Supplemental Material: Breaking global symmetries with locality-preserving operations

Here we provide additional details on the results stated in the main text.

CONTENTS

I. BOUND ON $U(1)$-ENTANGLEMENT ASYMMETRY FOR CLUSTERING STATES

In this section we provide additional details on the derivation of the asymmetry bound for clustering states, in the case of the $U(1)$ abelian symmetry. Next, we discuss the example of pure inhomogeneous product states, which saturate the inequality (9) of the main text.

A. Derivation of the bound

As in the main text, we denote the $U(1)$ charge by $Q = \sum_j q_j$, with the charge density $q_j = (\sigma_j^z + 1)/2$ supported on $\mathcal{H}_j$. We consider a state ρ satisfying the clustering decomposition

$$\langle O_i \, O_j \rangle_\rho - \langle O_i \rangle_\rho \langle O_j \rangle_\rho = 0, \quad \text{for} \quad \delta(i,j) > \Lambda \,, \tag{S1}$$

for some integer Λ, where $\langle O \rangle_\rho = \mathrm{Tr}[\rho O]$ and O_j is an arbitrary operator supported on $\mathcal{H}_j$. As discussed in the main text, Eq. (S1) is satisfied by the output of a LP channel acting on a product state.

We begin by putting a bound on the charge variance. To this end, we first note

$$|\langle q_x q_{x'} \rangle_c| = |\langle q_x q_{x'} \rangle - \langle q_x \rangle \langle q_{x'} \rangle| \leq |\langle q_x q_{x'} \rangle| + |\langle q_x \rangle| \cdot |\langle q_{x'} \rangle| \leq 2 \|q_0\|^2 = 2 \,, \tag{S2}$$

where $\langle q_x q_{x'} \rangle_c$ denotes the connected correlation function, while $\| \ldots \|$ is the operator norm. Here, we used $|\langle \mathcal{O} \rangle| \leq \|\mathcal{O}\|$, $\|\mathcal{O}_1 \mathcal{O}_2\| \leq \|\mathcal{O}_1\| \cdot \|\mathcal{O}_2\|$, and $||q_j|| = 1$. Using Eq. (S1) with $O_j = q_j$ and Eq.(S2), we can bound

$$\langle Q^2 \rangle_c = \left| \sum\nolimits_{x \in \{1,\ldots,N\},\, x' \in I_x^\Lambda} \langle q_x q_{x'} \rangle_c \right| \leq \sum_{x \in \{1,\ldots,N\},\, x' \in I_x^\Lambda} |\langle q_x q_{x'} \rangle_c| \leq 2 z_\Lambda N \,, \tag{S3}$$

where

$$I_x^\Lambda = \{ x' : \delta(x, x') \leq \Lambda \} \,, \tag{S4}$$

while z_Λ is the cardinality of I_x^Λ (which is independent of x). That is, the charge variance of a clustering state is at most extensive in the system size. Importantly, the variance of the charge is the second cumulant of the probability distribution $\{p_q\}$, where $p_q = \mathrm{Tr}[\rho P_q]$ and P_q is the projector onto the q-eigenspace of the charge Q. Namely,

$$\sigma^2 := \langle Q^2 \rangle_c = \sum_q p_q \, (q - \bar{q})^2 \,. \tag{S5}$$

Therefore, Eqs. (S3) and (S5) immediately yield a bound on the variance of the probability distribution $\{p_q\}$.

As a next step, we rewrite

$$\mathcal{G}[\rho] = \sum_{q=0}^{N} p_q \tilde{\rho}_q \,, \tag{S6}$$

where $\tilde{\rho}_q = P_q \rho P_q / p_q$, and recall the standard inequality [1]

$$S_V\left(\sum_a p_a \rho_a\right) \leq \sum_a p_a S_V(\rho_a) + H(p) \,, \tag{S7}$$

where $\{p_a\}$ is any discrete probability distribution, ρ_a are arbitrary mixed states, while $S_V(\rho) = -\mathrm{Tr}[\rho \log \rho]$ and $H(\{p_j\}) = -\sum_j p_j \log p_j$ are the von Neumann and Shannon entropies, respectively. Applying this inequality to (S6), we get

$$\Delta S_N^{U(1)}(\rho) \leq H(\{p_q\}) + \sum_q p_q S_V(\tilde{\rho}_q) - S_V(\rho) \,. \tag{S8}$$

Now, we observe that $\tilde{\rho}_q$ is obtained from ρ by measuring the charge operator and postselecting on the outcome q. On the other hand, the von Neumann entropy does not increase, on average, under measurements [2], and so $S_V(\rho) \geq \sum_q p_q S_V(\tilde{\rho}_q)$. Putting all together, we have

$$\Delta S_N^{U(1)}(\rho) \leq H(q) \,. \tag{S9}$$

Finally, we make use of a known result in classical probability theory, which is an upper bound on the Shannon entropy of an integer-valued random variable with fixed variance. To be precise, let x be an integer-valued random variable, with probability function p_x and support $A \subseteq \mathbb{Z}$, such that $0 < \sigma^2 < \infty$. Then [3, 4]

$$H(\{p_x\}) < \frac{1}{2} \log\left[2\pi e \left(\sigma^2 + \frac{1}{12}\right)\right] \,. \tag{S10}$$

Noting that Eq. (S10) is a monotonic function of σ, using Eqs. (S3) and (S5), we arrive at our final result

$$\Delta S_N^{U(1)}(\rho) \leq \frac{1}{2} \log\left[2\pi e \left(2 z_\Lambda N + \frac{1}{12}\right)\right] = \frac{1}{2} \log(N)(1 + o(1)) \,. \tag{S11}$$

B. Inhomogeneous product states

In Ref. [5] it was proven that the asymmetry of translation-invariant product states at large system size N is given by $\Delta S_N^{U(1)} \sim \frac{1}{2} \log N$. In this section, we explicitly obtain the bound (S11) for pure, inhomogeneous product states. The proof we provide is non-trivial, and it relies on the Shepp-Olkin theorem [6, 7]. The computations presented in this section can be considered as an independent check of Eq. (S11), but may also be interesting per se.

