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ON THE LOCAL REGULARITY OF THE GEVREY VECTORS FOR HÖRMANDER'S OPERATORS

GREGORIO CHINNI AND MAKHLOUF DERRIDJ

Abstract. We study the local regularity of the analytic/Gevrey vectors for the general class of Hörmander's operators, meaning those of degenerate elliptic type or of degenerate parabolic type

$$P(x,D) = -\sum_{\ell=1}^{m} (X_\ell(x,D))^2 + iX_0(x,D) + c(x),$$

thus improving the result obtained by M. Derridj in [26] . The optimal result in the degenerate elliptic case was obtained by the second author in [27] and we obtain in the present paper optimal result in the degenerate parabolic case.

"*In honor of Jorge Hounie for his many mathematical achievements*"

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1. Introduction

In [19] we dealt with Hörmander's operators of the first kind (or commonly known as "sums of squares of vector fields"), and considered the case of analytic vectors of operators with analytic coefficients. For that we used the method by F.B.I. transform taking advantage from the sharp subelliptic estimate obtained in ([1]), see also ([29]). Here we study the general class of Hörmander's operators, meaning those of degenerate elliptic type or of degenerate parabolic type i.e. operators of the form

$$P(x,D) = -\sum_{\ell=1}^{m} \left(X_\ell(x,D)\right)^2 + iX_0(x,D) + c(x), \tag{1.1}$$

in Ω, open neighborhood of the origin in $\mathbb{R}^n$, where $c(x)$ is a Gevrey function of order s ($s \geq 1$) in Ω and

$$X_\ell(x,D) = \sum_{j=1}^{n} a_{j,\ell}(x) D_j, \quad \ell = 0,1,\dots,m, \qquad \left(D_j = \frac{1}{\sqrt{-1}}\frac{\partial}{\partial x_j}\right)$$

where $a_{j,\ell}(x)$ are real valued Gevrey functions of order s on Ω, satisfying the Hörmander condition, see Definition 2.2.

This question of investigation of the Gevrey regularity of analytic-Gevrey vectors of partial differential operators goes back to the work by T.Kotake and M.Narasimhan ([40], 1962) where they considered analytic vectors of elliptic partial differential operators with analytic coefficients on an open set in R^n and showed there analyticity. This property on the iterates of elliptic operators was also called, after their work, "Kotake-Narasimhan property". (E. Nelson studied before analytic vectors of operators in the context of Lie group theory ([43], 1959). Then after more and more works followed, generalizing their result in various directions (systems, s-Gevrey regularity for s greater or equal to 1 ($s = 1$ corresponding to analytic case), non-elliptic operators, global, local or even microlocal situations. Let us mention a few of the numerous results in the field, as a guide leading at our result (we mention for that two surveys, one quite complete on results obtained before 1987 ([7]) and one very short, mainly on some results obtained in the eighties until 2016 ([24])). A result of G. Metivier obtained in 1978, [42], lightened the way in the study of the case of non elliptic operators and s-Gevrey situation, s greater or equal to 1, showing that the Kotake-Narasimhan property in s-Gevrey class,$s \geq 1$, of a partial differential operator P in an open set characterizes ellipticity of P in this open set, letting of course open the question on iterates of a non elliptic operators in case $s = 1$. A few years after , M.S. Baouendi and G.Metivier proved that the hypoelliptic operators of principal type with analytic coefficients satisfy the Kotake-Narasimhan property and gave condition on s' such that s-Gevrey vectors

are in s'-Gevrey space for $s > 1$ ([2]). In the case of systems let us just mention the case of vector fields: for real vector fields $(X_1, \dots, X_r)$ satisfying the Hörmander's condition introduced in his famous paper on hypoelliptic second order partial differential equations ([35]), two papers appeared in 1980, one by M.Damlakhi -B.Helffer proving "Kotake-Narasimhan property" for such systems with analytic coefficients ([21]), the other by B.Helffer-C.Mattera ([32]), proving analyticity for what they called reducible analytic vectors of the system (which is a less restrictive condition on the system); for complex vector fields, we mention two papers: in ([3]), 2011 and ([13]), 2013 for systems defining locally integrable structures. Since the paper of L. Hörmander's appeared, works on analytic or Gevrey regularity of associated analytic-Gevrey vectors began many years after: the first one was in the global context for a subclass of Hörmander's operators on a product of two tori ([12]) 2016 (in the global context the "Kotake-Narasimhan property" was studied also in [15]). The local version of such result for the general class of Hörmander's operators was established by the second author in two articles (([25]), 2018, ([26]), 2019, with an optimal version in the case of the so-called "Sum of squares" ([27]) and a more general operators of second order introduced by O.A. Oleĭnik and E.V. Radkevič in ([44]), ([28]). The microlocal version of such work was done by the two authors in the case of "Sum of squares with analytic coefficients", ([19]), 2022. We recall that in the case of second order operators of type Hörmander, subclass of that studied by Oleĭnik and Radkevič, the so called Oleĭnik and Radkevič-H örmander condition is weaker than that of Hörmander, in the sense that it does not allow obtaining the optimal subellipticity index, see the examples discussed in ([28]) and ([19]), related to the operators studied in ([16]), ([10]), ([11]), ([17]) and ([18]). We would like to mention some results using intensively the FBI transform (and generalization of it as in ([4]), ([5]), ([33]), ([34]), ([47]), and studying mainly operators (in particular with constant coefficients or of elliptic type) in more and more general classes of ultradifferentiable functions (see ([33]), ([31]), where there are many references).

2. Notations, definitions, preliminary facts and main result

Before stating the main result we recall some basic definitions and facts. First the definition of Gevrey class:

Definition 2.1. *If $\Omega \subset \mathbb{R}^n$ is an open set we say that the function u belongs to the Gevrey class of order $s \geq 1$, $G^s(\Omega)$, if $u \in C^\infty(\Omega)$ and for every compact set $K \subset U$ there exists a positive constant C_K such that*

$$\sup_K |\partial_x^\alpha u(x)| \leq C_K^{|\alpha|+1} \alpha!^s,$$

with $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$
When $s = 1$ we get the class of real analytic functions, also denoted by $C^\omega(U)$.

Remark 2.2. *The sup-norm in the above definition can be replaced by $L^p(K)$ norms, $p \geq 1$. In particular we have that $u \in G^s(\Omega)$ if and only if $u \in C^\infty(\Omega)$ and for every compact set $K \subset U$ there exists a positive constant $\widetilde{C}_K$ such that*

$$\|\partial_x^\alpha u\|_{L^2(K)} \leq \widetilde{C}_K^{|\alpha|+1} \alpha!^s.$$

Let $X(x,D)$ and $Y(x,D)$ be two vector fields with smooth coefficients. We write $\mathrm{ad}_X(Y) = [X,Y]$, where $[X,Y]$ is the Lie bracket or the commutator between $X(x,D)$ and $Y(x,D)$. Let $I = (i_1, i_2, \dots, i_k)$, $i_j \in \{0,1,2,\dots,m\}$, we denote by

$$X_I = \mathrm{ad}_{X_{i_1}} \mathrm{ad}_{X_{i_2}} \cdots \mathrm{ad}_{X_{i_{k-1}}} (X_{i_k}),$$

the k-times iterated commutator.
To the multi-index I we associate the weight length $|I|$ defined as

$$|I| = \#\{i_j \,|\, 1 \le i_j \le m,\, j = 1,\dots,k\} + 2\,\#\{i_j \,|\, i_j = 0,\, j = 1,\dots,k\}\,. \tag{2.1}$$

Definition 2.2. *We say that the system of vector fields $\{X_0(x,D), \dots, X_m(x,D)\}$ satisfies the Hörmander's condition in Ω, if for every $x_0 \in \Omega$, there is a neighborhood Ω_0 of x_0, $\Omega_0 \Subset \Omega$, and a positive integer r, depending on Ω_0, such that the vector fields $X_0, \dots, X_m$ and X_I, with $|I| \le r$, $|I|$ as in (2.1), generate the n-dimensional Lie algebra on Ω_0. We say that the system, $\{X_0(x,D), \dots, X_m(x,D)\}$, satisfies Hörmader's condition at step $\widetilde{r}$ in Ω, if $\widetilde{r} = \sup_{x_0 \in \Omega}\{r_{x_0}\}$, where r_{x_0} is the infimum of the integers such that the Hörmander's condition is satisfied at the point x_0.*

In [35], L. Hörmander showed that if the system of vector fields $\{X_0 \dots, X_m\}$ satisfies the Hörmander's condition in Ω then P is C^∞-hypoelliptic. Moreover in [46], Rothschild and Stein showed that the following a priori estimate holds

$$\|u\|_{\frac{2}{r}} + \sum_{\ell,j=1}^{m} \|X_\ell X_j u\|_0 + \|X_0 u\|_0 \le C\left(\|Pu\|_0 + \|u\|_0\right), \tag{2.2}$$

where $u \in \mathscr{D}(\Omega_0)$, $\Omega_0 \Subset \Omega$, r is as in Definition 2.2 and C is a suitable positive constant.

We recall the local notion of Gevrey vectors.

Definition 2.3. *Let $P(x,D)$ a differential operator of order N with Gevrey coefficients of order $s \ge 1$ in Ω open subset of $\mathbb{R}^n$. We denote by $G^s(\Omega;P)$, the space of the Gevrey vectors of order $s \ge 1$, in Ω, of P i.e.: the set of all distributions $u \in \mathscr{D}'(\Omega)$ such that for any compact subset K of Ω and every $k \in \mathbb{N}$, $P^k u$ is in $L^2(K)$ and there is a positive constant C_K such that*

$$\|P^k u\|_{L^2(K)} \le C_K^{k+1} (Nk!)^s. \tag{2.3}$$

When $s = 1$ we set $G^1(\Omega;P) = \mathscr{A}(\Omega;P)$ the set of the analytic vectors of P in Ω.

We state now the main result of the paper

Theorem 2.1. *Let $P(x,D)$ be as in (1.1). Let x_0 in Ω, Ω_0 a neighborhood of x_0 where the Hörmander's condition is satisfied at the step r, Definition 2.2, and $u \in G^s(\Omega;P)$. Then $u \in G^{rs}(\Omega_0)$.*

3. Estimates of a Gevrey vector of P

Let $x_0 \in \Omega$, Ω_0 a neighborhood of x_0, $\Omega_0 \Subset \Omega$, where the Hörmander's condition is satisfied at the step r and $u \in G^s(\Omega_0; P)$. Let M a given fixed integer which will be determined at the end of the section, having the form $M = pn + q$, p and q suitable integers. Let $\psi_N(x)$ be an Ehrenpreis-Hörmander sequence ([30], see also [37],[48]) associated to the couple (Ω_1, Ω_2), $x_0 \in \Omega_1$. More precisely $\psi_N(x) \equiv 1$ on Ω_1 and supported in Ω_2, with $\overline{\Omega}_1 \Subset \Omega_2 \Subset \Omega_0$, and there is a positive constant C_ψ such that

$$|D^\alpha \psi_N| \leq C_\psi^{|\alpha|+1} N^{(|\alpha|-M)^+}, \tag{3.1}$$

for all $\alpha \in \mathbb{Z}_+^n$ such that $|\alpha| \leq N$.
We will use the following convention: the Latin alphabet letters in the upper index will denote the derivatives with respect to the corresponding direction, i.e. $a^{(k)}(x) = D_k a(x)$, and the Greek alphabet letters in the upper index will denote the usual multi-index derivatives, i.e. $a^{(\alpha)}(x) = D^\alpha a(x) = D_1^{\alpha_1} \cdots D_n^{\alpha_n} a(x)$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$.
The purpose of the present section is to obtain a suitable estimate of $\|D^\alpha \psi_N^{(\beta)} P^k u\|_{p/r}^2$, $p = 1, \ldots, r$ [1].

3.1. Estimate in $H^{1/r}$ and $H^{2/r}$.

Our goal is to obtain a suitable estimate for $\|D^\alpha \psi_N^{(\beta)} P^k u\|_{1/r}$. In order to obtain it we use a modification of the Rothschild-Stein estimate (2.2) proved in [9]:

Lemma 3.1 (Lemma 2.1 in [9]). *The following a priori estimate holds*

$$\|v\|_{\frac{2}{r}} + \sum_{j=1}^m \left(\|\Lambda_r X_j v\|_0 + \|X_j \Lambda_r v\|_0\right) + \sum_{\ell,j=1}^m \|X_\ell X_j v\|_0 + \|X_0 v\|_0 \leq C\left(\|Pv\|_0 + \|v\|_0\right) \tag{3.2}$$

where $\Lambda_r = \left(1 + |D|^2\right)^{\frac{1}{2r}}$, $v \in \mathscr{D}(\Omega_0)$, $\Omega_0 \Subset \Omega$, r is as in Definition 2.2 and C is a suitable positive constant.

By the above Lemma, replacing v by $D^\alpha \psi_N^{(\beta)} P^k u$ in (3.2), the following estimate holds

$$\|D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{2}{r}} + \sum_{j=1}^m \|X_j D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{1}{r}} + \sum_{\ell,j=1}^m \|X_\ell X_j D^\alpha \psi_N^{(\beta)} P^k u\|_0 + \|X_0 D^\alpha \psi_N^{(\beta)} P^k u\|_0 \tag{3.3}$$

[1] The choice to use the set Ω_2 related to the Ehrenpreis-Hörmander sequences and not Ω_0, it will be more clear in the following when we will handle the case $p > 2$.

$$\leq C\left(\|PD^{\alpha}\psi_N^{(\beta)}P^k u\|_0 + \|D^{\alpha}\psi_N^{(\beta)}P^k u\|_0\right).$$

We begin to handle the first term on the right hand side. We have

$$\begin{aligned}(3.4)\quad \|PD^{\alpha}\psi_N^{(\beta)}P^k u\|_0 &\leq \|D^{\alpha}\psi_N^{(\beta)}P^{k+1}u\|_0 + \|[P, D^{\alpha}\psi_N^{(\beta)}]P^k u\|_0\\ &\leq \|D^{\alpha}\psi_N^{(\beta)}P^{k+1}u\|_0 + \sum_{\ell=1}^{m}\|[X_\ell^2, D^{\alpha}\psi_N^{(\beta)}]P^k u\|_0 + \|[X_0, D^{\alpha}\psi_N^{(\beta)}]P^k u\|_0\\ &\quad + \|[c, D^{\alpha}\psi_N^{(\beta)}]P^k u\|_0.\end{aligned}$$

In order to estimate the above terms we will take advantage from the following relation: let $f \in G^s(\Omega)$, then for every $\alpha \in \mathbb{Z}_+^n$ we have

$$(3.5)\qquad [f(x), D^{\gamma}] = -\sum_{\substack{\nu\leq\gamma,\\ \nu\neq 0}}\binom{\gamma}{\nu} f^{(\nu)}(x)D^{\gamma-\nu}.$$

Concerning the last term on the right hand side (3.4), using the above formula with $f = c$ and $\gamma = \alpha$, we get

$$\begin{aligned}(3.6)\quad \|[c, D^{\alpha}\psi_N^{(\beta)}]P^k u\|_0 &\leq \sum_{\substack{\nu\leq\alpha,\\ \nu\neq 0}}\binom{\alpha}{\nu}\|c^{(\nu)}D^{\alpha-\nu}\psi_N^{(\beta)}P^k u\|_0\\ &\leq \sum_{\substack{\nu\leq\alpha,\\ \nu\neq 0}}\frac{\alpha!}{(\alpha-\nu)!}C_1^{|\nu|+1}(\nu!)^{s-1}\|D^{\alpha-\nu}\psi_N^{(\beta)}P^k u\|_0.\end{aligned}$$

We remark that when $\nu = \alpha$ we have

$$C_1^{|\alpha|+1}(\alpha!)^s\|\psi_N^{(\beta)}P^k u\|_0.$$

We recall the following result:

Lemma 3.2 ([20])**.** *For every $N \in \mathbb{N}^*$, one has*

$$(3.7)\qquad k^j \leq B^j N^{(k-M)^+} \text{ for } B = \sup(M, 3),\ j \leq k \leq N.$$

Moreover the following elementary fact holds: if $p_1, \ldots, p_\ell$ and M are integers then

$$N^{(p_1-M)^+}\cdots N^{(p_\ell-M)^+} \leq N^{(p_1+\cdots+p_\ell-M)^+}.$$

Let K_0 be a compact set contained in Ω and containing all the supports of ψ_N, $\Omega_0 \subset K_0$. Since u is a G^s-vector for P in Ω we have $\|P^k u\|_{L^2(K_0)} \leq C_{K_0}^{2k+1}(2k)^{2sk}$. Taking advantage from the Lemma 3.2, we get

$$\begin{aligned}(3.8)\quad C_1^{|\alpha|+1}(\alpha!)^s\|\psi_N^{(\beta)}P^k u\|_0 &\leq C_1^{|\alpha|+1}(\alpha!)^s C_\psi^{|\beta|+1}N^{(|\beta|-M)^+}\|P^k u\|_{L^2(K_0)}\\ &\leq C_1^{|\alpha|+1}C_2^{|\beta|+2k+1}N^{s(|\alpha|+|\beta|+2k-M)^+}.\end{aligned}$$

Now, we consider the second to the last term on the right hand side of (3.4):

$$(3.9)\qquad \|[X_0, D^{\alpha}\psi_N^{(\beta)}]P^k u\|_0 \leq \sum_{j=1}^{n}\|[a_{j,0}(x)D_j, D^{\alpha}\psi_N^{(\beta)}]P^k u\|_0$$

$$\leq \sum_{j=1}^{n} \Big(\|a_{j,0}(x) D^\alpha \psi_N^{(\beta+j)} P^k u\|_0 + \|[a_{j,0}(x), D^\alpha] \psi_N^{(\beta+j)} P^k u\|_0 + \|[a_{j,0}(x), D^\alpha] D_j \psi_N^{(\beta)} P^k u\|_0 \Big)$$

$$\leq \sum_{j=1}^{n} \sum_{\substack{\nu \leq \alpha, \\ \nu \neq 0}} C_1^{|\nu|+1} \frac{\alpha!\ (\nu!)^{s-1}}{(\alpha-\nu)!} \Big(\|D^{\alpha-\nu} \psi_N^{(\beta+j)} P^k u\|_0 + \|D^{\alpha-\nu+j} \psi_N^{(\beta)} P^k u\|_0 \Big) + C_1 \sum_{j=1}^{n} \|D^\alpha \psi_N^{(\beta+j)} P^k u\|_0,$$

where we use (3.5) with $f(x) = a_{j,0}(x)$ and $\gamma = \alpha$ and the fact that

$$\sup_K |a_{\cdot,0}^{(\gamma)}(x)| \leq C_1^{|\gamma|+1} (\gamma!)^s, \quad \gamma \in \mathbb{Z}_+^n,$$

where K is a compact set contained in Ω and containing all the supports of ψ_N.

We focus on the terms in the sum on the right hand side of (3.4). We remark that

$$[X_\ell^2, D^\alpha \psi_N^{(\beta)}] = 2X_\ell [X_\ell, D^\alpha \psi_N^{(\beta)}] - [X_\ell, [X_\ell, D^\alpha \psi_N^{(\beta)}]], \qquad \ell = 1, 2, \ldots, m.$$

Then we have

$$\begin{aligned} (3.10) \quad & \|[X_\ell^2, D^\alpha \psi_N^{(\beta)}] P^k u\|_0 \\ & \leq 2\|X_\ell [X_\ell, D^\alpha \psi_N^{(\beta)}] P^k u\|_0 + \|[X_\ell, [X_\ell, D^\alpha \psi_N^{(\beta)}]] P^k u\|_0 \\ & = 2I_1 + I_2. \end{aligned}$$

We analyze separately these two terms.

Term I_1: we have

$$I_1 \leq \|X_\ell [X_\ell, D^\alpha] \psi_N^{(\beta)} P^k u\|_0 + \|X_\ell D^\alpha [X_\ell, \psi_N^{(\beta)}] P^k u\|_0.$$

Now, by (3.5), where $f = a_{j,\ell}$ and $\gamma = \alpha$, $j = 1, \ldots, n$, we have

$$(3.11) \qquad [X_\ell, D^\alpha] = \sum_{j=1}^{n} [a_{j,\ell} D_j, D^\alpha] = -\sum_{j=1}^{n} \sum_{\substack{\nu \leq \alpha, \\ \nu \neq 0}} \binom{\alpha}{\nu} a_{j,\ell}^{(\nu)}(x) D^{\alpha-\nu+j},$$

and

$$(3.12) \quad D^\alpha [X_\ell, \psi_N^{(\beta)}] = \sum_{j=1}^{n} \Big(a_{j,\ell}(x) D^\alpha \psi_N^{(\beta+j)} + \sum_{\substack{\nu \leq \alpha, \\ \nu \neq 0}} \binom{\alpha}{\nu} a_{j,\ell}^{(\nu)}(x) D^{\alpha-\nu} \psi_N^{(\beta+j)} \Big).$$

So, we obtain

$$I_1 \leq \sum_{j=1}^{n} \sum_{\substack{\nu \leq \alpha, \\ \nu \neq 0}} \binom{\alpha}{\nu} \Big(\|a_{j,\ell}^{(\nu)} X_\ell D^{\alpha-\nu+j} \psi_N^{(\beta)} P^k u\|_0 + \sum_{i=1}^{n} \|a_{p,\ell} a_{j,\ell}^{(\nu+i)} D^{\alpha-\nu+j} \psi_N^{(\beta)} P^k u\|_0 \Big)$$

$$+\sum_{j=1}^{n}\left(\|a_{j,\ell}X_\ell D^\alpha\psi_N^{(\beta+j)}P^ku\|_0+\sum_{i=1}^{n}\|a_{i,\ell}a_{j,\ell}^{(i)}D^\alpha\psi_N^{(\beta+j)}P^ku\|_0\right)$$
$$+\sum_{j=1}^{n}\sum_{\substack{\nu\le\alpha,\\ \nu\neq0}}\binom{\alpha}{\nu}\left(\|a_{j,\ell}^{(\nu)}X_\ell D^{\alpha-\nu}\psi_N^{(\beta+j)}P^ku\|_0+\sum_{i=1}^{n}\|a_{i,\ell}a_{j,\ell}^{(\nu+i)}D^{\alpha-\nu}\psi_N^{(\beta+j)}P^ku\|_0\right).$$

Let K be a compact set contained in Ω and containing all the supports of ψ_N, then the following estimate holds

$$\begin{aligned}&\sup_K|a_{\cdot,\ell}^{(\gamma)}(x)|\le C_1^{|\gamma|+1}[(\gamma)!]^s,\\ &\sup_K|a_{\cdot,\ell}^{(\gamma+p)}(x)|\le C_1^{|\gamma|+2}[(\gamma+e_p)!]^s\le C_1^{|\gamma|+2}2^{s[|\gamma|+1]}(\gamma!)^s\le C_2^{|\gamma|+2}(\gamma!)^s,\end{aligned}\tag{3.13}$$

for every $\gamma\in\mathbb{Z}_+^n$ and $1\le p\le n$.
We conclude that

$$\begin{aligned}&I_1\\ &\le\sum_{j=1}^{n}\sum_{\substack{\nu\le\alpha\\ \nu\neq0}}C_1^{|\nu|+1}\frac{\alpha!\,(\nu!)^{s-1}}{(\alpha-\nu)!}\Big[\|X_\ell D^{\alpha-\nu}\psi_N^{(\beta+j)}P^ku\|_0+\|X_\ell D^{\alpha-\nu+j}\psi_N^{(\beta)}P^ku\|_0\Big]\\ &+n\sum_{j=1}^{n}\sum_{\substack{\nu\le\alpha\\ \nu\neq0}}C_1C_2^{|\nu|+2}\frac{\alpha!\,(\nu!)^{s-1}}{(\alpha-\nu)!}\Big[\|D^{\alpha-\nu+j}\psi_N^{(\beta)}P^ku\|_0+\|D^{\alpha-\nu}\psi_N^{(\beta+j)}P^ku\|_0\Big]\\ &+\sum_{j=1}^{n}C_1\Big[\|X_\ell D^\alpha\psi_N^{(\beta+j)}P^ku\|_0+nC_1^2\|D^\alpha\psi_N^{(\beta+j)}P^ku\|_0\Big].\end{aligned}\tag{3.14}$$

Term I_2. We begin to observe that

$$\begin{aligned}&[X_\ell,[X_\ell,D^\alpha\psi_N^{(\beta)}]]\\ &\qquad=[X_\ell,[X_\ell,D^\alpha]]\psi_N^{(\beta)}+2[X_\ell,D^\alpha][X_\ell,\psi_N^{(\beta)}]+D^\alpha[X_\ell,[X_\ell,\psi_N^{(\beta)}]].\end{aligned}\tag{3.15}$$

