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OPTION PRICING GENERATORS

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Abstract. We characterize a class of option pricing models by their algebraic structure. Option prices are monoids, that is operators endowed with the commutativity and associativity property and an identity element. If the price of the underlying asset is bounded, the operator corresponds to the concept of t-conorm, while if it is defined on the positive real line the operator is a pseudo-addition. These operators have the same no-arbitrage properties as the classical option pricing models, but are also associative. Each model in this class is characterized by a univariate increasing function that is defined the *generator* of the model. The generator encodes a synthetic representation of the probability structure of the underlying asset. We provide no arbitrage conditions for the generators and practical guidelines to construct them.

1. **Introduction.** In a recent paper Carr and Torricelli (2021) propose a new strategy for the evaluation of options. The focus is shifted on a product paying $\max(S_T, K)$, where S_T is the price of the underlying asset at maturity time T and K is the strike price. The product is called a "married put", because it can be attained by holding the underlying asset and a put option, or, by put-call parity, holding a call option and a bond with face value equal to the strike.

The goal is to define a function

$$E_Q(\max(S_0, K)|\mathcal{F}_\tau) = m_\tau(S_\tau, K) \tag{1}$$

where $E_Q(.,|\mathcal{F}_\tau)$ denotes the conditional expectation given the information available τ periods before maturity under the risk neutral measure Q. Note that our notation substitutes the standard setting $E_Q(\max(S_T, K)|\mathcal{F}_t)$. Then, by put-call parity the function $m_\tau(S_\tau, K)$ can be used to price a call option, setting $f(S_\tau, \tau; K) = m_\tau(S_\tau, K) - B_\tau K$, where B_τ is the risk-free discount factor for term τ. The put option is obtained computing $p(S_\tau.\tau; K) = m_\tau(S_\tau, K) - S_\tau D_\tau$, where D_τ is the discounted value of the dividend yield.

Clearly, the price of the married put must abide by specific conditions, the most important being the requirement that at maturity $\tau = 0$ it must be worth $\max(S_0, K)$. Other requirements were first introduced by Merton (1973) who gave necessary no arbitrage conditions in his theory of rational option pricing. Necessary

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and sufficient conditions were analyzed in Carr and Madan (2005) and Davis and Hobson (2007).

Here we take an algebraic approach to analyze the features that may be possessed by the pricing function $m_\tau(S, K)$. Starting with the standard Black and Scholes model with zero interest rates and dividend yield, one can show that the price of the married put is constrained by

$$\max(S, K) \leq m_\tau(S, K) \leq S + K$$

In this case the bounds are standard aggregation operators, that is the standard sum and the $\max(S, K)$ operator. One is tempted to show under which conditions the functions $m_\tau(S, K)$ can be considered *pseudo-additions*. A pseudo-addition denotes a general aggregation operator which is increasing in both the arguments and endowed with the commutative and associative properties. Together with the definition of an identity element, this denotes a monoid. Formally, a monoid is defined as $(\mathcal{S}, \oplus)$, where $\mathcal{S}$ denotes a set and $\oplus$ an operation, with an element $e \in \mathcal{S}$ such that $x \oplus e = x, \forall x \in \mathcal{S}$.

One can easily show that $m_\tau(x, y)$ in the Black and Scholes formula is not a pseudo-addition because it fails to satisfy the associative property. Other models may instead qualify for a pseudo-addition representation. One is the Dagum option pricing model proposed by Carr and Torricelli (2021)

$$m_\tau(S, K) = \left(S^{1/b_\tau} + K^{1/b_\tau}\right)^{b_\tau}$$

with b_τ strictly increasing with limits 0 and 1. It is easy to check that this option pricing formula has the same properties as the one in the Black and Scholes model, but is also endowed with the associativity property.

Here we propose a general representation for associative option pricing models. The associativity property allows for a synthetic and elegant characterization of each pricing model by an increasing univariate function that is called the *generator* of the model. This generator completely encodes the probability structure of the model.

The structure of the paper is as follows. In section 2 we introduce the topic by analyzing the algebraic structure of the married put price in the Black and Scholes model with zero dividends and interest rates. In section 3 we analyze the option pricing problem in a setting with bounded price of the underlying asset, using t-conorm operators. In section 4 we extend the analysis to underlying assets with prices unbounded from above. Section 5 shows how the generator characterizes the implied probability of the underlying asset. Section 6 gives guidelines and examples about how to build no-arbitrage option pricing generators. In Section 7 we report examples of smile functions obtained by several generators. Section 8 concludes.

2. **Getting started: Black and Scholes.** Here we start our analysis with the most standard model, assuming no dividend payments and zero risk-free interest rates. Dropping these assumptions would not materially affect our analysis. In this setting, the price of the married put option in the Black and Scholes model is

$$m_\tau(S, K) = S\Phi(d_\tau + 0.5\sigma\sqrt{\tau}) + K\Phi(-d_\tau + 0.5\sigma\sqrt{\tau})$$

where S is the price of the underlying asset, K is the strike, τ the time to maturity, σ is volatility, $\Phi(x)$ denotes the standard normal distribution and

$$d_\tau(S/K) = \frac{\log(S/K)}{\sigma\sqrt{\tau}}$$

We then restrict our analysis to the algebraic properties of the function

$$\mu_\tau(x,y) = x\Phi(d_\tau(x/y) + 0.5\sigma\sqrt{\tau}) + y\Phi(-d_\tau(x/y) + 0.5\sigma\sqrt{\tau})$$

It is easy to prove that:

Proposition 2.1. *The married put price $m_\tau(S,K)$ in the Black and Scholes model is endowed with the following properties:*

1. *commutative: $m_\tau(x,y) = m_\tau(y,x)$*
2. *increasing in each place: $x' > x, y' > y$ implies $m_\tau(x',y') \geq m_\tau(x,y)$*
3. *0 is the identity: $m_\tau(x,0) = x$*
4. *limit at $\tau = 0$: $\lim_{\tau\to 0} m_\tau(x,y) = \max(x,y)$*
5. *limit at $\tau = \infty$: $\lim_{\tau\to\infty} m_\tau(x,y) = x + y$*

Proof. We have:

1. From $d_\tau(x/y) = -d_\tau(y/x)$:
$$\begin{aligned} m_\tau(x,y) &= x\Phi(d_\tau(x/y) + 0.5\sigma\sqrt{\tau}) + y\Phi(-d_\tau(x/y) + 0.5\sigma\sqrt{\tau}) \\ &= y\Phi(d_\tau(y/x) + 0.5\sigma\sqrt{\tau}) + x\Phi(-d_\tau(y/x) + 0.5\sigma\sqrt{\tau}) \\ &= m_\tau(y,x). \end{aligned} \tag{2}$$
2. Trivial.
3. From $\lim_{y\to 0} d_\tau(x/y) = \infty$ we get $\mu_\tau = x\Phi(\infty) + y\Phi(-\infty) = x$, the rest follows from the commutativity property.
4. $\lim_{\tau\to 0} d_\tau(x/y) = \infty$ if $x > y$, implying $m_\tau(x,y) = x$, while $x < y$ leads to $\lim_{\tau\to 0} d_\tau(x/y) = -\infty$, and $m_\tau(x,y) = y$.
5. Since $\lim_{\tau\to\infty} d_\tau(x/y) = \lim_{\tau\to\infty} -d_\tau(x/y) = 0$ we have $x\Phi(\infty) + y\Phi(\infty) = x + y$.

