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RESEARCH ARTICLE

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A small cusped hyperbolic 4-manifold

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Abstract

By gluing some copies of a polytope of Kerckhoff and Storm's, we build the smallest known orientable hyperbolic 4-manifold that is not commensurable with the ideal 24-cell or the ideal rectified simplex. It is cusped and arithmetic, and has twice the minimal volume.

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INTRODUCTION

There is a natural interest in hyperbolic manifolds of low volume, and this note addresses dimension four; see the survey [17].

Hyperbolic manifolds in the paper are understood to be complete and of finite volume. Two manifolds (or orbifolds) are *commensurable* if they are both finitely covered by a third manifold (or orbifold). Recall also that the generalised Gauss–Bonnet formula relates the volume of a hyperbolic 4-manifold M to its Euler characteristic $\chi(M)$ as follows:

$$\mathrm{Vol}(M) = \frac{4\pi^2}{3}\chi(M).$$

The smallest known closed hyperbolic 4-manifolds are non-orientable and have $\chi = 8$ [3, 13]. These and the few other explicit examples of closed manifolds that we could find in the literature [4, 6, 14] are tessellated by $2\pi/k$-angled 120-cells for $k = 3, 4, 5$, and are arithmetic and commensurable [9].

Concerning cusped manifolds, we found instead four commensurability classes: (we cite here the papers containing the smallest known manifolds in each class)

(1) $\chi = 1$ both orientable and not, arithmetic [5, 12, 21, 22, 25];
(2) $\chi = 1$ both orientable and not, arithmetic [2, 11, 24, 26];

(3) $\chi = 2$ non-orientable (so also $\chi = 4$ orientable), arithmetic [18, 24];
(4) $\chi = 3$ non-orientable and $\chi = 5$ orientable, non-arithmetic [24].

Each of these classes is represented by a hyperbolic Coxeter 4-polytope, for instance, (1) by the nine non-compact simplices, $\mathbb{H}^4/\mathrm{PO}(4,1;\mathbb{Z})$ and the ideal 24-cell, and (2) by the ideal rectified simplex .

We find here another commensurability class of low-volume hyperbolic 4-manifolds:

Theorem. *There exists an orientable, cusped, arithmetic, hyperbolic 4-manifold M with $\chi(M) = 2$, not belonging to any of the commensurability classes above.*

The manifold M is commensurable with a Coxeter polytope Q belonging to a continuous family of hyperbolic 4-polytopes discovered in 2010 by Kerckhoff and Storm [10]. Notably, classes (1) and (3) are represented by other Coxeter polytopes of the family, and the examples in (4) are hybrids of manifolds in (3) and (2). Our method applies to two additional Coxeter polytopes of the family, but giving less interesting examples from the viewpoint of this paper (see Subsection 2.4).

We build M explicitly, by gluing together some copies of a bigger polytope P of Kerckhoff and Storm's (Q is a quotient of P). The results in [18, 23, 24] are obtained in the same spirit, by gluing polytopes of the Kerckhoff–Storm family as well.

The polytope P has volume $(2/5) \cdot (4\pi^2/3)$ and octahedral symmetry. It has a few 2-faces with dihedral angle $2\pi/3$, while the remaining ones are right angled. These are the main features of P that will be exploited.

As a first step, we 'kill' in a natural way the $2\pi/3$ angles by gluing in pairs some facets of five copies of P, to get a particularly symmetric hyperbolic manifold X with right-angled corners. These objects have been fruitfully used in four-dimensional hyperbolic geometry in the very last years [1, 16, 19, 20].

A study of X and its symmetries then allows us to build M as follows. The facets of X are hyperbolic 3-manifolds with geodesic boundary (the corners), and are divided in two types. The corners, some punctured surfaces, are always the intersection of two facets of different type. We then close X up via two commuting, fixed-point free, isometric involutions of ∂X as gluing maps: one for each type of facet.

