. Here we give a new and more direct proof of it.

if M is non-orientable.

Proof. 1. In this case $M = L(b, 1)$ (see [O]), so the result follows from (2).

2. In this case $M = L(bp_1 + q_1, p_1)$ (see [O]), so the result follows from (2).

3. In this case $M = L(q_1, p_1)$ (see [O]), so the result follows from (2) since $M = L(q_1, r_1)$, where $r_1 \equiv p_1 \bmod q_1$ with $0 < r_1 < q_1$, and $S(q_1, r_1) = S(p_1, q_1) - \lfloor p_1/q_1 \rfloor$.

4. In this case $M = \mathbb{RP}^2 \times S^1$ and a spine for M is the polyhedron P_0 of the main block, which in this case has exactly one true vertex.

5. In this case $M = S^2 \widetilde{\times} S^1$ (see [O]). Let $S^2 \widetilde{\times} S^1 = (S^2 \times I)/\sim$, where $(x, 0) \sim (a(x), 1)$ being a the antipodal map of S^2, then an almost spine for M is $((S^2 \times \{0\}) \cup (\{P\} \times I))/\sim$, which is homeomorphic to a 2-sphere with a diameter, and therefore has no true vertices.

Now we turn to the proof of formulae (3) and (4).

If $\chi = 2$, $t = 1$ and $r = 0$, then the base space is a disk whose boundary is a reflector circle. A simple spine for M[8] is the union of the exceptional set (which is a torus T) and a section of the fibration (which is a disk D), being $T \cap D = \partial D$ a non-trivial simple closed curve on T. Of course the spine has no true vertices and therefore $c(M) = 0$, which proves (4).

From now on, let M be a Seifert fibre space different from the previous ones.

First suppose $b = 0$. Let $f : M \to B$ be the projection map and denote by $B_0 \subset B$ the surface obtained from B by removing disjoint open disks around each cone point and disjoint open collars around each reflector circle. Let $\partial_+ B_0 = \emptyset$, $\partial_- B_0 = \partial B_0$ and $(M_0, \partial_- M_0) = (f^{-1}(B_0), f^{-1}(\partial_- B_0))$, therefore $\partial_+ M_0 = \emptyset$. The block $(M_0, \partial_- M_0)$ is a main block with $s = t + r$ boundary components and $M \setminus \mathrm{int}(M_0)$ is the disjoint union of t exceptional blocks M_{E_i}

[8] It is easy to see that in this case $M = S^2 \widetilde{\times} S^1$.

(one for each component of $CE(M)$) and r fibred solid tori $T_1, \ldots, T_r$ (one for each isolated exceptional fibre).

For M_0 we take a skeleton P_0 as previously described, where the choice of the shifts depends on the following. The skeleton for T_j described in [FW] and having $S(p_j, q_j) - 3$ true vertices, have to be modified since in the closed case $P_0 \cap T_j$ is a theta curve, and no longer a simple closed curve. So we replace a transitional block (having no vertices) connecting a regular fibre with a theta graph denoted by θ' (according to the notation of [FW]), with either one or two flip blocks (see Figure 13), each having one true vertex, connecting the theta graph $P_0 \cap T_j$ to θ'. The number of the additional flip blocks depends on the shift of the corresponding component δ of A used in the construction of the main block. We call the shift of δ *regular* when a single flip is sufficient and *singular* when two flips are required (see Figure 16, where the shifted arcs are denoted by dotted lines). Since we want to have a skeleton P_0 for M_0 with $6(1-\chi)+3t+3r$ true vertices, all flips can be chosen regular if either $t > 0$ or $\chi < 2$ and all flips except one can be chosen regular otherwise (see Figure 17).

For M_{E_i} we take a skeleton P_{E_i} such that the theta graph $P_{E_i} \cap M_{E_i}$ coincides with the theta graph on the corresponding component of $M_0 \cap P_0$ for $i = 1, \ldots, t$. The skeleton has always 3 true vertices, since the choice of the shift on the corresponding component of A does not affect the number of its true vertices (see Figure 15).