The state we consider is:

$$|\{x_i\}\rangle = \bigotimes_{i=1}^{N} (\sqrt{x_i}\,|0\rangle_i + \sqrt{1 - x_i}\,|1\rangle_i), \quad x_i \in [0, 1]. \tag{S12}$$

Since the state is not translation-invariant, the local charges $q_i = (\sigma_i^z + 1)/2$ are not i.i.d, and the heuristic argument based on the classical central limit theorem cannot be invoked. Nonetheless, the clustering hypothesis hold, as $\langle q_i \rangle = x_i$, $\langle q_i q_j \rangle = (1 - \delta_{ij}) x_i x_j + \delta_{ij} x_i$, implying:

$$\langle q_i q_j \rangle = \begin{cases} x_i - x_i^2, & i = j \\ 0, & i \neq j \end{cases} . \tag{S13}$$

The charge probabilities $p_q = \langle \{x_i\}|\, P_q\, |\{x_i\}\rangle$ are obtained by taking the Fourier transform of the charge generating function:

$$\langle e^{i\alpha Q} \rangle = \prod_j (e^{i\alpha} x_j + (1 - x_j)). \tag{S14}$$

The above product is expanded in 2^N terms, $\binom{N}{q}$ of which contain a factor $e^{i\alpha q}$ that yields a non-vanishing contribution to p_q:

$$p_q = \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} e^{-i\alpha q} \prod_j (e^{i\alpha} x_j + (1 - x_j)) = \sum_{1 \le j_1 < \dots < j_q \le N} \prod_{\substack{j \in \{j_1, \dots, j_q\} \\ k \notin \{j_1, \dots, j_q\}}} x_j (1 - x_k). \tag{S15}$$

We observe that p_q is the probability mass function of a random variable $X = \sum_i X_i$, where X_i are independent Bernoulli variables with parameters x_i, $X_i \sim \mathcal{B}(x_i)$. Therefore, since the inequality (S9) is saturated, $\Delta S_N^{U(1)}(|\{x_i\}\rangle) = H(X)$, with $H(X) = -\sum_{q=0}^{N} p_q \log p_q$. The Shepp-Olkin theorem states that the Shannon entropy $H(X)$ is a concave function of the parameters x_i, and its unique maximum is reached when all $x_i = 1/2$, yielding $\Delta S_N^{U(1)}(|\{x_i\}\rangle) \le \frac{1}{2} \log(N)(1 + o(1))$.

II. BOUND ON $SU(2)$-ENTANGLEMENT ASYMMETRY FOR CLUSTERING STATES

In this section, we discuss the case of the non-abelian $SU(2)$ symmetry.

A. Preliminaries and notation

Following Refs. [8, 9], we begin by recalling a few notions about the $SU(2)$ representation theory and the $SU(2)$-asymmetry.

As in the main text, we choose the generators $Q^\alpha = S^\alpha = \sum_j \sigma_j^\alpha / 2$, and use the following conventions: the eigenvalues of the Casimir operator $\mathbf{S}^2 = (S^x)^2 + (S^y)^2 + (S^z)^2$ and of S^z are labeled, respectively by

$$\mathbf{S}^2 |s, m\rangle = s(s+1) |s, m\rangle \,, \tag{S16}$$

$$S^z |s, m\rangle = m |s, m\rangle \,. \tag{S17}$$

For concreteness, we will consider the case of N even. The allowed values for s and m are therefore

$$s = 0, 1, \dots \frac{N}{2} \,, \tag{S18}$$

$$m = -\frac{N}{2}, \dots, 0, \dots \frac{N}{2} \,. \tag{S19}$$

We further denote by P_s and P_m the orthogonal projectors onto the subspaces associated with the quantum numbers s and m (note that $[P_s, P_m] = 0$).

The integer s labels the $SU(2)$ irreducible representations and the Hilbert space $\mathcal{H}$ can be decomposed as follows

$$\mathcal{H} = \bigoplus_s \mathcal{M}_s \otimes \mathcal{N}_s \,, \tag{S20}$$

where $\mathcal{M}_s$ is the representation space corresponding to total spin s, while $\mathcal{N}_s$ is the associated multiplicity space. The dimension of $\mathcal{N}_s$, which corresponds to the multiplicity of the s-representation space, can be computed explicitly (see *e.g.* [10]) as

$$n_s = \dim(\mathcal{N}_s) = \binom{N}{N/2 - S} - \binom{N}{N/2 - S - 1}, \tag{S21}$$

which grows exponentially in N.

Using Schur's lemma, we can write an explicit form for the symmetrized density matrix, extending (S6) to the case of $SU(2)$. We have in particular [8, 9]

$$\mathcal{G}[\rho] = \sum_s p_s \frac{\mathbb{1}_{\mathcal{M}_s}}{\dim \mathcal{M}_s} \otimes \tilde{\rho}_s \,, \tag{S22}$$

where $p_s = \mathrm{Tr}[\rho P_s]$ is the probability distribution over the irreducible representations, while

$$\tilde{\rho}_s = \frac{1}{p_s} \mathrm{Tr}_{\mathcal{M}_s} [P_s \rho P_s] \,. \tag{S23}$$

By inspection of Eq. (S22), it is easy to determine the maximum dimension of the support of $\mathcal{G}[\rho]$ and, accordingly, derive the following general logarithmic bound [9]

$$\Delta S_N^{SU(2)}(\rho) \leq \log\left(\sum_s \dim(\mathcal{M}_s) \times \min\left(n_s, \dim(\mathcal{M}_s)\right)\right). \tag{S24}$$

In addition, Ref. [9] also exhibited states saturating the bound. Note that $\dim(\mathcal{M}_s) = 2s+1$, so that the RHS of Eq. (S24) scales as $\sim 3\log N$. In the following subsections, we will prove that, for any state ρ satisfying the clustering decomposition, $\Delta S_N^{SU(2)}(\rho) \leq 3/2 \log N + O(\log\log(N))$, namely the asymmetry scaling is bounded by half its maximnum value, as anticipated.

