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On the axioms of singquandles

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ABSTRACT

In this paper we deal with the notion of singquandles introduced in [CEHN17]. This is an algebraic structure that naturally axiomatizes Reidemeister moves for singular links, similarly to what happens for ordinary links and quandles. We present a new axiomatization that shows different algebraic aspects and simplifies applications. We also reformulate and simplify the axioms for affine singquandles (in particular in the idempotent case).

Keywords: Singular knots, link invariants, quandles.

Mathematics Subject Classification 2020: 20L05, 57K10

1. Introduction

Singular knot theory was introduced in 1990 by Vassiliev [Vas90] as an extension of classical knot theory allowing also immersions of $S^1 \to S^3$ with singularities; the aim was to get information on knots by studying the space of all their isotopy classes. Singular knots gave rise to a decreasing filtration on the infinite dimensional vector space generated by isotopy classes of knots. Together with the introduction of this extension, the notion of finite type (or Vassiliev) invariants, as invariants vanishing on some step of this filtration, was introduced producing a new point of view on knot theory. Since then, many knot invariants, as well as different knot representations and techniques have been extended to singular knots and links (see for example [Bir93,Fie10]). Recently, in [CEHN17], a singular link invariant having the form of a binary algebraic structure and called singquandle, was defined; as the name suggests, this structure extends to the singular case the quandle invariant for classical links. Quandles, or distributive groupoids, were introduced in the 1980s by Joyce [Joy82] and, independently, by Matveev [Mat82]: the fundamental quandle $Q(L)$ of a link L, axiomatizes the Reidemeister moves and it is a classifying invariant for prime knots. Even if comparing quandles is as difficult as comparing links, as for the case of Vassiliev theory, the introduction of quandles (and racks) in knot theory paved the way for the construction of new invariants and techniques. Moreover, beside the interest of quandles for knot theory, these structures are relevant in many other areas, as theoretical physics, for the study of the Yang-Baxter equation (see [AG03,ESS99,ESG01]) or abstract algebra itself (see [Sta15,BS21,BF21]). In [BEHY18] and [CCE21] the singquandle construction is done for the oriented case, while in [NOS19] the notion of psyquandles is introduced for the case of pseudoknots and singular knots and links as a generalization of biquandle structures for classical and virtual links [FJSK04].

In this paper we deal with algebraic structures associated to singular links. More precisely, we reformulate the definition of oriented and non-oriented singquandles, by using the language of binary operations: we simplify the axioms introduced in the above mentioned papers and we prove the independence of our axioms. With this new definition we are able to prove some algebraic properties of these structures and to simplify some associated constructions as, for example, *Alexander* singquandles introduced in [CEHN17] (in this paper we prefer the alternative terminology *affine* singquandles).

Starting from our new reformulation of the oriented singquandles $SQ(L)$ associated to a singular link L, we remark that, if $\bar{L}_+$ (resp. $\bar{L}_-$) is the link obtained by replacing each singular crossing of L with a positive (resp. negative) crossing, see the left (resp. right part) of Figure 9, then $Q(\bar{L}_+)$ (resp. $Q(\bar{L}_-)$) is a quotient of $SQ(L)$. This could be a starting point to explore the connection between $SQ(L)$ and $\{Q(\bar{L}_i)\}$, with $\{\bar{L}_i\}$, being the set of all regular links obtained from L by replacing a singular crossing with either a positive or negative crossing.

The definition of the (oriented) singquandle associated to a singular link is purely combinatorial. We wonder if there is also a topological construction for such an object as for the fundamental quandle for classical links [Joy82,Mat82] or alternatively for one of its proper quotients as for the fundamental biquandle of a link [Hor19].

In Section 2 we recall all the algebraic notions that will be used in the rest of the paper, as well as the definition of classical and singular links and their associated algebraic structures. In Section 3 we analyze the oriented case, reformulating the definition of oriented singquandle, while the non-oriented case is studied in Section 4. We conclude the paper by analyzing the case of affine singquandles.

In the paper we sometimes use the software [McC10] to generate examples and non examples of binary algebraic structures.

2. Preliminary results

2.1. *Binary structures and right quasigroups*

A binary operation $\cdot$ on a set X is a mapping

$$\cdot : X \times X \longrightarrow X, \quad (x, y) \mapsto x \cdot y$$

and a binary algebraic structure is a set X endowed with a set of binary operations. Let $(X, \cdot)$ be such a structure, the right multiplication by $x \in X$ is the map defined by setting

$$R_x : y \mapsto y \cdot x$$

and the *squaring mapping* is the map defined as

$$\sigma : X \longrightarrow X, \quad x \mapsto x \cdot x.$$

A (bijective) map $f : X \longrightarrow X$ is said to be an endomorphism (automorphism) of $(X, \cdot)$ if $f(x \cdot y) = f(x) \cdot f(y)$ for every $x, y \in X$. The group of automorphism of $(X, \cdot)$ is denoted by $\mathrm{Aut}(X, \cdot)$.

The structure $(X, \cdot)$ is a *right quasigroup* if R_x is a permutation for every $x \in A$. Clearly we can define the right division associated to $\cdot$ as $x/y = R_y^{-1}(x)$. Thus, for the scope of this paper, a right quasigroup can be alternatively be defined as a binary algebraic structure $(X, \cdot, /)$ such that

$$(x \cdot y)/y = x = (x/y) \cdot y.$$

Note that also $(X, /, \cdot)$ is a right quasigroup. A right quasigroup $(X, \cdot, /)$ is said to be:

(i) *permutation* if $x \cdot y = x \cdot z$ holds;
(ii) *idempotent* if $x \cdot x = x$ holds;
(iii) *involutory* if $(x \cdot y) \cdot y = x$ holds;
(iv) *right distributive* if $(x \cdot y) \cdot z = (x \cdot z) \cdot (y \cdot z)$ holds;

(v) 2-*divisible* if σ is bijective

for all $x, y, z \in X$. Idempotent permutation right quasigroups satisfy the identity $x \cdot y = x \cdot x = x$ and they are called *projection*. Idempotent right distributive right quasigroups are called (right) *quandle*.

Given a right quasigroup $(X, \cdot)$, the orbits with respect to the natural action of the group $\mathrm{RMlt}(X, \cdot) = \langle R_x,\, x \in X\rangle$ are called the *connected components* of $(X, \cdot)$ and we say that $(X, \cdot)$ is *connected* if such group is transitive on X. Note that X is the union of the connected components of its generators, therefore the following statement follows.

Lemma 2.1. *Let $(X, \cdot)$ be a right quasigroup generated by $S \subseteq X$. Then $(X, \cdot)$ is connected if and only if the element of S are in the same connected component.*

Let $(A, +)$ be an abelian group, $f \in \mathrm{Aut}(A, +)$, $g \in \mathrm{End}(A, +)$ and $c \in A$. The right quasigroup $(A, \cdot)$ defined by setting

$$x \cdot y = f(x) + g(y) + c$$

is called *affine* right quasigroup over A. We denote such right quasigroup by $\mathrm{Aff}(A, f, g, c)$.