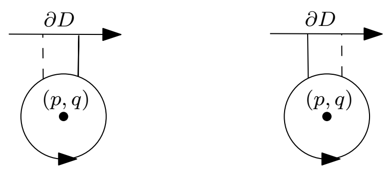

Figure 16: Regular shift (on the left) and singular shift (on the right).

By assembling P_{E_i} and the skeleton of $(T_j, \emptyset)$ with P_0 by the identity for $i = 1, \ldots, t$ and $j = 1, \ldots, r$, we obtain the desired spine S for M. When either $t > 0$ or $\chi < 2$, the number of true vertices of S is $6(1-\chi) + 3(t +$

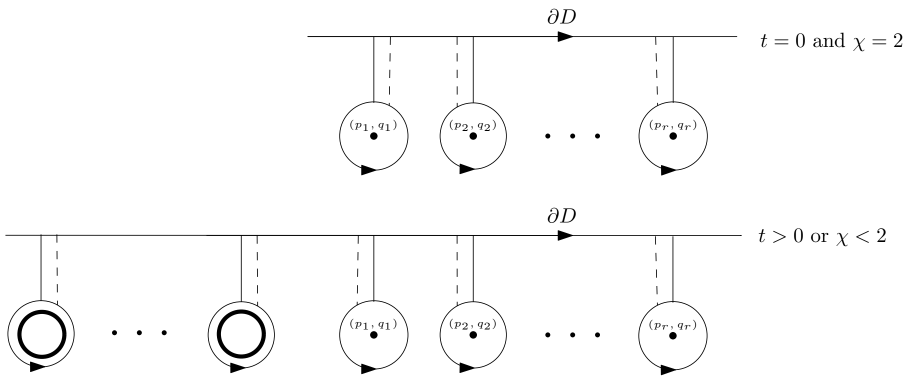

Figure 17: The choice of the shifts for the components of A intersecting ∂D.

$r) + 3t + \sum_{j=1}^{r} (S(p_j, q_j) - 2)$, which proves (3) and (4). When $\chi = 2$ and $t = 0$, the number of true vertices of S is $-6 + 3r + \sum_{j=1}^{r} (S(p_j, q_j) - 2) + 1$, and (3) is proved.

Let now $b \neq 0$ (and therefore $t = 0$). We prove (3) and (4) in different steps: $b = 1$, $b = -1$ and $|b| > 1$. Let M' be the Seifert fibre space with the same parameters of M but with $b = 0$, and let S' be the spine of M' constructed as before.

Figure 18

First suppose $b = 1$. In this case M can be obtained from M' by adding a non-trivial (non-exceptional) fibre of type $(1, 1)$. Namely, by removing from M' a trivially fibred solid torus Φ which is a fiber-neighborhood of a regular fibre ϕ, and by attaching back a solid torus $ST = \Delta \times S^1$ via a homeomorphism $\psi : \partial(ST) \to \partial\Phi$ such that $\psi(\partial\Delta \times \{1\})$ is a curve of type $(1, 1)$ on $\partial\Phi$. It is convenient to take the fibre ϕ corresponding to an internal point P of A and suppose that $f(\Phi)$ is a "small" disk intersecting the component δ of A containing P in an interval and being disjoint from $\partial B_0 \cup \partial D$ and from the other components of A. In this way $\delta \setminus \mathrm{int}(f(\Phi))$ is the disjoint union of two arcs δ' and δ'', where we can perform the shifts independently (see Figure 18). A spine S of M is obtained as follows.

First of all, remove from the spine S' of M' the internal part of Φ and possibly change the twist corresponding to either δ' or δ'' without increasing the number of true vertices of the main block. Let S'' be the polyhedron obtained in this way and set $S''' = S'' \cup \partial\Phi \cup_\psi (\Delta \times \{1\})$. Then $M \setminus S'''$ is the union of two open 3-balls, since $\Phi \setminus (\partial\Phi \cup_\psi (\Delta \times \{1\}))$ is an open 3-ball. Therefore, in order to obtain the spine S of M we have to remove from S''' a suitable open 2-cell on $\partial\Phi$. The space $\Gamma' = (\partial\Phi \cap S'') \cup \psi(\partial\Delta \times \{1\}))$ is a graph cellularly embedded in $\partial\Phi$ (see Figure 19, where the label 1 inside the disc stands for the fibre type $(1,1)$), so we delete the region R of $\partial\Phi \setminus \Gamma'$ having in the boundary the highest number of vertices of Γ'. If we take for δ' and δ'' the shifts induced by the one of δ, then we can choose R containing in its boundary all the true vertices of S belonging to $\partial\Phi$ with the exception of one (see the first two pictures of Figure 19) and S has one true vertex more than S'. On the contrary, if one of the two shifts is changed as in the third draw of Figure 19, then R can be chosen containing in its boundary all true vertices and therefore S and S' have the same number of true vertices.

