On Mean Value formulas for solutions to second order linear PDEs

ON MEAN VALUE FORMULAS FOR SOLUTIONS TO SECOND ORDER LINEAR PDEs

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Abstract. In this paper we give a general proof of Mean Value formulas for solutions to second order linear PDEs, only based on the local properties of their fundamental solution Γ. Our proof requires a kind of pointwise vanishing integral condition for the intrinsic gradient of Γ. Combining our Mean Value formulas with a "descent method" due to Kuptsov, we obtain formulas with improved kernels. As an application, we implement our general results to heat operators on stratified Lie groups and to Kolmogorov operators.

1. Introduction

The motivation of the present paper is twofold. On the one hand to provide a unified proof of Mean Value theorems for general linear second order PDEs, including both hypoelliptic degenerate elliptic operators and their evolution counterpart. On the other hand to give a proof only based on the *local properties* of the fundamental solutions.

We consider the linear second order partial differential operator with smooth coefficients

$$\mathcal{L}u(z) := \operatorname{div}(\mathcal{A}(z)\nabla u(z)) + \langle b(z), \nabla u(z)\rangle + c(z)u(z), \qquad z \in \mathbb{X}, \tag{1.1}$$

with $\mathbb{X}$ open set in $\mathbb{R}^N$, $N \geq 2$,

$$\mathcal{A}(z) = (a_{ij}(z)), \qquad z \in \mathbb{X},$$

real symmetric $N \times N$-matrix, $b : \mathbb{X} \to \mathbb{R}^N$, $c : \mathbb{X} \to \mathbb{R}$.

We assume

$$\mathcal{A}(z) \text{ positive semidefinite for every } z \in \mathbb{X}$$

and

$$\operatorname{div} b = 0 \text{ in } \mathbb{X}.$$

Fixed $z_0 \in \mathbb{X}$ we assume that there exists a lower semicontinuous function $\Gamma(z_0, \cdot) : \mathbb{X} \to [0, \infty]$,

$$\Gamma(z_0, \cdot) \in C^\infty(\mathbb{X} \setminus \{z_0\}; [0, \infty[), \qquad \limsup_{z \to z_0} \Gamma(z_0, z) = \infty,$$

such that $\Gamma(z_0, \cdot)$ is a *local fundamental solution* of $\mathcal{L}^*$ - the formal adjoint of $\mathcal{L}$ - with pole at z_0; i.e., $z \mapsto \Gamma(z_0, z) \in L^1_{\text{loc}}(\mathbb{X})$ and

$$\int_{\mathbb{X}} \Gamma(z_0, z)\mathcal{L}v(z)\, dz = -v(z_0) \qquad \forall\, v \in C^\infty_c(\mathbb{X}). \tag{1.2}$$

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For every $r>0$ we define the super-level set of Γ

$$\Omega_r(z_0) := \left\{ z \in \mathbb{X} \,:\, \Gamma(z_0,z) > \frac{1}{r} \right\}.$$

By the properties of Γ, $\Omega_r(z_0)$ is a not empty open set and $z_0 \in \overline{\Omega_r(z_0)}$. By Sard's Lemma, for a.e. $r>0$ such that $\partial\Omega_r(z_0)\setminus\{z_0\}$ is not empty, the set $\partial\Omega_r(z_0)\setminus\{z_0\}$ is a smooth manifold and

$$\partial\Omega_r(z_0)\setminus\{z_0\} = \left\{ z \in \mathbb{X}\setminus\{z_0\} \,:\, \Gamma(z_0,z) = \frac{1}{r} \right\}.$$

We now introduce our *pointwise vanishing integral condition* for the intrinsic gradient of the fundamental solution.

The *intrinsic gradient* of the fundamental solution is

$$|\nabla_{\mathcal{A}}\Gamma(z_0,z)| := \langle \mathcal{A}(z)\nabla\Gamma(z_0,z), \nabla\Gamma(z_0,z)\rangle^{\frac{1}{2}},$$

where $\langle\cdot,\cdot\rangle$ and ∇ stands for the usual *inner product* and *gradient* in $\mathbb{R}^N$.

As usual, we denote $B(z_0,\varepsilon)$ the Euclidean ball in $\mathbb{R}^N$ centered at z_0 with radius ε and $\mathcal{H}^{N-1}$ the $N-1$-dimensional Hausdorff measure.

If $\varrho>0$, we say that the property $(H(z_0,\varrho))$ holds if:

PROPERTY $(H(z_0,\varrho))$:

the set $\Omega_\varrho(z_0)$ is bounded, $\partial\Omega_\varrho(z_0)\setminus\{z_0\}$ is not empty and smooth and

$$\int_{\partial B(z_0,\varepsilon)\setminus\Omega_\varrho(z_0)} |\nabla_{\mathcal{A}}\Gamma(z_0,z)|\, d\mathcal{H}^{N-1}(z) = o(1) \qquad \text{as } \varepsilon\to 0. \tag{1.3}$$

Our main result is the following *Mean Value formula*.

Theorem 1.1. *Assume that $\mathcal{L}$ is the operator described above, and let $r>0$ be such that $(H(z_0,\varrho))$ holds for a.e. $\varrho\in]0,r[$. Then for every $u\in C^2(\overline{\Omega_r(z_0)})$ such that $\mathcal{L}u=0$ we have*

$$u(z_0) = \frac{1}{r}\int_{\Omega_r(z_0)} u(z) K_r(z_0,z)\, dz \tag{1.4}$$

where $K_r(z_0,z)$ is a kernel defined as follows:

$$K_r(z_0,z) = \left(|\nabla_{\mathcal{A}}\log\Gamma(z_0,z)|^2 \varphi\left(\frac{1}{r\Gamma(z_0,z)}\right) - c(z)\int_{\frac{1}{\Gamma(z_0,z)}}^{r} \frac{1}{t}\varphi\left(\frac{t}{r}\right)\, dt \right),$$

where $\varphi:]0,1[\to[0,\infty[$ is any continuous function such that $\displaystyle\int_0^1 \varphi(t)\,dt = 1$.

Formula (1.4) extends to the operators $\mathcal{L}$ the Gauss mean value formula for harmonic functions, see (vi) below. Formulas as (1.4) are sometimes referred to as weighted mean value formulas, see e.g. [12], or as representation formulas, see e.g. [5].

More comments are in order:

(i) $|\nabla_{\mathcal{A}}\log\Gamma(z_0,z)|^2 = \dfrac{\langle \mathcal{A}(z)\nabla\Gamma(z_0,z), \nabla\Gamma(z_0,z)\rangle}{\Gamma(z_0,z)^2}$,

(ii) if $c\le 0$ then $K_r(z_0,z)\ge 0$ for every $z\in\Omega_r(z_0)$,

(iii) if $\varphi=1$ then

$$K_r(z_0,z) = |\nabla_{\mathcal{A}}\log\Gamma(z_0,z)|^2 - c(z)\log(r\Gamma(z_0,z)),$$

(iv) the condition (1.3) is trivially satisfied for every ε sufficiently small if z_0 is an interior point of $\Omega_r(z_0)$. This happens if

$$\Gamma(z_0, z_0) = \lim_{\zeta \to z_0} \Gamma(z_0, \zeta) = \infty, \tag{1.5}$$

(v) if $\mathbb{X} = \mathbb{R}^N$ a sufficient condition to have $\Omega_r(z_0)$ bounded for every $r > 0$ is

$$\lim_{|\zeta| \to \infty} \Gamma(z_0, \zeta) = 0.$$

More generally, $\Omega_r(z_0)$ is bounded for every $r > 0$ if Γ is a $\mathcal{L}$-Green function for $\mathbb{X}$, satisfying

$$\lim_{\zeta \to z} \Gamma(z_0, \zeta) = 0$$

for every $z \in \partial\mathbb{X}$, and also for $z = \infty$ if $\mathbb{X}$ is unbounded,

(vi) if $\mathcal{L} = \Delta$ is the Euclidean Laplacian, formula (1.4) becomes the classical Gauss Mean Value Theorem for harmonic functions by choosing as $\Gamma(z_0, \cdot)$ the fundamental solution of Δ with pole at z_0 and as φ the function $\varphi(t) = (\alpha + 1)t^\alpha$ with $\alpha = \frac{2}{N-2}$,

(vii) if $\mathcal{L} = \Delta - \partial_t$ is the Heat operator in $\mathbb{R}^N = \mathbb{R}^n_x \times \mathbb{R}_t$, denoting $z_0 = (x_0, t_0)$ and $z = (x, t)$, by taking $\varphi = 1$ and Γ the fundamental solution of $\Delta - \partial_t$, formula (1.4) becomes

$$u(z_0) = \frac{1}{r} \int_{\Omega_r(z_0)} u(z) W(z_0 - z)\, dz, \tag{1.6}$$

where $W(z) = W(x, t) = \frac{1}{4} \frac{|x|^2}{t^2}$ is the Pini-Watson kernel.

(viii) if $\mathcal{L}u(z) = \operatorname{div}(\mathcal{A}(z)\nabla u(z))$, i.e. if $b = 0$ and $c = 0$, has a fundamental solution with the pole interior to its superlevel sets, as in the elliptic and sub-elliptic cases, formula (1.4) has been proved in [2] if the function φ is power-like.

For "evolution" equations and, more in general, in the case in which the pole z_0 is at the boundary of $\Omega_r(z_0)$, as it happens if

$$\liminf_{\zeta \to z_0} \Gamma(z_0, \zeta) = 0 \quad \text{and} \quad \limsup_{\zeta \to z_0} \Gamma(z_0, \zeta) = \infty, \tag{1.7}$$

the kernels appearing in (1.4) are usually unbounded: see, e.g., the Pini-Watson kernel in (1.6). For these operators, starting from Theorem 1.1, we obtain Mean Value formulas with *improved* kernels. Our technique to prove these mean formulas with "well behaved" kernels is based on a *method of descent* introduced by L.P. Kuptsov in [12] and subsequently used in [7] for parabolic equations with smooth coefficients and in [8] and [14] for Kolmogorov-Fokker-Planck operators.

To state this result we need some notation.

Let $\mathbb{O} \subseteq \mathbb{R}^n$ be an open set and $\mathbb{X} := \mathbb{O} \times \mathbb{R}$.

Consider the operator $\mathcal{L}$ with smooth coefficients

$$\mathcal{L} := \operatorname{div}_x(A(x)\nabla_x) + \langle b(x), \nabla_x \rangle_{\mathbb{R}^n} - \partial_t, \qquad x \in \mathbb{O}, \tag{1.8}$$

where

$$A(x) = (a_{ij}(x))$$

is a real symmetric positive semidefinite $n \times n$-matrix for every $x \in \mathbb{O}$.

We will refer to it as an *evolution* operator in $\mathbb{X} \subseteq \mathbb{R}^{n+1}_z = \mathbb{R}^n_x \times \mathbb{R}_t$.

Fixed $p \in \mathbb{N}$, denote

$$\mathcal{L}^{(p)} := \mathcal{L} + \Delta^{(p)}_y,$$

where $\Delta^{(p)}_y$ is the classical Laplace operator in $\mathbb{R}^p$ in the (new) variables $y = (y_1, \cdots, y_p)$.

Denoting z the point (x,t) in $\mathbb{X}$ and $\hat{z}$ the point (x,y,t), let $\Gamma(z_0,\cdot)$ be a *local fundamental solution* of $\mathcal{L}^*$ - the formal adjoint of $\mathcal{L}$ - with pole at z_0 and define

$$\Gamma^{(p)}(\hat{z}_0,\hat{z}) := \Gamma(z_0,z)G^{(p)}((y_0,t_0),(y,t)) \tag{1.9}$$

where

$$G^{(p)}((y_0,t_0),(y,t)) = \begin{cases} (4\pi(t_0-t))^{-p/2}\exp\left(-\frac{|y_0-y|^2}{4(t_0-t)}\right) & \text{if } t_0 > t \\ 0 & \text{if } t_0 \le t \end{cases} \tag{1.10}$$

is the fundamental solution of the Heat operator in $\mathbb{R}^{p+1}$

$$\Delta_y^{(p)} - \partial_t.$$

It can be proved in a standard way that $\Gamma^{(p)}(\hat{z}_0,\cdot)$ is a *local fundamental solution* of $(\mathcal{L}^{(p)})^*$.

