Alma Mater Studiorum Università di Bologna
Archivio istituzionale della ricerca

Frege on Compound Thoughts: A Wittgensteinian Diagnosis

(Article begins on next page)

03 September 2024

FRANCESCO BELLUCCI

Frege on Compound Thoughts: A Wittgensteinian Diagnosis

1. Introduction

Is the thought expressed by *A* the same as that expressed by *~~A* according to Frege? [1] In «Die Verneinung» (1918), the second part of the *Logische Untersuchungen*, he refers to «*A*» and to «the negation of the negation of *A*» as «two thoughts» that have the same truth-value[2]. In «Gedankengefüge» (1923), the third part of the series, he says that «not (not *B*)» has the same sense as «*B*»[3]. Since for Frege the sense of a sentence is the thought that it expresses, the two claims are in plain contradiction. A parallel contradiction arises with regard to a conditional and its contrapositive: in «Die Verneinung» Frege refers to sentences of the form $P \supset Q$ and $\sim Q \supset \sim P$ as two distinct «complex thoughts»[4], while in «Gedankengefüge» he says that in a conditional sentence we can replace the antecedent with the negation of the consequent and the consequent with the negation of the antecedent «without altering the sense of the whole»[5], i.e., since the sense of a sentence is the thought that it expresses, without passing from one thought to a distinct thought. The contradiction here detected can be expressed more generally by saying that in «Die Verneinung» Frege suggests that two truth-functionally equivalent but structurally distinct sentences express distinct thoughts, while in «Gedankengefüge» he says that such sentences express the same thought.

One could explain the contradiction by saying that Frege changed his mind between «Die Verneinung» and «Gedankengefüge»[6]. This explanation faces two serious problems. In the first place, the idea expressed in «Gedankengefüge», i.e. that two truth-functionally equivalent but structurally distinct sentences express the same thought, is also presented in some of Frege's writings and letters from 1906. Not only that, but in a piece written circa 1897/98 Frege had already said that the contraposition of a conditional sentence does not affect the sense of the sentence[7]. The idea set forth in «Gedankengefüge» as regard sameness of sentential sense is

---

[1] The following abbreviations for Frege's and Wittgenstein's works are used: CP *Collected Papers on Mathematics, Logic, and Philosophy*, ed. by B. McGuinness, Oxford, Blackwell, 1984; GG for *Grundgesetze der Arithmetik*, Band I/II, Jena, Herman Pohle, 1893/1903; transl. by P. Ebert and M. Rossberg (with C. Wright) as *Basic Laws of Arithmetic*, Oxford, Oxford University Press, 2013; PMC for *Philosophical and Mathematical Correspondence*, ed. by G. Gabriel, H. Hermes, F. Kambartel, C. Thiel, A. Veraart, Oxford, Blackwell, 1980: PW for *Posthumous Writings*, ed. by H. Hermes, F. Kambartel, F. Kaulbach, Oxford, Blackwell, 1979; TLP for *Tractatus logico-philosophicus*, London, Kegan Paul, Trench, Trubner & Co., 1922.

[2] CP, p. 389.

[3] CP, p. 399.

[4] CP, p. 376.

[5] CP, p. 403.

[6] This is suggested by Künne (W. Künne, Wittgenstein and Frege's *Logical Investigations*, in *Wittgenstein and Analytic Philosophy. Essays for P. M. S. Hacker*, edited by H.-J. Glock and J. Hyman, Oxford, Oxford University Press, 2009, pp. 26–62), who detects a possible influence of Wittgenstein on the *Logische Untersuchungen*. A similar suggestion is made in C. Penco, *Frege: Two Theses, Two Senses*, «History and Philosophy of Logic», XXIV, 2002, p. 10 and Id. *Frege, Sense, and Limited Rationality*, «The Review of Modern Logic», IX, 2003, pp. 61–62.

[7] PW, p. 154.

therefore scarcely attributable to a change of mind occurred after the composition of «Die Verneinung».

A second and more serious problem is that the idea that truth-functionally equivalent but structurally distinct sentences express the same thought is inconsistent with a principle that appears to have been crucial for Frege, namely the principle that the senses of the parts of a sentence are parts of the sense of the sentence; a statement of the principle of sense composition forms the *ouverture* of «Gedankengefüge», i.e. precisely where Frege makes claims about the sense of compound sentences which are incompatible with that principle. Unless it is hypothesized that the change of mind occurred, not between «Die Verneinung» and «Gedankengefüge», but while he was composing «Gedankengefüge» – a hypothesis consistent with that of a totally confused thinker – some other solution for our problem is called for.

In this paper I offer a diagnosis of the inconsistency manifested in Frege's writings between, on the one hand, the idea that truth-functionally equivalent but structurally distinct sentences express the same thought, and on the other hand, the principle of sense composition. I speak of a «diagnosis» rather than of a «solution» because I think that as far as Frege can be shown to have explicitly maintained both the horns of the dilemma no real conciliation of them is possible. My diagnosis is, as the reader will discover, based on one of the fundamental ideas of Wittgenstein's *Tractatus*, i.e. that the logical components of a sentence do not characterize its sense. I argue that it was under the influence of an analogy between the sense of logical and that of non-logical components that Frege could maintain both that the sense of a logical component of a sentence is part of the sense of that sentence and that truth-functionally equivalent sentences with distinct logical components express the same sense.

The contrast in question is a particular case of the more general tension between the Fregean idea that structurally distinct sentences can express the same thought and the Fregean principle of sense composition. Frege does say that distinct sentences may have the same sense[8]. As David Bell has observed[9], if with «different sentences» it is meant «different tokens of the same sentence type», or «distinct but intensionally isomorphic sentences», no problem arises. But if «different sentences» means «differently structured sentences», then the thesis that distinct sentences may have the same sense is incompatible with the principle of sense composition. In an excellent review of the matter, Silver Bronzo offers an overview of the examples that illustrate this more general tension[10]. He distinguishes five categories of pairs of «structurally distinct sentences» that may express the same thought. The third of Bronzo's categories is the one discussed in the present paper, i.e. the case of truth-functionally distinct sentences. As none of the pairs of sentences belonging to the other four categories manifest a difference in structure that can be explained in terms of truth-functional complexity, the diagnosis that I offer in this paper is inapplicable to them.