B. Bounding the asymmetry by a classical Shannon entropy

The decomposition (S22) allows us to write

$$\Delta S_N^{SU(2)}(\rho) = S_V(\mathcal{G}[\rho]) - S_V(\rho) = \sum_s p_s \log(2s+1) - \sum_s p_s \log p_s - \sum_s p_s \mathrm{Tr}\left(\tilde{\rho}_s \log \tilde{\rho}_s\right) - S_V(\rho)\,. \tag{S25}$$

In this subsection, we bound the right-hand side in terms of a probability distribution function over the quantum numbers m and s.

We first observe that the projector P_m is block-diagonal with respect to the decomposition (S20), acting as the identity on the multiplicity space $\mathcal{N}_s$. In particular, $P_m = \sum_s P_{m|s}$ where $P_{m|s} := \tilde{P}_{m|s} \otimes \mathbb{1}_{\mathcal{N}_s}$ and $\tilde{P}_{m|s}$ is the restriction of P_m onto the subspace $\mathcal{M}_s$. Inserting now appropriate resolutions of the identity, we can rewrite

$$\tilde{\rho}_s = \frac{1}{p_s}\mathrm{Tr}_{\mathcal{M}_s}\left[\left(\sum_{s'}\sum_m P_{m|s'}\right) P_s \rho P_s \left(\sum_{s'}\sum_m P_{m|s'}\right)\right] = \frac{1}{p_s}\sum_m \mathrm{Tr}_{\mathcal{M}_s}(P_{m|s}P_s\rho P_s P_{m|s}) = \sum_m p(m|s)\tilde{\omega}_{s,m}\,; \tag{S26}$$

here, we have defined the conditional probability distribution $p(m|s) = \frac{p_{m,s}}{p_s}$, where $p_{m,s} = \mathrm{Tr}[\rho P_s P_m]$, and the normalized states

$$\tilde{\omega}_{s,m} = \frac{\mathrm{Tr}_{\mathcal{M}_s}(P_{m|s}P_s\rho P_s P_{m|s})}{p_{m,s}}\,. \tag{S27}$$

Next, we apply the inequality (S7) to the density matrix $\tilde{\rho}_s$, yielding

$$S_V(\tilde{\rho}_s) \leq \sum_m p(m|s) S_V(\tilde{\omega}_{s,m}) + H(\{p(m|s)\}_m)\,. \tag{S28}$$

Plugging into (S25) and using $p(m|s)p_s = p_{m,s}$, we obtain

$$\begin{aligned}\Delta S_N^{SU(2)}(\rho) &\leq \sum_s p_s \log(2s+1) - \sum_s p_s \log p_s + \sum_s p_s H(\{p(m|s)\}_m) + \sum_{s,m} p_{s,m} S_V(\tilde{\omega}_{s,m}) - S_V(\rho) \\ &= \sum_s p_s \log(2s+1) - \sum_{s,m} p_{s,m}\log p_{s,m} + \sum_{s,m} p_{s,m} S_V(\tilde{\omega}_{s,m}) - S_V(\rho)\,.\end{aligned} \tag{S29}$$

Now we claim

$$\sum_{s,m} p_{s,m} S_V(\tilde{\omega}_{s,m}) - S_V(\rho) \leq 0\,. \tag{S30}$$

To see this, we first note that $S_V(\tilde{\omega}_{s,m}) = S_V(\hat{\omega}_{s,m})$, where

$$\hat{\omega}_{s,m} = \frac{P_{m|s}P_s\rho P_s P_{m|s}}{p_{m,s}}\,. \tag{S31}$$

Since $\tilde{P}_{m|s}$ is a rank-one projector, $\tilde{P}_{m|s} = |v_m\rangle\langle v_m|$ for some $|v_m\rangle \in \mathcal{M}_s$, then $\hat{\omega}_{s,m} = |v_m\rangle\langle v_m| \otimes \tilde{\omega}_{s,m}$, and $S_V(\hat{\omega}_{s,m}) = S_V(\tilde{\omega}_{s,m})$.

On the other hand, Eq. (S31) can be rewritten as $\hat{\omega}_{s,m} = \frac{P_m P_s \rho P_s P_m}{p_{m,s}}$. That is, $\hat{\omega}_{s,m}$ is obtained from ρ by measuring both the Casimir operator $\mathbf{S}^2 = \sum_\alpha (S^\alpha)^2$ and Q^z, and postselecting on the outcomes s and m, respectively. Then, Eq. (S30) follows from the fact that the von Neumann entropy does not increase, on average, under measurements [2].

Putting all together, we arrive at the final result

$$\Delta S_N^{SU(2)}(\rho) \le \sum_s p_s \log(2s+1) - \sum_{s,m} p_{s,m} \log p_{s,m} \,. \tag{S32}$$

Eq. (S32) generalizes Eq. (S9) to the non-abelian case. It is our starting point to derive the asymmetry bound for states satisfying the cluster property, as detailed in the next subsection.

C. Bound for clustering states

Given a state ω satisfying the cluster property with range Λ, we can always find $U = u^{\otimes N}$ such that the state $\rho = U\omega U^\dagger$ satisfies

$$\langle S^x \rangle_\rho = \langle S^y \rangle_\rho = 0 \,. \tag{S33}$$

This is because $U = u^{\otimes N}$ can implement arbitrary rotations of the vector $(\langle S^x\rangle_\omega, \langle S^y\rangle_\omega, \langle S^z\rangle_\omega)$. We will put a bound on the asymmetry of ρ, which is the same as that of ω. Importantly, ρ is also a quantum state satisfying the clustering property with range Λ.

For clarity, we will often use the notation $p(s,m)$ in place of $p_{s,m}$. Due to Eq. (S32), it is enough to put a bound on the functional

$$\mathcal{F}[\{p(s,m)\}] = \sum_{s,m} p(s,m) \log(2s+1) - \sum_{s,m} p(s,m) \log p(s,m) \,. \tag{S34}$$

As the proof is rather technical, we will split it into a series of intermediate steps.

