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BLOCK NUMBER, DESCENTS AND SCHUR POSITIVITY
OF FULLY COMMUTATIVE ELEMENTS IN B_n

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ABSTRACT. The distribution of Coxeter descents and block number over the set of fully commutative elements in the hyperoctahedral group B_n, $\mathrm{FC}(B_n)$, is studied in this paper. We prove that the associated Chow quasi-symmetric generating function is equal to a non-negative sum of products of two Schur functions. The proof involves a decomposition of $\mathrm{FC}(B_n)$ into a disjoint union of two-sided Barbash-Vogan combinatorial cells, a type B extension of Rubey's descent preserving involution on 321-avoiding permutations and a detailed study of the intersection of $\mathrm{FC}(B_n)$ with S_n-cosets which yields a new decomposition of $\mathrm{FC}(B_n)$ into disjoint subsets called fibers. We also compare two different type B Schur-positivity notions, arising from works of Chow and Poirier.

1. INTRODUCTION

1.1. **Outline.** An element w in a Coxeter group W is fully commutative if any reduced expression for w in Coxeter generators can be obtained from any other using only commutation relations. The study of these elements was motivated by generalizations of the Temperley–Lieb algebra to all Coxeter types. Fan [13] and Graham [16] proved that for every Coxeter group W, the associated Temperley–Lieb algebra admits a linear basis indexed by the fully commutative elements in W. Various combinatorial characterizations, enumeration and connections with enriched P-partitions and Schur's Q-functions were studied in a series of papers by Stembridge [33, 34, 35]. Compatibility of the Kazhdan–Lusztig cell decomposition of a Coxeter group W with the set of fully commutative elements was studied by Green and Losonczy [17].

The graded ring of quasi-symmetric functions, introduced by Gessel [15], has many applications to enumerative combinatorics, as well as to other branches of mathematics; see, e.g., [29, Ch. 7]. A quasi-symmetric function is a formal power series $f(x_1, x_2, \dots,)$ of bounded degree such that for each fixed k-tuple $(\alpha_1, \dots, \alpha_k)$ of nonnegative integers, with $k \in \mathbb{N}$, all the monomials in f of the form $x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}$, where $i_1 < i_2 < \cdots < i_k$, share the same coefficient. The vector space of all quasi-symmetric functions which are homogeneous of degree n, has a distinguished basis $\{F_J \mid J \subseteq [n-1]\}$, where $[n-1] := \{1, 2, \dots, n-1\}$ and F_J is the Gessel fundamental quasi-symmetric function indexed by $J \subseteq [n-1]$.

The block number of a permutation π in S_n, which was studied in [30] as the cardinality of the connectivity set of π, is equal to the maximal number of summands in an expression of π as a direct sum of smaller permutations. It was shown recently that the quasi-symmetric generating function of the descent set statistic over the set of 321-avoiding permutations with prescribed block number is Schur-positive [1]. Actually the 321-avoiding permutations in the symmetric group S_n are in one-to-one correspondence with the fully commutative elements in the Coxeter group of type A_{n-1} [6]. Similarly, the set of fully commutative elements in the Coxeter group of type B_n has an explicit combinatorial description in terms of several forbidden patterns in signed permutations [34].

The concept of quasi-symmetric functions has been extended to Coxeter group of type B_n in two different ways. Chow's construction applies the presentation of B_n as a Coxeter group; Chow's fundamental quasi-symmetric functions are indexed by type B Coxeter descent sets [12]. Poirier's construction applies the presentation of B_n as a wreath product, or equivalently as a colored permutation group; Poirier's fundamental basis elements are indexed by signed descent sets [26]. For discussion and comparison of these two families of quasi-symmetric functions of type B see [25, 24, 3]. In the current paper, we study the type B

quasi-symmetric functions determined by the Coxeter descent sets of fully commutative elements in B_n. It turns out that while Poirier's approach is not useful in this setting, Chow's provides a nice description. In particular, we give an explicit expansion of Chow's quasi-symmetric generating functions over the subset of fully commutative elements with a prescribed block number in the Coxeter groups of type B_n in terms of Schur functions and show that the coefficients are non-negative.

1.2. **Main results.** For a positive integer n and W the Coxeter group of type A_{n-1} or B_n, let $FC(W)$ be its subset of fully commutative elements (see Section 2.1 below for precise definitions). For an integer partition λ, denote by s_λ the associated Schur function. The *block number* of a permutation $\pi = [\pi_1, \ldots, \pi_n]$ in the symmetric group S_n is defined by

$$\mathrm{bl}(\pi) := \#\{i \mid (\forall j < i) \ \pi_j \leq i\} = 1 + \#\{1 \leq i \leq n-1 \mid \max(\pi_1, \ldots, \pi_i) < \min(\pi_{i+1}, \ldots, \pi_n)\}.$$

Let $F_{\mathrm{Des}(\pi)}$ be Gessel's fundamental quasi-symmetric function indexed by the (right) descent set $\mathrm{Des}(\pi)$, as defined in Section 2.1. For a pair of partitions λ, μ such that $\mu \subseteq \lambda$, denote the set of standard Young tableaux of skew shape λ/μ by $\mathrm{SYT}(\lambda/\mu)$ (see [29] for definitions of these classical objects). For a standard Young tableau $T \in \mathrm{SYT}(\lambda/\mu)$, let $\mathrm{ldes}(T)$ be the maximal descent of T; if the descent set is empty we set $\mathrm{ldes}(T) := 0$ (see Section 2 for more details).

The following Schur-positivity result is a reformulation of [1, Theorem 1.2].

Theorem 1.1. *For any positive integer n, we have*

$$\sum_{\pi \in FC(S_n)} q^{\mathrm{bl}(\pi)} F_{\mathrm{Des}(\pi)} = \sum_{k=0}^{\lfloor n/2 \rfloor} \left(\sum_{j=0}^{n} a_{n,k,j} \ q^j \right) s_{(n-k,k)}, \tag{1.1}$$

where s_λ is the Schur function corresponding to the partition λ and

$$a_{n,k,j} := \#\{T \in \mathrm{SYT}(n-k,k) \mid \mathrm{ldes}(T) = n-j\},$$

which is thus non-negative.

The main goal of the present work is to prove a type B analogue of the above result. The block number of a signed permutation $w = [w_1, \ldots, w_n] \in B_n$ is defined by

$$\mathrm{bl}(w) := 1 + \#\{1 \leq i \leq n-1 \mid \max(w_1, \ldots, w_i) < \min(w_{i+1}, \ldots, w_n)\}.$$

Let $s_\lambda(x_I)$ be the Schur function in the set of indeterminates indexed by the elements in the ordered set I, and let $F^B_{\mathrm{Des}_B(w)}$ be Chow's fundamental quasi-symmetric function indexed by the type B (right) descent set $\mathrm{Des}_B(w)$, see Section 2.2 for detailed definitions. Consider the natural embedding of S_n as a maximal parabolic subgroup of B_n.

Our main result is the following type B analogue of Theorem 1.1.

Theorem 1.2. *For any positive integer n, we have*

$$\sum_{w \in FC(B_n) \setminus FC(S_n)} q^{\mathrm{bl}(w^{-1})} F^B_{\mathrm{Des}_B(w)} = \sum_{k=1}^{n} \left(\sum_{j=0}^{n} b_{n,k,j} q^j \right) s_{(k)}(x_1, x_2, \ldots) \ s_{(n-k)}(x_0, x_1, \ldots), \tag{1.2}$$

where

$$b_{n,k,j} := \#\{T \in \mathrm{SYT}((n,k)/(k)) \mid \mathrm{ldes}(T) = n-j\},$$

which is thus non-negative.

To prove Theorem 1.2, we will combine two new explicit decompositions of $FC(B_n)$, one as a disjoint union of fibers (see Theorem 4.2), and one as a disjoint union of Barbash-Vogan combinatorial cells (see Theorem 5.7), together with an equidistribution phenomenon which takes the following form.

For a subset $J \subseteq \{0, 1, \ldots, n\}$ let $\mathbf{x}^J := \prod_{i \in J} x_i$.

Theorem 1.3. *For any positive integer n we have the following equidistribution on $FC(B_n)$:*

$$\sum_{w \in FC(B_n)} \mathbf{x}^{\mathrm{Des}_B(w)} \mathbf{z}^{\mathrm{Neg}(w)} q^{\mathrm{bl}(w^{-1})} t^{n-\mathrm{ldes}(w^{-1})} = \sum_{w \in FC(B_n)} \mathbf{x}^{\mathrm{Des}_B(w)} \mathbf{z}^{\mathrm{Neg}(w)} q^{n-\mathrm{ldes}(w^{-1})} t^{\mathrm{bl}(w^{-1})}.$$

Here, Neg and ldes denote the negative set and the last descent of a signed permutation, respectively (see Section 2.2 for precise definitions).

Finally, we compare the two different notions of type B Schur-positivity, based on Chow's and Poirier's approaches, studied in [24] and [3], respectively. It is shown that every Poirier type B Schur-positive set is a Chow type B Schur-positive set (see Theorem 8.9). The converse does not hold: the set of fully commutative elements in B_n, $\mathrm{FC}(B_n)$, is Chow type B Schur-positive but not Poirier type B Schur-positive, as concluded in Remark 8.11.

The paper is organized as follows. In Sections 2 and 3 we provide the necessary background: Section 2 is devoted to Coxeter groups, fully commutative elements, quasi-symmetric functions associated with the hyperoctahedral group, and the different kinds of tableaux and statistics that will be in use; Section 3 deals with the theory of heaps as defined by Viennot [37]. Using these heaps, we prove in Section 4 our decomposition of the set $\mathrm{FC}(B_n)$ into fibers. In Section 5, we describe the cellular structure of $\mathrm{FC}(B_n)$. In Section 6, we prove the equidistribution result given in Theorem 1.3 above, using the results of Section 4 and an involution due to Rubey [27]. In Section 7 we prove Theorem 1.2 and Section 8 ends the paper with a discussion on the above mentioned two notions of type B Schur-positivity.

2. BACKGROUND

2.1. **Coxeter groups and fully commutative elements.** Let (W, S) be a Coxeter system with Coxeter matrix $M = (m_{st})_{s,t \in S}$. We recall that the finite set of generators S is subject only to relations of the form $(st)^{m_{st}} = 1$, where $m_{ss} = 1$, and $m_{st} = m_{ts} \geq 2$, for $s \neq t \in S$. If st has infinite order we set $m_{st} = \infty$. These relations can be rewritten more explicitly as $s^2 = 1$ for all $s \in S$, and

$$\underbrace{sts\cdots}_{m_{st}} = \underbrace{tst\cdots}_{m_{ts}},$$

where $m_{st} < \infty$. These are the so-called *braid relations*. When $m_{st} = 2$, they are named *commutation relations*, $st = ts$. This information is encoded in the *Dynkin diagram*, which is a graph with one vertex for each $s \in S$ and in which an edge connects two elements $s, t \in S$ if and only if $m_{st} \geq 3$. When $m_{st} > 3$, we write the number m_{st} above the edge connecting s and t.
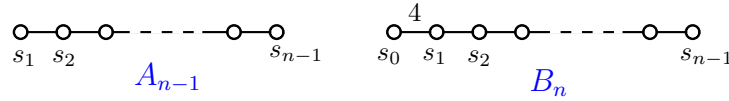
FIGURE 1. The Dynkin diagrams of types A_{n-1} and B_n.

For $w \in W$, the *length* of w, denoted $\ell(w)$, is the minimum length ℓ of any expression of w as a product $s_{i_1} \cdots s_{i_\ell}$ with $s_{i_j} \in S$. These expressions of length ℓ are called *reduced* and denoted with a bold symbol $\mathbf{w} = s_{i_1} \cdots s_{i_\ell}$. Denote by $\mathcal{R}(w)$ the set of all reduced expressions of w.

The *right descent set* of $w \in W$ is

$$\mathrm{Des}(w) := \{s \in S \mid \ell(ws) < \ell(w)\}. \tag{2.1}$$

If $\mathbf{w} = s_{i_1} \cdots s_{i_\ell}$ is a reduced expression for w, then a reduced expression for w^{-1} is given by $s_{i_\ell} \cdots s_{i_1}$. It follows that

$$\mathrm{Des}^L(w) := \{s \in S \mid \ell(sw) < \ell(w)\} = \mathrm{Des}(w^{-1}), \tag{2.2}$$

known as the *left descent set* of w.

For $J \subseteq S$, denote by W_J the *parabolic subgroup* of W generated by J, and by

$$W^J := \{w \in W \mid \mathrm{Des}(w) \subseteq S \setminus J\},$$

the *set of minimal coset representatives*, or *quotient*. The next result is well known, see for example [7, Proposition 2.4.4].

Proposition 2.1. *For every $J \subseteq S$ the following holds.*

(i) *Every $w \in W$ has a unique factorization $w = w^J \cdot w_J$ such that $w^J \in W^J$ and $w_J \in W_J$.*

(ii) *For this factorization $\ell(w) = \ell(w^J) + \ell(w_J)$.*

The well-known *Matsumoto-Tits word property* ensures that any reduced expression of $w \in W$ can be obtained from any other using only braid relations (see for instance [19]). The concept of full commutativity is a strengthening of this property.

Definition 2.2. An element w is *fully commutative* (FC) if any reduced expression for w can be obtained from any other one by using only commutation relations.

The following characterization of FC elements, originally due to Stembridge, is particularly useful for checking whether a given element is FC.

Proposition 2.3. [33, Prop. 2.1] *An element $w \in W$ is fully commutative if and only if for all s, t such that $3 \leq m_{st} < \infty$, there is no reduced expression for w that contains the factor $\underbrace{sts\cdots}_{m_{st}}$.*

We let S^* be the free monoid generated by S. Define the following equivalence relation on S^*: two words $a, b \in S^*$ are equivalent if b can be obtained from a by a finite sequence of commutation relations. The equivalence classes of this relation are usually called *commutation classes*. By definition, for a FC element w, the set $\mathcal{R}(w)$ of *reduced expressions* of w forms a single commutation class ; we will see in Section 3 that the concept of heap, as originally defined by Viennot [37], helps to capture the notion of full commutativity.

2.2. **The hyperoctahedral group.** In this section we fix a positive integer $n \geq 2$. Recall that the Coxeter group of type A_{n-1} is isomorphic to the symmetric group S_n, i.e. the group of bijections from the set $[n]$ onto itself. Similarly, the Coxeter group of type B and rank n may be realized as the *group of signed permutations* B_n, that is the group of all bijections w of the set $[\pm n] := \{\pm 1, \pm 2, \ldots, \pm n\}$ onto itself such that

$$w(-i) = -w(i)$$

for every $1 \leq i \leq n$, with composition as the group operation. This group is also known as the *hyperoctahedral group* of rank n. We identify S_n as a subgroup of B_n, and B_n as a subgroup of $S_{[\pm n]}$ in the natural ways.

If $w \in B_n$, we write $w = [w_1, \ldots, w_n]$ to mean that $w(i) = w_i$ for $1 \leq i \leq n$, and we set

$$\mathrm{Neg}(w) := \{i \in [n] \mid w_i < 0\},$$

the *negative set* of w.