Let $\hat{\Omega}_r^{(p)}(\hat{z}_0)$ be the super-level set

$$\hat{\Omega}_r^{(p)}(\hat{z}_0) := \big\{\hat{z} \,:\, \Gamma^{(p)}(\hat{z}_0,\hat{z}) > \frac{1}{r}\big\}.$$

If we denote $\mathcal{A}^{(p)}$ the matrix of the second order part of $\mathcal{L}^{(p)}$, then $\mathcal{A}^{(p)}(\hat{z})$ is the $(n+p+1)\times(n+p+1)$ real matrix

$$\mathcal{A}^{(p)}(\hat{z}) = \mathcal{A}^{(p)}(x,y,t) = \mathcal{A}^{(p)}(x) := \begin{bmatrix} A^{(p)}(x) & 0 \\ 0 & 0 \end{bmatrix},$$

with $A^{(p)}$ the $(n+p)\times(n+p)$ matrix

$$A^{(p)}(x) := \begin{bmatrix} A(x) & 0 \\ 0 & I_p \end{bmatrix}.$$

We also define

$$\Phi_p(z_0,z) := \frac{\Gamma(z_0,z)}{(4\pi(t_0-t))^{\frac{p}{2}}}, \tag{1.11}$$

$$R_r(z_0,z) := \sqrt{4\pi(t_0-t)\log(r\Phi_p(z_0,z))} \tag{1.12}$$

and

$$\Omega_r^{(p)}(z_0) = \big\{z=(x,t) \,:\, \Phi_p(z_0,z) > \frac{1}{r}\big\}.$$

Then the following mean value formula holds.

Theorem 1.2. *Assume that $\mathcal{L}$ is the operator* (1.8) *and let $r>0$ be such that $\Gamma^{(p)}$ satisfies the analogue of $(H(z_0,\varrho))$ for a.e. $\varrho \in]0,r[$, i.e.,*

the set $\hat{\Omega}_\varrho^{(p)}(\hat{z}_0)$ is bounded, $\partial\hat{\Omega}_\varrho^{(p)}(\hat{z}_0)\setminus\{\hat{z}_0\}$ is not empty and smooth and

$$\int_{\partial B(\hat{z}_0,\varepsilon)\setminus\hat{\Omega}_\varrho^{(p)}(\hat{z}_0)} |\nabla_{\mathcal{A}^{(p)}}\Gamma^{(p)}(\hat{z}_0,\hat{z})|\, d\mathcal{H}^{n+p}(\hat{z}) = o(1) \qquad \text{as } \varepsilon\to 0.$$

Then

$$u(z_0) = \frac{1}{r}\int_{\Omega_r^{(p)}(z_0)} u(z)W_r^{(p)}(z_0,z)\,dz \tag{1.13}$$

for every $u\in C^2(\overline{\Omega_r^{(p)}(z_0)})$ such that $\mathcal{L}u=0$.

Here

$$W_r^{(p)}(z_0,z) := \omega_p R_r^p(z_0,z)\left\{W(z_0,z) + \frac{p}{4(p+2)}\Big(\frac{R_r(z_0,z)}{t_0-t}\Big)^2\right\}, \tag{1.14}$$

where

$$W(z_0,z) := \frac{\langle A(x)\nabla_x\Gamma(z_0,z), \nabla_x\Gamma(z_0,z)\rangle_{\mathbb{R}^n}}{\Gamma(z_0,z)^2}.$$

We remark that the presence in the kernel (1.14) of the coefficient $R_r^p(z_0,z)$, containing the positive power $(t_0-t)^{\frac{p}{2}}$, improves the behaviour of the kernel close to z_0.

We notice that if $p=0$, (1.13) becomes

$$u(z_0) = \frac{1}{r}\int_{\Omega_r(z_0)} u(z)\frac{\langle A(x)\nabla_x\Gamma(z_0,z), \nabla_x\Gamma(z_0,z)\rangle_{\mathbb{R}^n}}{\Gamma(z_0,z)^2}\,dz,$$

that coincides with (1.4), taking into account that in this case $c=0$, $\varphi=1$ and $\mathcal{A}(z)=\mathcal{A}(x,t)=\mathcal{A}(x)$ is the $(n+1)\times(n+1)$ real matrix

$$\mathcal{A}(x,t)=\mathcal{A}(x)=\begin{bmatrix} A(x) & 0 \\ 0 & 0 \end{bmatrix}.$$

As it is well known there is a wide literature concerning the Mean Value formulas for solutions to linear second order elliptic, parabolic and even subelliptic PDE's.

The Mean Value formula for caloric functions on the superlevel sets of the fundamental solution of the heat operator has been first proved by Pini in [18], see also Montaldo [16]. Formula (1.6) is due to Watson [23]. Fabes and Garofalo extended (1.6) to the solutions of parabolic equations with smooth coefficients in [6]. Citti, Garofalo and the second author proved in [4] a version of Theorem 1.1 for Hörmander operators "sum of squares of vector fields", hence, in particular, for the sub-Laplacians on stratified Lie groups, so widely extending a Mean Value formula proved by Gaveau [9] for the Kohn Laplacian on the Heisenberg-Weyl group. More generally, when $\mathcal{L}$ is an hypoelliptic operator endowed with a "well behaved" fundamental solution with pole $z_0\in\Omega_r(z_0)$ (see the remark (iv) above), and the coefficients b and c are zero, formula (1.4) with $\varphi(t)=(\alpha+1)t^\alpha$, $\alpha>-1$, has been proved in [2].

Mean value formulas on the level sets of the fundamental solutions of Kolmogorov-Fokker-Planck operators were proved in [8] and in [14], while the paper [13] contains analogous formulas for hypoelliptic second order evolution operators which agree with the classical heat operator in a neighborhood of *infinity*. We want to stress that all the proofs of the just mentioned Mean Value formulas for evolution equations rest on a *global property* of the involved fundamental solutions: their so called stochastic completeness (see e.g. [20] for the definition of this last notion and its applications to Sobolev-type inequalities). On the contrary our proof rests on a *local property* of the fundamental solutions: a kind of *pointwise vanishing integral condition* for the *intrinsic gradient* of the local fundamental solutions: see (1.3). Due to the local nature of this condition, our method allows to obtain Mean Value formulas on the super level sets of Green functions.

The plan of the paper is the following. In Section 2 we give the proof of Theorem 1.1 and in Section 3 the proof of Theorem 1.2. In Section 4 we apply Theorem 1.1 and Theorem 1.2 to the Heat operators on stratified Lie groups, see Theorems 4.1 and 4.3, respectively. We remark that in Theorem 4.3 we prove Mean Value formulas with bounded kernels. In Section 5 we apply Theorems 1.1 and 1.2 to Kolmogorov-Fokker-Planck operators, see Theorems 5.1 and 4.3, respectively, so obtaining Mean Value formulas already stated in [14]. We remark that the application of Theorem 1.1 to Kolmogorov-Fokker-Planck operators requires a precise estimate of the intrinsic gradient of the logarithm of the fundamental solution. This estimate is proved in Lemma 5.4 and it seems to have an independent interest in its own right. This estimate is the analogous to the ones proved in [15] and [19] for the Heat operators on Heisenberg's type groups.

As usual, a list of references closes the paper. Due to the wide literature on the object of our research, our list is far from being exhaustive. For more hints we refer to the bibliography of the papers here cited, to the excellent survey by Netuka and Veselý [17] and to the recent [5].

2. Proof of Theorem 1.1

To prove Theorem 1.1, we first prove a Poisson-Jensen formula.

Proposition 2.1 (Poisson-Jensen formula)**.** *Assume that $\mathcal{L}$ is the operator* (1.1)*, and let $r > 0$ be such that $(H(z_0, r))$ holds. Then for a.e. $r > 0$ such that $\Omega_r(z_0)$ is bounded and smooth*

$$u(z_0) = \int_{\partial\Omega_r(z_0)} u(z)\langle \mathcal{A}(z)\nabla\Gamma(z_0, z), \frac{\nabla\Gamma(z_0, z)}{|\nabla\Gamma(z_0, z)|}\rangle \, d\mathcal{H}^{N-1}(z) - \int_{\Omega_r(z_0)} \left(\Gamma(z_0, z) - \frac{1}{r}\right) \mathcal{L}u \, dz - \frac{1}{r}\int_{\Omega_r(z_0)} cu \, dz$$

for every $u \in C^2(\overline{\Omega_r(z_0)})$.

To prove Proposition 2.1 we will use the following lemma.

Lemma 2.2. *Let $\mathcal{L}$ be the operator* (1.1)*. If $u \in C^2(\Omega)$, $\Omega \subseteq \mathbb{X}$ bounded smooth open set, then*

$$\int_\Omega \mathcal{L}u \, dz = \int_{\partial\Omega} \left(\langle \mathcal{A}\nabla u, \nu\rangle + u\langle b, \nu\rangle\right) d\mathcal{H}^{N-1}(z) + \int_\Omega cu \, dz.$$

Proof. Since $\operatorname{div} b = 0$,

$$\int_\Omega \langle b, \nabla u\rangle \, dz = \int_\Omega \operatorname{div}(bu) \, dz - \int_\Omega u \operatorname{div} b \, dz = \int_\Omega \operatorname{div}(bu) \, dz.$$

Therefore by the divergence theorem we get

$$\int_\Omega \mathcal{L}u \, dz = \int_\Omega \big(\operatorname{div}(\mathcal{A}\nabla u + bu) + cu\big) \, dz = \int_{\partial\Omega} \left(\langle \mathcal{A}\nabla u, \nu\rangle + u\langle b, \nu\rangle\right) d\mathcal{H}^{N-1}(z) + \int_\Omega cu \, dz.$$

□

We now give the proof of Proposition 2.1.

Proof of Proposition 2.1. Consider $u \in C^2(\overline{\Omega_r(z_0)})$. Then there exists an open smooth set $O \subset \mathbb{X}$, $\overline{\Omega_r(z_0)} \subset O$ such that $u \in C^2(O)$.

Fixed $\varepsilon_0 > 0$ small enough so that $\overline{B(z_0, \varepsilon_0)} \subseteq O$, we denote

$$\tilde{\Omega}_{r,\varepsilon}(z_0) := \Omega_r(z_0) \cup B(z_0, \varepsilon) \qquad \varepsilon \in]0, \varepsilon_0[,$$

where $B(z_0, \varepsilon)$ is the Euclidean ball in $\mathbb{X}$. The open set $\tilde{\Omega}_{r,\varepsilon}(z_0)$ has C^1-piecewise boundary and $|\partial\tilde{\Omega}_{r,\varepsilon}(z_0)| = 0$.

Notice that if $z_0 \in \Omega_r(z_0)$ then we can assume $\tilde{\Omega}_{r,\varepsilon}(z_0) = \Omega_r(z_0)$ and many of the computations below become trivial (see (iv) in Introduction).

Consider

$$\varphi \in C_c^\infty(O), \quad \text{such that } \ \varphi(z) = 1 \text{ in } \overline{\tilde{\Omega}_{r,\varepsilon}(z_0)} \tag{2.1}$$

and satisfying $0 \le \varphi(z) \le 1$ for all z.