- **Changing variables**. As a first step, we introduce a new stochastic variable, ξ, which is a function of the random variables s, m. It is defined by

$$s(s+1) = m^2 + \xi^2 \,. \tag{S35}$$

Since $m \le s$, we have $s \le \xi^2 \le s(s+1)$. In principle, ξ could be as large as $\sim N$. However, we will show now that clustering implies an effective bound $\xi \le O(\sqrt{N})$. In order to make this precise, we first show

$$\mathbb{E}\left[\xi^2\right] \le c(\Lambda) N, \tag{S36}$$

where $c(\Lambda) = O(1)$ is a function of Λ.

To derive Eq. (S36), we first observe that the clustering property of ρ implies

$$\langle \mathbf{S}^2 \rangle_\rho - \sum_{\alpha=x,y,z} \langle S^\alpha \rangle_\rho^2 \le c(\Lambda) N \,, \tag{S37}$$

where $c(\Lambda) = O(1)$ is a function of Λ (independent of N). Because of Eq. (S33), we get

$$\langle \mathbf{S}^2 \rangle_\rho - \langle S^z \rangle_\rho^2 \le c(\Lambda) N \,. \tag{S38}$$

Now, for any Hermitian operator A and density matrix ρ, one has $\langle A^2 \rangle_\rho \ge \langle A \rangle_\rho^2$, which follows from the Cauchy-Schwarz inequality. Therefore

$$\langle \mathbf{S}^2 \rangle_\rho - \langle (S^z)^2 \rangle_\rho \le \langle \mathbf{S}^2 \rangle_\rho - \langle S^z \rangle_\rho^2 \le c(\Lambda) N \,. \tag{S39}$$

Finally, we recognize $\langle \mathbf{S}^2 \rangle_\rho - \langle (S^z)^2 \rangle_\rho = \mathbb{E}\left[s(s+1) - m^2\right] = \mathbb{E}\left[\xi^2\right]$, which concludes the proof of Eq. (S36).

Now, the values of (m,s) uniquely specify the pair (m,ξ), with $\xi > 0$. The converse is also true, as

$$s = -\frac{1}{2} + \left(\frac{1}{4} + m^2 + \xi^2\right)^{1/2} \,. \tag{S40}$$

Therefore, we can parametrize $\{p(s,m)\} \to \{p(m,\xi)\}$, and rewrite (S34) as

$$\mathcal{F}[\{p(m,\xi)\}] = \frac{1}{2} \sum_{m,\xi} p(m,\xi) \log\left[1 + 4(m^2+\xi^2)\right] - \sum_{m,\xi} p(m,\xi) \log p(m,\xi) \,. \tag{S41}$$

- **Restricting the variable range**. Next, we want to use (S36) to induce an effective cutoff on the values of ξ to be considered in the sums appearing in Eq. (S41). Choose $\ell > 0$ (for the moment, ℓ is arbitrary), and rewrite (S36) as [omitting the dependence of c on Λ]

$$\sum_{\xi<\ell\sqrt{cN}} p(\xi)\xi^2 + \sum_{\xi\geq\ell\sqrt{cN}} p(\xi)\xi^2 \leq cN\,. \tag{S42}$$

Defining $T = \sum_{\xi\geq\ell\sqrt{cN}} p(\xi)$ it is immediate to show

$$T(\ell^2 cN) \leq \sum_{\xi\geq\ell\sqrt{cN}} p(\xi)\xi^2 \leq cN\,, \tag{S43}$$

and so

$$T \leq \frac{1}{\ell^2}\,. \tag{S44}$$

In the following, we denote by $\sum'_{m,\xi}$ and $\sum''_{m,\xi}$, the sums restricted to $\xi < \ell\sqrt{cN}$ and $\xi \geq \ell\sqrt{cN}$, respectively. We have

$$\begin{aligned}\frac{1}{2}\sum_{m,\xi} p(m,\xi)\log\left[1+4(m^2+\xi^2)\right] =& \frac{1}{2}\sum_{m,\xi}{}' p(m,\xi)\log\left[1+4(m^2+\xi^2)\right] + \frac{1}{2}\sum_{m,\xi}{}'' p(m,\xi)\log\left[1+4(m^2+\xi^2)\right] \\ \leq& \frac{1}{2}\sum_{m,\xi}{}' p(m,\xi)\log\left[1+4(m^2+\xi^2)\right] + \frac{1}{2}T\log[1+N^2+N(N+2)] \\ \leq& \frac{1}{2}\sum_{m,\xi}{}' p(m,\xi)\log\left[1+4(m^2+\xi^2)\right] + T\log(2N)\,.\end{aligned} \tag{S45}$$

Similarly,

$$\begin{aligned}-\sum_{m,\xi}{}'' p(m,\xi)\log p(m,\xi) &= -T\sum_{m,\xi}{}'' \frac{p(m,\xi)}{T}\log\left[\frac{p(m,\xi)}{T}T\right] \\ &= -T\log T - T\sum_{m,\xi}{}'' \frac{p(m,\xi)}{T}\log\left[\frac{p(m,\xi)}{T}\right]\,.\end{aligned} \tag{S46}$$

In the second term, we recognize the entropy for the probability distribution $q(m,\xi) = p(m,\xi)/T$ (it is positive and sums to 1 by construction). Now, the total number of values that the pair (m,ξ) can take is upper bounded by $(N/2+1)(N+1) \leq N^2$ (assuming $N \geq 4$). Therefore,

$$-\sum_{m,\xi}{}'' \frac{p(m,\xi)}{T}\log\left[\frac{p(m,\xi)}{T}\right] \leq \log(N^2)\,. \tag{S47}$$

Putting all together

$$\begin{aligned}-\sum_{m,\xi} p(m,\xi)\log p(m,\xi) &= -\sum_{m,\xi}{}' p(m,\xi)\log p(m,\xi) - \sum_{m,\xi}{}'' p(m,\xi)\log p(m,\xi) \\ &\leq -\sum_{m,\xi}{}' p(m,\xi)\log p(m,\xi) - T\log T + T\log(N^2)\,.\end{aligned} \tag{S48}$$