In the paper we usually deal with algebraic structure with two binary operations denoted by $(X, \cdot, *)$. In the sequel, we denote the operation $\cdot$ just by juxtaposition and the right multiplication mappings by $x \in X$ respectively by $R_x : y \mapsto y \cdot x$ and by $\rho_x : y \mapsto y * x$.

2.2. *S-right quasigroups*

Let us introduce a class of right quasigroups that will be relevant in the present paper in connection with coloring invariants of singular knots. The right quasigroups satisfying the identity

$$x/y = x(yx) \tag{S}$$

will be called *S-right quasigroups.*

Lemma 2.2. *Let $(X, \cdot)$ be a binary algebraic structure. The following are equivalent:*

(i) *$(X, \cdot)$ is a S-right quasigroup.*
(ii) *The identity*

$$(yx)(x(yx)) = y \tag{2.1}$$

holds.
(iii) *The identity*

$$(x(yx))y = x \tag{2.2}$$

holds.

Proof. (i) $\Rightarrow$ (ii) We have

$$y = (yx)/x \overset{(S)}{=} (yx)(x(yx)).$$

(ii) $\Rightarrow$ (iii) Using the identity (2.1) twice, we have

$$(x(yx))y = (x(yx))((yx)(x(yx))) = x.$$

(iii) $\Rightarrow$ (ii) Using the identity (2.2) twice, we have

$$(yx)(x(yx)) = (y((x(yx))y))(x(yx)) = y.$$

(ii), (iii) $\Rightarrow$ (i) Let us denote $x \bullet y = x(yx)$. The identities (2.1) and (2.2) read

$$(xy) \bullet y = (x \bullet y)y = x,$$

therefore $\bullet$ is the right division with respect to $\cdot$. $\square$

It is easy to prove that the identity (S) is equivalent to

$$xy = x/(y/x). \tag{S'}$$

Indeed, it is enough to replace y by y/x in order to get the identity (S') from (S) and replace y by yx conversely.

Proposition 2.3. *Let $(X, \cdot, /)$ be a right quasigroup. The following are equivalent:*

(i) The map

$$\varphi : X \times X \longrightarrow X \times X, \quad (x, y) \mapsto (xy, y/x)$$

is an involution.

(ii) $(X, \cdot, /)$ is a S-right quasigroup.

(iii) $(X, /, \cdot)$ is a S-right quasigroup.

Proof. The items (ii) and (iii) are equivalent because of the previous remark.

(i) $\Leftrightarrow$ (ii), (iii) Since

$$\varphi^2(x, y) = \varphi(xy, y/x) = ((xy)(y/x), (y/x)/(xy))$$

then φ is an involution if and only if

$$(xy)(y/x) = x, \qquad (y/x)/(xy) = y,$$

hold. Namely (S) and (S') hold. $\square$

Let $t = s$ be an identity that follows from (S). According to Proposition 2.3, then also the identity $t' = s'$ where $\cdot$ and $/$ are interchanged follows. For instance the identity $x = (x(y((x(yx))y)))y$ is a consequence of (S) and so also $x = (x/(y/((x/(y/x))/y)))/y$ holds for S-right quasigroups.

Let us show some examples of S-right quasigroups.

Example 2.4.

(i) Let $(X, \cdot)$ be an involutory right quasigroup. Then $(X, \cdot)$ is a S-right quasigroup if and only if

$$x(yx) = xy \tag{2.3}$$

holds. Thus, if $R_x = R_y$ whenever x and y are in the same connected component of $(X, \cdot)$ the identity (2.3) is satisfied. For instance, involutory permutation right quasigroups and 2-reductive involutory (right) quandles have such property (see [JPSZD15] for the construction of such quandles).

(ii) Let $X = \mathrm{Aff}(A, f, g, c)$. Then X is an S-right quasigroup if and only if

$$fg^2 + f^2 - 1 = g + fgf = (f + fg + 1)(c) = 0. \tag{2.4}$$

Then $(A, +)$ has a $\mathbb{Z}[t, t^{-1}, u]/(tut + u, t^2 + tu^2 - 1)$-module structure. On the other hand, given a module M over such ring, $(M, \cdot)$ where

$$x \cdot y = tx + uy + c$$

is a S-right quasigroup if and only if $(1 + t + tu)c = 0$ (e.g. $c = 0$).

(iii) A quandle $(Q, \cdot)$ is a S-right quasigroup if and only if

$$x = ((xy)x)y$$

holds. Let G be a group and Q be the conjugation quandle associated to G. Then Q is a S-right quasigroup if and only if

$$[x, y^2] = 1$$

holds, i.e. $\{y^2 : y \in G\} \subseteq Z(G)$ (G is also said to be 2-central).

(iv) A group $(G, \cdot)$ is a right quasigroup and $x/y = xy^{-1}$. Therefore, $(G, \cdot)$ is a S-right quasigroup if and only if

$$xy^{-1} = x(yx) \Leftrightarrow x = y^{-2}$$

holds. Such an identity is satisfied only by the trivial group.

2.3. *Regular and Singular links, quandles and singquandles*

An oriented link in S^3 is an embedding $S^1 \sqcup S^1 \sqcup \cdots \sqcup S^1$ of μ disjoint copies of S^1 in S^3, together with the choice of an orientation on every connected component. Each oriented link can be represented by means of a regular diagram on a plane, that is a plane quadrivalent directed graph having vertices colorated with overcrossing and undercrossing (see Figure 1). As oriented links in S^3 are

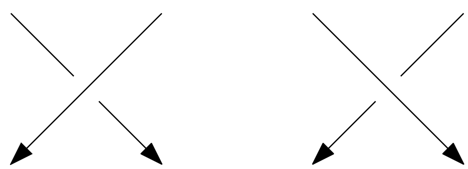

Fig. 1. The two possible colorings of a vertex in a diagram of a link.

considered up to isotopy, their diagrams are considered up to planar isotopy (preserving colorings) and the classical Reidemeister moves depicted in Figure 2.

A usual way to construct link invariants is to define them on diagrams in a way that ensure invariance under Reidemeister moves. With respect to this approach, algebraic invariants are provided by quandles. Indeed, the axioms satisfied by a quandle structure, idempotency, the right-quasigroup property and right-distributivity, correspond exactly to the Reidemeister moves. More precisely given an non-empty set Q and a binary operation $*$, suppose to color each arc of a link diagram by an element of Q as depicted in Figure 3: requiring the invariance under the Reidemeister moves, correspond to imposing the quandle axioms to the binary operation. So, to each oriented link L we can associate a quandle $Q(L)$, called *fundamental quandle* of L, by taking the quotient of the free quandle generated by the arcs of a diagram of L modulo the crossing relations represented in Figure 3. A coloring of a link by a quandle T is a quandle homomorphism $f \colon Q(L) \to T$ or equivalently a coloring of the edges of a link diagram with elements of T such that the crossing relations in Figure 3 are satisfied; so a quandle could color a link L if and only if it is a quotient of the fundamental quandle of L.