By (1.2) and (2.1) we get

$$\begin{aligned} u(z_0) = u(z_0)\varphi(z_0) &= -\int_O \Gamma(z_0,z)\mathcal{L}(u\varphi)(z)\,dz \\ &= -\int_{O\setminus\overline{\tilde\Omega_{r,\varepsilon}(z_0)}} \Gamma(z_0,z)\mathcal{L}(u\varphi)(z)\,dz - \int_{\tilde\Omega_{r,\varepsilon}(z_0)} \Gamma(z_0,z)\mathcal{L}u(z)\,dz. \end{aligned}$$

By definition of $\mathcal{L}$ we obtain

$$\begin{aligned} u(z_0) = &-\int_{O\setminus\overline{\tilde\Omega_{r,\varepsilon}(z_0)}} \Gamma(z_0,z)\,\mathrm{div}(\mathcal{A}(z)\nabla(u\varphi)(z))\,dz \\ &-\int_{O\setminus\overline{\tilde\Omega_{r,\varepsilon}(z_0)}} \Gamma(z_0,z)\langle b(z),\nabla(u\varphi)(z)\rangle\,dz - \int_{O\setminus\overline{\tilde\Omega_{r,\varepsilon}(z_0)}} c(z)(u\varphi)(z)\Gamma(z_0,z)\,dz \\ &-\int_{\tilde\Omega_{r,\varepsilon}(z_0)} \Gamma(z_0,z)\mathcal{L}u(z)\,dz. \end{aligned} \tag{2.2}$$

Let us consider the first integral at the right hand side.
By the symmetry of $\mathcal{A}$

$$-\Gamma(z_0,z)\,\mathrm{div}(\mathcal{A}(z)\nabla(u\varphi)(z)) = -\,\mathrm{div}(\Gamma(z_0,z)\mathcal{A}(z)\nabla(u\varphi)(z)) + \langle\mathcal{A}(z)\nabla\Gamma(z_0,z),\nabla(u\varphi)(z)\rangle$$

and, by the divergence theorem and the properties of φ,

$$-\int_{O\setminus\overline{\tilde\Omega}_{r,\varepsilon}(z_0)} \mathrm{div}(\Gamma(z_0,z)\mathcal{A}(z)\nabla(u\varphi)(z))\,dz = \int_{\partial\tilde\Omega_{r,\varepsilon}(z_0)} \Gamma(z_0,z)\langle\mathcal{A}(z)\nabla u(z),\nu\rangle\,d\mathcal{H}^{N-1}(z),$$

where ν is the outward normal vector to the set $\tilde\Omega_{r,\varepsilon}(z_0)$. Therefore

$$\begin{aligned} &-\int_{O\setminus\overline{\tilde\Omega}_{r,\varepsilon}(z_0)} \Gamma(z_0,z)\,\mathrm{div}(\mathcal{A}(z)\nabla(u\varphi)(z))\,dz \\ &= \int_{\partial\tilde\Omega_{r,\varepsilon}(z_0)} \Gamma(z_0,z)\langle\mathcal{A}(z)\nabla u(z),\nu\rangle\,d\mathcal{H}^{N-1}(z) + \int_{O\setminus\overline{\tilde\Omega_{r,\varepsilon}(z_0)}} \langle\mathcal{A}(z)\nabla\Gamma(z_0,z),\nabla(u\varphi)(z)\rangle\,dz \end{aligned}$$

Thus, the equality (2.2) can be written as follows.

$$\begin{aligned} u(z_0) = &\int_{\partial\tilde\Omega_{r,\varepsilon}(z_0)} \Gamma(z_0,z)\langle\mathcal{A}(z)\nabla u(z),\nu\rangle\,d\mathcal{H}^{N-1}(z) \\ &+\int_{O\setminus\overline{\tilde\Omega_{r,\varepsilon}(z_0)}} \langle\mathcal{A}(z)\nabla\Gamma(z_0,z),\nabla(u\varphi)(z)\rangle\,dz \\ &-\int_{O\setminus\overline{\tilde\Omega_{r,\varepsilon}(z_0)}} \Gamma(z_0,z)\langle b,\nabla(u\varphi)\rangle\,dz - \int_{O\setminus\overline{\tilde\Omega_{r,\varepsilon}(z_0)}} c(z)(u\varphi)(z)\Gamma(z_0,z)\,dz \\ &-\int_{\tilde\Omega_{r,\varepsilon}(z_0)} \Gamma(z_0,z)\mathcal{L}u(z)\,dz \\ &=: J_1 + J_2 + J_3 + J_4 + J_5. \end{aligned} \tag{2.3}$$

Estimate of J_1.
We have

$$J_1 = \int_{\partial\tilde\Omega_{r,\varepsilon}(z_0)} \left(\Gamma(z_0,z) - \frac{1}{r}\right)\langle\mathcal{A}(z)\nabla u(z),\nu\rangle\,d\mathcal{H}^{N-1}(z) + \frac{1}{r}\int_{\partial\tilde\Omega_{r,\varepsilon}(z_0)} \langle\mathcal{A}(z)\nabla u(z),\nu\rangle\,d\mathcal{H}^{N-1}(z).$$

By Lemma 2.2 applied to the set $\tilde{\Omega}_{r,\varepsilon}(z_0)$ we get

$$\int_{\partial\tilde{\Omega}_{r,\varepsilon}(z_0)} \langle \mathcal{A}(z)\nabla u(z),\nu\rangle\, d\mathcal{H}^{N-1}(z) = -\int_{\partial\tilde{\Omega}_{r,\varepsilon}(z_0)} u(z)\langle b(z),\nu\rangle\, d\mathcal{H}^{N-1}(z) - \int_{\tilde{\Omega}_{r,\varepsilon}(z_0)} c(z)u(z)\,dz + \int_{\tilde{\Omega}_{r,\varepsilon}(z_0)} \mathcal{L}u(z)\,dz.$$

Therefore

$$\begin{aligned} J_1 = &\int_{\partial\tilde{\Omega}_{r,\varepsilon}(z_0)} \left(\Gamma(z_0,z) - \frac{1}{r}\right) \langle \mathcal{A}(z)\nabla u(z),\nu\rangle\, d\mathcal{H}^{N-1}(z) \\ &- \frac{1}{r}\int_{\partial\tilde{\Omega}_{r,\varepsilon}(z_0)} u(z)\langle b(z),\nu\rangle\, d\mathcal{H}^{N-1}(z) - \frac{1}{r}\int_{\tilde{\Omega}_{r,\varepsilon}(z_0)} c(z)u(z)\,dz \\ &+ \frac{1}{r}\int_{\tilde{\Omega}_{r,\varepsilon}(z_0)} \mathcal{L}u(z)\,dz. \end{aligned}$$

Taking into account that

$$\partial\tilde{\Omega}_{r,\varepsilon}(z_0) = \big(\partial\Omega_r(z_0)\setminus B(z_0,\varepsilon)\big) \cup \big(\partial B(z_0,\varepsilon)\setminus \overline{\Omega_r(z_0)}\big) \tag{2.4}$$

and

$$\left|\Gamma(z_0,z) - \frac{1}{r}\right| \le \begin{cases} 0 & \text{in } \partial\Omega_r(z_0)\setminus\{z_0\} \\ \frac{1}{r} & \text{in } \overline{B(z_0,\varepsilon)}\setminus\overline{\Omega_r(z_0)}, \end{cases} \tag{2.5}$$

we get

$$\begin{aligned} &\left|\int_{\partial\tilde{\Omega}_{r,\varepsilon}(z_0)} \left(\Gamma(z_0,z) - \frac{1}{r}\right) \langle \mathcal{A}(z)\nabla u(z),\nu\rangle\, d\mathcal{H}^{N-1}(z)\right| \\ &\le \frac{1}{r}\int_{\partial B(z_0,\varepsilon)\setminus\overline{\Omega_r(z_0)}} |\mathcal{A}(z)\nabla u(z)|\, d\mathcal{H}^{N-1}(z) = o(1) \quad \text{as } \varepsilon\to 0. \end{aligned}$$

We have so proved that

$$\begin{aligned} J_1 = &-\frac{1}{r}\int_{\partial\tilde{\Omega}_{r,\varepsilon}(z_0)} u(z)\langle b(z),\nu\rangle\, d\mathcal{H}^{N-1}(z) - \frac{1}{r}\int_{\tilde{\Omega}_{r,\varepsilon}(z_0)} c(z)u(z)\,dz \\ &+ \frac{1}{r}\int_{\tilde{\Omega}_{r,\varepsilon}(z_0)} \mathcal{L}u(z)\,dz + o(1) \quad \text{as } \varepsilon\to 0. \end{aligned} \tag{2.6}$$

Estimate of J_2.

Let us now consider J_2, that is

$$J_2 := \int_{O\setminus\overline{\tilde{\Omega}_{r,\varepsilon}(z_0)}} \langle \mathcal{A}(z)\nabla\Gamma(z_0,z), \nabla(u\varphi)(z)\rangle\, dz.$$

By (1.2),

$$\mathcal{L}^*\Gamma(z_0,z) = 0 \qquad \forall\, z\in O\setminus\overline{\tilde{\Omega}_{r,\varepsilon}(z_0)}, \tag{2.7}$$

where $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$. Therefore

$$\operatorname{div}(\mathcal{A}(z)\nabla\Gamma(z_0,z)) = \langle b, \nabla\Gamma(z_0,z)\rangle - c(z)\Gamma(z_0,z) \qquad z\in O\setminus\overline{\tilde{\Omega}_{r,\varepsilon}(z_0)}.$$

This implies that in $O\setminus\overline{\tilde{\Omega}_{r,\varepsilon}(z_0)}$ the following equalities hold:

$$\begin{aligned} \langle \mathcal{A}(z)\nabla\Gamma(z_0,z), \nabla(u\varphi)(z)\rangle &= \operatorname{div}((u\varphi)(z)\mathcal{A}(z)\nabla\Gamma(z_0,z)) - (u\varphi)(z)\operatorname{div}(\mathcal{A}(z)\nabla\Gamma(z_0,z)) \\ &= \operatorname{div}((u\varphi)(z)\mathcal{A}(z)\nabla\Gamma(z_0,z)) - (u\varphi)(z)\langle b, \nabla\Gamma(z_0,z)\rangle \\ &\quad + (u\varphi)(z)c(z)\Gamma(z_0,z). \end{aligned}$$

By the divergence theorem and the properties of φ,

$$\int_{O\setminus\overline{\tilde{\Omega}_{r,\varepsilon}(z_0)}} \operatorname{div}((u\varphi)(z)\mathcal{A}(z)\nabla\Gamma(z_0,z))\,dz = -\int_{\partial\tilde{\Omega}_{r,\varepsilon}(z_0)} u(z)\langle\mathcal{A}(z)\nabla\Gamma(z_0,z),\nu\rangle\,d\mathcal{H}^{N-1}(z),$$

then

$$\begin{aligned} J_2 = &-\int_{\partial\tilde{\Omega}_{r,\varepsilon}(z_0)} u(z)\langle\mathcal{A}(z)\nabla\Gamma(z_0,z),\nu\rangle\,d\mathcal{H}^{N-1}(z) - \int_{O\setminus\overline{\tilde{\Omega}_{r,\varepsilon}(z_0)}}(u\varphi)(z)\langle b,\nabla\Gamma(z_0,z)\rangle\,dz \\ &+\int_{O\setminus\overline{\tilde{\Omega}_{r,\varepsilon}(z_0)}}(u\varphi)(z)c(z)\Gamma(z_0,z)\,dz. \end{aligned} \tag{2.8}$$

By (2.4) we can split the first integral at the right hand side of (2.8) so obtaining

$$\begin{aligned} J_2 = &-\int_{\partial\Omega_r(z_0)} u(z)\langle\mathcal{A}(z)\nabla\Gamma(z_0,z),\nu\rangle\left(1-\chi_{B(z_0,\varepsilon)}(z)\right)\,d\mathcal{H}^{N-1}(z) \\ &-\int_{\partial B(z_0,\varepsilon)\setminus\overline{\Omega_r(z_0)}} u(z)\langle\mathcal{A}(z)\nabla\Gamma(z_0,z),\nu\rangle\,d\mathcal{H}^{N-1}(z) \\ &-\int_{O\setminus\overline{\tilde{\Omega}_{r,\varepsilon}(z_0)}}(u\varphi)(z)\langle b,\nabla\Gamma(z_0,z)\rangle\,dz \\ &+\int_{O\setminus\overline{\tilde{\Omega}_{r,\varepsilon}(z_0)}}(u\varphi)(z)c(z)\Gamma(z_0,z)\,dz. \end{aligned}$$