The paper is divided as follows. In §2 I expose Frege's most explicit statements about the criteria for sameness of sentential sense. I also hope to show in this section that the notorious problem concerning the «logical» and the «epistemic» criteria that Frege set forth in some writings of 1906 can be solved by a more careful reading of what Frege says in presenting his allegedly «epistemic» criterion. In §3 I discuss

---

[8] Cf. e.g. CP, p. 188; PW, p. 269.
[9] D. Bell, *The Formation of Concepts and the Structure of Thoughts*, «Philosophy and Phenomenological Research», LVI, 1996, p. 584.
[10] S. Bronzo, *Frege on Multiple Analyses and the Essential Articulatedness of Thought*, «Journal for the History of Analytic Philosophy», V, 2017, pp. 1–34.

Frege's principle of sense composition, and explain how it conflicts with the 1906 criterion. In §4 I argue that one of the solutions that have been proposed in the literature, namely that the principle of sense composition only applies to formalized languages, is unable to solve the inconsistency regarding truth-functionally equivalent but distinct sentences. In §5 I illustrate my Wittgensteinian diagnosis of the problem.

2. Equipollence in 1906

In some unpublished diary notes composed in August 1906 Frege wrote:

If a whole is composed of two sentences connected by «and», each of which expresses a thought, then the sense of the whole is also to be construed as a thought, for this sense is either true or false […] If we call the thought of the whole the conjunction of the two component thoughts, then the conjunction too has its opposite thought, as does every thought. Now it is clear what the opposite of a conjunction of the opposite of a thought *A* with a thought *B* is. It is what I express by means of the conditional stroke. The sentence expressing thought *A* is the consequent, that expressing thought *B* the antecedent. But the whole sentence expressing the opposite of the conjunction of the opposite of *A* with *B* may be called the hypothetical sentence whose consequent expresses *A* and whose antecedent expresses *B*[11].

Frege here says that the *thought* that the negation of the conjunction of the negation of *A* with *B* is true is the thought expressed by the conditional sentence «If *B*, then *A*». But since he also speaks of a «whole […] composed of two sentences connected by "and"», this «whole» can only be itself a *sentence*, indeed a conjunctive sentence. When this «whole» is itself negated, to the thought that so results there will correspond a sentence – to deny this latter point would be to deny that Frege is here excluding the possibility of constructing a sentence of the form $\sim(\xi \,\&\, \zeta)$. Therefore, what Frege is here implying is that two sentences of the form $\sim(\sim A \,\&\, B)$ and $B \supset A$ express the same thought, which is a thesis consistent with what he says in «Gedankengefüge» and inconsistent with what he says in «Die Verneinung».

Sometime in 1906 Husserl sends to Frege his own review of Anton Marty's «Über subjektlose Sätze und das Verhältnis der Grammatik zur Logik und Psychologie», in which Marty had maintained that the sentences «If *A* then *B*» and «It is not the case that *A* without *B*», though not «congruent» (i.e. structurally isomorphic), are yet «equipollent» (truth-functionally equivalent)[12]. According to Husserl, these sentences are neither congruent nor equipollent, because their respective negations are not equipollent:

Wenn A so B ≅ Es ist (gilt) nicht A ohne B. Die Kongruenz wird man mit Grund bestreiten dürfen. Es besteht nicht einmal Äquivalenz (Äquipollenz). Die Probe der Negation, die doch wieder Gleichwertiges liefern müsste, versagt. Die Negation auf der linken Seite ergibt: *Es kann* A gelten, ohne dass B gilt, die auf der rechten Seite: *Es gilt* A, ohne dass B gilt[13].

Husserl is here construing the conditional proposition as a *strict* implication («It is not possible that *A* holds without *B* holding»), not as a *material* implication, and this is the reason why the «proof of negation» gives a different result in the two cases[14]. As

[11] PW, p. 188.

[12] A. Marty, *Über subjektlose Sätze und das Verhältnis der Grammatik zur Logik und Psychologie*, «Vierteljahrsschrift für wissenschaftliche Philosophie» XIX, 1895, p. 304n.

[13] E. Husserl, *Bericht über Deutsche Schriften zur Logik in den Jahren 1895–99. Fünfter Artikel*. «Archiv für systematische Philosophie», X, 1904, p. 122; Now in *Husserliana XXII. Aufsätze und Rezensionen (1890–1910)*, edited by B. Rang. The Hague: M. Nijhoff, 1979, p. 255.

[14] See the editorial note in PMC, p. 68n3.

Husserl makes it clear, «Äquipollenz» is Marty's term for what he at other places calls «Äquivalenz»: two sentences are «equivalent» in Husserl's sense if they are truth-functionally equivalent: «Äquivalente Sätze sind bedeutungsverschiedene Sätze, die aber in Bezug auf Wahrheit und Falschheit gleichwertig sind»[15].

In a letter to Husserl dated October 30–November 1, 1906, Frege replies that the conditional «If *A* then *B*» has to be taken in the sense that of the four truth-possibilities of *A* and *B* only that in which *A* is true and *B* is false is negated by the conditional. By negating the conditional, then, we get «*A* is true and *B* is false», or «*A* holds without *B* holding», which is precisely the negation of Husserl's «It is not the case that *A* holds without *B* holding». Thus, the two sentences are «equipollent» in Marty's sense, i.e. «equivalent» in Husserl's[16]. It should be clear that Frege's use of «equipollence» in his October letter accords with Husserl's use of that term in his review of Marty: «equipollence» just means truth-functional equivalence.

But do equipollent sentences express the same thought? In the letter to Husserl of October 1906, Frege specifies that «one must decide to regard equipollent propositions as differing only according to form. […] equipollent propositions have something in common in their content, and this is what I call the thought they express»[17]. And in «A brief Survey of my logical Doctrines», written after August 1906 and thus substantially coeval with the October letter to Husserl, he says:

> Now two sentences *A* and *B* can stand in such a relation that anyone who recognizes the content of *A* as true must thereby also recognize the content of *B* as true and, conversely, that anyone who accepts the content of *B* must straightaway accept that of *A*. (*Equipollence*). It is here being assumed that there is no difficulty in grasping the content of *A* and *B*. [...] I assume there is nothing in the content of either of the two equipollent sentences *A* and *B* that would have to be immediately accepted as true by anyone who had grasped it properly. […] So one has to separate off from the content of a sentence the part that alone can be accepted as true or rejected as false. I call this part the thought expressed by the sentence. It is the same in equipollent sentences of the kind given above. It is only with this part of the content that logic is concerned[18].