By construction, M is commensurable with the orbifold Q. Being Q arithmetic, Maclachlan's work [15] allows us to distinguish its commensurability class. One similarly gets a few non-orientable manifolds with $\chi = 2$ in the same class (Remark 2.4).

Structure of the paper

The polytopes Q and P are introduced in Section 1, while the manifolds X and M are built in Section 2.

1 | THE POLYTOPES

We introduce here two polytopes from [10]: the Coxeter polytope Q in Subsection 1.1, and the bigger polytope P in Subsection 1.2. The commensurability class of Q is distinguished in Proposition 1.1.

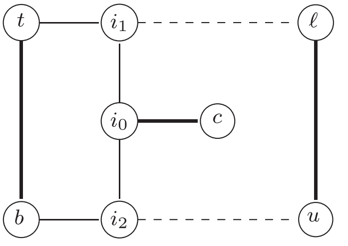

FIGURE 1 The Coxeter diagram of the reflection group Γ of Q. If two nodes are joined by a thin, thick, or dashed edge, then the two corresponding bounding hyperplanes meet with angle $\pi/3$, are tangent at infinity, are ultraparallel, respectively. There is no edge joining two nodes if the corresponding hyperplanes are orthogonal.

We refer the reader to the book [27] for the general theory of hyperbolic Coxeter polytopes and reflection groups, including arithmeticity.

1.1 | The Coxeter polytope

The Coxeter diagram in Figure 1 represents a non-compact hyperbolic Coxeter 4-polytope $Q \subset \mathbb{H}^4$ of finite volume [10, Proposition 13.1], called Q_{t_3} in [10].

The associated reflection group

$$\Gamma = \langle\, t,\, b,\, u,\, \ell,\, c,\, i_0,\, i_1,\, i_2 \,\rangle < \mathrm{Isom}(\mathbb{H}^4)$$

is arithmetic [10, Theorem 13.2].

The symbols t, b, u, ℓ, c and i are chosen to remind the words 'top', 'bottom', 'upper', 'lower', 'central' and 'internal', respectively. This may help later.

Recall that the path graph with $n-1$ vertices is a Coxeter diagram for the symmetric group $\mathfrak{S}_n$ [27, table 1]. In the following sections, we will be interested in the edge I of Q corresponding to the reflection subgroup

$$G_I = \langle i_0, i_1, i_2 \rangle \cong \mathfrak{S}_4, \tag{1}$$

and its vertices $V, V' \in I$, called *top* and *bottom* vertices, corresponding to the reflection groups

$$G_V = \langle i_0, i_1, i_2, t \rangle \cong \mathfrak{S}_5, \quad G_{V'} = \langle i_0, i_1, i_2, b \rangle \cong \mathfrak{S}_5. \tag{2}$$

We conclude the section distinguishing the commensurability class of Q.

Proposition 1.1. *The arithmetic orbifold $Q = \mathbb{H}^4/\Gamma$ does not belong to any of the commensurability classes (1), (2), (3), (4) mentioned in the introduction.*

Proof. By Maclachlan's work [15], the ramification set of the quadratic form associated to Γ is a commensurability invariant of arithmetic reflection groups. We compute this set as explained in [8]. For a quick description of the computation without proofs, one may consult [18, section 4.5].

Up to isometry of $\mathbb{H}^4$ in its hyperboloid model, the bounding hyperplanes of the polytope $Q \subset \mathbb{H}^4$ (coherently oriented) are dually represented by these spacelike vectors of $\mathbb{R}^{1,4}$:

$$t = \left(\sqrt{2}, 1, 1, 1, \sqrt{7}\right), \ b = \left(\sqrt{2}, 1, 1, -1, -\sqrt{7}\right),$$
$$u = \left(\sqrt{2}, 1, 1, 1, -\sqrt{7}/7\right), \ \ell = \left(\sqrt{2}, 1, 1, -1, \sqrt{7}/7\right),$$
$$c = \left(1, \sqrt{2}, 0, 0, 0\right), \ i_0 = (0, -1, 1, 0, 0),$$
$$i_1 = (0, 0, -1, -1, 0), \ i_2 = (0, 0, -1, 1, 0)$$