As Coxeter generating set for B_n we take $S := \{s_i \mid 0 \leq i < n\}$, where $s_0 := [-1, 2, 3, 4, \ldots, n]$ and, for $1 \leq i < n$, $s_i := [1, \ldots, i-1, i+1, i, i+2, \ldots, n]$, see Figure 1, right.

It is well known, [7, Proposition 8.1.2], that by letting $w_0 := 0$, the right descent set, defined in (2.1), is identified for a signed permutation w to the set of indices

$$\mathrm{Des}_B(w) := \{0 \leq i \leq n-1 \mid w_i > w_{i+1}\}. \tag{2.3}$$

Similarly, for a permutation $\pi \in S_n$ the right descent set is identified with the set

$$\mathrm{Des}(\pi) := \{1 \leq i \leq n-1 \mid \pi_i > \pi_{i+1}\}. \tag{2.4}$$

For $w \in B_n$, let $\mathrm{ldes}(w)$ be the *maximal descent* in $\mathrm{Des}_B(w)$; if the descent set is empty we set $\mathrm{ldes}(w) := 0$.

2.3. **Chow's quasi-symmetric functions, domino tableaux and bi-tableaux.** For an infinite set of formal variables $x_1, x_2, \ldots$, the *Gessel fundamental quasi-symmetric function* indexed by a set $J \subseteq [n-1]$ is defined as

$$F_J(x_1, x_2, \ldots) := \sum_{\substack{0 < i_1 \leq i_2 \leq \cdots \leq i_n \\ j \in J \Rightarrow i_j < i_{j+1}}} x_{i_1} \cdots x_{i_n}.$$

The descent set of a standard Young tableau T of size n is defined as

$$\mathrm{Des}(T) := \{0 < i < n \mid i+1 \text{ is in a lower row than } i\}.$$

The above quasi-symmetric functions are related to the classical symmetric Schur functions by the following result.

Theorem 2.4. [29, Theorem 7.19.7] *For every partition $\lambda \vdash n$,*

$$\sum_{T \in \mathrm{SYT}(\lambda)} F_{\mathrm{Des}(T)} = s_\lambda,$$

where $s_\lambda = s_\lambda(x_1, x_2, \dots)$ is the Schur function indexed by λ.

This mechanism has been extended to the framework of the hyperoctahedral group B_n in two different ways. One approach was introduced by Poirier [26], determining the signed quasi-symmetric functions, see also [25, 3]. In this paper we follow the second approach, based on the work of Chow [12], which is relevant to our purposes. After defining Chow's type B quasi-symmetric functions, we will recall the necessary background on tableaux used in this theory.

Definition 2.5. For an infinite set of formal variables $x_0, x_1, x_2, \dots$, *Chow's type B fundamental quasi-symmetric function* indexed by $J \subseteq \{0\} \cup [n-1]$ is defined as

$$F_J^B(x_0, x_1, \dots) := \sum_{\substack{0 \le i_1 \le i_2 \le \cdots \le i_n \\ j \in J \Rightarrow i_j < i_{j+1}}} x_{i_1} \cdots x_{i_n},$$

where $i_0 := 0$.

Example 2.6. For $n = 3$, we have $F_{\{1,2\}}^B = \displaystyle\sum_{0 \le i < j < k} x_i x_j x_k$, and $F_{\{0,2\}}^B = \displaystyle\sum_{1 \le i \le j < k} x_i x_j x_k$.

The Chow type B fundamental quasi-symmetric functions are intimately related to domino tableaux.

Definition 2.7. Let $\lambda \vdash 2n$ be a partition.

1. A *standard domino tableau* of shape λ consists of a tiling of the Young diagram of λ by dominoes which are labelled by $1, 2, \dots, n$, such that the entries are strictly increasing along rows when read from left to right and along columns when read from top to bottom. Denote by $P^0(n)$ the set of partitions $\lambda \vdash 2n$ that can be filled by dominoes and by $\mathrm{SDT}(\lambda)$ the set of standard domino tableaux of shape λ.
2. If the dominoes are labelled by non-negative integers, and entries are weakly increasing along the rows and strictly increasing along the columns, the domino tableau is *semi-standard*. Denote by $\mathrm{SSDT}(\lambda)$ the set of semi-standard domino tableaux of shape λ, which satisfy the following additional condition: if the upper leftmost domino is vertical then it cannot be labelled by 0. The *content* of a semi-standard domino tableau T is defined to be $w(\mathsf{T}) = (\mu_0, \mu_1, \dots)$ where for each i, μ_i is the number of appearances of the number i in T.

Domino tableaux will be denoted in serif mode (for instance T) to distinguish them from classical tableaux and bi-tableaux (for instance T).

Generating functions for domino tableaux, or *domino functions* are well studied, see e.g. [20]. Here we use a modified version due to Mayorova and Vassilieva [23].

Definition 2.8. Let $\lambda \in P^0(n)$. The *domino function* of λ is the generating function

$$\mathcal{G}_\lambda(\mathbf{x}) := \sum_{\mathsf{T} \in \mathrm{SSDT}(\lambda)} \mathbf{x}^{w(\mathsf{T})},$$

where $\mathbf{x}^{w(\mathsf{T})} := \prod_{i \ge 0} x_i^{\mu_i}$.

The *standard descent set* of a standard domino tableau T consists of all letters $1 \le i < n$, such that the northeast cell filled by $i+1$ is in a lower row than the northeast cell filled by i. Denote the letter in the (i,j) cell of T by $\mathsf{T}_{i,j}$. The *type B descent set* of a standard domino tableau T of size n is defined as

$$\mathrm{Des}_B(\mathsf{T}) := \begin{cases} \mathrm{Des}(\mathsf{T}) \sqcup \{0\} & \text{if } \mathsf{T}_{2,1} = 1, \\ \mathrm{Des}(\mathsf{T}) & \text{if } \mathsf{T}_{1,2} = 1. \end{cases}$$

Example 2.9. Here are two domino tableaux $\mathsf{T}, \mathsf{P} \in \mathrm{SDT}(4, 4, 2)$

$$\mathsf{T} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & 3 \\ \hline 2 & 4 & 5 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array} \quad , \quad \mathsf{P} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 & 5 \\ \hline 1 & 3 & 3 & 5 \\ \hline 4 & 4 \\ \cline{1-2} \end{array}$$

with descent sets $\mathrm{Des}(\mathsf{T}) = \mathrm{Des}_B(\mathsf{T}) = \{1, 3\}$ and $\mathrm{Des}(\mathsf{P}) = \{2, 3\} \subsetneq \mathrm{Des}_B(\mathsf{P}) = \{0, 2, 3\}$.

The following type B analogue of Theorem 2.4 holds.

Proposition 2.10. [23, Prop. 3.9] *For every partition* $\lambda \in P^0(n)$,

$$\sum_{\mathsf{T} \in \mathrm{SDT}(\lambda)} F^B_{\mathrm{Des}_B(\mathsf{T})} = \mathcal{G}_\lambda.$$

Recall the hook formula for the number of domino tableaux of given shape. Denote $f^\lambda := \#\mathrm{SYT}(\lambda)$ and $f_2^\lambda := \#\mathrm{SDT}(\lambda)$. Let $[\lambda]$ be the Young diagram of shape λ and $h_{i,j}$ be the *hook length* of the cell $(i, j) \in [\lambda]$, that is the number of cells in the i-th row and j-th column minus $i + j$.

Theorem 2.11. [2, Theorem 14.9.18] *For every partition* $\lambda \in P^0(n)$

$$f_2^\lambda = \frac{n!}{\prod\limits_{\substack{c \in [\lambda] \\ h_c \ is \ even}} \frac{h_c}{2}}. \tag{2.5}$$

Corollary 2.12. *For every* $n \geq 0$, *we have:*

$$\sum_{k=0}^{n} \left(f_2^{(2n-k,k)} \right)^2 = \binom{2n}{n};$$

$$\sum_{k=1}^{\lfloor n/2 \rfloor} \left(f_2^{(2n-2k,2k-1,1)} \right)^2 = \frac{1}{n+1} \binom{2n}{n} - 1.$$

Proof. From (2.5), by considering the parity of k, we derive

$$f_2^{(2n-k,k)} = \binom{n}{\lfloor k/2 \rfloor},$$

and the first formula follows by using

$$\sum_{k=0}^{n} \binom{n}{\lfloor k/2 \rfloor}^2 = \sum_{k=0}^{n} \binom{n}{k}^2.$$

Comparison of (2.5) with the hook formula for SYT [2, Theorem 14.5.3] yields

$$f_2^{(2n-2k,2k-1,1)} = f^{(n-k,k)},$$

for every $n \geq 2k \geq 2$. Recall that the *Catalan number* $C_n := \frac{1}{n+1} \binom{2n}{n}$ counts the set of 321-avoiding permutations in S_n, which in turn corresponds via RSK to the set of pairs of tableaux of the same shape $\lambda \vdash n$ containing at most 2 rows, see [29, Corollary 7.23.12]. Therefore $\sum_{k=0}^{\lfloor n/2 \rfloor} (f^{(n-k,k)})^2 = C_n$, and the second equation follows. $\qquad\square$

A family of skew shapes which plays an important role in the type B theory is the following. A *bi-shape* $(\lambda^-, \lambda^+) \vdash n$ is a pair of partitions of total size n. We draw the bi-shape (λ^-, λ^+) so that the southwest corner of the component of shape λ^+ is identified with the northeast corner of the component of shape λ^-. Denote *the set of standard Young tableaux of bi-shape* (λ^-, λ^+) by $\mathrm{BSYT}(\lambda^-, \lambda^+)$. For $T \in \mathrm{BSYT}(\lambda^-, \lambda^+)$, let $T_{\lambda^-}, T_{\lambda^+}$ be the components of T of shape λ^- and λ^+ respectively.

The *standard descent set* of a standard Young bi-tableau T of size n is defined as

$$\mathrm{Des}(T) := \{0 < i < n \mid i + 1 \text{ is in a lower row than } i\}, \tag{2.6}$$

while the *type B descent set* of a standard Young bi-tableau T of size n is defined as

$$\mathrm{Des}_B(T) := \begin{cases} \mathrm{Des}(T) \sqcup \{0\} & \text{if } 1 \in T_{\lambda^-}, \\ \mathrm{Des}(T) & \text{if } 1 \in T_{\lambda^+}. \end{cases} \tag{2.7}$$

Example 2.13. Here are two standard bi-tableau of shape $((2),(2,1))$ with descent sets $\mathrm{Des}(T) = \mathrm{Des}_B(T) = \{1,3\}$ and $\mathrm{Des}(P) = \{2,3\} \subsetneq \mathrm{Des}_B(P) = \{0,2,3\}$:

$$T = \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline \end{array} \begin{array}{|c|} \hline 2 \\ \hline \end{array} \quad , \quad P = \quad \begin{array}{|c|c|} \hline 2 & 5 \\ \hline \end{array} \begin{array}{|c|} \hline 3 \\ \hline \end{array} \quad .$$

$$\begin{array}{|c|c|} \hline 4 & 5 \\ \hline \end{array} \qquad \qquad \begin{array}{|c|c|} \hline 1 & 4 \\ \hline \end{array}$$

2.4. **From domino tableaux to bi-tableaux.** There exists a well-known bijection from (semi)-standard domino tableaux of shape $\lambda \in P^0(n)$, to bi-(semi)-standard-Young tableaux of corresponding bi-shape $(\lambda^-, \lambda^+) \vdash n$, due to Carré–Leclerc, see [11, Algorithm 6.1]. The bijection associates each (semi)-standard domino tableau T of shape λ with a pair (T^-, T^+) of (semi)-standard Young tableaux of shapes (λ^-, λ^+). The tableaux T^- and T^+ are constructed as follows : assign to each (single) box of T a sign $-$ or $+$ such that the upper leftmost box is assigned a $-$ and two adjacent boxes have opposite signs. The component T^- (resp. T^+) is then obtained from the sub-tableau of T composed of the dominoes with upper rightmost box filled with $-$ (resp. $+$).

Note that the resulting shapes (λ^-, λ^+) only depend on the shape λ, and they correspond to the *2-quotient* obtained from λ by the *Littlewood decomposition* (see [29, page 468] or [2, Chapter 14.9]); in this case the 2-core of λ is empty. We will denote this particular case of the Littlewood decomposition by $\psi(\lambda) := (\lambda^-, \lambda^+)$.

Example 2.14. Let T be the following domino tableau:

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 4 \\ \hline 1 & 2 \\ \cline{1-2} 3 & 3 \\ \cline{1-2} \end{array}$$

We assign the $\pm$ signs to get the following:

$$\begin{array}{|c|c|c|c|} \hline 1^- & 2^+ & 4^- & 4^+ \\ \hline 1^+ & 2^- \\ \cline{1-2} 3^- & 3^+ \\ \cline{1-2} \end{array}$$

According to the algorithm, the corresponding standard Young bi-tableau is :

$$\begin{array}{|c|c|} \hline 2 & 4 \\ \hline \end{array} \begin{array}{|c|} \hline 3 \\ \hline \end{array}$$
$$\begin{array}{|c|} \hline 1 \\ \hline \end{array}$$

Consider the case of semi-standard domino tableaux. Note that the condition that 0 must not occupy a vertical upper leftmost domino implies that in the semi-standard bi-tableau associated with a domino semi-standard tableau, 0 will not appear in the lower component. Now, the Carré-Leclerc bijection from semi-standard domino tableaux to semi-standard bi-tableaux is content-preserving, implying that

$$\sum_{\mathsf{T} \in \mathrm{SSDT}(\lambda)} \mathbf{x}^{w(\mathsf{T})} = s_{\lambda^-}(x_1, x_2, \dots) s_{\lambda^+}(x_0, x_1, \dots).$$

Thus, by Proposition 2.10, we derive the following.

Proposition 2.15. [23, Prop. 3.13] *For every $\lambda \in P^0(n)$,*

$$\sum_{\mathsf{T} \in \mathrm{SDT}(\lambda)} F^B_{\mathrm{Des}_B(\mathsf{T})}(x_0, x_1, \dots) = s_{\lambda^-}(x_1, x_2, \dots) \; s_{\lambda^+}(x_0, x_1, x_2, \dots).$$

Comparing Proposition 2.15 with [3, Prop. 4.2], Mayorova and Vassilieva deduce the following result.

Lemma 2.16. [24, Lemma 3] *For every $\lambda \in P^0(n)$ there exists an implicit Des_B-preserving bijection from the set of standard domino tableaux of shape $\lambda \vdash 2n$ to the set of standard bi-tableaux of bi-shape (λ^-, λ^+).*

In particular, we have the following remark that will be used in Section 7.