According to both the October 1906 letter to Husserl and the slightly earlier «Survey», equipollent sentences have the same sense. According to the letter to Husserl, logically equivalent sentences are equipollent. Since all truth-functionally equivalent sentences are logically equivalent[19], the inference is immediate: truth-functionally equivalent sentences have the same sense. This idea is confirmed by what Frege writes to Husserl in December:

> Now it seems to me that the only possible means of deciding whether proposition *A* expresses the same thought as proposition *B* is the following, and here I assume that neither of the two propositions contains a logically self-evident component part in its sense. If *both* the assumption that the content of *A* is false and that of *B* true *and* the assumption that the content of *A* is true and that of *B* false lead to a logical contradiction, and if this can be established without knowing whether the content of *A* or *B* is true or false, and without requiring other than purely logical laws for this purpose, then nothing can belong to the content of *A* as far as it is capable of being judged true or false, which does not belong to the content of *B*[20].

---

[15] E. Husserl, *Logik. Vorlesung 1902/1903*, edited by E. Schuhmann. Dordrecht: Springer, 2001, p. 136.

[16] PMC, pp. 68–69.

[17] PMC, p. 67.

[18] PW, pp. 197–198.

[19] Though the reverse is, plainly enough, not true: the sentences flanking the = in Basic Law V in *Grundgesetze* are logically equivalent without being truth-functionally equivalent.

[20] PMC, p. 70.

If both the assumption of the falsity of *A* and of the truth of *B*, and the assumption of the truth of *A* and of the falsity of *B* lead to a logical contradiction, then *A* and *B* are logically equivalent. Since truth-functionally equivalent sentences are logically equivalent, this criterion – on the basis of what Frege says in both his letters to Husserl – amounts to saying that truth-functionally equivalent sentences express the same thought.

According to several commentators, the two parallel passages in the «Suvery» and in the December letter to Husserl offer two distinct and irreducible criteria for the sameness of thoughts. According to the latter passage, two sentences express the same thought if they are *logically* equivalent. According to the former, two sentences express the same thought if they are *epistemically* equivalent, with which it is meant that it is not possible to recognize the one as true without thereby recognizing the other as true. The first criterion offers a coarse-grained notion of sense, and fits with what Frege says in «Gedankengefüge»; the second criterion offers a finer-grained notion of sense, and fits with what Frege says in «Die Verneinung»[21].

That these are really distinct criteria may be doubted on several grounds. In the first place, is it credible that between the August and the December of the same year Frege should have stated so distinct and irreducible criteria for such a fundamental feature of his philosophy of logic as the notion of sense? Since that in the December letter to Husserl is presented by Frege as *the only possible* criterion of sameness of sentential sense, if the other criterion were really distinct and irreducible to it should we not be justified in thinking of him as little more than a blundering thinker? In the second place, both criteria are qualified by Frege by saying that neither sentence should contain a self-evident truth[22]. The qualification is intended to prevent tautologies, or sentences containing tautologies, from being regarded as expressing the same sense[23]. Now, the fact that two criteria are qualified in exactly the same manner should make us suspicious about their alleged difference. Could it not be the case that Frege is proposing one and the same criterion, and that he is simply formulating it in different ways?

In fact, the criterion presented in the «Survey» (the «epistemic» criterion) contains an additional qualification that the criterion presented in the December letter to Husserl (the «logical» criterion) lacks: in the «Survey» Frege says that he is assuming «that there is no difficulty in grasping the content of» the two sentences. What is the role of this further qualification? Frege believes that what is capable of being judged true or false in the content of a sentence is the thought expressed by that sentence. As he puts it in *Grundgesetze*, the sense of a sentence is «the thought that its truth-conditions are fulfilled»[24]. Suppose that someone who has studied some elementary logic and thus knows the meaning of the logical constants, grasps the sense, i.e. recognizes the truth-conditions, of $P \supset Q$, and that she also grasps the sense, i.e. recognizes the truth-conditions, of $\sim(P \mathbin{\&} \sim Q)$. Can she thereby fail to recognize that $P \supset Q$ and $\sim(P \mathbin{\&} \sim Q)$ have the same truth-conditions? Surely, if she knows the sense of the logical constants, she cannot. If she does fail to recognize the

[21] See M. Beaney, *Frege. Making Sense*. London: Duckworth, 1996, p. 229; C. Penco, *Frege, Sense, and Limited Rationality*, cit.; S. Schellenberg, *Sameness of Fregean Sense*, «Synthese», CLXXXIX, 2012, pp. 163–175.

[22] This was perceptively noticed by J. van Heijenoort, *Frege on Sense Identity*, «Journal of Philosophical Logic», VI, 1977, pp. 106–107.

[23] This has been sufficiently explained by M. Beaney, *Frege*, cit., pp. 229–230 and S. Schellenberg, *Sameness of Fregean Sense*, cit.

[24] GG §32.

sameness of their truth-conditions, then we should conclude that *there is* some sort of difficulty in grasping the sense of one or both of the sentences. And this is precisely what Frege's additional qualification excludes. In other words, the additional qualification reduces the «epistemic» criterion to the «logical» one: since to grasp the sense of a sentence is to know its truth-conditions, to say that anyone who recognizes the content of *A* as true must thereby also recognize the content of *B* as true *is* to say that anyone who knows the truth-conditions of *A* must thereby also know the truth-conditions of *B*, if *A* and *B* have the same truth-conditions. As the presence in *both* passages of one and the same qualification should suggest, we are here confronted with one and the same criterion of sameness of sentential sense; this criterion can be formulated either as Frege does in the December letter to Husserl directly in terms of identity of truth-conditions, or as he does in the «Survey» in terms of identity of recognition of sense, provided however that to this latter formulation the qualification is added that no difficulty in grasping the sense of the sentences is involved.