□

Some of the properties above are recognized to be the conditions for rational option pricing pointed out in Merton (1973). From $f(S,\tau;K) = m_\tau(S,K) - K$ condition (3) yields: i) $m_\tau(S,0) = S = f(S,\tau;0)$; ii) $f(0,\tau;K) = m_\tau(0,K) - K = 0$. Moreover, condition (5) yields Theorem 3 in Merton (1973): the value of a perpetual warrant is $f(S,\infty;K) = m_\infty(S,K) - K = S + K - K = S$. However, it is not difficult to find examples in which the perpetual warrant condition is not required. Instead, the condition (4) is one of the no arbitrage conditions imposed by Carr and Madan (2005): when the time to maturity tends to zero the price must converge to the payoff. For call options we have $f(S,0;K) = \lim_{\tau\to 0} m_\tau(S,K) - K = \max(S,K) - K = \max(S-K,0)$. For put options, $p(S,0;K) = \lim_{\tau\to 0} m_\tau(S,K) - S = \max(S,K) - S = \max(K-S,0)$.

Carr and Madan (2005) and Davis and Hobson (2007) stated sufficient conditions for no arbitrage. Namely, a pricing formula for a call (put) option is arbitrage free if and only if: i) it is convex and decreasing (increasing) in the strike price with derivative uniformly bounded below by -1 (above by $+1$) (no horizontal arbitrage); ii) increasing with maturity (no vertical arbitrage); iii) the limit of the maturity tending to 0 is the payoff. We collect all the no arbitrage requirements in the proposition below:

Proposition 2.2. *In an economy with no dividends and zero interest rates, in order to avoid arbitrage opportunities the function $m_\tau(S,K)$ must satisfy the conditions:*

1. *grounded call and put prices:* $m_\tau(S,0) = S$,$m_\tau(0,K) = K$*;*
2. *terminal condition:* $\lim_{\tau\to 0} m_\tau(S,K) = \max(S,K)$*;*
3. *avoiding horizontal arbitrage:*
$$0 \leq \frac{\partial m_\tau(S,K)}{\partial K} \leq 1; \tag{3}$$
4. *avoiding vertical arbitrage:*
$$0 \leq \frac{\partial m_\tau(S,K)}{\partial \tau}. \tag{4}$$

Proof. Condition 1) makes sure that the price of put options be zero when the strike is zero: $m_\tau(S,0)-S=0$ and call options must be zero when the underlying is worth zero: $m_\tau(0,K)-K=0$. Condition 2) is evident. As for the horizontal arbitrage condition, the price of put option must be increasing in the strike

$$\frac{\partial p(S,\tau;K)}{\partial K} = \frac{\partial (m_\tau(S,K)-S)}{\partial K} \geq 0$$

and the price of a call option must be decreasing in the strike

$$\frac{\partial f(S,\tau;K)}{\partial K} = \frac{\partial (m_\tau(S,K)-K)}{\partial K} = \frac{\partial m_\tau(S,K)}{\partial K} - 1 \leq 0$$

The case of vertical arbitrage avoids calendar arbitrage opportunities. □

3. **Option pricing generators: Bounded underlying.** We now extend the analysis above to cases in which the hypothesis of the Black and Scholes model about the dynamics of the underlying asset cannot be maintained. A straightforward example is when the price of the underlying asset is bounded both from below and above. This is obviously the case when the underlying asset is a bond. However, the hypothesis that the price is bounded has been also used in general asset pricing models: it was made by Long (1990) in the analysis of the numeraire portfolio and by Carr and Yu (2012) in their proof of the Ross recovery theorem.

If the price of the underlying asset is bounded, we can rescale it without loss of generality to the unit interval. Keeping in mind Proposition 2.1 it is natural to revisit the definition of t-conorm. The concepts of t-norm and t-conorm were first proposed by Menger (1942) and developed by Schweitzer and Sklar (1961) in their studies of statistical and probability metric spaces. In finance, this concept has been typically used in the literature on non-additive decomposable capacities (Chateauneuf, 1996, Cherubini, 1997, Cherubini and Mulinacci, 2020). We report here the definition:

Definition 3.1. A function $\perp$: $[0,1]\times[0,1] \to [0,1]$ is called a *t-conorm* (or *s-norm*) if

1. $\perp(a,b) = \perp(b,a)$ for all $a,b \in [0,1]$,
2. it is non-decreasing in each argument,
3. $\perp(a,0) = a$ for all $a \in [0,1]$,
4. $\perp(\perp(a,b),c) = \perp(a,\perp(b,c))$ for all $a,b,c \in [0,1]$.

Moreover, if $\perp(a,a) > a$ for all $a \in (0,1)$ then the t-conorm is called *Archimedean* and if $\perp(.,.)$ is strictly increasing in $(0,1)\times(0,1)$ the t-conorm is called *strict*.

Note that the points (1) through (3) of the definition are the same as in Proposition 2.1. Condition (4) is the associative property, and it is not a property satisfied by the Black and Scholes algebra. It is then natural to merge them with Proposition 2.2 to provide an axiomatic characterization of a no-arbitrage option pricing function.

Proposition 3.2. *Assume an underlying asset price* $S \in [0,1]$. *Take a function* $m_\tau(x,y)$ *endowed with the following properties*

1. $m_\tau(x,y)$ *is an Archimedean* t*-conorm,*
2. *terminal condition at* $\tau = 0$*:* $\lim_{\tau \to 0} m_\tau(x,y) = \max(x,y)$
3. *no horizontal arbitrage:*
$$0 \leq \frac{\partial m_\tau(x,y)}{\partial y} \leq 1$$
4. *no vertical arbitrage:*
$$0 \leq \frac{\partial m_\tau(x,y)}{\partial \tau}$$

Then, $m_\tau(S_\tau, K)$ *is the no-arbitrage price of a married put option.*

While all the conditions have been already discussed, it is important to stress the Archimedean requirement. This is a technical requirement that is quite natural to impose in an option pricing application, since it amounts to assume that at-the-money call and put option values must be positive. In fact, we have

$$f(S,\tau;S) = m_\tau(S,S) - S = p(S,\tau;S) > 0$$

for $\tau > 0$ and $S \in (0,1)$. It is clear that if the Archimedean property fails S_τ has to be a constant.

The Archimedean property, together with the associative property that we included using the t-conorm definition, brings an important feature into the picture: the option pricing generator.

Theorem 3.3. *A function* $\perp\colon [0,1] \times [0,1] \to [0,1]$ *is a continuous Archimedean* t*-conorm if and only if there exists a strictly increasing function* $\psi : [0,1] \to [0,+\infty]$ *with* $\psi(0) = 0$ *such that*

$$a \perp b = \psi^{[-1]}(\psi(a) + \psi(b))$$

where $\psi^{[-1]}$ *is the pseudo-inverse of* ψ *defined as*

$$\psi^{[-1]}(x) = \begin{cases} \psi^{-1}(x), & \text{if } x \leq \psi(1) \\ 1, & \text{if } x > \psi(1) \end{cases}$$

Moreover, ψ *is called the generator of* $\perp$ *and* $\perp$ *is* strict *if and only if* $\psi(1) = +\infty$.