(see [10, section 4] for i_0, i_1, i_2 and [10, table 3] with $t = t_3 = \sqrt{7}/7$ for the remaining vectors). The Gram matrix of the corresponding vectors of unit Minkowski norm is:

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ -1 & 1 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & -\frac{\sqrt{7}}{2} \\ 0 & 0 & -1 & 1 & 0 & 0 & -\frac{\sqrt{7}}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 & -\frac{\sqrt{7}}{2} & 0 & -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & -\frac{\sqrt{7}}{2} & 0 & 0 & -\frac{1}{2} & 0 & 1 \end{pmatrix}.$$

For some basis, a matrix representing the associated quadratic form over $\mathbb{Q}$ is:

$$\begin{pmatrix} 7 & 0 & 0 & \frac{7}{2} & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{7}{2} & 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 1 \end{pmatrix}.$$

Diagonalising the form, we get the matrix $\mathrm{diag}(7/2,\ 1/2,\ 1/6,\ -3/8,\ 1/8)$. From this datum, with the help of [8, Propositions 4.8, 4.13, 4.15], it is straightforward to compute the ramification set of the form (as done in [18, Proposition 4.25]), which is $\{2, 7\}$.

The ramification sets of the arithmetic commensurability classes (1), (2) and (3) are instead $\emptyset$, $\{3, \infty\}$ and $\{2, 5\}$, respectively (see [24, Proposition 2.5], and the references therein), while (4) is non-arithmetic. So, the five classes are distinct. □

1.2 | The bigger polytope

Recall Formula (1). By reflecting the Coxeter polytope Q around its edge I, we get a bigger polytope

$$P = \bigcup_{g \in G_I} g(Q) \subset \mathbb{H}^4$$

tessellated by $4! = 24$ copies of $Q \cong P/G_I$.

We resume here all the needed information on P , referring to [10, 18] for proofs and further details (see, especially, [18, Proposition 3.16]).

A remarkable property of P is its antipodal symmetry. We call *antipodal map* the inversion $a \in \mathrm{Isom}(P)$ through the centre of P (the midpoint of the edge I of Q).

The polytope P has 22 facets, partitioned up to symmetry of P into three sets:[†]

(E) the *extremal facets*, divided into *top* and *bottom* facets

$$E_1, E_2, E_3, E_4 \text{ and } E'_1, E'_2, E'_3, E'_4,$$

tessellated by the copies of the facet t, resp. b, of Q, where $E'_i = a(E_i)$;

(H) the *half-height facets*, divided into *upper* and *lower* facets

$$H_1, H_2, H_3, H_4 \text{ and } H'_1, H'_2, H'_3, H'_4,$$

tessellated by the copies of the facet u, resp. ℓ, of Q, where $H'_i = a(H_i)$;

(C) the *central facets*

$$C_{12}, C_{13}, C_{14}, C_{23}, C_{24}, C_{34},$$

tessellated by the copies of the facet e of Q, where C_{ij} is the unique such facet of P that intersects the top facets E_i and E_j.

The ideal vertices of P all lie in the ideal boundary of a hyperplane $H^3 \subset \mathbb{H}^4$ [18, Proposition 3.19]. By definition, the names of the facets agree on whether they are contained in one of the two half-spaces bounded by H^3 ('top/bottom' and 'upper/lower') or not ('central'). The group G_I preserves the two half-spaces, while a exchanges them.

With this conventions, all is well defined up to co-orientation of H^3 ('top/bottom' and 'upper/lower') and permutation of the facet indices, thanks to the following lemma. Let $\mathfrak{A}_n < \mathfrak{S}_n$ denote the alternating subgroup.