Collecting all terms,

$$\begin{aligned}\mathcal{F}[\{p(m,\xi)\}] \leq& -T\log T + T\log(2N^3) \\ &+\frac{1}{2}\sum_{m,\xi}{}' p(m,\xi)\log\left[1+4(m^2+\xi^2)\right] - \sum_{m,\xi}{}' p(m,\xi)\log p(m,\xi)\,,\end{aligned} \tag{S49}$$

and exploiting (S44), we arrive at

$$\mathcal{F}[\{p(m,\xi)\}] \leq \frac{1}{2}\log 2 + (\ell^{-2})\log(2N^3) + \frac{1}{2}\sum_{m,\xi}{}' p(m,\xi)\log\left[1+4(m^2+\xi^2)\right] - \sum_{m,\xi}{}' p(m,\xi)\log p(m,\xi)\,, \tag{S50}$$

where we also used $-T\log T \leq (1/2)\log 2$ for $0 \leq T \leq 1$.

- **Bounding the restricted sums**. As a last step, we put a bound on the terms appearing in the second line of Eq. (S50). Although this might appear as hard as bounding the functional (S34), the task is now greatly simplified because, by construction, the random variable ξ takes value in a relatively small set.

  We now take the most delicate step of the proof. We first rewrite

$$\begin{aligned} -\sum_{m,\xi}{}' p(m,\xi)\log p(m,\xi) &= -\sum_m \sum_{s\in\mathcal{S}_{\ell,m}} p(s,m)\log p(s,m) \\ &= -\sum_m p(m)\sum_{s\in\mathcal{S}_{\ell,m}} \frac{p(s,m)}{p(m)}\log\left[\frac{p(s,m)}{p(m)}\right] - \sum_m\sum_{s\in\mathcal{S}_{\ell,m}} p(s,m)\log p(m) \\ &\leq -\sum_m p(m)\log p(m) - \sum_m p(m)\sum_{s\in\mathcal{S}_{\ell,m}} \frac{p(s,m)}{p(m)}\log\left[\frac{p(s,m)}{p(m)}\right]. \end{aligned} \tag{S51}$$

  Here, we define $\mathcal{S}_{\ell,m} = \{s : \xi(s,m) < \ell\sqrt{c(\Lambda)N}\}$, and we used

$$\sum_m p(m)\log p(m) - \sum_m\sum_{s\in\mathcal{S}_{\ell,m}} p(s,m)\log p(m) \leq 0\,. \tag{S52}$$

  Next, define $q_m(s) = \frac{p(s,m)}{p(m)}$. Clearly, $0 \leq q_m(s) \leq 1$ and $\sum_{s\in\mathcal{S}_{\ell,m}} q_m(s) \leq 1$. We can then apply the following fact, whose proof is immediate: for any $\{q_k\}_{k=1}^R$ such that $0 \leq q_k \leq 1$ and $\sum_k q_k \leq 1$, it holds that $-\sum_{k=1}^R q_k\log q_k \leq \log(R+1)$. In order to apply this result to Eq. (S51), we need to bound the number of s-values that $q_m(s)$ is supported on, for a given m. This can be done precisely because we have a cutoff on the allowed values of ξ.

  Explicitly, we start from Eq. (S40) and rewrite it as

$$s = -\frac{1}{2} + \left(m^2+\frac{1}{4}\right)^{1/2} + \left\{\left(\frac{1}{4}+m^2+\xi^2\right)^{1/2} - \left(m^2+\frac{1}{4}\right)^{1/2}\right\}. \tag{S53}$$

  From this expression, it is clear that, for a fixed m, the number of values that s can take is bounded by the maximum value that

$$\Delta(\xi) = \left\{\left(\frac{1}{4}+m^2+\xi^2\right)^{1/2} - \left(m^2+\frac{1}{4}\right)^{1/2}\right\}, \tag{S54}$$

  can take, as a function of ξ. This is because $\Delta(\xi) \geq 0$ and $\Delta(\xi)$ can only take discrete values, separated by 1. Defining $a = \left(m^2+1/4\right)^{1/2}$, we get

$$\Delta(\xi) = \left\{\left(a^2+\xi^2\right)^{1/2} - |a|\right\} = |a|\left\{(1+\xi^2/a^2)^{1/2} - 1\right\}, \tag{S55}$$

  and using $\sqrt{1+x^2} \leq 1 + (x^2/(1+x))$ for $x > 0$,

$$\Delta(\xi) \leq |a|\left\{\left(1+\frac{\xi^2/a^2}{1+(\xi/a)}\right) - 1\right\} = \frac{\xi^2}{\xi + (m^2+1/4)^{1/2}}\,. \tag{S56}$$

  Recalling $\xi \leq \ell\sqrt{cN}$, we arrive at

$$\Delta(\xi) \leq \frac{\ell^2 cN}{\ell\sqrt{cN} + (m^2+1/4)^{1/2}}\,. \tag{S57}$$

Finally, putting all together, Eq. (S50) yields

$$
\begin{aligned}
&-\sum_m \sum_{s\in\mathcal{S}_{\ell,m}} p(s,m)\log p(s,m) \\
&\leq -\sum_m p(m)\log p(m) + \sum_m p(m)\log\left(1+\frac{\ell^2 cN}{\ell\sqrt{cN}+(m^2+1/4)^{1/2}}\right).
\end{aligned} \tag{S58}
$$

Similarly, the term of the second line in Eq. (S50) can be bounded by

$$
\frac{1}{2}{\sum_{m,\xi}}' p(m,\xi)\log\left[1+4(m^2+\xi^2)\right] \leq \frac{1}{2}\sum_m p(m)\log[1+4(m^2+\ell^2 cN)]\,, \tag{S59}
$$

where we used ${\sum}'_{\xi} p(m,\xi) \leq p(m)$.