I conclude, on the basis of this reading of Frege's alleged «epistemic» criterion, that the criterion of sameness of sentential sense set forth in Frege's writings and letters of 1906 is that logically equivalent sentences have the same sense. That this is the most natural interpretation of Frege's criterion is a direct consequence of taking seriously what Frege says in the October 1906 letter to Husserl and in the slightly earlier «Survey» (equipollent sentences have the same sense) and what he says in the same letter as a reply to Husserl's example of non-equipollent sentences (logically equivalent sentences are equipollent). The claim that logically equivalent sentences have the same sense is also repeated in the clearest of terms in the December letter to Husserl. Truth-functionally equivalent sentences being logically equivalent, we can say with some confidence that Frege's writings of 1906 commit him to the thesis that truth-functionally equivalent sentences express the same thought, in agreement with «Gedankengefüge» and in disagreement with «Die Verneinung».

3. Composition of senses

As Frege scholars know too well, the idea that truth-functionally equivalent but structurally distinct sentences express the same thought is patently in conflict with a principle that Frege quite clearly endorsed, i.e. that the sense of a part of a sentence is a part of the thought that the sentence expresses. This principle is enunciated at various junctures of Frege's philosophical career, and is commonly considered a cornerstone of his entire philosophy of logic. The principle is clearly formulated in *Grundgesetze*, in the «Einführung in die Logik» of 1906, in «Logik in der Mathematik» of 1914, in «Die Verneinung» of 1918, in the notes composed for L. Darmstaedter in 1919, and in the *ouverture* of «Gedankengefüge»[25]. At some point, Frege seems to have toyed with the idea that a parallel principle holds for the reference of a part of a sentence. But he subsequently expressly withdraws it (Mars is not part of the truth-value of the sentence «Mars is half the diameter of the Earth»), while continuing to affirm that the principle holds for senses[26].

The principle of sense composition implies that truth-functionally equivalent but structurally distinct sentences cannot have the same sense. To see this we have to consider what Michael Dummett has labeled «principle K», namely the principle that if one sentence involves a concept that another sentence does not, the two sentences

---

[25] GG §32; PW, pp. 191–192; PW, p. 225; CP, p. 378; PW, p. 255; CP, p. 390.
[26] Dummett takes this to be a good indication that Frege maintained the principle of sense composition even after having abjured the principle of reference composition.

cannot express the same thought[27]. In order for principle K to exclude that truth-functionally equivalent but structurally distinct sentences have the same sense we have to show that sentential operators have, just like the other parts of which a sentence may be composed (typically, proper names and predicates or relational expressions), a sense which is part of the sense of the sentence in which it occurs. If this is so, then by principle K a sentence that involves a given sentential operator cannot have the same sense as a sentence in which that operator does not occur.

Frege says that the reference of a predicate is a concept, and a concept is a function that maps objects onto truth-values. The sense of a predicate is a mode of presentation of a concept, that is, a mode of presentation of a function that maps objects onto truth-values. To entertain the thought expressed by a sentence is to think of its truth-value as being the True: that is, it is to think of an object that is given in a certain way as being mapped to the True by a function that is given in a certain way. As Dummett has stressed, the determination of the truth-value of a sentence goes via the *referent*, not the *sense*, of the proper names that it contains[28]. For the sense of a proper name is the way in which the referent of that name is given, i.e. the manner in which its reference is specified, and that sense contributes to the sense of the sentence. The contribution that the sense of the proper name makes to the sense of the sentence, then, is by specifying that reference, for otherwise it would remain unexplained how the sense of the name could be both a contribution to the sense of the sentence and the manner in which its reference is specified. That is, if it does both things (it both contributes to the sense of the sentence and specifies the reference of the name), it must do one thing (contribute to the sense of the sentence) *by doing* the other thing (specify the reference of the name).

For predicates and relational expressions the matter is similar. The sense of a predicate or relational expression is a way in which a function from the referent of a proper name (an object) to the referent of a sentence (a truth-value) is given. But can we regard the sense of the predicate itself as a *function* from the sense of the proper name to the sense of the sentence? Peter Geach maintained that we can so regard it: «for every predicate there will be a corresponding sense-function – from senses of names to thoughts, e.g. starting from the (most common) sense of the name "Julius Caesar" as argument the predicate "— conquered Gaul" would determine as value of the function the thought that Julius Caesar conquered Gaul»[29]. Dummett argued that we cannot. One of his reasons was that this thesis conflicts with Frege's principle of sense composition: since the sense of the sentence is made up of the senses of its parts, and since in general neither a function not an argument are parts of the value of that function for that argument, it follows that the senses of the predicate and of the proper name that make up a complete sentence are not related as function and argument, and thus that a predicate is not a function from the sense of a proper name to the sense of a sentence[30]. The sense of the predicate and the sense of the proper name do determine the sense of the sentence; not, however, as a function and its

---

[27] M. Dummett, *More about Thoughts*, «Modern Logic», XXX, 1989, pp. 1–19.

[28] M. Dummett, *The Interpretation of Frege's Philosophy*, London, Duckworth, 1981, pp. 250–251.

[29] P.T. Geach, *Review of M. Dummett, Frege: Philosophy of Language*, «Mind», LXXXV, 1976, pp. 436–449. Other arguments in favor of the thesis that the sense of a predicate is a function from senses to senses are in H. Jackson, *Frege on Sense-Functions*, «Analysis», XXIII, 1963, pp. 84–87; G.P. Backer and P.M.S Hacker, *Frege: Logical Excavations*, Oxford, Oxford University Press, 1984, pp. 324–326; A. Diller, *On the Sense of Unsaturated Expressions*, «Philosophical Papers», XXII, 1993, pp. 71–79; K.C. Klement, *Frege and the Logic of Sense and Reference*, London, Routledge, 2001, pp. 65–76.

[30] M. Dummett, *The Interpretation*, cit. pp. 482, 251–253.

argument determine the value of that function for that argument, but as the *parts* determine the whole. A sentence expresses a true thought in case the referent of a name satisfies the condition expressed by the predicate, i.e. in case the predicate is true of the object which is the referent of the name; but it makes little or no sense to say that a thought is true in case the *sense* of a name satisfies the condition expressed by the predicate, i.e. in case the predicate is true of the manner in which the object which is the referent of the name is given. The predicate, having a sense, is a particular manner in which a function from objects to truth-values is given; but this sense is not itself a function from senses to senses. The function from objects to truth-values is the referent, not the sense, of the predicate. Its sense is only a particular manner in which such a function is given, but is not itself a function.