So, if either $\chi = 2$ (and therefore $r \geq 2$) or $\chi = 1$ and $\epsilon = n_2$, we take as δ any arc of A and use for δ' and δ'' the shifts induced by the one of δ. Then (3) is proved.

If $\chi < 2$ and $\epsilon = n_1$ when $\chi = 1$, it is always possible to choose an arc δ of A not intersecting ∂B_0 and to choose the shifts for δ' and δ'' as depicted in the third draw of Figure 19 without increasing the number of true vertices of the main block. Then (3) and (4) are proved.

In this way (4) is proved for all cases.

Let now $b = -1$ (and therefore $r \geq 2$). The procedure to obtain M from M' and to construct the spine S is the same as in case $b = 1$, but this time

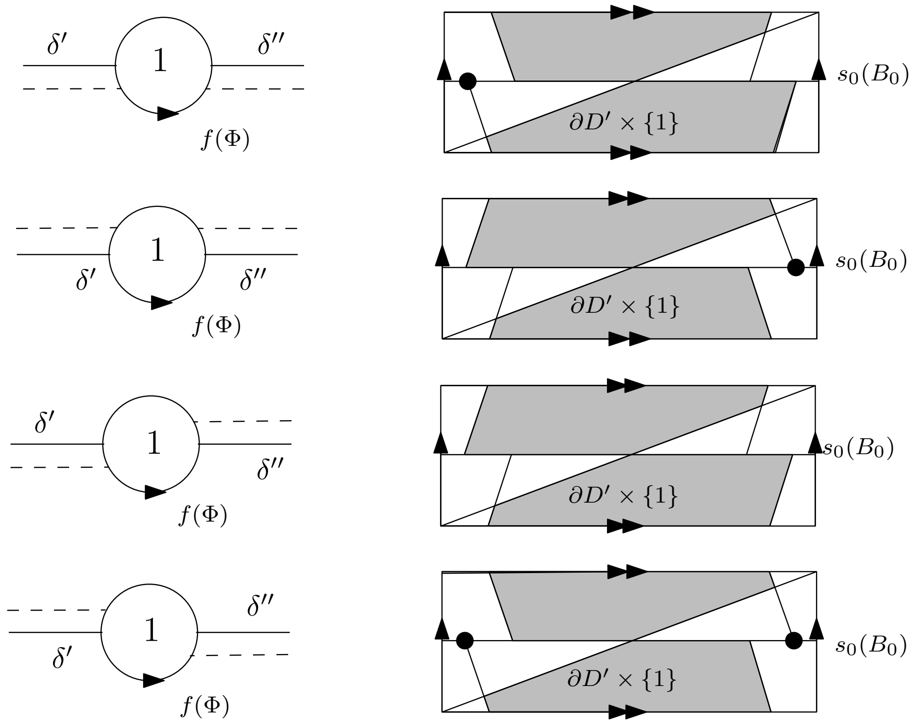

Figure 19: The graph Γ' embedded in the torus $\partial\Phi$ with different choices of the shifts for δ' and δ''.

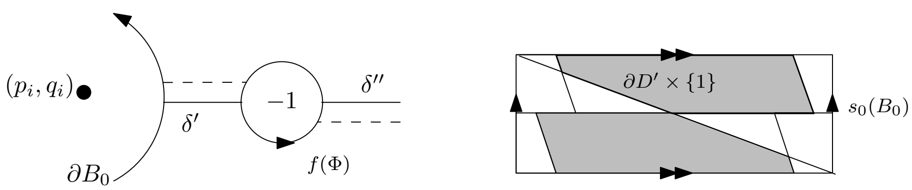

Figure 20

adding a non-trivial fibre of type $(1, -1)$.