- **The final result**. In summary, we arrived at the following result: for any $\ell > 0$, we have

$$
\mathcal{F}[\{p(m,\xi)\}] \leq \frac{1}{2}\log 2 + (\ell^{-2})\log(2N^3) - \sum_m p(m)\log p(m) + \sum_m p(m) f_\ell(m)\,, \tag{S60}
$$

where

$$
f_\ell(m) = \frac{1}{2}\log[1+4(m^2+\ell^2 cN)] + \log\left(1+\frac{\ell^2 cN}{\ell\sqrt{cN}+(m^2+1/4)^{1/2}}\right). \tag{S61}
$$

So far, we have not specified ℓ. Let us now fix $\ell = \log(N)$. With this choice, the second term in the RHS. of Eq. (S60) is trivial. Indeed $(\ell^{-2})\log(2N^3) = O([\log N]^{-1}) < 1$, for sufficiently large N. Next, the equation $\partial f_{\log N}(m)/\partial m = 0$ has three real solutions, $m = 0$ (local maximum) and $m = \pm m_*$ (local minima). The analytic expression of $m_* = m_*(N,c)$ is quite cumbersome and we do not report it here. Thus, it holds:

$$
f_{\log N}(m) \leq \max\{f_{\log N}(0),\, f_{\log N}(N/2)\}. \tag{S62}
$$

We see that both $f_{\log N}(0) = \log N + O(\log\log N)$, $f_{\log N}(N/2) = \log N + O(\log\log N)$, with $|f_{\log N}(0) - f_{\log N}(N/2)| = O(\log\log N)$. It follows that $f_{\log N}(m) \leq \log N + O(\log\log N)$ and so

$$
\sum_m p(m) f_\ell(m) \leq \log N + O(\log\log N)\,. \tag{S63}
$$

Finally, when studying the $U(1)$ case, we have already shown that the cluster property of ρ implies $-\sum_m p(m)\log p(m) \leq (1/2)\log N + d$, where $d = O(1)$ depends on Λ.

Putting all together, we have therefore proven

$$
\mathcal{F}[\{p(m,\xi)\}] \leq \frac{3}{2}\log N + O[\log\log(N)]\,, \tag{S64}
$$

which is the anticipated result.

III. MAXIMALLY ASYMMETRIC STATES

In this section we provide additional details on pure states displaying maximal asymmetry, focusing on the case of $U(1)$.

At any finite N, the maximal $U(1)$-asymmetry is obtained when the charge distribution p_q is flat [11], yielding $\Delta S_N^{U(1)} = \log(N+1)$. We now show that the same leading scaling of ΔS can be obtained whenever the probability distribution of the charge density Q/N is continuous in the thermodynamic limit. It is important to stress that such a scenario is not typical in the context of gapped ground states or thermal states (in the absence of spontaneous symmetry breaking). For the aforementioned cases, clustering of correlations holds, implying the existence of a scaled cumulant generating function in the limit $N \to \infty$

$$\frac{1}{N}\log\langle e^{i\alpha Q}\rangle = \sum_{n=0}^{\infty}\frac{(i\alpha)^n}{n!}\langle Q^n\rangle_c/N, \tag{S65}$$

with $\langle Q^n\rangle_c$ the n-th cumulant of Q. In other words, the cumulants of Q are extensive, and $(Q-\langle Q\rangle)/\sqrt{N}$ is normal-distributed in the large N limit, with a variance denoted by σ^2. The generating function can be expanded at leading order as

$$\langle e^{i\alpha Q}\rangle \simeq e^{i\alpha\langle Q\rangle}e^{-\frac{N\alpha^2\sigma^2}{2}} \tag{S66}$$

and the probability distribution is

$$\begin{aligned} p_q = \langle\psi| P_q |\psi\rangle &= \int_{-\pi}^{\pi}\frac{d\alpha}{2\pi}e^{-i\alpha q}\langle e^{i\alpha Q}\rangle \\ &\underset{N \text{ large}}{\simeq} \int_{-\infty}^{\infty}\frac{d\alpha}{2\pi}e^{i\alpha(q-\langle Q\rangle)}e^{-\frac{N\alpha^2\sigma^2}{2}} = \frac{1}{\sqrt{2\pi N\sigma^2}}\exp\left[-\frac{(q-\langle Q\rangle)^2)}{2N\sigma^2}\right]. \end{aligned} \tag{S67}$$

As a consequence, one can show that the asymmetry of a pure state, being the Shannon entropy of the probability distribution, satisfies $\Delta S = H(\{p_q\}) \simeq 1/2\log N$.

We now discuss a general mechanism that leads to the saturation of the bound (S11) at the leading order. Let us consider the generating function of the charge density in the large N limit

$$\lim_{N\to\infty}\left\langle e^{i\omega\frac{Q}{N}}\right\rangle = f(\omega). \tag{S68}$$

We assume that the Fourier transform of $f(\omega)$, which gives the probability distribution of the charge density, exists and it is continuous: we stress that this is not the case when central limit theorem applies, since the charge density is δ-distributed (in the $N\to\infty$ limit).

Given $p(u)$ the Fourier transform of $f(\omega)$, we can approximate $p_q \simeq p(q/N)/N$ in the large N limit and estimate the asymmetry as

$$\Delta S_N^{U(1)} = -\sum_{q=0}^{N}p_q\log p_q \simeq \log N - \int_0^1 du\ p(u)\log p(u), \tag{S69}$$

where we used the fact that $\int_0^1 du\, p(u) = \sum_q p_q = 1$. It is worth noting that, in many interesting cases, $p(u)$ might show edge singularities: as a consequence $\int_0^1 du\ p(u)^n$ is not necessarily finite (that is, $p(u)\in L^1([0,1])$ but in general $p(u)\notin L^n([0,1])$ for $n>1$). In those cases, subleading logarithmic contributions can arise in the computation of the nth Rényi asymmetry, for $n>1$.

Here, we analyze two paradigmatic examples: the "kink" and Dicke states. Kink states have a flat probability distribution of the charge, and thus they have maximal asymmetry. Dicke states, on the other hand, are symmetric, but because of their long-range correlations, we show how it is possible to achieve maximal asymmetry $\Delta S_N^{U(1)} \simeq \log N$ by applying local rotations.