What about sentential operators, then? Frege explains them as functions from truth-values to truth-values. In the sentential fragment of Frege's logical system only two sentential operators are employed: negation and the conditional. Negation is a first-level function of one argument; the conditional is a first-level function of two arguments; the signs that represent these functions are names, i.e. «function-names». These names have both a reference and a sense. The reference of the function-name $\sim\xi$ is a unary function whose value is the opposite truth-value of the truth-value of its argument; the reference of the function-name $\xi \supset \zeta$ is a binary function whose value is the False in case the truth-values of its two arguments are the True and the False, respectively, and the True in all other cases[31].

Given that the referent of the name of a sentential operator is a function from truth-values to truth-values, what is the *sense* of these function-names? It is the manner in which the function that constitutes the function-name's referent is given. Dummett said: «From Frege's viewpoint, sentential operators contribute to the senses of sentences in which they occur in the same general way as do expressions of other kinds»[32]. The sense of the name of a sentential operator is a particular manner in which the function denoted by that name is given; this sense contributes to the sense of the sentence in which it occurs because it determines under what conditions the value of that function is the one or the other of the two truth-values. Just as one and the same object may be given in different ways, these ways corresponding to the distinct senses that we associate to distinct names of that object, so one and the same truth-function can be given in different ways, these ways corresponding to the senses of distinct function-names. For instance, the two sentences $P \supset Q$ and $\sim(P \& \sim Q)$ are truth-functionally equivalent, and therefore have the same reference. But the senses of these two sentences – the thoughts that they express – are different, because each displays a distinct method of recognizing that one or the other of the two truth-values is the truth-value of the sentence, i.e., because to each a distinct computation procedure is associated. It is important to notice that in general it is the *pattern of occurrence* of a sentential operator that determines the difference in sense between two names of one and the same truth-function. In and of itself, the name of a sentential operator has just one sense, which is the way in which its referent – a truth-function – is given, and two distinct function-names have not only different senses, but also different *referents* – they denote distinct truth-functions. So, each of the function-names $\supset$, $\sim$, and $\lor$ has a distinct truth-function for its referent. But when $\sim$ and $\&$ occur in the pattern $\sim\xi \lor \zeta$, this pattern becomes the name of the same truth-function which is the referent of $\supset$. If our language contained as *primitive* a sentential

[31] Cf. GG §32.

[32] M. Dummett, *Frege: Philosophy of Language*, London, Duckworth, 1981 (1st ed. 1973), p. 323.

operator of the form «Not-ξ or ζ», then the sense of this primitive sentential operator would be different from that of ⊃, because to each a distinct computation procedure is associated, but they would be names of one and the same truth-function[33].

Since the sense of a sentence is the manner in which its truth-value is determined, the sense of a sentential operator is the contribution that it makes to the sense of any sentence in which it occurs, i.e. the contribution to the determination of the truth-value of that sentence. Just like, in general, a function from objects as arguments to objects as values is not part of the value of that function for that argument, so the truth-function denoted by the name of a sentential operator is not part of the truth-value of the sentence in which it occurs. By contrast, just like, in general, the sense of a proper name or of a predicate is part of the sense of the sentence in which they occur, so the sense of the name of a sentential operator is part of the sense of the sentence in which it occurs. In this perspective, sentential operators behave just like any other sentential component: their referent is a function from truth-values to truth-values which contributes to the truth-value of the sentence, and their sense is the manner in which that truth-function is given and is part of the sense of the sentence. By principle K, then, two distinct but truth-functionally equivalent sentences cannot have the same sense. By principle K, the sense of ⊃ is part of the sense of $P \supset Q$; but if the sense of $P \supset Q$ were the same as the sense of $\sim(P \,\&\, \sim Q)$, we would have that the sense of ⊃ is part of the sense of a sentence in which it does not occur, which is of course absurd. If the sense of a sentential operator is part of the sense of any sentence in which it occurs, then truth-functionally equivalent but structurally distinct sentences cannot have the same sense. This is in neat contrast not only with what Frege says in «Gedankengefüge» and elsewhere about the sameness of sense of truth-functionally equivalent but structurally distinct sentences, but also with the criterion of sameness of sentential sense enunciated in 1906.

The contrast is all the more evident in «Die Verneinung». He says: «[i]f one thought contradicts another, then from a sentence whose sense is the one it is easy to construct a sentence expressing the other. Consequently the thought that contradicts another thought appears as made up of that thought and negation»[34]. At first sight, this seems plausible: «Trump was not celebrated», which contradicts «Trump was celebrated», is made up of this latter thought and negation. Problems emerge as soon as we consider that not only does «Trump was not celebrated» contradict «Trump was celebrated», but also that, conversely, «Trump was celebrated» contradicts «Trump was not celebrated». As Frege explains, thoughts cannot be divided into negative and affirmative thoughts[35]. Assuming then that «Trump was celebrated» contradicts «Trump was not celebrated», it would follow from Frege's thesis that the thought that contradicts another thought is made up of that latter thought and negation, that «Trump was celebrated» is made up of «Trump was not celebrated» and negation. This is absurd. What Frege might be taken to say is that the *sentence* made up of the negation particle and «Trump was not celebrated», i.e. the sentence «It is not true that Trump was not celebrated», is a distinct sentence expressing the same thought as «Trump was celebrated». As distinct sentences, they are made up of distinct sentence-parts: «It is not true that Trump was not celebrated» contains two occurrences of the negation particle, while «Trump was celebrated» contains none. But the *thought* that both express is not made up of the corresponding parts of the sentences that express it,

---

[33] See M. Dummett, *Frege*, cit. p. 323.
[34] CP, p. 386.
[35] CP, p. 380.

for otherwise we should conclude that one and the same thought contains and does not contain negation. This absurd conclusion, however, is simply a consequence of the application of the principle of sense composition to the sense of sentential operators. This application is consistent with the assertion that *A* and «the negation of the negation of *A*» have the same sense, but is incompatible with the idea that if one thought is the negation of another thought, then the former contains the latter thought plus negation.