Using (3.5) with $f=a_{\cdot,\ell}$ or $f=a_{\cdot,\ell}^{(\cdot)}$ and $\gamma=\alpha$ or $\gamma=\alpha-\nu$, we get

$$\begin{aligned}[X_\ell,[X_\ell,D^\alpha]]\psi_N^{(\beta)}&=-\sum_{j=1}^{n}\sum_{i=1}^{n}\sum_{\substack{\nu\le\alpha,\\ \nu\neq0}}\binom{\alpha}{\nu}[a_{i,\ell}D_i,a_{j,\ell}^{(\nu+i)}D^{\alpha-\nu+j}]\psi_N^{(\beta)}\\ &=-\sum_{j=1}^{n}\sum_{i=1}^{n}\sum_{\substack{\nu\le\alpha,\\ \nu\neq0}}\binom{\alpha}{\nu}a_{i,\ell}\,a_{j,\ell}^{(\nu+i)}D^{\alpha-\nu+j}\psi_N^{(\beta)}+\sum_{j=1}^{n}\sum_{i=1}^{n}a_{j,\ell}^{(\alpha)}a_{i,\ell}^{(j)}\,D_i\psi_N^{(\beta)}\\ &\qquad-\sum_{j=1}^{n}\sum_{i=1}^{n}\sum_{\substack{\nu<\alpha,\\ \nu\neq0}}\binom{\alpha}{\nu}a_{j,\ell}^{(\nu)}[a_{i,\ell},D^{\alpha-\nu+j}]D_i\psi_N^{(\beta)};\end{aligned}$$

since $[a_{i,\ell},D^{\alpha-\nu+j}]=[a_{i,\ell},D^{\alpha-\nu}]D_j-a_{i,\ell}^{(j)},D^{\alpha-\nu}-[D^{\alpha-\nu},a_{i,\ell}^{(j)}]$, we obtain

$$
\begin{aligned}
(3.16)\quad [X_\ell,[X_\ell,D^\alpha]]\psi_N^{(\beta)} &= -\sum_{j=1}^{n}\sum_{i=1}^{n}\sum_{\substack{\nu\le\alpha,\\ \nu\neq 0}}\binom{\alpha}{\nu}a_{i,\ell}\,a_{j,\ell}^{(\nu+i)}D^{\alpha-\nu+j}\psi_N^{(\beta)}\\
&+\sum_{j,i=1}^{n}\sum_{\substack{\nu<\alpha,\\ \nu\neq 0}}\sum_{\substack{\mu\le\alpha-\nu,\\ \mu\neq 0}}\binom{\alpha}{\nu}\binom{\alpha-\nu}{\mu}a_{j,\ell}^{(\nu)}\left(a_{i,\ell}^{(\mu)}D^{\alpha-\nu-\mu+j+i}+a_{i,\ell}^{(\mu+j)}D^{\alpha-\nu-\mu+i}\right)\psi_N^{(\beta)}\\
&+\sum_{j,i=1}^{n}\sum_{\substack{\nu\le\alpha,\\ \nu\neq 0}}\binom{\alpha}{\nu}a_{j,\ell}^{(\nu)}a_{i,\ell}^{(j)}D^{\alpha-\nu+i}\psi_N^{(\beta)}.
\end{aligned}
$$

Acting in the same way, we have

$$
\begin{aligned}
(3.17)\quad & 2[X_\ell,D^\alpha][X_\ell,\psi_N^{(\beta)}]\\
&=-2\sum_{j,i=1}^{n}\sum_{\substack{\nu\le\alpha,\\ \nu\neq 0}}\binom{\alpha}{\nu}a_{j,\ell}^{(\nu)}\left(a_{i,\ell}D^{\alpha-\nu+j}+a_{i,\ell}^{(j)}D^{\alpha-\nu}\right)\psi_N^{(\beta+i)}\\
&-2\sum_{j,i=1}^{n}\sum_{\substack{\nu<\alpha,\\ \nu\neq 0}}\sum_{\substack{\mu\le\alpha-\nu,\\ \mu\neq 0}}\binom{\alpha}{\nu}\binom{\alpha-\nu}{\mu}a_{j,\ell}^{(\nu)}\left(a_{i,\ell}^{(\mu+j)}D^{\alpha-\nu-\mu}+a_{i,\ell}^{(\mu)}D^{\alpha-\nu-\mu+j}\right)\psi_N^{(\beta+i)};
\end{aligned}
$$

and

$$
\begin{aligned}
(3.18)\quad D^\alpha[X_\ell,[X_\ell,\psi_N^{(\beta)}]] &= \sum_{j,i=1}^{n}a_{i,\ell}\left(a_{j,\ell}^{(i)}D^\alpha\psi_N^{(\beta+j)}+a_{j,\ell}D^\alpha\psi_N^{(\beta+j+i)}\right)\\
&+\sum_{j,i=1}^{n}\sum_{\substack{\nu\le\alpha,\\ \nu\neq 0}}\binom{\alpha}{\nu}\Big[\left(a_{i,\ell}a_{j,\ell}^{(\nu+i)}+a_{i,\ell}^{(\nu)}a_{j,\ell}^{(i)}\right)D^{\alpha-\nu}\psi_N^{(\beta+j)}\\
&\qquad+\left(a_{i,\ell}a_{j,\ell}^{(\nu)}+a_{i,\ell}^{(\nu)}a_{j,\ell}\right)D^{\alpha-\nu}\psi_N^{(\beta+j+i)}\Big]\\
&+\sum_{j,i=1}^{n}\sum_{\substack{\nu<\alpha,\\ \nu\neq 0}}\sum_{\substack{\mu\le\alpha-\nu,\\ \mu\neq 0}}\binom{\alpha}{\nu}\binom{\alpha-\nu}{\mu}a_{i,\ell}^{(\nu)}\Big[a_{j,\ell}^{(\mu+i)}\,D^{\alpha-\nu-\mu}\psi_N^{(\beta+j)}\\
&\qquad+a_{j,\ell}^{(\mu)}\,D^{\alpha-\nu-\mu}\psi_N^{(\beta+j+i)}\Big].
\end{aligned}
$$

By (3.13) and the identities, (3.15), (3.16), (3.17) and (3.18), we obtain

$$
\begin{aligned}
(3.19)\quad & I_2\\
&\le 4n\sum_{j=1}^{n}\sum_{\substack{\nu\le\alpha,\\ \nu\neq 0}}C_2^{|\nu|+3}\frac{\alpha!\,(\nu!)^{s-1}}{(\alpha-\nu)!}\left(\|D^{\alpha-\nu}\psi_N^{(\beta+j)}P^ku\|_0+\|D^{\alpha-\nu+j}\psi_N^{(\beta)}P^ku\|_0\right)\\
&+2\sum_{i,j=1}^{n}\sum_{\substack{\nu\le\alpha,\\ \nu\neq 0}}C_1^{|\nu|+2}\frac{\alpha!\,(\nu!)^{s-1}}{(\alpha-\nu)!}\left(\|D^{\alpha-\nu}\psi_N^{(\beta+j+i)}P^ku\|_0+\|D^{\alpha-\nu+j}\psi_N^{(\beta+i)}P^ku\|_0\right)
\end{aligned}
$$

$$+3n\sum_{j=1}^{n}\sum_{\substack{\nu<\alpha,\\ \nu\neq 0}}\sum_{\substack{\mu\leq\alpha-\nu,\\ \mu\neq 0}} C_1^{|\nu|+1}C_2^{|\mu|+2}\frac{\alpha!\,(\nu!)^{s-1}\,(\mu!)^{s-1}}{(\alpha-\nu-\mu)!}\Big[\|D^{\alpha-\nu-\mu}\psi_N^{(\beta+j)}P^ku\|_0 + \|D^{\alpha-\nu-\mu+j}\psi_N^{(\beta)}P^ku\|_0\Big]$$

$$+2\sum_{i,j=1}^{n}\sum_{\substack{\nu<\alpha,\\ \nu\neq 0}}\sum_{\substack{\mu\leq\alpha-\nu,\\ \mu\neq 0}} C_1^{|\nu|+|\mu|+2}\frac{\alpha!\,(\nu!)^{s-1}\,(\mu!)^{s-1}}{(\alpha-\nu-\mu)!}\Big[\|D^{\alpha-\nu-\mu}\psi_N^{(\beta+j+i)}P^ku\|_0 + \|D^{\alpha-\nu-\mu+j}\psi_N^{(\beta+i)}P^ku\|_0 + \|D^{\alpha-\nu-\mu+j+i}\psi_N^{(\beta)}P^ku\|_0\Big]$$

$$+nC_1^3\sum_{j=1}^{n}\|D^{\alpha}\psi_N^{(\beta+j)}P^ku\|_0 + C_1^2\sum_{j=1}^{n}\sum_{i=1}^{n}\|D^{\alpha}\psi_N^{(\beta+j+i)}P^ku\|_0.$$

Summing up, by (3.3), (3.4), (3.6), (3.8), (3.9), (3.10), (3.14) and (3.19), we conclude that there are two new positive constants C_1 and C_2, independent of α, β, k and N, such that

$$\begin{aligned}(3.20)\quad &\|D^{\alpha}\psi_N^{(\beta)}P^ku\|_{\frac{2}{r}} + \sum_{j=1}^{m}\|X_jD^{\alpha}\psi_N^{(\beta)}P^ku\|_{\frac{1}{r}} \\ &\qquad + \sum_{\ell,j=1}^{m}\|X_\ell X_jD^{\alpha}\psi_N^{(\beta)}P^ku\|_0 + \|X_0D^{\alpha}\psi_N^{(\beta)}P^ku\|_0 \\ &\leq C_2\Big[\|D^{\alpha}\psi_N^{(\beta)}P^{k+1}u\|_0 + \|D^{\alpha}\psi_N^{(\beta)}P^ku\|_0 + \sum_{\ell=1}^{m}\sum_{j=1}^{n}\|X_\ell D^{\alpha}\psi_N^{(\beta+j)}P^ku\|_0 \\ &\qquad + \sum_{j=1}^{n}\|D^{\alpha}\psi_N^{(\beta+j)}P^ku\|_0 + \sum_{j=1}^{n}\sum_{i=1}^{n}\|D^{\alpha}\psi_N^{(\beta+j+i)}P^ku\|_0 \\ &+\sum_{\ell=1}^{m}\sum_{j=1}^{n}\sum_{\substack{\nu\leq\alpha,\\ \nu\neq 0}} C_1^{|\nu|+1}\frac{\alpha!\,(\nu!)^{s-1}}{(\alpha-\nu)!}\Big(\|X_\ell D^{\alpha-\nu}\psi_N^{(\beta+j)}P^ku\|_0 + \|X_\ell D^{\alpha-\nu+j}\psi_N^{(\beta)}P^ku\|_0\Big) \\ &+\sum_{j=1}^{n}\sum_{\substack{\nu\leq\alpha,\\ \nu\neq 0}} C_1^{|\nu|+1}\frac{\alpha!\,(\nu!)^{s-1}}{(\alpha-\nu)!}\Big(\|D^{\alpha-\nu}\psi_N^{(\beta+j)}P^ku\|_0 + \|D^{\alpha-\nu+j}\psi_N^{(\beta)}P^ku\|_0\Big) \\ &+\sum_{j,i=1}^{n}\sum_{\substack{\nu\leq\alpha,\\ \nu\neq 0}} C_1^{|\nu|+1}\frac{\alpha!\,(\nu!)^{s-1}}{(\alpha-\nu)!}\Big(\|D^{\alpha-\nu}\psi_N^{(\beta+j+i)}P^ku\|_0 + \|D^{\alpha-\nu+j}\psi_N^{(\beta+i)}P^ku\|_0\Big) \\ &+\sum_{j=1}^{n}\sum_{\substack{\nu<\alpha,\\ \nu\neq 0}}\sum_{\substack{\mu\leq\alpha-\nu,\\ \mu\neq 0}} C_1^{|\nu|+|\mu|+1}\frac{\alpha!\,(\nu!)^{s-1}\,(\mu!)^{s-1}}{(\alpha-\nu-\mu)!}\Big(\|D^{\alpha-\nu-\mu}\psi_N^{(\beta+j)}P^ku\|_0 \\ &\qquad + \|D^{\alpha-\nu-\mu+j}\psi_N^{(\beta)}P^ku\|_0\Big)\end{aligned}$$

$$+\sum_{i,j=1}^{n}\sum_{\substack{\nu<\alpha,\\ \nu\neq 0}}\sum_{\substack{\mu\leq\alpha-\nu,\\ \mu\neq 0}} C_1^{|\nu|+|\mu|+2}\frac{\alpha!\,(\nu!)^{s-1}\,(\mu!)^{s-1}}{(\alpha-\nu-\mu)!}\Big(\|D^{\alpha-\nu-\mu}\psi_N^{(\beta+j+i)}P^k u\|_0$$
$$+\|D^{\alpha-\nu-\mu+j}\psi_N^{(\beta+i)}P^k u\|_0+\|D^{\alpha-\nu-\mu+j+i}\psi_N^{(\beta)}P^k u\|_0\Big)$$
$$+\sum_{\substack{\nu<\alpha,\\ \nu\neq 0}} C_1^{|\nu|+1}\frac{\alpha!\,(\nu!)^{s-1}}{(\alpha-\nu)!}\|D^{\alpha-\nu}\psi_N^{(\beta)}P^k u\|_0+C_1^{|\alpha|+1}C_2^{|\beta|+2k}N^{s(|\alpha|+|\beta|+2k-M)^+}\Big].$$

We remark that when $\nu=\alpha$ or $\mu=\alpha-\nu$ the first term in the multiple-sums can be directly estimated. More precisely, setting $\nu=\alpha$ and using the same strategy adopted to estimate the last term in (3.6), see (3.8), the first term of second and third multiple-sum can be bounded as follows

$$\sum_{j=1}^{n} C_1^{|\alpha|+1}\,(\alpha!)^s\,\|\psi_N^{(\beta+j)}P^k u\|_0 \leq nC_1^{|\alpha|+1}\widetilde{C}_2^{|\beta|+2k+2}N^{s(|\alpha|+|\beta|+2k+1-M)^+}; \tag{3.21}$$
$$\sum_{j=1}^{n}\sum_{i=1}^{n} C_1^{|\alpha|+1}\,(\alpha!)^s\,\|\psi_N^{(\beta+j+i)}P^k u\|_0 \leq n^2C_1^{|\alpha|+1}\widetilde{C}_2^{|\beta|+2k+3}N^{s(|\alpha|+|\beta|+2k+2-M)^+}.$$

In the same way, setting $\mu=\alpha-\nu$ in the first term of the fourth and fifth multiple-sum involving μ, and recalling that the number of the multi-index ν with $|\nu|\leq|\alpha|$ is equal to $\binom{|\alpha|+n}{|\alpha|}$, smaller than $2^{|\alpha|+n}$, we have

$$\sum_{j=1}^{n}\sum_{\substack{\nu<\alpha,\\ \nu\neq 0}} C_1^{|\alpha|+1}\alpha!\,(\nu!)^{s-1}\,[(\alpha-\nu)!]^{s-1}\,\|\psi_N^{(\beta+j)}P^k u\|_0 \leq n2^{|\alpha|+n}C_1^{|\alpha|+1}\widetilde{C}_2^{|\beta|+2k+2}N^{s(|\alpha|+|\beta|+2k+1-M)^+}; \tag{3.22}$$
$$\sum_{j=1}^{n}\sum_{i=1}^{n}\sum_{\substack{\nu<\alpha,\\ \nu\neq 0}} C_1^{|\alpha|+1}\alpha!\,(\nu!)^{s-1}\,[(\alpha-\nu)!]^{s-1}\,\|\psi_N^{(\beta+i+j)}P^k u\|_0 \leq n^2 2^{|\alpha|+n}C_1^{|\alpha|+1}\widetilde{C}_2^{|\beta|+2k+3}N^{s(|\alpha|+|\beta|+2k+2-M)^+}.$$

Concerning the first term in the first multiple-sum on the right hand side of (3.20), setting $\nu=\alpha$, it becomes

$$\sum_{\ell=1}^{m}\sum_{j=1}^{n} C_1^{|\alpha|+1}\,(\alpha!)^s\,\|X_\ell\psi_N^{(\beta+j)}P^k u\|_0. \tag{3.23}$$

In order to handle it, we recall that the following a priori estimate holds

$$\sum_{\ell=1}^{m}\|X_\ell v\|_0^2\leq\widetilde{C}\left(|\langle Pv,v\rangle|+\|v\|^2\right),\qquad\forall\,v\in\mathscr{D}(\Omega_0).$$

Replacing v by $\psi_N^{(\beta+j)}P^k u$, we obtain

$$\sum_{\ell=1}^{m}\|X_\ell\psi_N^{(\beta+j)}P^k u\|_0^2 \le \widetilde{C}\left(|\langle P\psi_N^{(\beta+j)}P^k u,\psi_N^{(\beta+j)}P^k u\rangle| + \|\psi_N^{(\beta+j)}P^k u\|^2\right).$$

Since

$$P\psi_N^{(\beta+j)}P^k u = \psi_N^{(\beta+j)}P^{k+1}u + \sum_{\ell=1}^{m}\left(2X_\ell[X_\ell,\psi_N^{(\beta+j)}] - [X_\ell,[X_\ell,\psi_N^{(\beta+j)}]]\right) + [X_0,\psi_N^{(\beta+j)}],$$

where

$$[X_\ell,\psi_N^{(\beta+j)}] = \sum_{i=1}^{n} a_{i,\ell}\psi_N^{(\beta+j+i)}, \qquad \ell = 0,1,\dots,m;$$

$$[X_\ell,[X_\ell,\psi_N^{(\beta+j)}]] = \sum_{i=1}^{n}\sum_{p=1}^{n} a_{p,\ell}\left(a_{i,\ell}^{(p)}\psi_N^{(\beta+j+i)} + a_{i,\ell}\psi_N^{(\beta+j+i+p)}\right), \qquad \ell = 1,\dots,m;$$

and keeping in mind that $X_\ell^* = X_\ell + \sum_{i=1}^{n} a_{i,\ell}^{(i)}$, we obtain

$$\begin{aligned}\sum_{\ell=1}^{m}\|X_\ell\psi_N^{(\beta+j)}P^k u\|_0^2 \le \widetilde{C}_1\Bigg[&\|\psi_N^{(\beta+j)}P^{k+1}u\|_0\|\psi_N^{(\beta+j)}P^k u\|_0 + \|\psi_N^{(\beta+j)}P^k u\|_0^2\\ &+\sum_{\ell=0}^{m}\sum_{i=1}^{n}\|a_{i,\ell}\psi_N^{(\beta+j+i)}P^k u\|_0^2 + \sum_{\ell=1}^{m}\sum_{i=1}^{n}\|a_{i,\ell}^{(i)}\psi_N^{(\beta+j)}P^k u\|_0^2\\ &+\sum_{\ell=1}^{m}\sum_{i=1}^{n}\sum_{p=1}^{n}\left(\|a_{p,\ell}a_{i,\ell}^{(p)}\psi_N^{(\beta+j+i)}P^k u\|_0 + \|a_{p,\ell}a_{i,\ell}\psi_N^{(\beta+j+i+p)}P^k u\|_0\right)\|\psi_N^{(\beta+j)}P^k u\|_0\Bigg].\end{aligned}$$

So, using the same strategy adopted to estimate the last term in (3.6) (see (3.8)), we conclude that

$$\sum_{\ell=1}^{m}\|X_\ell\psi_N^{(\beta+j)}P^k u\|_0^2 \le \widetilde{C}_2^{2(|\beta|+2k+4)}N^{2s(|\beta|+2k+3-M)^+}.$$

The (3.23) can be bounded by

$$C_1^{|\alpha|+1}\widetilde{C}_3^{|\beta|+2k+4}N^{s(|\alpha|+|\beta|+2k+3-M)^+}, \tag{3.24}$$

where $\widetilde{C}_3$ is a suitable positive constant independent of α, β, k and N.

We end this subsection recalling that by interpolation the following inequality holds

$$\|D^\alpha\psi_N^{(\beta)}P^k u\|_{\frac{1}{r}} \le \left(\|D^\alpha\psi_N^{(\beta)}P^k u\|_{\frac{2}{r}}\right)^{1/2}\left(\|D^\alpha\psi_N^{(\beta)}P^k u\|_0\right)^{1/2}. \tag{3.25}$$

3.2. **Estimate in $H^{p/r}$, $p = 3,\dots,r$.**

The purpose of the present subsection is to obtain a suitable estimate of $\|v\|_{p/r}$, $\|X_j v\|_{(p-1)/r}$ and $\|X_0 v\|_{(p-2)/r}$, $j = 1,\dots,m$, $p = 3,\dots,r$, where $v = D^\alpha\psi_N^{(\beta)}P^k u$. We denote by Λ_r^ℓ the pseudodifferential operator with symbol $(1+|\xi|^2)^{\ell/2r}$. Let

$\widetilde{\psi}$ be in $\mathscr{D}(\Omega_3)$, $\overline{\Omega}_3 \Subset \Omega_0$, such that $\widetilde{\psi} \equiv 1$ on Ω_2, $\overline{\Omega}_2 \Subset \Omega_3$. We recall that $u \in G^s(\Omega_0; P)$, where Ω_0 is a neighborhood of x_0, $\Omega_0 \Subset \Omega$, where the Hörmander condition is satisfied at the step r , and that ψ_N is the Ehrenpreis-Hörmander sequence associated to the couple (Ω_1, Ω_2), $x_0 \in \Omega_1$, so $\widetilde{\psi}\psi_N = \psi_N$.
We have

$$
\begin{aligned}
(3.26)\quad \|D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p}{r}} &+ \sum_{j=1}^m \|X_j D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-1}{r}} + \|X_0 D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-2}{r}} \\
&+ \sum_{\ell,j=1}^m \|X_\ell X_j D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-2}{r}} \\
= \|\widetilde{\psi} D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p}{r}} &+ \sum_{j=1}^m \|\widetilde{\psi} X_j D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-1}{r}} + \|\widetilde{\psi} X_0 D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-2}{r}} \\
&+ \sum_{\ell,j=1}^m \|\widetilde{\psi} X_\ell X_j D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-2}{r}}.
\end{aligned}
$$

We analyze separately the terms on the right hand side. About the first term:

$$
\begin{aligned}
\|\widetilde{\psi} D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p}{r}} &= \|\Lambda_r^{p-2} \widetilde{\psi} D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{2}{r}} \\
&\le \|\widetilde{\psi} \Lambda_r^{p-2} D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{2}{r}} + \|[\widetilde{\psi}, \Lambda_r^{p-2}] D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{2}{r}}.
\end{aligned}
$$

Since $[\Lambda_r^{p-2}, \widetilde{\psi}]$ is a pseudodifferential operator of order $\frac{p-2}{r} - 1 < 0$, then the symbol associated to the operator $\Lambda_r^2 [\Lambda_r^{p-2}, \widetilde{\psi}]$ satisfies the assumptions of the Theorem 18.1.1′ in [38], we conclude that

$$
(3.27)\qquad \|\widetilde{\psi} D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p}{r}} \le \|\widetilde{\psi} \Lambda_r^{p-2} D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{2}{r}} + C_0 \|D^\alpha \psi_N^{(\beta)} P^k u\|.
$$

Before to trite the other terms on the right hand side of (3.27), for technical reasons, i.e. in order to handle the commutators of X_j, $j = 0, 1, \ldots, m$, with the pseudo-differential operator Λ_r^ℓ, we introduce a new Ehrenpreis-Hörmander sequence φ_N associated to the couple (Ω_3, Ω_4), $\overline{\Omega}_3 \Subset \Omega_4 \Subset \Omega_0$. So, φ_N is identically one on the supports of ψ_N and $\widetilde{\psi}$, $\varphi_N \psi_N = \psi_N$ and $\varphi_N \widetilde{\psi} = \widetilde{\psi}$. We set

$$
(3.28)\qquad \widetilde{X}_j(x, D) = \sum_{\ell=1}^n \widetilde{a}_{N,\ell,j}(x) D_\ell, \qquad j = 0, 1, \ldots, m,
$$

where $\widetilde{a}_{N,\ell,j}(x) = \varphi_N a_{\ell,j}(x)$; we remark: $\widetilde{\psi}\widetilde{X}_j = \widetilde{\psi} X_j$ and $[\widetilde{\psi}, \widetilde{X}_j] = [\widetilde{\psi}, X_j]$.
For every j, $j = 1, \ldots, m$, we have

(3.29)

$$
\begin{aligned}
\|\widetilde{\psi} X_j D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-1}{r}} &= \|\widetilde{\psi}\widetilde{X}_j D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-1}{r}} = \|\Lambda_r^{p-2} \widetilde{\psi}\widetilde{X}_j D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{1}{r}} \\
&\le \|\widetilde{\psi}\widetilde{X}_j \Lambda_r^{p-2} D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{1}{r}} + \|[\Lambda_r^{p-2}, \widetilde{\psi}\widetilde{X}_j] D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{1}{r}} \\
&\le \|\widetilde{X}_j \widetilde{\psi} \Lambda_r^{p-2} D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{1}{r}} + \|[\widetilde{\psi}, \widetilde{X}_j] \Lambda_r^{p-2} D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{1}{r}} \\
&\quad + \|[\Lambda_r^{p-2}, \widetilde{\psi}\widetilde{X}_j] D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{1}{r}}.
\end{aligned}
$$