Lemma 1.2. *The action of $\mathfrak{S}_4$ on the facet indices is induced by an isomorphism*

$$\mathrm{Isom}(P) \cong \mathbb{Z}/2\mathbb{Z} \times \mathfrak{S}_4,$$

which restricts to

$$\langle a \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \{\mathrm{id}\}, \quad G_I \cong \{0\} \times \mathfrak{S}_4, \quad \textit{and} \quad \mathrm{Isom}^+(P) \cong \mathbb{Z}/2\mathbb{Z} \times \mathfrak{A}_4.$$

Moreover, the quotient $P/\mathrm{Isom}(P)$ is isometric to the orbifold $Q/\mathrm{Isom}(Q)$.

Proof. The same argument of [24, Proposition 2.4] applies, replacing the words 'upper tetrahedral facet' with 'link of the top vertex V' (see also [18, Lemma 4.15]). The key point is that the unique non-trivial symmetry of Q (an order-two rotation corresponding to the reflection along the edge $\{i_0, c\}$ of the diagram in Figure 1) writes as the composition of a with an element of G_I. □

[†] In [10, 18], these are called: the 'odd' and 'even' 'positive walls', the 'even' and 'odd' 'negative walls' and the 'letter walls', respectively.

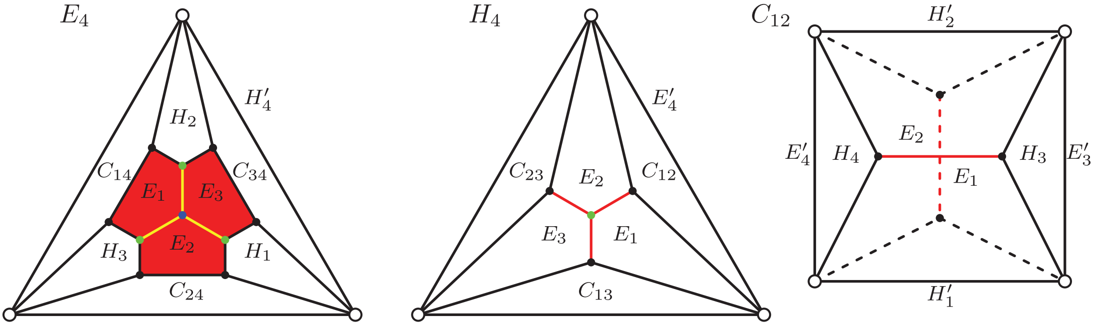

FIGURE 2 The top, upper and central facets E_4, H_4 and C_{12} of P. The ideal vertices are drawn as white dots, while the top vertex V and the type-2 and type-3 vertices are in blue, green and black, respectively. The black, red and yellow edges have dihedral angle $\pi/2$, $2\pi/3$ and $\arccos(-1/3)$, respectively. The red pentagons are $2\pi/3$-angled ridges of P, while the other 2-faces in the picture are right angled.

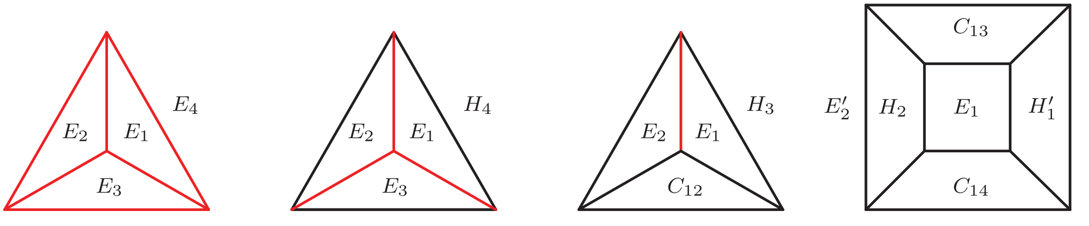

FIGURE 3 The vertex links of P up to symmetry: three spherical tetrahedra for the finite vertices (from left to right, of type 1, 2 and 3) and a Euclidean parallelepiped for the ideal vertices. The black edges are right angled, while the red ones are $2\pi/3$ angled.