Remark 2.17. The 2-quotient of the domino shape $\lambda = (2n - k, k)$ is the bi-shape $((k/2), (n - k/2))$ if k is even, and $((n - (k-1)/2), ((k-1)/2))$ if k is odd. The 2-quotient of the domino shape $\lambda = (2n - 2k, 2k - 1, 1)$ is $(\emptyset, (n - k, k))$. By Lemma 2.16, there exist Des_B-preserving maps

(1) from $\mathrm{SDT}(2n - 2k, 2k)$ to $\mathrm{BSYT}((k), (n - k))$ for $0 \leq k \leq \lfloor n/2 \rfloor$;
(2) from $\mathrm{SDT}(2n - 2k - 1, 2k + 1)$ to $\mathrm{BSYT}((n - k), (k))$, for $1 \leq k \leq \lfloor (n - 1)/2 \rfloor$;
(3) from $\mathrm{SDT}(2n - 2k, 2k - 1, 1)$ to $\mathrm{BSYT}(\emptyset, (n - k, k)))$, for $1 \leq k \leq \lfloor n/2 \rfloor$.

Note that the Carré-Leclerc bijection is not Des_B-preserving in general, but it is for domino shapes of the form $\lambda = (2n - 2k, 2k - 1, 1)$.

3. Heaps and full commutativity

3.1. **Types A and B.**

We briefly describe a way to define the above mentioned heaps and their relation with full commutativity, for more details see for instance [5] and the references cited there.

Let (W, S) be a Coxeter system, and fix a word $\mathbf{w} = s_{a_1} \cdots s_{a_l}$ in S^*. Define a partial ordering $\prec$ on the index set $\{1, \ldots, l\}$ as follows: set $i \prec j$ if $i < j$ and s_{a_i}, s_{a_j} do not commute, and extend by transitivity. We denote by $\mathrm{Heap}(\mathbf{w})$ this poset together with a labeling map $\epsilon : i \mapsto s_{a_i}$. Heaps are well-defined up to commutation classes [37], that is, if $\mathbf{w}$ and $\mathbf{w}'$ are two reduced expressions for $w \in W$, that are in the same commutation class, then the corresponding labeled heaps are isomorphic. Therefore, when w is FC we can define $\mathrm{Heap}(w) := \mathrm{Heap}(\mathbf{w})$, where $\mathbf{w}$ is any reduced expression for w. Another important feature in heaps theory is that the linear extensions of $\mathrm{Heap}(w)$ are in bjiection with the reduced expressions of w, see [33, Proposition 2.2].

Example 3.1. Consider $w = [4, 1, 5, 2, 3] \in \mathrm{FC}(A_4) = \mathrm{FC}(S_5)$. Its heap is represented in Figure 2, left. In the Hasse diagram of $\mathrm{Heap}(w)$, elements with the same labels are drawn in the same column. We recall that each vertex is labeled by the corresponding generator, but we do not write those labels for visibility reasons.

Its set of reduced expressions $\mathcal{R}(w) = \{s_3 s_2 s_1 s_4 s_3, s_3 s_2 s_4 s_1 s_3, s_3 s_2 s_4 s_3 s_1, s_3 s_4 s_2 s_3 s_1, s_3 s_4 s_2 s_1 s_3\}$ is obtained by listing the labels of each linear extension of $\mathrm{Heap}(w)$.
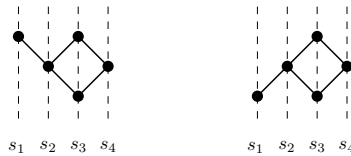

FIGURE 2. Two FC heaps of type A_4.

Given a heap $H = \mathrm{Heap}(w)$ for $w \in \mathrm{FC}(W)$ and a subset $I \subset S$, we denote by H_I the sub-poset induced by all elements of H with labels in I.

Definition 3.2. Let (W, S) be a Coxeter system, $w \in \mathrm{FC}(W)$, and $H := \mathrm{Heap}(w)$. We say that H is *alternating* if for each non commuting generators s, t in S, the chain $H_{\{s,t\}}$ has alternating labels s and t from bottom to top.

Note that if $\mathrm{Heap}(w)$ is alternating, then any reduced expression $\mathbf{w}$ of w is *alternating* in the sense that for each non commuting generators $s, t \in S$, the occurrences of s alternate with those of t in $\mathbf{w}$. In this case we say that $w \in \mathrm{FC}(W)$ is alternating.

We now recall the descriptions of FC heaps corresponding to the Dynkin diagrams of types A_{n-1} and B_n which were given for instance in [5].

Proposition 3.3 (Classification of FC heaps in type $\boldsymbol{A_{n-1}}$). *An element $w \in A_{n-1}$ is fully commutative if and only if* $\mathrm{Heap}(w)$ *is alternating. More precisely, in* $\mathrm{Heap}(w)$,

(a) *There is at most one occurrence of s_1 (resp. s_{n-1});*

(b) *For each $i \in \{1, \ldots, n-2\}$, the elements with labels s_i, s_{i+1} form an alternating chain.*

As already mentioned, such elements are in bijection with 321-avoiding permutations in S_n, that are counted by the Catalan number C_n. In Figure 3, the heaps of three FC elements of type A are depicted.
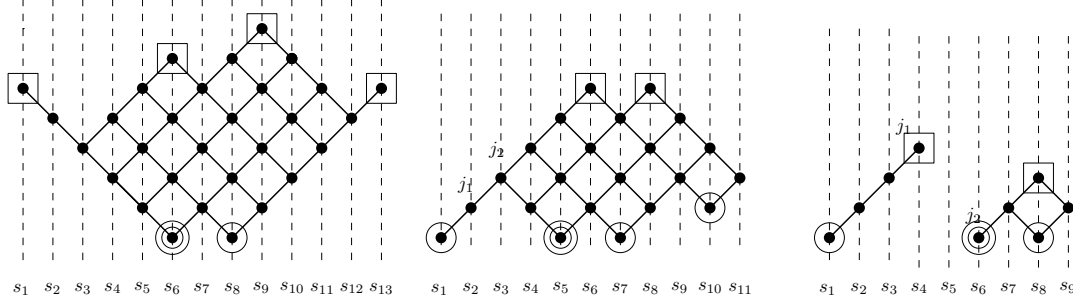

FIGURE 3. Three alternating heaps of type A.

Now we will need the following second family of heaps, which, in addition to alternating heaps, describes FC heaps of type B (see, e.g. [5]).

Definition 3.4. A *left-peak*, associated with the Dynkin diagram of type B_n, is a heap such that there exists a unique $j \in \{1, \ldots, n-1\}$ satisfying:

(a) $\mathrm{Heap}_{\{s_0, \ldots, s_j\}} = \mathrm{Heap}(s_j \cdots s_1 s_0 s_1 \cdots s_j)$;

(b) $\mathrm{Heap}_{\{s_j, s_{j+1}\}} = s_j s_j$ or $s_{j+1} s_j s_j s_{j+1}$ for $j < n-1$, and $s_{n-1} s_{n-1}$ for $j = n-1$;

(c) $\mathrm{Heap}_{\{\hat{s}_j, s_{j+1} \ldots, s_{n-1}\}}$ is alternating, where $\hat{s}_j$ means that one occurrence of s_j is deleted.

An element $w \in \mathrm{FC}(B_n)$ for which $\mathrm{Heap}(w)$ is a left-peak will also be called a *left-peak element*.

Example 3.5. In Figure 4, left, there is an example of an alternating heap of type B: note that in contrast to the type A case (having at most one vertex labeled s_1), it can have any finite number (between 0 and n) of vertices labeled s_0. In the left-peak of Figure 4, right, we have $j = 2$.
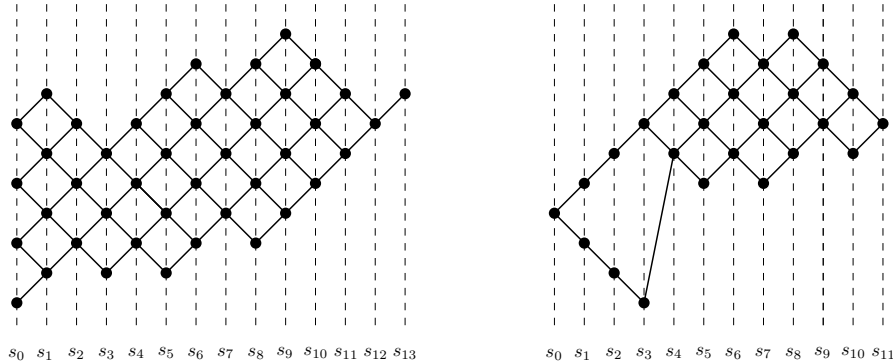

FIGURE 4. Left: an alternating heap of type B_{14}. Right: a left-peak of type B_{12}

From [5, Theorem 3.10 and Section 4.4], we have the following result.

Proposition 3.6 (Classification of FC heaps in type $\boldsymbol{B_n}$). *An element $w \in B_n$ is fully commutative if and only if* $\mathrm{Heap}(w)$ *is either an alternating heap or a left-peak.*

Remark 3.7. The set of alternating FC elements in B_n coincides with the set of FC top elements defined in Theorem 4.1 of [34], while the set of FC left peak elements is exactly the set of FC bottom elements which are not top elements. Therefore by [34, Theorem 5.9], the number of alternating FC elements is $\binom{2n}{n}$, and the number of FC left peaks is $C_n - 1$. Thus the total number of FC elements in B_n is

$$\frac{n+2}{n+1}\binom{2n}{n} - 1.$$

Stembridge provided a characterization of FC elements in B_n by using pattern avoidance: as for $\mathrm{FC}(S_n)$ these elements are 321-avoiding but they also have to avoid other patterns.

Proposition 3.8. [34, Theorem 5.1] *A signed permutation* $w \in B_n$ *is fully commutative if and only if w avoids the pattern* $[-1, -2]$ *and all patterns* $[a, b, c]$ *such that* $|a| > b > c$ *or* $-b > |a| > c$.

3.2. **Reduced expressions.** Let $w \in W$ be a FC element of type A_{n-1} or B_n. By definition $i \in \mathrm{Des}(w)$ if and only if there exists a peak in $\mathrm{Heap}(w)$ labeled by s_i, where by a *peak* we mean a vertex having all its neighbors vertices below it. Moreover, $i \in \mathrm{Des}^L(w)$ if and only if there exists a valley in $\mathrm{Heap}(w)$ labeled by s_i, where by a *valley* we mean a vertex having all its neighbors above it. Indeed, notice that w is FC if and only if w^{-1} is FC. Moreover, one can see that $\mathrm{Heap}(w^{-1})$ is the dual heap of $\mathrm{Heap}(w)$, i.e. the heap of w with the reverse order, that can be obtained from $\mathrm{Heap}(w)$ by a horizontal reflection. In Figure 2 a heap and its dual are depicted.

For our purposes, it will be useful to introduce a new statistic on $\mathrm{FC}(S_n)$.

Definition 3.9. For any $\pi \in \mathrm{FC}(S_n)$ we define the *first valley*, denoted $v(\pi)$, as follows :

$$v(\pi) := \begin{cases} \min\left\{\mathrm{Des}(\pi^{-1}) \setminus \{1\}\right\} & \text{if } \mathrm{Des}(\pi^{-1}) \setminus \{1\} \neq \emptyset; \\ n & \text{if } \mathrm{Des}(\pi^{-1}) \setminus \{1\} = \emptyset. \end{cases} \tag{3.1}$$

Note that $v(\pi) = n$ if and only if $\pi = e$ or $\mathrm{Des}(\pi^{-1}) = \{1\}$. Moreover, for $\pi \in \mathrm{FC}(S_n)$, we have that

$$1 \in \mathrm{Des}(\pi^{-1}) \Leftrightarrow \pi(1) = 2. \tag{3.2}$$

In Figure 3, the descents of the three elements are surrounded by a square, the valleys by a circle, and the first valley by a double circle.

As we mentioned above, if w is FC then the set of linear extensions of $\mathrm{Heap}(w)$ is in bijection with the set of reduced expressions of w. It will be helpful in the sequel to consider a particular reduced expression for each $w \in \mathrm{FC}(S_n) \cup \mathrm{FC}(B_n)$.

Definition 3.10. The *diagonal reduced expression* of $w \in \mathrm{FC}(S_n) \cup \mathrm{FC}(B_n)$ is obtained by reading the labels of the vertices in the "diagonals" of $\mathrm{Heap}(w)$, directed from south east to north west of $\mathrm{Heap}(w)$, starting from the leftmost diagonal. Each such diagonal contributes a factor of the form $(s_i s_{i-1} \cdots s_j)$ with $0 \leq j \leq i \leq n-1$. It is easy to see that such an expression corresponds to a linear extension of $\mathrm{Heap}(w)$.

More precisely, consider first $e \neq w \in \mathrm{FC}(S_n)$. Then the diagonal reduced expression for w is of the form

$$\mathbf{w} = (s_{v_0} s_{v_0-1} \cdots s_{j_0})(s_{v_1} s_{v_1-1} \cdots s_{j_1}) \cdots (s_{v_k} s_{v_k-1} \cdots s_{j_k}), \tag{3.3}$$

where $1 \leq v_0 < v_1 < \ldots < v_k \leq n-1$ and $1 \leq j_0 < j_1 < \ldots < j_k \leq n-1$. We have $v_0 = v(\pi)$ if $1 \notin \mathrm{Des}(\pi^{-1})$ and $v_0 = 1$ otherwise.

Example 3.11. The diagonal reduced expression for the element on the left of Figure 3 is

$$(s_6 \cdots s_1)(s_8 \cdots s_4)(s_9 \cdots s_5)(s_{10} \cdots s_6)(s_{11} \cdots s_8)(s_{12} \cdots s_9)(s_{13}),$$

while for the element on the right it is $(s_1)(s_2)(s_3)(s_4)(s_6)(s_8 s_7)(s_9 s_8)$.

In $\mathrm{FC}(B_n)$ there are two possibilities:

- **Alternating:** If w_1 is alternating then its diagonal reduced expression takes the form

$$\mathbf{w}_1 = (s_{v_0} s_{v_0-1} \cdots s_{j_0})(s_{v_1} s_{v_1-1} \cdots s_{j_1}) \cdots (s_{v_k} s_{v_k-1} \cdots s_{j_k}), \tag{3.4}$$

where $0 \leq v_0 < v_1 < \ldots < v_k \leq n-1$ and $0 \leq j_0 \leq j_1 \leq \ldots \leq j_k \leq n-1$ with the condition that equality between two j_i's occurs only if both are 0.

- **Left-peak:** If w_2 is a left-peak then the diagonal reduced expression has the form

$$\mathbf{w}_2 = (s_{v_0}s_{v_0-1}\cdots s_0)(s_1)\cdots(s_{j_1-1})(s_{v_1}s_{v_1-1}\cdots s_{j_1})\cdots(s_{v_k}s_{v_k-1}\cdots s_{j_k}), \tag{3.5}$$

where $0 < v_0 < v_1 < \ldots < v_k \leq n-1$ and $1 < j_1 < \ldots < j_k \leq n-1$.