This result is very well known, and was proved for t-norms and t-conorms by Schweizer and Sklar (1961) and Ling (1965). In this model, all the information content is contained in a univariate increasing function that we call the *option pricing generator*. The generator allows then a characterization of the option pricing function.

Theorem 3.4. *The price of a married put option on a bounded underlying* $S \in [0,1]$ *satisfying Proposition 3.2 may be written as:*

$$m_\tau(S,K) = \psi_\tau^{[-1]}(\psi_\tau(S) + \psi_\tau(K))$$

where $\psi_\tau(\cdot)$ *is called the* generator *of the option pricing model.*

4. **Option pricing generators: Underlying unbounded from above.** We now extend the analysis to the case in which the price of the underlying asset is unbounded from above. Monoids of increasing, commutative and associative operators defined on the positive real line are called *pseudo-additions*. The extension was first proposed by Sugeno and Murofushi (1987). We start with the definition of pseudo-addition:

Definition 4.1. Given a set $\mathcal{S} \subseteq [0, \infty]$ a *pseudo-addition* is an operator $\oplus : \mathcal{S}\ X\ \mathcal{S} \to \mathcal{S}$ which:

1. is commutative: $x \oplus y = y \oplus x$
2. is non decreasing in each argument: $x' \geq x, y' \geq y \to x' \oplus y' \geq x \oplus y$
3. has 0 as identity item: $0 \oplus x = x \oplus 0 = x$
4. is associative: $x \oplus (y \oplus z) = (x \oplus y) \oplus z$
5. satisfies continuity: $x_n \to x, y_n \to y$ implies $x_n \oplus y_n \to x \oplus y$.

We are clearly interested in the case $\mathcal{S} = [0, \infty]$. The strategy to get there is to first partition the set $[0, \infty]$ into subsets, and on these to define sub-monoids with respect to $([0, \infty], \oplus)$. Then, given further requirements on the operation $\oplus$ we will look for a pseudo-addition defined on the whole monoid. We start with the formal definition of these sub-monoids in Sugeno and Murofushi (1987).

Definition 4.2. Let $\{(\alpha_k, \beta_k) : k \in \mathcal{K}\}$ be a family of disjoint open intervals in $[0, \infty]$ indexed by a countable set $\mathcal{K}$. For each $k \in \mathcal{K}$, associate a continuous and strictly increasing function

$$g_k : [\alpha_k, \beta_k] \to [0, \infty].$$

We say that a binary operation $\oplus$ has a representation

$$\{\langle(\alpha_k, \beta_k), g_k\rangle : k \in \mathcal{K}\}$$

if and only if

$$x \oplus y = \begin{cases} g_k^{[-1]}(g_k(x) + g_k(y)), & \text{if } (x, y) \in [\alpha_k, \beta_k]^2 \\ \max(x, y), & \text{otherwise} \end{cases}$$

where $g_k^{[-1]}$ is the pseudo-inverse of g_k defined by

$$g_k^{[-1]}(x) = g^{-1}(\min(x, g_k(\beta_k)))$$

So, a representation of pseudo-addition can be obtained for every sub-monoid, along the same lines followed for the bounded case.

Theorem 4.3. *A binary operation is a pseudo-addition if and only if it has a representation* $\{\langle(\alpha_k, \beta_k), g_k\rangle : k \in \mathcal{K}\}$.

This is all that can be obtained if we stick to the conditions in definition 4.1. In the general case, then, the best that can be done is to get a set of generators g_k. The representation is largely simplified if we add the Archimedean property. On this we follow Mesiar and Ribáric (1993).

Definition 4.4. A pseudo-addition on $\mathcal{S} = [0, \infty]$ satisfies the conditions

1. commutativity: $x \oplus y = y \oplus x$
2. non decreasing in each argument: $x' \geq x, y' \geq y \to x' \oplus y' \geq x \oplus y$
3. 0 as identity item: $0 \oplus x = x \oplus 0 = x$
4. associativity: $x \oplus (y \oplus z) = (x \oplus y) \oplus z$

5. continuity: $x_n \to x, y_n \to y$ implies $x_n \oplus y_n \to x \oplus y$
6. Archimedean property: $x \oplus x > x, \forall x \in (0, \infty)$
7. finite sum of finite elements: if $x < \infty$ and $y < \infty$ then $x \oplus y < \infty$.

We can now prove the result that we are going to use.

Theorem 4.5. *A binary operation* $\oplus$ *on* $[0, \infty]$ *is a pseudo-addition if and only if there exists a continuous and strictly increasing function*

$$g : [0, \infty] \to [0, \infty]. \quad g(0) = 0 \quad g(\infty) = \infty$$

such that

$$x \oplus y = g^{-1}(g(x) + g(y)) \quad \textit{for every} \quad x, y \in [0, \infty]$$

Proof. The proof is in Mesiar and Ribáric (1993), but we give here a sketch of it because it clarifies what is going on.

1. By theorem 4.3 the solution must have the $\{\langle(\alpha_k, \beta_k), g_k\rangle : k \in \mathcal{K}\}$ representation. Then, we must study the set $\mathcal{K}$. If $\mathcal{K} = \emptyset$, we have $x \oplus y = \max(x, y), \forall x, y$, that violates the Archimedean property. Moreover, the Archimedean property is also violated at all points α_k, β_k except $\alpha_1 = 0$ and $\beta_1 = \infty$. Then, the set $\mathcal{K}$ is a singleton identifying a single function: $\{\langle(0, \infty), g\rangle\}$. The function is of course unique up to multiplication by a positive constant.
2. By contradiction we prove that it cannot be: $g(\infty) = M < \infty$. If this is the case one may in fact find u such that $2g(u) > M$. Then, $u \oplus u = g^{[-1]}(2g(u)) = \infty$ violates condition (7) in Definition 4.4 which would require instead $u \oplus u < \infty$.
3. The direct part can be easily proved verifying that the function $x \oplus y$ satisfies all the conditions for a pseudo-addition.

□

Note that the Archimedean property allows to identify a single monoid $([0, \infty], \oplus)$ associated to a single function g, unique up to multiplication by a positive constant. Without the Archimedean property, then, we would remain with an arbitrary set of sub-monoids. We remind that this is not at all a restriction in our application, since the Archimedean property is a requirement to ensure positive at-the-money option prices. We also note a difference with respect to the case in which the price of the underlying is bounded. The pricing function is strictly increasing and the generator is always strict.

We are now in a position to characterize the set of option prices.

Proposition 4.6. *Assume an underlying asset price* $S \in [0, \infty]$*. Take a function* $m_\tau(x, y)$ *endowed with the following properties:*

1. $m_\tau(x, y)$ *is an Archimedean pseudo-addition;*
2. *boundary condition at* $\tau = 0$*:* $\lim_{\tau \to 0} m_\tau(x, y) = \max(x, y)$
3. *no horizontal arbitrage:*
$$0 \leq \frac{\partial m_\tau(x, y)}{\partial y} \leq 1;$$
4. *no vertical arbitrage:*
$$0 \leq \frac{\partial m_\tau(x, y)}{\partial \tau}.$$

Then, $m_\tau(S_\tau, K)$ *is the no arbitrage price of a married put.*

Finally, we can provide a general characterization for married put prices in this set of models in terms of generators:

Theorem 4.7. *The price of a married put option on an unbounded underlying S in $\in [0,\infty]$ satisfying Proposition 4.6 is given by:*

$$\mu_\tau(S,K) = g_\tau^{-1}(g_\tau(S) + g_\tau(K))$$

where $g_\tau(\cdot)$ is called the generator *of the option pricing model.*

5. **Generators and implied probabilities.** If option prices can be uniquely represented by generators, they encode all the information about the model, that is the probability distribution of the underlying asset. This can be easily extracted using to the seminal paper by Breeden and Litzenberger (1978).