Some facets of P are drawn in Figure 2, and some vertex links in Figure 3. From these two pictures and the respective captions, one recovers the whole combinatorics and geometry of P (like, for instance, the adjacency graph of the facets) by means of Lemma 1.2. We record here a few consequences that will be relevant for us.

First, the top (resp., bottom) facets intersect each other with angle $2\pi/3$, and share the top (resp., bottom) vertex $V = E_1 \cap \ldots \cap E_4$ (resp., $V' = a(V) = E'_1 \cap \ldots \cap E'_4$). Second, the central, resp. half-height, facets are pairwise disjoint. Third, if two non-isometric facets intersect, then they are orthogonal. Fourth, the antipodal map acts on the central facets as follows:

$$a(C_{ij}) = C_{kl} \quad \text{for all distinct } i, j, k, l. \tag{3}$$

2 | THE CONSTRUCTION

In this section, we build the manifold M by gluing some copies of P. Recall that the top and bottom vertices of P have as link the spherical tetrahedron with dihedral angles $2\pi/3$. The latter tiles S^3 in the regular honeycomb combinatorially equivalent to the triangulation of the boundary of the 4-simplex. Therefore, we need at least five copies of P to build a manifold, and we will see that five is enough.

We proceed as follows. In Subsection 2.1, we build a manifold with corners X by gluing in pairs the extremal facets of five copies of P. In Subsection 2.2, we study X and its symmetries. In Subsection 2.3, we close X up by pairing the remaining facets via two symmetries of X, and conclude our proof. Finally, in Subsection 2.4, we give some additional information on how to apply the construction to two more polytopes of Kerckhoff and Storm's.

Before beginning with the construction, let us introduce here some terminology. A *hyperbolic n-manifold with 'pure' right-angled corners* is a hyperbolic manifold X with boundary, locally modelled on the intersection in $\mathbb{H}^n$ of two closed half-spaces bounded by orthogonal hyperplanes. The boundary ∂X is naturally stratified into maximal connected, totally geodesic, submanifolds: the $(n-1)$-dimensional *facets* have (possibly empty) totally geodesic boundary, while the $(n-2)$-dimensional *corners* have no boundary. We say that ∂X is *bicolourable* if there are two unions of facets A and B such that $\partial X = A \cup B$ and the corners of X are the connected components of $A \cap B$. This implies that each corner is contained in exactly two facets: one in A, and one in B. In particular, the facets of X are isometrically embedded hyperbolic manifolds with totally geodesic boundary (made of the corners of X).

2.1 | Extremal gluing

We build here X by gluing isometrically in pairs all the extremal facets of five copies of P: the top ones first and the bottom ones after. There is in fact a unique way to perform the first gluing, and a preferred one for the second.

Recall Formula (2). By reflecting the smaller Coxeter polytope $Q \subset P$ around its top vertex $V = E_1 \cap \ldots \cap E_4$, we get a big polytope

$$\tilde{P} = \bigcup_{g \in G_V} g(Q) \subset \mathbb{H}^4.$$

As $G_I < G_V$ with index $5 = 5!/4!$, the polytope $\tilde{P}$ contains P and is tessellated by five isometric copies of P . These are glued altogether around V (the centre of $\tilde{P}$), in the pattern of the triangulation of S^3 induced by the boundary of the 4-simplex.

Note that we can alternatively think of $\tilde{P}$ as built by pairing isometrically all the top facets of five abstract copies of P as described in Figure 4. We now build X from $\tilde{P}$ by pairing all the bottom facets of these five copies of P as follows. Let $\varphi : F \to G$ be a pairing map between two top facets of two copies P_x and P_y of P . Then we want to glue $a_x(F)$ to $a_y(G)$ via $a_y \circ \varphi \circ a_x$, where a_z denotes the antipodal map of P_z. This is done in Figure 4.