We set $[\widetilde{\psi}, \widetilde{X}_j] = \sum_{\ell=1}^n \widetilde{a}_{N,\ell,j} \widetilde{\psi}^{(\ell)} = f$; since

$$
\left(1 + |\xi|^2\right)^{t/2} \le (1 + |\eta|)^t \left(1 + |\xi - \eta|^2\right)^{t/2}, \qquad t \ge 0,
$$

we have

$$\begin{aligned}\|[\widetilde{\psi},\widetilde{X}_j]\Lambda_r^{p-2}D^\alpha\psi_N^{(\beta)}P^k u\|_{\frac{1}{r}} &= \|f\Lambda_r^{p-2}D^\alpha\psi_N^{(\beta)}P^k u\|_{\frac{1}{r}}\\ &\le \left(\int (1+|\xi|)^{\frac{1}{r}}\,|\widehat{f}(\xi)|\,d\xi\right)\|D^\alpha\psi_N^{(\beta)}P^k u\|_{\frac{p-1}{r}}\\ &\le C^{n+3}N^{(n+2-M)^+}\|D^\alpha\psi_N^{(\beta)}P^k u\|_{\frac{p-1}{r}}.\end{aligned}$$

In the same way we obtain

$$\begin{aligned}\|\widetilde{X}_j\widetilde{\psi}\Lambda_r^{p-2}D^\alpha\psi_N^{(\beta)}P^k u\|_{\frac{1}{r}} &= \|\varphi_N X_j\widetilde{\psi}\Lambda_r^{p-2}D^\alpha\psi_N^{(\beta)}P^k u\|_{\frac{1}{r}}\\ &\le \left(\int (1+|\xi|)^{\frac{1}{r}}\,|\widehat{\varphi}_N(\xi)|\,d\xi\right)\|X_j\widetilde{\psi}\Lambda_r^{p-2}D^\alpha\psi_N^{(\beta)}P^k u\|_{\frac{1}{r}}\\ &\le C^{n+3}N^{(n+2-M)^+}\|X_j\widetilde{\psi}\Lambda_r^{p-2}D^\alpha\psi_N^{(\beta)}P^k u\|_{\frac{1}{r}}.\end{aligned}$$

Now, we handle the last term on the right hand side of (3.29):

$$[\Lambda_r^{p-2},\widetilde{\psi}\widetilde{X}_j] = [\Lambda_r^{p-2},\widetilde{\psi}]\widetilde{X}_j + \widetilde{\psi}[\Lambda_r^{p-2},\widetilde{X}_j];$$

moreover

$$[\Lambda_r^{p-2},\widetilde{X}_j] = \sum_{\ell_1=1}^n\sum_{\ell_2=1}^n \widetilde{a}_{N,\ell_1,j}^{(\ell_2)}(\Lambda_r^{p-2})^{(\ell_2)}D_{\ell_1} + \sum_{\ell_1=1}^n \mathscr{R}_1\left([\Lambda_r^{p-2},\widetilde{a}_{N,\ell_1,j}]D_{\ell_1}\right), \tag{3.30}$$

where the symbol associated to $\mathscr{R}_1$ is

$$\begin{aligned}&\mathscr{R}_1\left([\Lambda_r^{p-2},\widetilde{a}_{N,\ell_1,j}]D_{\ell_1}\right)(x,\xi)\\ &= \frac{2}{(2\pi)^n}\sum_{|\mu|=2}\frac{1}{\mu!}\int e^{i(\eta-\xi)(x-y)}\widetilde{a}_{N,\ell_1,j}^{(\mu)}(y)\,\xi_{\ell_1}\int_0^1(1-t)\left(\Lambda_r^{p-2}\right)^{(\mu)}(\xi+t(\eta-\xi))\,dt\,d\zeta dy.\end{aligned}$$

We remark that $\mathscr{R}_1$ is a pseudodifferential operator of order $\frac{p-2}{r}-1$.
We conclude that

$$\begin{aligned}\|[\Lambda_r^{p-2},\widetilde{\psi}\widetilde{X}_j]D^\alpha\psi_N^{(\beta)}P^k u\|_{\frac{1}{r}} &\le \|[\Lambda_r^{p-2},\widetilde{\psi}]\widetilde{X}_j D^\alpha\psi_N^{(\beta)}P^k u\|_{\frac{1}{r}}\\ &\quad + \|\widetilde{\psi}[\Lambda_r^{p-2},\widetilde{X}_j]D^\alpha\psi_N^{(\beta)}P^k u\|_{\frac{1}{r}}\\ &\le C_1\|X_jD^\alpha\psi_N^{(\beta)}P^k u\|_{\frac{p-1}{r}-1} + C_2\|D^\alpha\psi_N^{(\beta)}P^k u\|_{\frac{p-1}{r}} + C_3\|\widetilde{\psi}D^\alpha\psi_N^{(\beta)}P^k u\|_{\frac{p-1}{r}-1}.\end{aligned}$$

Without loss of generality we assumed that $M > n+3$. By (3.29) and the above consideration we obtain

$$\begin{aligned}\|\widetilde{\psi}X_jD^\alpha\psi_N^{(\beta)}P^k u\|_{\frac{p-1}{r}} \le C\Big(&\|X_j\widetilde{\psi}\Lambda_r^{p-2}D^\alpha\psi_N^{(\beta)}P^k u\|_{\frac{1}{r}} + \|D^\alpha\psi_N^{(\beta)}P^k u\|_{\frac{p-1}{r}}\\ &+ \|X_jD^\alpha\psi_N^{(\beta)}P^k u\| + \|\widetilde{\psi}D^\alpha\psi_N^{(\beta)}P^k u\|\Big),\end{aligned} \tag{3.31}$$

where C is a suitable positive constant.
Using the same strategy adopted above, we conclude that the third term on the right hand side of (3.26) can be estimated as follow

$$\begin{aligned}\|\widetilde{\psi}X_0D^\alpha\psi_N^{(\beta)}P^k u\|_{\frac{p-2}{r}} \le C\Big(&\|X_0\widetilde{\psi}\Lambda_r^{p-2}D^\alpha\psi_N^{(\beta)}P^k u\| + \|D^\alpha\psi_N^{(\beta)}P^k u\|_{\frac{p-2}{r}}\\ &+ \|X_0D^\alpha\psi_N^{(\beta)}P^k u\| + \|D^\alpha\psi_N^{(\beta)}P^k u\|\Big).\end{aligned} \tag{3.32}$$

Lastly, we have to estimate the last term on the right hand side of (3.26). For every j and ℓ, $1 \leq j\,, \ell \leq m$, we have

$$\begin{aligned}
\|\widetilde{\psi}X_\ell X_j D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-2}{r}} &= \|\Lambda_r^{p-2}\widetilde{\psi}\widetilde{X}_\ell\widetilde{X}_j D^\alpha \psi_N^{(\beta)} P^k u\| \\
&\leq \|[\Lambda_r^{p-2},\widetilde{\psi}]\widetilde{X}_\ell\widetilde{X}_j D^\alpha \psi_N^{(\beta)} P^k u\| + \|\widetilde{\psi}\Lambda_r^{p-2}\widetilde{X}_\ell\widetilde{X}_j D^\alpha \psi_N^{(\beta)} P^k u\| \\
&\leq \|[\Lambda_r^{p-2},\widetilde{\psi}]\widetilde{X}_\ell\widetilde{X}_j D^\alpha \psi_N^{(\beta)} P^k u\| + \|\widetilde{\psi}[\Lambda_r^{p-2},\widetilde{X}_\ell]\widetilde{X}_j D^\alpha \psi_N^{(\beta)} P^k u\| \\
&\qquad + \|\widetilde{\psi}\widetilde{X}_\ell\Lambda_r^{p-2}\widetilde{X}_j D^\alpha \psi_N^{(\beta)} P^k u\| \\
&\leq \|[\Lambda_r^{p-2},\widetilde{\psi}]\widetilde{X}_\ell\widetilde{X}_j D^\alpha \psi_N^{(\beta)} P^k u\| + \|\widetilde{\psi}[\Lambda_r^{p-2},\widetilde{X}_\ell]\widetilde{X}_j D^\alpha \psi_N^{(\beta)} P^k u\| \\
&\quad + \|\widetilde{X}_\ell\widetilde{X}_j\widetilde{\psi}\Lambda_r^{p-2} D^\alpha \psi_N^{(\beta)} P^k u\| + \|\widetilde{\psi}[\Lambda_r^{p-2},\widetilde{X}_j]\widetilde{X}_\ell D^\alpha \psi_N^{(\beta)} P^k u\| \\
&\qquad + \|\widetilde{\psi}[\widetilde{X}_\ell,[\Lambda_r^{p-2},\widetilde{X}_j]]D^\alpha \psi_N^{(\beta)} P^k u\|.
\end{aligned}$$

Since $[\Lambda_r^{p-2},\widetilde{\psi}]$ is a pseudodifferential operator of order $\frac{p-2}{r}-1$, we have

$$\|[\Lambda_r^{p-2},\widetilde{\psi}]\widetilde{X}_\ell\widetilde{X}_j D^\alpha \psi_N^{(\beta)} P^k u\| \leq C_0\|X_\ell X_j D^\alpha \psi_N^{(\beta)} P^k u\|.$$

Without fall in details, using the same strategy adopted previously to obtain (3.31), we conclude that

$$\|\widetilde{\psi}[\Lambda_r^{p-2},\widetilde{X}_\ell]\widetilde{X}_j D^\alpha \psi_N^{(\beta)} P^k u\| \leq C_1\|\widetilde{X}_j D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-2}{r}} + C_2\|\widetilde{X}_j D^\alpha \psi_N^{(\beta)} P^k u\|;$$

$$\|\widetilde{\psi}[\Lambda_r^{p-2},\widetilde{X}_j]\widetilde{X}_\ell D^\alpha \psi_N^{(\beta)} P^k u\| \leq C_1\|\widetilde{X}_\ell D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-2}{r}} + C_2\|\widetilde{X}_\ell D^\alpha \psi_N^{(\beta)} P^k u\|;$$

and

$$\|\widetilde{\psi}[\widetilde{X}_\ell,[\Lambda_r^{p-2},\widetilde{X}_j]]D^\alpha \psi_N^{(\beta)} P^k u\| \leq C_3\|D^\alpha \psi_N^{(\beta)} P^k u\|.$$

So

$$\begin{aligned}
\|\widetilde{\psi}X_\ell X_j D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-2}{r}} &\leq C\Big(\|X_\ell X_j\widetilde{\psi}\Lambda_r^{p-2} D^\alpha \psi_N^{(\beta)} P^k u\| \\
&\quad + \|X_j D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-2}{r}} + \|X_\ell D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-2}{r}} + \|D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-2}{r}} \\
&\quad + \|X_\ell X_j D^\alpha \psi_N^{(\beta)} P^k u\| + \|X_j D^\alpha \psi_N^{(\beta)} P^k u\| + \|X_\ell D^\alpha \psi_N^{(\beta)} P^k u\| \\
&\quad + \|D^\alpha \psi_N^{(\beta)} P^k u\|\Big),
\end{aligned} \tag{3.33}$$

where C is a suitable positive constant independent of α, β, k and N.
Summing up by (3.26), (3.27), (3.31), (3.32) and (3.33), there is a suitable positive constant, C_1, independent of α, β, k and N, such that

$$\begin{aligned}
&\|D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p}{r}} + \sum_{j=1}^m \|X_j D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-1}{r}} + \|X_0 D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-2}{r}} \\
&\qquad + \sum_{\ell,j=1}^m \|X_\ell X_j D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-2}{r}} \\
&\leq C_1\Big(\|\widetilde{\psi}\Lambda_r^{p-2} D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{2}{r}} + \sum_{j=1}^m \|X_j\widetilde{\psi}\Lambda_r^{p-2} D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{1}{r}} \\
&\quad + \|X_0\widetilde{\psi}\Lambda_r^{p-2} D^\alpha \psi_N^{(\beta)} P^k u\| + \sum_{\ell,j=1}^m \|X_\ell X_j\widetilde{\psi}\Lambda_r^{p-2} D^\alpha \psi_N^{(\beta)} P^k u\|
\end{aligned} \tag{3.34}$$

$$+\|D^\alpha\psi_N^{(\beta)}P^ku\|_{\frac{p-1}{r}}+\|D^\alpha\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}+\sum_{j=1}^m\|X_jD^\alpha\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}$$
$$+\sum_{j=1}^m\|X_jD^\alpha\psi_N^{(\beta)}P^ku\|+\|X_0D^\alpha\psi_N^{(\beta)}P^ku\|+\sum_{\ell,j=1}^m\|X_\ell X_jD^\alpha\psi_N^{(\beta)}P^ku\|$$
$$+\|D^\alpha\psi_N^{(\beta)}P^ku\|\Big).$$

We have to estimate the first four terms on the right hand side; the other terms are "good" terms for the induction argument used in the last section to gain the main result. Since $\widetilde{\psi}\Lambda_r^{p-2}D^\alpha\psi_N^{(\beta)}P^ku\in\mathscr{D}(\Omega_0)$, we take advantage from the estimate (3.2), Lemma 3.1, replacing v by $\widetilde{\psi}\Lambda_r^{p-2}D^\alpha\psi_N^{(\beta)}P^ku\in\mathscr{D}(\Omega_0)$. We have

$$\|\widetilde{\psi}\Lambda_r^{p-2}D^\alpha\psi_N^{(\beta)}P^ku\|_{\frac{2}{r}}+\sum_{j=1}^m\|X_j\widetilde{\psi}\Lambda_r^{p-2}D^\alpha\psi_N^{(\beta)}P^ku\|_{\frac{1}{r}}$$
$$+\|X_0\widetilde{\psi}\Lambda_r^{p-2}D^\alpha\psi_N^{(\beta)}P^ku\|+\sum_{\ell,j=1}^m\|X_\ell X_j\widetilde{\psi}\Lambda_r^{p-2}D^\alpha\psi_N^{(\beta)}P^ku\|$$
$$\le C\left(\|P\widetilde{\psi}\Lambda_r^{p-2}D^\alpha\psi_N^{(\beta)}P^ku\|+\|\widetilde{\psi}\Lambda_r^{p-2}D^\alpha\psi_N^{(\beta)}P^ku\|\right). \tag{3.35}$$

We remark that

$$\|\widetilde{\psi}\Lambda_r^{p-2}D^\alpha\psi_N^{(\beta)}P^ku\|\le\|D^\alpha\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}. \tag{3.36}$$

We focus on the first term on the right hand side of (3.35). We proceed as in the case $H^{2/r}$ with the difference that in this case we have to handle new ingredients, in particular, the presence of the operator Λ_r^{p-2}. For technical reasons, we introduce

$$P_N=\sum_{\ell=1}^m\widetilde{X}_\ell^2+\widetilde{X}_0+\widetilde{c}_N, \tag{3.37}$$

where $\widetilde{X}_j$, $j=0,1,\dots,m$, have the form described in (3.28). We point out that since φ_N is identically one on the support of $\widetilde{\psi}$ and φ_N, the identities $P_N(\widetilde{\psi}v)=P(\widetilde{\psi}v)$ and $\psi_N^{(\beta)}P_Nv=\psi_N^{(\beta)}Pv$ hold.
We have

$$\|P\widetilde{\psi}\Lambda_r^{p-2}D^\alpha\psi_N^{(\beta)}P^ku\|=\|P_N\widetilde{\psi}\Lambda_r^{p-2}D^\alpha\psi_N^{(\beta)}P^ku\|$$
$$\le\|\widetilde{\psi}\Lambda_r^{p-2}D^\alpha\psi_N^{(\beta)}P^{k+1}u\|+\|[P_N,\widetilde{\psi}\Lambda_r^{p-2}D^\alpha\psi_N^{(\beta)}]P^ku\|$$
$$\le\|D^\alpha\psi_N^{(\beta)}P^{k+1}u\|_{\frac{p-2}{r}}+\|[P_N,\widetilde{\psi}\Lambda_r^{p-2}]D^\alpha\psi_N^{(\beta)}P^ku\|$$
$$+\|\widetilde{\psi}\Lambda_r^{p-2}[P_N,D^\alpha\psi_N^{(\beta)}]P^ku\|. \tag{3.38}$$

We focus on the second term on the right hand side:

$$[P_N,\widetilde{\psi}\Lambda_r^{p-2}]=\sum_{j=1}^m[\widetilde{X}_j^2,\widetilde{\psi}\Lambda_r^{p-2}]+[\widetilde{X}_0,\widetilde{\psi}\Lambda_r^{p-2}]+[\widetilde{c}_N,\widetilde{\psi}\Lambda_r^{p-2}]$$
$$=\sum_{j=1}^m\Big(2[\widetilde{X}_j,\widetilde{\psi}\Lambda_r^{p-2}]\widetilde{X}_j+[\widetilde{X}_j,[\widetilde{X}_j,\widetilde{\psi}\Lambda_r^{p-2}]]\Big)+[\widetilde{X}_0,\widetilde{\psi}\Lambda_r^{p-2}]+[\widetilde{c}_N,\widetilde{\psi}\Lambda_r^{p-2}].$$

So, we have

$$\begin{aligned}(3.39)\quad &\|[P_N,\widetilde{\psi}\Lambda_r^{p-2}]D^\alpha\psi_N^{(\beta)}P^k u\|\\ &\le\sum_{j=1}^m\Big(2\|[\widetilde{X}_j,\widetilde{\psi}\Lambda_r^{p-2}]\widetilde{X}_jD^\alpha\psi_N^{(\beta)}P^k u\|+\|[\widetilde{X}_j,[\widetilde{X}_j,\widetilde{\psi}\Lambda_r^{p-2}]]D^\alpha\psi_N^{(\beta)}P^k u\|\Big)\\ &\qquad+\|[\widetilde{X}_0,\widetilde{\psi}\Lambda_r^{p-2}]D^\alpha\psi_N^{(\beta)}P^k u\|+\|[\widetilde{c}_N,\widetilde{\psi}\Lambda_r^{p-2}]D^\alpha\psi_N^{(\beta)}P^k u\|\\ &\qquad\qquad=\sum_{k=1}^4 I_k.\end{aligned}$$

We analyze separately the term on the right hand side.
Term I_4: since $[\widetilde{c}_N,\widetilde{\psi}\Lambda_r^{p-2}]=\widetilde{\psi}[\widetilde{c}_N,\Lambda_r^{p-2}]$ and $[\widetilde{c}_N,\Lambda_r^{p-2}]$ is a pseudodifferential operator of order $\frac{p-1}{r}-1$, we have

$$I_4\le C_1\|D^\alpha\psi_N^{(\beta)}P^k u\|. \tag{3.40}$$

The above estimate is consequence of the Theorem 18.1.1′ in [38], we remark that since we are assuming that $M>n+2$, then C_1 is independent of N.
Term I_1: since

$$[\widetilde{X}_j,\widetilde{\psi}\Lambda_r^{p-2}]=[\widetilde{X}_j,\widetilde{\psi}]\Lambda_r^{p-2}+\widetilde{\psi}[\widetilde{X}_j,\Lambda_r^{p-2}],$$

where $[\widetilde{X}_j,\Lambda_r^{p-2}]=-[\Lambda_r^{p-2},\widetilde{X}_j]$ (explicitly given in (3.30)), we conclude that

$$I_1\le C_2\sum_{\ell=1}^m\Big(\|X_jD^\alpha\psi_N^{(\beta)}P^k u\|_{\frac{p-2}{r}}+\|X_jD^\alpha\psi_N^{(\beta)}P^k u\|\Big). \tag{3.41}$$

Term I_3: it can be handled as the terms in I_1, we have

$$I_3\le C_3\Big(\|D^\alpha\psi_N^{(\beta)}P^k u\|_{\frac{p-2}{r}}+\|D^\alpha\psi_N^{(\beta)}P^k u\|\Big). \tag{3.42}$$

Term I_2: observing that

$$[X_j,[X_j,\widetilde{\psi}\Lambda_r^{p-2}]]=[X_j,[X_j,\widetilde{\psi}]]\Lambda_r^{p-2}+2[X_j,\widetilde{\psi}][\widetilde{X}_j,\Lambda_r^{p-2}]+\widetilde{\psi}[\widetilde{X}_j,[\widetilde{X}_j,\Lambda_r^{p-2}]],$$

where in particular

$$\begin{aligned}[\widetilde{X}_j,[\widetilde{X}_j,\Lambda_r^{p-2}]]=\sum_{\ell=1}^n\sum_{k=1}^n\Big(&\widetilde{a}_{k,j}[\widetilde{a}_{\ell,j}^{(k)},\Lambda_r^{p-2}]D_\ell+[\widetilde{a}_{k,j},[\widetilde{a}_{\ell,j},\Lambda_r^{p-2}]]D_kD_\ell\\ &+[\widetilde{a}_{k,j},[\widetilde{a}_{\ell,j}^{(k)},\Lambda_r^{p-2}]]D_\ell+\widetilde{a}_{k,j}^{(\ell)}[\widetilde{a}_{\ell,j},\Lambda_r^{p-2}]D_k-[\widetilde{a}_{k,j}^{(\ell)},[\widetilde{a}_{\ell,j},\Lambda_r^{p-2}]]D_k\Big),\end{aligned}$$

with $[\widetilde{a}_{\ell,j}^{(k)},\Lambda_r^{p-2}]$ and $[\widetilde{a}_{k,j},[\widetilde{a}_{\ell,j},\Lambda_r^{p-2}]]$, $[\widetilde{a}_{k,j},[\widetilde{a}_{\ell,j}^{(k)},\Lambda_r^{p-2}]]$ and $[\widetilde{a}_{k,j}^{(\ell)},[\widetilde{a}_{\ell,j},\Lambda_r^{p-2}]]$ pseudodifferential operators of order $\frac{p-2}{r}-1$ and $\frac{p-2}{r}-2$ respectively.
We have

$$\|[X_j,[X_j,\widetilde{\psi}]]\Lambda_r^{p-2}D^\alpha\psi_N^{(\beta)}P^k u\|\le C\|D^\alpha\psi_N^{(\beta)}P^k u\|_{\frac{p-2}{r}},$$

and, by (3.30) and the above considerations, that

$$\begin{aligned}&\|[X_j,\widetilde{\psi}][\widetilde{X}_j,\Lambda_r^{p-2}]D^\alpha\psi_N^{(\beta)}P^k u\|+\|\widetilde{\psi}[\widetilde{X}_j,[\widetilde{X}_j,\Lambda_r^{p-2}]]D^\alpha\psi_N^{(\beta)}P^k u\|\\ &\qquad\le C\Big(\|D^\alpha\psi_N^{(\beta)}P^k u\|_{\frac{p-2}{r}}+\|D^\alpha\psi_N^{(\beta)}P^k u\|\Big).\end{aligned}$$

We conclude that

$$I_2 \leq C_4 \left(\|D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-2}{r}} + \|D^\alpha \psi_N^{(\beta)} P^k u\| \right). \tag{3.43}$$

Summing up, by (3.39), (3.40), (3.41), (3.42) and (3.43) we obtain

$$\begin{aligned} &\|[P_N, \widetilde{\psi}\Lambda_r^{p-2}] D^\alpha \psi_N^{(\beta)} P^k u\| \\ &\leq C \Bigg(\sum_{\ell=1}^m \|X_j D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-2}{r}} + \|X_0 D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-2}{r}} + \|D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-2}{r}} \\ &\qquad + \sum_{\ell=1}^m \|X_j D^\alpha \psi_N^{(\beta)} P^k u\| + \|X_0 D^\alpha \psi_N^{(\beta)} P^k u\| + \|D^\alpha \psi_N^{(\beta)} P^k u\| \Bigg), \end{aligned} \tag{3.44}$$

where C is a new suitable constant independent of N.
Now, we have to estimate the last term on the right hand side of (3.38):

$$\|\widetilde{\psi}\Lambda_r^{p-2}[P_N, D^\alpha \psi_N^{(\beta)}] P^k u\|.$$

This term is similar to that one treated in the previous subsection, in order to make the paper more readable we will cover it in quite detail anyway. Since

$$\begin{aligned} [P_N, D^\alpha \psi_N^{(\beta)}] = \sum_{j=1}^m \Big(2\widetilde{X}_j [\widetilde{X}_j, D^\alpha \psi_N^{(\beta)}] - [\widetilde{X}_j, [\widetilde{X}_j, D^\alpha \psi_N^{(\beta)}]] \Big) \\ + [\widetilde{X}_0, D^\alpha \psi_N^{(\beta)}] + [\widetilde{c}_N, D^\alpha \psi_N^{(\beta)}], \end{aligned}$$

we have

$$\begin{aligned} \|\widetilde{\psi}\Lambda_r^{p-2}[P_N, D^\alpha \psi_N^{(\beta)}] P^k u\| &\leq 2 \sum_{j=1}^m \|\widetilde{\psi}\Lambda_r^{p-2} \widetilde{X}_j [\widetilde{X}_j, D^\alpha \psi_N^{(\beta)}] P^k u\| \\ &+ \sum_{j=1}^m \|\widetilde{\psi}\Lambda_r^{p-2} [\widetilde{X}_j, [\widetilde{X}_j, D^\alpha \psi_N^{(\beta)}]] P^k u\| + \|\widetilde{\psi}\Lambda_r^{p-2} [\widetilde{X}_0, D^\alpha \psi_N^{(\beta)}] P^k u\| \\ &+ \|\widetilde{\psi}\Lambda_r^{p-2} [\widetilde{c}_N, D^\alpha \psi_N^{(\beta)}] P^k u\| = H_{2,1} + H_{2,2} + H_{2,3} + H_{2,4}. \end{aligned} \tag{3.45}$$