As it is well known, the probability distribution implied by a set of no-arbitrage prices of options can be recovered as the limit of call and put spreads, corresponding to the derivative of the option price with respect to the strike price. Coherently, the density is recovered as the limit of butterfly spreads, that is the second derivative of option prices. In our setting, in which prices are completely determined by generators, the probability distribution is also completely determined by them. Moreover, writing the distribution in terms of generators would allow us to identify the requirements needed for a monoid to be a consistent arbitrage free option price and for the corresponding generator to be eligible as an option pricing generator.

Formally, let us assume that the price of a married put could be written as

$$m_\tau(S_\tau, K) = g_\tau^{-1}(g_\tau(S_\tau) + g_\tau(K))$$

Then, the cumulative probability distribution (CPD) of the underlying asset S at maturity can be recovered from

$$Q_\tau(S_0 \le K) = \frac{\partial m_\tau(S_\tau, K)}{\partial K} = \frac{g'_\tau(K)}{g'_\tau(m_\tau(S_\tau, K))}$$

where $g'_\tau(\cdot)$ denotes the first derivative of $g_\tau(\cdot)$. If we assume that the generator is twice differentiable, the density is given by the second derivative of $m_\tau(S_\tau, K)$ and we compute:

$$\begin{aligned}
\frac{\partial^2 m_\tau(S_\tau, K)}{\partial K^2} &= \frac{g''(K)}{g'_\tau(m_\tau(S_\tau, K))} - (g'_\tau(K))^2 \frac{g''_\tau(m_\tau(S_\tau, K))}{g'_\tau(m_\tau(S_\tau, K))^3} \\
&= \frac{g'_\tau(K)}{g'_\tau(m_\tau(S_\tau, K))} \left(\frac{g''_\tau(K)}{g'_\tau(K)} - \frac{g'_\tau(K)}{g'_\tau(m_\tau(S_\tau, K))} \frac{g''_\tau(m_\tau(S_\tau, K))}{g'_\tau(m_\tau(S_\tau, K))} \right) \\
&= Q_\tau(S_0 \le K) \left(\frac{g''_\tau(K)}{g'_\tau(K)} - Q_\tau(S_0 \le K) \frac{g''_\tau(m_\tau(S_\tau, K))}{g'_\tau(m_\tau(S_\tau, K))} \right)
\end{aligned}$$

where $g''_\tau(\cdot)$ denotes the second derivatives of $g_\tau(\cdot)$.

Having characterized the implied probability distribution of the underlying asset we can gather all the results to ensure that the generator used in the pricing model satisfies the no-arbitrage requirement. The conditions are reported in the theorem below.

Theorem 5.1. *A function $g_\tau(x)$ satisfies the no arbitrage conditions in Theorem 8.1 if the following conditions hold:*

1. *$g_\tau(x)$ is a smooth pseudo-addition generator;*

2. *for any* $x < y$

$$\lim_{\tau \to 0} \frac{g_\tau(x)}{g_\tau(y)} = 0 \tag{5}$$

3. *the function* $1/g'_\tau(x)$ *is convex*
4. *for* $\tau_1 > \tau_2$ *we have* $g_{\tau_1}(x) > g_{\tau_2}(x)$

The proof is reported in Appendix. It is quite intuitive that condition 2) restricts the choice of generators to make sure that the married put option converges to its payoff at maturity. Moreover, the Breeden and Litzenberger approach suggests that the smoothness requirement in condition 1) and the convexity condition 3) have to do with implied probability. Finally, condition 4) is clearly linked to the vertical no arbitrage condition.

6. **How to build generators.** Here we give practical guidelines and examples of option pricing generators and models based on them. We have already made clear that not all increasing functions are eligible as option pricing generators. For example, $\mathcal{G}(x) = \log(x)$ is a generator of the monoid $([0, \infty], x \times y)$. Of course, this cannot work as a generator for a pseudo-sum. In fact, the range of the function is the whole real line instead of the positive part of it (see Theorem (4.5)). The question is whether and how this generator of a product can be turned into the generator of a sum. A first suggestion comes from the observation that the identity element of the product is $e = 1$ and that $\log(x + 1)$ qualifies as the generator of a pseudo-sum. Moreover, it is clear that if a function qualifies as a generator of a sum, its exponentiation to a proper exponent may maintain that property.

Here we give guidelines to transform a generator into an option pricing generator. In practice, we introduce two operations on the generator to satisfy the no arbitrage requirements:

1. shift the identity element of a generator: given a generator $\mathcal{G}(x)$ with $\mathcal{G}(e) = 0$ we define a generator $g(x)$ with $g(0) = 0$
2. power of a generator: given a generator $\mathcal{G}(x)$ we define $g(x) = \mathcal{G}(x)^{1/\theta}$

We start defining the identity-shifting operation:

Proposition 6.1. *Given an Archimedean monoid* $(\mathcal{S}, \oplus)$, $\mathcal{S} \subseteq [0, \infty]$ *with generator* $\mathcal{G}(x)$, *and identity element* e, *the function* $g(x) = \mathcal{G}(x + e)$ *generates a monoid with identity element* 0 *and operation* $\oplus$ *defined as*

$$x \oplus y = \mathcal{G}^{-1}(\mathcal{G}(x + e) + \mathcal{G}(y + e)) - e \tag{6}$$

Proof. If $y = 0$, from $\mathcal{G}(e) = 0$ we have $\mathcal{G}^{-1}(\mathcal{G}(x + e)) - e = x$. □

We now address the problem of exponentiation of the generator to make it consistent with the boundary condition $\max(S, K)$.

Proposition 6.2. *Assume an Archimedean monoid* $(\mathcal{S}, \oplus)$, $\mathcal{S} \subseteq [0, \infty]$ *with generator* $\mathcal{G}(x)$, *and identity element* e, *and a function* $1/b_\tau$, $b_\tau \in (0, 1]$, *strictly increasing in* τ *and such that* $\lim_{\tau \to 0} b_\tau = 0$. *Then*

$$m_\tau = \mathcal{G}^{-1}\left(\mathcal{G}(S + e)^{1/b_\tau} + \mathcal{G}(K + e)^{1/b_\tau}\right)^{b_\tau} - e \tag{7}$$

is the no-arbitrage price of a European married put option.