More concretely, let r_i be the reflection along the top face E_i of P, and define $P_i = r_i(P) \subset \mathbb{H}^4$ (so that $a_i = r_i \circ a \circ r_i$). To build X from

$$\tilde{P} = P \cup P_1 \cup \ldots \cup P_4 \subset \mathbb{H}^4,$$

we glue the facet E'_i of P to the facet $r_i(E'_i)$ of P_i via r_i, and, for $i \neq j$, the facet $r_i(E'_j)$ of P_i to the facet $r_j(E'_i)$ of P_j via the reflection $r_{ij} \in G_I$ corresponding to $(ij) \in \mathfrak{S}_4$ (recall Lemma 1.2 and see Figure 5).

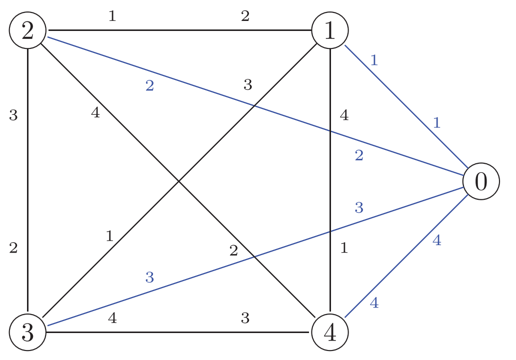

FIGURE 4 The complete graph K_5, with its nodes and half -edges labelled to remind the 'more abstract' construction of X and $\tilde{P}$, which follows here. To build X (resp., $\tilde{P}$), we glue the extremal (resp., top) facets of five abstract copies $P_0, \dots, P_4$ of P as follows. Consider the edge of K_5 joining the nodes i and j, where $i < j$. If $i = 0$ (blue edge), glue the facets E_j and E'_j (resp., the facet E_j) of P_0 to the corresponding facets of P_j via the map induced by the identity of P. If instead $i \neq 0$ (black edge), glue the facets E_j and E'_j (resp., the facet E_j) of P_i to the facets E_i and E'_i (resp., the facet E_i) of P_j via the map induced by the reflection $(ij) \in \mathfrak{S}_4 = G_I$ of P.

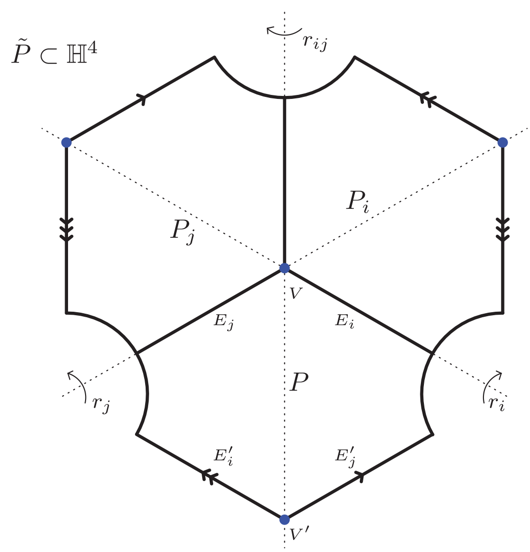

FIGURE 5 A schematic picture of the 'more concrete' construction of X from $\tilde{P} = P \cup P_1 \cup \dots \cup P_4 \subset \mathbb{H}^4$.

2.2 | The manifold with corners

We study here the space X just constructed.

Let H and C denote the union of the copies of the half-height and central facets of P in X, respectively. These facets are the unpaired ones.

Proposition 2.1. *The resulting complex X is an orientable hyperbolic 4-manifold with pure right-angled corners and boundary $\partial X = H \cup C$ bicoloured by H and C.*

Proof. We analyse the effect of the gluing on the vertex links of the copies of P along their extremal facets.