Let us consider the second integral at the right hand side. By Chauchy-Schwarz inequality, we have

$$|\langle\mathcal{A}(z)\nabla\Gamma(z_0,z),\nu\rangle| \le \langle\mathcal{A}(z)\nabla\Gamma(z_0,z),\nabla\Gamma(z_0,z)\rangle^{\frac{1}{2}}\langle\mathcal{A}(z)\nu,\nu\rangle^{\frac{1}{2}}.$$

Therefore, there exists C, depending on $\|A\|_{L^\infty(B(z_0,\varepsilon))}$ and $\|u\|_{L^\infty(B(z_0,\varepsilon))}$, such that

$$\begin{aligned} &\left|\int_{\partial B(z_0,\varepsilon)\setminus\overline{\Omega_r(z_0)}} u(z)\langle\mathcal{A}(z)\nabla\Gamma(z_0,z),\nu\rangle\,d\mathcal{H}^{N-1}(z)\right| \\ &\le C\int_{\partial B(z_0,\varepsilon)\setminus\overline{\Omega_r(z_0)}}\langle\mathcal{A}(z)\nabla\Gamma(z_0,z),\nabla\Gamma(z_0,z)\rangle^{\frac{1}{2}}\,d\mathcal{H}^{N-1}(z). \end{aligned}$$

Thus, by (1.3)

$$\int_{\partial B(z_0,\varepsilon)\setminus\overline{\Omega_r(z_0)}} u(z)\langle\mathcal{A}(z)\nabla\Gamma(z_0,z),\nu\rangle\,d\mathcal{H}^{N-1}(z) = o(1) \qquad \text{as } \varepsilon\to 0.$$

Therefore

$$\begin{aligned} J_2 = &-\int_{\partial\Omega_r(z_0)} u(z)\langle\mathcal{A}(z)\nabla\Gamma(z_0,z),\nu\rangle\left(1-\chi_{B(z_0,\varepsilon)}(z)\right)\,d\mathcal{H}^{N-1}(z) \\ &-\int_{O\setminus\overline{\tilde{\Omega}_{r,\varepsilon}(z_0)}}(u\varphi)(z)\langle b,\nabla\Gamma(z_0,z)\rangle\,dz \\ &+\int_{O\setminus\overline{\tilde{\Omega}_{r,\varepsilon}(z_0)}}(u\varphi)(z)c(z)\Gamma(z_0,z)\,dz + o(1) \qquad \text{as } \varepsilon\to 0^+. \end{aligned} \tag{2.9}$$

Conclusion.

Collecting (2.3), (2.6) and (2.9) and observing that the integral

$$\int_{O\setminus\overline{\tilde{\Omega}_{r,\varepsilon}(z_0)}}(u\varphi)(z)c(z)\Gamma(z_0,z)\,dz$$

appears in (2.3) and in (2.9), but with opposite sign, we get

$$\begin{aligned} u(z_0) = &- \int_{\partial\Omega_r(z_0)} u(z)\langle \mathcal{A}(z)\nabla\Gamma(z_0,z),\nu\rangle \left(1-\chi_{B(z_0,\varepsilon)}(z)\right)\, d\mathcal{H}^{N-1}(z) \\ &- \int_{O\setminus\overline{\tilde\Omega_{r,\varepsilon}(z_0)}} \left((u\varphi)(z)\langle b,\nabla\Gamma(z_0,z)\rangle + \Gamma(z_0,z)\langle b,\nabla(u\varphi)\rangle\right)\, dz \\ &- \frac{1}{r}\int_{\partial\tilde\Omega_{r,\varepsilon}(z_0)} u(z)\langle b(z),\nu\rangle\, d\mathcal{H}^{N-1}(z) \\ &- \frac{1}{r}\int_{\tilde\Omega_{r,\varepsilon}(z_0)} c(z)u(z)\, dz \\ &- \int_{\tilde\Omega_{r,\varepsilon}(z_0)} \left(\Gamma(z_0,z)-\frac{1}{r}\right)\mathcal{L}u(z)\, dz + o(1) \quad \text{as } \varepsilon\to 0. \end{aligned} \tag{2.10}$$

By assumption $\operatorname{div} b = 0$, so

$$\begin{aligned} &(u\varphi)(z)\langle b(z),\nabla\Gamma(z_0,z)\rangle + \Gamma(z_0,z)\langle b,\nabla(u\varphi)(z)\rangle \\ &= \operatorname{div}(\Gamma(z_0,z)b(z)(u\varphi)(z)) - \Gamma(z_0,z)(u\varphi)(z)\operatorname{div} b \\ &= \operatorname{div}(\Gamma(z_0,z)b(u\varphi)(z)). \end{aligned}$$

Therefore, using the divergence theorem and recalling that $\varphi = 0$ on ∂O and $\varphi = 1$ on $\partial\tilde\Omega_{r,\varepsilon}(z_0)$, we get

$$\begin{aligned} &- \int_{O\setminus\overline{\tilde\Omega}_{r,\varepsilon}(z_0)} \left((u\varphi)(z)\langle b,\nabla\Gamma(z_0,z)\rangle + \Gamma(z_0,z)\langle b,\nabla(u\varphi)(z)\rangle\right)\, dz \\ &= - \int_{O\setminus\overline{\tilde\Omega}_{r,\varepsilon}(z_0)} \operatorname{div}(\Gamma(z_0,z)b(u\varphi)(z))\, dz = \int_{\partial\tilde\Omega_{r,\varepsilon}(z_0)} \Gamma(z_0,z)u(z)\langle b,\nu\rangle\, d\mathcal{H}^{N-1}(z). \end{aligned} \tag{2.11}$$

By (2.10), (2.11) and taking into account that

$$\lim_{\varepsilon\to 0}\int_{\tilde\Omega_{r,\varepsilon}(z_0)} c(z)u(z)\, dz = \int_{\Omega_r(z_0)} c(z)u(z)\, dz,$$

we get

$$\begin{aligned} u(z_0) = &- \int_{\partial\Omega_r(z_0)} u(z)\langle \mathcal{A}(z)\nabla\Gamma(z_0,z),\nu\rangle \left(1-\chi_{B(z_0,\varepsilon)}(z)\right)\, d\mathcal{H}^{N-1}(z) \\ &+ \int_{\partial\tilde\Omega_{r,\varepsilon}(z_0)} \left(\Gamma(z_0,z)-\frac{1}{r}\right) u(z)\langle b,\nu\rangle\, d\mathcal{H}^{N-1}(z) \\ &- \frac{1}{r}\int_{\Omega_r(z_0)} c(z)u(z)\, dz \\ &- \int_{\tilde\Omega_{r,\varepsilon}(z_0)} \left(\Gamma(z_0,z)-\frac{1}{r}\right)\mathcal{L}u(z)\, dz + o(1) \quad \text{as } \varepsilon\to 0. \end{aligned} \tag{2.12}$$

Using (2.4) and (2.5),

$$\begin{aligned} &\left| \int_{\partial\tilde\Omega_r(z_0)} \left(\Gamma(z_0,z)-\frac{1}{r}\right) u(z)\langle b,\nu\rangle\, d\mathcal{H}^{N-1}(z)\right| \\ &= \left| \int_{\partial B(z_0,\varepsilon)\setminus\overline{\Omega_r(z_0)}} \left(\Gamma(z_0,z)-\frac{1}{r}\right) u(z)\langle b,\nu\rangle\, d\mathcal{H}^{N-1}(z)\right| \end{aligned}$$

$$\leq \frac{1}{r}\int_{\partial B(z_0,\varepsilon)} |u(z)\langle b,\nu\rangle|\, d\mathcal{H}^{N-1}(z) = o(1) \quad \text{as } \varepsilon \to 0,$$

then (2.12) implies

$$\begin{aligned} u(z_0) = &-\int_{\partial\Omega_r(z_0)} u(z)\langle \mathcal{A}(z)\nabla\Gamma(z_0,z),\nu\rangle \left(1-\chi_{B(z_0,\varepsilon)}(z)\right)\, d\mathcal{H}^{N-1}(z)\\ &-\frac{1}{r}\int_{\Omega_r(z_0)} c(z)u(z)\,dz\\ &-\int_{\tilde\Omega_{r,\varepsilon}(z_0)}\left(\Gamma(z_0,z)-\frac{1}{r}\right)\mathcal{L}u(z)\,dz + o(1) \quad \text{as } \varepsilon\to 0. \end{aligned} \tag{2.13}$$

We claim that as ε goes to 0 then the above equality gives:

$$\begin{aligned} u(z_0) = &\int_{\partial\Omega_r(z_0)} u(z)\langle \mathcal{A}(z)\nabla\Gamma(z_0,z), \frac{\nabla\Gamma(z_0,z)}{|\nabla\Gamma(z_0,z)|}\rangle\, d\mathcal{H}^{N-1}(z)\\ &-\frac{1}{r}\int_{\Omega_r(z_0)} c(z)u(z)\,dz - \int_{\Omega_r(z_0)}\left(\Gamma(z_0,z)-\frac{1}{r}\right)\mathcal{L}u\,dz. \end{aligned} \tag{2.14}$$

To prove this, we begin by applying (2.13) to the constant function $u\equiv 1$. Since $\mathcal{L}1 = c$, we get

$$\begin{aligned} 1 = &-\int_{\partial\Omega_r(z_0)} \langle \mathcal{A}(z)\nabla\Gamma(z_0,z),\nu\rangle \left(1-\chi_{B(z_0,\varepsilon)}(z)\right)\, d\mathcal{H}^{N-1}(z)\\ &-\frac{1}{r}\int_{\Omega_r(z_0)} c(z)\,dz\\ &-\int_{\tilde\Omega_{r,\varepsilon}(z_0)}\left(\Gamma(z_0,z)-\frac{1}{r}\right)c(z)\,dz + o(1) \quad \text{as } \varepsilon\to 0. \end{aligned} \tag{2.15}$$

Since

$$\nu(z) = -\frac{\nabla\Gamma(z_0,z)}{|\nabla\Gamma(z_0,z)|} \qquad z\in\partial\Omega_r(z_0)\setminus\{z_0\},$$

then (2.15) implies

$$\begin{aligned} &\int_{\partial\Omega_r(z_0)} \langle \mathcal{A}(z)\nabla\Gamma(z_0,z), \frac{\nabla\Gamma(z_0,z)}{|\nabla\Gamma(z_0,z)|}\rangle \left(1-\chi_{B(z_0,\varepsilon)}(z)\right)\, d\mathcal{H}^{N-1}(z)\\ =&1+\frac{1}{r}\int_{\Omega_r(z_0)} c(z)\,dz\\ &+\int_{\tilde\Omega_{r,\varepsilon}(z_0)}\left(\Gamma(z_0,z)-\frac{1}{r}\right)c(z)\,dz + o(1) \quad \text{as } \varepsilon\to 0. \end{aligned} \tag{2.16}$$

As far as the last integral is concerned, notice that

$$\begin{aligned} \int_{\tilde\Omega_{r,\varepsilon}(z_0)}\left(\Gamma(z_0,z)-\frac{1}{r}\right)c(z)\,dz = &\int_{\Omega_r(z_0)}\left(\Gamma(z_0,z)-\frac{1}{r}\right)c(z)\,dz\\ &+\int_{B(z_0,\varepsilon)\setminus\Omega_r(z_0)}\left(\Gamma(z_0,z)-\frac{1}{r}\right)c(z)\,dz \end{aligned}$$

and that by (2.5)

$$\int_{B(z_0,\varepsilon)\setminus\Omega_r(z_0)}\left(\Gamma(z_0,z)-\frac{1}{r}\right)c(z)\,dz = o(1) \qquad \text{as } \varepsilon\to 0.$$