Example 3.12. The diagonal reduced expression for the element on the left of Figure 4 is

$$(s_0)(s_1s_0)(s_3s_2s_1s_0)(s_5\cdots s_0)(s_6\cdots s_1)(s_8\cdots s_4)(s_9\cdots s_5)(s_{10}\cdots s_6)(s_{11}\cdots s_8)(s_{12}\cdots s_9)(s_{13})$$

while the diagonal reduced expression for the element on the right is

$$(s_3s_2s_1s_0)(s_1)(s_2)(s_5s_4s_3)(s_7\cdots s_4)(s_8\cdots s_5)(s_{10}\cdots s_6)(s_{11}\cdots s_8).$$

Remark 3.13. Note that if w is a left-peak then we must have $w(1) = 1$. Moreover, there must be unique $i > 1$ and $k > 1$ such that $w(i) = -k$.

Remark 3.14. Observe that in the three above expressions (3.3)–(3.5), the left descents of w belong to the set $\{v_0, v_1, \ldots, v_k\}$ of initial indices of the factors exhibited in the diagonal expressions. More precisely, $v_0 \in \mathrm{Des}^L(w)$, and for $i \geq 1$, $v_i \in \mathrm{Des}^L(w)$ if and only if $v_i - v_{i-1} \geq 2$.

4. Decomposition of $\mathrm{FC}(B_n)$ into fibers

In this section we let $W = B_n$ and $J = S \setminus \{s_0\}$. The parabolic subgroup $(B_n)_J$ is isomorphic to S_n and the quotient has the form

$$(B_n)^J := \{\mu \in B_n \mid \mathrm{Des}_B(\mu) \subseteq \{0\}\} = \{\mu \in B_n \mid \mu(1) < \ldots < \mu(n)\}. \tag{4.1}$$

By Proposition 2.1, every $w \in B_n$ has a unique decomposition

$$w = \mu \cdot \pi \tag{4.2}$$

where $\mu \in (B_n)^J, \pi \in (B_n)_J$, and

$$\ell_B(w) = \ell_B(\mu) + \ell_B(\pi). \tag{4.3}$$

Notice that μ can be written as the ascending reordering of w, and the counterpart π is the permutation in S_n which records the letters $w_1, \ldots, w_n$ in the relative standard order. The permutation π is called the *standardization* of the signed permutation w. For example, $[1, -3, -2, 4] = [-3, -2, 1, 4] \cdot [3, 1, 2, 4]$.

We can characterize precisely the reduced expressions of the elements in $(B_n)^J$. By defining

$$\delta_i := s_{i-1} \cdots s_2 s_1 s_0$$

for integers i such that $1 \leq i \leq n$, we have

$$(B_n)^J = \{\mu \in B_n \mid \mu = \mu_1 \cdots \mu_n, \ \mu_i \in \{e, \delta_i\}\}. \tag{4.4}$$

Indeed, take an element $\mu \in (B_n)^J$. If $\mathrm{Neg}(\mu) = \emptyset$ then $\mu = e$, otherwise set $\mathrm{Neg}(\mu) = \{i_1, \ldots, i_k\}$ with $-i_1 < \cdots < -i_k < 0$. We have $\mu(1) = -i_1, \ldots, \mu(k) = -i_k$, while $\mu(k+1), \ldots, \mu(n)$ have to be positive and in increasing order, therefore μ has a reduced expression $\boldsymbol{\mu} = \delta_{i_k} \ldots \delta_{i_1}$. Conversely, one checks that any element with reduced expression $\delta_{i_k} \cdots \delta_{i_1}$ satisfies the inequalities in (4.1).

This implies the following description that will be used in Section 6.

Observation 4.1. Every $\mu \in (B_n)^J$ is an increasing sequence of n letters from $[\pm n]$ with distinct absolute values. Hence μ^{-1} is a shuffle of $[-k, -k+1, \ldots, -1]$ with $[k+1, k+2, \ldots, n]$ for some $0 \leq k \leq n$.

As clearly no *long braid type factor*, that is a factor of the form $s_0s_1s_0s_1$, $s_1s_0s_1s_0$ or $s_is_{i\pm1}s_i$, can occur in all the reduced expressions of the elements in the set (4.4), one deduces from Proposition 2.3 that each element in $(B_n)^J$ is FC, namely

$$\mathrm{FC}((B_n)^J) := (B_n)^J \cap \mathrm{FC}(B_n) = (B_n)^J. \tag{4.5}$$

This also follows from Proposition 3.8, since any $\mu \in (B_n)^J$ is an increasing sequence. Moreover, note that the reduced expressions for the elements in $(B_n)^J$ given in (4.4) are the diagonal reduced expressions from Definition 3.10 of the corresponding heaps.

Note that the heap of any element in $(B_n)^J$ can be depicted as a sub-poset of the "triangular" heap corresponding to the element $\delta_1 \cdots \delta_n$ having maximal length, see Figure 5.

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7 \qquad\qquad \delta_1\delta_2\delta_4\delta_6 \qquad\qquad \delta_7$$
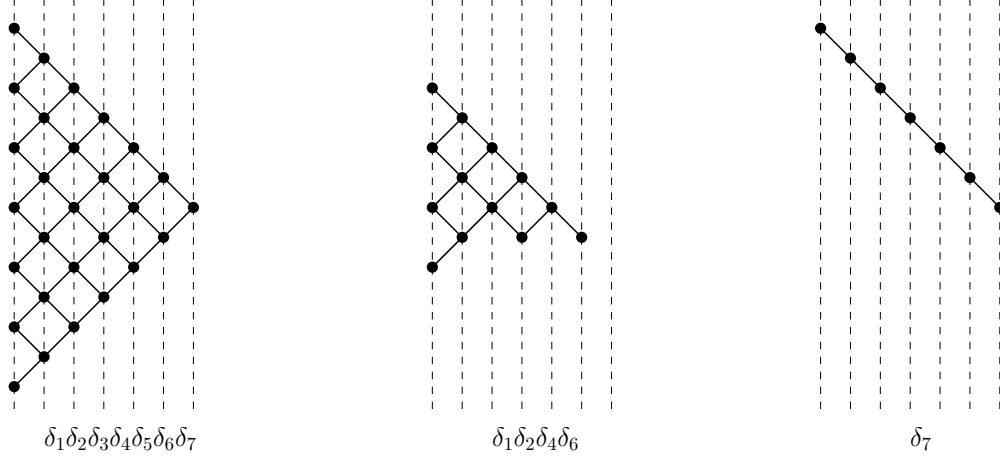
FIGURE 5. Left: the heap of the element with maximal length of B_7^J, that is the element with diagonal reduced expression $\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7$. Center: the heap of the element in B_7^J with diagonal reduced expression $\delta_1\delta_2\delta_4\delta_6$. Right: the heap of the element with diagonal reduced expression δ_7.

Now we consider the restriction of the decomposition (4.2) to FC elements, $w = \mu \cdot \pi$ for $w \in \mathrm{FC}(B_n)$. Then all reduced expressions of w contain no braid relation, and thanks to (4.3) above, it implies that both μ and π are FC. Therefore we obtain the following inclusion

$$\mathrm{FC}(B_n) \subset \mathrm{FC}((B_n)^J) \times \mathrm{FC}(S_n) = (B_n)^J \times \mathrm{FC}(S_n). \tag{4.6}$$

It is easy to show that this inclusion is strict (take for instance $\mu = s_0s_1s_0$ and $\pi = s_1$).

Our next result refines the previous inclusion by exhibiting for any fixed FC permutation in S_n the corresponding subset of $(B_n)^J$.

Theorem 4.2. *We have the following decomposition*

$$\mathrm{FC}(B_n) = \biguplus_{\pi \in \mathrm{FC}(S_n)} B_n(\pi) \cdot \pi, \tag{4.7}$$

where

$$B_n(\pi) := \begin{cases} \{\mu \in B_n \mid \mu = \mu_1 \cdots \mu_{v(\pi)},\ \mu_i \in \{e, \delta_i\}\} & \text{if } 1 \notin \mathrm{Des}(\pi^{-1}); \\ \{\mu \in B_n \mid \mu \in \{e, \delta_1, \ldots, \delta_{v(\pi)}\}\} & \text{if } 1 \in \mathrm{Des}(\pi^{-1}), \end{cases}$$

and $v(\pi)$ is the first valley of π from Definition 3.9.

Proof. First note that the sets on the right-hand side of (4.7) are disjoint by uniqueness of the decomposition (4.2).

Let us now consider an element $w \in \mathrm{FC}(B_n)$ and write it uniquely as $w = \mu \cdot \pi \in (B_n)^J \times \mathrm{FC}(S_n)$, according to the decomposition (4.2) and the inclusion (4.6). We need to show that $\mu \in B_n(\pi)$, and to this aim we consider three cases. Along the proof we set $v := v(\pi)$; note that by definition $v \geq 2$.

(1) If $\pi = e$, we get the result by (4.4).
(2) If $\pi \neq e$ and $1 \notin \mathrm{Des}(\pi^{-1})$, then there exists a reduced expression $\boldsymbol{\pi}$ of π starting with a factor $s_v s_{v-1} \cdots s_j$, for an integer j satisfying $1 \leq j \leq v$. For the sake of a contradiction, assume that the rightmost factor in the reduced expression $\boldsymbol{\mu} = \mu_1 \cdots \mu_n$ of μ is δ_i with $i > v$. It suffices to assume that $i = v + 1$. Then me may write a reduced expression of w as

$$\mathbf{w} = \mathbf{u}(s_v s_{v-1} \cdots s_1 s_0) \cdot (s_v s_{v-1} \cdots s_j)\tilde{\mathbf{u}},$$

where $\mathbf{u}$ (respectively $\tilde{\mathbf{u}}$) is a left (respectively right) factor of $\boldsymbol{\mu}$ (respectively $\boldsymbol{\pi}$). Now between the two above occurrences of s_v there is no occurrence of s_{v+1}, hence by applying commutation relations to $\mathbf{w}$ we obtain a reduced expression containing the factor $s_v s_{v-1} s_v$, a contradiction for a FC element in B_n (as $v \geq 2$). An example of this case is depicted in Figure 6, left.

(3) If $1 \in \mathrm{Des}(\pi^{-1})$, then we discuss two cases.

Assume first that $\mathrm{Des}(\pi^{-1}) = \{1\}$, which means by definition that $v = n$. Equivalently, the one-line notation of π is $[2, \ldots, 1, \ldots]$ where the elements represented by the dots are in increasing order. This means that π has a reduced expression of the form $s_1 s_2 \cdots s_j$ for some $j \in \{1, \ldots, n-1\}$. For the sake of a contradiction, suppose that no reduced expression $\boldsymbol{\mu}$ of μ belongs to $\{e, \delta_1, \ldots, \delta_n\}$. Then by (4.4), $\boldsymbol{\mu}$ contains at least two factors δ_i. Let us consider the two rightmost factors in $\boldsymbol{\mu}$, say $\delta_{i_1}, \delta_{i_2}$ with $1 \leq i_1 < i_2 \leq n$. Hence

$$\mathbf{w} = \mathbf{u}(s_{i_1-1} \cdots s_1 s_0)(s_{i_2-1} \cdots s_2 s_1 s_0) \cdot (s_1) \cdots (s_j).$$

Now the occurrence of s_0 in the factor δ_{i_1} commutes with all the generators in δ_{i_2} on its right up to s_2 included; so we can move it until the occurrence of s_1 in δ_{i_2}, which would give a reduced expression of w that contains a factor $s_0 s_1 s_0 s_1$. This is a contradiction since w is fully commutative.

Next assume that $\{1\} \subsetneq \mathrm{Des}(\pi^{-1})$. Since π^{-1} is FC, $2 \notin \mathrm{Des}(\pi^{-1})$ so we must have $v \geq 3$. (See an example in Figure 6, right.) The diagonal reduced expression of Definition 3.10 of π starts with the factors $(s_1)(s_2) \cdots (s_{j_1})(s_v s_{v-1} \cdots s_{j_2})$, where $1 \leq j_1 < j_2 \leq v$. Note that if $j_2 = j_1 + 1$ then $v > j_2$ (see Figure 3, center). For the sake of a contradiction, suppose that each reduced expression of μ satisfies $\boldsymbol{\mu} \notin \{e, \delta_1, \ldots, \delta_v\}$. Then by (4.4), $\boldsymbol{\mu}$ contains either the product of at least two different factors of the form δ_i, or a single δ_i with $i > v$. In the first situation, let us consider the two rightmost such factors in $\boldsymbol{\mu}$, say $\delta_{i_1}, \delta_{i_2}$ with $1 \leq i_1 < i_2 \leq n$. Hence

$$\mathbf{w} = \mathbf{u}(s_{i_1-1} \cdots s_1 s_0)(s_{i_2-1} \cdots s_2 s_1 s_0) \cdot (s_1)(s_2) \cdots (s_{j_1})(s_v s_{v-1} \cdots s_{j_2})\tilde{\mathbf{u}}.$$

Now by commutation relations we obtain the same contradiction as above.

In the second situation, take the rightmost factor δ_i in $\boldsymbol{\mu}$ with $i > v$: without loss of generality, one can take $i = v + 1$. We get

$$\mathbf{w} = (s_v s_{v-1} \cdots s_1 s_0) \cdot (s_1)(s_2) \cdots (s_{j_1})(s_v s_{v-1} \cdots s_{j_2})\tilde{\mathbf{u}}.$$

If $j_2 > j_1 + 1$, then $v \geq j_2 > j_1 + 1$ (see e.g. Figure 3, right), therefore the second occurrence of s_v in this expression commutes with all generators on its left up to $s_v s_{v-1}$, thus commutation relations would yield a factor $s_v s_{v-1} s_v$, a contradiction. If $j_2 = j_1 + 1$, we can conclude in the same way, thanks to the condition $v > j_2 > j_1$ in this case.
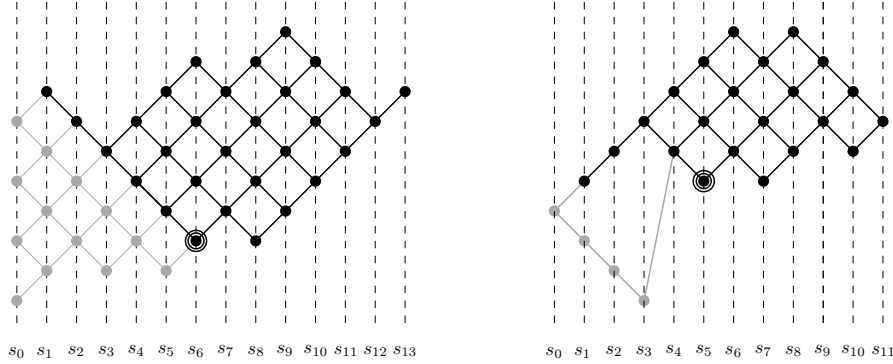

FIGURE 6. Two products $\mu \cdot \pi$, one yielding an alternating element (left), the other giving a left-peak (right). The heap of μ is in grey.

Let us show the opposite inclusion, by considering again three cases.

(1) Taking $\pi = e$, by (4.5) we have $B_n(e) = (B_n)^J \subset \mathrm{FC}(B_n)$.