We refer to Figures 3 and 4. By construction, the gluing graphs of the type-k vertex links correspond to the subgraphs of K_5 spanned by $6 - k$ nodes (repeated twice to deal with the antipodal vertices). Precisely, the type-1 links are glued so as to tessellate two copies of S^3 (thus giving points in the interior of X) like the boundary of the 4-simplex, the type-2 links tessellate some copies of a half-sphere of S^3 (giving internal points of the facets of X), and the type-3 links tessellate some copies of a right-angled bigon of S^3 (giving points of the corners of X). The links of the ideal vertices are glued to form flat 3-manifolds with right-angled corners, stratum-preserving homoeomorphic to $[0,1]^2$-bundles over S^1 (giving the cusps of X).

We have shown that the vertex links glue to form spherical or flat 3-manifolds with pure right-angled corners, so X is a hyperbolic 4-manifold with pure right-angled corners. It is orientable, being obtained from $\tilde{P}$ via orientation-reversing gluing maps.

By construction $\partial X = H \cup C$ and H and C are unions of facets. The previous analysis also shows that the corners are contained in the intersection of copies of a half-height and a central facet of P (see the gluing of the type-3 vertex links). Therefore, H and C define a bicolouring of ∂X, and this concludes the proof. □

It is not difficult to show that H consists of two disjoint facets (an 'upper' one and a 'lower' one), while C is a facet. We omit the proof because this fact is not needed.

Recall now Lemma 1.2 and Formulae (1) and (2). In the sequel, we shall think of $\mathfrak{S}_4 \subset \mathfrak{S}_5$ as the stabiliser of 0 in the permutation group $\mathfrak{S}_5$ of $\{0, \dots, 4\}$.

Lemma 2.2. *Every symmetry of P is the restriction of an isometry of X.*

Proof. This is true for a thanks to the 'a-equivariance' of the gluing maps producing X, and for each $\sigma \in \mathfrak{S}_4 \subset \mathfrak{S}_5 \cong \mathrm{Aut}(K_5)$ because it induces a permutation (σ itself!) of the half-edge labels of the graph K_5 in Figure 4. □

As, moreover, $G_V \cong \mathfrak{S}_5$ permutes the five copies of P in X , we can write:

$$\mathrm{Isom}(P) = \mathbb{Z}/2\mathbb{Z} \times \mathfrak{S}_4 \subset \mathbb{Z}/2\mathbb{Z} \times \mathfrak{S}_5 \subset \mathrm{Isom}(X).$$

In the following section, we will implicitly use Lemma 2.2 and adopt the above convention.

2.3 | Half-height and central gluing

We are finally ready to build M from X.

Let $r_{ij} \in \mathrm{Isom}(X)$ correspond to the reflection $(ij) \in \mathfrak{S}_4 \subset \mathrm{Isom}(P)$, and recall that $\partial X = H \cup C$. Now pick X, glue H to itself via $h = a \circ r_{12}$ and C to itself via $c = a \circ r_{34}$, and call M the resulting space.

Proposition 2.3. *The resulting complex M is an orientable hyperbolic 4-manifold with $\chi(M) = 2$, commensurable with the orbifold $Q = \mathbb{H}^4/\Gamma$.*