Thus, (2.16) gives

$$\begin{aligned}&\int_{\partial\Omega_r(z_0)}\langle\mathcal{A}(z)\nabla\Gamma(z_0,z),\frac{\nabla\Gamma(z_0,z)}{|\nabla\Gamma(z_0,z)|}\rangle\left(1-\chi_{B(z_0,\varepsilon)}(z)\right)\,d\mathcal{H}^{N-1}(z)\\ =&1+\frac{1}{r}\int_{\Omega_r(z_0)}c(z)\,dz+\int_{\Omega_r(z_0)}\left(\Gamma(z_0,z)-\frac{1}{r}\right)c(z)\,dz+o(1)\\ =&1+\int_{\Omega_r(z_0)}\Gamma(z_0,z)c(z)\,dz+o(1)\quad\text{as }\varepsilon\to0.\end{aligned}$$

The integrand at the left hand side is non-negative, because $\mathcal{A}$ is positive semidefinite, thus, by the monotone convergence theorem, we get

$$\int_{\partial\Omega_r(z_0)}\langle\mathcal{A}(z)\nabla\Gamma(z_0,z),\frac{\nabla\Gamma(z_0,z)}{|\nabla\Gamma(z_0,z)|}\rangle\,d\mathcal{H}^{N-1}(z)=1+\int_{\Omega_r(z_0)}\Gamma(z_0,z)c(z)\,dz.$$

Since $\Gamma(z_0,\cdot)\in L^1_{\rm loc}$, then the right hand side is finite and consequently the same is for the left hand side. This allows to use the dominated convergence theorem to compute the limit of the first integral in (2.13). Precisely,

$$\begin{aligned}&-\lim_{\varepsilon\to0}\int_{\partial\Omega_r(z_0)}u(z)\langle\mathcal{A}(z)\nabla\Gamma(z_0,z),\nu\rangle\left(1-\chi_{B(z_0,\varepsilon)}(z)\right)\,d\mathcal{H}^{N-1}(z)\\ &=\lim_{\varepsilon\to0}\int_{\partial\Omega_r(z_0)}u(z)\langle\mathcal{A}(z)\nabla\Gamma(z_0,z),\frac{\nabla\Gamma(z_0,z)}{|\nabla\Gamma(z_0,z)|}\rangle\left(1-\chi_{B(z_0,\varepsilon)}(z)\right)\,d\mathcal{H}^{N-1}(z)\\ &=\int_{\partial\Omega_r(z_0)}u(z)\langle\mathcal{A}(z)\nabla\Gamma(z_0,z),\frac{\nabla\Gamma(z_0,z)}{|\nabla\Gamma(z_0,z)|}\rangle\,d\mathcal{H}^{N-1}(z).\end{aligned}$$

Moreover,

$$\begin{aligned}\int_{\tilde\Omega_{r,\varepsilon}(z_0)}\left(\Gamma(z_0,z)-\frac{1}{r}\right)\mathcal{L}u(z)\,dz=&\int_{\Omega_r(z_0)}\left(\Gamma(z_0,z)-\frac{1}{r}\right)\mathcal{L}u(z)\,dz\\ &+\int_{B(z_0,\varepsilon)\setminus\Omega_r(z_0)}\left(\Gamma(z_0,z)-\frac{1}{r}\right)\mathcal{L}u(z)\,dz.\end{aligned}$$

By (2.5),

$$\left|\int_{B(z_0,\varepsilon)\setminus\Omega_r(z_0)}\left(\Gamma(z_0,z)-\frac{1}{r}\right)\mathcal{L}u(z)\,dz\right|=o(1)\qquad\text{as }\varepsilon\to0.$$

Therefore, as ε goes to 0 in (2.13) we get (2.14).

□

We easily get Theorem 1.1 from Proposition 2.1 and the use of the coarea formula.

Proof of Theorem 1.1. Let $r>0$ be such that $(H(z_0,\varrho))$ holds for a.e. $\varrho\in]0,r[$. For every continuous function $\varphi:]0,1[\to[0,\infty[$, $\int_0^1\varphi(t)\,dt=1$, let us define $\varphi_r(t)=\frac{1}{r}\varphi\big(\frac{t}{r}\big)$.

By Proposition 2.1, if $\mathcal{L}u=0$ then

$$\begin{aligned}u(z_0)&=\int_0^r\varphi_r(t)u(z_0)\,dt\\ &=\int_0^r\varphi_r(t)\left(\int_{\partial\Omega_t(z_0)}u(z)\langle\mathcal{A}(z)\nabla\Gamma(z_0,z),\frac{\nabla\Gamma(z_0,z)}{|\nabla\Gamma(z_0,z)|}\rangle\,d\mathcal{H}^{N-1}(z)\right)dt\end{aligned}$$

$$- \int_0^r \frac{\varphi_r(t)}{t} \int_{\Omega_t(z_0)} c(z)u(z)\,dz\,dt. \tag{2.17}$$

As far as the first integral at the right hand side is concerned, by the change of variable $t = \frac{1}{\varrho}$ and the coarea formula we get

$$\begin{aligned}
&\int_0^r \left(\int_{\partial\Omega_t(z_0)} u(z) \langle \mathcal{A}(z)\nabla\Gamma(z_0,z), \frac{\nabla\Gamma(z_0,z)}{|\nabla\Gamma(z_0,z)|}\rangle \, d\mathcal{H}^{N-1}(z) \right) dt \\
&= \int_{\frac{1}{r}}^{\infty} \left(\varphi_r\big(\frac{1}{\varrho}\big)\varrho^{-2} \int_{\{\Gamma(z_0,z)=\varrho\}} u(z) \langle \mathcal{A}(z)\nabla\Gamma(z_0,z), \frac{\nabla\Gamma(z_0,z)}{|\nabla\Gamma(z_0,z)|}\rangle \, d\mathcal{H}^{N-1}(z) \right) d\varrho \\
&= \int_{\Omega_r(z_0)} u(z) \frac{\langle \mathcal{A}(z)\nabla\Gamma(z_0,z), \nabla\Gamma(z_0,z)\rangle}{\Gamma(z_0,z)^2} \varphi_r\big(\frac{1}{\Gamma(z_0,z)}\big)\,dz.
\end{aligned}$$

Let us discuss the last integral in (2.17). Since $\Omega_t(z_0) \subseteq \Omega_r(z_0)$ for every $t \in]0,r[$,

$$\int_0^r \frac{\varphi_r(t)}{t} \int_{\Omega_t(z_0)} c(z)u(z)\,dz\,dt = \int_0^r \frac{\varphi_r(t)}{t} \int_{\Omega_r(z_0)} \chi_{\Omega_t(z_0)}(z)c(z)u(z)\,dz\,dt. \tag{2.18}$$

To do the change in the integration order, we now prove that

$$(z,t) \mapsto \frac{\varphi_r(t)}{t} \chi_{\Omega_t(z_0)}(z)c(z)u(z) \in L^1(\Omega_r(z_0)\times[0,r]).$$

By definition of $\Omega_t(z_0)$, we have

$$\begin{aligned}
&\int_{\Omega_r(z_0)} \int_0^r \frac{\varphi_r(t)}{t} \chi_{\Omega_t(z_0)}(z)|c(z)u(z)|\,dz\,dt \\
&\le \|cu\|_{L^\infty(\Omega_r(z_0))} \int_0^r \frac{\varphi_r(t)}{t} |\Omega_t(z_0)|\,dt \\
&\le \|cu\|_{L^\infty(\Omega_r(z_0))} \int_0^r \varphi_r(t) \int_{\Omega_t(z_0)} \Gamma(z_0,z)\,dzdt \\
&\le \|cu\|_{L^\infty(\Omega_r(z_0))} \int_0^r \varphi_r(t) \int_{\Omega_r(z_0)} \Gamma(z_0,z)\,dzdt \\
&= \|cu\|_{L^\infty(\Omega_r(z_0))} \|\Gamma(z_0,\cdot)\|_{L^1(\Omega_r(z_0))} < +\infty.
\end{aligned}$$

This allows us to exchange the integration order in the last integral in (2.18):

$$\begin{aligned}
&\int_0^r \frac{\varphi_r(t)}{t} \int_{\Omega_r(z_0)} \chi_{\Omega_t(z_0)}(z)c(z)u(z)\,dz\,dt \\
&= \int_{\Omega_r(z_0)} c(z)u(z) \int_{\frac{1}{\Gamma(z_0,z)}}^r \frac{\varphi_r(t)}{t}\,dt\,dz \\
&= \frac{1}{r} \int_{\Omega_r(z_0)} c(z)u(z) \int_{\frac{1}{\Gamma(z_0,z)}}^r \frac{1}{t}\varphi\left(\frac{t}{r}\right) dt\,dz.
\end{aligned}$$

The conclusion follows. □

3. Proof of Theorem 1.2.

In this section we prove Theorem 1.2.

Proof of Theorem 1.2. We apply Theorem 1.1 to $\mathcal{L}^{(p)}$. This is possible, because the operator $\mathcal{L}^{(p)}$ is in the class of operators (1.1) with $c = 0$. If we denote $\mathcal{A}^{(p)}$ the matrix of the second order part of $\mathcal{L}^{(p)}$, then $\mathcal{A}^{(p)}(x, y, t) = \mathcal{A}^{(p)}(x)$ is the $(n + p + 1) \times (n + p + 1)$ real matrix

$$\mathcal{A}^{(p)}(x, y, t) = \mathcal{A}^{(p)}(x) := \begin{bmatrix} A^{(p)}(x) & 0 \\ 0 & 0 \end{bmatrix},$$

with $A^{(p)}$ the $(n + p) \times (n + p)$ matrix

$$A^{(p)}(x) := \begin{bmatrix} A(x) & 0 \\ 0 & I_p \end{bmatrix}. \tag{3.1}$$

Fixed $\hat{z}_0 = (x_0, y_0, t_0)$, with $(x_0, t_0) \in \mathbb{X}$, and $y_0 \in \mathbb{R}^p$, let $\Gamma^{(p)}$ be the fundamental solution defined in (1.9) and let $\hat{\Omega}_r^{(p)}(\hat{z}_0)$ be the super-level set

$$\hat{\Omega}_r^{(p)}(\hat{z}_0) := \big\{\hat{z} \,:\, \Gamma^{(p)}(\hat{z}_0, \hat{z}) > \frac{1}{r}\big\}. \tag{3.2}$$

If $\hat{u} \in C^2(\overline{\hat{\Omega}_r^{(p)}(\hat{z}_0)})$, then by Theorem 1.1 applied to $\mathcal{L}^{(p)}$ with $\varphi = 1$ we have

$$u(\hat{z}_0) = \frac{1}{r} \int_{\hat{\Omega}_r^{(p)}(\hat{z}_0)} u(\hat{z}) \frac{\langle A^{(p)}(x, y)\nabla_{x,y}\Gamma^{(p)}(\hat{z}_0, \hat{z}), \nabla_{x,y}\Gamma^{(p)}(\hat{z}_0, \hat{z})\rangle_{\mathbb{R}^{n+p}}}{\Gamma^{(p)}(\hat{z}_0, \hat{z})^2}\, d\hat{z}. \tag{3.3}$$

By (1.9), (1.10) and (3.1), formula (3.3) becomes

$$u(\hat{z}_0) = \frac{1}{r} \int_{\hat{\Omega}_r^{(p)}(\hat{z}_0)} u(\hat{z}) \left(W(z_0, z) + \frac{|y_0 - y|^2}{4(t_0 - t)^2} \right) d\hat{z}, \tag{3.4}$$

where

$$W(z_0, z) := \frac{\langle A(x)\nabla_x\Gamma(z_0, z), \nabla_x\Gamma(z_0, z)\rangle_{\mathbb{R}^n}}{\Gamma(z_0, z)^2}$$

and $\dfrac{|y_0 - y|^2}{4(t_0 - t)^2}$ is the classical Pini-Watson kernel in $\mathbb{R}^{p+1}$.