(2) Now, let $\pi \in \mathrm{FC}(S_n)$ such that $\pi \neq e$ and $1 \notin \mathrm{Des}(\pi^{-1})$ (see Figure 6, left). The diagonal reduced expression of π takes the form (3.3). Now let μ be any element in $B_n(\pi)$. First notice that μ is FC, as $B_n(\pi) \subset (B_n)^J$. Moreover the diagonal reduced expression for μ is made of some factors chosen from the product $(s_0)(s_1 s_0) \cdots (s_{v-1} s_{v-2} \cdots s_2 s_1 s_0)$. Assume that the rightmost of these factors is $(s_i s_{i-1} \cdots s_2 s_1 s_0)$, with $i \leq v - 1$. Then we can concatenate the two expressions to get

$$\mathbf{w} = (s_0)^{\pm}(s_1 s_0)^{\pm} \cdots (s_i s_{i-1} \cdots s_2 s_1 s_0) \cdot (s_v s_{v-1} \cdots s_{j_0})(s_{v_1} s_{v_1-1} \cdots s_{j_1}) \cdots (s_{v_k} s_{v_k-1} \cdots s_{j_k}),$$

where $(\cdot)^{\pm}$ means that the expression between the parentheses might appear or not appear. Recall that, as both μ and π are separately FC, the two reduced expressions above for μ and π do not contain any long braid type factor. We will show that their product is also a reduced expression for w with no braid type factor.

If a *nil factor* $s_i s_i$, or a long braid type factor appears in the product, it has to involve generators in the last factor of μ: $(s_i s_{i-1} \cdots s_1 s_0)$ and in the first of π: $(s_v s_{v-1} \cdots s_{j_0})$. Let us consider two consecutive occurrences of a generator s_q in $\mathbf{w}$, one in μ and the other in π, therefore with $j_0 \leq q \leq i$. By definition, the generator s_{q-1} follows s_q in the rightmost factor of μ. Moreover, since $q \leq i \leq v-1$ we have that s_{q+1} appears before s_q in π. This implies that between the two occurrences of s_q in w there are both occurrences of s_{q+1} and s_{q-1}. Hence neither nil nor braid type factor may appear in any expression of w, and $w \in \mathrm{FC}(B_n)$.

(3) Finally consider a FC element π such that $1 \in \mathrm{Des}(\pi^{-1})$ (see Figure 6, right). As observed before, either $\boldsymbol{\pi} = (s_1)(s_2)\cdots(s_j)$ or $\boldsymbol{\pi} = (s_1)(s_2)\cdots(s_{j_1})(s_v s_{v-1} \cdots s_{j_2})\tilde{\mathbf{u}}$. If $\mu = e$, then $\mu \cdot \pi = \pi$ is FC. If $\boldsymbol{\mu} = \delta_i$ for $1 \leq i \leq v$, then the corresponding reduced expression for the product $\mu \cdot \pi$ is either equal to

$$(s_{i-1} \cdots s_1 s_0) \cdot (s_1)(s_2) \cdots (s_j)$$

or

$$(s_{i-1} s_{i-2} \cdots s_1 s_0) \cdot (s_1)(s_2) \cdots (s_{j_1})(s_v s_{v-1} \cdots s_{j_2})\tilde{\mathbf{u}}.$$

In both cases, this product is FC, since it contains neither nil nor braid factor.

$\square$

We call the set $B_n(\pi) \cdot \pi := \{\mu \cdot \pi \mid \mu \in B_n(\pi)\}$ the *fiber* associated to $\pi \in \mathrm{FC}(S_n)$. We can characterize alternating and left-peak elements by using fibers.

Corollary 4.3. *Let $w \in \mathrm{FC}(B_n)$ be written in the form $w = \mu \cdot \pi \in (B_n)^J \times \mathrm{FC}(S_n)$ according to (4.7). Then we have the following characterizations:*

- *w is a left-peak if and only if $\pi(1) = 2$ and $\mu = \delta_i$ for some $i \in \{2, \ldots, v(\pi)\}$;*
- *w is alternating if and only if either $\pi(1) \neq 2$ or ($\pi(1) = 2$ and $\mu = e$ or $\mu = \delta_1$).*

Proof. We start with the first assertion and assume that $\mathrm{Heap}(w)$ is a left-peak. Then the diagonal reduced expression (see Definition 3.10 and (3.5)) for w is of the form

$$\boldsymbol{w} = s_i \cdots s_1 s_0 s_1 \boldsymbol{\pi}_0,$$

where $i \geq 1$ and $s_1 \boldsymbol{\pi}_0$ is a reduced expression of an element in $\mathrm{FC}(A_{n-1})$. By uniqueness of the decomposition (4.2), we derive that reduced expressions for μ and π are given by $\boldsymbol{\mu} = \delta_{i+1}$ and $\boldsymbol{\pi} = s_1 \boldsymbol{\pi}_0$, respectively. Therefore $1 \in \mathrm{Des}^L(\pi) = \mathrm{Des}(\pi^{-1})$, which by (3.2) is equivalent to $\pi(1) = 2$. Setting $j = i + 1 \geq 2$, we get the desired reduced expression for $\boldsymbol{\mu}$ by Theorem 4.2.

Conversely, again (3.2) implies that $\boldsymbol{\pi} = s_1 \boldsymbol{\pi}_0$. Therefore

$$\boldsymbol{w} = s_i \cdots s_1 s_0 s_1 \boldsymbol{\pi}_0,$$

where $1 \leq i \leq v(\pi) - 1$. As there is no s_2 between the above two occurrences of s_1, we deduce that $\mathrm{Heap}(w)$ is not alternating, so it is a left-peak.

The second assertion is a consequence of the first one and Theorem 4.2, together with (3.2). $\square$

To illustrate Theorem 4.2, we end this section by giving two examples.

Example 4.4. Let $\pi = [1, 5, 2, 3, 4] = s_4 s_3 s_2$, therefore $\pi(1) \neq 2$ so $1 \notin \mathrm{Des}(\pi^{-1})$. In the following table, for each element $\mu \cdot \pi$, both the one line notation and the diagonal reduced expression $\boldsymbol{\mu}$ are shown. All elements in the fiber are alternating.

$B_5(\pi)\cdot\pi$	$\mu\cdot\pi$
$[1,5,2,3,4]$	$(s_4 s_3 s_2)$
$[\bar{1},5,2,3,4]$	$s_0\cdot(s_4 s_3 s_2)$
$[\bar{2},5,1,3,4]$	$s_1 s_0\cdot(s_4 s_3 s_2)$
$[\bar{3},5,1,2,4]$	$s_2 s_1 s_0\cdot(s_4 s_3 s_2)$
$[\bar{4},5,1,2,3]$	$s_3 s_2 s_1 s_0\cdot(s_4 s_3 s_2)$
$[\bar{2},5,\bar{1},3,4]$	$s_0 s_1 s_0\cdot(s_4 s_3 s_2)$
$[\bar{3},5,\bar{1},2,4]$	$s_0 s_2 s_1 s_0\cdot(s_4 s_3 s_2)$
$[\bar{4},5,\bar{1},2,3]$	$s_0 s_3 s_2 s_1 s_0\cdot(s_4 s_3 s_2)$
$[\bar{3},5,\bar{2},1,4]$	$s_1 s_0 s_2 s_1 s_0\cdot(s_4 s_3 s_2)$
$[\bar{4},5,\bar{2},1,3]$	$s_1 s_0 s_3 s_2 s_1 s_0\cdot(s_4 s_3 s_2)$
$[\bar{4},5,\bar{3},1,2]$	$s_2 s_1 s_0 s_3 s_2 s_1 s_0\cdot(s_4 s_3 s_2)$
$[\bar{3},5,\bar{2},\bar{1},2]$	$s_0 s_1 s_0 s_2 s_1 s_0\cdot(s_4 s_3 s_2)$
$[\bar{4},5,\bar{2},\bar{1},3]$	$s_0 s_1 s_0 s_3 s_2 s_1 s_0\cdot(s_4 s_3 s_2)$
$[\bar{4},5,\bar{3},\bar{1},2]$	$s_2 s_1 s_0 s_3 s_2 s_1 s_0\cdot(s_4 s_3 s_2)$
$[\bar{4},5,\bar{3},\bar{1},2]$	$s_2 s_1 s_0 s_3 s_2 s_1 s_0\cdot(s_4 s_3 s_2)$
$[\bar{4},5,\bar{3},\bar{1},\bar{2}]$	$s_2 s_1 s_0 s_3 s_2 s_1 s_0\cdot(s_4 s_3 s_2)$

Let $\pi = [2,4,5,1,3] = s_1 s_3 s_2 s_4 s_3$, therefore $\pi(1) = 2$ so $1\in\mathrm{Des}(\pi^{-1})$. In this case the fiber is made of two alternating elements (the first two) and two left-peaks.

$B_5(\pi)\cdot\pi$	$\mu\cdot\pi$
$[2,4,5,1,3]$	$(s_1 s_3 s_2 s_4 s_3)$
$[2,4,5,\bar{1},3]$	$s_0\cdot(s_1 s_3 s_2 s_4 s_3)$
$[1,4,5,\bar{2},3]$	$s_1 s_0\cdot(s_1 s_3 s_2 s_4 s_3)$
$[1,4,5,\bar{3},2]$	$s_2 s_1 s_0\cdot(s_1 s_3 s_2 s_4 s_3)$

5. Cellular structure

Recall the classical RSK bijection from permutations in S_n to pairs of standard Young tableaux of the same shape, see e.g. [29, §7.11]. This algorithm was extended to signed permutations in several ways, see, e.g., [32, 14]. Here we describe Barbash-Vogan's extension of the RSK algorithm which associates with each signed permutation $w\in B_n$ a pair of domino tableaux of the same shape [4, 14]. We follow the exposition of [36]. We start with the following definition.

Definition 5.1. Let $w = [w_1,\ldots,w_n]\in B_n$. The *palindromic representation* or the 0-*core representation* of w is $w^0 := [-w_n,\ldots,-w_1,w_1,\ldots,w_n]\in S_{[\pm n]}$.

The first step of the algorithm applies the usual RSK algorithm on w^0 with respect to the natural order $-n < \cdots < -1 < 1 < \cdots < n$ to get a pair of standard tableaux $P_0(w)$ and $Q_0(w)$. In the second step we apply jeu de taquin slides to vacate each negative number $-i$ (starting from $-n$) in each of the Young tableaux $P_0(w)$ and $Q_0(w)$ until $-i$ becomes adjacent to i, in which case it loses its sign and becomes i. The resulting domino tableaux will be respectively denoted $\mathsf{P}(w)$ and $\mathsf{Q}(w)$.

Here is an example which illustrates this algorithm.

Example 5.2. Let $w = [-3,1,2]$. Then $w^0 = [-2,-1,3,-3,1,2]\in S_6$. By applying the RSK algorithm we get

$$w^0\longmapsto (P_0(w),Q_0(w)) = \left(\begin{array}{|c|c|c|c|}\hline -3 & -1 & 1 & 2 \\\hline -2 & 3 \\\cline{1-2}\end{array}\quad,\quad \begin{array}{|c|c|c|c|}\hline -3 & -2 & -1 & 3 \\\hline 1 & 2 \\\cline{1-2}\end{array}\right).$$

Now, the following process sticks the negative numbers to their positive counterparts by using jeu de taquin slides, starting by vacating -3, in both tableaux.

$$
\begin{array}{|c|c|c|c|}\hline -3 & -1 & 1 & 2 \\\hline -2 & 3 \\\cline{1-2}\end{array} \rightarrow
\begin{array}{|c|c|c|c|}\hline -2 & -1 & 1 & 2 \\\hline 3 & 3 \\\cline{1-2}\end{array} \rightarrow
\begin{array}{|c|c|c|c|}\hline -1 & -2 & 1 & 2 \\\hline 3 & 3 \\\cline{1-2}\end{array} \rightarrow
\begin{array}{|c|c|c|c|}\hline -1 & 1 & 2 & 2 \\\hline 3 & 3 \\\cline{1-2}\end{array} \rightarrow
\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\\hline 3 & 3 \\\cline{1-2}\end{array} = \mathsf{P}(w)
$$

and

$$
\begin{array}{|c|c|c|c|}\hline -3 & -2 & -1 & 3 \\\hline 1 & 2 \\\cline{1-2}\end{array} \rightarrow
\begin{array}{|c|c|c|c|}\hline -2 & -3 & -1 & 3 \\\hline 1 & 2 \\\cline{1-2}\end{array} \rightarrow
\begin{array}{|c|c|c|c|}\hline -2 & -1 & 3 & 3 \\\hline 1 & 2 \\\cline{1-2}\end{array} \rightarrow
\begin{array}{|c|c|c|c|}\hline 1 & -2 & 3 & 3 \\\hline 1 & 2 \\\cline{1-2}\end{array} \rightarrow
\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 3 \\\hline 1 & 2 \\\cline{1-2}\end{array} = \mathsf{Q}(w).
$$

Proposition 5.3. [36, Propositions 2.7 and 2.9] *The above extension of the RSK algorithm is a bijection between B_n and pairs of domino tableaux such that for each $w \in B_n$, which satisfies*

 (1) $\mathsf{P}(w^{-1}) = \mathsf{Q}(w)$ *and* $\mathsf{Q}(w^{-1}) = \mathsf{P}(w)$,
 (2) $\mathrm{Des}_B(w) = \mathrm{Des}_B(\mathsf{Q}(w))$ *and* $\mathrm{Des}_B(w^{-1}) = \mathrm{Des}_B(\mathsf{P}(w))$.

Definition 5.4. The *two-sided combinatorial cell* of shape $\lambda \vdash 2n$ is the class
$$
\mathcal{C}_\lambda := \{w \in B_n \mid \mathrm{shape}(\mathsf{P}(w)) = \lambda\}.
$$

For an intensive discussion of these cells and their relations to the combinatorial description of the Kazhdan–Lusztig cells for type B_n with unequal parameters, see [10, 36, 9].

Recall that for $J \subseteq \{0, 1, \ldots, n\}$ we denote $\mathbf{x}^J := \prod_{i \in J} x_i$ and $\mathbf{y}^J := \prod_{i \in J} y_i$.

Theorem 5.5. *For every partition $\lambda \in P^0(n)$*
$$
\sum_{w \in \mathcal{C}_\lambda} \mathbf{x}^{\mathrm{Des}_B(w)} \mathbf{y}^{\mathrm{Des}_B(w^{-1})} \quad = \sum_{(\mathsf{P},\mathsf{Q}) \in \mathrm{SDT}(\lambda) \times \mathrm{SDT}(\lambda)} \mathbf{x}^{\mathrm{Des}_B(\mathsf{Q})} \mathbf{y}^{\mathrm{Des}_B(\mathsf{P})}. \tag{5.1}
$$

Proof. It follows from Proposition 5.3.   □

Green and Losonczy proved that the set $\mathrm{FC}(B_n)$ is a disjoint union of two-sided Kazhdan-Lusztig cells [17, Thm. 3.1.1]. We need a combinatorial analogue of this theorem.

Definition 5.6. The two following kinds of domino shapes will be called *admissible domino shapes*:

 • $\lambda = (2n - k, k)$ for $0 \le k \le n$,
 • $\lambda = ((2n - 2k, 2k - 1, 1)$ for $1 \le k \le \lfloor n/2 \rfloor$.