Remark 2.5. Given a coloring of a link L by a quandle T, the colors (i.e. the elements of T) that color the edges within the same connected component of L are also within the same connected component of T. Moreover, for a knot any pair of colors at an arbitrary crossing completely determines the whole coloration. So, all quandles that color a knot diagram are 2-generated and connected according to Lemma 2.1.

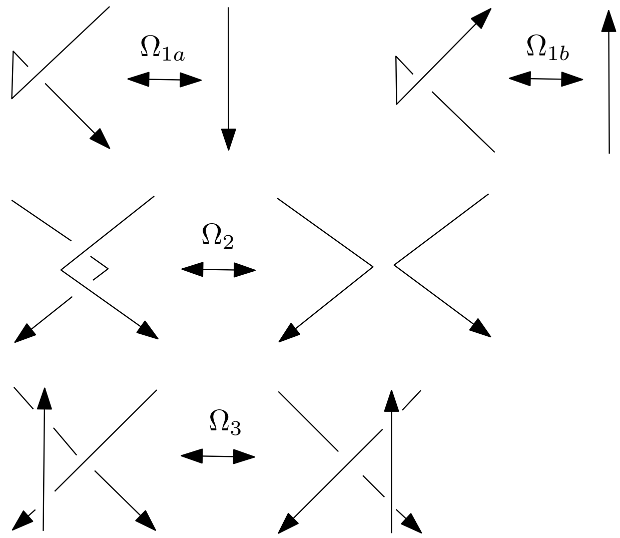

Fig. 2. The classical Reidemeister moves.

In [FR92], the fundamental quandle was introduced in a topological way using paths in the link complement. Using this approach it is easy to see that the fundamental quandle in fact depends on the orientation of the pair (S^3, L): if both the orientation of the 3-sphere and that of the link L are changed the quandle structure does not change, while if only one of them is changed the quandle $(Q(L), \cdot, /)$ and $(Q(L'), /, \cdot)$ are isomorphic where $Q(L')$ is the fundamental quandle associated to the pair with just one orientation reversed.

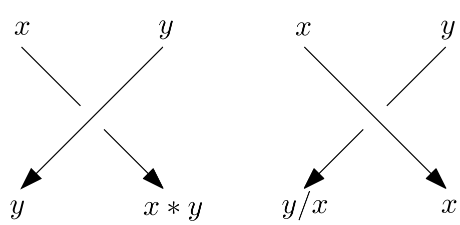

Fig. 3. The coloring of crossings in the fundamental quandle.

The above way of reasoning could be generalized to the case of oriented singular links. An oriented singular link is the immersion $S^1 \sqcup S^1 \sqcup \cdots \sqcup S^1$ of μ disjoint copies of S^1 in S^3, together with the choice of an orientation on every connected component. Taking a combinatorial point of view, an equivalent definition of a singular link is an equivalence class of plane quadrivalent directed graphs having some vertices colorated with overcrossing and undercrossing, modulo the equivalence relation generated by planar isotopy (preserving colorings) and the Reidemeister moves depicted in Figure 2 and in Figure 4 (see [BEHY18]).

The colorated crossings are called regular crossing, and the others are called singular crossing. The set of moves displayed in both Figures 2 and 4 are called singular Reidemeister moves. In [BEHY18] an algebraic structure having three binary operations $(*, R_1, R_2)$, associated to a singular link and generalizing the fundamental quandle associated to classical links, is constructed. It is called the fundamental oriented singquandle and it is defined as follows.

Suppose to color the arcs of the diagram of a singular links as in Figure 5; as before, imposing

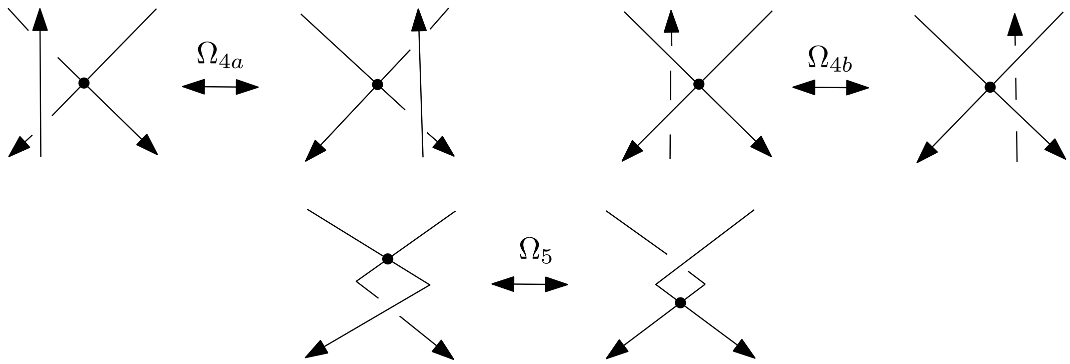

Fig. 4. The Reidemeister moves for singular crossings.

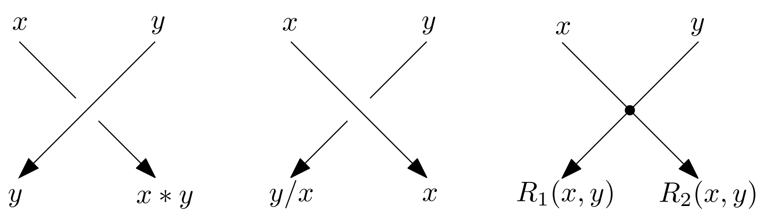

Fig. 5. The coloring of crossings of a singular diagram.

the invariance with respect to the classical Reidemeister moves implies that the operation $*$ is a quandle, while the invariance under the other generalized Reidemeister moves, imposes some further axioms (see Figure 6 for Ω_5). After a suitable change of variables we can rewrite them as

$$R_1(x, y) * z = R_1(x * z, y * z) \tag{OS1}$$

$$R_2(x, y) * z = R_2(x * z, y * z) \tag{OS2}$$

$$(y * x) * z = (y * R_1(x, z)) * R_2(x, z) \tag{OS3}$$

$$R_1(x, y) * R_2(x, y) = R_2(y, x * y). \tag{OS4}$$

$$R_2(x, y) = R_1(y, x * y) \tag{OS5}$$

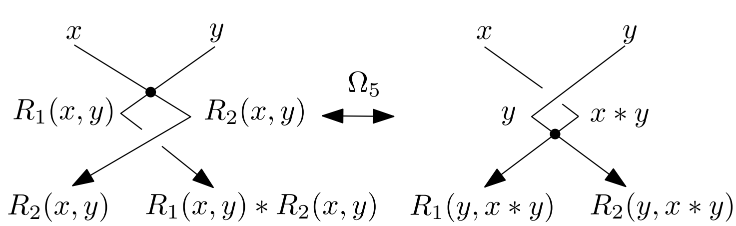

Fig. 6. The axioms associated to Ω_5.