From (3.2) and (1.9), we get

$$\hat{\Omega}_r^{(p)}(\hat{z}_0) = \left\{ \hat{z} = (x, y, t) \,:\, r\frac{\Gamma(z_0, z)}{(4\pi(t_0 - t))^{\frac{p}{2}}} > \exp\big(\frac{|y_0 - y|^2}{4(t_0 - t)}\big) \right\}$$

in the right hand side of (3.4) we can eliminate the variable y by integration as follows:

$$\begin{aligned} &\int_{\hat{\Omega}_r^{(p)}(\hat{z}_0)} u(z) \left(W(z_0, z) + \frac{|y_0 - y|^2}{4(t_0 - t)^2}) \right) d\hat{z} \\ &= \int_{\Omega_r^{(p)}(z_0)} u(z) \left(\int_{\{|y_0 - y| < R_r(z_0, z)\}} \big(W(z_0, z) + \frac{|y_0 - y|^2}{4(t_0 - t)^2}\big)\, dy \right) dz, \end{aligned} \tag{3.5}$$

where we recall that

$$\Omega_r^{(p)}(z_0) = \big\{z = (x, t) \,:\, \Phi_p(z_0, z) > \frac{1}{r}\big\}$$

and

$$R_r(z_0, z) := \sqrt{4\pi(t_0 - t)\log(r\Phi_p(z_0, z))},$$

see (1.11) and (1.12).

A straightforward computation of the inner integral in (3.5) leads to the mean value formula (1.13).

□

4. Applications to Heat operators on stratified Lie groups

In this section we consider $\mathbb{R}^N = \mathbb{R}^{n+1}$. A vector $z \in \mathbb{R}^{n+1}$ will be written as $(x,t) \in \mathbb{R}^n \times \mathbb{R}$.

4.1. **Preliminaries.** We call Heat operator on a stratified Lie group in $\mathbb{R}^n$ every linear second order partial differential operator of the type

$$\mathcal{L} := \mathcal{L}_0 - \partial_t,$$

where $\mathcal{L}_0$ is a sub-Laplacian in $\mathbb{R}^n$; i.e. $\mathcal{L}_0$ is a sum of squares

$$\mathcal{L}_0 = \sum_{i=1}^{m} X_j^2, \qquad 2 \le m \le n,$$

satisfying the following conditions.

- (H1) The X_j's are smooth vector fields in $\mathbb{R}^n$ and generate a Lie algebra $\mathfrak{a}$ satisfying $\operatorname{rank} \mathfrak{a}(x) = \dim \mathfrak{a} = n$ at any point $x \in \mathbb{R}^n$.
- (H2) There exists a group of dilations $(\delta_\lambda)_{\lambda>0}$ in $\mathbb{R}^n$ such that every vector field X_j is δ_λ-homogeneous of degree one.

A group of dilations in $\mathbb{R}^n$ is a family of diagonal linear functions $(\delta_\lambda)_{\lambda>0}$ of the kind

$$\delta_\lambda(x_1, \dots, x_n) = (\lambda^{\varepsilon_1} x_1, \dots, \lambda^{\varepsilon_n} x_n),$$

where the ε_j's are natural numbers.

Due to the rank condition in (H1), the operator $\mathcal{L}_0$ and $\mathcal{L}$ are hypoelliptic, see [11], so that the solutions to $\mathcal{L}_0 u = 0$ ($\mathcal{L}u = 0$) are smooth.

Conditions (H1) and (H2) imply the existence of a group law $\circ$ making $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_\lambda)$ a stratified Lie group on which every vector field X_j is left translation invariant, see [1]. The natural number $Q := \varepsilon_1 + \ldots + \varepsilon_n$ is called the homogeneous dimension of $\mathbb{G}$. Since $\varepsilon_i \ge 1$ for every i and $n \ge 3$, then $Q \ge 3$.

The operator $\mathcal{L}$ can be written as

$$\mathcal{L} = \operatorname{div}(\mathcal{A}(z)\nabla) - \partial_t,$$

with $\mathcal{A}(z) = \mathcal{A}(x,t) = \mathcal{A}(x)$ the $(n+1) \times (n+1)$ real matrix

$$\mathcal{A}(x,t) = \mathcal{A}(x) = \begin{bmatrix} A(x) & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$A(x) = \begin{bmatrix} X_1 & \cdots & X_m \end{bmatrix} \begin{bmatrix} X_1^T \\ \vdots \\ X_m^T \end{bmatrix} \tag{4.1}$$

symmetric and positive semidefinite $n \times n$ real matrix, see [3, Chapter 1, Sect. 5]. The matrix A is t-independent, therefore we can write

$$\mathcal{L} = \operatorname{div}_x(A(x)\nabla_x) - \partial_t. \tag{4.2}$$

Since $\mathcal{L}_0$ is left translation invariant in $\mathbb{G}$, then $\mathcal{L} = \mathcal{L}_0 - \partial_t$ is left translation invariant in $\mathbb{G} \oplus \mathbb{R}$. For the sake of simplicity, we will use the same symbol $\circ$ for the Group law in $\mathbb{G} \oplus \mathbb{R}$:

$$z^{-1} \circ z_0 = (x^{-1} \circ x_0, t_0 - t).$$

If $(x,t) \mapsto \gamma(x,t)$ denotes the fundamental solution of $\mathcal{L}$ with pole at the origin, we have $\gamma(x,t) = 0$ if $t \le 0$. Moreover, for every $z_0 = (x_0, t_0)$ and $z = (x,t)$ in $\mathbb{R}^n \times \mathbb{R}$, the function

$$z \mapsto \Gamma(z_0, z) := \gamma(z^{-1} \circ z_0)$$

is the fundamental solution with pole at z_0 of the adjoint operator of $\mathcal{L}$

$$\mathcal{L}^* := \operatorname{div}_x(A(x)\nabla_x) + \partial_t.$$

Γ is a smooth function satisfying (1.2) and (1.7).

4.2. **Mean value formula.** Theorem 1.1 applies to the operator $\mathcal{L}$ taking the following form.

Theorem 4.1. *Assume that $\mathcal{L}$ is the operator described above. For every $z_0 = (x_0, t_0)$ and for every $r > 0$ we have*

$$u(z_0) = \frac{1}{r} \int_{\Omega_r(z_0)} u(z) W(z_0^{-1} \circ z)\, dz \tag{4.3}$$

for every $u \in C^2(\overline{\Omega_r(z_0)})$ such that $\mathcal{L}u = 0$.

Here

$$W(z) := \frac{\langle A(x)\nabla_x \Gamma(e,z), \nabla_x \Gamma(e,z)\rangle}{\Gamma(e,z)^2} = \frac{\sum_{j=1}^{n} |X_j \Gamma(e,z)|^2}{\Gamma(e,z)^2}. \tag{4.4}$$

where $z = (x,t)$, $e = (0,0)$ and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$.

We remark again that if $\mathcal{L}$ is the classical Heat operator, then

$$W(z) = W(x,t) = \frac{1}{4}\frac{|x|^2}{t^2}, \tag{4.5}$$

the so-called Pini-Watson kernel.

The kernel W in (4.4) is unbounded, since it is homogeneous of degree -2 with respect to the dilations $D_\lambda(x,t) := (\delta_\lambda(x), \lambda^2 t)$. In Section 4.3 we prove Mean value formulas with bounded kernels.

The fundamental solution Γ satisfies Gaussian estimates in terms of the control distance d related to the vector fields X_j's. Precisely, let $\mathcal{C}$ be the set of all absolutely continuous paths $s : [0,1] \to \mathbb{R}^n$, satisfying

$$s'(\tau) = \sum_{j=1}^{m} a_j(\tau) X_j(s(\tau)), \qquad \text{for almost every } \tau \in [0,1].$$

Put

$$|s| = \int_0^1 \left(\sum_{j=1}^{m} a_j^2(\tau) \right)^{\frac{1}{2}} d\tau,$$

and, for $x, y \in \mathbb{R}^n$,

$$d(x,y) := \inf\{|s| \,:\, s \in \mathcal{C},\ s(0) = x,\ s(1) = y\}.$$

The function d is a distance, called the *control distance* related to the vector fields X_j's.

The fundamental solution γ with pole at the origin satisfies the following estimates for $t > 0$:

for every $\sigma \in]0,1[$ there exists $C = C_{\sigma,Q} > 0$ such that for every $z = (x,t) \in \mathbb{R}^{n+1}$, $t > 0$,

$$\frac{t^{-\frac{Q}{2}}}{C} \exp\left(-\frac{d^2(x)}{4(1-\sigma)t}\right) \le \gamma(z) \le C t^{-\frac{Q}{2}} \exp\left(-\frac{d^2(x)}{4(1+\sigma)t}\right) \tag{4.6}$$

and for every $j \in \{1, \cdots, m\}$

$$|X_j \gamma(z)| \le C t^{-\frac{1+Q}{2}} \exp\left(-\frac{d^2(x)}{4(1+\sigma)t}\right), \tag{4.7}$$

with $d(x) := d(0,x)$, see [21], [22, Theorem IV.4.2] and [22, page 61].

Notice that (4.6) implies that, for any $r > 0$, $\Omega_r(z_0)$ is an open bounded set in $\mathbb{R}^n \times \mathbb{R}$.

In the following lemma we prove that the fundamental solution Γ satisfies (1.3).

Lemma 4.2. *Under the above assumptions, for every $z_0 \in \mathbb{R}^n$ and a.e. $r > 0$, Γ satisfies (1.3), that in the present case it reduces to*

$$\int_{\Sigma_{r,\varepsilon}(z_0)} \langle A(x)\nabla_x\Gamma(z_0,z), \nabla_x\Gamma(z_0,z)\rangle^{\frac{1}{2}}\, d\mathcal{H}^n(z) = o(1) \qquad \text{as } \varepsilon \to 0^+, \tag{4.8}$$

where

$$\Sigma_{r,\varepsilon}(z_0) := \{(x,t) \in \partial B(z_0,\varepsilon) \setminus \overline{\Omega_r(z_0)} \,:\, t < t_0\}.$$

Proof. Since $\mathcal{L}$ is a left translation invariant operator on a Lie group in $\mathbb{R}^{n+1}$, with unit $e = (0,0)$, without loss of generality we can assume $z_0 = e$.

By (4.6) and (4.7) for every $\sigma \in]0,1[$ there exist $C_\sigma, K_\sigma > 0$ such that for every $z = (x,t) \in \mathbb{R}^{n+1}$, $t < 0$,

$$|\langle A(x)\nabla_x\Gamma(e,z), \nabla_x\Gamma(e,z)\rangle|^{\frac{1}{2}} \le C_\sigma |t|^{-\frac{1+Q}{2}} \exp\left(-\frac{d^2(x)}{4(1+\sigma)|t|}\right) \tag{4.9}$$

and

$$\Gamma(e,z) \ge K_\sigma |t|^{-\frac{Q}{2}} \exp\left(-\frac{d^2(x)}{4(1-\sigma)|t|}\right). \tag{4.10}$$

Fixed $r > 0$ such that $\Omega_r(e)$ is smooth, for every $\varrho \ge r$, if $z = (x,t) \in \{\Gamma(e,z) = \frac{1}{\varrho}\}$ by (4.10)

$$K_\sigma |t|^{-\frac{Q}{2}} \exp\left(-\frac{d^2(x)}{4(1-\sigma)|t|}\right) \le \frac{1}{\varrho},$$

that is equivalent to

$$\exp\left(-\frac{d^2(x)}{4(1+\sigma)|t|}\right) \le \left(\frac{|t|^{\frac{Q}{2}}}{K_\sigma \varrho}\right)^{\frac{1-\sigma}{1+\sigma}}.$$

Thus by (4.9) we have

$$|\langle A(x)\nabla_x\Gamma(e,z), \nabla_x\Gamma(e,z)\rangle|^{\frac{1}{2}} \le C_\sigma |t|^{-\frac{1+Q}{2}} \left(\frac{|t|^{\frac{Q}{2}}}{K_\sigma \varrho}\right)^{\frac{1-\sigma}{1+\sigma}}.$$