If $\chi = 2$ take δ as the arc with non-regular shift. Then the shift of δ' and δ'' can be chosen as in Figure 20 (no true vertices out of the boundary of the gray region). Since the shift of the new arc which intersect ∂B_0 (say δ') becomes regular, the spine S has one true vertex less than S' (namely it has $-6 + 3r + \sum_{j=1}^{r} (S(p_j, q_j) - 2)$ true vertices) and (3) is proved.

If $\chi < 2$ take as δ an arc intersecting ∂B_0 and having a regular shift.

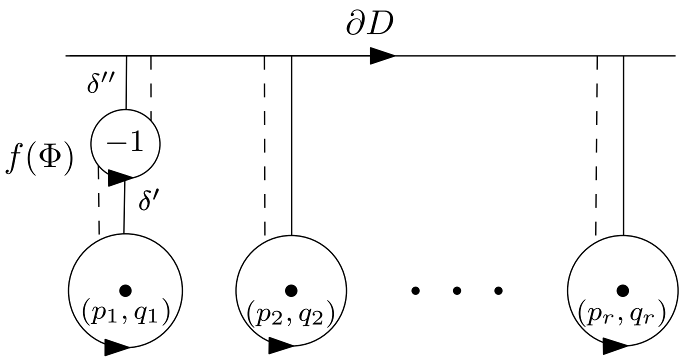

Figure 21

Then the shifts of δ' and δ'' can be chosen as in Figure 21: the main block does not increases the number of true vertices and the spine has the same number of vertices of the one of M', so proving (3).

Finally, let $|b| > 1$. In this case M can be obtained from M' by replacing $|b|$ trivial fibres with $|b|$ fibres of type $(1, \operatorname{sign}(b))$. Again it is convenient to choose the fibres corresponding to internal points of A and to remove disjoint fibre-neighborhoods of the chosen fibres with the same properties as before.

If $b < -1$ take all points in different arcs δ_i which intersect ∂B_0 (it is possible since $r \geq -2b = 2|b| > |b|$) and the shifts of the new arcs as depicted in Figure 22 (so $\partial\Phi_i$ is as in Figure 20). Moreover, if $\chi = 2$ the first point has to be taken in the arc with non-regular shift. In this way the shifts of all new arcs still intersecting ∂B_0 (say δ_i'') are regular, and the number of true vertices of the main block does not increase.

Therefore the spine S has the same number of true vertices of the case $b = -1$, which proves (3).

If $b > 1$ then it is possible to take $1 - \chi$ points in different arcs δ_i not intersecting ∂B_0 and to choose the shifts of δ_i' and δ_i'' in such a way that (i) $\partial\Phi_i$ is as in the third draw of Figure 19 and (ii) the number of true vertices of the main block does not increase (see the upper picture of Figure 23). The remaining $b - 1 + \chi$ points are chosen outside $f(\Phi_i)$ for all i, with the shifts of the new edges induced by those of the old ones as depicted in the bottom

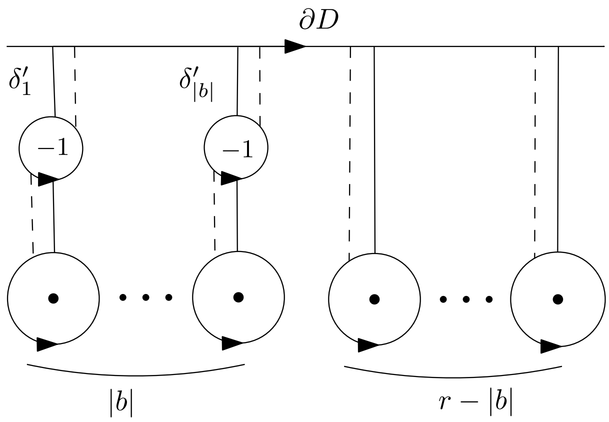

Figure 22

part of Figure 23. In this way (3) is proved.