We analyze separately the terms on the right hand side.
Term $H_{2,4}$. By (3.5) we have

$$[\widetilde{c}_N, D^\alpha \psi_N^{(\beta)}] = [\widetilde{c}_N, D^\alpha] \psi_N^{(\beta)} = - \sum_{\substack{\nu \leq \alpha, \\ \nu \neq 0}} \binom{\alpha}{\nu} \widetilde{c}_N^{(\nu)} D^{\alpha - \nu} \psi_N^{(\beta)},$$

then

$$\begin{aligned} &\|\widetilde{\psi}\Lambda_r^{p-2}[\widetilde{c}_N, D^\alpha \psi_N^{(\beta)}] P^k u\| \\ &\qquad \leq 2^{\frac{p-2}{4r}} \sum_{\substack{\nu \leq \alpha, \\ \nu \neq 0}} \binom{\alpha}{\nu} \left(\int (1+|\zeta|)^{\frac{p-2}{r}} |\widehat{\widetilde{c}_N^{(\nu)}}(\zeta)| \, d\zeta \right) \|D^{\alpha-\nu} \psi_N^{(\beta)} P^k u\|_{\frac{p-2}{r}}. \end{aligned}$$

Let's proceed to estimate in detail the product of the first two factors in the sum. Later on we will use the same strategy several times, but we will omit the details in order to do not weigh down the discussion. Let q equal to 1 or 0 if n is odd or even. We have

$$\int (1+|\zeta|)^{\frac{p-2}{r}} |\widehat{\widetilde{c}_N^{(\nu)}}(\zeta)|\, d\zeta \le \int \frac{1}{(1+|\zeta|)^{n+1}} (1+|\zeta|)^{n+2+q} |\widehat{\widetilde{c}_N^{(\nu)}}(\zeta)|\, d\zeta$$
$$\le \int \frac{1}{(1+|\zeta|)^{n+1}} 2^{n+2+q} \left(1+|\zeta|^{n+2+q}\right) |\widehat{\widetilde{c}_N^{(\nu)}}(\zeta)|\, d\zeta$$
$$= \int \frac{1}{(1+|\zeta|)^{n+1}} 2^{n+2+q} \left(|\widehat{\widetilde{c}_N^{(\nu)}}(\zeta)| + \sum_{\mu=q'} \frac{q'!}{\mu_1!\cdots\mu_n!} |\widehat{\widetilde{c}_N^{(\nu+2\mu)}}(\zeta)| \right) d\zeta,$$

where $q' = n+2+q$. So

$$\binom{\alpha}{\nu} |\widetilde{c}_N^{(\nu+2\mu)}| \le \binom{\alpha}{\nu} \sum_{\gamma\le\nu+2\mu} \binom{\nu+2\mu}{\gamma} |\varphi_N^{(\nu+2\mu-\gamma)}||c^{(\gamma)}|$$
$$\le 2^{|\nu|+n+2+q}(2\mu!)\frac{\alpha!}{(\alpha-\nu)!}$$
$$\times \sum_{\gamma\le\nu+2\mu} \frac{1}{(\nu+2\mu-\gamma)!} C_1^{|\nu|-|\gamma|+n+q+3} N^{(|\nu|+2|\mu|-|\gamma|-M)^+} C_2^{|\gamma|+1} (\gamma!)^{s-1}.$$

By the Young inequality for products; assuming that $M > 2|\mu|+1 = n+3+q$ and since the number of the multi-indexes γ with $|\gamma| \le |\nu|+2|\mu|$ is smaller or equal to $2^{|\nu|+n+2+q}$, we obtain

$$\binom{\alpha}{\nu} |\widetilde{c}_N^{(\nu+2\mu)}| \le 2^{2(|\nu|+n+2+q)}(2\mu!) C_3^{|\nu|+n+q+3} \frac{\alpha!}{(\alpha-\nu)!} \left(N^{|\nu|-1} + |\nu|^{(s-1)|\nu|} \right),$$

and

$$\binom{\alpha}{\nu} |\widetilde{c}_N^{(\nu)}| \le 2^{2(|\nu|)} C_3^{|\nu|+1} \frac{\alpha!}{(\alpha-\nu)!} \left(N^{|\nu|-1} + |\nu|^{(s-1)|\nu|} \right).$$

Since $\sum_{\mu=q'} \frac{q'!}{\mu_1!\cdots\mu_n!} = n^{n+2+q}$, we have

$$\binom{\alpha}{\nu} | \int (1+|\zeta|)^{\frac{p-2}{r}} |\widehat{\widetilde{c}_N^{(\nu)}}(\zeta)|\, d\zeta \le C_4^{|\nu|+1} \frac{\alpha!}{(\alpha-\nu)!} \left(N^{|\nu|-1} + |\nu|^{(s-1)|\nu|} \right), \tag{3.46}$$

where C_4 is a suitable positive constant independent of ν and N.
We conclude that

$$H_{2,4} = \|\widetilde{\psi}\Lambda_r^{p-2}[\widetilde{c}_N, D^\alpha \psi_N^{(\beta)}] P^k u\| \tag{3.47}$$
$$\le \sum_{\substack{\nu\le\alpha,\\ \nu\neq 0}} C_5^{|\nu|+1} \frac{\alpha!}{(\alpha-\nu)!} \left(N^{|\nu|-1} + |\nu|^{(s-1)|\nu|} \right) \|D^{\alpha-\nu}\psi_N^{(\beta)} P^k u\|_{\frac{p-2}{r}}.$$

Term $H_{2,3}$. By (3.5), we have

$$[\widetilde{X}_0, D^\alpha \psi_N^{(\beta)}] = [\widetilde{X}_0, D^\alpha]\psi_N^{(\beta)} + D^\alpha [\widetilde{X}_0, \psi_N^{(\beta)}]$$
$$= \sum_{\ell=1}^n \sum_{\substack{\nu\le\alpha,\\ \nu\neq 0}} \binom{\alpha}{\nu} \left(\widetilde{a}_{N,\ell,0}^{(\nu)} D^{\alpha-\nu}\psi_N^{(\beta+\ell)} - \widetilde{a}_{N,\ell,0}^{(\nu)} D^{\alpha-\nu+\ell}\psi_N^{(\beta)} \right) + \sum_{\ell=1}^n \widetilde{a}_{N,\ell,0} D^\alpha \psi_N^{(\beta+\ell)},$$

so

$$H_{2,3} \le \sum_{\ell=1}^{n} \|\widetilde{\psi}\Lambda_r^{p-2}\widetilde{a}_{N,\ell,0} D^\alpha \psi_N^{(\beta+\ell)} P^k u\|$$

$$+\sum_{\ell=1}^{n}\sum_{\substack{\nu\le\alpha,\\ \nu\neq 0}}\binom{\alpha}{\nu}\left(\|\widetilde{\psi}\Lambda_r^{p-2}\widetilde{a}^{(\nu)}_{N,\ell,0} D^{\alpha-\nu+\ell}\psi_N^{(\beta)} P^k u\| + \|\widetilde{\psi}\Lambda_r^{p-2}\widetilde{a}^{(\nu)}_{N,\ell,0} D^{\alpha-\nu}\psi_N^{(\beta+\ell)} P^k u\|\right)$$

$$\le 2^{\frac{p-2}{4r}}\sum_{\ell=1}^{n}\left(\int (1+|\zeta|)^{\frac{p-2}{r}}\,|\widehat{\widetilde{a}_{N,\ell,0}}(\zeta)|\,d\zeta\right)\|D^\alpha\psi_N^{(\beta+\ell)}P^k u\|_{\frac{p-2}{r}}$$

$$+2^{\frac{p-2}{4r}}\sum_{\ell=1}^{n}\sum_{\substack{\nu\le\alpha,\\ \nu\neq 0}}\binom{\alpha}{\nu}\left(\int (1+|\zeta|)^{\frac{p-2}{r}}\,|\widehat{\widetilde{a}^{(\nu)}_{N,\ell,0}}(\zeta)|\,d\zeta\right)$$
$$\times\left(\|D^{\alpha-\nu+\ell}\psi_N^{(\beta)}P^k u\|_{\frac{p-2}{r}} + \|D^{\alpha-\nu}\psi_N^{(\beta+\ell)}P^k u\|_{\frac{p-2}{r}}\right).$$

Using the same strategy adopted to obtain (3.46), we conclude that

$$\begin{aligned}(3.48)\quad H_{2,3} &= \|\widetilde{\psi}\Lambda_r^{p-2}[\widetilde{X}_0, D^\alpha\psi_N^{(\beta)}]P^k u\|\\ &\le \sum_{\ell=1}^{n} C_1^{n+3} N^{s(n+2-M)^+}\|D^\alpha\psi_N^{(\beta+\ell)}P^k u\|_{\frac{p-2}{r}}\\ &+\sum_{\ell=1}^{n}\sum_{\substack{\nu\le\alpha,\\ \nu\neq 0}} C_1^{|\nu|+1}\frac{\alpha!}{(\alpha-\nu)!}\left(N^{|\nu|-1}+|\nu|^{(s-1)|\nu|}\right)\\ &\qquad\times\left(\|D^{\alpha-\nu+\ell}\psi_N^{(\beta)}P^k u\|_{\frac{p-2}{r}} + \|D^{\alpha-\nu}\psi_N^{(\beta+\ell)}P^k u\|_{\frac{p-2}{r}}\right).\end{aligned}$$

Term $H_{2,1}$. For every $j=1,\dots,m$, we observe that

$$\begin{aligned}&\widetilde{X}_j[\widetilde{X}_j, D^\alpha\psi_N^{(\beta)}]\\ &\quad=\sum_{\ell=1}^{n}\sum_{\substack{\nu\le\alpha,\\ \nu\neq 0}}\binom{\alpha}{\nu}\left(\widetilde{a}^{(\nu)}_{N,\ell,j}\widetilde{X}_j D^{\alpha-\nu}\psi_N^{(\beta+\ell)} - \widetilde{a}^{(\nu)}_{N,\ell,j}\widetilde{X}_j D^{\alpha-\nu+\ell}\psi_N^{(\beta)}\right)\\ &\quad+\sum_{\ell=1}^{n}\sum_{i=1}^{n}\sum_{\substack{\nu\le\alpha,\\ \nu\neq 0}}\binom{\alpha}{\nu}\left(\widetilde{a}_{N,i,j}\widetilde{a}^{(\nu+i)}_{N,\ell,j} D^{\alpha-\nu}\psi_N^{(\beta+\ell)} - \widetilde{a}_{N,i,j}\widetilde{a}^{(\nu+i)}_{N,\ell,j} D^{\alpha-\nu+\ell}\psi_N^{(\beta)}\right)\\ &\qquad+\sum_{\ell=1}^{n}\widetilde{a}_{N,\ell,j}\widetilde{X}_j D^\alpha\psi_N^{(\beta+\ell)} + \sum_{\ell=1}^{n}\sum_{i=1}^{n}\widetilde{a}_{N,i,j}\widetilde{a}^{(i)}_{N,\ell,j} D^\alpha\psi_N^{(\beta+\ell)}.\end{aligned}$$

So, we have

$$\begin{aligned}\frac{1}{2}H_{2,1} &\le \sum_{j=1}^{m}\sum_{\ell=1}^{n}\sum_{\substack{\nu\le\alpha,\\ \nu\neq 0}}\binom{\alpha}{\nu}\Big(\|\widetilde{\psi}\Lambda_r^{p-2}\widetilde{a}^{(\nu)}_{N,\ell,j}\widetilde{X}_j D^{\alpha-\nu+\ell}\psi_N^{(\beta)}P^k u\|\\ &\qquad+\|\widetilde{\psi}\Lambda_r^{p-2}\widetilde{a}^{(\nu)}_{N,\ell,j}\widetilde{X}_j D^{\alpha-\nu}\psi_N^{(\beta+\ell)}P^k u\|\Big)\\ &+\sum_{j=1}^{m}\sum_{i,\ell=1}^{n}\sum_{\substack{\nu\le\alpha,\\ \nu\neq 0}}\binom{\alpha}{\nu}\Big(\|\widetilde{\psi}\Lambda_r^{p-2}\widetilde{a}_{N,i,j}\widetilde{a}^{(\nu+i)}_{N,\ell,j} D^{\alpha-\nu+\ell}\psi_N^{(\beta)}P^k u\|\\ &\qquad+\|\widetilde{\psi}\Lambda_r^{p-2}\widetilde{a}_{N,i,j}\widetilde{a}^{(\nu+i)}_{N,\ell,j} D^{\alpha-\nu}\psi_N^{(\beta+\ell)}P^k u\|\Big)\end{aligned}$$

$$+\sum_{j=1}^{m}\sum_{\ell=1}^{n}\left(\|\widetilde{\psi}\Lambda_r^{p-2}\widetilde{a}_{N,\ell,j}\widetilde{X}_jD^\alpha\psi_N^{(\beta+\ell)}P^ku\|+\sum_{i=1}^{n}\|\widetilde{\psi}\Lambda_r^{p-2}\widetilde{a}_{N,i,j}\widetilde{a}_{N,\ell,j}^{(i)}D^\alpha\psi_N^{(\beta+\ell)}P^ku\|\right)$$

$$\leq\sum_{j=1}^{m}\sum_{\ell=1}^{n}\sum_{\substack{\nu\leq\alpha,\\ \nu\neq0}}\binom{\alpha}{\nu}\left(\int(1+|\zeta|)^{\frac{p-2}{r}}\,|\widehat{\widetilde{a}_{N,\ell,j}^{(\nu)}}(\zeta)|\,d\zeta\right)\left(\int(1+|\zeta|)^{\frac{p-2}{r}}\,|\widehat{\varphi}_N(\zeta)|\,d\zeta\right)$$
$$\times\left[\|X_jD^{\alpha-\nu+\ell}\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}+\|X_jD^{\alpha-\nu}\psi_N^{(\beta+\ell)}P^ku\|_{\frac{p-2}{r}}\right]$$

$$+\sum_{j=1}^{m}\sum_{i,\ell=1}^{n}\sum_{\substack{\nu\leq\alpha,\\ \nu\neq0}}\binom{\alpha}{\nu}\left(\int(1+|\zeta|)^{\frac{p-2}{r}}\,|\widehat{\widetilde{a}_{N,\ell,j}^{(\nu+i)}}(\zeta)|\,d\zeta\right)\left(\int(1+|\zeta|)^{\frac{p-2}{r}}\,|\widehat{\widetilde{a}_{N,i,j}}(\zeta)|\,d\zeta\right)$$
$$\times\left[\|D^{\alpha-\nu+\ell}\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}+\|D^{\alpha-\nu}\psi_N^{(\beta+\ell)}P^ku\|_{\frac{p-2}{r}}\right]$$

$$+\sum_{j=1}^{m}\sum_{\ell=1}^{n}\left(\int(1+|\zeta|)^{\frac{p-2}{r}}\,|\widehat{\widetilde{a}_{N,\ell,j}}(\zeta)|\,d\zeta\right)\left(\int(1+|\zeta|)^{\frac{p-2}{r}}\,|\widehat{\varphi}_N(\zeta)|\,d\zeta\right)$$
$$\times\|X_jD^\alpha\psi_N^{(\beta+\ell)}P^ku\|_{\frac{p-2}{r}}$$

$$+\sum_{j=1}^{m}\sum_{i,\ell=1}^{n}\left(\int(1+|\zeta|)^{\frac{p-2}{r}}\,|\widehat{\widetilde{a}_{N,\ell,j}^{(i)}}(\zeta)|\,d\zeta\right)\left(\int(1+|\zeta|)^{\frac{p-2}{r}}\,|\widehat{\widetilde{a}_{N,i,j}}(\zeta)|\,d\zeta\right)$$
$$\times\|D^\alpha\psi_N^{(\beta+\ell)}P^ku\|_{\frac{p-2}{r}}.$$

Now, using suitably the same strategy adopted to obtain (3.46), we conclude that

$$H_{2,1}=2\sum_{j=1}^{m}\|\widetilde{\psi}\Lambda_r^{p-2}\widetilde{X}_j[\widetilde{X}_j,D^\alpha\psi_N^{(\beta)}]P^ku\| \tag{3.49}$$
$$\leq 2\sum_{j=1}^{m}\sum_{\ell=1}^{n}\sum_{\substack{\nu\leq\alpha,\\ \nu\neq0}}C_1^{|\nu|+1}\frac{\alpha!}{(\alpha-\nu)!}\left(N^{|\nu|-1}+|\nu|^{(s-1)|\nu|}\right)N^{s(n+3-M)^+}$$
$$\times\left[\|X_jD^{\alpha-\nu+\ell}\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}+\|X_jD^{\alpha-\nu}\psi_N^{(\beta+\ell)}P^ku\|_{\frac{p-2}{r}}\right]$$
$$+2\sum_{\ell=1}^{n}\sum_{\substack{\nu\leq\alpha,\\ \nu\neq0}}C_1^{|\nu|+1}\frac{\alpha!}{(\alpha-\nu)!}\left(N^{|\nu|-1}+|\nu|^{(s-1)|\nu|}\right)N^{s(n+3-M)^+}$$
$$\times\left[\|D^{\alpha-\nu+\ell}\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}+\|D^{\alpha-\nu}\psi_N^{(\beta+\ell)}P^ku\|_{\frac{p-2}{r}}\right]$$
$$+2C_1^{2n+6}N^{s(2n+4-M)^+}\sum_{j=1}^{m}\sum_{\ell=1}^{n}\|X_jD^\alpha\psi_N^{(\beta+\ell)}P^ku\|_{\frac{p-2}{r}}$$
$$+2C_1^{2n+7}N^{s(2n+5-M)^+}\sum_{\ell=1}^{n}\|D^\alpha\psi_N^{(\beta+\ell)}P^ku\|_{\frac{p-2}{r}}.$$

Term $H_{2,2}$. For every $j=1,\dots,m$, we have

$$\begin{aligned}[\widetilde{X}_j,[\widetilde{X}_j,D^\alpha\psi_N^{(\beta)}]] &= [\widetilde{X}_j,[\widetilde{X}_j,D^\alpha]\psi_N^{(\beta)}]+[\widetilde{X}_j,D^\alpha[\widetilde{X}_j,\psi_N^{(\beta)}]]\\ &= [\widetilde{X}_j,[\widetilde{X}_j,D^\alpha]]\psi_N^{(\beta)}+2[\widetilde{X}_j,D^\alpha][\widetilde{X}_j,\psi_N^{(\beta)}]+D^\alpha[\widetilde{X}_j,[\widetilde{X}_j,\psi_N^{(\beta)}]].\end{aligned}$$

We obtain

$$\begin{aligned}H_{2,2} &= \sum_{j=1}^m \|\widetilde{\psi}\Lambda_r^{p-2}[\widetilde{X}_j,[\widetilde{X}_j,D^\alpha\psi_N^{(\beta)}]]P^ku\| \qquad (3.50)\\ &\le \sum_{j=1}^m\Big(\|\widetilde{\psi}\Lambda_r^{p-2}[\widetilde{X}_j,[\widetilde{X}_j,D^\alpha]]\psi_N^{(\beta)}P^ku\|+2\|\widetilde{\psi}\Lambda_r^{p-2}[\widetilde{X}_j,D^\alpha][\widetilde{X}_j,\psi_N^{(\beta)}]P^ku\|\\ &\qquad +\|\widetilde{\psi}D^\alpha[\widetilde{X}_j,[\widetilde{X}_j,\psi_N^{(\beta)}]]\Lambda_r^{p-2}P^ku\|\Big)\\ &= H_{2,2,1}+H_{2,2,2}+H_{2,2,3}.\end{aligned}$$

We proceed to estimate the three terms separately.
Term $H_{2,2,1}$. Using the same steps to obtain (3.16), we have

$$\begin{aligned}[\widetilde{X}_j,[\widetilde{X}_j,D^\alpha]] &= -\sum_{\ell,i=1}^n\sum_{\substack{\nu\le\alpha,\\ \nu\neq0}}\binom{\alpha}{\nu}\widetilde{a}_{N,i,j}\,\widetilde{a}_{N,\ell,j}^{(\nu+i)}D^{\alpha-\nu+\ell}\\ &+\sum_{\ell,i=1}^n\sum_{\substack{\nu\le\alpha,\\ \nu\neq0}}\sum_{\substack{\mu\le\alpha-\nu\\ \mu\neq0}}\binom{\alpha}{\nu}\binom{\alpha-\nu}{\mu}\widetilde{a}_{N,\ell,j}^{(\nu)}\left(\widetilde{a}_{N,i,j}^{(\mu)}D^{\alpha-\nu-\mu+\ell+i}+\widetilde{a}_{N,i,j}^{(\mu+\ell)}D^{\alpha-\nu-\mu+i}\right)\\ &+\sum_{\ell,i=1}^n\sum_{\substack{\nu\le\alpha,\\ \nu\neq0}}\binom{\alpha}{\nu}\widetilde{a}_{N,\ell,j}^{(\nu)}\widetilde{a}_{N,i,j}^{(\ell)}D^{\alpha-\nu+i}.\end{aligned}$$

So, using the strategy adopted to handle the previous terms, in particular the strategy to obtain (3.46), and assuming that M is greater than $n+6$, we have

$$\begin{aligned}H_{2,2,1} &= \sum_{j=1}^m\|\widetilde{\psi}\Lambda_r^{p-2}[\widetilde{X}_j,[\widetilde{X}_j,D^\alpha]]\psi_N^{(\beta)}P^ku\| \qquad (3.51)\\ &\le \sum_{j=1}^m\sum_{\ell,i=1}^n\sum_{\substack{\nu\le\alpha,\\ \nu\neq0}}\binom{\alpha}{\nu}\left(\int(1+|\zeta|)^{\frac{p-2}{r}}\,|\widehat{\widetilde{a}}_{N,i,j}(\zeta)|\,d\zeta\right)\\ &\qquad\times\left(\int(1+|\zeta|)^{\frac{p-2}{r}}\,|\widehat{\widetilde{a}}_{N,\ell,j}^{(\nu+i)}(\zeta)|\,d\zeta\right)\|D^{\alpha-\nu+\ell}\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}\\ &+\sum_{j=1}^m\sum_{\ell,i=1}^n\sum_{\substack{\nu\le\alpha,\\ \nu\neq0}}\binom{\alpha}{\nu}\left(\int(1+|\zeta|)^{\frac{p-2}{r}}\,|\widehat{\widetilde{a}}_{N,\ell,j}^{(\nu)}(\zeta)|\,d\zeta\right)\\ &\qquad\times\left(\int(1+|\zeta|)^{\frac{p-2}{r}}\,|\widehat{\widetilde{a}}_{N,i,j}^{(\ell)}(\zeta)|\,d\zeta\right)\|D^{\alpha-\nu+i}\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}\end{aligned}$$

$$+\sum_{j=1}^{m}\sum_{\ell,i=1}^{n}\sum_{\substack{\nu\le\alpha,\\ \nu\neq 0}}\sum_{\substack{\mu\le\alpha-\nu\\ \mu\neq 0}}\binom{\alpha}{\nu}\binom{\alpha-\nu}{\mu}\Big[\Big(\int(1+|\zeta|)^{\frac{p-2}{r}}\,|\widehat{\widetilde{a}}^{(\nu)}_{N,\ell,j}(\zeta)|\,d\zeta\Big)$$
$$\times\Big(\int(1+|\zeta|)^{\frac{p-2}{r}}\,|\widehat{\widetilde{a}}^{(\mu+\ell)}_{N,i,j}(\zeta)|\,d\zeta\Big)\,\|D^{\alpha-\nu-\mu+i}\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}$$
$$+\Big(\int(1+|\zeta|)^{\frac{p-2}{r}}\,|\widehat{\widetilde{a}}^{(\nu)}_{N,\ell,j}(\zeta)|\,d\zeta\Big)\Big(\int(1+|\zeta|)^{\frac{p-2}{r}}\,|\widehat{\widetilde{a}}^{(\mu)}_{N,i,j}(\zeta)|\,d\zeta\Big)$$
$$\times\|D^{\alpha-\nu-\mu+\ell+i}\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}\Big]$$

$$\le\sum_{\ell=1}^{n}\sum_{\substack{\nu\le\alpha,\\ \nu\neq 0}}C_1^{|\nu|+2}\frac{\alpha!}{(\alpha-\nu)!}\Big(N^{|\nu|-1}+|\nu|^{(s-1)|\nu|}\Big)\,\|D^{\alpha-\nu+\ell}\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}$$
$$+\sum_{i=1}^{n}\sum_{\substack{\nu\le\alpha,\\ \nu\neq 0}}\sum_{\substack{\mu\le\alpha-\nu\\ \mu\neq 0}}C_1^{|\nu|+|\mu|+2}\frac{\alpha!}{(\alpha-\nu-\mu)!}\Big(N^{|\nu|-1}+|\nu|^{(s-1)|\nu|}\Big)$$
$$\times\Big(N^{|\mu|-1}+|\mu|^{(s-1)|\mu|}\Big)\,\|D^{\alpha-\nu-\mu+i}\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}$$
$$+\sum_{\ell,i=1}^{n}\sum_{\substack{\nu\le\alpha,\\ \nu\neq 0}}\sum_{\substack{\mu\le\alpha-\nu\\ \mu\neq 0}}C_1^{|\nu|+|\mu|+1}\frac{\alpha!}{(\alpha-\nu-\mu)!}\Big(N^{|\nu|-1}+|\nu|^{(s-1)|\nu|}\Big)$$
$$\times\Big(N^{|\mu|-1}+|\mu|^{(s-1)|\mu|}\Big)\,\|D^{\alpha-\nu-\mu+i+\ell}\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}.$$