Theorem 5.7. *The set $\mathrm{FC}(B_n)$ is a disjoint union of two-sided combinatorial cells of admissible domino shapes $(2n - k, k)$, $0 \le k \le n$, and $(2n - 2k, 2k - 1, 1)$, $1 \le k \le \lfloor n/2 \rfloor$.*

Proof. For every element $w \in B_n$ the domino shape of w is the common shape of the corresponding pair of domino tableaux. This is, in turn, the shape of the SYT corresponding to its palindromic 0-core representation $w^0 \in S_{[\pm n]}$ under the RSK bijection. We have to show that $w \in \mathrm{FC}(B_n)$ if and only if its shape is admissible. Here, by the shape of w we mean the shape of its image under the above extension of the RSK algorithm.

First, we will show that the domino shape of any $w \in \mathrm{FC}(B_n)$ is admissible, namely the height of the domino shape is ≤ 3 and the length of the second column is ≤ 2. Since the height of the shape of $\pi \in S_{2n}$ is equal to the maximal length of decreasing subsequence in π (see [28]), it suffices to prove Claim 1 below. By [18], the total length of the first two columns is the maximal size of two disjoint decreasing subsequence in w^0. Since no decreasing subsequence of length 4 appears in w^0, it suffices to show that there are no two disjoint subsequences of length 3 in w^0. This will be proved in Claims 2 and 3.

Claim 1. For every $w \in \mathrm{FC}(B_n)$, there is no decreasing subsequence of length 4 in w^0.

Proof of Claim 1. Let (d, c, b, a) be a decreasing subsequence of length 4 in w^0. If the position of b is in the right half then if $b < 0$, the subsequence (b, a) violates Proposition 3.8, since the pattern $[-1, -2]$ is forbidden; if $b > 0$ then $0 > -c > -d$ and $(-c, -d)$ is a forbidden subsequence in the right half. If the position of b is in the left half then $(-b, -c, -d)$ is a decreasing subsequence in the right half, violating Proposition 3.8, since the pattern $[3, 2, 1]$ is forbidden. This proves that for every $w \in \mathrm{FC}(B_n)$ the height of the corresponding domino shape of w is ≤ 3. Proof of Claim 1 is completed.

Claim 2. For every $w \in \mathrm{FC}(B_n)$, any decreasing subsequence of length 3 in w^0 is contained in a subsequence whose pattern is $[2, -1, 1, -2]$ and having its last two letters in the right half of w^0.

Proof of Claim 2. Notice that for every decreasing subsequence of length 3 in w^0 (c, b, a), $c > b > a$, the sequence $(-a, -b, -c)$ is also a decreasing subsequence of length 3 in w^0. Without loss of generality, (c, b, a) contains a decreasing subsequence of length 2 in the right half, since by definition of w^0, if the first two letters in (c, b, a) are in the left half then $(-a, -b, -c)$ is a decreasing subsequence in w^0 with two letters in the right half. Since $[3, 2, 1]$ is forbidden in w, we may assume that the subsequence (c, b, a) contains one letter in the left half of w^0. Since $[-1, -2]$ is forbidden, the second letter b in the subsequence is positive, thus $0 < b < c$. If $c \neq -a$ then the right half contains a shuffle of $-c$ with (b, a), where $c > b > a$, violating Proposition 3.8. Thus $-a = c > b$ and both (c, b, a) and $(-a, -b, -c)$ are contained in $[-a, -b, b, a]$ with two letters in the right half. This completes the proof of Claim 2.

Claim 3. For every $w \in \mathrm{FC}(B_n)$, there are no two disjoint subsequences of length 3 in w^0.

Proof of Claim 3. By Claim 2, the existence of two disjoint decreasing subsequences of length 3 forces existence of two disjoint subsequences $(a, -b)$ and $(x, -y)$, with $0 < a < b$ and $0 < x < y$, in the right half. Otherwise, either $y = b$ or $x = a$. If $y = b$ both decreasing subsequences end with $-b = -y$, thus not disjoint. If $x = a$ then, without loss of generality, the right half contains the subsequence $(a, -b, -y)$ of patterns $[1, -2, -3]$ or $[1, -3, -2]$; by Proposition 3.8, both patterns are forbidden.

Consider all shuffles of $(a, -b)$ with $(x, -y)$ in the right half. Without loss of generality, the first letter in the shuffle is a, thus there exists a subsequence in the right half of w^0 of one of the following forms:

$$(a, -b, x, -y) \text{ or } (a, x, -b, -y) \text{ or } (a, x, -y, -b).$$

Since $[3, 2, 1]$ is forbidden, we are forced to assume that $a < x$. Also, $[-1, -2]$ is forbidden, forcing $b > y$ in the two first above forms and $y > b$ in the third one. Altogether we deduce that $a < x < y < b$ in the first two forms; and $a < b < y$, $a < x < y$ in the last one. Thus, the first two forms are of patterns $[1, -4, 2, -3]$ and $[1, 2, -4, -3]$ respectively, and the pattern of the last form is either $[1, 2, -4, -3]$ or $[1, 3, -4, -2]$. In all cases, the last three letters violate one of the forbidden patterns in Proposition 3.8. This completes the proof of Claim 3.

We have proved that every $w \in \mathrm{FC}(B_n)$ has an admissible shape. To finish the proof, notice that by Corollary 2.12, the sum of the squares of the number of domino tableaux of all admissible shapes is equal to

$$\sum_{k=0}^{n} (f_2^{(2n-k,k)})^2 + \sum_{k=1}^{\lfloor n/2 \rfloor} (f_2^{(2n-2k,2k-1,1)})^2 = \frac{n+2}{n+1}\binom{2n}{n} - 1, \tag{5.2}$$

which is equal by Remark 3.7 to the number of elements in $\mathrm{FC}(B_n)$. One concludes that the total size of combinatorial two-sided cells of admissible shapes in B_n is equal to the size of $\mathrm{FC}(B_n)$. Since all FC elements are of admissible shapes, this shows that there are no elements in $B_n \setminus \mathrm{FC}(B_n)$ whose shape is admissible, completing the proof.

$\square$

Remark 5.8. It was conjectured by Bonnafé, Geck, Inacu, and Lam [10, Conjecture A(c)] that Kazhdan-Lusztig two-sided cells of type B_n with unequal parameters are two-sided combinatorial cells, see also [9]. By Green-Losonczy Theorem [17, Thm. 3.1.1], Theorem 5.7 (with no explicit description of the shapes) follows from this conjecture.

We conclude now that the Barbash–Vogan bijection described above preserves the division of the set of FC elements into alternating and left-peak elements.

Corollary 5.9. *We have the following equivalences:*
- *$w \in \mathrm{FC}(B_n)$ is a left-peak if and only if $shape(\mathsf{P}(w)) = (2n - 2k, 2k - 1, 1)$, for some k.*
- *$w \in \mathrm{FC}(B_n)$ is alternating if and only if $shape(\mathsf{P}(w)) = (2n - k, k)$, for some k.*

Proof. Let w be a left-peak. By Remark 3.13, the one line notation of w is $w = [1, \ldots, -k, \ldots]$ for an integer $k > 1$. Thus $w^0 = [\ldots, k, \ldots, -1, 1, \ldots, -k, \ldots]$, containing a $[3, 2, 1]$-pattern, hence the height of the shape of $\mathsf{P}(w)$ is at least 3. By Theorem 5.7 and the cardinality arguments given in Remark 3.7, we get the first equivalence. The second assertion is then a consequence of the first one and Theorem 5.7. $\square$

Remark 5.10. Three different decompositions of $\mathrm{FC}(B_n)$ into disjoint subsets are considered in the present paper. The first one into Kazhdan-Lusztig cells is due to Green-Losonczy [17]; the second one into Barbash-Vogan combinatorial cells is given in Theorem 5.7; the third one into fibers is shown in Theorem 4.2. Comparing Corollary 5.9 with Corollary 4.3, one deduces that fibers are, in general, different from combinatorial cells. A remaining open problem is whether Kazhdan-Lusztig cells and combinatorial cells, restricted to $\mathrm{FC}(B_n)$, coincide or not. A positive answer to this question would solve a special case of [10, Conjecture A(c)] mentioned in Remark 5.8.

6. Equidistribution

In this section we prove Theorem 1.3. In order to do this, we introduce an involution on $\mathrm{FC}(B_n)$, relying on the decomposition into fibers from Section 4 and on the properties of an involution due to Rubey. Throughout this section, for a signed permutation w, we set $\mathrm{Des}(w)$ as in (2.4).

In [27], Rubey defines an involution $f : S_n(321) \longrightarrow S_n(321)$ satisfying the following properties.

Proposition 6.1. *For each $\pi \in \mathrm{FC}(S_n)$, we have*
  i) $\mathrm{Des}(\pi) = \mathrm{Des}(f(\pi))$;
  ii) $\mathrm{bl}(f(\pi)^{-1}) = n - \mathrm{ldes}(\pi^{-1})$, *equivalently* $\mathrm{bl}(\pi^{-1}) = n - \mathrm{ldes}(f(\pi)^{-1})$.

We extend the involution f to a mapping $\Phi : \mathrm{FC}(B_n) \longrightarrow B_n$ by defining

$$\Phi(w) := f(\pi^{-1}) \cdot \mu^{-1}, \tag{6.1}$$

where $w = \pi^{-1} \cdot \mu^{-1}$, or equivalently $w^{-1} = \mu \cdot \pi$, is the decomposition in (4.2). Note that, as recalled in Section 3.2, $\pi^{-1} \in S_n(321)$, so the map Φ is well defined.

Observation 6.2. Let $J = S \setminus \{s_0\}$ and recall the definition of the quotient $(B_n)^J$ from (4.1). For every $\mu \in (B_n)^J$, $\mathrm{Des}(\mu) = \emptyset$, so left multiplication by μ is order preserving. Hence, for every π in the parabolic subgroup $(B_n)_J \cong S_n$

$$\mathrm{Des}(\mu \cdot \pi) = \mathrm{Des}(\pi),$$

thus $\mathrm{ldes}(\mu \cdot \pi) = \mathrm{ldes}(\pi)$.

Lemma 6.3. *For every $\pi \in \mathrm{FC}(S_n)$, $B_n(\pi) = B_n(f(\pi^{-1})^{-1})$.*

Proof. By definition, $\mathrm{Des}^L(\pi) = \mathrm{Des}(\pi^{-1})$, thus by Proposition 6.1, $\mathrm{Des}^L(\pi) = \mathrm{Des}^L(f(\pi^{-1})^{-1})$. Since, by Theorem 4.2, $B_n(\pi)$ depends only on the left descent set of π, the statement holds. $\qquad\square$

Lemma 6.4. *For every $\pi \in \mathrm{FC}(S_n)$ and $\mu \in (B_n)^J$,*

$$\mathrm{Des}_B(\pi^{-1} \cdot \mu^{-1}) = \mathrm{Des}_B(f(\pi^{-1}) \cdot \mu^{-1}),$$

equivalently, $\mathrm{Des}_B^L(\mu \cdot \pi) = \mathrm{Des}_B^L(\mu \cdot f(\pi^{-1})^{-1})$.

Proof. Let $\mu \in (B_n)^J$. Observe that for each $\pi \in S_n$, if $\mu^{-1}(i) < 0$ and $\mu^{-1}(i+1) > 0$ ($\mu^{-1}(i) > 0$ and $\mu^{-1}(i+1) < 0$) then $i \notin \mathrm{Des}(\pi \cdot \mu^{-1})$ ($i \notin \mathrm{Des}(\pi \cdot \mu^{-1})$) independently of $\pi \in S_n$. On the other hand, if $\mu^{-1}(i)$ and $\mu^{-1}(i+1)$ have the same sign then by Observation 4.1, $\mu^{-1}(i), \mu^{-1}(i+1)$ have consecutive values. Then for each $\pi \in S_n$, $i \in \mathrm{Des}(\pi \cdot \mu^{-1})$ if and only if $0 < \mu^{-1}(i) \in \mathrm{Des}(\pi)$ or $0 < -\mu^{-1}(i+1) \in \mathrm{Des}(\pi)$. It follows that in the last case, $i \in \mathrm{Des}(\pi^{-1} \cdot \mu^{-1})$ if and only $\mu^{-1}(i) \in \mathrm{Des}(\pi^{-1})$. By Proposition 6.1, this is true if and only if $\mu^{-1}(i) \in \mathrm{Des}(f(\pi^{-1}))$. We deduce:

$$\mathrm{Des}(\pi^{-1} \cdot \mu^{-1}) = \mathrm{Des}(f(\pi^{-1}) \cdot \mu^{-1}).$$

To conclude, notice that

$$0 \in \mathrm{Des}_B(\pi^{-1} \cdot \mu^{-1}) \Longleftrightarrow \mu^{-1}(1) < 0 \Longleftrightarrow 0 \in \mathrm{Des}_B(f(\pi^{-1}) \cdot \mu^{-1}).$$

$\qquad\square$

Proposition 6.5. *For every $w \in \mathrm{FC}(B_n)$*
  (i) $\Phi(w) \in \mathrm{FC}(B_n)$;
  (ii) $\mathrm{Des}_B(w) = \mathrm{Des}_B(\Phi(w))$;
  (iii) $\mathrm{bl}(w^{-1}) = n - \mathrm{ldes}(\Phi(w)^{-1})$;
  (iv) $\mathrm{Neg}(w) = \mathrm{Neg}(\Phi(w))$.

Proof.

(i) Following (4.7), we write uniquely $w^{-1} = \mu \cdot \pi$ with $\mu \in B_n(\pi)$. By Lemma 6.3,

$$\mu \in B_n(\pi) = B_n(f(\pi^{-1})^{-1}).$$

By definition,

$$\Phi(w)^{-1} = \mu \cdot f(\pi^{-1})^{-1} \in B_n(\pi) \cdot f(\pi^{-1})^{-1} = B_n(f(\pi^{-1})^{-1}) \cdot f(\pi^{-1})^{-1} \subseteq \mathrm{FC}(B_n).$$

The last containment follows from Theorem 4.2. Hence, as mentionned in Section 3.2, $\Phi(w) \in \mathrm{FC}(B_n)$.

(ii) One can write

$$\mathrm{Des}_B(w) = \mathrm{Des}_B^L(w^{-1}) = \mathrm{Des}_B^L(\mu \cdot \pi) = \mathrm{Des}_B^L(\mu \cdot f(\pi^{-1})^{-1}) = \mathrm{Des}_B((f(\pi^{-1}) \cdot \mu^{-1}) = \mathrm{Des}_B(\Phi(w)),$$

where the third equality follows from Lemma 6.4 .

(iii) The following equalities are derived from Proposition 6.1 and Observation 6.2:

$$\begin{aligned}
\mathrm{bl}(w^{-1}) = \mathrm{bl}(\mu \cdot \pi) = \mathrm{bl}(\pi) &= n - \mathrm{ldes}(f(\pi^{-1})^{-1}) \\
&= n - \mathrm{ldes}(\mu \cdot f(\pi^{-1})^{-1}) \\
&= n - \mathrm{ldes}(\Phi(w)^{-1}).
\end{aligned}$$

(iv) Multiplying a signed permutation on the left by a permutation in S_n does not change the positions of the negative entries. Hence the result follows from the definition of Φ.