4. Formal languages

Frege's most recurring examples of distinct sentences with the same sense are pairs of corresponding sentences in the active and passive voice: «The sentences "M gave document A to N", "Document A was given to N by M" […] express exactly the same thought»[36]. It is probably with this kind of examples in mind that some authors have proposed to emendate Frege's principle of sense composition so as to render it compatible with the thesis that distinct sentences may have the same sense. For example, according to Mark Textor[37], while in a natural language it is not always the case that the manner in which a sentence is structured into parts may be taken as an indication of the manner in which the thought expressed by that sentence is structured into parts, in a regimented language like Frege's Begriffsschrift it is possible to take the structure of a sentence as an indication of the structure of the thought it expresses. Therefore, Textor argues, while Frege's principle of sense composition does not hold generally for any language, it does hold for a regimented language as the Begriffschrift. In such a language, sentences that in natural language would be distinctly structured, as «*m* gave document *a* to *n*» and «Document *a* was given to *n* by *m*» are, would be symbolized by one and the same formula: G*anm*.

This proposal is, clearly enough, unsatisfying for our present purposes. In the Begriffsschrift, and in any formalized language for the sentential calculus, the sentences of natural language «It rains» and «It is not the case that it does not rain» may be formalized as *A* and ~~*A*, respectively. ~~*A* contains the negation sign ~ as part, and thus the thought that it expresses should contain a corresponding thought-part. But the thought that ~~*A* expresses is – at least according to Frege's 1906 writings and «Gedankengefüge» – the same as that expressed by *A*, and it is absurd to say that this thought should contain the thought-part corresponding to the negation sign. Therefore, Textor's restricted principle of sense composition is unable to account for the sameness of sense of sentences in which sentential operators occur.

On similar lines, Dummett restricts the possibility that distinct sentences may express the same thought to the sentences of natural language. But two distinct sentences of a formal language cannot express the same thought, according to Dummett, because this would violate principle K, and Frege is not in the position to repudiate principle K. In fact, Dummett thinks that principle K is compelling: «no one can be said to grasp the thought that a certain politician is dishonest, for example, if he lacks the concept of dishonesty»[38]. Accordingly, if Frege's idea was that distinct sentences of a formal language can express the same thought, it has to be rejected, and Frege's occasional statements of the thesis that distinct sentences express the same thought are either to be construed, as Textor does, as being restricted to natural

---

[36] PW, p. 141.

[37] M. Textor, *Frege's recognition criterion for thoughts and its problems*, «Synthese», CXCV, 2017, pp. 2677–2696.

[38] M. Dummett, *More about Thoughts*, cit., p. 298.

languages, or, if they really concern formal languages as well, they have to be dismissed as blunders on Frege's part.

I do not think that Dummett's solution is completely satisfying. For one thing, Frege does say, not only in the later «Gedankengefüge» but also in the earlier «The Argument for my stricter Canons of Definitions», that truth-functionally equivalent but differently structured sentences express the same thought. These claims obviously also apply to formal languages, and it is not so clear that they can be dismissed as blunders. Also, we have seen that his 1906 criterion of equipollence allows for the sameness of sense of truth-functionally but differently structured sentences. This circumstance is also hardly explainable as a confusion of thought. But in the second place, Dummett's argument to the effect that the thesis that distinct sentences of a formal language can express the same thought violates principle K, while seeming to hold for atomic sentences (sentence that do not involve sentential operators), yet constitutes a *petitio principii* when it is construed as holding of molecular sentences. It is true that no one can be said to grasp the thought expressed by «If it rains, then it thunders» if he lacks the concept of rain; but can one be said to grasp that thought if he lacks the concept of the conditional relation while grasping the senses of negation and of logical conjunction? To deny that one can be said to grasp the sense of the thought expressed by «If it rains, then it thunders» if he lacks the concept of the conditional relation is to *presuppose* that the senses of «If it rains, then it thunders» and «It is not the case that it rains without thundering» are distinct. And since the difference of sentential sense of molecular sentences is presupposed by it, Dummett's argument to the effect that sameness of sentential sense of truth-functionally equivalent but structurally distinct sentences would violate principle K cannot be used to prove such difference.

5. A Wittgensteinian diagnosis

In 1906, Frege offers a criterion of sameness of sense according to which two sentences have the same sense if they are logically equivalent. This conflicts with what he says in «Die Verneinung» about the senses of A and of $\sim\sim A$, namely that they have different senses; the 1906 criterion accords nonetheless with what Frege says in «Gedankengefüge» about these sentences, namely that they have the same sense. That Frege did not change his mind between the composition of «Die Verneinung» and the composition of «Gedankengefüge» is shown by the fact that the 1906 criterion accords with the *later*, not with the earlier, statement concerning A and $\sim\sim A$. However, if the 1906 criterion had been Frege's criterion of sameness of sentential sense throughout, then the contrast between «Die Verneinung» and «Gedankengefüge» could be explained as a momentary repudiation of that criterion at the time of «Die Verneinung». But this cannot be the right explanation, because the 1906 criterion is inconsistent with a principle to which Frege was certainly committed throughout, i.e. the principle of sense composition. I have argued that even if we restrict the principle of sense composition to sentences of a formal language, the problem of explaining the sameness of sense of A and of $\sim\sim A$ remains. Was Frege an inconsistent thinker?

I think we can diagnose what the source of the inconsistency is if we restrict the principle of sense composition in the following manner. While in general the sense of a part of a sentence is part of the sense of the sentence, the sense of a sentential operator is not part of the sense of the sentence of which it is a part. If taken with this restriction, the principle of sense composition only holds of sentential components

other than sentential operators. Sentential operators do not behave like other sentential components: their referent is a function from truth-values to truth-values which contributes to the truth-value of the sentence, their sense is the manner in which that truth-function is given, but that sense is not part of the sense of the sentence. This restriction allows us to regard the sense of «If Venus is a planet, then the Earth is a planet» as being composed of the senses of the proper names «Venus» and «the Earth», and of the sense of the predicate «ξ is a planet», but not of sense of the sentential operator «If ξ, then ζ»; and this is why a sentence composed of the same atomic parts but of different sentential operators (such as «Either Venus is not a planet or the Earth is a planet») can have the same sense that it has.