Proof. Observe that $h, c \in \mathrm{Isom}(P)$ are both the composition of an inversion through a point with a reflection through a hyperplane containing the point, so their fixed-point set is a line. In both cases, this line joins two ideal vertices of P:

$$\overline{E_1} \cap \overline{E_2'} \cap \overline{H_2} \cap \overline{H_1'} \cap \overline{C_{13}} \cap \overline{C_{14}} \quad \text{and} \quad \overline{E_2} \cap \overline{E_1'} \cap \overline{H_1} \cap \overline{H_2'} \cap \overline{C_{23}} \cap \overline{C_{24}},$$

$$\overline{E_3} \cap \overline{E_4'} \cap \overline{H_4} \cap \overline{H_3'} \cap \overline{C_{13}} \cap \overline{C_{23}} \quad \text{and} \quad \overline{E_4} \cap \overline{E_3'} \cap \overline{H_3} \cap \overline{H_4'} \cap \overline{C_{14}} \cap \overline{C_{24}},$$

respectively, as it is easily checked by Lemma 1.2 and Formula (3). Therefore, the fixed-point sets of $h, c \in \mathrm{Isom}(X)$ are contained in the interior of $P \subset X$.

As H and C define a bicolouring of ∂X (Proposition 2.1) and the gluing maps h and c are two distinct, commuting, fixed-point free, isometric involutions of ∂X, it follows that M is a hyperbolic manifold: near the points corresponding to the internal points of the facets of X because h and c are fixed-point free isometric involutions of ∂X, and near the points corresponding to the corners of X because moreover h and c are distinct and commute (so each corner cycle has length 4 and trivial return map).

The manifold M is orientable because h and c are orientation reversing by Lemma 1.2 (recall that X is orientable by Proposition 2.1).

The orbifold Euler characteristic of Q is $\chi^{\mathrm{orb}}(Q) = 1/60$ (for a quick check one may use [7]), and M is tessellated into $5! = 120$ copies of Q. Therefore, $\chi(M) = 120/60 = 2$.

Finally, note that the gluing maps used to build M from the copies of P are induced by symmetries of P. Therefore M covers the orbifold $P/\mathrm{Isom}(P) \cong Q/\mathrm{Isom}(Q)$ (recall Lemma 1.2), and so M and Q are commensurable. □

We conclude the paper with some additional information.

Remark 2.4. One builds a few non-orientable manifolds with $\chi = 2$ in the same commensurability class, just by choosing as gluing maps h and c other pairs of distinct, commuting, isometric involutions of X with no fixed point on ∂X, without requiring they both reverse the orientation. Any two among a, $a \circ r_{12}$, $a \circ r_{34}$ and $a \circ r_{12} \circ r_{34}$ work.

2.4 | More Kerckhoff–Storm polytopes

The method applies to two additional Coxeter polytopes of Kerckhoff and Storm's with the same combinatorics of Q, called Q_{t_4} and Q_{t_5} in [10]. One lies in the commensurability class (1) and has $\chi^{\mathrm{orb}} = 5/192$, while the other one is non-arithmetic and has $\chi^{\mathrm{orb}} = 241/7200$ (see [10, Theorem 13.2] and [18, Propositions 3.22 and 4.25]).

The main difference is that now the pentagonal faces of P have dihedral angle $\pi/2$ (resp., $2\pi/5$) in place of $2\pi/3$, therefore the link of the top vertex V tessellates S^3 like the boundary of the regular 16-cell (resp., 600-cell) R in place of the simplex Δ. So, to build X, one has to glue 16 (resp., 600) copies of P in place of 5. This time $\mathfrak{S}_4$ acts on $P \subset X$ like a facet stabiliser in $\mathrm{Isom}(R)$, in place of $\mathrm{Isom}(\Delta) \cong \mathfrak{S}_5$.

The analogous construction gives an orientable manifold M with $\chi = 10$ (resp., 482). Moreover, in this case, M has an orientation-preserving isometric involution ι without fixed points, so the quotient $M/\langle\iota\rangle$ is a twice-smaller orientable manifold. One gets ι by composing the map induced by a with that induced by the antipodal map of R (a symmetry that Δ does not enjoy!).

In conclusion, one gets two more cusped, orientable, hyperbolic 4-manifolds: one in the class (1) of $\mathbb{H}^4/\mathrm{PO}(4,1;\mathbb{Z})$ with $\chi = 5$, and one non-arithmetic with $\chi = 241$.

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