Term $H_{2,2,2}$. By (3.17), for every j, $j=1,\dots,m$, we have

$$2[\widetilde{X}_j,D^\alpha][\widetilde{X}_j,\psi_N^{(\beta)}]$$
$$=-2\sum_{\ell,i=1}^{n}\sum_{\substack{\nu\le\alpha,\\ \nu\neq 0}}\binom{\alpha}{\nu}\widetilde{a}^{(\nu)}_{N,\ell,j}\Big(\widetilde{a}_{N,i,j}D^{\alpha-\nu+\ell}+\widetilde{a}^{(\ell)}_{N,i,j}D^{\alpha-\nu}\Big)\,\psi_N^{(\beta+i)}$$
$$-2\sum_{\ell,i=1}^{n}\sum_{\substack{\nu\le\alpha,\\ \nu\neq 0}}\sum_{\substack{\mu\le\alpha-\nu+\ell\\ \mu\neq 0}}\binom{\alpha}{\nu}\binom{\alpha-\nu}{\mu}\widetilde{a}^{(\nu)}_{N,\ell,j}\Big(\widetilde{a}^{(\mu+\ell)}_{N,i,j}D^{\alpha-\nu-\mu}$$
$$+\widetilde{a}^{(\mu)}_{N,i,j}D^{\alpha-\nu-\mu+\ell}\Big)\,\psi_N^{(\beta+i)}.$$

We conclude that

$$H_{2,2,2}=2\sum_{j=1}^{m}\|\widetilde{\psi}\Lambda_r^{p-2}[\widetilde{X}_j,D^\alpha][\widetilde{X}_j,\psi_N^{(\beta)}]P^ku\| \tag{3.52}$$
$$\le\sum_{i=1}^{n}\sum_{\substack{\nu\le\alpha,\\ \nu\neq 0}}C_1^{|\nu|+2}\frac{\alpha!}{(\alpha-\nu)!}\Big(N^{|\nu|-1}+|\nu|^{(s-1)|\nu|}\Big)\,\|D^{\alpha-\nu}\psi_N^{(\beta+i)}P^ku\|_{\frac{p-2}{r}}$$

$$+\sum_{\ell,i=1}^{n}\sum_{\substack{\nu\leq\alpha,\\ \nu\neq 0}} C_1^{|\nu|+1}\frac{\alpha!}{(\alpha-\nu)!}\left(N^{|\nu|-1}+|\nu|^{(s-1)|\nu|}\right)\|D^{\alpha-\nu+\ell}\psi_N^{(\beta+i)}P^k u\|_{\frac{p-2}{r}}$$

$$+\sum_{i=1}^{n}\sum_{\substack{\nu\leq\alpha,\\ \nu\neq 0}}\sum_{\substack{\mu\leq\alpha-\nu\\ \mu\neq 0}} C_1^{|\nu|+|\mu|+3}\frac{\alpha!}{(\alpha-\nu-\mu)!}\left(N^{|\nu|-1}+|\nu|^{(s-1)|\nu|}\right)$$
$$\times\left(N^{|\mu|-1}+|\mu|^{(s-1)|\mu|}\right)\|D^{\alpha-\nu-\mu}\psi_N^{(\beta+i)}P^k u\|_{\frac{p-2}{r}}$$

$$+\sum_{\ell,i=1}^{n}\sum_{\substack{\nu\leq\alpha,\\ \nu\neq 0}}\sum_{\substack{\mu\leq\alpha-\nu\\ \mu\neq 0}} C_1^{|\nu|+|\mu|+2}\frac{\alpha!}{(\alpha-\nu-\mu)!}\left(N^{|\nu|-1}+|\nu|^{(s-1)|\nu|}\right)$$
$$\times\left(N^{|\mu|-1}+|\mu|^{(s-1)|\mu|}\right)\|D^{\alpha-\nu-\mu+\ell}\psi_N^{(\beta+i)}P^k u\|_{\frac{p-2}{r}}.$$

Term $H_{2,2,3}$. Taking advantage from (3.18), for every j, $j=1,\dots,m$, we have

$$D^\alpha[\widetilde{X}_j,[\widetilde{X}_j,\psi_N^{(\beta)}]]=\sum_{j,i=1}^{n}\widetilde{a}_{N,i,j}\left(\widetilde{a}_{N,\ell,j}^{(i)}D^\alpha\psi_N^{(\beta+\ell)}+\widetilde{a}_{N,\ell,j}D^\alpha\psi_N^{(\beta+\ell+i)}\right)$$
$$+\sum_{\ell,i=1}^{n}\sum_{\substack{\nu\leq\alpha\\ \nu\neq 0}}\binom{\alpha}{\nu}\Big[\left(\widetilde{a}_{N,i,j}\widetilde{a}_{N,\ell,j}^{(\nu+i)}+\widetilde{a}_{N,i,j}^{(\nu)}\widetilde{a}_{N,\ell,j}^{(i)}\right)D^{\alpha-\nu}\psi_N^{(\beta+\ell)}$$
$$+\left(\widetilde{a}_{N,i,j}\widetilde{a}_{N,\ell,j}^{(\nu)}+\widetilde{a}_{N,i,j}^{(\nu)}\widetilde{a}_{N,\ell,j}\right)D^{\alpha-\nu}\psi_N^{(\beta+\ell+i)}\Big]$$
$$+\sum_{\ell,i=1}^{n}\sum_{\substack{\nu<\alpha\\ \nu\neq 0}}\sum_{\substack{\mu\leq\alpha-\nu\\ \mu\neq 0}}\binom{\alpha}{\nu}\binom{\alpha-\nu}{\mu}\widetilde{a}_{N,i,j}^{(\nu)}\Big[\widetilde{a}_{N,\ell,j}^{(\mu+i)}D^{\alpha-\nu-\mu}\psi_N^{(\beta+\ell)}$$
$$+\widetilde{a}_{N,\ell,j}^{(\mu)}D^{\alpha-\nu-\mu}\psi_N^{(\beta+\ell+i)}\Big].$$

From this identity, we obtain

$$H_{2,2,3}=\sum_{j=1}^{m}\|\widetilde{\psi}\Lambda_r^{p-2}D^\alpha[\widetilde{X}_j,[\widetilde{X}_j,\psi_N^{(\beta)}]]P^k u\| \tag{3.53}$$
$$\leq\sum_{\ell=1}^{n}C_1^3\|D^\alpha\psi_N^{(\beta+\ell)}P^k u\|_{\frac{p-2}{r}}+\sum_{j,i=1}^{n}C_1^2\|D^\alpha\psi_N^{(\beta+\ell+i)}P^k u\|_{\frac{p-2}{r}}$$
$$+\sum_{\ell=1}^{n}\sum_{\substack{\nu\leq\alpha\\ \nu\neq 0}}C_1^{|\nu|+3}\frac{\alpha!}{(\alpha-\nu)!}\left(N^{|\nu|-1}+|\nu|^{(s-1)|\nu|}\right)\|D^{\alpha-\nu}\psi_N^{(\beta+\ell)}P^k u\|_{\frac{p-2}{r}}$$
$$+\sum_{\ell,i=1}^{n}\sum_{\substack{\nu\leq\alpha\\ \nu\neq 0}}C_1^{|\nu|+2}\frac{\alpha!}{(\alpha-\nu)!}\left(N^{|\nu|-1}+|\nu|^{(s-1)|\nu|}\right)\|D^{\alpha-\nu}\psi_N^{(\beta+\ell+i)}P^k u\|_{\frac{p-2}{r}}$$
$$+\sum_{\ell=1}^{n}\sum_{\substack{\nu<\alpha\\ \nu\neq 0}}\sum_{\substack{\mu\leq\alpha-\nu\\ \mu\neq 0}}C_1^{|\nu|+|\mu|+3}\frac{\alpha!}{(\alpha-\nu-\mu)!}\left(N^{|\nu|-1}+|\nu|^{(s-1)|\nu|}\right)$$

$$\times \left(N^{|\mu|-1}+|\mu|^{(s-1)|\mu|}\right)\|D^{\alpha-\nu-\mu}\psi_N^{(\beta+\ell)}P^ku\|_{\frac{p-2}{r}}$$

$$+\sum_{\ell,i=1}^{n}\sum_{\substack{\nu<\alpha\\ \nu\neq 0}}\sum_{\substack{\mu\leq\alpha-\nu\\ \mu\neq 0}} C_1^{|\nu|+|\mu|+2}\frac{\alpha!}{(\alpha-\nu-\mu)!}\left(N^{|\nu|-1}+|\nu|^{(s-1)|\nu|}\right)$$

$$\times\left(N^{|\mu|-1}+|\mu|^{(s-1)|\mu|}\right)\|D^{\alpha-\nu-\mu}\psi_N^{(\beta+\ell+i)}P^ku\|_{\frac{p-2}{r}}.$$

By (3.50), (3.51),(3.52) and(3.53), we have

$$H_{2,2}=\sum_{j=1}^{m}\|\widetilde{\psi}\Lambda_r^{p-2}[\widetilde{X}_j,[\widetilde{X}_j,D^\alpha\psi_N^{(\beta)}]]P^ku\| \tag{3.54}$$

$$\leq\sum_{\ell=1}^{n}C_1^3\|D^\alpha\psi_N^{(\beta+\ell)}P^ku\|_{\frac{p-2}{r}}+\sum_{j,i=1}^{n}C_1^2\|D^\alpha\psi_N^{(\beta+\ell+i)}P^ku\|_{\frac{p-2}{r}}$$

$$+\sum_{\ell=1}^{n}\sum_{\substack{\nu\leq\alpha\\ \nu\neq 0}}C_1^{|\nu|+3}\frac{\alpha!}{(\alpha-\nu)!}\left(N^{|\nu|-1}+|\nu|^{(s-1)|\nu|}\right)\left[\|D^{\alpha-\nu+\ell}\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}\right.$$

$$\left.+\|D^{\alpha-\nu}\psi_N^{(\beta+\ell)}P^ku\|_{\frac{p-2}{r}}+\sum_{i=1}^{n}\left(\|D^{\alpha-\nu+\ell}\psi_N^{(\beta+i)}P^ku\|_{\frac{p-2}{r}}+\|D^{\alpha-\nu}\psi_N^{(\beta+\ell+i)}P^ku\|_{\frac{p-2}{r}}\right)\right]$$

$$+\sum_{\ell=1}^{n}\sum_{\substack{\nu<\alpha\\ \nu\neq 0}}\sum_{\substack{\mu\leq\alpha-\nu\\ \mu\neq 0}}C_1^{|\nu|+|\mu|+3}\frac{\alpha!}{(\alpha-\nu-\mu)!}\left(N^{|\nu|-1}+|\nu|^{(s-1)|\nu|}\right)$$

$$\times\left(N^{|\mu|-1}+|\mu|^{(s-1)|\mu|}\right)\left[\|D^{\alpha-\nu-\mu+\ell}\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}+\|D^{\alpha-\nu-\mu}\psi_N^{(\beta+\ell)}P^ku\|_{\frac{p-2}{r}}\right.$$

$$+\sum_{i=1}^{n}\left(\|D^{\alpha-\nu-\mu+\ell+i}\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}+\|D^{\alpha-\nu-\mu+\ell}\psi_N^{(\beta+i)}P^ku\|_{\frac{p-2}{r}}\right.$$

$$\left.\left.\left.+\|D^{\alpha-\nu-\mu}\psi_N^{(\beta+\ell+i)}P^ku\|_{\frac{p-2}{r}}\right)\right].$$

Summing up, by (3.34), (3.35), (3.36), (3.38), (3.44), (3.45), (3.47), (3.48), (3.49) and (3.54) we conclude that there are suitable constants, C_1 and C_2, independent of α, β and N, such that

$$\|D^\alpha\psi_N^{(\beta)}P^ku\|_{\frac{p}{r}}+\sum_{j=1}^{m}\|X_jD^\alpha\psi_N^{(\beta)}P^ku\|_{\frac{p-1}{r}}+\|X_0D^\alpha\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}} \tag{3.55}$$

$$+\sum_{\ell,j=1}^{m}\|X_\ell X_jD^\alpha\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}$$

$$\leq C_2\left\{\|D^\alpha\psi_N^{(\beta)}P^{k+1}u\|_{\frac{p-2}{r}}+\sum_{j=1}^{m}\|X_jD^\alpha\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}\right.$$

$$+\|D^\alpha\psi_N^{(\beta)}P^ku\|_{\frac{p-1}{r}}+\|D^\alpha\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}+\|D^\alpha\psi_N^{(\beta)}P^ku\|$$

$$\left.+\sum_{j=1}^{m}\|X_jD^\alpha\psi_N^{(\beta)}P^ku\|+\|X_0D^\alpha\psi_N^{(\beta)}P^ku\|+\sum_{\ell,j=1}^{m}\|X_\ell X_jD^\alpha\psi_N^{(\beta)}P^ku\|\right.$$

$$
\begin{aligned}
&+ C_1 N^{s(2n+5-M)^+}\left(\sum_{\ell=1}^{n}\|D^\alpha\psi_N^{(\beta+\ell)}P^k u\|_{\frac{p-2}{r}} + \sum_{\ell,i=1}^{n}\|D^\alpha\psi_N^{(\beta+\ell+i)}P^k u\|_{\frac{p-2}{r}}\right.\\
&\qquad\qquad\left. + \sum_{j=1}^{m}\sum_{\ell=1}^{n}\|X_j D^\alpha\psi_N^{(\beta+\ell)}P^k u\|_{\frac{p-2}{r}}\right)\\
&+ \sum_{j=1}^{m}\sum_{\ell=1}^{n}\sum_{\substack{\nu\le\alpha\\ \nu\neq 0}} C_1^{|\nu|+1}\frac{\alpha!}{(\alpha-\nu)!}\left(N^{|\nu|-1}+|\nu|^{(s-1)|\nu|}\right)N^{s(n+3-M)^+}\\
&\qquad\times\left(\|X_j D^{\alpha-\nu+\ell}\psi_N^{(\beta)}P^k u\|_{\frac{p-2}{r}} + \|X_j D^{\alpha-\nu}\psi_N^{(\beta+\ell)}P^k u\|_{\frac{p-2}{r}}\right)\\
&+ \sum_{\substack{\nu\le\alpha\\ \nu\neq 0}} C_1^{|\nu|+1}\frac{\alpha!}{(\alpha-\nu)!}\left(N^{|\nu|-1}+|\nu|^{(s-1)|\nu|}\right)\|D^{\alpha-\nu}\psi_N^{(\beta)}P^k u\|_{\frac{p-2}{r}}\\
&+ \sum_{\ell=1}^{n}\sum_{\substack{\nu\le\alpha\\ \nu\neq 0}} C_1^{|\nu|+1}\frac{\alpha!}{(\alpha-\nu)!}\left(N^{|\nu|-1}+|\nu|^{(s-1)|\nu|}\right)N^{s(n+3-M)^+}\\
&\qquad\times\left(\|D^{\alpha-\nu+\ell}\psi_N^{(\beta)}P^k u\|_{\frac{p-2}{r}} + \|D^{\alpha-\nu}\psi_N^{(\beta+\ell)}P^k u\|_{\frac{p-2}{r}}\right)\\
&+ \sum_{\ell,i=1}^{n}\sum_{\substack{\nu\le\alpha\\ \nu\neq 0}} C_1^{|\nu|+1}\frac{\alpha!}{(\alpha-\nu)!}\left(N^{|\nu|-1}+|\nu|^{(s-1)|\nu|}\right)\\
&\qquad\times\left(\|D^{\alpha-\nu+\ell}\psi_N^{(\beta+i)}P^k u\|_{\frac{p-2}{r}} + \|D^{\alpha-\nu}\psi_N^{(\beta+\ell+i)}P^k u\|_{\frac{p-2}{r}}\right)\\
&+ \sum_{\ell=1}^{n}\sum_{\substack{\nu<\alpha\\ \nu\neq 0}}\sum_{\substack{\mu\le\alpha-\nu\\ \mu\neq 0}} C_1^{|\nu|+|\mu|+3}\frac{\alpha!}{(\alpha-\nu-\mu)!}\left(N^{|\nu|-1}+|\nu|^{(s-1)|\nu|}\right)\\
&\times\left(N^{|\mu|-1}+|\mu|^{(s-1)|\mu|}\right)\left[\|D^{\alpha-\nu-\mu+\ell}\psi_N^{(\beta)}P^k u\|_{\frac{p-2}{r}} + \|D^{\alpha-\nu-\mu}\psi_N^{(\beta+\ell)}P^k u\|_{\frac{p-2}{r}}\right.\\
&\qquad+\sum_{i=1}^{n}\left(\|D^{\alpha-\nu-\mu+\ell+i}\psi_N^{(\beta)}P^k u\|_{\frac{p-2}{r}} + \|D^{\alpha-\nu-\mu+\ell}\psi_N^{(\beta+i)}P^k u\|_{\frac{p-2}{r}}\right.\\
&\qquad\qquad\left.\left.\left.+\|D^{\alpha-\nu-\mu}\psi_N^{(\beta+\ell+i)}P^k u\|_{\frac{p-2}{r}}\right)\right]\Bigg\}.
\end{aligned}
$$

We point out that now we can choose M equal to $2n+7$.

4. Local Gevrey regularity of a Gevrey vector of P

In this section we prove our main theorem concerning the regularity of the Gevrey vectors of $P(x,D)$, (1.1). About that, we begin to prove a couple of results consequence of the estimates established in the previous section.

Proposition 4.1. *Let P be as in* (1.1)*, x_0 in Ω and Ω_0 a neighborhood of x_0 where the Hörmander's condition is satisfied at the step r, Definition 2.2. Let ψ_N be as in the previous section,* (3.1)*, and u a Gevrey-vector of order s for P. There exist constants A_1 and B_1 such that, if:*

$$(4.1)\quad (1)_0 \begin{cases} \|D^\alpha \psi_N^{(\beta)} P^k u\| \le A_1^{|\beta|+2k+1} B_1^{2r|\alpha|+1} N^{s[r|\alpha|+|\beta|+2k]}, \\ \|X_\ell D^\alpha \psi_N^{(\beta)} P^k u\| \le A_1^{|\beta|+2k+1} B_1^{2r(|\alpha|+(2r)^{-1})+1} N^{s[r(|\alpha|+r^{-1})+|\beta|+2k]}, \\ \ell = 1,\dots,m, \ \textit{for } 2r|\alpha| + (|\beta|-M) + 2k \le N. \end{cases}$$

Then, one has for $p=1$

$$(4.2)\quad (1)_1 \begin{cases} \|D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{1}{r}} \le A_1^{|\beta|+2k+1+1} B_1^{2r|\alpha|+1+1} N^{s[r|\alpha|+|\beta|+2k+1]}, \\ \textit{for } 2r|\alpha| + (|\beta|-M) + 2k \le N-4, \end{cases}$$

and for $2 \le p \le r$

$$(4.3)\quad (1)_p \begin{cases} \|D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p}{r}} \le A_1^{|\beta|+2k+p+1} B_1^{2r|\alpha|+p+1} N^{s[r|\alpha|+|\beta|+2k+p]}, \\ \|X_\ell D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-1}{r}} \le A_1^{|\beta|+2k+p+1} B_1^{2r|\alpha|+p+1} N^{s[r|\alpha|+|\beta|+2k+p]}, \\ \|X_0 D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-2}{r}} \le A_1^{|\beta|+2k+p+1} B_1^{2r|\alpha|+p+1} N^{s[r|\alpha|+|\beta|+2k+p]}, \\ \|X_j X_\ell D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-2}{r}} \le A_1^{|\beta|+2k+p+1} B_1^{2r|\alpha|+p+1} N^{s[r|\alpha|+|\beta|+2k+p]}, \\ \ell, j \in \{1,\dots,m\}, \ \textit{for } \ 2r|\alpha| + (|\beta|-M) + 2k \le N-2p. \end{cases}$$

We recall that the constant M is fixed, more precisely $M = 2n+7$.

Proof. The result is obtained by induction on p. The main tools are the estimates (3.20) and (3.55) obtained in the previous section.

Step $p = 1, 2$. We want to show (4.2) and (4.3), case $p=1$ and $p=2$ respectively. We begin from the case $p=2$; the case $p=1$ will be obtained using (3.25).
We recall the estimate (3.20) slightly modified in view of the considerations made at the end of the subsection "Estimate in $H^{1/r}$ and $H^{2/r}$":

$$(4.4)\quad \|D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{2}{r}} + \sum_{j=1}^m \|X_j D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{1}{r}}$$

$$
\begin{aligned}
&+\sum_{\ell,j=1}^{m}\|X_\ell X_j D^\alpha\psi_N^{(\beta)}P^k u\|_0+\|X_0 D^\alpha\psi_N^{(\beta)}P^k u\|_0\\
&\le C_2\Bigg\{\|D^\alpha\psi_N^{(\beta)}P^{k+1}u\|_0+\|D^\alpha\psi_N^{(\beta)}P^k u\|_0+\sum_{\ell=1}^{m}\sum_{j=1}^{n}\|X_\ell D^\alpha\psi_N^{(\beta+j)}P^k u\|_0\\
&\qquad+\sum_{j=1}^{n}\|D^\alpha\psi_N^{(\beta+j)}P^k u\|_0+\sum_{j=1}^{n}\sum_{i=1}^{n}\|D^\alpha\psi_N^{(\beta+j+i)}P^k u\|_0\\
&+\sum_{\ell=1}^{m}\sum_{j=1}^{n}\sum_{\substack{\nu<\alpha,\\ \nu\neq 0}}C_1^{|\nu|+1}\frac{\alpha!\,(\nu!)^{s-1}}{(\alpha-\nu)!}\Big(\|X_\ell D^{\alpha-\nu}\psi_N^{(\beta+j)}P^k u\|_0+\|X_\ell D^{\alpha-\nu+j}\psi_N^{(\beta)}P^k u\|_0\Big)\\
&+\sum_{j=1}^{n}\sum_{\substack{\nu<\alpha,\\ \nu\neq 0}}C_1^{|\nu|+1}\frac{\alpha!\,(\nu!)^{s-1}}{(\alpha-\nu)!}\Big(\|D^{\alpha-\nu}\psi_N^{(\beta+j)}P^k u\|_0+\|D^{\alpha-\nu+j}\psi_N^{(\beta)}P^k u\|_0\Big)\\
&+\sum_{j=1}^{n}\sum_{i=1}^{n}\sum_{\substack{\nu<\alpha,\\ \nu\neq 0}}C_1^{|\nu|+1}\frac{\alpha!\,(\nu!)^{s-1}}{(\alpha-\nu)!}\Big(\|D^{\alpha-\nu}\psi_N^{(\beta+j+i)}P^k u\|_0+\|D^{\alpha-\nu+j}\psi_N^{(\beta+i)}P^k u\|_0\Big)\\
&+\sum_{j=1}^{n}\sum_{\substack{\nu<\alpha,\\ \nu\neq 0}}\sum_{\substack{\mu<\alpha-\nu,\\ \mu\neq 0}}C_1^{|\nu|+|\mu|+2}\frac{\alpha!\,(\nu!)^{s-1}\,(\mu!)^{s-1}}{(\alpha-\nu-\mu)!}\Bigg[\|D^{\alpha-\nu-\mu}\psi_N^{(\beta+j)}P^k u\|_0\\
&\qquad+\|D^{\alpha-\nu-\mu+j}\psi_N^{(\beta)}P^k u\|_0+\sum_{i=1}^{n}\Bigg(\|D^{\alpha-\nu-\mu}\psi_N^{(\beta+j+i)}P^k u\|_0\\
&\qquad+\|D^{\alpha-\nu-\mu+j}\psi_N^{(\beta+i)}P^k u\|_0+\|D^{\alpha-\nu-\mu+j+i}\psi_N^{(\beta)}P^k u\|_0\Bigg)\Bigg]\\
&+C_1^{|\alpha|+1}(\alpha!)^s\sum_{j=1}^{n}\Bigg[\sum_{\ell=1}^{m}\|X_\ell D_j\psi_N^{(\beta)}P^k u\|_0+\left(2^{|\alpha|+n}+1\right)\|D_j\psi_N^{(\beta)}P^k u\|_0\\
&\qquad+\left(2^{|\alpha|+n}+1\right)\sum_{i=1}^{n}\Big(\|D_j\psi_N^{(\beta+i)}P^k u\|_0+\|D_jD_i\psi_N^{(\beta)}P^k u\|_0\Big)\Bigg]\\
&+\sum_{\substack{\nu<\alpha,\\ \nu\neq 0}}C_1^{|\nu|+1}\frac{\alpha!\,(\nu!)^{s-1}}{(\alpha-\nu)!}\|D^{\alpha-\nu}\psi_N^{(\beta)}P^k u\|_0+C_1^{|\alpha|+1}C_2^{|\beta|+2k}N^{s(|\alpha|+|\beta|+2k)}\Bigg\}.
\end{aligned}
$$