Proof. First, using proposition 6.1 it is clear that the identity element of the monoid is 0. Then, we can assume $e = 0$ without loss of generality and prove:

$$\lim_{b_\tau \to 0} m_\tau(S, K) = \max(S, K)$$

From

$$\lim_{b_\tau \to 0} \left(\mathcal{G}(S)^{1/b_\tau} + \mathcal{G}(K)^{1/b_\tau} \right)^{b_\tau} = \max(\mathcal{G}(S), \mathcal{G}(K))$$

it follows

$$\begin{aligned} \lim_{b_\tau \to 0} m_\tau(S, K) &= \lim_{b_\tau \to 0} \mathcal{G}^{-1} \left(\mathcal{G}(S)^{1/b_\tau} + \mathcal{G}(K)^{1/b_\tau} \right)^{b_\tau} \\ &= \mathcal{G}^{-1}(\max(\mathcal{G}(S), \mathcal{G}(K)) \\ &= \max(S, K) \end{aligned}$$

□

Note that the case $\mathcal{G}(x) = x$ gives the conjugate Dagum model used in Carr and Torricelli (2021).

A technical point that needs to be addressed is when the exponentiation is applied to a generator that may take values in the negative region. This is hardly the case in standard option pricing applications, when the price is defined over positive values and the generator is shifted so that it cannot produce negative values (setting $\mathcal{G}(x + e)$ in place of $\mathcal{G}(x)$). Nevertheless, the problem may emerge in option pricing applications in which the underlying can take negative values. Examples are options on interest rates, that in the European market have taken on negative values for quite a long time, or derivatives on correlation, that can take values in $[-1, 1]$. And moreover, the problem of well-defined exponentiation of a monoid generator is an interesting and challenging task in itself. If the generator $g(x)$ has a symmetric structure for positive and negative values, a natural way to perform this extension would be to make sure that the exponentiation distortion would preserve such symmetry. We would call this operation: *flipping* of the generator. The operation would consist in pasting the value on the positive axis on the corresponding value of the negative part.

Of course, it is clear that in order to define the flipping operation we should first define how to map any point of the positive range of the generator on the corresponding points in the negative region. This we do in the following definition of symmetry of a generator.

Definition 6.3. A generator $\mathcal{G}(x)$, $x \in \mathcal{S}$ is symmetric if:

1. there exists $e \in \mathcal{S}$ such that $\mathcal{G}(e) = 0$;
2. there exists a function $f(x)$ such that: i) $f(e) = e$; ii) $\mathcal{G}(f(x)) = -\mathcal{G}(x)$.

Based on this we can then extend the exponentiation operation over negative values as follows:

Definition 6.4. Given a generator $\mathcal{G}(x)$, $\mathcal{G}(e) = 0$, $x, e \in \mathcal{S}$, symmetric around e with function $f(x)$ we define the *Power Generator* $\mathcal{G}(x)^{[1/b]}$as

$$\mathcal{G}(x)^{[1/b]} = \begin{cases} \mathcal{G}(x)^{1/b}, & x \geq e \\ -\left(\mathcal{G}(f(x))^{1/b_\tau}\right), & x < e \end{cases} \tag{8}$$

6.1. **Power of Lamperti transforms.** As a first example of how to build an option pricing generator, we resort to *Lamperti transform*. We first remind that the Lamperti transform is the operator that applied to a Ito process reduces it to a unit variance arithmetic Brownian motion.

Definition 6.5. Given a Ito process $(x_t)_t$ with diffusion term given by the function $a(x_t)$ the Lamperti transform is defined as

$$\mathcal{G}(x) = \int^x \frac{1}{a(u)} du \tag{9}$$

so that the process $\mathcal{G}(x_t)$ is an Ito process such that

$$d\mathcal{G}(x_t) = \textit{drift} + dW_t$$

where W_t is a Wiener process.

Note that the transformation changes the drift of the Ito process, but this is not the focus of our analysis. Our goal here is to use the transform $\mathcal{G}(x)$ as an option pricing generator, defining $g(x) = \mathcal{G}(x+e)^{1/b_\tau}$, $b_\tau \in (0,1]$. Since Lamperti transforms yield a Wiener process under a suitable change of measure, they are clearly symmetric around a suitable function $f(x)$, as the following examples show.

Example 6.6. Assume $a(x) = x$, $x \in (0,\infty)$ so that the Ito process is a geometric Brownian motion. It is very well known that in this case the Lamperti transform is $\mathcal{G}(x) = \log(x)$. We then have $e = 1$ and $f(x)$ is given by

$$f(x) = \mathcal{G}^{-1}(-\mathcal{G}(x)) = \exp(-\log(x)) = \frac{1}{x}$$

Example 6.7. Assume $a(x) = x(1-x)$. Then, the Lamperti transform is

$$\mathcal{G}(x) = \log\left(\frac{x}{1-x}\right)$$

and it is clear that $e = 1/2$. As for the function $f(x)$, we compute

$$\mathcal{G}^{-1}(y) = \frac{1}{1+\exp(-y)}$$

and

$$f(x) = \mathcal{G}^{-1}(-\mathcal{G}(x)) = \frac{1}{1+\exp(\log(\frac{x}{1-x}))} = 1-x$$

6.2. **Archimedean copula generators.** Generators are used in copula theory in a set of models called Archimedean copula functions (see Nelsen, 2006 and Cherubini et al. 2004 for applications in finance). The structure of this class of copulas is in fact given by

$$C(x,y) = \phi^{-1}(\phi(x) + \phi(y)) \tag{10}$$

with $x, y \in [0,1]$. It is well known that x, y represent uniform random variables and the function $C(x,y)$ is the corresponding joint distribution. The function $\phi(\cdot)$ is called the generator of the copula function. It is a decreasing convex function with $\phi(0) = 1$ and $\phi(\infty) = 0$. Clearly, from an algebraic point of view, an Archimedean copula function is a commutative and associative operation with identity element $e = 1$. We may then wonder whether these functions could be used as option pricing generators. We first observe that the identity element e should be changed from 1 to 0. The other problem is that the maximum copula function is $\min(x,y)$ while we are interested in a sequence of functions reaching $\max(x,y)$. For underlying bounded in

$[0,1]$ there is a well known result that allows to achieve a suitable generator. This is known as Generalized De Morgan Law, that gives a link between t-norms (of which Archimedean copula functions are a special case) and t-conorms, that are option pricing generators as discussed in section 3.

Theorem 6.8. *Given a t-norm $t(x,y), x, y \in [0,1]$ the function*

$$x \oplus y = 1 - t(1-x, 1-y) \tag{11}$$

is a t-conorm.

We remind that t-norms are defined just like t-conorms in definition 3.1, with the difference that the identity element is 1 instead of 0. Moreover, it can be checked that $\min(x,y)$ is a t-norm, and the dual t-conorm is $\max(x,y)$. So, the Generalized De Morgan Law delivers all the changes needed to transform a t-norm, and a copula function, into an option pricing generator. It is also possible to establish a link between the generator of the t-norm (and the copula) and the t-conorm (and the option pricing generator). More specifically, it is easy to check that if $\phi(x)$ is the generator of the copula function, $\psi(x) = \phi(1-x)$ is the generator of the t-conorm.

Here we want to carry out a parallel analysis for the case in which the underlying is unbounded from above. A casual observation of standard Archimedean copulas reveals that the same link between copula and t-conorms can be extended to the positive real line as follows

Proposition 6.9. *Denote $C_\theta(u,v)$ an Archimedean copula function with generator $\phi(x)$ indexed by a parameter θ such that $\lim_{\theta\to\theta^*} C_\theta(x,y) = \min(x,y)$. Assume $S, K \in [0,\infty]$. Then,*

$$\lim_{\theta\to\theta^*} S \oplus_\theta K = C_\theta\left(\frac{1}{S+1}, \frac{1}{K+1}\right)^{-1} - 1 \tag{12}$$

with generator

$$g(x) = \phi\left(\frac{1}{x+1}\right) \tag{13}$$

is a pseudo-addition.