$\square$

Remark 6.6. As f is an involution, by Theorem 6.5(i), the map Φ is an involution on $\mathrm{FC}(B_n)$.

Now we are ready to prove our equidistribution result given in Theorem 1.3.

Proof of Theorem 1.3. By Remark 6.6 together with Proposition 6.5(ii)–(iv), Φ is an involution on $\mathrm{FC}(B_n)$ which maps the left hand side to the right hand side. $\square$

7. Proof of the main theorem

Applying the vector space homomorphism from the ring of quasi-symmetric functions to the multilinear subspace of the formal power series ring $\mathbb{Z}[x_1, x_2, \ldots]$, defined by $F_{n,J} \mapsto \mathbf{x}^J$, and using the fact that for every $\pi \in S_n$, $\mathrm{bl}(\pi^{-1}) = \mathrm{bl}(\pi)$, Theorem 1.1 is equivalent to the following equation

$$\sum_{\pi \in \mathrm{FC}(S_n)} \mathbf{x}^{\mathrm{Des}(\pi)} q^{\mathrm{bl}(\pi^{-1})} = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{(P,Q) \in \mathrm{SYT}^2(n-k,k)} \mathbf{x}^{\mathrm{Des}(Q)} q^{n-\mathrm{ldes}(P)}, \tag{7.1}$$

where the right sum is over pairs of tableaux of shape $(n-k, k)$.

In this section we use Theorem 1.3 to prove the type B analogue, namely Theorem 1.2. The first step is to present the corresponding equidistribution for type B in the language of domino tableaux.

Theorem 7.1. *For any positive integer n we have*

$$\sum_{w \in \mathrm{FC}(B_n)} \mathbf{x}^{\mathrm{Des}_B(w)} q^{\mathrm{bl}(w^{-1})} = \sum_{k=0}^{n} \sum_{(\mathsf{P},\mathsf{Q}) \in \mathrm{SDT}^2(2n-k,k)} \mathbf{x}^{\mathrm{Des}_B(\mathsf{Q})} q^{n-\mathrm{ldes}(\mathsf{P})}.$$

$$+ \sum_{k=1}^{\lfloor n/2 \rfloor} \sum_{(\mathsf{P},\mathsf{Q}) \in \mathrm{SDT}^2(2n-2k,2k-1,1)} \mathbf{x}^{\mathrm{Des}_B(\mathsf{Q})} q^{n-\mathrm{ldes}(\mathsf{P})}. \tag{7.2}$$

Proof. Recall from Theorem 5.7 that the set $\mathrm{FC}(B_n)$ is a union of combinatorial cells corresponding to the domino tableaux of the shapes $(2n-k, k)$ and $(2n-2k, 2k-1, 1)$.

By Theorem 5.7 together with Equation (5.1),

$$
\sum_{w\in\mathrm{FC}(B_n)} \mathbf{x}^{\mathrm{Des}_B(w)}\mathbf{y}^{\mathrm{Des}_B(w^{-1})} = \sum_{k=0}^{n} \sum_{(\mathsf{P},\mathsf{Q})\in\mathrm{SDT}^2(2n-k,k)} \mathbf{x}^{\mathrm{Des}_B(\mathsf{Q})}\mathbf{y}^{\mathrm{Des}_B(\mathsf{P})}
$$
$$
+ \sum_{k=1}^{\lfloor n/2\rfloor} \sum_{(\mathsf{P},\mathsf{Q})\in\mathrm{SDT}^2(2n-2k,2k-1,1)} \mathbf{x}^{\mathrm{Des}_B(\mathsf{Q})}\mathbf{y}^{\mathrm{Des}_B(\mathsf{P})}.
$$

Applying the map $\mathbf{y}^J \mapsto q^{n-j_t}$, where $J = \{j_1 < j_2 < \cdots < j_t\} \subseteq [0, n-1]$, together with Theorem 1.3 completes the proof. $\qquad\square$

Next, we deduce the following consequence, which is the translation of Theorem 7.1 to the language of bi-tableaux. In order to give a more elegant version of this result, we consider here the equidistribution over $\mathrm{FC}(B_n) \setminus \mathrm{FC}(S_n)$ rather than over $\mathrm{FC}(B_n)$.

Corollary 7.2. *For any positive integer n we have*

$$
\sum_{w\in\mathrm{FC}(B_n)\setminus\mathrm{FC}(S_n)} \mathbf{x}^{\mathrm{Des}_B(w)}q^{\mathrm{bl}(w^{-1})} = \sum_{k=1}^{\lfloor n/2\rfloor} \sum_{(P,Q)\in\mathrm{BSYT}^2((k),(n-k))} \mathbf{x}^{\mathrm{Des}_B(Q)}q^{n-\mathrm{ldes}(P)}
$$
$$
+ \sum_{k=0}^{\lfloor (n-1)/2\rfloor} \sum_{(P,Q)\in\mathrm{BSYT}^2((n-k),(k))} \mathbf{x}^{\mathrm{Des}_B(Q)}q^{n-\mathrm{ldes}(P)}. \quad (7.3)
$$

Proof. By Remark 2.17,

$$
\sum_{k=0}^{n} \sum_{(\mathsf{P},\mathsf{Q})\in\mathrm{SDT}^2(2n-k,k)} \mathbf{x}^{\mathrm{Des}_B(\mathsf{Q})}q^{n-\mathrm{ldes}(\mathsf{P})} = \sum_{k=0}^{\lfloor n/2\rfloor} \sum_{(\mathsf{P},\mathsf{Q})\in\mathrm{SDT}^2(2n-2k,2k)} \mathbf{x}^{\mathrm{Des}_B(\mathsf{Q})}q^{n-\mathrm{ldes}(\mathsf{P})}
$$
$$
+ \sum_{k=0}^{\lfloor (n-1)/2\rfloor} \sum_{(\mathsf{P},\mathsf{Q})\in\mathrm{SDT}^2(2n-2k-1,2k+1)} \mathbf{x}^{\mathrm{Des}_B(\mathsf{Q})}q^{n-\mathrm{ldes}(\mathsf{P})}
$$
$$
= \sum_{k=0}^{\lfloor n/2\rfloor} \sum_{(P,Q)\in\mathrm{BSYT}^2((k),(n-k))} \mathbf{x}^{\mathrm{Des}_B(Q)}q^{n-\mathrm{ldes}(P)}
$$
$$
+ \sum_{k=0}^{\lfloor (n-1)/2\rfloor} \sum_{(P,Q)\in\mathrm{BSYT}^2((n-k),(k))} \mathbf{x}^{\mathrm{Des}_B(Q)}q^{n-\mathrm{ldes}(P)}
$$

and

$$
\sum_{k=1}^{\lfloor n/2\rfloor} \sum_{(P,Q)\in\mathrm{SDT}^2(2n-2k,2k-1,1)} \mathbf{x}^{\mathrm{Des}_B(Q)}q^{n-\mathrm{ldes}(P)} = \sum_{k=1}^{\lfloor n/2\rfloor} \sum_{(P,Q)\in\mathrm{BSYT}^2(\emptyset,(n-k,k))} \mathbf{x}^{\mathrm{Des}_B(Q)}q^{n-\mathrm{ldes}(P)}
$$
$$
= \sum_{k=1}^{\lfloor n/2\rfloor} \sum_{(P,Q)\in\mathrm{SYT}^2(n-k,k)} \mathbf{x}^{\mathrm{Des}_B(Q)}q^{n-\mathrm{ldes}(P)},
$$

where the last equality is due to the obvious descent-preserving bijection between $\mathrm{BSYT}(\emptyset, (n-k, k))$ and $\mathrm{SYT}(n-k, k)$. Now, by Theorem 7.1 and Equation (7.1), we obtain

$$
\begin{aligned}
\sum_{w\in\mathrm{FC}(B_n)\backslash\mathrm{FC}(S_n)} \mathbf{x}^{\mathrm{Des}_B(w)}q^{\mathrm{bl}(w^{-1})} &= \sum_{w\in\mathrm{FC}(B_n)} \mathbf{x}^{\mathrm{Des}_B(w)}q^{\mathrm{bl}(w^{-1})} - \sum_{w\in\mathrm{FC}(S_n)} \mathbf{x}^{\mathrm{Des}_B(w)}q^{\mathrm{bl}(w^{-1})} \\
&= \sum_{k=0}^{\lfloor n/2\rfloor}\; \sum_{(P,Q)\in\mathrm{BSYT}^2((k),(n-k))} \mathbf{x}^{\mathrm{Des}_B(Q)}q^{n-\mathrm{ldes}(P)} \\
&\quad + \sum_{k=0}^{\lfloor (n-1)/2\rfloor}\; \sum_{(P,Q)\in\mathrm{BSYT}^2((n-k),(k))} \mathbf{x}^{\mathrm{Des}_B(Q)}q^{n-\mathrm{ldes}(P)} \\
&\quad + \sum_{k=1}^{\lfloor n/2\rfloor}\; \sum_{(P,Q)\in\mathrm{SYT}^2(n-k,k)} \mathbf{x}^{\mathrm{Des}_B(Q)}q^{n-\mathrm{ldes}(P)} \\
&\quad - \sum_{k=0}^{\lfloor n/2\rfloor}\; \sum_{(P,Q)\in\mathrm{SYT}^2(n-k,k)} \mathbf{x}^{\mathrm{Des}_B(Q)}q^{n-\mathrm{ldes}(P)},
\end{aligned}
$$

which is equal to the RHS of Equation (7.3). $\qquad\square$

Proof of Theorem 1.2. By Corollary 7.2 we have

$$
\begin{aligned}
\sum_{w\in\mathrm{FC}(B_n)\backslash\mathrm{FC}(S_n)} \mathbf{x}^{\mathrm{Des}_B(w)}q^{\mathrm{bl}(w^{-1})} &= \sum_{k=1}^{\lfloor n/2\rfloor}\left(\sum_{P\in\mathrm{BSYT}((k),(n-k))} q^{n-\mathrm{ldes}(P)}\right)\left(\sum_{Q\in\mathrm{BSYT}((k),(n-k))} \mathbf{x}^{\mathrm{Des}_B(Q)}\right) \\
&\quad + \sum_{k=0}^{\lfloor (n-1)/2\rfloor}\left(\sum_{P\in\mathrm{BSYT}((n-k),(k))} q^{n-\mathrm{ldes}(P)}\right)\left(\sum_{Q\in\mathrm{BSYT}((n-k),(k))} \mathbf{x}^{\mathrm{Des}_B(Q)}\right).
\end{aligned}
$$

Remark 2.17 then transforms both sums over Q on the right-hand side of the above identity to sums over $\mathsf{Q} \in \mathrm{SDT}(2n-2k, 2k)$ and $\mathsf{Q} \in \mathrm{SDT}(2n-2k-1, 2k+1)$, respectively. Applying the vector space homomorphism from the multi-linear subspace of the formal power series ring $\mathbb{Z}[x_0, x_1, x_2, \ldots]$ to the ring of Chow's type B quasi-symmetric functions, defined by $\mathbf{x}^J \mapsto F^B_{n,J}$, to both sides of the resulting formula, we can then use Proposition 2.15 to transform the sums over Q and get:

$$
\begin{aligned}
\sum_{w\in\mathrm{FC}(B_n)\backslash\mathrm{FC}(S_n)} q^{\mathrm{bl}(w^{-1})}F^B_{\mathrm{Des}_B(w)} &= \sum_{k=1}^{\lfloor n/2\rfloor}\left(\sum_{P\in\mathrm{BSYT}((k),(n-k))} q^{n-\mathrm{ldes}(P)}\right) s_{(k)}(x_1, x_2, \ldots)\, s_{(n-k)}(x_0, x_1, \ldots) \\
&\quad + \sum_{k=0}^{\lfloor (n-1)/2\rfloor}\left(\sum_{P\in\mathrm{BSYT}((n-k),(k))} q^{n-\mathrm{ldes}(P)}\right) s_{(n-k)}(x_1, x_2, \ldots)\, s_{(k)}(x_0, x_1, \ldots).
\end{aligned}
$$

The conclusion follows by replacing k by $n-k$ in the second sum and noting that every $P \in \mathrm{BSYT}((k), (n-k))$ may be identified with a $T \in \mathrm{SYT}((n,k)/(k))$ with same ldes. $\qquad\square$

8. Two notions of type B Schur-positivity

A subset $A \subseteq S_n$ is *Schur-positive* if the quasisymmetric function $\mathcal{Q}(A) := \sum_{w\in A} F_{\mathrm{Des}(w)}$ is symmetric and Schur-positive. Here $\{F_J \mid J \subseteq [n-1]\}$ are Gessel's fundamental quasi-symmetric functions and $\mathrm{Des}(w)$ is the standard descent set from (2.4). Determining whether a given symmetric function is Schur-positive is a major problem in contemporary algebraic combinatorics [31].

As mentioned in the introduction, the concept of quasi-symmetric functions has been extended to Coxeter groups of type B in two different ways. The two associated notions of type B Schur-positivity follow.

Recall Chow's type B fundamental quasi-symmetric functions $\{F^B_J \mid J \subseteq \{0\}\cup[n-1]\}$ from Definition 2.5 and the type B right descent set $\mathrm{Des}_B(\pi)$ from (2.3). Definition 2.8 of domino functions leads to the following type B Schur-positivity notion, introduced in [24].

Definition 8.1. A subset $A \subseteq B_n$ is *Chow type B Schur-positive* if the Chow quasi-symmetric function

$$\mathcal{Q}^C(A) := \sum_{w \in A} F^B_{\mathrm{Des}_B(w)}$$

can be written as a non-negative sum of domino functions.

Proposition 8.2. *For every $n \geq j \geq 1$, the set $\{w \in \mathrm{FC}(B_n)| \mathrm{bl}(w^{-1}) = j\}$ is Chow type B Schur-positive.*

We will first prove the following lemma. Consider the natural embedding of S_n in B_n.

Lemma 8.3. *Let $A \subseteq S_n \subseteq B_n$ and $\mathcal{Q}(A) = \sum_{w \in A} F_{\mathrm{Des}(w)}(x_1, x_2, \dots)$. If $\mathcal{Q}(A)$ is symmetric in $x_1, x_2, \dots$ then $\mathcal{Q}^C(A)$ is symmetric in $x_0, x_1, \dots$, and for every $\lambda \vdash n$*

$$\langle \mathcal{Q}^C(A), s_\lambda(x_0, x_1, \dots)\rangle = \langle \mathcal{Q}(A), s_\lambda(x_1, x_2, \dots)\rangle,$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product on symmetric functions.

