Team equivalences for finite-state machines with silent moves

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Team Equivalences for Finite-State Machines with Silent Moves

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Abstract

Finite-state machines, a simple class of finite Petri nets, were equipped in [16] with an efficiently decidable, truly concurrent, bisimulation-based, behavioral equivalence, called *team equivalence*, which conservatively extends classic bisimulation equivalence on labeled transition systems and which is checked in a distributed manner, without necessarily building a global model of the overall behavior. This paper addresses the problem of defining variants of this equivalence which are insensitive to silent moves. We define (rooted) weak team equivalence and (rooted) branching team equivalence as natural transposition to finite-state machines of Milner's weak bisimilarity [25] and of van Glabbeek and Weijland's branching bisimilarity [12] on labeled transition systems. The process algebra CFM [15] is expressive enough to represent all and only the finite-state machines, up to net isomorphism. Here we first prove that the rooted versions of these equivalences are congruences for the operators of CFM, then we show some algebraic properties, and, finally, we provide finite, sound and complete, axiomatizations for rooted weak team equivalence and rooted branching team equivalence over CFM.

Keywords: Petri nets, finite-state machines, truly-concurrent semantics, weak bisimulation, branching bisimulation, equivalence checking, axiomatization.

1. Introduction

By finite-state machine (FSM, for short) we mean a simple type of finite Petri net [15, 29, 33] whose transitions have singleton pre-set and singleton, or empty, post-set. The name originates from the fact that a net of this kind is isomorphic to a nondeterministic finite automaton (NFA), usually called a finite-state machine as well. However, semantically, our FSMs are richer than NFAs because, as their initial marking may be not a singleton, these nets can also exhibit concurrent behavior, while NFAs are strictly sequential. FSMs are also very similar to finite-state, labeled transition systems (LTSs,

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for short) [22], a class of models that are suitable for describing sequential, nondeterministic systems, and are also widely used as a semantic model for process algebras (see, e.g., [14]). On this class of models, there is widespread agreement that a very natural and convenient equivalence relation is bisimulation equivalence [28, 25]. If the LTS contains silent transitions, i.e., transitions labeled by the invisible action τ, then Milner proposed *weak bisimulation* equivalence [25] as a natural extension of bisimulation equivalence to this setting. However, van Glabbeek and Weijland in [12] argued that weak bisimilarity does not completely respect the branching structure of processes and so they proposed *branching bisimulation* equivalence as a suitable equivalence in the presence of silent moves.

In [16] we defined a new truly-concurrent equivalence relation over FSMs (without silent moves), called *team equivalence*, that can be computed in a distributed manner, without resorting to a global model of the overall behavior of the analyzed marked net. Since an FSM is so similar to an LTS, the basic idea we started with was to define bisimulation equivalence directly over the set of places of the *unmarked* net. The advantage is that bisimulation equivalence is a relation on places, rather than on markings (as it is customary for Petri nets), and so much more easily computable; more precisely, if m is the number of net transitions and n is the number of places, checking whether two places are bisimilar can be done in $O(m \log{(n+1)})$ time, by adapting the optimal algorithm in [30] for standard bisimulation on LTSs. After the bisimulation equivalence over the set of places has been computed, we can define, in a purely structural way, that two markings m_1 and m_2 are *team equivalent* if they have the same cardinality, say $|m_1| = k = |m_2|$, and there is a bisimulation-preserving, bijective mapping between the two markings, so that each of the k pairs of places (s_1, s_2), with $s_1 \in m_1$ and $s_2 \in m_2$, is such that s_1 and s_2 are bisimilar. Team equivalence is a truly concurrent behavioral equivalence as it is sensitive to the size of the distributed state; as a matter of fact, it relates markings of the same size, only. The name *team equivalence* reminds us that two distributed systems, composed of a team of non-cooperating, sequential processes, are equivalent if it is possible to match each sequential component of the first system with one bisimulation-equivalent, sequential component of the other one, as in any sports where two competing (distributed) teams have the same number of (sequential) players. Once bisimilarity on places has been computed, checking whether two markings of size k are team equivalent can be computed in $O(k^2)$ time (or $O(n)$, cf. Remark 1).

Note that to check whether two markings are team equivalent we need not to construct an LTS describing the global behavior of the whole system, but only to find a suitable, bisimulation-preserving match among the local, sequential states (i.e., the elements of the markings). Nonetheless, we proved that team equivalence is coherent with the global behavior of the net. More precisely, we showed in [16] that team equivalence is finer than interleaving bisimilarity, actually it coincides with *strong place bisimilarity* [1] and so it respects the causal semantics of nets.