This restriction highlights an asymmetry between how logical and non-logical components behave at the level of sense. Sentential operators do contribute to the determination of the sense of any sentence in which they occur, but they are not parts of the sense of that sentence. Frege could affirm both his 1906 criterion of sameness of sentential sense and the principle of sense composition (in its unrestricted form) because he was under the influence of a false analogy between the sense of a non-logical component (the parts of an atomic sentence) and the sense of a logical component of a sentence. According to this analogy, just as the sense of a non-logical component is part of the sense of any sentence of which it is part, so the sense of a logical component (a sentential operator) is part of any sentence in which it occurs. That the analogy is a false analogy is shown by the fact that Frege's claim that truth-functionally equivalent but structurally distinct sentences may express the same sense is incompatible with the unrestricted principle of sense composition, while it is compatible with the restricted version I have suggested above.

The first to see the inconsistency of Frege's doctrine of the sense of sentential operators was, of course, Wittgenstein. In the *Tractatus* he declares the proposition that the «logical constants» do not represent to be his *Grundgedanke*, his fundamental thought in the philosophy of logic[39]. The *Grundgedanke* is explicitly presented as an objection to the philosophy of logic of Frege (and Russell): «Here is becomes clear that there are no such things as "logical objects" or "logical constants" (in the sense of Frege and Russell)»[40]. One of the arguments in support of the *Grundgedanke* is the following. If ~ were an object, then ~~p would treat of ~ as an object; this is impossible, because ~~p says the same as p, and p does not treat of ~[41]. The argument that I have used above in order to show that the principle of sense composition, in its unrestricted version, is incompatible with the 1906 criterion of sameness of sentential sense, can be regarded as an application of this argument of Wittgenstein's to the realm of Fregean senses: in Fregean terms, if the sense of ~ were part of the sense of any sentence in which it occurs, then ~~p would have the sense of ~ as one of its parts. This is impossible if the 1906 criterion is correct, because by that criterion ~~p has the same sense as p, and to say that the sense of ~ is part of the sense of p is an absurdity. No such absurdity emerges as soon as we accept the restricted version of the principle of sense composition that I have proposed above.

A second argument that Wittgenstein offers for his *Grundgedanke* is that the sentential operators are not relations in the sense of right and left[42]. Peter Sullivan argues, correctly in my opinion, that the target of this criticism is Frege: «In *Grundgesetze* all complete expressions, simple and complex singular terms and

[39] TLP 4.0312.
[40] TLP 5.4.
[41] Cf. TLP 5.44.
[42] Cf. TLP 5.42.

propositions, are of the same logical category. Since the logical category of all other expressions is fixed relatively to complete expressions, there is no logical difference between an expression that takes two terms to make a proposition and an expression that takes two propositions to make a bigger proposition»[43]. There is no difference for Frege between a predicate or relational expression and a sentential operator: a predicate or relational expression is the name of a function from objects to truth-values, a sentential operator the name of a function from truth-values to truth-values. Just as the relational expression «ξ is to the left of ζ» maps the ordered pair <Tom; Jack> to the True if Tom is to the left of Jack, so the sentential operator «If ξ, then ζ» maps the ordered pair <It rains; It thunders> to the True if it does not rain without thundering; and just as «ξ is to the left of ζ» is part of the sense of «Tom is to the left of Jack», so «If ξ, then ζ» is part of the sentence «If it rains, it thunders».

Perhaps the best of Wittgenstein's arguments against Frege's treatment of sentential operators as relations (which is also an argument in support of the *Grundgedanke*) is contained in the following remark from the *Notes on Logic*: «Logical indefinables cannot be predicates or relations, because propositions, owing to sense, cannot have predicates or relations»[44]. Sullivan explains this remark as follows[45]. For Wittgenstein, a proposition is a fact that represents a fact, not the name of a fact. What symbolizes in the proposition «Tom is to the left of Jack», symbolizable as *tLj*, is the fact that *L*, which is a relational expression, occurs between the names *t* and *j*. Now, if the sentential operator $\supset$ were a relation in the same sense as *L*, we should say that what symbolizes in *tLj $\supset$ jRt* is the fact that $\supset$ occurs between *tLj* and *jRt*. But since both *tLj* and *jRt* are facts, not names, we would be saying that what symbolizes in *tLj $\supset$ jRt* is the fact that $\supset$ occurs between two facts, which is nonsense. An atomic proposition, *owing to sense*, i.e. owing to its being a fact about certain symbols, cannot in its turn be considered as the name of an object that a *molecular* proposition is about.

For Wittgenstein, a sentential operator represents an «operation», not a function. An operation does not characterize the sense of the sentences in which it occurs. This is evident: $\supset$ does not characterize the sense of $P \supset Q$. If it did, it would equally characterize the sense of $\sim(P \mathbin{\&} \sim Q)$, because these sentences have, for Wittgenstein, the same sense. An operation is «the expression of a relation between the structures of its results and of its bases»[46], i.e. between the structure of the molecular sentence in which it occurs and the structure of the atomic sentences on which it operates in order to obtain that molecular sentence as result. In expressing a relation between the structures of sentences, an operation «is not the mark of a form, but only of a difference in forms»[47]: the operation $\supset$ is not the mark of $P \supset Q$; it is, rather, the mark of a difference between *P* and *Q* on the one hand, and $P \supset Q$ on the other. As Peter Hylton puts it, «an operation may express the difference between two senses, without at the same time characterising either sense»[48], i.e. without at the same time being part of either. By contrast, a function, if conceived in the Fregean manner, is, in Wittgensteinian terms, a mark of a form, i.e. the form shared by all the sentences that

[43] P.M. Sullivan, *The Totality of Facts*, «Proceedings of the Aristotelian Society», C, 2000, p. 178.
[44] L. Wittgenstein, *Notes on Logic*, in M. Potter, *Wittgenstein's Notes on Logic*, Oxford, Oxford University Press, 2008, p. 285.
[45] P.M. Sullivan, *The Totality of Facts*, cit., pp. 178–179.
[46] TLP 5.22.
[47] TLP 5.241.
[48] P. Hylton, *Propositions, Functions, and Analysis. Selected Essays on Russell's Philosophy*, Oxford, Clarendon Press, 2005, p. 150.

may be the value of that function for some argument. By being a «mark of a form», a function does characterize (i.e. is a part of) the sense of any sentence having that form («ξ is a philosopher» characterizes the sense of «Socrates is a philosopher», «Plato is a philosopher», «Aristotle is a philosopher», etc.).