Proof. First note that with $\lim_{\theta\to\theta^*} C_\theta(x,y) = \min(x,y)$ we have

$$\begin{aligned} x \oplus y &= \frac{1}{\min\left(\frac{1}{x+1}, \frac{1}{y+1}\right)} - 1 \\ &= \frac{1}{\frac{1}{\max(x+1,y+1)}} - 1 \\ &= \max(x+1, y+1) - 1 = \max(x,y) \end{aligned}$$

As for the generator:

$$g(x) = \phi\left(\frac{1}{x+1}\right)$$

we have

$$g^{-1}(s) = \frac{1}{\phi^{-1}(s)} - 1$$

We then compute

$$x \oplus y = g^{-1}(g(x) + g(y))$$

$$= \frac{1}{\phi^{-1}\left(\phi\left(\frac{1}{x+1}\right)+\phi\left(\frac{1}{y+1}\right)\right)} - 1$$
$$= C_\theta\left(\frac{1}{x+1}, \frac{1}{y+1}\right)^{-1} - 1.$$

□

It may be appropriate to give a clarification of the reason why we need to include a limit with respect to the parameter θ requiring that the copula functions reaches its maximum value $\min(x, y)$. The reason is that this limit is not defined for all copula functions. For cases in which the maximum copula is not reached the pseudo-sum generator cannot reach $\max(x, y)$ and one of the no arbitrage conditions is not satisfied.

We give here examples of these generators using the most popular Archimedean copula functions.

Example 6.10. Consider Gumbel-Hougaard copula with generator:

$$\phi(x) = (-\log x)^\theta$$

with $\theta \in [1, \infty)$. Then,

$$g(x) = (\log(x+1))^\theta \qquad g^{-1}(s) = \exp\left(s^{1/\theta}\right) - 1.$$

We then have

$$x \oplus y = \exp\left(\left((\log(1+x))^\theta + (\log(1+y))^\theta\right)^{1/\theta}\right) - 1.$$

Note that setting $b = 1/\theta$ this gives

$$x \oplus y = \exp\left(\left((\log(1+x))^{1/b} + (\log(1+y))^{1/b}\right)^b\right) - 1$$

with $b \in (0, 1]$ that is the same generator obtained by exponentiation of the Lamperti transform of a Geometric Brownian Motion.

Example 6.11. The Clayton copula is characterized by the generator

$$\phi(x) = \frac{x^{-\theta} - 1}{\theta}$$

with $\theta \in [-1, \infty]$. Then,

$$g(x) = \frac{(x+1)^\theta - 1}{\theta} \qquad g^{-1}(s) = (1 + \theta s)^{1/\theta} - 1.$$

Note that this generator may be also considered as the logarithm in Tsallis algebra (Tsallis, 1988). For this reason we call this model Clayton-Tsallis. We then have:

$$x \oplus y = \left((x+1)^\theta + (y+1)^\theta - 1)\right)^{1/\theta} - 1.$$

Table 1. Models

Generator g_τ	Married Put m_τ	Implied Probability $Q_\tau(S_0 \le K)$
x^{1/b_τ}	$\left(S^{1/b_\tau} + K^{1/b_\tau}\right)^{b_\tau}$	$\left(1 + \left(\frac{S}{K}\right)^{-1/b_\tau}\right)^{1-b_\tau}$
$(\log(x+1))^{1/b_\tau}$	$\exp\left(\left((\log(1+S))^{1/b_\tau} + (\log(1+K))^{1/b_\tau}\right)^{b_\tau}\right) - 1$	$\left(\frac{\log(K+1)}{\log(m_\tau+1)}\right)^{1/b_\tau - 1} \frac{m_\tau+1}{K+1}$
$\frac{(x+1)^{b_\tau}-1}{b_\tau}$	$\left((S+1)^{b_\tau} + (K+1)^{b_\tau} - 1)\right)^{1/b_\tau} - 1$	$\left(\frac{K+1}{m_\tau+1}\right)^{b_\tau - 1}$

6.3. **Hybrid models: Black-Scholes-Dagum.** It is now time to compare the associative option pricing models, for which an option pricing generator exists, and the standard Black and Scholes model. Moreover, we can show that generators can be applied to other models, including the Black ad Scholes one.

Associative married put option pricing models share the representation

$$m_\tau(S,K) = g_\tau^{-1}(g_\tau(S) + g_\tau(K))$$

where $g(\cdot)$ is an increasing function (strictly increasing if the function is in $[0,\infty]$).

Note that the common feature of these models is that there exists a transformation of the married put price m_τ that yields a linear relationship to the same transformation of the variables S and K. Formally,

$$g_\tau(m_\tau(S,K)) = g_\tau(S) + g_\tau(K)$$

Let us now compare this structure with the Black and Scholes formula:

$$m_\tau(S,K) = S\Phi(d_\tau + 0.5\sigma\sqrt{\tau}) + K\Phi(-d_\tau + 0.5\sigma\sqrt{\tau})$$

with

$$d_\tau(S/K) = \frac{\log(S/K)}{\sigma\sqrt{\tau}}$$

It is clear that this function cannot be directly re-conducted to an associative function in S and K in $[0,\infty]$. However, a closer inspection at the pricing formula shows that it can be seen as associative on a set of *functions of* S and K. In fact, defining the function

$$D_\tau(x) = x\Phi\left(d_\tau(x/y) + 0.5\sigma\sqrt{\tau}\right)$$

we note that the Black and Scholes married put price can be written

$$m_\tau(S,K) = D_\tau(S) + D_\tau(K)$$

This derives from the function

$$\mu_\tau(y) = x\Phi(d_\tau(x/y) + 0.5\sigma\sqrt{\tau}) + y\Phi(-d_\tau(x/y) + 0.5\sigma\sqrt{\tau})$$

and $-d(x/y) = d(y/x)$. The financial meaning of the representation is very well known: $D_\tau(S)$ is the price of an *asset-or-nothing* (AoN) digital option and $D_\tau(K)$ is the price of a *cash-or-nothing* (CoN) digital option paying K dollars.

As for the algebraic properties of the married put function, we note that the Black and Scholes function is commutative in $D_\tau(x)$. Moreover, it is clear that it is additive in the digital prices with generator $g(x) = x$. It then makes sense to conceive that the pseudo-addition version of the Black and Scholes formula could be extended to other generators:

$$m_\tau(S,K) = g^{-1}\left(g(D_\tau(S)) + g(D_\tau(K))\right)$$

Note that this representation extends the linear no-arbitrage representation of options in terms of AoN and CoN digital options. As a representative element of this hybrid class of models, it is natural to nest the Black and Scholes model in the Dagum one.