So we get the following definition.

Definition 2.6. [BEHY18] An *oriented singquandle* is a triple $(X, *, R_1, R_2)$ where $(X, *)$ is a (right) quandle and

$$R_1,\, R_2 : X \times X \longrightarrow X$$

such that (OS1), (OS2), (OS3), (OS4), (OS5) hold.

In [CCE21], the fundamental oriented singquandle $SQ(L)$ associated to a singular link L is defined as the quotient of the free singquandle generated by the arcs of any diagram of L modulo the crossing relations represented in Figure 5. As for classical knots, colorings of a link L by an oriented singquandle T correspond to morphisms from $SQ(L)$ onto T.

Let us see what is going to happen if we forget about orientation: the quandle structure will be involutive and the other two binary operations will have to respect a rotational symmetry. More precisely, following [CEHN17], we get the following definition.

Definition 2.7. [CEHN17] Let $(X, *)$ be an involutive quandle and let $R_1, R_2 : X \times X \longrightarrow X$. The triple $(X, *, R_1, R_2)$ is called a singquandle if the following axioms hold:

$$x = R_1(y, R_2(x, y)) = R_2(R_2(x, y), R_1(x, y)) \tag{S1a}$$
$$y = R_2(R_1(x, y), x) = R_1(R_2(x, y), R_1(x, y)) \tag{S1b}$$
$$(R_1(x, y), R_2(x, y)) = (R_2(y, R_2(x, y)), R_1(R_1(x, y), x)) \tag{S1c}$$
$$(y * z) * R_2(x, z) = (y * x) * R_1(x, z) \tag{S2}$$
$$R_1(x, y) = R_2(y * x, x) \tag{S3}$$
$$R_2(x, y) = R_1(y * x, x) * R_2(y * x, x) \tag{S4}$$
$$R_1(x * y, z) * y = R_1(x, z * y) \tag{S5}$$
$$R_2(x * y, z) = R_2(x, z * y) * y \tag{S6}$$

Notice that the axioms (S1a), (S1b), (S1c) are those corresponding to a rotational symmetry of $\pi/2$, π and $3/2\pi$ of the coloration of a singular crossing (see Figure 7); hence it is enough to set the symmetry with respect to a rotation of $\pi/2$ degree and so set only the axiom

$$(R_2(x, y), y) = (R_1(R_1(x, y), x), R_2(R_1(x, y), x)). \tag{S1}$$

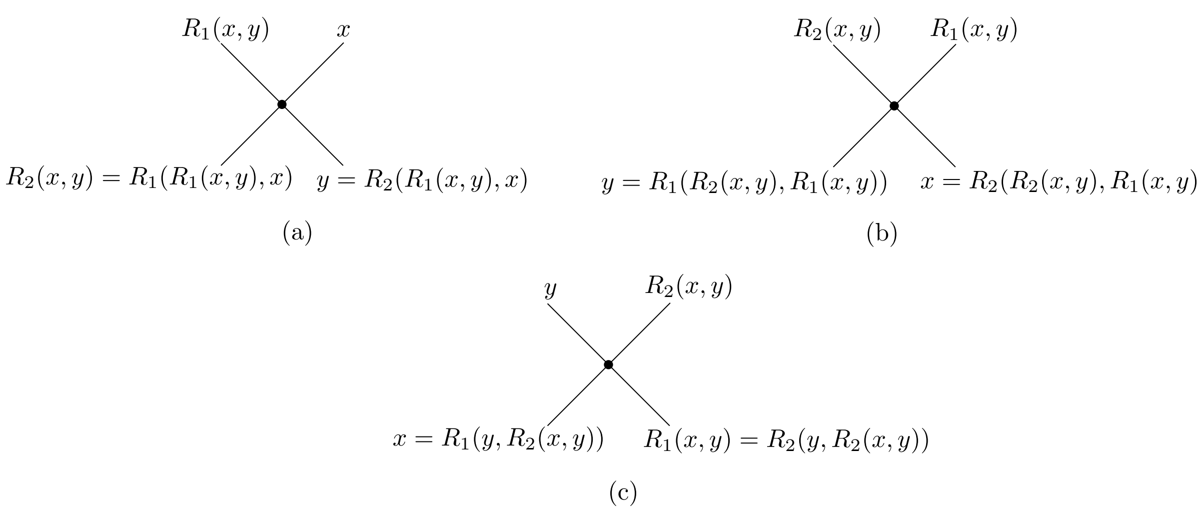

Fig. 7. Rotational simmetry and corresponding singquandle axioms.

The definition of fundamental singquandle associated to a (non-oriented) singular link is analogue to the one of oriented singquandle.

3. Oriented singquandles

In this section we deal with binary algebraic structure with several binary operations as $(X, \cdot, *)$. and both $(X, \cdot)$ and $(X, *)$ are right quasigroups. We denote by $/$ the right division associated to $\cdot$ and by $/^*$ the right division associated to $*$.

If we set $yx = R_1(x, y)$ and we take (OS5) as the definition of R_2 by $\cdot$ and $*$ as $R_2(x, y) = (x*y)y$, we can rewrite Definition 2.6 as follows.

Definition 3.1. An *oriented singquandle* is a binary algebraic structure $(X, \cdot, *)$ where $(X, *)$ is a (right) quandle and such that

$$(yx) * z = (y * z)(x * z) \tag{OS1'}$$
$$(x(yx)) * z = (x * z)((y * z)(x * z)) \tag{OS2'}$$
$$(y * x) * z = (y * (zx)) * ((x * z)z) \tag{OS3'}$$
$$(yx) * ((x * y)y) = (y * (x * y))(x * y) \tag{OS4'}$$

It is clear that (OS1') implies (OS2'), so we can omit (OS2') from the definition of oriented singquandles.

If $(X, \cdot, *)$ satisfies the axioms in Definition 3.1, then $(X, *, R_1, R_2)$ where $R_1(x, y) = y \cdot x$ and $R_2(x, y) = (x \cdot y) * y$ is an oriented singquandle in the sense of Definition 2.6. Thus, we stick to Definition 3.1 as the definition of oriented singquandles.

Proposition 3.2. *Let $(X, \cdot, *)$ be a binary algebraic structure. The following are equivalent:*

(i) $(X, \cdot, *)$ *is an oriented singquandle.*
(ii) $(X, *)$ *is a quandle,* $\rho_x \in \mathrm{Aut}(X, \cdot)$ *and*

$$\rho_x \rho_y = \rho_{(y*x)x} \rho_{xy}. \tag{3.1}$$

for every $x, y \in X$.
(iii) $(X, *)$ *is a quandle and the following identities hold:*

$$(xy) * z = (x * z)(y * z), \tag{3.2}$$
$$(z * y) * x = (z * (xy)) * ((y * x)x). \tag{3.3}$$

Proof.