Wittgenstein identified the sense of a proposition with its truth-conditions. The sense of a proposition is its agreement and disagreement with the truth-possibilities of the atomic facts[49], an atomic fact is that which is represented in an elementary proposition[50], and an elementary proposition is a connection of names[51]. Wittgenstein of course has an account of how atomic propositions represent atomic facts: by being pictures of them. An atomic proposition is a picture of a fact because to each of the elements of the fact there corresponds an element in the proposition and to the structure of the fact there corresponds the structure of the proposition. But Wittgenstein's picture theory applies directly only to elementary propositions. It applies to molecular propositions only *indirectly*, because the sense of a molecular proposition is its agreement and disagreement with the truth-possibilities of elementary propositions. The sense of a molecular proposition is not, on this account, a picture composed of the senses of the elementary propositions plus the senses of the sentential operators, for a sentential operator is an operation, not a function, and thus does not characterize the sense of the sentences in which it occurs. As Hylton observes, «The fundamental connection between language and the world is set up simply at the level of elementary propositions, and our ways of compounding elementary propositions into non-elementary propositions do not require any further connections of this sort»[52].

By contrast, Frege assumes that just as an atomic sentence is a picture of an atomic thought in which to each element of the sentence there corresponds an element of the thought, so a molecular sentence is a picture of a molecular thought in which to each element of the sentence, *sentential operators included*, there corresponds an element of the thought. He evidently considers the principle of sense composition as a picture theory of thought – as Textor has suggested[53] –, a picture theory that does not distinguish between atomic and molecular sentences. Under the influence of the analogy between the sense of logical components and that of non-logical ones, Frege could thus maintain that the sense of a sentential operator is part of the sense of any sentence in which it occurs. Frege apparently thought that the only way to express the difference between two senses is by being part of one of them (and, presumably, by not being part of the other). He never considered the possibility that the way in which the sense of a sentential operator contributes to the sense of any sentence in which it occurs is not by being a part of that sense, but, like Wittgenstein's operations, by merely marking a difference between senses. The false analogy between the sense of logical components and that of non-logical ones brought him to say that the sense of a sentential operator is part of the sense of any sentence in which it occurs, thus incurring in the troubles caused by his holding, at the same time, that logically equivalent but structurally distinct sentences have the same sense. The inconsistency between Frege's criterion of sameness of sentential sense and his principle of sense composition is based on a certain assumption about sense (that the analogy between the sense of logical components and the sense of non-logical components holds), an

---

[49] TLP 4.2.
[50] TLP 4.21.
[51] TLP 4.22.
[52] P. Hylton, *Propositions*, cit., p. 147.
[53] M. Textor, *Frege's recognition criterion*, cit.

assumption that, as Wittgenstein's work shows, is by no means inevitable[54].

In sum, my Wittgensteinian diagnosis of the tensions in Frege's doctrine of compound thoughts is the following. Frege's doctrine of sense manifests a tension between his criterion of sameness of truth-functionally complex sense and his principle of sense composition because he failed to see the difference between the way in which a non-logical component contributes to the sense of any sentence in which it occurs, i.e. by being a part of that sense, and the way in which a logical component contributes to the sense of any molecular sentence that may be constructed by means of it, i.e. by being not a part of the sense of that sentence, but only a way for marking a difference between senses.

Of course, my diagnosis is not thoroughly Wittgensteinian. For Wittgenstein, *no* sub-sentential element has a sense: a sentence is a concatenation of names; names have a *Bedeutung* without having a *Sinn*, while the sentence has a *Sinn* without having a *Bedeutung*. A thoroughly Wittgensteinian reformulation of Frege's doctrine of compound thoughts would therefore require not only abandoning any talk of truth-functional sense composition, but also any talk of sub-sentential sense composition, and with it the principle of sense composition itself. Such a thoroughly Wittgensteinian reformulation of Frege's doctrine would probably coincide with the *Tractatus* itself, or at least with the manner in which that work develops Frege's philosophy of logic to its extreme consequences. My purpose in this paper was more modest: it was to identify the source of a main tension in Frege's doctrine of compound thoughts and to offer a rectification of it whose best and most consequent justification is to be found in the Wittgenstein's *Grundgedanke* and in the arguments that support it.

Summary. Frege on Compound Thoughts: A Wittgensteinian Diagnosis

In this paper I offer a diagnosis of the inconsistency manifested in Frege's writings between the idea that truth-functionally equivalent but structurally distinct sentences express the same thought and the principle of sense composition. My diagnosis is that Frege was under the influence of an analogy between how the sense of logical and that of non-logical components contributes to the sentence of which it is part. I show that in order to avoid the inconsistency, he should have restricted the principle of sense composition in the following way: the sense of any part of a sentence other than a sentential operator is part of the sense of that sentence. The diagnosis is based on one of the fundamental ideas of the *Tractatus* of Wittgenstein, i.e. that the sense of a logical component of a sentence does not characterize the sense of that sentence.

Keywords: Frege, Sense, Sentential operators, Sense composition, Wittgenstein, Operations.

---

[54] Following R.G. Heck and R. May (*The Composition of Thoughts*, «Noûs», XLV, 2011, pp. 126–166), one could distinguish the *strong* claim that the sense of a part of a sentence is part of the sense of the sentence from the *weaker* claim that the sense of a part of a sentence contributes to the determination of the sense of the sentence. That there is a conceptual difference between the two claims is indeed evidenced by the comparison with the case of reference: although the references of the parts of a sentence determine the reference of the sentence (i.e. its truth-value), yet the references of the parts of a sentence are not parts of the reference of the sentence (Mars is not part of the truth-value of the sentence «Mars is half the diameter of the Earth»). Frege is clearly committed to the strong claim (which amounts to his principle of sense composition). Wittgenstein can accept neither, because no sub-sentential component has sense for him.

FRANCESCO BELLUCCI is Senior Assistant Professor at the Department of Philosophy and Communication Studies, University of Bologna, Via Azzo Gardino 23, 40122 Bologna.
e-mail: francesco.bellucci4@unibo.it