Proposition 6.12. *The Black-Scholes-Dagum model is defined as*

$$m_\tau = \left(D_\tau(S)^{1/b(\tau)} + D_\tau(K)^{1/b(\tau)}\right)^{b(\tau)} \qquad (14)$$

with

$$D_\tau(x) \quad = \quad x\Phi\left(d_\tau(x/y) + 0.5\sigma\sqrt{\tau}\right)$$

FIGURE 1. Smiles: comparison of models. Short Term-Low Volatility

$$d_\tau(x/y) = \frac{\log(x/y)}{\sigma\sqrt{\tau}}$$

with $b(\tau)$ *increasing in* $[0,1]$ *and* $\sigma \in (0,\infty)$. *The Black and Scholes model is obtained with* $b(\tau) = 1, \forall\tau$ *and the Dagum model is obtained as* $\sigma \to 0$ *or* $\sigma \to \infty$.

7. **Three examples.** Here we give three examples of models characterized by generators. Please consider that this is not meant to be a full-fledged empirical analysis of these option pricing models, but just a first attempt to understand the possible shapes of *smile* functions that can be generated. It is only for the sake of realism that we calibrate the models on the *at-the-money* value of traded options in a trading day. The models were calibrated on the one year maturity option on the Italian FTSE-MIB stock index. The first model considered is the Dagum model. The other two are those proposed in the previous section: one is the logarithm exponentiated in model 6.10 and the other is the logarithm in the Tsallis algebra, model 6.11. Details of the models are reported in Table 1. The models are compared with the Black and Scholes and all of them are calibrated to have the same value for the *at-the-money* option.

The comparison of the models is reported in Figure (1) in terms of implied volatilities. The parameter b_τ generates a smile for all the models. The Dagum and Tsallis-Clayton models produce results that are very similar, with a smile curve that is steeper for the lower tail than for the upper tail. The smile of the power-log model lies below the Dagum model in the lower tail, and above it in the upper tail.

An interesting question is how the shape of the smile would evolve with maturity. Figure (2) reports the smiles for the Dagum and the power of log model for a value of the parameters three times higher, corresponding to a value of integrated variance in the Black and Scholes model equal to 0.581424. We note that while the shape of the smile for the Dagum model remains about the same as that for the shorter maturity,

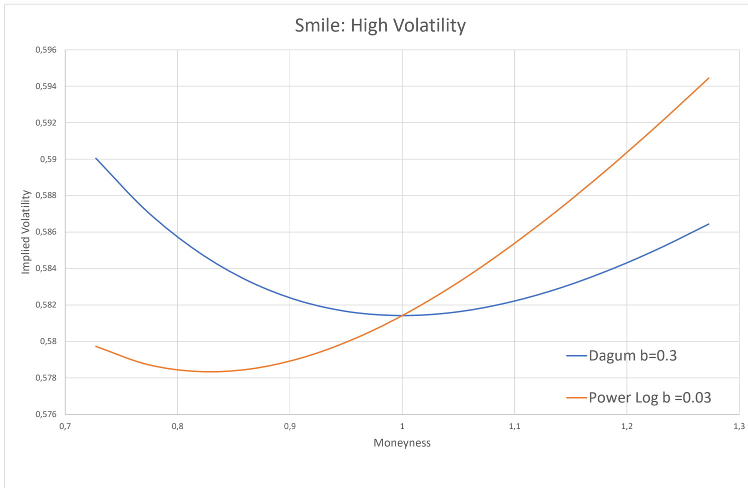

FIGURE 2. Smiles: comparison of the Dagum and power of log models. Long Term-High Volatility

that one of the power of log model is substantially different. More precisely, it is shifted well below the Dagum smile for the out-of-the money part of the curve and above the line for in-the-money options. In other words, the smile of power-log is rotated while being shifted up, while the Dagum model is only shifted up. In this sense, the Dagum model represents a natural extension of the Black and Scholes model, with the difference that the smile is not flat.

8. **Conclusions.** In this paper we have characterized a class of option pricing models that are completely described by a one-place increasing function grounded at zero, called the *generator* of the option pricing model. The analysis of this generator is made possible by focusing on the pricing function of a married put option, that is a position consisting of the underlying and a protective put. The task is then to find a set of functions that evolve towards the payoff function $\max(S, K)$ at maturity.

Existence of the generator is granted if one assumes that this pricing function is commutative, associative and Archimedean. The Archimedean assumption is particularly relevant in the option pricing application, because it amounts to requiring that at-the-money options must have positive value. Violation of this assumption would result in a degenerate distribution. So, under the assumption that the price of the underlying is bounded, without loss of generality in $[0, 1]$ the option pricing formula is a t-conorm, while if the underlying is in $[0, \infty]$ the option pricing formula is a pseudo-addition. In both cases, the pricing functions admit the existence of an option pricing generator.

We showed how to select generators compliant with the no-arbitrage requirements. In particular, we showed that using exponentiation of some well known generators found in stochastic process theory (Lamperti transform), or in multivariate distribution theory (Archimedean copula functions generators) can be suitably adjusted to give possible candidates for an option price generator.

Once the option pricing generator, the information concerning the risk-adjusted density of the underlying asset is completely encoded in the option pricing generator, and can be recovered using the standard Breeden and Litzenberger (1978) machinery. Cumulative distribution and probability density functions are written in terms of first and second derivatives of the option pricing generators.

We provide examples of option pricing generator comparing the power generator using the power function, leading to the Dagum distribution, a generator using exponentiation of the logarithm function, and one using the Tsallis definition of the logarithm in the q-algebra. The difference between the Dagum and the power of log models is getting larger for longer maturities. In particular, the Dagum model preserves about the same smile shape for different maturities, while the power of log model exhibits a twist.

Appendix. Here we give proof of Theorem (5.1). The theorem is proved by the following lemmas. We give credit to Lin Yang for most of this proof, that we embedded in our monoid generator analysis. We start writing the option pricing function and the no-arbitrage requirements in terms of the generator.

Theorem 8.1. *Let us define a monoid* $([0, \infty], \oplus)$ *where* $\oplus$ *is an Archimedean pseudo-addition with generator* $g_\tau(x)$*, continuous and twice differentiable. Then, the function*

$$m_\tau(S_\tau, K) \equiv S_\tau \oplus K = g_\tau^{-1}(g_\tau(S) + g_\tau(K)) \tag{15}$$

denotes the arbitrage free value of a European married put option written on the underlying S *with strike price* K *and time to maturity* τ *if an only if it satisfies the following:*

1. *terminal condition:*

$$\lim_{\tau \to 0} g_\tau^{-1}(g_\tau(S_\tau) + g_\tau(K)) = \max(S_0, K) \tag{16}$$

2. *horizontal arbitrage condition:*

$$\frac{\partial m_\tau(S_\tau, K)}{\partial K} = Q_\tau(S_0 \leq K) = \frac{g'_\tau(K)}{g'_\tau(m_\tau(S_\tau, K))} \leq 1 \tag{17}$$

$$\frac{\partial^2 m_\tau(S_\tau, K)}{\partial K^2} = Q_\tau(S_0 \leq K)\left(\frac{g''_\tau(K)}{g'_\tau(K)} - Q_\tau(S_0 \leq K)\frac{g''_\tau(m_\tau(S_\tau, K))}{g'_\tau(m_\tau(S_\tau, K))}\right) > 0 \tag{18}$$

3. *vertical arbitrage condition:* $g_\tau(\cdot)$ *must be such that for any* $\tau_1 > \tau_2 \geq 0$*:*

$$m_{\tau_1}(S, K) > m_{\tau_2}(S, K) \tag{19}$$

Now we report a set of lemmas linking the requirements in Theorem (8.1) with the conditions in Theorem (5.1).