The equivalence between (ii) and (iii) is straightforward. Let us show the equivalence between (i) and (ii).

- The identity (OS1') is equivalent to have that $\rho_z \in \mathrm{Aut}(X, \cdot)$ for every $z \in X$. In particular, R_z and ρ_z commute: indeed $\rho_z R_z(x) = (xz) * z = (x * z)(z * z) = (x * z)z = R_z \rho_z(x)$ for every $x \in X$.
- The identity (OS3') is equivalent to
$$\rho_x \rho_y = \rho_{(y*x)x} \rho_{xy}. \tag{3.4}$$
- Using that $\rho_x \in \mathrm{Aut}(X, \cdot)$ and $\rho_{x*y} = \rho_y \rho_x \rho_y^{-1}$, the identity (OS4') can be written as
$$\rho_{(x*y)y} R_x(y) = R_{x*y} \rho_{x*y}(y) = \rho_y R_x \rho_y^{-1} \rho_y \rho_x \rho_y^{-1}(y) = \rho_y R_x \rho_x(y).$$
Then using (3.4) we have $\rho_{(x*y)y} = \rho_y \rho_x \rho_{yx}^{-1}$ and so
$$\rho_{(x*y)y} R_x(y) = \rho_y \rho_x \rho_{yx}^{-1}(yx) = \rho_y \rho_x R_x(y) = \rho_y R_x \rho_x(y)$$
holds since ρ_x and R_x commute. Therefore (OS4') follows from (3.2) and (3.3).

□

Let $(X, \cdot, *)$ be a binary algebraic structure.

(i) If $(X, *)$ is projection then $(X, \cdot, *)$ is an oriented singquandle (the corresponding maps are $R_1(x, y) = yx$ and $R_2(x, y) = xy$). Therefore, given any binary structure $(X, \cdot)$ we can color the diagram of a singular knot as as in Figure 8.

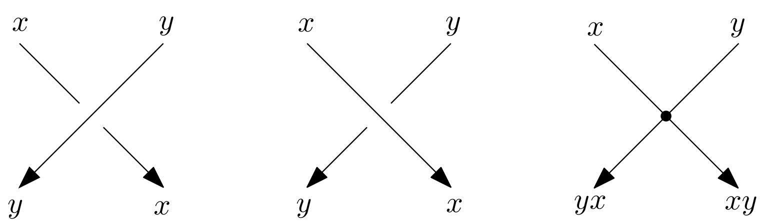

Fig. 8.

(ii) If $(X, \cdot)$ is a projection right quasigroup then $\rho_x \in \mathrm{Aut}(X, \cdot) = \mathrm{Sym}(X)$ for every $x \in X$ and (3.1) turns out to be right distributivity for $(X, *)$ (the corresponding maps are $R_1(x, y) = y$ and $R_2(x, y) = x * y$). Hence $(X, \cdot, *)$ is an oriented singquandle for any quandle $(X, *)$ and we can color the diagram of a singular knot as in Figure 9.

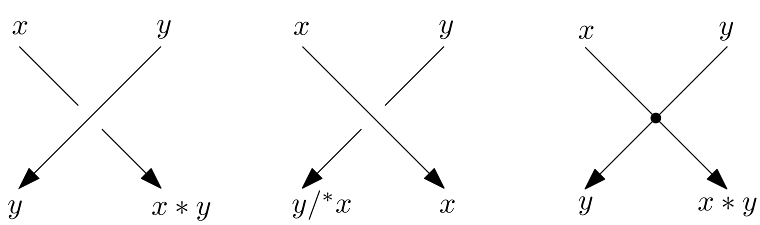

Fig. 9.

Moreover, if $\bar{L}_+$ is the link obtained by replacing each singular crossing with the regular crossing on the left of Figure 9, then $Q(\bar{L}_+)$ is a quotient of $SQ(L)$. Indeed the link L is colorable by the oriented singquandle X generated by the arcs modulo the crossing relations as in Figure 9 that is isomorphic to $Q(\bar{L}_+)$ (the $\cdot$ operation in X is trivial and the generators of X satisfy the very same relations satisfied by the generators of $Q(\bar{L}_+)$). Therefore we have the canonical morphism

$$SQ(L) \longrightarrow X \cong Q(\bar{L}_+)$$

that identifies the canonical generators.

(iii) Let $(X, *)$ be a quandle. Then $(X, /^*, *)$ is an oriented singquandle. Indeed clearly $\rho_x \in \mathrm{Aut}(X, /^*) = \mathrm{Aut}(X, *)$ for every $x \in X$ and

$$\begin{aligned}(y * (z/^*x)) * ((x * z)/^*z) &= (y * (z/^*x)) * x\\ &= (y * x) * ((z/^*x) * x)\\ &= (y * x) * z,\end{aligned}$$

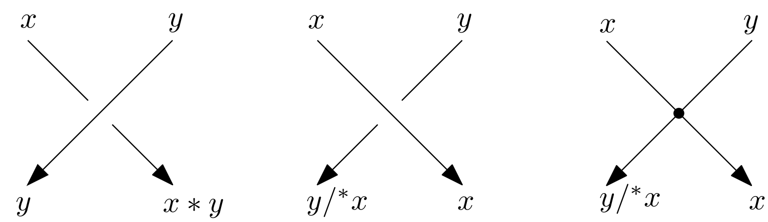

Fig. 10.

i.e. (3.3) holds. In this case the crossing relations look like those in Figure 10.

Similarly to the previous case, if $\bar{L}_-$ is the link obtained by replacing each singular crossing with the regular crossing on the right of Figure 10, then $Q(\bar{L}_-)$ is a quotient of $SQ(L)$.

The axioms in Proposition 3.2(iii) are independent, as we can see from the following computer generated examples, computed by using Mace 4 [McC10].

- The binary algebraic structure $(X, \cdot, *)$ where

$$(X,\cdot) = \begin{array}{|ccc|} \hline 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 3 & 3 \\ \hline \end{array}, \qquad (X,*) = \begin{array}{|ccc|} \hline 1 & 3 & 1 \\ 2 & 2 & 2 \\ 3 & 1 & 3 \\ \hline \end{array},$$

  satisfies (3.2) but not (3.3).
- The binary algebraic structure $(X, \cdot, *)$ where

$$(X,\cdot) = \begin{array}{|ccc|} \hline 2 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \\ \hline \end{array}, \qquad (X,*) = \begin{array}{|ccc|} \hline 1 & 1 & 1 \\ 3 & 2 & 2 \\ 2 & 3 & 3 \\ \hline \end{array},$$

  satisfies (3.3) but not (3.2).