We remark that all the $2n+1-$tuples of the form $(\alpha',\beta',k')\in\mathbb{N}^{2n+1}$ associated to each term on the right hand side of (4.4) satisfy the condition (4.1).
In order to make more readable and flowing the writing of the proof, we introduce the following notation: let us denote the powers of A_1, B_1 and N appearing in (4.1), (4.2) and (4.3), respectively by:

$$
\begin{aligned}
&S_p\doteq S_p(|\beta|,k)=|\beta|+2k+p+1;\\
&T_p\doteq T_p(|\alpha|)=2r|\alpha|+p+1;\\
&U_p\doteq U_p(|\alpha|,|\beta|,k)=s\left[r|\alpha|+|\beta|+2k+p\right].
\end{aligned}
\tag{4.5}
$$

We point out that $S_{p+1} = S_p + 1$, $T_{p+1} = T_p + 1$ and $U_{p+1} = U_p + s$. With this notation the assumption (4.1) can be rewritten as

$$(1)_0 \begin{cases} \|D^\alpha \psi_N^{(\beta)} P^k u\| \leq A_1^{S_0} B_1^{T_0} N^{U_0}, \\ \|X_\ell D^\alpha \psi_N^{(\beta)} P^k u\| \leq A_1^{S_0} B_1^{T_1} N^{U_1}, \\ \ell = 1, \ldots, m, \text{ for } T_0 + S_0 - M \leq N + 2. \end{cases} \tag{4.6}$$

So, at this step, $p = 2$, our purpose is to show that for $T_2 + S_2 - M \leq N + 2$ the left hand side of (4.4) is bounded by $A_1^{S_2} B_1^{T_2} N^{U_2}$. This will be gain showing that all the terms on the right hand side of (4.4) are smaller than $A_1^{S_2} B_1^{T_2} N^{U_2}$ times a factor depending on negative power of A_1 or B_1 or N. A suitable choice of A_1 and B_1 will yield the summand of all this factors less than one.
We begin to estimate the first five terms on the right hand side of (4.4):

$$\begin{aligned} C_2 \Bigg(& \|D^\alpha \psi_N^{(\beta)} P^{k+1} u\|_0 + \|D^\alpha \psi_N^{(\beta)} P^k u\|_0 + \sum_{\ell=1}^m \sum_{j=1}^n \|X_\ell D^\alpha \psi_N^{(\beta+j)} P^k u\|_0 \\ & + \sum_{j=1}^n \|D^\alpha \psi_N^{(\beta+j)} P^k u\|_0 + \sum_{j=1}^n \sum_{i=1}^n \|D^\alpha \psi_N^{(\beta+j+i)} P^k u\|_0 \Bigg) \\ \leq C_2 \Big(& A_1^{S_2} B_1^{T_0} N^{U_2} + A_1^{S_0} B_1^{T_0} N^{U_0} + nm A_1^{S_1} B_1^{T_1} N^{U_2} + n A_1^{S_1} B_1^{T_0} N^{U_1} \\ & + n^2 A_1^{S_2} B_1^{T_0} N^{U_2} \Big) \\ \leq A_1^{S_2} B_1^{T_2} N^{U_2} \times C_2 \big(& B_1^{-2} + A_1^{-2} B_1^{-2} N^{-2s} \\ & + nm A_1^{-1} B_1^{-1} + n A_1^{-1} B_1^{-2} N^{-s} + n^2 B_1^{-2} \big). \end{aligned} \tag{4.7}$$

Now, we handle the multiple-sums in ν and μ. We stress that $|\nu|$ and $|\mu|$ are strictly greater than 0. We have

$$\begin{aligned} & C_2 \sum_{\ell=1}^m \sum_{j=1}^n \sum_{\substack{\nu < \alpha, \\ \nu \neq 0}} C_1^{|\nu|+1} \frac{\alpha! \, (\nu!)^{s-1}}{(\alpha - \nu)!} \Big(\|X_\ell D^{\alpha-\nu} \psi_N^{(\beta+j)} P^k u\|_0 \\ & \qquad + \|X_\ell D^{\alpha-\nu+j} \psi_N^{(\beta)} P^k u\|_0 \Big) \\ & \leq nm C_2 C_1 \sum_{\substack{\nu < \alpha, \\ \nu \neq 0}} C_1^{|\nu|} \frac{\alpha! \, (\nu!)^{s-1}}{(\alpha - \nu)!} \Big(A_1^{S_2 - 1} B_1^{T_2 - 1 - 2r|\nu|} N^{U_2 - sr|\nu|} \\ & \qquad + A_1^{S_2 - 2} B_1^{T_2 - 2r(|\nu|-1} N^{U_2 - s[r(|\nu|-1)+2]} \Big). \end{aligned}$$

Since $\frac{\alpha!}{(\alpha-\nu)!} \leq |\alpha|^{|\nu|} \leq N^{|\nu|}$ and $(\nu!)^{s-1} \leq N^{(s-1)|\nu|}$ the right hand side of the above estimate can be bounded as follow

$$\begin{aligned} A_1^{S_2} B_1^{T_2} N^{U_2} \Bigg(& nm C_2 C_1 A_1^{-1} B^{-1} \sum_{p=1}^{|\alpha|-1} \sum_{|\nu|=p} C_1^{|\nu|} B_1^{-2r|\nu|} N^{-s(r-1)|\nu|} \\ & + nm C_2 C_1^2 A_1^{-2} \sum_{p=1}^{|\alpha|-1} \sum_{|\nu|=p} C_1^{|\nu|-1} B_1^{-2r(|\nu|-1)} N^{-s(|\nu|-1)(r-1)-1} \Bigg). \end{aligned}$$

Now, $|\nu| \geq 1$ and $r > 1$, so $-s(r-1)|\nu|$ and $-s(|\nu|-1)(r-1)-1$ are less than zero; moreover the number of multi-indexes ν with $|\nu| = p$ is given by $\binom{p+n-1}{n-1}$, bounded by $2^{|p|+n-1}$. The above quantity can be estimated by

$$A_1^{S_2} B_1^{T_2} N^{U_2} \left(nm 2^{n-1} C_2 C_1 A_1^{-1} B^{-1} \sum_{p=1}^{|\alpha|-1} \left(2C_1 B_1^{-2r}\right)^p \right.$$
$$\left. + nm 2^n C_2 C_1^2 A_1^{-2} \sum_{p=1}^{|\alpha|-1} \left(2C_1 B_1^{-2r}\right)^{p-1} \right).$$

Choosing B_1 such that $2C_1 B_1^{-2r} < 2^{-1}$ we have

$$\sum_{p=1}^{|\alpha|-1} \left(2C_1 B_1^{-2r}\right)^p \leq 1 \quad \text{and} \quad \sum_{p=1}^{|\alpha|-1} \left(2C_1 B_1^{-2r}\right)^{p-1} \leq 2.$$

We obtain

$$C_2 \sum_{\ell=1}^{m} \sum_{j=1}^{n} \sum_{\substack{\nu<\alpha,\\ \nu\neq 0}} C_1^{|\nu|+1} \frac{\alpha! \, (\nu!)^{s-1}}{(\alpha-\nu)!} \left(\|X_\ell D^{\alpha-\nu} \psi_N^{(\beta+j)} P^k u\|_0 \right.$$
$$\left. + \|X_\ell D^{\alpha-\nu+j} \psi_N^{(\beta)} P^k u\|_0 \right) \tag{4.8}$$
$$\leq A_1^{S_2} B_1^{T_2} N^{U_2} \times \left(2^{n-1} nm C_2 C_1\right) A_1^{-1} \left(B^{-1} + 4C_1 A_1^{-1}\right).$$

In the same way we conclude that

$$C_2 \sum_{j=1}^{n} \sum_{\substack{\nu<\alpha,\\ \nu\neq 0}} C_1^{|\nu|+1} \frac{\alpha! \, (\nu!)^{s-1}}{(\alpha-\nu)!} \left(\|D^{\alpha-\nu} \psi_N^{(\beta+j)} P^k u\|_0 \right.$$
$$\left. + \|D^{\alpha-\nu+j} \psi_N^{(\beta)} P^k u\|_0 \right) \tag{4.9}$$
$$\leq A_1^{S_2} B_1^{T_2} N^{U_2} \times \left(2^{n-1} n C_2 C_1\right) \left(N^{-s} + 4C_1\right) A_1^{-2} B^{-2};$$

and

$$C_2 \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{\substack{\nu<\alpha,\\ \nu\neq 0}} C_1^{|\nu|+1} \frac{\alpha! \, (\nu!)^{s-1}}{(\alpha-\nu)!} \left(\|D^{\alpha-\nu} \psi_N^{(\beta+j+i)} P^k u\|_0 \right.$$
$$\left. + \|D^{\alpha-\nu+j} \psi_N^{(\beta+i)} P^k u\|_0 \right) \tag{4.10}$$
$$\leq A_1^{S_2} B_1^{T_2} N^{U_2} \times \left(2^{n-1} n^2 C_2 C_1\right) \left(B_1^{-2} + +4C_1 A_1^{-1}\right).$$

Concerning the other multiple-sums on the right hand side of (4.4) in ν and μ we proceed analogously.
More precisely: since

$$\frac{\alpha!}{(\alpha-\nu-\mu)!} \leq N^{|\nu|+|\mu|},$$

and $(r-1)(|\nu|+|\mu|) > 0$, $(r-1)(|\nu|-1)+(r-1)|\mu| > 0$ and $(r-1)(|\nu|+|\mu|-2) \geq 0$, $r \geq 2$ and $|\nu|$, $|\mu| \geq 1$, we obtain

$$
\begin{aligned}
(4.11)\quad & C_2\sum_{j=1}^{n}\sum_{\substack{\nu<\alpha,\\ \nu\neq 0}}\sum_{\substack{\mu<\alpha-\nu,\\ \mu\neq 0}} C_1^{|\nu|+|\mu|+2}\frac{\alpha!\,(\nu!)^{s-1}\,(\mu!)^{s-1}}{(\alpha-\nu-\mu)!}\Big[\|D^{\alpha-\nu-\mu}\psi_N^{(\beta+j)}P^k u\|_0\\
&\qquad+\|D^{\alpha-\nu-\mu+j}\psi_N^{(\beta)}P^k u\|_0+\sum_{i=1}^{n}\Big(\|D^{\alpha-\nu-\mu}\psi_N^{(\beta+j+i)}P^k u\|_0\\
&\qquad\qquad+\|D^{\alpha-\nu-\mu+j}\psi_N^{(\beta+i)}P^k u\|_0+\|D^{\alpha-\nu-\mu+j+i}\psi_N^{(\beta)}P^k u\|_0\Big)\Big]\\
&\leq A_1^{S_2}B_1^{T_2}N^{U_2}\times\left(2^{2(n-1)}nC_2C_1^2B_1^{-2}\right)\left(n+(2n+1)A_1^{-1}+2(2n+1)A_1^{-2}\right);
\end{aligned}
$$

We point out that, as before, we chosen B_1 large enough in order that $2C_1B_1^{-2r}<2$.

Concerning the last terms on the right hand side of (4.4), we have

$$
\begin{aligned}
(4.12)\quad & C_2\Bigg\{C_1^{|\alpha|+1}\,(\alpha!)^s\sum_{j=1}^{n}\Bigg[\sum_{\ell=1}^{m}\|X_\ell D_j\psi_N^{(\beta)}P^k u\|_0+\left(2^{|\alpha|+n}+1\right)\|D_j\psi_N^{(\beta)}P^k u\|_0\\
&\qquad+\left(2^{|\alpha|+n}+1\right)\sum_{i=1}^{n}\left(\|D_j\psi_N^{(\beta+i)}P^k u\|_0+\|D_jD_i\psi_N^{(\beta)}P^k u\|_0\right)\Bigg]\\
&\quad+\sum_{\substack{\nu<\alpha,\\ \nu\neq 0}}C_1^{|\nu|+1}\frac{\alpha!\,(\nu!)^{s-1}}{(\alpha-\nu)!}\|D^{\alpha-\nu}\psi_N^{(\beta)}P^k u\|_0+C_1^{|\alpha|+1}C_2^{|\beta|+2k}N^{s(|\alpha|+|\beta|+2k)}\Bigg\}.\\
&\leq A_1^{S_2}B_1^{T_2}N^{U_2}\times\Big\{nC_2C_1^2A_1^{-1}B_1^{-1}\Big[mA_1^{-1}\left(C_1B^{-2r}\right)^{|\alpha|-1}N^{-s(|\alpha|-1)(r-1)}\\
&\qquad+2^{n+1}(n+N^{-s})A_1^{-1}B_1^{-1}\left(2C_1B^{-2r}\right)^{|\alpha|-1}N^{-s(|\alpha|-1)(r-1)}\\
&\quad+2^{n+1}nC_1\,(A_1B_1)^{-1}\left(2C_1B^{-2r}\right)^{|\alpha|-2}N^{-s(|\alpha|-2)(r-1)}\Big]+C_2C_1 2^{n-1}\,(A_1B_1)^{-2}\\
&\qquad+C_1B_1^{-3}A_1^{-2}\left(C_1B_1^{-2r}\right)^{|\alpha|}\left(C_2A_1^{-1}\right)^{|\beta|+2k+1}N^{-s[|\alpha|(r-1)+2]}\Big\}.
\end{aligned}
$$

We point out that since $r\geq 2$ and we are assuming $|\alpha|\geq 2$ (as ν is non zero and ν is strictly less than α we get $|\alpha|$ greater or equal to 2), then $(|\alpha|-1)(r-1)$ and $(|\alpha|-2)(r-1)$ are greater or equal to zero.

Summing up, if A_1 and B_1 are chosen large enough, the sum of the second factor on the right hand side of (4.7), (4.8), (4.9), (4.10), (4.11) and (4.12) can be made smaller than one. By (4.4) and above considerations we obtain the desired estimate:

$$
\begin{aligned}
(4.13)\quad & \|D^\alpha\psi_N^{(\beta)}P^k u\|_{\frac{2}{r}}+\sum_{j=1}^{m}\|X_jD^\alpha\psi_N^{(\beta)}P^k u\|_{\frac{1}{r}}\\
&\qquad+\sum_{\ell,j=1}^{m}\|X_\ell X_jD^\alpha\psi_N^{(\beta)}P^k u\|_0+\|X_0D^\alpha\psi_N^{(\beta)}P^k u\|_0\\
&\qquad\qquad\leq A_1^{S_2}B_1^{T_2}N^{U_2}.
\end{aligned}
$$

We then obtained (4.3) in the case $p=2$.

Step $p = 1$. We want to show that $\|v\|_{1/r} \leq A_1^{S_1} B_1^{T_1} N^{U_1}$ for $T_1 + S_1 \leq N$, $v = D^\alpha \psi_N^{(\beta)} P^k u$. We recall the estimate (3.25):

$$\|D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{1}{r}} \leq \left(\|D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{2}{r}}\right)^{1/2} \left(\|D^\alpha \psi_N^{(\beta)} P^k u\|_0\right)^{1/2}.$$

We point out that in this case, since $T_1+S_1 \leq N$, i.e. $2r|\alpha|+(|\beta|-M)+2k \leq N-4$, the $(2n+1)$-tuple (α, β, k) satisfies the assumptions in (4.3), case $p = 2$, and those in (4.1).
By (4.13), we obtain

$$\begin{aligned}
(4.14)\quad \|D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{1}{r}} &\leq \left(\|D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{2}{r}}\right)^{1/2} \left(\|D^\alpha \psi_N^{(\beta)} P^k u\|_0\right)^{1/2} \\
&\leq \left(A_1^{S_2} B_1^{T_2} N^{U_2}\right)^{1/2} \left(A_1^{S_0} B_1^{T_0} N^{U_0}\right)^{1/2} \\
&= \left(A_1^{|\beta|+2k+2+1} B_1^{2r|\alpha|+2+1} N^{s[r|\alpha|+|\beta|+2k+2]}\right)^{\frac{1}{2}} \\
&\qquad \times \left(A_1^{|\beta|+2k+1} B_1^{2r|\alpha|+1} N^{s[r|\alpha|+|\beta|+2k]}\right)^{\frac{1}{2}} \\
&= A_1^{|\beta|+2k+1+1} B_1^{2r|\alpha|+1+1} N^{s[r|\alpha|+|\beta|+2k+1]} = A_1^{S_1} B_1^{T_1} N^{U_1}.
\end{aligned}$$

So, we showed (4.2).

Step $p \geq 3$. Using the notation introduced in (4.5), we assume

$$(4.15)\qquad (1)_{p-1} \begin{cases} \|D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-1}{r}} \leq A_1^{S_{p-1}} B_1^{T_{p-1}} N^{U_{p-1}}, \\ \|X_\ell D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-2}{r}} \leq A_1^{S_{p-1}} B_1^{T_{p-1}} N^{U_{p-1}}, \\ \|X_0 D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-3}{r}} \leq A_1^{S_{p-1}} B_1^{T_{p-1}} N^{U_{p-1}}, \\ \|X_j X_\ell D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-3}{r}} \leq A_1^{S_{p-1}} B_1^{T_{p-1}} N^{U_{p-1}}, \\ \ell, j \in \{1, \dots, m\}, \text{ for } S_{p-1} + T_{p-1} - M \leq N + 2. \end{cases}$$

We want to show, via the estimate (3.55), that $(1)_p$ holds. We recall the estimate (3.55):

$$\begin{aligned}
(4.16)\quad &\|D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p}{r}} + \sum_{j=1}^m \|X_j D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-1}{r}} + \|X_0 D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-2}{r}} \\
&\qquad + \sum_{\ell,j=1}^m \|X_\ell X_j D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-2}{r}} \\
&\leq C_2 \Bigg\{ \|D^\alpha \psi_N^{(\beta)} P^{k+1} u\|_{\frac{p-2}{r}} + \sum_{j=1}^m \|X_j D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-2}{r}} + \|D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-2}{r}} \\
&\qquad + \|D^\alpha \psi_N^{(\beta)} P^k u\| + \sum_{j=1}^m \|X_j D^\alpha \psi_N^{(\beta)} P^k u\| + \|X_0 D^\alpha \psi_N^{(\beta)} P^k u\| \\
&\qquad + \sum_{\ell,j=1}^m \|X_\ell X_j D^\alpha \psi_N^{(\beta)} P^k u\| + \|D^\alpha \psi_N^{(\beta)} P^k u\|_{\frac{p-1}{r}}
\end{aligned}$$

$$
\begin{aligned}
&+C_1\sum_{\ell=1}^{n}\Big(\|D^\alpha\psi_N^{(\beta+\ell)}P^ku\|_{\frac{p-2}{r}}+\sum_{i=1}^{n}\|D^\alpha\psi_N^{(\beta+\ell+i)}P^ku\|_{\frac{p-2}{r}}+\sum_{j=1}^{m}\|X_jD^\alpha\psi_N^{(\beta+\ell)}P^ku\|_{\frac{p-2}{r}}\Big)\\
&\qquad+\sum_{j=1}^{m}\sum_{\ell=1}^{n}\sum_{\substack{\nu\le\alpha\\ \nu\neq0}}C_1^{|\nu|+1}\frac{\alpha!}{(\alpha-\nu)!}\left(N^{|\nu|-1}+|\nu|^{(s-1)|\nu|}\right)\\
&\qquad\qquad\times\left(\|X_jD^{\alpha-\nu+\ell}\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}+\|X_jD^{\alpha-\nu}\psi_N^{(\beta+\ell)}P^ku\|_{\frac{p-2}{r}}\right)\\
&\quad+\sum_{\substack{\nu\le\alpha\\ \nu\neq0}}C_1^{|\nu|+1}\frac{\alpha!}{(\alpha-\nu)!}\left(N^{|\nu|-1}+|\nu|^{(s-1)|\nu|}\right)\Bigg[\|D^{\alpha-\nu}\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}\\
&\qquad+\sum_{\ell=1}^{n}\left(\|D^{\alpha-\nu+\ell}\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}+\|D^{\alpha-\nu}\psi_N^{(\beta+\ell)}P^ku\|_{\frac{p-2}{r}}\right)\\
&\qquad+\sum_{\ell,i=1}^{n}\left(\|D^{\alpha-\nu+\ell}\psi_N^{(\beta+i)}P^ku\|_{\frac{p-2}{r}}+\|D^{\alpha-\nu}\psi_N^{(\beta+\ell+i)}P^ku\|_{\frac{p-2}{r}}\right)\Bigg]\\
&\quad+\sum_{\ell=1}^{n}\sum_{\substack{\nu<\alpha\\ \nu\neq0}}\sum_{\substack{\mu\le\alpha-\nu\\ \mu\neq0}}C_1^{|\nu|+|\mu|+3}\frac{\alpha!}{(\alpha-\nu-\mu)!}\left(N^{|\nu|-1}+|\nu|^{(s-1)|\nu|}\right)\\
&\times\left(N^{|\mu|-1}+|\mu|^{(s-1)|\mu|}\right)\Bigg[\|D^{\alpha-\nu-\mu+\ell}\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}+\|D^{\alpha-\nu-\mu}\psi_N^{(\beta+\ell)}P^ku\|_{\frac{p-2}{r}}\\
&\qquad+\sum_{i=1}^{n}\Big(\|D^{\alpha-\nu-\mu+\ell+i}\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}+\|D^{\alpha-\nu-\mu+\ell}\psi_N^{(\beta+i)}P^ku\|_{\frac{p-2}{r}}\\
&\qquad\qquad+\|D^{\alpha-\nu-\mu}\psi_N^{(\beta+\ell+i)}P^ku\|_{\frac{p-2}{r}}\Big)\Bigg]\Bigg\}.
\end{aligned}
$$

The purpose is to show that, for $T_p+S_p\le N+2$, all the terms on the right hand side of (4.16) are smaller than $A^{S_p}B_1^{T_p}N^{U_p}$ times a factor depending on negative powers of A_1 or B_1. As before, a suitable choice of the constants A_1 and B_1 will yield the sum of all these factors less than one. We point out that the $2n+1$-tuples of the form $(\alpha',\beta',k')\in\mathbb{N}^{2n+1}$ associated to each term on the right hand side of (4.16) satisfy the condition in (4.15). To gain the result we will essentially use the same strategies adopted in the case $p=2$. We begin to estimate the terms in first three lines on the right hand side of (4.16):

$$
\begin{aligned}
(4.17)\quad&\|D^\alpha\psi_N^{(\beta)}P^{k+1}u\|_{\frac{p-2}{r}}+\sum_{j=1}^{n}\|X_jD^\alpha\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}\\
&\qquad+\|D^\alpha\psi_N^{(\beta)}P^ku\|+\sum_{j=1}^{n}\|X_jD^\alpha\psi_N^{(\beta)}P^ku\|+\|X_0D^\alpha\psi_N^{(\beta)}P^ku\|\\
&\quad+\sum_{\ell,j=1}^{m}\|X_\ell X_jD^\alpha\psi_N^{(\beta)}P^ku\|+\|D^\alpha\psi_N^{(\beta)}P^ku\|_{\frac{p-1}{r}}+\|D^\alpha\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}\\
&\qquad\le A_1^{S_p}B_1^{T_p}N^{U_p}\times\left(B_1^{-2}+A_1^{-1}B_1^{-1}N^{-s}\left(1+3A_1^{-1}B_1^{-1}N^{-s}\right)\right.
\end{aligned}
$$

$$+ A_1^{-p+1}B_1^{-p+1}N^{-s(p-1)}\left(1+2B_1^{-1}N^{-s}\right)\Big).$$

Moreover, about the terms in the forth line, we have

$$\begin{aligned}(4.18)\quad C_1\sum_{\ell=1}^{n}\Big(&\|D^\alpha\psi_N^{(\beta+\ell)}P^ku\|_{\frac{p-2}{r}}+\sum_{i=1}^{n}\|D^\alpha\psi_N^{(\beta+\ell+i)}P^ku\|_{\frac{p-2}{r}}\\&+\sum_{j=1}^{m}\|X_jD^\alpha\psi_N^{(\beta+\ell)}P^ku\|_{\frac{p-2}{r}}\Big)\\&\le A_1^{S_p}B_1^{T_p}N^{U_p}\times nC_1B_1^{-2}\left(A_1^{-1}N^{-s}+n+mA_1^{-1}N^{-s}\right)\end{aligned}$$

Now, we analyze the multi-index sums in ν and μ. We adopt the same strategy used to obtain (4.8), we consider

$$\begin{aligned}(4.19)\quad &\sum_{j=1}^{m}\sum_{\ell=1}^{n}\sum_{\substack{\nu\le\alpha\\ \nu\neq0}}C_1^{|\nu|+1}\frac{\alpha!}{(\alpha-\nu)!}\left(N^{|\nu|-1}+|\nu|^{(s-1)|\nu|}\right)\\&\times\left(\|X_jD^{\alpha-\nu+\ell}\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}+\|X_jD^{\alpha-\nu}\psi_N^{(\beta+\ell)}P^ku\|_{\frac{p-2}{r}}\right).\end{aligned}$$

By inductive hypothesis we have

$$\begin{aligned}&\|X_jD^{\alpha-\nu+\ell}\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}\le A_1^{S_p}B_1^{T_p}N^{U_p}\left(A_1^{-2}B_1^{-2-2r(|\nu|-1)}N^{-2s-sr(|\nu|-1)}\right);\\&\|X_jD^{\alpha-\nu}\psi_N^{(\beta+\ell)}P^ku\|_{\frac{p-2}{r}}\le A_1^{S_p}B_1^{T_p}N^{U_p}\left(A_1^{-1}B_1^{-2-2r|\nu|}N^{-s-sr|\nu|}\right).\end{aligned}$$