Since $\varrho \ge r$ we get

$$|\langle A(x)\nabla_x\Gamma(e,z), \nabla_x\Gamma(e,z)\rangle|^{\frac{1}{2}} \le \bar{C}_\sigma\, r^{\frac{\sigma-1}{\sigma+1}} |t|^{-\frac{1}{2}-\frac{Q\sigma}{1+\sigma}}. \tag{4.11}$$

This inequality implies

$$\begin{aligned}\int_{\Sigma_{r,\varepsilon}(e)} \langle A(x)\nabla_x\Gamma(e,z), \nabla_x\Gamma(e,z)\rangle^{\frac{1}{2}}\, d\mathcal{H}^n(z) &\le \int_{\Sigma_{r,\varepsilon}(e)} \frac{\bar{C}_\sigma\, r^{\frac{\sigma-1}{\sigma+1}}}{|t|^{\frac{1}{2}+\frac{\sigma Q}{1+\sigma}}}\, d\mathcal{H}^n(z) \\ &\le \int_{\partial B(e,\varepsilon)} \frac{\bar{C}_\sigma\, r^{\frac{\sigma-1}{\sigma+1}}}{|t|^{\frac{1}{2}+\frac{\sigma Q}{1+\sigma}}}\, d\mathcal{H}^n(z).\end{aligned}$$

If we choose σ in such a way that $\frac{1}{2} + \frac{\sigma Q}{1+\sigma} < 1$ we get

$$\int_{\partial B(e,\varepsilon)} \frac{1}{|t|^{\frac{1}{2}+\frac{\sigma Q}{1+\sigma}}}\, d\mathcal{H}^n(z) = o(1) \qquad \text{as } \varepsilon \to 0$$

and the conclusion follows. □

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. By Lemma 4.2 we can use Theorem 1.1 with $\varphi = 1$ and $c = 0$. Then for every $u \in C^2(\overline{\Omega_r(z_0)})$ such that $\mathcal{L}u = 0$ and for $z_0 = e = (0,0)$, we have

$$u(e) = \frac{1}{r}\int_{\Omega_r(e)} u(z)K_r(e,z)\,dz,$$

with

$$K_r(e,z) = \frac{\langle A\nabla_x\Gamma(e,z), \nabla_x\Gamma(e,z)\rangle}{\Gamma(e,z)^2} = W(z).$$

An easy computation based on the invariance of the operator $\mathcal{L}$ with respect to the composition law in the Lie group $\mathbb{G}\oplus\mathbb{R}$ allows to conclude. □

4.3. **Mean value formula with improved kernels.** Let $\mathcal{L}$ be as above, i.e.,

$$\mathcal{L} := \sum_{i=1}^{m} X_j^2 - \partial_t \qquad 2 \le m \le n, \tag{4.12}$$

with vector fields X_j satisfying (H1) and (H2).

Fixed $p \in \mathbb{N}$, denote

$$\mathcal{L}^{(p)} := \mathcal{L} + \Delta_y^{(p)},$$

where $\Delta_y^{(p)}$ is the classical Laplace operator in $\mathbb{R}^p$ in the (new) variables $y = (y_1, \cdots, y_p)$.

Obviously $\mathcal{L}^{(p)}$ is a Heat operator on the stratified Lie group $\mathbb{G}\oplus\mathbb{R}^p$, therefore, by what previously proved, for this new operator Theorem 1.2 applies.

This leads to the mean value formula (4.13) below, where the kernels are bounded if $p \ge 3$.

Theorem 4.3. *Assume that $\mathcal{L}$ is the operator* (4.12) *described above. For every $r > 0$ we have*

$$u(z_0) = \frac{1}{r}\int_{\Omega_r^{(p)}(z_0)} u(z)W_r^{(p)}(z_0^{-1}\circ z)\,dz \tag{4.13}$$

for every $u \in C^2(\overline{\Omega_r^{(p)}(z_0)})$ such that $\mathcal{L}u = 0$.

Here

$$W_r^{(p)}(z) := \omega_p R_r^p(e,z)\left\{W(z) + \frac{p}{4(p+2)}\Big(\frac{R_r(e,z)}{t}\Big)^2\right\},$$

with $R_r^p(e,z)$ defined in (1.12).

Moreover, for every $p \in \mathbb{N}$, $p \ge 3$, $z \mapsto W_r^{(p)}(z_0^{-1}\circ z)$ is bounded in $\Omega_r^{(p)}(z_0)$.

Proof. Formula (4.13) comes from Theorem 1.2.

It remains to prove that the kernel of the mean value formula above is bounded for $p \ge 3$. Without loss of generality we can assume $z_0 = e = (0,0)$.

Using (4.7) and the first inequality in (4.6), with elementary computations we obtain that for every $\sigma \in]0,1[$ there exists $c_\sigma > 0$ such that

$$W(z) \le \frac{c_\sigma}{|t|}\exp\left(\frac{\sigma}{1-\sigma^2}\frac{d^2(x)}{|t|}\right) \qquad \forall t < 0.$$

On the other hand, if $z = (x,t) \in \Omega_r^{(p)}(e)$, then the right inequality in (4.6) implies

$$\exp\left(\frac{\sigma}{1-\sigma^2}\frac{d^2(x)}{|t|}\right) \le c(r,\sigma)|t|^{-2(p+Q)\frac{\sigma}{1-\sigma}}.$$

Therefore

$$W(z) \leq c(r,\sigma)|t|^{-1-2(p+Q)\frac{\sigma}{1-\sigma}} \qquad \forall z=(x,t)\in \Omega_r^{(p)}(e). \tag{4.14}$$

By the right inequality in (4.6), there exists $c>0$ such that

$$\Gamma(e,z) \leq c|t|^{-\frac{Q}{2}},$$

that implies

$$\Phi_p(e,z) \leq \frac{c}{(4\pi)^{\frac{p}{2}}}|t|^{-\frac{p+Q}{2}}.$$

Therefore

$$R_r(e,z) \leq c(r,p,Q)\left(|t|\log\frac{1}{|t|}\right)^{\frac{1}{2}} \qquad \forall z\in\Omega_r^{(p)}(e). \tag{4.15}$$

By (4.14) and (4.15) we obtain that for every $\varepsilon>0$

$$W_r^{(p)}(z) \leq c(\varepsilon,r,p,Q)|t|^{\frac{p}{2}-1-\varepsilon} \qquad \forall z\in\Omega_r^{(p)}(e).$$

By choosing ε small enough we get that $z\mapsto W_r^{(p)}(z)$ is bounded in $\Omega_r^{(p)}(e)$ if $p\geq 3$. □

5. Application to Kolmogorov operators

5.1. **Preliminaries.** Consider the Kolmogorov-type operator

$$\mathcal{L} := \operatorname{div}(A\nabla) + \langle Bx, \nabla\rangle - \partial_t, \qquad (x,t)\in\mathbb{R}^n\times\mathbb{R} \tag{5.1}$$

where ∇ and $\langle\cdot\,,\cdot\,\rangle$ denote the gradient and the inner product in $\mathbb{R}^n$ and A, B are $n\times n$ constant matrices, such that

(i) A is symmetric and positive semidefinite
(ii) $\operatorname{tr} B = 0$
(iii) letting $E(s) := \exp(-sB)$, the matrix

$$C(t) := \int_0^t E(s)AE^T(s)\,ds \tag{5.2}$$

is strictly positive definite for every $t>0$.

The operator $\mathcal{L}$ is of the type (1.8) with $b:\mathbb{R}^n\to\mathbb{R}^n$, $b(x)=Bx$. Notice that $\operatorname{div} b = 0$, because $\operatorname{tr} B = 0$.

The operator $\mathcal{L}$ is hypoelliptic in $\mathbb{R}^{n+1}$, see [14], and it is left-translation invariant on the Lie group $\mathbb{K}=(\mathbb{R}^{n+1},\circ)$ whose composition law is defined as follows:

$$(x,t)\circ(x',t') = (x'+E(t')x, t+t').$$

A fundamental solution of $\mathcal{L}^*$ satisfying (1.2) and (1.7), is given by

$$z\mapsto \Gamma(z_0,z) := \gamma(z^{-1}\circ z_0),$$

where z^{-1} denotes the inverse of z in $\mathbb{K}$ and

$$\gamma(z) = \gamma(x,t) = \begin{cases} 0 & \text{if } t\leq 0 \\ \frac{(4\pi)^{-n/2}}{\sqrt{\det C(t)}}\exp\left(-\frac{1}{4}\left\langle C^{-1}(t)\,x,x\right\rangle\right) & \text{if } t>0. \end{cases}$$

5.2. **Mean value formula.** In Lemma 5.3 we will prove that Γ satisfies (1.3). Therefore, all the assumptions of Theorem 1.1 hold true for the operator $\mathcal{L}$ in (5.1). Due to the particular structure of this operator, the kernel of the mean value formula can be written with an explicit dependence on the matrices A and C. Indeed, as proved in [14, Remark 4.1], if $z_0 = e := (0,0) \in \mathbb{R}^n \times \mathbb{R}$ and $z = (x,t)$, then

$$\frac{\langle A\nabla_x \Gamma(e,z), \nabla_x \Gamma(e,z)\rangle_{\mathbb{R}^n}}{\Gamma(e,z)^2} = \frac{\langle AC^{-1}(t)x, C^{-1}(t)x\rangle_{\mathbb{R}^n}}{4}. \tag{5.3}$$

By the left translation invariance of the operator $\langle A\nabla_x, \nabla_x\rangle$ on the Lie group $\mathbb{K}$, we will be able to prove the following Mean Value formula.

Theorem 5.1. *Assume that $\mathcal{L}$ is the operator described above. For every $r > 0$ such that $\Omega_r(z_0)$ is a bounded set and for every $u \in C^2(\overline{\Omega_r(z_0)})$ such that $\mathcal{L}u = 0$, then*

$$u(z_0) = \frac{1}{r}\int_{\Omega_r(z_0)} u(z) W(z_0^{-1} \circ z)\, dz, \tag{5.4}$$

where

$$W(x,t) = \frac{\langle AC^{-1}(t)x, C^{-1}(t)x\rangle_{\mathbb{R}^n}}{4}. \tag{5.5}$$

Remark 5.2. *If $\mathcal{L}$ is the classical Heat operator (i.e., $A = I_n$, $B = 0$), then the kernel $(x,t) \mapsto W(x,t)$ gives back the classical Pini-Watson kernel* (4.5).

To prove the mean value formula we need to show that Γ satisfies (1.3). By virtue of the left translation invariance of the operator $\langle A\nabla_x, \nabla_x\rangle$ on the Lie group $\mathbb{K}$, it suffices to prove (1.3) for $z_0 = e$.

Lemma 5.3. *For a.e. $r > 0$, the fundamental solution $\Gamma(e,\cdot)$ satisfies* (1.3) *for $z_0 = e$, that in the present case it reduces to*

$$\int_{\Sigma_{r,\varepsilon}(e)} W(x,t)^{\frac{1}{2}} \Gamma(e,z)\, d\mathcal{H}^n(z) = o(1) \qquad \text{as } \varepsilon \to 0^+, \tag{5.6}$$

where $W(x,t)$ is as in (5.5) *and*

$$\Sigma_{r,\varepsilon}(e) := \{(x,t) \in \partial B(e,\varepsilon) \setminus \overline{\Omega_r(e)} \,:\, t < 0\}.$$

To prove this lemma we need to remind some properties of the operator $\mathcal{L}$ proved in [14] and an estimate of the derivatives Γ_x, see Lemma 5.4 below.

The assumption (iii) is equivalent to

$$C(t) < 0 \qquad \forall t < 0$$

and it implies that, for some basis of $\mathbb{R}^n$, the matrices A, B take the following form:

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{5.7}$$

with $A_0 = (a_{ij})_{i,j=1}^{p_0}$ $p_0 \times p_0$ constant matrix ($p_0 \le n$), symmetric and strictly positive definite;

$$B^T = \begin{bmatrix} * & B_1 & 0 & \dots & 0 \\ * & * & B_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & B_r \\ * & * & * & \dots & * \end{bmatrix} \tag{5.8}$$

where B_j is a $p_{j-1}\times p_j$ block with rank $p_j, j=1,2,...,r$, $p_0\geq p_1\geq ...\geq p_r\geq 1$ and $p_0+p_1+...+p_r=n$.