4. Singquandles

Let $(X, *, R_1, R_2)$ be a singquandle and let us define $yx = R_1(x, y)$ and, according to (S1), $R_2(x, y) = R_1(R_1(x, y), x) = x(yx)$ (see Figure 11). Let us show Definition 2.7 in terms of identities satisfied by $\cdot$ and $*$.

$$y = (yx)(x(yx)) \tag{S1'}$$
$$(y * z) * (x(zx)) = (y * x) * (zx) \tag{S2'}$$
$$yx = (y * x)(x(y * x)) \tag{S3'}$$
$$x(yx) = (x(y * x)) * ((y * x)(x(y * x)) \tag{S4'}$$
$$(z(x * y)) * y = (z * y)x \tag{S5'}$$
$$(x * y)(z(x * y)) = (x((z * y)x)) * y \tag{S6'}$$

Proposition 4.1. *Let $(X, \cdot, *)$ be a binary algebraic structure. The following are equivalent:*

(i) $(X, \cdot, *)$ is a singquandle.
(ii) $(X, \cdot)$ is a S-right quasigroup, $x * y = (xy)y$ and $(X, \cdot, *)$ is an oriented singquandle.

(iii) The following identities hold

$$(x(yx))y = x \tag{4.1}$$
$$x * y = (xy)y \tag{4.2}$$
$$x * x = x \tag{4.3}$$
$$((xy)z)z = ((xz)z)((yz)z) \tag{4.4}$$
$$(x * z) * y = (x * (yz)) * (z/y) \tag{4.5}$$

Proof. Let us first point out some observations:

- According to Proposition 2.2, the identity (S1') is equivalent to have that $(X, \cdot)$ is a S-right quasigroup.
- The identity (S2') is equivalent to

$$\rho_{zx}\rho_{x(zx)} = \rho_x\rho_z. \tag{4.6}$$

(i) $\Rightarrow$ (ii) Using $\rho_x^2 = 1$ and replacing x by $x * y$ in (S5') we have

$$(z((x * y) * y)) * y = (zx) * y = (z * y)(x * y)$$

namely $\rho_x \in \mathrm{Aut}(X, \cdot)$. In particular, R_x and ρ_x commute.

Using that $\rho_x \in \mathrm{Aut}(X, \cdot)$ in (S3') we have

$$yx = (y * x)(x(y * x)) = (y * x)((x * x)(y * x)) = (y(xy)) * x = (y/x) * x.$$

Thus, $R_x = \rho_x R_x^{-1}$, i.e. $\rho_x = \rho_x^{-1} = R_x^2$ and so $x * y = (xy)y$.

Using that $(x * z)z = x/z = x(zx)$ in (4.6) we have that (3.1) follows.

Therefore $(X, \cdot, *)$ is an oriented singquandle according to Proposition 3.2.

(ii) $\Rightarrow$ (iii) The mapping $\rho_x^2 \in \mathrm{Aut}(X, \cdot)$ and so (4.4) holds. $(X, *)$ is a quandle and so $x * x = (xx)x = x$. Finally (4.5) follows by (3.1) just by replacing the definition of $*$.

(iii) $\Rightarrow$ (i) Let us first show that $(X, *)$ is an involutory quandle. Since $\rho_x \in \mathrm{Aut}(X, \cdot)$ then $\rho_x \in \mathrm{Aut}(X, *)$ and $(X, *)$ is idempotent by (4.3). Thus $(X, *)$ is a quandle.

Note that

$$(x/y)(yx) \overset{(S)}{=} (x(yx))(yx) = x * (yx), \tag{4.7}$$
$$x/(yx) \overset{(S)}{=} x((yx)x) = x(y * x) \overset{(4.3)}{=} (x * x)(y * x) \overset{(4.4)}{=} (xy) * x. \tag{4.8}$$

Therefore we have

$$(x * (yx)) * (x/y) \overset{(4.7)}{=} ((x/y)(yx)) * (x/y) \overset{(4.8)}{=} (x/y)/((yx)(x/y)) \overset{(S')}{=} (x/y)/y. \tag{4.9}$$

Finally, we have

$$x * y \overset{(4.3)}{=} (x * x) * y \overset{(4.5)}{=} (x * (yx)) * (x/y) \overset{(4.9)}{=} (x/y)/y.$$

Therefore $\rho_y = R_y^2 = R_y^{-2} = \rho_y^{-1}$, i.e. $(X, *)$ is involutory.

Under this assumption (S5') is equivalent to $\rho_y \in \mathrm{Aut}(X, \cdot)$ and (S6') follows from the same argument. Also (S3') follows as in the first part of the proof, by using that ρ_x is an involutory automorphism of $(X, \cdot)$. The identity (S2') is equivalent to (4.4) modulo the identity (4.2).

Let us check the identity (S4'). Indeed, using that $y * x = (yx)x$ and that $x/y = x(yx)$ we have

$$\begin{aligned}(x(y * x)) * ((y * x)(x(y * x)) &= (x((yx)x)) * ((y * x)/x)\\ &= (x/(yx)) * (yx) = x(yx).\end{aligned}$$

□

Singquandles are 2-divisible.

Corollary 4.2. *Singquandles are 2-divisible and the squaring mapping is an involution.*

Proof.
Let $(X, \cdot)$ be a singquandle and $x \in X$. It is enough to prove that $\sigma^2(x) = (xx)(xx) = x$. According to (4.3) we have $xx = x/x$. Hence by (S) it follows that

$$(xx)(xx) \overset{(4.3)}{=} (xx)(x/x) \overset{(S)}{=} x. \qquad \square$$

Note that the proof of Corollary 4.2 actually uses just (S) and (4.3), so also S-right quasigroups such that $(xx)x = x$ holds are 2-divisible.

The set of axioms given in Proposition 4.1(iii) is independent. We can consider singquandles as a right quasigroup using (4.2) as the definition of $*$ and rewrite all the axioms accordingly as:

$$(x(yx))y = x, \tag{4.10}$$

$$(xx)x = x, \tag{4.11}$$

$$((xy)z)z = ((xz)z)((yz)z), \tag{4.12}$$

$$(((xz)z)y)y = (((x(yz))(yz))(z/y))(z/y). \tag{4.13}$$

Let us show that the axioms above are independent by examples generated by the software Mace4:

- Involutory S-right quasigroups are singquandles. Indeed, if $(X, \cdot)$ is involutory then $(X, *)$ is projection and so (4.11), (4.12) and (4.13) are trivially satisfied. The involutory right quasigroup

$$(X, \cdot) = \begin{array}{|cc|} \hline 2 & 1 \\ 1 & 2 \\ \hline \end{array},$$

does not satisfy (4.10).
- The right quasigroup

$$(X, \cdot) = \begin{array}{|cccc|} \hline 2 & 3 & 3 & 2 \\ 4 & 1 & 1 & 4 \\ 1 & 4 & 4 & 1 \\ 3 & 2 & 2 & 3 \\ \hline \end{array},$$

satisfies all the axioms but (4.11).
- The right quasigroup

$$(X, \cdot) = \begin{array}{|cccccc|} \hline 1 & 4 & 6 & 1 & 1 & 1 \\ 3 & 2 & 2 & 3 & 3 & 3 \\ 2 & 3 & 3 & 2 & 2 & 2 \\ 4 & 1 & 5 & 5 & 5 & 4 \\ 5 & 6 & 4 & 4 & 4 & 5 \\ 6 & 5 & 1 & 6 & 6 & 6 \\ \hline \end{array},$$

satisfies all the axioms but (4.12).
- The right quasigroup

$$(X, \cdot) = \begin{array}{|ccccc|} \hline 1 & 4 & 5 & 1 & 1 \\ 3 & 2 & 2 & 3 & 3 \\ 2 & 3 & 3 & 2 & 2 \\ 4 & 5 & 1 & 4 & 4 \\ 5 & 1 & 4 & 5 & 5 \\ \hline \end{array},$$

satisfies all the axioms but (4.13).