Since $\frac{\alpha}{(\alpha-\nu)!}\le|\alpha|^{|\nu|}\le N^{|\nu|}$ we have

$$\begin{aligned}&\frac{\alpha!}{(\alpha-\nu)!}\left(N^{|\nu|-1}+|\nu|^{(s-1)|\nu|}\right)N^{-2s-sr(|\nu|-1)}\\&\qquad\le N^{-(sr-2)(|\nu|-1)-2s+1}+N^{-s(r-1)(|\nu|-1)-s}.\end{aligned}$$

Moreover, we observe that since $s\ge1$ and $r\ge2$ then $-(sr-2)(|\nu|-1)-2s+1<0$ and $-s(r-1)(|\nu|-1)-s<0$. The right hand side of the above inequality can be bounded by 2. Modulo $A_1^{S_p}B_1^{T_p}N^{U_p}$, (4.19) can be bounded by

$$\begin{aligned}&2nmC_1A_1^{-1}B_1^{-2}\left(A^{-1}\sum_{\substack{\nu\le\alpha\\ \nu\neq0}}(C_1B_1^{-2r})^{|\nu|-1}+\sum_{\substack{\nu\le\alpha\\ \nu\neq0}}(C_1B_1^{-2r})^{|\nu|}\right)\\&\quad=2nmC_1A^{-1}B_1^{-2}\left(A^{-1}\sum_{p>0}\sum_{|\nu|=p}(C_1B_1^{-2r})^{|p|-1}+\sum_{p>0}\sum_{|\nu|=p}(C_1B_1^{-2r})^{|p|}\right)\\&\quad=2nmC_1A^{-1}B_1^{-2}\left(A_1^{-1}\sum_{p>0}\binom{p+n-1}{n-1}(C_1B_1^{-2r})^{|p|-1}\right.\\&\qquad\left.+\sum_{p>0}\binom{p+n-1}{n-1}(C_1B_1^{-2r})^{|p|}\right)\end{aligned}$$

$$\leq 2^{n-1}nmC_1A_1^{-1}B_1^{-2}\left(2A_1^{-1}\sum_{p\geq 0}(2C_1B_1^{-2r})^{|p|}+\sum_{p>0}(2C_1B_1^{-2r})^{|p|}\right).$$

Choosing B_1 such that $2C_1B_1^{-2r}\leq 2^{-1}$ we conclude that (4.19) is estimated by

$$A_1^{S_p}B_1^{T_p}N^{U_p}\times 2^{n-1}nmC_1A_1^{-1}B_1^{-2}\left(4A_1^{-1}+1\right). \tag{4.20}$$

In order to handle the other terms on the right hand side of (4.16) we adopt essentially the same strategy, in order to not bore the reader we will omit the details.
We have

$$\begin{aligned}
&\sum_{\substack{\nu\leq\alpha\\ \nu\neq 0}} C_1^{|\nu|+1}\frac{\alpha!}{(\alpha-\nu)!}\left(N^{|\nu|-1}+|\nu|^{(s-1)|\nu|}\right) \qquad (4.21)\\
&\times\left[\|D^{\alpha-\nu}\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}+\sum_{\ell=1}^{n}\left(\|D^{\alpha-\nu+\ell}\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}+\|D^{\alpha-\nu}\psi_N^{(\beta+\ell)}P^ku\|_{\frac{p-2}{r}}\right)\right.\\
&\left.\sum_{\ell=1}^{n}\sum_{i=1}^{n}\left(\|D^{\alpha-\nu+\ell}\psi_N^{(\beta+i)}P^ku\|_{\frac{p-2}{r}}+\|D^{\alpha-\nu}\psi_N^{(\beta+\ell+i)}P^ku\|_{\frac{p-2}{r}}\right)\right]\\
&\leq A_1^{S_p}B_1^{T_p}N^{U_p}\\
&\times\sum_{\substack{\nu\leq\alpha\\ \nu\neq 0}} C_1^{|\nu|+1}\left[A_1^{-2}B_1^{-2-2r|\nu|}\left(N^{-(sr-2)|\nu|-1-2s}+N^{-s(r-1)|\nu|-2s}\right)\right.\\
&+nA_1^{-2}B_1^{-2-2r(|\nu|-1)}\left(N^{-(sr-2)(|\nu|-1)-2s+1}+N^{-s(r-1)(|\nu|-1)-s}\right)\\
&+nA_1^{-1}B_1^{-2-2r|\nu|}\left(N^{-(sr-2)|\nu|-s-1}+N^{-s(r-1)|\nu|-s}\right)\\
&+n^2A_1^{-1}B_1^{-2-2r(|\nu|-1)}\left(N^{-(sr-2)(|\nu|-1)-s+1}+N^{-s(r-1)(|\nu|-1)}\right)\\
&\left.+n^2B_1^{-2-2r|\nu|}\left(N^{-(sr-2)|\nu|-1}+N^{-s(r-1)|\nu|}\right)\right].
\end{aligned}$$

As $s\geq 1$, $r\geq 2$ and $|\nu|\geq 1$, all the exponents of N are less or equal to zero. Using the same strategy adopted to handle (4.19), we conclude that the right hand side of (4.21) is bounded by

$$\begin{aligned}
&A_1^{S_p}B_1^{T_p}N^{U_p} \qquad (4.22)\\
&\times 2^nC_1B_1^{-2}\left(A_1^{-2}+2nC_1A_1^{-2}+nA_1^{-1}+2n^2C_1A_1^{-1}+2n^2A_1^{-1}+n\right).
\end{aligned}$$

Concerning the last multi-indexes-sum on the right hand side of (4.16), remarking that $\frac{\alpha!}{(\alpha-\nu-\mu)!}\leq N^{|\nu|+\mu}$, we get

$$\sum_{\ell=1}^{n}\sum_{\substack{\nu<\alpha\\ \nu\neq 0}}\sum_{\substack{\mu\leq\alpha-\nu\\ \mu\neq 0}} C_1^{|\nu|+|\mu|+3}\frac{\alpha!}{(\alpha-\nu-\mu)!}\left(N^{|\nu|-1}+|\nu|^{(s-1)|\nu|}\right) \tag{4.23}$$

$$\times \left(N^{|\mu|-1}+|\mu|^{(s-1)|\mu|}\right)\Bigg[\|D^{\alpha-\nu-\mu+\ell}\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}+\|D^{\alpha-\nu-\mu}\psi_N^{(\beta+\ell)}P^ku\|_{\frac{p-2}{r}}$$
$$+\sum_{i=1}^{n}\Bigg(\|D^{\alpha-\nu-\mu+\ell+i}\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}+\|D^{\alpha-\nu-\mu+\ell}\psi_N^{(\beta+i)}P^ku\|_{\frac{p-2}{r}}$$
$$+\|D^{\alpha-\nu-\mu}\psi_N^{(\beta+\ell+i)}P^ku\|_{\frac{p-2}{r}}\Bigg)\Bigg]$$
$$\leq A_1^{S_p}B_1^{T_p}N^{U_p}\times n\sum_{\substack{\nu<\alpha\\ \nu\neq 0}}\sum_{\substack{\mu\leq\alpha-\nu\\ \mu\neq 0}}C_1^{|\nu|+|\mu|+3}$$
$$\times\Bigg[A_1^{-2}B_1^{-2-2r(|\nu|-1)-2r|\mu|}\left(N^{-(sr-2)|\mu|-1}+N^{-s(r-1)|\mu|}\right)$$
$$\times\left(N^{-(sr-2)(|\nu|-1)-s+1}+N^{-s(r-1)(|\nu|-1)}\right)+A_1^{-1}B_1^{-2-2r|\nu|-2r|\mu|}N^{-s}$$
$$\times\left(N^{-(sr-2)|\mu|-1}+N^{-s(r-1)|\mu|}\right)\left(N^{-(sr-2)|\nu|-1}+N^{-s(r-1)|\nu|}\right)$$
$$+nA_1^{-1}B_1^{-2-2r(|\nu|-1)-2r(|\mu|-1)}\left(N^{-(sr-2)(|\nu|-1)-s+1}+N^{-s(r-1)(|\nu|-1)}\right)$$
$$\times\left(N^{-(sr-2)(|\mu|-1)-s+1}+N^{-s(r-1)(|\mu|-1)}\right)+nA_1^{-1}B_1^{-2-2r(|\nu|-1)-2r|\mu|}$$
$$\times\left(N^{-(sr-2)(|\nu|-1)-s+1}+N^{-s(r-1)(|\nu|-1)}\right)\left(N^{-(sr-2)|\mu|-1}+N^{-s(r-1)|\mu|}\right)$$
$$+nA_1^{-1}B_1^{-2-2r|\nu|-2r|\mu|}N^{-s}\left(N^{-(sr-2)|\mu|-1}+N^{-s(r-1)|\mu|}\right)$$
$$\times\left(N^{-(sr-2)|\nu|-1}+N^{-s(r-1)|\nu|}\right)\Bigg].$$

Since $|\nu|$, $|\mu|$ and s are greater than 1 and r is greater than 2 all the exponents of N are less or equal to zero. Applying the same strategies to handle the previous multi-indexes sum, we obtain that the right hand side of the above inequality is bounded by

$$A_1^{S_p}B_1^{T_p}N^{U_p}\leq 2^{2n+2}n^2B_1^{-2}\times\left[(2nC_1)^{-1}A_1^{-2}+\left(4nC_1^2\right)^{-1}A_1^{-1}+A_1^{-1}+(2C_1)^{-1}A_1^{-1}+(2C_1)^{-2}\right]. \tag{4.24}$$

Summing up, enlarging A_1 and B_1 if necessary, the summand of the second factor of (4.20), (4.22) and (4.24) and the second factor on the right hand side of (4.17) and (4.18) can be made smaller than C_2^{-1}, we conclude

$$\|D^\alpha\psi_N^{(\beta)}P^ku\|_{\frac{p}{r}}+\sum_{j=1}^{m}\|X_jD^\alpha\psi_N^{(\beta)}P^ku\|_{\frac{p-1}{r}}+\|X_0D^\alpha\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}$$
$$+\sum_{\ell,j=1}^{m}\|X_\ell X_jD^\alpha\psi_N^{(\beta)}P^ku\|_{\frac{p-2}{r}}\leq A_1^{S_p}B_1^{T_p}N^{U_p}. \tag{4.25}$$

We obtained (4.3).
This conclude the proof of the Proposition (4.1). □

Proposition 4.2. *There exist positive constants A_1 and B_1 such that* (4.1) *in Proposition 4.1 is true.*

Proof. We use the induction on $|\alpha|$. Case $|\alpha| = 0$. Let K_0 be a compact set contained in Ω and containing all the supports of ψ_N, $\Omega_0 \subset K_0$. Since u is a G^s-vector for P in Ω we have $\|P^k u\|_{L^2(K_0)} \leq C_{K_0}^{2k+1} k^{2sk}$. Then, taking advantage from the Lemma 3.2 we obtain

$$\begin{aligned}\|\psi_N^{(\beta)} P^k u\|_0 &\leq C_\psi^{|\beta|+1} N^{(|\beta|-M)^+} \|P^k u\|_{L^2(K_0)} \\ &\leq C_1^{|\alpha|+1} C_2^{|\beta|+2k+1} N^{s(|\alpha|+|\beta|+2k-M)^+}.\end{aligned}$$

Since M is a fixed constant depending only on n and greater than one, we conclude that there are two positive constants A_1 and B_1, independent of β and k, such that

$$\|\psi_N^{(\beta)} P^k u\| \leq A_1^{|\beta|+2k+1} B_1 N^{s[|\beta|+2k]}. \tag{4.26}$$

In order to gain the second assertion in (4.1), we use the following estimate

$$\sum_{\ell=1}^m \|X_\ell v\|_0^2 \leq \widetilde{C} \left(|\langle Pv, v\rangle| + \|v\|^2 \right), \qquad \forall\, v \in \mathscr{D}(\Omega_0). \tag{4.27}$$

Replacing v by $\psi_N^{(\beta)} P^k u$ we have

$$\sum_{\ell=1}^m \|X_\ell \psi_N^{(\beta)} P^k u\|_0^2 \leq \widetilde{C} \left(|\langle P\psi_N^{(\beta)} P^k u, \psi_N^{(\beta)} P^k u\rangle| + \|\psi_N^{(\beta)} P^k u\|^2 \right).$$

Since

$$P\psi_N^{(\beta)} P^k u \,=\, \psi_N^{(\beta)} P^{k+1} u + \sum_{\ell=1}^m \left(2X_\ell [X_\ell, \psi_N^{(\beta)}] - [X_\ell, [X_\ell, \psi_N^{(\beta)}]] \right) + [X_0, \psi_N^{(\beta)}],$$

where

$$\begin{aligned}[X_\ell, \psi_N^{(\beta)}] &= \sum_{i=1}^n a_{i,\ell} \psi_N^{(\beta+i)}, \qquad \ell = 0, 1, \ldots, m; \\ [X_\ell, [X_\ell, \psi_N^{(\beta)}]] &= \sum_{i=1}^n \sum_{j=1}^n a_{j,\ell} \left(a_{i,\ell}^{(j)} \psi_N^{(\beta+i)} + a_{i,\ell} \psi_N^{(\beta+i+j)} \right), \qquad \ell = 1, \ldots, m;\end{aligned}$$

and keeping in mind that $X_\ell^* = X_\ell - i \sum_{i=1}^n a_{i,\ell}^{(i)}$, we obtain

$$\begin{aligned}\sum_{\ell=1}^m \|X_\ell \psi_N^{(\beta)} P^k u\|_0^2 \leq \widetilde{C}_1 \Bigg[&\|\psi_N^{(\beta)} P^{k+1} u\|_0 \|\psi_N^{(\beta)} P^k u\|_0 + \|\psi_N^{(\beta)} P^k u\|_0^2 \\ &+ \sum_{\ell=0}^m \sum_{i=1}^n \|a_{i,\ell} \psi_N^{(\beta+i)} P^k u\|_0^2 + \sum_{\ell=1}^m \sum_{i=1}^n \|a_{i,\ell}^{(i)} \psi_N^{(\beta)} P^k u\|_0^2 \\ &+ \sum_{\ell=1}^m \sum_{i=1}^n \sum_{p=1}^n \left(\|a_{p,\ell} a_{i,\ell}^{(p)} \psi_N^{(\beta+i)} P^k u\|_0 + \|a_{p,\ell} a_{i,\ell} \psi_N^{(\beta+i+p)} P^k u\|_0 \right) \|\psi_N^{(\beta)} P^k u\|_0 \Bigg].\end{aligned}$$

So, there is a new positive constant $\widetilde{C}_2$ such that

$$\sum_{\ell=1}^{m}\|X_\ell\psi_N^{(\beta)}P^ku\|_0^2\le\widetilde{C}_2\bigg[\|\psi_N^{(\beta)}P^{k+1}u\|_0\|\psi_N^{(\beta)}P^ku\|_0+\|\psi_N^{(\beta)}P^ku\|_0^2$$
$$+\sum_{i=1}^{n}\|\psi_N^{(\beta+i)}P^ku\|_0^2+\sum_{i=1}^{n}\|\psi_N^{(\beta+i)}P^ku\|_0\|\psi_N^{(\beta)}P^ku\|_0$$
$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\|\psi_N^{(\beta+i+j)}P^ku\|_0\|\psi_N^{(\beta)}P^ku\|_0\bigg].$$

By this estimates and the (4.26), for every $\ell\in\{1,\dots,m\}$, we obtain

$$\|X_\ell\psi_N^{(\beta)}P^ku\|_0\le A_1^{|\beta|+2k+1}B_1^2N^{s(1+|\beta|+2k)}$$
$$\times\widetilde{C}_2^{\frac{1}{2}}B_1^{-1}\bigg(A_1+N^{-s}+\sqrt{n}A_1+\sqrt{n}A_1^{\frac{1}{2}}N^{-\frac{s}{2}}+nA_1\bigg).$$

Taking B_1 large enough compared to A_1, we have

$$\widetilde{C}_2^{\frac{1}{2}}B_1^{-1}\bigg(A_1+N^{-s}+\sqrt{n}A_1+\sqrt{n}A_1^{\frac{1}{2}}N^{-\frac{s}{2}}+nA_1\bigg)\le 1.$$

We conclude that

$$\|X_\ell\psi_N^{(\beta)}P^ku\|_0\le A_1^{|\beta|+2k+1}B_1^2N^{s(1+|\beta|+2k)},\qquad \ell=1,\dots,m,$$

i.e. the second assertion in (4.1).

Let $m_0\in\mathbb{N}$, we assume that

$$\begin{cases}\|D^\alpha\psi_N^{(\beta)}P^ku\|\le A_1^{|\beta|+2k+1}B_1^{2r|\alpha|+1}N^{s[r|\alpha|+|\beta|+2k]},\\ \|X_\ell D^\alpha\psi_N^{(\beta)}P^ku\|\le A_1^{|\beta|+2k+1}B_1^{2r(|\alpha|+(2r)^{-1})+1}N^{s[r(|\alpha|+r^{-1})+|\beta|+2k]},\ \ell=1,\dots,m,\end{cases}$$

for every α such that $|\alpha|\le m_0$ and $2r|\alpha|+(|\beta|-M)+2k\le N$. Our goal is to show that the above inequalities are true for α such that $|\alpha|=m_0+1$.
By the Proposition (4.1) we know that

$$\|D^\alpha\psi_N^{(\beta)}P^ku\|_{\frac{p}{r}}\le A_1^{|\beta|+2k+p+1}B_1^{2r|\alpha|+p+1}N^{s[r|\alpha|+|\beta|+2k+p]},$$

holds for $2\le p\le r$, $|\alpha|\le m_0$ and $2r|\alpha|+(|\beta|-M)+2k\le N-2p$. In particular it holds when $p=r$:

$$\|D^\alpha\psi_N^{(\beta)}P^ku\|_1\le A_1^{|\beta|+2k+r+1}B_1^{2rm_0+r+1}N^{s[rm_0+|\beta|+2k+r]}.$$

Let $(\alpha,\beta,k)\in\mathbb{N}^{2n+1}$, with $|\alpha|=m_0+1$, such that $2r|\alpha|+(|\beta|-M)+2k\le N$, then $2r(m_0+1)+(|\beta|-M)+2k\le N$, i.e. $2rm_0+(|\beta|-M)+2k\le N-2r$. So $\alpha=\alpha_0+e_j$, $e_j=(0,\dots,0,1,0,\dots,0)$, with $|\alpha_0|=m_0$. Since

$$2rm_0+(|\beta|-M)+2k=2r(m_0+1)+(|\beta|-M)+2k\le N-2r,$$

by inductive hypothesis, (4.1), in Proposition 4.1 is true for (α_0,β,k), so by Proposition 4.1, (4.3) holds for $p=r$ and (α_0,β,k), we conclude that

$$\|D^\alpha\psi_N^{(\beta)}P^ku\|_0=\|D_jD^{\alpha_0}\psi_N^{(\beta)}P^ku\|_0 \tag{4.28}$$
$$\le\|D^{\alpha_0}\psi_N^{(\beta)}P^ku\|_1\le A_1^{|\beta|+2k+r+1}B_1^{2rm_0+r+1}N^{s[rm_0+|\beta|+2k+r]}$$
$$=A_1^{|\beta|+2k+1}B_1^{2r|\alpha|+1}N^{s[r|\alpha|+|\beta|+2k]}\times(A_1B_1^{-1})^r.$$

As before, assuming B_1 large compared to A_1 the factor $A_1B_1^{-1}$ is less than one. We have obtained what we want.
Now, we have to show the second assertion, i.e. that

$$\|X_\ell D^\alpha \psi_N^{(\beta)} P^k u\| \leq A_1^{|\beta|+2k+1} B_1^{2r(|\alpha|+(2r)^{-1})+1} N^{s[r(|\alpha|+r^{-1})+|\beta|+2k]},$$

is true for $|\alpha| = m_0 + 1$.
By (4.27), replacing v by $D^\alpha \psi_N^{(\beta)} P^k u$, we obtain the following estimate

$$\begin{aligned}
(4.29)\quad \sum_{\ell=1}^m \|X_\ell D^\alpha \psi_N^{(\beta)} P^k u\|_0^2 &\leq \widetilde{C}\Big[\|D^\alpha \psi_N^{(\beta)} P^{k+1} u\|_0 \|D^\alpha \psi_N^{(\beta)} P^k u\|_0 \\
&\qquad + \|D^\alpha \psi_N^{(\beta)} P^k u\|_0^2 + |\langle [P, D^\alpha \psi_N^{(\beta)}] P^k u, D^\alpha \psi_N^{(\beta)} P^k u\rangle|\Big] \\
&\leq \widetilde{C}\Big[\|D^\alpha \psi_N^{(\beta)} P^{k+1} u\|_0 \|D^\alpha \psi_N^{(\beta)} P^k u\|_0 + \|D^\alpha \psi_N^{(\beta)} P^k u\|_0^2 \\
&\quad + \sum_{\ell=1}^m \Big(2|\langle X_\ell [X_\ell, D^\alpha \psi_N^{(\beta)}] P^k u, D^\alpha \psi_N^{(\beta)} P^k u\rangle| \\
&\qquad\qquad + |\langle [X_\ell, [X_\ell, D^\alpha \psi_N^{(\beta)}]] P^k u, D^\alpha \psi_N^{(\beta)} P^k u\rangle|\Big) \\
&\quad + |\langle [X_0, D^\alpha \psi_N^{(\beta)}] P^k u, D^\alpha \psi_N^{(\beta)} P^k u\rangle| + |\langle [c, D^\alpha] \psi_N^{(\beta)} P^k u, D^\alpha \psi_N^{(\beta)} P^k u\rangle|\Big].
\end{aligned}$$

Now, since $X_\ell^* = X_\ell + f_\ell$, $f_\ell = \sum_{j=1}^n a_{j,\ell}^{(j)}$, and

$$\begin{aligned}
\sum_{\ell=1}^m 2|\langle X_\ell [X_\ell, D^\alpha \psi_N^{(\beta)}] P^k u, D^\alpha \psi_N^{(\beta)} P^k u\rangle| &\leq \frac{1}{2\widetilde{C}} \sum_{\ell=1}^m \|X_\ell D^\alpha \psi_N^{(\beta)} P^k u\|_0^2 \\
&\quad + 4\widetilde{C} \sum_{\ell=1}^m \|[X_\ell, D^\alpha \psi_N^{(\beta)}] P^k u\|_0^2 + 2\sum_{\ell=1}^m |\langle [X_\ell, D^\alpha \psi_N^{(\beta)}] P^k u, f_\ell D^\alpha \psi_N^{(\beta)} P^k u\rangle|,
\end{aligned}$$

then (4.29) can be rewritten as follow

$$\begin{aligned}
(4.30)\quad \sum_{\ell=1}^m \|X_\ell D^\alpha \psi_N^{(\beta)} P^k u\|_0^2 &\leq 2\widetilde{C}\Big[\|D^\alpha \psi_N^{(\beta)} P^{k+1} u\|_0 \|D^\alpha \psi_N^{(\beta)} P^k u\|_0 \\
&\qquad + \|D^\alpha \psi_N^{(\beta)} P^k u\|_0^2 + 2\sum_{\ell=1}^m |\langle [X_\ell, D^\alpha \psi_N^{(\beta)}] P^k u, f_\ell D^\alpha \psi_N^{(\beta)} P^k u\rangle| \\
&\quad + 4\widetilde{C} \sum_{\ell=1}^m \|[X_\ell, D^\alpha \psi_N^{(\beta)}] P^k u\|_0^2 + \sum_{\ell=1}^m |\langle [X_\ell, [X_\ell, D^\alpha \psi_N^{(\beta)}]] P^k u, D^\alpha \psi_N^{(\beta)} P^k u\rangle| \\
&\qquad + |\langle [X_0, D^\alpha \psi_N^{(\beta)}] P^k u, D^\alpha \psi_N^{(\beta)} P^k u\rangle| + |\langle [c, D^\alpha] \psi_N^{(\beta)} P^k u, D^\alpha \psi_N^{(\beta)} P^k u\rangle|\Big].
\end{aligned}$$

We analyze separately the term on the right hand side.
We begin to remark that by (4.28) ,we know that

$$(4.31)\quad \|D^\alpha \psi_N^{(\beta)} P^k u\|_0^2$$

$$\leq \left(A_1^{|\beta|+2k+1} B_1^{2r(|\alpha|+(2r)^{-1})+1} N^{s[r(|\alpha|+r^{-1})+|\beta|+2k]} \right)^2 \times B_1^{-2} N^{-2s}.$$

Concerning the first term on the right hand side we have:

$$\|D^\alpha \psi_N^{(\beta)} P^{k+1} u\|_0 \|D^\alpha \psi_N^{(\beta)} P^k u\|_0 \leq \|D^{\alpha_0} \psi_N^{(\beta)} P^{k+1} u\|_1 \|D^\alpha \psi_N^{(\beta)} P^k u\|_0,$$

now we remark that

$$2r|\alpha_0| + |\beta| + 2k + 2 = 2r(|\alpha| - 1) + |\beta| + 2k + 2 \leq 2r|\alpha| + |\beta| + 2k;$$