1. *Kink states*

We define a kink state as

$$|K\rangle := \frac{1}{\sqrt{N}}\sum_{j=1}^{N}|j\rangle, \quad |j\rangle = |0\dots 0\underset{j}{1}\dots 1\rangle, \tag{S70}$$

which is a uniform superposition of states with charges $j=1,\dots,N$. The charge probability distribution is uniform:

$$p_q = \langle K| P_q |K\rangle = \frac{1}{N}, \quad \forall\, q = 1,\dots,N. \tag{S71}$$

Hence, kink states have maximal asymmetry $\Delta S_N^{U(1)} = \log(N+1)$ at any finite size N. To make contact with the mechanism explained previously, we compute the continuous distribution $p(u)$ from the generating function of Q:

$$\langle K| e^{i\alpha Q} |K\rangle = \frac{1}{N}\sum_{j=1}^{N} e^{i\alpha(j-1)} = \frac{e^{i\alpha N}-1}{N(e^{i\alpha}-1)}. \tag{S72}$$

In the thermodynamic limit $N\to\infty$ with $\omega = \alpha N$ fixed, we obtain

$$\langle K| e^{i\alpha Q} |K\rangle = \frac{e^{i\omega}-1}{i\omega} + O\left(\frac{1}{N}\right). \tag{S73}$$

We then Fourier transform the function above (at leading order), obtaining a flat probability distribution of the charge density u

$$p(u) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega u}\cdot \frac{e^{i\omega}-1}{i\omega} = \chi_{[0,1]}(u). \tag{S74}$$

2. *Rotated Dicke states*

Finally, we show how Dicke states, which are entangled states with definite spin in the z-direction and thus have zero asymmetry with respect to the charge Q^z, can be mapped into maximally asymmetric states by acting on them with a one-layer circuit consisting of local unitaries. Namely, it is only required to rotate the state into a Dicke state in the x-direction, which -at large system size- has maximal asymmetry with respect to spin measurements in the z-direction. We provide two complementary proofs of this: first, we explicitly derive the discrete charge distribution at finite size and compute the large-volume limit of the asymmetry. Then, we show how the limiting distribution of the charge density in the large-volume limit is obtained from the generating function of the $U(1)$ charge.

Dicke states in the z-direction on an N-sites lattice are defined in the following way:

$$|D_k^z\rangle = \binom{N}{k}^{-1/2} \sum_{1\le j_1<\cdots<j_k\le N} \sigma_{j_1}^x\cdots\sigma_{j_k}^x |0\rangle^{\otimes N}, \quad k=0,\dots,N, \tag{S75}$$

that is, $|D_k^z\rangle$ is the normalized sum of all the distinct product states made with k qubits $|1\rangle$ and $N-k$ qubits $|0\rangle$. A Dicke state in the x-direction, denoted by $|D_k^x\rangle$, is obtained by rotating each qubit with a Hadamard matrix:

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}, \quad H|0\rangle = |+\rangle, \quad H|1\rangle = |-\rangle, \tag{S76}$$

where $|+\rangle$, $|-\rangle$ are the eigenstates of σ^x:

$$|\pm\rangle = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}, \quad \sigma^x|\pm\rangle = \pm|\pm\rangle. \tag{S77}$$

In [12] it was shown that each Dicke state in the x-direction is a linear combination of Dicke states in the z-direction:

$$|D_k^x\rangle = H^{\otimes N}|D_k^z\rangle = \frac{1}{2^{N/2}}\sum_{i=0}^{N}\sqrt{\binom{N}{i}\binom{N}{k}}K_i\left(k;\frac{1}{2},N\right)|D_i^z\rangle, \tag{S78}$$

where $K_i\left(k;\frac{1}{2},N\right)$ are symmetric Krawtchouk polynomials

$$K_i\left(k;\frac{1}{2},N\right) := {}_2F_1(-i,-k,-N;2) = \binom{N}{i}^{-1}\sum_{j=0}^{i}(-1)^n\binom{N-k}{i-j}\binom{k}{j}, \tag{S79}$$

that satisfy the orthogonality relation [13]:

$$\sum_{i=0}^{N}\binom{N}{i}K_i\left(k;\frac{1}{2},N\right)K_i\left(l;\frac{1}{2},N\right) = 2^N\delta_{k,l}\binom{N}{k}^{-1}. \tag{S80}$$

As a simple check of the above transformation law, one can for instance verify that (here $N=3$)

$$|D_1^x\rangle = \sqrt{\frac{3}{8}}\,|D_0^z\rangle + \sqrt{\frac{1}{8}}\,|D_1^z\rangle - \sqrt{\frac{1}{8}}\,|D_2^z\rangle - \sqrt{\frac{3}{8}}\,|D_3^z\rangle\,,$$

as obtained from a direct expansion of the states $|+ + -\rangle\,, |+ - +\rangle\,, |- + +\rangle$ in the $\{|0\rangle\,, |1\rangle\}$ basis.

We are interested in the probability $p_k(q;N)$ of obtaining a value $q=0,\dots,N$ when performing a measurement of the charge Q^z in the rotated Dicke state $|D_k^x\rangle$. Let us denote as usual by P_q the projector in the subspace of fixed charge q. Then $P_q\,|D_i^z\rangle = \delta_{q,i}\,|D_i^z\rangle$, and the probability $p_k(q;N)$ is given by:

$$\begin{aligned} p_k(q;N) = \langle D_k^x|P_q|D_k^x\rangle &= \frac{1}{2^N}\binom{N}{k}\sum_{i,j=0}^{N}\sqrt{\binom{N}{i}\binom{N}{j}}K_i\left(k;\frac{1}{2},N\right)K_j\left(k;\frac{1}{2},N\right)\langle D_i^z|P_q|D_j^z\rangle \\ &= \frac{1}{2^N}\binom{N}{k}\binom{N}{q}K_q\left(k;\frac{1}{2},N\right)K_q\left(k;\frac{1}{2},N\right), \end{aligned} \tag{S81}$$

where (S80) implies:

$$\sum_{q=0}^{N} p_k(q;N) = 1. \tag{S82}$$

Obtaining the asymptotic scaling of the asymmetry:

$$\Delta S_N^{U(1)} = -\sum_{q=0}^{N} p_k(q;N)\log p_k(q;N), \tag{S83}$$

at large N and for arbitrary values of k is a hard task to perform analytically, due to the fast oscillatory behavior of Krawtchouk polynomials, which results in asymptotic expansions of $K_q\left(k;\frac{1}{2},N\right)$ strongly dependent on the scaling of k with N [14, 15]. However, we observe numerically that, when k is fixed in the large-volume limit, $\Delta S \simeq \frac{1}{2}\log N$, whereas if the ratio k/N is fixed $\Delta S \simeq \log N$. Some simplifications occur in the case $k=N/2$, making the large-volume limit analytically tractable. Indeed, if $N=2M$, $k=M$:

$$p_M(q;2M) = \frac{1}{2^{2M}}\binom{2M}{M}\binom{2M}{q}^{-1}\left[\sum_{j=0}^{q}(-1)^j\binom{M}{q-j}\binom{M}{j}\right]^2 = \begin{cases} 0 & q\,\text{odd} \\ \frac{1}{2^{2M}}\binom{2M}{M}\binom{2M}{q}^{-1}\binom{M}{q/2}^2 & q\,\text{even} \end{cases}. \tag{S84}$$

Thus, in this case, the asymmetry reduces to

$$\Delta S_N^{U(1)} = -\sum_{q=0}^{M} p_M(2q;2M)\log p_M(2q;2M). \tag{S85}$$

By fixing $q=uM$, $u=0,1/M,2/M,\dots,1$, the Stirling approximation yields the large-N leading behavior:

$$p_M(2uM;2M) \simeq \frac{1}{\pi M\sqrt{u(1-u)}}, \tag{S86}$$

which implies

$$\Delta S_N^{U(1)} \simeq -M\int_0^1 \mathrm{d}u\, p_M(2uM;2M)\log p_M(2uM;2M) = \frac{1}{\pi}\int_0^1 \mathrm{d}u\,\frac{\log(\pi M\sqrt{u(1-u)})}{\sqrt{u(1-u)}} = \log N + \log\frac{\pi}{4}. \tag{S87}$$

The integral is convergent and it is easily performed via the change of variable $u=\sin^2\theta$. We numerically verified that the result above reproduces the entanglement asymmetry with an error $O(10^{-4})$ at $N=1000$.

We now follow a complementary approach and obtain the charge density $p(u)$ in the limit of large N by computing the Fourier transform of the charge generating function [16]:

$$\langle e^{i\alpha Q^x}\rangle := \langle D_k^z|\,e^{i\alpha Q^x}\,|D_k^z\rangle\,,\quad Q^x = \sum_j \frac{\sigma_j^x+1}{2}. \tag{S88}$$

In order to do so, we observe that the Dicke state in (S75) can be written as:

$$|D_k^z\rangle = \binom{N}{k}^{-1/2} (\sigma^-)^k |0\rangle^{\otimes N}, \quad \sigma^\pm = \sum_j \sigma_j^\pm := \sum_j \frac{\sigma_j^x \pm i\sigma_j^y}{2}. \tag{S89}$$

As before, we are interested in the limit where k/N is kept fixed in the thermodynamic limit, as it is in this regime that the state $|D_k^z\rangle$ displays long-range correlations.

We compute the generating function of Q^x as

$$\langle e^{i\alpha Q^x}\rangle \propto \oint \frac{d\bar{\zeta}'}{2\pi i\bar{\zeta}'} \oint \frac{d\zeta}{2\pi i\zeta} \zeta^{-k}\bar{\zeta}'^{-k} \langle\Uparrow| \exp(\bar{\zeta}'\sigma^+) \exp(i\alpha Q) \exp(\zeta\sigma^-) |\Uparrow\rangle, \tag{S90}$$

where $|\Uparrow\rangle := |0\rangle^{\otimes L}$ and one can easily check that the proportionality constant in the right-hand side is $N!\,k!\,(N-k)!$. In addition, a simple calculation shows

$$\langle\Uparrow| \exp(\bar{\zeta}'\sigma^+) \exp(i\alpha Q^x) \exp(\zeta\sigma^-) |\Uparrow\rangle = \left(\frac{e^{i\alpha}+1}{2}(1+\bar{\zeta}'\zeta) + \frac{e^{i\alpha}-1}{2}(\bar{\zeta}'+\zeta)\right)^N. \tag{S91}$$

Since the right-hand side of the above equation is a polynomial in $\zeta, \bar{\zeta}'$, the computation of (S90) boils down to the identification of the coefficient of $\zeta^k\bar{\zeta}'^k$. Similarly, $\langle D_k^z| e^{i\alpha Q^x} |D_z^k\rangle$ is a polynomial in $e^{i\alpha}$ and $p_k(q;N)$ is identified by the coefficient of $e^{i\alpha q}$.

We calculate (S90) in the thermodynamic limit $N\to\infty$, with αN fixed. In this limit, one employs saddle-point techniques (as in Ref. [17]): in particular, for $k/N = 1/2$ the integral localizes at $\zeta' = \zeta$ and $|\zeta| = 1$, yielding

$$\langle D_k^z| e^{i\alpha Q^x} |D_z^k\rangle \simeq \oint_{|\zeta|=1} \frac{d\zeta}{2\pi i\zeta} e^{i\alpha N(1-\mathrm{Re}(\zeta))/2} = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\alpha N(1-\cos\theta)/2} = e^{i\alpha N/2} J_0(\alpha N/2). \tag{S92}$$

Thus, the probability distribution of the charge density, obtained as the Fourier transform of the generating function, is

$$p(u) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\omega(u(\theta)-u)} = \int_0^{2\pi} \frac{d\theta}{2\pi} \delta(u-u(\theta)) = \frac{1}{\pi\sqrt{u(1-u)}}, \tag{S93}$$

where $u(\theta) = (1-\cos\theta)/2$. As expected, $p(u)/M = p_M(2uM; 2M)$, with $M = N/2$ and $p_M(2uM; 2M)$ given in (S86).

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