Corollary 4.3. *Let $(X,\cdot)$ be right quasigroups. If $(X,\cdot)$ is a singquandle then $(X,/)$ is a singquandle.*

Proof. According to Proposition 2.3, $(X,/)$ is a S-right quasigroup and so (4.10) hold for $(X,/)$. Note that we can write (4.12) and (4.13) in terms of the right multiplication mappings as

$$(4.12) \Leftrightarrow R_x^2 \in \mathrm{Aut}(X,\cdot)$$
$$(4.13) \Leftrightarrow R_y^2 R_z^2 = R_{z/y}^2 R_{yz}^2.$$

Thus, using that $R_z^4 = 1$ and $\mathrm{Aut}(X,\cdot) = \mathrm{Aut}(X,/)$, we have

$$(xx)x = x \Leftrightarrow (x/x)/x = x,$$
$$R_x^2 \in \mathrm{Aut}(X,\cdot) \Leftrightarrow R_x^{-2} \in \mathrm{Aut}(X,/)$$
$$R_y^2 R_z^2 = R_{z/y}^2 R_{yz}^2 \Leftrightarrow R_z^{-2} R_y^{-2} = R_{yz}^{-2} R_{z/y}^{-2},$$

namely all the axioms of singquandles hold for $(X,/)$. □

Note that the quandle associated to $(X,\cdot)$ and to $(X,/)$ is the same.

Finally, note that colorings of non-oriented singular links by singquandles are obtained as in Figure 11.

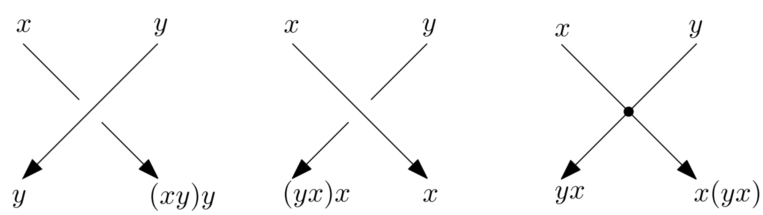

Fig. 11.

Note that, since singquandles are right quasigroups, Remark 2.5 holds also for such colorings.

4.1. *Affine singquandles*

Let us turn our attention to the family of affine singquandles.

Proposition 4.4. *Let $(X,\cdot) = \mathrm{Aff}(A,f,g,c)$ be an affine right quasigroup. The following are equivalent:*

(i) *$(X,\cdot)$ is a singquandle.*
(ii) *The identities* (S), (4.11) *and $(((xy)y)y)y = x$ hold.*
(iii) *The following identities hold:*

$$fg^2 + f^2 - 1 = 0, \tag{A1}$$
$$g + fgf = 0, \tag{A2}$$
$$1 - f^4 = 0 \tag{A3}$$
$$(1+f)g = 1 - f^2 \tag{A4}$$
$$(1+f)(c) = g(c) = 0. \tag{A5}$$

Proof. (i) ⇒ (ii) According to Proposition 4.1, (S) and (4.12) hold and $(X,*)$ where $x*y = (xy)y$ is an involutory quandle. Then $(x*y)*y = (((xy)y)y)y = x$ holds.

(ii) $\Rightarrow$ (iii) We compute the conditions on f, g and c that need to be satisfied in order to have that (S), (4.11) and $(((xy)y)y)y = x$ hold.

- The identity (S) holds if and only if the identities (2.4) hold, i.e. (A1), (A2) and $(1+fg+f)(c) = 0$. hold.
- The identity (4.11) holds if and only if

$$(xx)x = c + g(x) + f(c) + fg(x) + f^2(x) = x,$$

namely

$$(1+f)(c) = 0,\ (1+f)g = 1 - f^2. \tag{4.14}$$

Using the first equation of (4.14) and that $(1+fg+f)(c) = 0$ we have that $(1+f)(c) = g(c) = 0$. Therefore, (A4) and (A5) hold.
- We have that $((xy)y)y)y = x$ if and only if

$$\begin{aligned}(((xy)y)y)y &= (1+f^2)(1+f)(c) + (1+f^2)(1+f)g(y) + f^4(x)\\ &\overset{\text{(A5)}}{=} (1+f^2)(1+f)g(y) + f^4(x)\\ &\overset{\text{(A4)}}{=} (1+f^2)(1-f^2)(y) + f^4(x)\\ &= (1-f^4)(y) + f^4(x) = x\end{aligned}$$

i.e. (A3) holds.

(iii) $\Rightarrow$ (i) Let us prove that such conditions are sufficient for the other axioms of singquandles. We have already showed that (S) and (4.11) are equivalent to the equations (A1), (A2), (A3), (A4) and (A5). Let us check the other identities.

- Since

$$\begin{aligned}((xy)z)z &= f^2(c) + (g+fg)(z) + f^2g(y) + f^3(x)\\ ((xz)z)((yz)z) &= f^2(c) + (g^2+gfg+fg+f^2g)(z) + (gf^2)(y) + f^3(x),\end{aligned}$$

the identity (4.12) holds if and only if

$$g = g^2 + gfg + f^2g, \tag{4.15}$$

$$f^2g = gf^2 \tag{4.16}$$

Since $f^4 = 1$ then by (A2) we have

$$f^2g = -fgf^{-1} = gf^{-2} = gf^2.$$

Moreover

$$g^2 + gfg + f^2g - g \overset{(4.16)}{=} g^2 + gfg + gf^2 - g = g(g + fg + f^2 - 1) \overset{\text{(A4)}}{=} 0.$$

- It is easy to compute that

$$(((xz)z)y)y = (1-f^2)(y-z) + x.$$

So we have

$$\begin{aligned}(((x(yz))(yz))(z/y))(z/y) &= (1-f^2)(z/y - yz) + x\\ &= (1-f^2)(f^{-1}(z-g(y)-c) - c - g(z) - f(y)) + x\\ &\overset{\text{(A5)}}{=} (1-f^2)((f^{-1}-g)(z) - (f^{-1}g+f)(y)) + x.\end{aligned}$$