We denote by B_0 the matrix obtained by annihilating every $*$ block in (5.8):

$$B_0^T=\begin{bmatrix} 0 & B_1 & 0 & \dots & 0\\ 0 & 0 & B_2 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & B_r\\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \tag{5.9}$$

with B_j as in (5.8). In [14] the operator

$$\mathcal{L}_0=\operatorname{div}(A\nabla)+\langle B_0x,\nabla\rangle-\partial_t \tag{5.10}$$

is called the *principal part* of $\mathcal{L}$. This operator is homogeneous of degree two with respect to a group of anisotropic dilations on $\mathbb{R}^{n+1}$, whose matrix is

$$\delta(\lambda):=diag\left(\lambda I_{p_0},\lambda^3 I_{p_1},...,\lambda^{2r+1}I_{p_r},\lambda^2\right),\qquad \lambda>0,$$

where I_{p_j} denotes the $p_j\times p_j$ identity matrix.

We denote $D(\lambda)$ the diagonal $n\times n$ matrix

$$D(\lambda):=diag\left(\lambda I_{p_0},\lambda^3 I_{p_1},...,\lambda^{2r+1}I_{p_r}\right),\qquad \lambda>0.$$

Note that

$$\det(\delta(\lambda))=\lambda^{Q+2}\quad\text{and}\quad\det(D(\lambda))=\lambda^Q,$$

where

$$Q:=p_0+3p_1+...+(2r+1)p_r.$$

The operator $\mathcal{L}$ is left translation invariant on the Lie group $\mathbb{K}$, with unit $e=(0,0)$.

If $z=(x,t)\in\mathbb{R}^{n+1}\setminus\{e\}$, then

$$\Gamma(e,z)=\gamma(z^{-1})=\begin{cases}\gamma(-E(-t)x,-t) & \text{if } t<0\\ 0 & \text{if } t\geq 0.\end{cases}$$

By the last formula in [14, page 50],

$$\Gamma(e,z)=\frac{(4\pi)^{-n/2}}{\sqrt{\det C(-t)}}\exp\left(\frac{1}{4}\langle C^{-1}(t)x,x\rangle\right),\qquad z=(x,t),\ t<0. \tag{5.11}$$

In the proof of Lemma 5.3 we need the following estimate of the derivatives of Γ.

Lemma 5.4. *There exist $t_0>0$ and $c>0$ such that*

$$\frac{\langle A(x)\nabla_x\Gamma(e,z),\nabla_x\Gamma(e,z)\rangle}{\Gamma(e,z)^2}\leq c\frac{|\xi|^2}{|t|}$$

for every $z=(x,t)$, $t\in]-t_0,0[$, and $\xi=D\left(\frac{1}{\sqrt{-t}}\right)x$.

Proof. Let us denote $C_0(t)$ the matrix analogous to $C(t)$, with B replaced by B_0, see (5.2) and (5.9). Then, by [14, Lemma 3.2], that holds true for $t<0$ by replacing $D(\sqrt{t})$ with $D(\sqrt{-t})$, we have

$$C(t)=C_0(t)+D(\sqrt{-t})M(t)D(\sqrt{-t})$$

with $M(t)=O(t)$ as $t\to 0^-$. Since

$$C_0(t)=D(\sqrt{-t})C_0(-1)D(\sqrt{-t})$$

(see [14, Proposition 3.2]) we get

$$C^{-1}(t) = D\left(\frac{1}{\sqrt{-t}}\right) G(t) D\left(\frac{1}{\sqrt{-t}}\right), \tag{5.12}$$

where G is a symmetric matrix such that

$$G(t) = (I + O(t))(C_0(-1))^{-1}. \tag{5.13}$$

By (5.5) and (5.12),

$$W(x,t) = \frac{1}{4}\langle \tilde{A}\xi, \xi\rangle \tag{5.14}$$

with

$$\tilde{A} := G(t) D\left(\frac{1}{\sqrt{-t}}\right) A D\left(\frac{1}{\sqrt{-t}}\right) G(t)$$

and

$$\xi := D\left(\frac{1}{\sqrt{-t}}\right) x. \tag{5.15}$$

Taking into account the definition of the matrix A, see (5.7), and noting that

$$D\left(\frac{1}{\sqrt{-t}}\right) = \begin{bmatrix} \frac{1}{\sqrt{-t}} I_{p_0} & 0 \\ 0 & * \end{bmatrix},$$

we get

$$D\left(\frac{1}{\sqrt{-t}}\right) A D\left(\frac{1}{\sqrt{-t}}\right) = \frac{1}{|t|} A,$$

so obtaining

$$\tilde{A} = \frac{1}{|t|} G(t) A G(t) \qquad t < 0. \tag{5.16}$$

By (5.14), (5.16) and (5.13), we obtain that there exist $t_0 > 0$ and $c > 0$ such that

$$W(x,t) \leq c \frac{|\xi|^2}{|t|}$$

for every $z = (x,t)$, $t \in]-t_0, 0[$, and $\xi = D\left(\frac{1}{\sqrt{-t}}\right) x$, see (5.15). The thesis follows by (5.3), since

$$\langle A(x)\nabla_x \Gamma(e,z), \nabla_x \Gamma(e,z)\rangle^{\frac{1}{2}} = W(x,t)^{\frac{1}{2}} \Gamma(e,z).$$

□

We are now ready to prove Lemma 5.3.

Proof of Lemma 5.3. The estimate (5.6) holds if there exist $T > 0$, $c > 0$ and $\theta \in]0,1[$, such that

$$W(x,t)^{\frac{1}{2}} \Gamma(e,z) \leq \frac{c}{|t|^\theta} \qquad \forall t \in]-T, 0[\tag{5.17}$$

on the set $\{\Gamma(e,z) \leq 1\}$. We recall that $W(x,t)$ is defined as in (5.5).

By Lemma 5.4 and (5.11), there exist $t_0 > 0$ and $c > 0$ such that

$$W(x,t)^{\frac{1}{2}} \Gamma(e,z) \leq \frac{c}{\sqrt{|t|}} |\xi| \exp\big(-\varepsilon|\xi|^2\big) \frac{(4\pi)^{-n/2}}{\sqrt{\det C(-t)}} \exp\big(\frac{1}{4}\langle C^{-1}(t)x, x\rangle + \varepsilon|\xi|^2\big).$$

for every $z = (x,t)$, $t \in]-t_0, 0[$, and $\xi = D\left(\frac{1}{\sqrt{-t}}\right) x$, and for every $\varepsilon > 0$.

Thus, there exists $c(\varepsilon, n) > 0$ such that

$$W(x,t)^{\frac{1}{2}}\Gamma(e,z) \leq \frac{c(\varepsilon,n)}{\sqrt{|t|}\sqrt{\det C(-t)}} \exp\big(\frac{1}{4}\langle C^{-1}(t)x, x\rangle + \varepsilon|\xi|^2\big). \tag{5.18}$$

We claim that for every $\sigma \in]0,1[$ there exists $\varepsilon > 0$, and $t_1 \in]0, t_0]$ such that

$$\frac{1}{4}\langle C^{-1}(t)x, x\rangle + \varepsilon|\xi|^2 \leq \frac{\sigma}{4}\langle C^{-1}(t)x, x\rangle \tag{5.19}$$

for every $x \in \mathbb{R}^n$ and every $t \in]-t_1, 0[$. If the claim holds, that by (5.18) we obtain

$$\begin{aligned} W(x,t)^{\frac{1}{2}}\Gamma(e,z) &\leq \frac{c(\varepsilon,n)}{\sqrt{|t|}\sqrt{\det C(-t)}} \exp\big(\ \frac{\sigma}{4}\langle C^{-1}(t)x, x\rangle\big) \\ &\leq \frac{c(\varepsilon,n)}{\sqrt{|t|}} \left(\frac{1}{\sqrt{\det C(-t)}}\right)^{1-\sigma} \Gamma(e,z)^{\sigma}. \end{aligned}$$

Thus, by [14, Eq. (1.21), (1.22), (3.14)] there exists a dimensional positive constant c such that

$$\det C(-t) = c(-t)^Q(1 + o(1)) \quad \text{as } t \to 0^-.$$

Thus, there exists $T \in]0, t_1]$, such that on the set $\{\Gamma(e,z) \leq 1\}$ the following estimate holds:

$$W(x,t)^{\frac{1}{2}}\Gamma(e,z) \leq \frac{\tilde{c}(\varepsilon,n)}{|t|^{\frac{1}{2}(1+Q(1-\sigma))}}, \qquad t \in]-t_2, 0[.$$

The estimate (5.17) follows if we choose σ close to 1 so that $\theta := \frac{1}{2}(1 + Q(1-\sigma)) < 1$.

It remains to prove the claim (5.19). By (5.12), (5.19) is equivalent to prove that there exists $t_1 \in]0, t_0]$ such that

$$\frac{1-\sigma}{4}\langle G(t)\xi, \xi\rangle \leq -\varepsilon|\xi|^2, \tag{5.20}$$

where $\xi := D\left(\frac{1}{\sqrt{-t}}\right)x$, see (5.15), and $t \in]0, t_1]$.

By (5.13), and since $C(-1)^{-1}$ is negative definite, there exist $\varepsilon_0 > 0$ and $t_1 \in]0, t_0]$ such that

$$\langle G(t)\xi, \xi\rangle \leq -\varepsilon_0|\xi|^2 \qquad t \in]-t_1, 0[.$$

Therefore

$$\frac{1-\sigma}{4}\langle G(t)\xi, \xi\rangle \leq -\frac{\varepsilon_0(1-\sigma)}{4}|\xi|^2 \qquad t \in]-t_1, 0[$$

and (5.20) follows, with $\varepsilon := \frac{\varepsilon_0(1-\sigma)}{4}$.

□

Now the proof of Theorem 5.1 immediately follows.

Proof of Theorem 5.1 . By Lemma 5.3 and Theorem 1.1, we get Theorem 5.1 with $z_0 = e$. By the left translation invariance of the operator $\langle A\nabla_x, \nabla_x\rangle$ on the Lie group $\mathbb{K}$, we obtain (5.4) for any z_0. □

5.3. **Mean value formula with improved kernels.** Let $\mathcal{L}$ be as above,

$$\mathcal{L} := \operatorname{div}(A\nabla) + \langle Bx, \nabla\rangle - \partial_t, \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}$$

Fixed $p \in \mathbb{N}$, denote

$$\mathcal{L}^{(p)} := \mathcal{L} + \Delta_y^{(p)},$$

where $\Delta_y^{(p)}$ is the classical Laplace operator in $\mathbb{R}^p$ in the (new) variables $y = (y_1, \cdots, y_p)$.

Obviously $\mathcal{L}^{(p)}$ is a Kolmogorov operator and, by what previously proved, for this new operator Theorem 1.2 applies.

This leads to the mean value formula (5.21) below.

Theorem 5.5. *Assume that $\mathcal{L}$ is the operator* (4.12) *described above. For every $r > 0$ we have*

$$u(z_0) = \frac{1}{r} \int_{\Omega_r^{(p)}(z_0)} u(z) W_r^{(p)}(z_0^{-1} \circ z)\, dz \tag{5.21}$$

for every $u \in C^2(\overline{\Omega_r^{(p)}(z_0)})$ such that $\mathcal{L}u = 0$.

Here

$$W_r^{(p)}(z) := \omega_p R_r^p(e,z) \left\{ W(z) + \frac{p}{4(p+2)} \Big(\frac{R_r(e,z)}{t} \Big)^2 \right\},$$

with $W(z)$ and $R_r(e,z)$ defined, respectively, in (5.5) *and in* (1.12).

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