Proof. For all $\lambda \vdash n$, consider

$$c_\lambda := \langle \mathcal{Q}(A), s_\lambda(x_1, x_2, \dots)\rangle.$$

By assumption, $\mathcal{Q}(A)$ is symmetric in $x_1, x_2, \dots$. Thus, by Theorem 2.4,

$$\sum_{J \subseteq [n-1]} a_{A,J} F_J = \mathcal{Q}(A) = \sum_{\lambda \vdash n} c_\lambda \, s_\lambda(x_1, x_2, \dots) = \sum_{\lambda \vdash n} c_\lambda \sum_{T \in \mathrm{SYT}(\lambda)} F_{\mathrm{Des}(T)} = \sum_{\lambda \vdash n} c_\lambda \sum_{J \subseteq [n-1]} b_{\lambda,J} F_J,$$

where $a_{A,J} := \#\{w \in A \mid \mathrm{Des}(w) = J\}$ and $b_{\lambda,J} := \#\{T \in \mathrm{SYT}(\lambda) \mid \mathrm{Des}(T) = J\}$. It follows that

$$a_{A,J} = \sum_{\lambda \vdash n} c_\lambda b_{\lambda,J}.$$

Now notice that for every $w \in S_n$, $0 \notin \mathrm{Des}_B(\pi)$, thus $\mathrm{Des}_B(w) = \mathrm{Des}(w)$ and $\#\{w \in A \mid \mathrm{Des}_B(w) = J\} = a_{A,J}$. It follows that

$$\begin{aligned}
\mathcal{Q}^C(A) &= \sum_{w \in A} F^B_{\mathrm{Des}_B(w)} = \sum_{J \subseteq [n-1]} a_{A,J} F^B_J = \sum_{J \subseteq [n-1]} \sum_{\lambda \vdash n} c_\lambda b_{\lambda,J} F^B_J \\
&= \sum_{\lambda \vdash n} c_\lambda \sum_{J \subseteq [n-1]} b_{\lambda,J} F^B_J = \sum_{\lambda \vdash n} c_\lambda \, s_\lambda(x_0, x_1, \dots).
\end{aligned}$$

The last equality follows from Proposition 2.15, by noticing that $\mathrm{SYT}(\lambda)$ can be identified with $\mathrm{BSYT}(\emptyset, \lambda)$, and then with $\mathrm{SDT}(\mu)$, where $(\emptyset, \lambda) = \psi(\mu)$ and μ is an empty 2-core, see Section 2.4. $\qquad\square$

Proof of Proposition 8.2. First notice that for every $w \in S_n$, $\mathrm{bl}(w) = \mathrm{bl}(w^{-1})$, while for $w \in B_n \setminus S_n$ this is not necessarily the case. Combining this with Theorem 1.2, Theorem 1.1 and Lemma 8.3, we obtain

$$\begin{aligned}
\sum_{w \in \mathrm{FC}(B_n)} q^{\mathrm{bl}(w^{-1})} F^B_{\mathrm{Des}_B(w)} &= \sum_{w \in \mathrm{FC}(B_n) \setminus \mathrm{FC}(S_n)} q^{\mathrm{bl}(w^{-1})} F^B_{\mathrm{Des}_B(w)} + \sum_{w \in \mathrm{FC}(S_n)} q^{\mathrm{bl}(w^{-1})} F^B_{\mathrm{Des}_B(w)} \\
&= \sum_{k=1}^{n} \left(\sum_{j=0}^{n} b_{n,k,j} q^j \right) s_{(k)}(x_1, x_2, \dots) \, s_{(n-k)}(x_0, x_1, \dots) \\
&\quad + \sum_{k=0}^{\lfloor n/2 \rfloor} \left(\sum_{j=0}^{n} a_{n,k,j} \, q^j \right) s_{(n-k,k)}(x_0, x_1, \dots),
\end{aligned}$$

with non-negative integer coefficients $b_{n,k}$ and $a_{n,k}$. Equivalently,

$$\begin{aligned}
\mathcal{Q}^C(\{w \in \mathrm{FC}(B_n)| \mathrm{bl}(w^{-1}) = j\}) = \sum_{k=1}^{n} b_{n,k,j} \, s_{(k)}(x_1, x_2, \dots) \, s_{(n-k)}(x_0, x_1, \dots) \\
+ \sum_{k=0}^{\lfloor n/2 \rfloor} a_{n,k,j} \, s_{(n-k,k)}(x_0, x_1, \dots).
\end{aligned}$$

By Propositions 2.10 and 2.15, for every $\lambda \in P^0(n)$, $\mathcal{G}_\lambda = s_{\lambda^-}(x_1, x_2, \dots) \, s_{\lambda^+}(x_0, x_1, \dots)$, thus the right hand side is a non-negative sum of domino functions, completing the proof. $\qquad\square$

Another definition of type B Schur-positivity was suggested in [3], using Poirier's type B quasi-symmetric functions, which were introduced in [26]. The following definition reformulates [26, 3]. Let $\mathrm{X} := (x_1, x_2, \ldots)$ and $\mathrm{Y} := (y_1, y_2, \ldots)$ be two infinite sets of formal variables.

Definition 8.4. We define the following.

1. Let $<_r$ be the order on $[\pm n]$

$$-1 <_r -2 <_r \cdots <_r -n <_r 1 <_r 2 <_r \cdots <_r n.$$

The *r-descent set* of $w \in B_n$ is

$$\mathrm{rDes}(w) := \{1 \le i < n \mid w_i >_r w_{i+1}\}.$$

2. The *Poirier type B quasi-symmetric function*, associated with $w \in B_n$ is

$$F_w^P(\mathrm{X}, \mathrm{Y}) := \sum_{\substack{1 \le i_1 \le i_2 \le \ldots \le i_n \\ j \in \mathrm{rDes}(\sigma) \Rightarrow i_j < i_{j+1}}} z_{i_1} z_{i_2} \cdots z_{i_n}$$

where $z_{i_j} = x_{i_j}$ if $j \notin \mathrm{Neg}(w)$, and $z_{i_j} = y_{i_j}$ if $j \in \mathrm{Neg}(w)$.

3. For a subset $A \subseteq B_n$ let

$$\mathcal{Q}^P(A) := \sum_{w \in A} F_w^P(\mathrm{X}, \mathrm{Y}).$$

A subset $A \subseteq B_n$ is *Poirier type B Schur-positive* if $\mathcal{Q}^P(A)$ is symmetric in X, Y and can be expanded as a non-negative sum in the basis $s_\lambda(\mathrm{X})s_\mu(\mathrm{Y})$.

Example 8.5. Let $w = [-3, -1, 2] \in B_3$. Then $\mathrm{rDes}(w) = \{1\}$ and thus $F_w^P(X, Y) = \sum_{i_1 < i_2 \le i_3} y_{i_1} y_{i_2} x_{i_3}$.

Remark 8.6. The *signed descent set* of a signed permutation $w \in B_n$ is the pair $(\mathrm{rDes}(w), \mathrm{Neg}(w))$. The *signed descent set* of a bi-tableau $T = (T^-, T^+)$ of bi-shape (λ^-, λ^+) is the pair $(\mathrm{Des}(T), \mathrm{Neg}(T))$ where $\mathrm{Des}(T)$ is the descent set of T defined in (2.6), and $\mathrm{Neg}(T)$ is the set of entries in T^-. By [3, Cor. 3.7], a subset $A \subseteq B_n$ is Poirier type B Schur-positive if and only if the distribution of the signed descent set over A is equal to its distribution over bi-tableaux of some multiset of bi-shapes. Furthermore, in this case,

$$\mathcal{Q}^P(A) = \sum_{\lambda \in P^0(n)} c_\lambda \, s_{\lambda^-}(\mathrm{X}) s_{\lambda^+}(\mathrm{Y})$$

if and only if

$$\sum_{w \in A} \mathbf{x}^{\mathrm{rDes}(w)} \mathbf{y}^{\mathrm{Neg}(w)} = \sum_{\lambda \in P^0(n)} c_\lambda \sum_{T \in \mathrm{BSYT}(\lambda^-, \lambda^+)} \mathbf{x}^{\mathrm{Des}(T)} \mathbf{y}^{\mathrm{Neg}(T)}.$$

Here we use the notation from Section 2.3, $P^0(n)$ for the set of partitions of $2n$ with empty 2-core, and (λ^-, λ^+) for the 2-quotient of a partition $\lambda \in P^0(n)$.

Remark 8.7. Examples of Poirier type B Schur-positive sets include conjugacy classes [26, Theorem 16] and inverse signed descent classes $\{w \in B_n \mid \mathrm{Des}(w^{-1}) = I, \mathrm{Neg}(w^{-1}) = J\}$ [3, Proposition 5.5.1]. For more examples see [3].

Proposition 8.8. *For every $n > 2$, $\mathrm{FC}(B_n)$ is not Poirier type B Schur-positive.*

Proof. Observe that for any $\lambda \vdash n-1$ the number of standard bi-tableaux of bi-shape $((1), \lambda)$ with $\mathrm{Neg}(T) = \{i\}$ is independent of i. Combining this with Remark 8.6, we deduce that for every Poirier type B Schur-positive set $A \subseteq B_n$, the cardinality of the set $\{w \in A \mid \mathrm{Neg}(w) = \{i\}\}$ is independent of i. For $n \ge 3$, the set $\mathrm{FC}(B_n)$ violates this condition as follows. By Proposition 3.8, $\#\{w \in \mathrm{FC}(B_n) \mid \mathrm{Neg}(w) = \{n\}\} = n$, since $w = [w_1, \ldots, w_n] \in \mathrm{FC}(B_n)$ avoids a decreasing subsequence of order 3, thus for every j, the only signed permutation in $\mathrm{FC}(B_n)$ with $w_n = -j$ is $[1, 2, \ldots, j-1, j+1, \ldots, n, -j]$. On the other hand, $\#\{w \in \mathrm{FC}(B_n) \mid \mathrm{Neg}(w) = \{1\}\} \ge 2n - 2$, since for every $1 \le j < n-1$ there are at least two signed permutations in $\mathrm{FC}(B_n)$ with $w_1 = -j$, $[-j, 1, 2, \ldots, j-1, j+1, \ldots, n-2, n, n-1]$ and $[-j, 1, 2, \ldots, j-1, j+1, \ldots, n]$, the latter is in $\mathrm{FC}(B_n)$ for $j = n-1, n$ as well. $\square$

Theorem 8.9. *We have the following.*

1. *Every Poirier type B Schur-positive set A is a Chow type B Schur-positive set.*

2. *In this case, if*

$$\mathcal{Q}^P(A) = \sum_{\lambda \in P^0(n)} c_\lambda \, s_{\lambda^-}(\mathrm{X})s_{\lambda^+}(\mathrm{Y})$$

then

$$\mathcal{Q}^C(A) = \sum_{\lambda \in P^0(n)} c_\lambda \, s_{(\lambda^-)'}(x_1, x_2, \dots)s_{\lambda^+}(x_0, x_1, \dots),$$

where (λ^-, λ^+) is the 2-quotient of λ, and $(\lambda^-)'$ is the conjugate partition of λ^-.

Proof. Let $A \subseteq B_n$ be a type B Poirier Schur-positive set. By definition,

$$\mathcal{Q}^P(A) = \sum_{\lambda \in P^0(n)} c_\lambda \, s_{\lambda^-}(\mathrm{X})s_{\lambda^+}(\mathrm{Y})$$

with non-negative integer coefficients $c_\lambda \geq 0$. By Remark 8.6,

$$\sum_{w \in A} \mathbf{x}^{\mathrm{rDes}(w)}\mathbf{y}^{\mathrm{Neg}(w)} = \sum_{\lambda \in P^0(n)} c_\lambda \sum_{T \in \mathrm{BSYT}(\lambda^-, \lambda^+)} \mathbf{x}^{\mathrm{Des}(T)}\mathbf{y}^{\mathrm{Neg}(T)}. \tag{8.1}$$

Let $T' = ((T^-)', T^+) \in \mathrm{BSYT}((\lambda^-)', \lambda^+)$ be the standard Young bi-tableau obtained by transposing T^-. Note that for every $w \in B_n$

$$\mathrm{Des}(T) = \mathrm{rDes}(w) \text{ and } \mathrm{Neg}(T) = \mathrm{Neg}(w) \iff \mathrm{Des}(T') = \mathrm{Des}(w) \text{ and } \mathrm{Neg}(T') = \mathrm{Neg}(w).$$

We conclude that Equation (8.1) is equivalent to the following.

$$\sum_{w \in A} \mathbf{x}^{\mathrm{Des}(w)}\mathbf{y}^{\mathrm{Neg}(w)} = \sum_{\lambda \in P^0(n)} c_\lambda \sum_{T \in \mathrm{BSYT}(\lambda^-, \lambda^+)} \mathbf{x}^{\mathrm{Des}(T')}\mathbf{y}^{\mathrm{Neg}(T')}$$

$$= \sum_{\lambda \in P^0(n)} c_\lambda \sum_{T \in \mathrm{BSYT}((\lambda^-)', \lambda^+)} \mathbf{x}^{\mathrm{Des}(T)}\mathbf{y}^{\mathrm{Neg}(T)}.$$

Setting $y_1 = x_0$ and $y_2 = \cdots = y_n = 1$ we obtain

$$\sum_{w \in A} \mathbf{x}^{\mathrm{Des}_B(w)} = \sum_{\lambda \in P^0(n)} c_\lambda \sum_{T \in \mathrm{BSYT}((\lambda^-)', \lambda^+)} \mathbf{x}^{\mathrm{Des}_B(T)},$$

By Lemma 2.16, for every $\lambda \in P^0(n)$ the distribution of Des_B over $\mathrm{SDT}(\lambda)$ is equal to its distribution over $\mathrm{BSYT}(\psi(\lambda))$, where ψ is the Littlewood decomposition defined in Section 2.4. Combining this with Propositions 2.10 and 2.15, the last equation then implies

$$\mathcal{Q}^C(A) = \sum_{w \in A} F^B_{\mathrm{Des}_B(w)} = \sum_{\lambda \in P^0(n)} c_\lambda \sum_{T \in \mathrm{BSYT}((\lambda^-)', \lambda^+)} F^B_{\mathrm{Des}_B(T)} = \sum_{\lambda \in P^0(n)} c_\lambda \sum_{\mathsf{T} \in \mathrm{SDT}(\psi^{-1}((\lambda^-)', \lambda^+))} F^B_{\mathrm{Des}_B(\mathsf{T})}$$

$$= \sum_{\lambda \in P^0(n)} c_\lambda \mathcal{G}_{\psi^{-1}((\lambda^-)', \lambda^+)} = \sum_{\lambda \in P^0(n)} c_\lambda \, s_{(\lambda^-)'}(x_1, x_2, \dots)s_{\lambda^+}(x_0, x_1, \dots),$$

as desired. $\square$

Remark 8.10. Combining Remark 8.7 with Theorem 8.9.1, one deduces that conjugacy classes and inverse signed (or unsigned) descent classes in B_n are Chow type B Schur-positive.

Remark 8.11. The converse of Theorem 8.9.1 does not hold. Indeed, by Proposition 8.2, $\mathrm{FC}(B_n)$ is Chow type B Schur-positive, while by Proposition 8.8, it is not Poirier type B Schur-positive.

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