Thus, the identity (4.13) holds if and only if

$$(1-f)(1+f)(1+f^{-1}g+f)=0$$
$$(1-f)(1+f)(f^{-1}-g+1)=0$$

Using that $(1+f)g=1-f^2$ and that $1-f^4=0$ we have

$$\begin{aligned}(1-f)(1+f)(1+f^{-1}g+f) &= (1-f)(1+f+f^3(1-f^2)+f+f^2)\\ &= (1-f)(1+f+f^2+f^3)=1-f^4=0\\ (1-f)(1+f)(f^{-1}-g+1) &= (1-f)(f^3+1-(1-f^2)+1+f)\\ &= (1-f)(1+f+f^2+f^3)=1-f^4=0.\end{aligned}$$

Thus the identity (4.5) holds. □

The construction of Alexander singquandles given in [CEHN17, Proposition 4.3] defined as a binary algebraic structure over an abelian group A using $t, B \in \mathrm{Aut}(A,+)$ by setting

$$x*y=tx+(1-t)y, \qquad R_1(x,y)=(1+t-B)x+(t+B)y, \qquad R_2(x,y)=(1-B)x+By,$$

provides exactly affine idempotent singquandles (the relation between f, g and B is $g=B(1-B)$ and $f=B^2-B+1$ and the pair B and t satisfy $1-(1-B)^4=B(1+(1-B)^2)=(1-B)^2-t=0$).

It is easy to check that the affine right quasigroup $\mathrm{Aff}(A,f,g,c)$ is a idempotent if and only if

$$g=1-f, \quad c=0. \tag{4.17}$$

Note that idempotent affine right quasigroup are quandles and so (4.4) holds and according to Example 2.4(iii), the identity (S) is equivalent to the identity $((xy)x)y=x$. Thus, under the assumptions (4.17) and using that $1-f^4=(1+f)(1-f)(1+f^2)$ the identities in Proposition 4.4(iii) reduce to

$$(1-f)(1+f^2)=0.$$

So, we have the following result.

Corollary 4.5. *Let $X=\mathrm{Aff}(A,f,g,c)$ be an affine right quasigroup. The following are equivalent:*

(i) X is an idempotent singquandle.
(ii) The identities $xx=x$ and $((xy)x)y=x$ hold.
(iii) $g=1-f$, $(1-f)(1+f^2)=0$ and $c=0$.

According to Corollary 4.5, affine idempotent singquandles are endowed with a module structure over the ring $R=\mathbb{Z}[t,t^{-1}]/((1-t)(1+t^2))$. Conversely, every module M over R is an idempotent affine singquandle with the operation

$$x\cdot y=(1-t)x+ty$$

for $x, y\in M$. In particular, given an affine quandle $Q=\mathrm{Aff}(A,1-f,f,0)$ we can consider the right quasigroup $Q'=\mathrm{Aff}(A/\left((1-f)(1+f^2)A\right),1-f',f',0)$ where f' is the automorphism induced by f on the quotient group $A/\left((1-f)(1+f^2)A\right)$. Then Q' is a singquandle.

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References

[AG03] Nicolás Andruskiewitsch and Matías Graña, *From racks to pointed Hopf algebras*, Adv. Math. **178** (2003), no. 2, 177–243. MR 1994219 (2004i:16046)

[BEHY18] Khaled Bataineh, Mohamed Elhamdadi, Mustafa Hajij, and William Youmans, *Generating sets of reidemeister moves of oriented singular links and quandles*, J. Knot Theory Ramifications **27** (2018), no. 14.

[BF21] Marco Bonatto and Stefano Fioravanti, *Mal'cev classes of left-quasigroups and Quandles*, Quasigroups and related systems **29** (2021), no. 2.

[Bir93] Joan Birman, *New points of view in knot theory*, Bull. Amer. Math. Soc. **29** (1993), no. 2, 253–287.

[BS21] Marco Bonatto and David Stanovský, *Commutator theory for racks and quandles*, J. Math. Soc. Japan **1** (2021), 41–75.

[CCE21] Jose Ceniceros, Indu R. U. Churchill, and Mohamed Elhamdadi, *Polynomial invariants of singular knots and links*, J. Knot Theory Ramifications **30** (2021), no. 1.

[CEHN17] Indu R. U. Churchill, Mohamed Elhamdadi, Mustafa Hajij, and Sam Nelson, *Singular knots and involutive quandles*, J. Knot Theory Ramifications **26** (2017), no. 14.

[ESG01] Pavel Etingof, Alexander Soloviev, and Robert Guralnick, *Indecomposable set-theoretical solutions to the quantum Yang-Baxter equation on a set with a prime number of elements*, J. Algebra **242** (2001), no. 2, 709–719. MR 1848966 (2002e:20049)

[ESS99] Pavel Etingof, Travis Schedler, and Alexandre Soloviev, *Set-theoretical solutions to the quantum Yang-Baxter equation*, Duke Math. J. **100** (1999), no. 2, 169–209. MR 1722951 (2001c:16076)

[Fie10] Thomas Fiedler, *The Jones and Alexander polynomial for singular links*, J. Knot Theory Ramifications **19** (2010), no. 7, 859–866.

[FJSK04] Roger Fenn, Natsumi Jordan-Santana, and Luis Kauffman, *Biquandles and virtual links*, Topology Appl. **145** (2004), 157–175.

[FR92] Roger Fenn and Colin Rourke, *Racks and links in codimension two*, J. Knot Theory Ramifications **1** (1992), no. 4, 343–406. MR 1194995 (94e:57006)

[Hor19] Eva Horvat, *The topological biquandle of a link*, Pacific J. Math. **302** (2019), no. 2, 627–644.

[Joy82] David Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra **23** (1982), no. 1, 37–65. MR 638121 (83m:57007)

[JPSZD15] Přemysl Jedlička, Agata Pilitowska, David Stanovský, and Anna Zamojska-Dzienio, *The structure of medial quandles*, J. Algebra **443** (2015), 300–334. MR 3400403

[Mat82] Sergei Matveev, *Distributive groupoids in knot theory*, Mat. Sb. (N.S.) **119(161)** (1982), no. 1, 78–88, 160. MR 672410

[McC10] William McCune, *Prover9 and mace4*, `http://www.cs.unm.edu/~mccune/prover9/`, 2005–2010.

[NOS19] Sam Nelson, Natsumi Oyamaguchi, and Radmila Sazdanovic, *Psyquandles, singular knots and pseudoknots*, Tokyo J. Math. **42** (2019), no. 2, 405–429.

[Sta15] David Stanovský, *A guide to self-distributive quasigroups, or Latin quandles*, Quasigroups Relat. Syst. **23** (2015), no. 1, 91–128. MR 3353113

[Vas90] Victor Anatolyevich Vassiliev, *Coomology of knot spaces*, 1990, pp. 23–69. MR 1089670