Lemma 8.2. *If* $g_\tau(x)$ *is a smooth generator of a pseudo-addition (condition 1 and condition 2 in Theorem (5.1)) holds, then*

$$\lim_{\tau \to 0} m_\tau(x, y) = \max(x, y)$$

Proof. First, remind that $g(x)$ is positive and increasing, and $g_\tau^{-1}(x)$ is also increasing, being the generator of a pseudo-addition (Theorem 4.5). Now, set $\delta > 0$ and $y > x > 0$. Condition 2) gives

$$\lim_{\tau\to 0} \frac{g_\tau(y)}{g_\tau(y+\delta)} = 0$$

and

$$\lim_{\tau\to 0} \frac{g_\tau(x)}{g_\tau(y+\delta)} = 0$$

from which

$$\lim_{\tau\to 0} \frac{g_\tau(x) + g_\tau(y)}{g_\tau(y+\delta)} = 0$$

So, for all $\delta > 0$, $y > x > 0$, and for any $\varepsilon > 0$ there exists τ_0 such that for $\tau < \tau_0$

$$\left| \frac{g_\tau(y) + g_\tau(x)}{g_\tau(y+\delta)} \right| < \varepsilon$$

Since $g(x)$ is positive, we get, for $\varepsilon = 1$

$$g_\tau(x) + g_\tau(y) < g_\tau(y+\delta)$$

and

$$g_\tau(y) < g_\tau(x) + g_\tau(y) < g_\tau(y+\delta)$$

Since $g_\tau^{-1}(\cdot)$ is increasing, we obtain

$$y < g_\tau^{-1}(g_\tau(x) + g_\tau(y)) < y + \delta$$

So, for all $\delta > 0$ there exists $\tau < \tau_0$ such that:

$$\left| g_\tau^{-1}(g(x) + g(y)) - y \right| < \delta$$

and we have proved

$$\lim_{\tau\to 0} g_\tau^{-1}(g(x) + g(y)) = y > x$$

□

Lemma 8.3. *Condition 2) in Theorem (5.1) implies that $g(x)$ is convex.*

Proof. By contradiction, assume that, there exist $y > x$ and $t \in (0,1)$ such that for all $\tau > 0$

$$g_\tau(tx + (1-t)y) > tg_\tau(x) + (1-t)g_\tau(y)$$

Since $g_\tau(x)$ is positive,

$$\frac{g_\tau(tx + (1-t)y)}{g_\tau(y)} > \frac{tg_\tau(x) + (1-t)g_\tau(y)}{g_\tau(y)}$$

Applying condition 2, and reminding $y > x$ we obtain

$$\lim_{\tau\to 0} \frac{g_\tau(tx + (1-t)y)}{g_\tau(y)} = 0,$$

but the same condition yields

$$\lim_{\tau\to 0} \frac{tg_\tau(x) + (1-t)g_\tau(y)}{g_\tau(y)} = 1 - t > 0,$$

a contradiction. □

Lemma 8.4. *If $m_\tau(x,y) \equiv g_\tau^{-1}(g_\tau(x)+g_\tau(y))$, condition 2 in Theorem (5.1) implies*

$$0 \le \frac{\partial m_\tau(x,y)}{\partial y} \le 1 \tag{20}$$

Proof. First, we compute the derivative of inverse function

$$\frac{\partial m_\tau(x,y)}{\partial y} = \frac{g'_\tau(y)}{g'_\tau(m_\tau(x,y))}$$

and since $g'_\tau(\cdot)$ is positive we have

$$\frac{\partial m_\tau(x,y)}{\partial y} \geq 0$$

Next, from Lemma 8.3 we have that $g(\cdot)$ is convex, so that $g'(\cdot)$ is increasing. So, from $m_\tau(x,y) \geq y$, we obtain

$$\frac{\partial m_\tau(x,y)}{\partial y} = \frac{g'_\tau(y)}{g'_\tau(m_\tau(x,y))} \leq 1$$

□

Lemma 8.5. *$m_\tau(x,y)$ is convex in y if and only if $1/g'_\tau(\cdot)$ is convex (condition 3 in Theorem 5.1).*

Proof. From

$$\frac{\partial m_\tau}{\partial y} = \frac{g'_\tau(y)}{g'_\tau(m_\tau)}$$

we compute

$$\frac{\partial^2 m_\tau(x,y)}{\partial y^2} = \frac{g''_\tau(y)g'_\tau(m_\tau) - g'_\tau(y)(\partial g'_\tau(m_\tau)/\partial y)}{[g'_\tau(m_\tau)]^2}.$$

The condition for convexity then is

$$g''_\tau(y)g'_\tau(m_\tau) - g'_\tau(y)\frac{\partial g'_\tau(m_\tau)}{\partial y} \geq 0$$

This can be written a

$$g''_\tau(y)g'_\tau(m_\tau) - g'_\tau(y)g''_\tau(m_\tau)\frac{\partial m_\tau}{\partial y} \geq 0$$

from which

$$g''_\tau(y)g'_\tau(m_\tau) - g'_\tau(y)g''_\tau(m_\tau)\frac{g'_\tau(y)}{g'_\tau(m_\tau)} \geq 0.$$

This yields the condition:

$$\frac{g''_\tau(y)}{[g'_\tau(y)]^2} \geq \frac{g''_\tau(m_\tau)}{[g'_\tau(m_\tau)]^2}.$$

Since $m_\tau \geq y$ this implies that the function

$$\frac{g''_\tau(z)}{[g'_\tau(z)]^2}$$

must be decreasing. But since we have

$$\frac{\partial(1/g'_\tau(z))}{\partial z} = -\frac{g''_\tau(z)}{[g'_\tau(z)]^2}$$

the function $1/g'_\tau(z)$ must be convex. The same proof can be run backward. □

Lemma 8.6. *If $\tau_1 > \tau_2$, $m_{\tau_1} > m_{\tau_2}$ iff $g_{\tau_1}(z) > g_{\tau_2}(z), \forall z$.*

Proof. First note that $g_{\tau_1}(\cdot) \lesseqgtr g_{\tau_2}(\cdot)$ implies

$$g_{\tau_1}\left(g_{\tau_2}^{-1}(z)\right) \lesseqgtr g_{\tau_2}\left(g_{\tau_2}^{-1}(z)\right) = z$$

Now if $m_{\tau_1}(x,y) > m_{\tau_2}(x,y)$ for $\tau_1 > \tau_1$ we have

$$\begin{aligned} g_{\tau_1}(m_{\tau_1}(x,y)) &= g_{\tau_1}(x) + g_{\tau_1}(y) \\ &> g_{\tau_1}\left(m_{\tau_2}(x,y)\right) \\ &= g_{\tau_1}\left(g_{\tau_2}^{-1}(g_{\tau_2}(x) + g_{\tau_2}(y))\right) \\ &> g_{\tau_2}(x) + g_{\tau_2}(y) \end{aligned}$$

which implies $g_{\tau_1}(\cdot) > g_{\tau_2}(\cdot)$. □

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