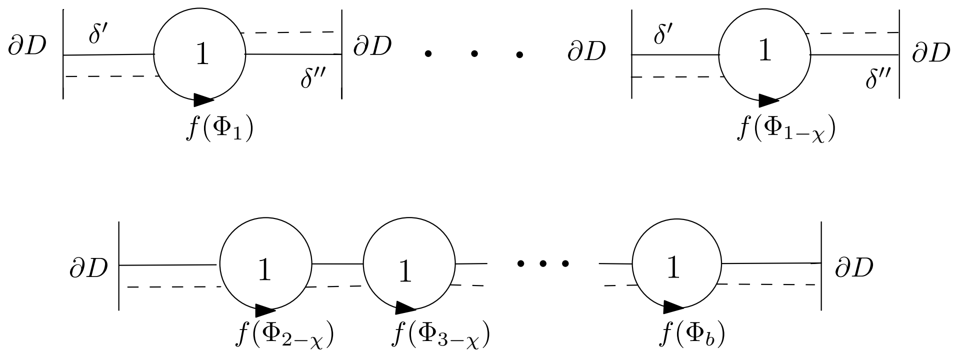

Figure 23

$\square$

Remark 4. In the non-orientable (closed) case, exact values of complexity are listed in [AM] (up to complexity 7), in [B] (up to complexity 10) and at the web page `https://regina-normal.github.io` (up to complexity 11). For all the Seifert fibre spaces included in those lists, that are about 350, the complexity estimation given by (4) is sharp, except in the cases: (i) $\chi = 1$, $\epsilon = n_1$, $t = 0$, $r = 1$ and (ii) $\chi = 2$, $t = r = 1$. Note that these cases concern different Seifert fibre structures of $\mathbb{RP}^2 \times S^1$ and $S^2 \widetilde{\times} S^1$, respectively, whose correct estimation is given in 4. and 5. of Theorem 5.

It is worth noting that Burton in [B] and `https://regina-normal.github.io` uses in some cases non-normalized parameters for Seifert fibre spaces: the space $\{1; (\epsilon, g, (0,0)); (\mid); ((p_1, q_1), \ldots, (p_{r-1}, q_{r-1}), (p_r, q_r))\}$, with $\epsilon \in \{o_2, n_1, n_3, n_4\}$, appears there as $\{0; (\epsilon, g, (0,0)); (\mid); ((p_1, q_1), \ldots, (p_{r-1}, q_{r-1}), (p_r, p_r - q_r))\}$.

In the orientable (closed) case, exact values of complexity are listed in [M2] (up to complexity 6 and partially up to complexity 7), in [MP2] (up to complexity 6 and partially up to complexity 9) and in the web page `http://matlas.math.csu.ru/?page=search` (up to complexity 12). For all the Seifert fibre spaces included in those lists the complexity estimation given by (3) is sharp except for the following cases:

(i) manifolds of the form $M = \{-1; (o_1, 0, (0,0)); (\mid); ((2,1), (n,1), (m,1))\}$ with $2 \le n \le m$, where the estimation of (3) exceeds the exact value by one or two.[9] In particular, if $n = 2$ then M also admits the Seifert fibre structure $M = \{m; (n_2, 1, (0,0)); (\mid);\}$ (see [O]), and in this case (3) gives the sharper value of complexity $c(M) \le m$;

(ii) manifolds of the form $M = \{-1; (o_1, 0, (0,0)); (\mid); ((2,1), (3,1), (p,q))\}$, with $p/q > 5$ and $p/q \notin \mathbb{Z}$, where the estimation of (3) exceeds the exact value by one.

The sharpness of formula (4) in all known cases justifies the following conjecture.

Conjecture. Let $M = \{b; (\epsilon, g, (t,k)); (\mid); ((p_1, q_1), \ldots, (p_r, q_r))\}$ be a non-orientable closed irreducible and $\mathbb{P}^2$-irreducible Seifert fibre space, then

$$c(M) = 6(1-\chi) + 6t + \sum_{j=1}^{r} (S(p_j, q_j) + 1).$$

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