$2(1-r)$ is negative as $r \geq 2$. Since $|\alpha_0| = m_0$, $(\alpha_0, \beta, k+1)$ satisfies the condition in (4.1), then by Proposition 4.1, (4.3) is true for $p = r$ for the $2n+1$-tuples $(\alpha_0, \beta, k+1)$; we have

$$\|D^{\alpha_0} \psi_N^{(\beta)} P^{k+1} u\|_1 \leq A_1^{|\beta|+2k+2+r+1} B_1^{2r|\alpha_0|+r+1} N^{s[r|\alpha_0|+|\beta|+2k+2+r]}.$$

By (4.31) and the above inequality we conclude that

$$\begin{aligned} (4.32) \quad & \|D^\alpha \psi_N^{(\beta)} P^{k+1} u\|_0 \|D^\alpha \psi_N^{(\beta)} P^k u\|_0 \\ & \leq \left(A_1^{|\beta|+2k+1} B_1^{2r(|\alpha|+(2r)^{-1})+1} N^{s[r(|\alpha|+r^{-1})+|\beta|+2k]} \right)^2 \times A_1^{r+2} B_1^{-r-2}. \end{aligned}$$

We focus on the third term on the right hand side of (4.30). By (3.5), (3.11), (3.12) and (3.13) we have

$$\begin{aligned} (4.33) \quad & \sum_{\ell=1}^m |\langle [X_\ell, D^\alpha \psi_N^{(\beta)}] P^k u, f_\ell D^\alpha \psi_N^{(\beta)} P^k u \rangle| \\ & \leq \sum_{\ell=1}^m \sum_{j=1}^n C_1^3 \|D^{\alpha_0} \psi_N^{(\beta+j)} P^k u\|_1 \|D^\alpha \psi_N^{(\beta)} P^k u\|_0 \\ & + \sum_{\ell=1}^m \sum_{j=1}^n \sum_{\substack{\nu \leq \alpha \\ \nu \neq 0}} C_1^{|\nu|+3} \frac{\alpha!}{(\alpha-\nu)!} (\nu!)^s \|D^{\alpha-\nu+j} \psi_N^{(\beta)} P^k u\|_0 \|D^\alpha \psi_N^{(\beta)} P^k u\|_0 \\ & + \sum_{\ell=1}^m \sum_{j=1}^n \sum_{\substack{\nu \leq \alpha \\ \nu \neq 0}} C_1^{|\nu|+3} \frac{\alpha!}{(\alpha-\nu)!} (\nu!)^s \|D^{\alpha-\nu} \psi_N^{(\beta+j)} P^k u\|_0 \|D^\alpha \psi_N^{(\beta)} P^k u\|_0. \end{aligned}$$

In order to handle the second factor in the first sum we remark that

$$2r|\alpha_0| + |\beta| + 2k + 1 = 2r(|\alpha| - 1) + |\beta| + 2k + 1 \leq 2r|\alpha| + |\beta| + 2k;$$

$1 - 2r$ is negative, $r \geq 2$. Since $|\alpha_0| = m_0$ and $(\alpha_0, \beta + j, k)$, satisfies the condition in (4.1), then by Proposition 4.1, (4.3) is true for $p = r$ for the $2n+1$-tuples $(\alpha_0, \beta + j, k)$; we have

$$\|D^{\alpha_0} \psi_N^{(\beta+j)} P^k u\|_1 \leq A_1^{|\beta|+1+2k+r+1} B_1^{2r|\alpha_0|+r+1} N^{s[r|\alpha_0|+|\beta|+1+2k+r]}.$$

By (4.31) and the above inequality, we obtain

$$\begin{aligned} & \|D^{\alpha_0} \psi_N^{(\beta+j)} P^k u\|_1 \|D^\alpha \psi_N^{(\beta)} P^k u\|_0 \\ & \leq \left(A_1^{|\beta|+2k+1} B_1^{2r(|\alpha|+(2r)^{-1})+1} N^{s[r(|\alpha|+r^{-1})+|\beta|+2k]} \right)^2 \times A_1^{r+1} B_1^{-r-2} N^{-s}. \end{aligned}$$

In order to handle the terms in the second multiple-sum we have to distinguish two cases, $|\nu| = 1$ and $|\nu| > 1$; we remark that in the second case, we have that $|\alpha - \nu + j| \leq m_0$, so we can use directly the inductive hypothesis. Concerning the case $|\nu| = 1$ we have

$$\|D^{\alpha-\nu+j}\psi_N^{(\beta)}P^k u\|_0 \leq \|D^{\alpha_0-\nu+j}\psi_N^{(\beta)}P^k u\|_1.$$

Now

$$2r(|\alpha_0| - |\nu| + 1) + |\beta| + 2k = 2r(|\alpha| - |\nu|) + |\beta| + 2k < 2r|\alpha| + |\beta| + 2k;$$

$|\nu| = 1$. Since $|\alpha_0| = m_0$ and $(\alpha_0 - \nu + j, \beta, k)$, satisfies the condition in (4.1), then by Proposition 4.1, (4.3) is true for $p = r$ for the $2n+1$-tuples $(\alpha_0, \beta + j, k)$. We have

$$\|D^{\alpha_0-\nu+j}\psi_N^{(\beta)}P^k u\|_1 \leq A_1^{|\beta|+2k+r+1}B_1^{2r|\alpha_0|-2r|\nu|+2r+r+1}N^{s[r|\alpha_0|-r|\nu|+r+|\beta|+2k+r]}.$$

Recalling that $|\nu| = 1$, by (4.31) and the above inequality, we obtain

$$\begin{aligned}\|D^{\alpha_0-\nu+j}\psi_N^{(\beta)}P^k u\|_1&\|D^{\alpha}\psi_N^{(\beta)}P^k u\|_0\\ &\leq \left(A_1^{|\beta|+2k+1}B_1^{2r(|\alpha|+(2r)^{-1})+1}N^{s[r(|\alpha|+r^{-1})+|\beta|+2k]}\right)^2 \times A_1^r B_1^{-r-2}N^{-2s}.\end{aligned}$$

As $\frac{\alpha!}{(\alpha-\nu)!} \leq |\alpha|^{|\nu|} \leq N^{|\nu|}$, we have

$$\begin{aligned}\sum_{\ell=1}^{m}\sum_{j=1}^{n}\sum_{\substack{\nu\leq\alpha\\ \nu\neq 0}} C_1^{|\nu|+3}\frac{\alpha!}{(\alpha-\nu)!}(\nu!)^s&\|D^{\alpha-\nu+j}\psi_N^{(\beta)}P^k u\|_0\|D^{\alpha}\psi_N^{(\beta)}P^k u\|_0\\ &\leq \sum_{\ell=1}^{m}\sum_{j=1}^{n}\sum_{|\nu|=1} C_1^4 N\|D^{\alpha_0-\nu+j}\psi_N^{(\beta)}P^k u\|_1\|D^{\alpha}\psi_N^{(\beta)}P^k u\|_0\\ &+\sum_{\ell=1}^{m}\sum_{j=1}^{n}\sum_{\substack{\nu\leq\alpha\\ |\nu|\geq 2}} C_1^{|\nu|+3}N^{s|\nu|}\|D^{\alpha-\nu+j}\psi_N^{(\beta)}P^k u\|_0\|D^{\alpha}\psi_N^{(\beta)}P^k u\|_0\\ &\leq \left(A_1^{|\beta|+2k+1}B_1^{2r(|\alpha|+(2r)^{-1})+1}N^{s[r(|\alpha|+r^{-1})+|\beta|+2k]}\right)^2\\ \times nmC_1^4&\left[A_1^r B_1^{-r-2}N^{-2s} + B_1^{-1}N^{-s}\sum_{\substack{\nu\leq\alpha\\ |\nu|\geq 2}}\left(C_1B_1^{-2r}\right)^{|\nu|-1}N^{-s(r-1)(|\nu|-1)}\right].\end{aligned}$$

Since we chosen B_1 large enough in order that $2C_1B_1^{-2r} < 2$ and $r \geq 2$, we conclude that

$$\begin{aligned}\sum_{\ell=1}^{m}\sum_{j=1}^{n}\sum_{\substack{\nu\leq\alpha\\ \nu\neq 0}} C_1^{|\nu|+3}\frac{\alpha!}{(\alpha-\nu)!}(\nu!)^s&\|D^{\alpha-\nu+j}\psi_N^{(\beta)}P^k u\|_0\|D^{\alpha}\psi_N^{(\beta)}P^k u\|_0\\ &\leq \left(A_1^{|\beta|+2k+1}B_1^{2r(|\alpha|+(2r)^{-1})+1}N^{s[r(|\alpha|+r^{-1})+|\beta|+2k]}\right)^2\\ &\times nmC_1^4\left[A_1^r B_1^{-r-2}N^{-2s} + 2^{n-1}B_1^{-1}N^{-s}\right].\end{aligned}$$

Concerning the last multiple-sum on the right hand side of (4.33), we observe that since $|\nu| \geq 1$ we can use directly the inductive hypothesis. Using the same strategy adopted above, we obtain

$$\sum_{\ell=1}^{m}\sum_{j=1}^{n}\sum_{\substack{\nu\leq\alpha\\ \nu\neq 0}} C_1^{|\nu|+3}\frac{\alpha!}{(\alpha-\nu)!}(\nu!)^s\|D^{\alpha-\nu}\psi_N^{(\beta+j)}P^k u\|_0\|D^\alpha\psi_N^{(\beta)}P^k u\|_0$$
$$\leq \left(A_1^{|\beta|+2k+1}B_1^{2r(|\alpha|+(2r)^{-1})+1}N^{s[r(|\alpha|+r^{-1})+|\beta|+2k]}\right)^2 \times 2^{n-1}nmC_1^3A_1B_1^{-2}N^{-s}.$$

Summing up, by the above consideration, we conclude that

$$\sum_{\ell=1}^{m}|\langle[X_\ell, D^\alpha\psi_N^{(\beta)}]P^k u, f_\ell D^\alpha\psi_N^{(\beta)}P^k u\rangle| \tag{4.34}$$
$$\leq \left(A_1^{|\beta|+2k+1}B_1^{2r(|\alpha|+(2r)^{-1})+1}N^{s[r(|\alpha|+r^{-1})+|\beta|+2k]}\right)^2$$
$$\times\left[A_1^rB_1^{-r-2}N^{-2s}+nm2^{n-1}C_1^4B_1^{-1}N^{-s}\left(A_1^rB_1^{-r-1}2^{-n+1}N^{-s}+1+A_1B_1^{-1}\right)\right].$$

Concerning the second to last term on the right hand side of (4.30): using the same strategy adopted to handle the previous term, we have

$$|\langle[X_0, D^\alpha\psi_N^{(\beta)}]P^k u, D^\alpha\psi_N^{(\beta)}P^k u\rangle| \tag{4.35}$$
$$\leq \left(A_1^{|\beta|+2k+1}B_1^{2r(|\alpha|+(2r)^{-1})+1}N^{s[r(|\alpha|+r^{-1})+|\beta|+2k]}\right)^2$$
$$\times\left[A_1^rB_1^{-r-2}N^{-2s}+nm2^{n-1}C_1^2B_1^{-1}N^{-s}\left(A_1^rB_1^{-r-1}2^{-n+1}N^{-s}+1+A_1B_1^{-1}\right)\right].$$

We handle now the last term on the right hand side of (4.30). By (3.6) we have

$$|\langle[c, D^\alpha]\psi_N^{(\beta)}P^k u, D^\alpha\psi_N^{(\beta)}P^k u\rangle|$$
$$\leq \sum_{\substack{\nu\leq\alpha,\\ \nu\neq 0}} C_1^{|\nu|+1}\frac{\alpha!}{(\alpha-\nu)!}(\nu!)^{s-1}\|D^{\alpha-\nu}\psi_N^{(\beta)}P^k u\|_0\|D^\alpha\psi_N^{(\beta)}P^k u\|_0.$$

We observe that since $|\nu| \geq 1$, $|\alpha-\nu| \leq m_0$. We can use directly the inductive hypothesis. Arguing as before to handle the other multi-index sums, we obtain

$$|\langle[c, D^\alpha]\psi_N^{(\beta)}P^k u, D^\alpha\psi_N^{(\beta)}P^k u\rangle| \tag{4.36}$$
$$\leq \left(A_1^{|\beta|+2k+1}B_1^{2r(|\alpha|+(2r)^{-1})+1}N^{s[r(|\alpha|+r^{-1})+|\beta|+2k]}\right)^2 \times\left(2^{n-1}C_1B_1^{-2}N^{-2s}\right).$$

There remains one last term to handle, the third to last term on the right hand side of (4.30). By (3.13) and the identities (3.15), (3.16), (3.17) and (3.18), we have

$$
\begin{aligned}
(4.37)\quad & \sum_{\ell=1}^{m} |\langle [X_\ell,[X_\ell, D^\alpha \psi_N^{(\beta)}]]P^k u, D^\alpha \psi_N^{(\beta)} P^k u\rangle| \\
&\le \Big\{ 4n \sum_{j=1}^{n} \sum_{\substack{\nu\le\alpha,\\ \nu\neq 0}} C_1^{|\nu|+3} \frac{\alpha!\,(\nu!)^{s-1}}{(\alpha-\nu)!} \Big(\|D^{\alpha-\nu}\psi_N^{(\beta+j)} P^k u\|_0 + \|D^{\alpha-\nu+j}\psi_N^{(\beta)} P^k u\|_0 \Big) \\
&\quad + 2\sum_{i,j=1}^{n} \sum_{\substack{\nu\le\alpha,\\ \nu\neq 0}} C_1^{|\nu|+2} \frac{\alpha!\,(\nu!)^{s-1}}{(\alpha-\nu)!} \Big(\|D^{\alpha-\nu}\psi_N^{(\beta+j+i)} P^k u\|_0 + \|D^{\alpha-\nu+j}\psi_N^{(\beta+i)} P^k u\|_0 \Big) \\
&\quad + 3n \sum_{j=1}^{n} \sum_{\substack{\nu<\alpha,\\ \nu\neq 0}} \sum_{\substack{\mu\le\alpha-\nu,\\ \mu\neq 0}} C_1^{|\nu|+|\mu|+3} \frac{\alpha!\,(\nu!)^{s-1}\,(\mu!)^{s-1}}{(\alpha-\nu-\mu)!} \Big[\|D^{\alpha-\nu-\mu}\psi_N^{(\beta+j)} P^k u\|_0 \\
&\qquad\qquad + \|D^{\alpha-\nu-\mu+j}\psi_N^{(\beta)} P^k u\|_0 \Big] \\
&\quad + 2\sum_{i,j=1}^{n} \sum_{\substack{\nu<\alpha,\\ \nu\neq 0}} \sum_{\substack{\mu\le\alpha-\nu,\\ \mu\neq 0}} C_1^{|\nu|+|\mu|+2} \frac{\alpha!\,(\nu!)^{s-1}\,(\mu!)^{s-1}}{(\alpha-\nu-\mu)!} \Big[\|D^{\alpha-\nu-\mu}\psi_N^{(\beta+j+i)} P^k u\|_0 \\
&\qquad\qquad + \|D^{\alpha-\nu-\mu+j}\psi_N^{(\beta+i)} P^k u\|_0 + \|D^{\alpha-\nu-\mu+j+i}\psi_N^{(\beta)} P^k u\|_0 \Big] \\
&\quad + nC_1^3 \sum_{j=1}^{n} \|D^\alpha \psi_N^{(\beta+j)} P^k u\|_0 + C_1^2 \sum_{j,i=1}^{n} \|D^\alpha \psi_N^{(\beta+j+i)} P^k u\|_0 \Big\} \|D^\alpha \psi_N^{(\beta)} P^k u\|_0.
\end{aligned}
$$

We begin to analyze the terms in the last two sums. We have

$$
\begin{aligned}
& C_1^2 \Big(nC_1 \sum_{j=1}^{n} \|D^\alpha \psi_N^{(\beta+j)} P^k u\|_0 + \sum_{j,i=1}^{n} \|D^\alpha \psi_N^{(\beta+j+i)} P^k u\|_0 \Big) \|D^\alpha \psi_N^{(\beta)} P^k u\|_0 \\
&\quad \le nC_1^3 \Big(\sum_{j=1}^{n} \|D^{\alpha_0} \psi_N^{(\beta+j)} P^k u\|_1 + \sum_{j,i=1}^{n} \|D^{\alpha_0} \psi_N^{(\beta+j+i)} P^k u\|_1 \Big) \|D^\alpha \psi_N^{(\beta)} P^k u\|_0.
\end{aligned}
$$

We remark that

$$
\begin{aligned}
& 2r|\alpha_0| + |\beta| + 2k + 1 = 2r(|\alpha|-1) + |\beta| + 2k + 1 \le 2r|\alpha| + |\beta| + 2k; \\
& 2r|\alpha_0| + |\beta| + 2k + 2 = 2r(|\alpha|-1) + |\beta| + 2k + 2 \le 2r|\alpha| + |\beta| + 2k;
\end{aligned}
$$

$1-2r$ and $2(1-r)$ are negative, $r \ge 2$. Since $|\alpha_0| = m_0$ and $(\alpha_0, \beta+j, k)$, and $(\alpha_0, \beta+j+i, k)$ satisfy the condition in (4.1), then by Proposition 4.1, (4.3) is true for $p = r$ for these tow $2n+1$-tuples. We have

$$
\begin{aligned}
& \|D^{\alpha_0}\psi_N^{(\beta+i)} P^k u\|_1 \le A_1^{|\beta|+1+2k+r+1} B_1^{2r|\alpha_0|+r+1} N^{s[r|\alpha_0|+|\beta|+1+2k+r]}; \\
& \|D^{\alpha_0}\psi_N^{(\beta+i+j)} P^k u\|_1 \le A_1^{|\beta|+2+2k+r+1} B_1^{2r|\alpha_0|+r+1} N^{s[r|\alpha_0|+|\beta|+2+2k+r]}.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
& C_1^2 \Big(nC_1 \sum_{j=1}^{n} \|D^\alpha \psi_N^{(\beta+j)} P^k u\|_0 + \sum_{j,i=1}^{n} \|D^\alpha \psi_N^{(\beta+j+i)} P^k u\|_0 \Big) \|D^\alpha \psi_N^{(\beta)} P^k u\|_0 \\
&\qquad \le \Big(A_1^{|\beta|+2k+1} B_1^{2r(|\alpha|+(2r)^{-1})+1} N^{s[r(|\alpha|+r^{-1})+|\beta|+2k]} \Big)^2
\end{aligned}
$$

$$\times\, n^3 C_1^3 A_1^{r+2} B_1^{-r-2}\left(A_1^{-1}N^{-s}+1\right).$$

About the other terms on the right hand side of (4.36), we remark that, except the second term in the first two sums, when $|\nu|=1$, and the third term in the last multi-indexes sum, when $|\nu|=|\mu|=1$, since $|\nu|$, $|\mu|$ are greater than one, we can use the inductive hypothesis to handle all the terms in multi-indexes sums. We point out that in the cases $|\nu|=1$ and $|\nu|=|\mu|=1$, we have to cross the same problem run into to handle the first term in the first multi-index sum on the right hand side of (4.33).
Case, $|\nu|=1$. By (4.28), we have

$$\|D^{\alpha-\nu+j}\psi_N^{(\beta)}P^k u\|_0 \le A_1^{|\beta|+2k+1}B_1^{2r|\alpha|+1}N^{s[r|\alpha|+|\beta|+2k]}\times(A_1B_1^{-1})^r;$$

then

$$\|D^{\alpha-\nu+j}\psi_N^{(\beta)}P^k u\|_0\|D^{\alpha}\psi_N^{(\beta)}P^k u\|_0 \le \left(A_1^{|\beta|+2k+1}B_1^{2r(|\alpha|+(2r)^{-1})+1}N^{s[r(|\alpha|+r^{-1})+|\beta|+2k]}\right)^2\times A_1^r B_1^{-r-2}N^{-2s}.$$

Concerning the second term in the second multi-index sum we have

$$\sum_{j=1}^{n}\|D^{\alpha-\nu+j}\psi_N^{(\beta+i)}P^k u\|_0\|D^{\alpha}\psi_N^{(\beta)}P^k u\|_0 = \|D^{\alpha-\nu}\psi_N^{(\beta+i)}P^k u\|_1\|D^{\alpha}\psi_N^{(\beta)}P^k u\|_0,$$

where $|\alpha-\nu|$ is equal to m_0. Since

$$2rm_0+(|\beta|-M)+2k = 2r(m_0+1)+(|\beta|-M)+2k \le N-2r,$$

(4.1), in Proposition 4.1 is true for $(\alpha-\nu,\beta+j,k)$, so by Proposition 4.1, (4.3) holds for $p=r$ and $(\alpha-\nu,\beta+j,k)$, We have

$$\|D^{\alpha-\nu}\psi_N^{(\beta+i)}P^k u\|_1 \le A_1^{|\beta|+2k+1}B_1^{2r|\alpha|+1}N^{s[r|\alpha|+|\beta|+2k+1]}\times\left(A_1^{r+1}B_1^{-r-1}\right).$$

We conclude that

$$\|D^{\alpha-\nu}\psi_N^{(\beta+i)}P^k u\|_1\|D^{\alpha}\psi_N^{(\beta)}P^k u\|_0 \le \left(A_1^{|\beta|+2k+1}B_1^{2r(|\alpha|+(2r)^{-1})+1}N^{s[r(|\alpha|+r^{-1})+|\beta|+2k]}\right)^2\times A_1^{2r+1}B_1^{-2r-2}N^{-s}.$$

Case $|\nu|=|\mu|=1$:

$$\sum_{i=1}^{n}\|D^{\alpha-\nu-\mu+j+i}\psi_N^{(\beta)}P^k u\|_0\|D^{\alpha}\psi_N^{(\beta)}P^k u\|_0 = \|D^{\alpha-\nu-\mu+j}\psi_N^{(\beta)}P^k u\|_1\|D^{\alpha}\psi_N^{(\beta)}P^k u\|_0.$$

where $|\alpha-\nu-\mu+j|$ is equal to m_0. Using the same argument adopted above we have

$$\|D^{\alpha-\nu-\mu+j}\psi_N^{(\beta)}P^k u\|_1\|D^{\alpha}\psi_N^{(\beta)}P^k u\|_0 \le \left(A_1^{|\beta|+2k+1}B_1^{2r(|\alpha|+(2r)^{-1})+1}N^{s[r(|\alpha|+r^{-1})+|\beta|+2k]}\right)^2\times A_1^{2r}B_1^{-3r-2}N^{-s}.$$

So, by the above considerations and using the same strategies adopted, several times, previously to handle the multi-index sums, we conclude that

$$\sum_{\ell=1}^{m}|\langle[X_\ell,[X_\ell,D^\alpha\psi_N^{(\beta)}]]P^ku,D^\alpha\psi_N^{(\beta)}P^ku\rangle| \tag{4.38}$$
$$\leq\left(A_1^{|\beta|+2k+1}B_1^{2r(|\alpha|+(2r)^{-1})+1}N^{s[r(|\alpha|+r^{-1})+|\beta|+2k]}\right)^2$$
$$\times 2^{2n}n^2C_1^4A_1^{r+1}B_1^{-r-1}\left[2B_1^{-1}+4A_1^{-1}B_1^{-1}+2A_1^rB_1^{-r-1}+C_1^{-4}+C_1^{-1}AB^{-1}\right].$$

Enlarging B_1 if necessary and taking it large enough compared to A_1, the second factor of (4.31), (4.32), (4.34), (4.35), (4.36) and (4.37) times $4\widetilde{C}^2$ can be made strictly less than 6^{-1}. We conclude that

$$\|X_\ell D^\alpha\psi_N^{(\beta)}P^ku\|_0\leq A_1^{|\beta|+2k+1}B_1^{2r|\alpha|+1}N^{s[r|\alpha|+1+|\beta|+2k]},$$

where $|\alpha|=m_0+1$.
By induction we have obtained that (4.1) is true for $m\in\mathbb{N}$ and $(\alpha,\beta,k)\in\mathbb{N}^{2n+1}$ such that $2r|\alpha|+(|\beta|-M)+2k\leq N$.
This conclude the proof of the Proposition. □

Corollary 4.1. *The following estimate holds*

$$\|D^\alpha\psi_Nu\|_0\leq B^{|\alpha|+1}N^{sr|\alpha|}, \tag{4.39}$$

where B is independent of α and N.

We recall the following result due to Hörmander ([36])

Proposition 4.3. *Let $x_0\in\Omega_0\subset\mathbb{R}^n$ and $u\in\mathscr{D}'(\Omega)$. Then $u\in G^s$, $s\geq 1$, in a neighborhood of x_0 if and only if for some Ω_1 of x_0 there there is a sequence $u_N\in\mathscr{E}'(\Omega_0)$ which is equal to u in Ω_1 and satisfies the estimates*

$$|\widehat{u}_N(\xi)|\leq C\left(CN^s/|\xi|\right)^N,\qquad|\xi|\geq 1,\qquad N=1,2,\ldots, \tag{4.40}$$

for some constant C.

Setting $u_N=\psi_Nu$ and $N=2r|\alpha|$ the estimate (4.39) is equivalent to (4.40) in the above Proposition, we conclude that $u\in G^{sr}(\Omega_0)$. We obtained the Theorem 2.1.

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