The main goal of this paper is to extend our previous definition of (strong) bisimulation on places (and of team equivalence) to FSMs *with silent moves*, taking inspiration from Milner's weak bisimulation [25, 14] and van Glabbeek and Weijland's branching bisimulation [12, 14]. Therefore, we first define *weak bisimulation on places* and its associated *weak team equivalence*, together with the variants requiring the so-called rootedness condition (i.e., an initial silent move can be matched only by a nonempty sequence of silent moves, as in rooted weak bisimilarity [25, 14]). Then, we define *branching bisimulation on places* and its associated *branching team equivalence*, together with the variants requiring the rootedness condition (i.e., the first move is to be matched strongly, as in rooted branching bisimilarity [12, 14]). The originality of our proposal is not in the technical definition of (rooted) weak/branching bisimulations on places (which are, indeed, almost identical to the original ones on LTSs), rather on the fact that these relations are defined over the *places of an unmarked net*, rather than on the *reachable markings of a marked net*. Moreover, the main originality of our proposal is in the definition of (rooted) weak/branching *team* equivalences; these equivalences are all computed in a structural way, without building a model of the global behavior. Nonetheless, we will prove that these are coherent with the global behavior; in particular, they are finer than the corresponding (rooted) weak/branching *interleaving* bisimulation equivalences over FSMs (see Section 2 for details), which are equivalences relations defined over the net markings. The key feature common to all the new team equivalences we present in this paper, is that, contrary to the weak/branching *interleaving* bisimulation equivalences, to a silent move of a single sequential component of the marking m_1, the marking m_2 may reply only with a (possibly empty) sequence of silent transitions which are *local* to one of its sequential components.

In [15] we proved that the class of FSMs can be "alphabetized" by means of the process algebra CFM: not only the net semantics of each CFM term is an FSM, but also, given a FSM N, we can single out a CFM term p_N such that its net semantics is an FSM isomorphic to N. This means that we can define team equivalences also over CFM process terms. CFM is a simple process algebra: it is actually a slight extension to finite-state CCS [25] and a subcalculus of both regular CCS and BPP [14].

Based on our previous work [17], where we provided a sound and complete axiomatization for (strong) team equivalence over CFM, the goals of the second part of this paper are three: we want (i) to prove that rooted weak/branching team equivalence is a congruence for the CFM operators, (ii) to study the algebraic properties of these equivalences and, finally, (iii) to provide them with a sound and complete, finite axiomatization for CFM. These axiomatizations are not too surprising: it is enough to add to (a slighty revised version of) the finite axiomatization of rooted weak/branching bisimulation equivalence for finite-state CCS [26, 11], three axioms for parallel composition stating that it is associative, commutative and with **0** as neutral element. However, the technical treatment is different (and somehow simpler) than [26, 11], as we base our axiomatization on guarded process constants (e.g., $C \doteq a.C$) rather than on the recursion construct (with possible unguarded variables; e.g., $\mu X a.X + X$). To the best of our knowledge, these are the first axiomatizations of a truly concurrent behavioral equivalence in the presence of silent moves.

The paper is organized as follows. Section 2 introduces the basic definitions about finite-state machines and some well-known behavioral equivalences on them: (rooted) weak interleaving bisimilarity and (rooted) branching interleaving bisimilarity. Section 3 copes with the distributed equivalence checking problem for (rooted) weak equivalence; first, (rooted) weak bisimulation over places of an unmarked net is defined; then, (rooted) weak team equivalence is introduced and some examples are presented, together with a proof that it is finer than (rooted) weak interleaving bisimilarity; moreover, the minimization of an FSM w.r.t. weak bisimilarity on places is defined. Section

4 copes with the similar distributed equivalence checking problem for (rooted) branching bisimilarity: we define first (rooted) branching bisimilarity on places, then (rooted) branching team equivalence and its minimized net. Section 5 introduces the process algebra CFM, its syntax, its net semantics and recalls the so-called *representability theorem* from [15]. Section 6 shows that rooted weak/branching team equivalences are congruences for the CFM operators and studies their algebraic properties. Section 7 presents the finite axiomatizations of rooted weak/branching team equivalences for CFM, proving that they are sound and complete. Finally, Section 8 discusses some related literature and future research.

2. Basic Definitions and Behavioral Equivalences

Definition 1. (Multiset) Let $\mathbb{N}$ be the set of natural numbers. Given a finite set S, a *multiset* over S is a function $m : S \to \mathbb{N}$. The *support* set $dom(m)$ of m is the set $\{s \in S \mid m(s) \neq 0\}$. The set of all multisets over S, denoted by $\mathscr{M}(S)$, is ranged over by m, possibly indexed. We write $s \in m$ if $m(s) > 0$. The *multiplicity* of s in m is given by the number $m(s)$. The *size* of m, denoted by $|m|$, is the number $\sum_{s \in S} m(s)$, i.e., the total number of its elements. A multiset m such that $dom(m) = \emptyset$ is called *empty* and is denoted by θ. We write $m \subseteq m'$ if $m(s) \leq m'(s)$ for all $s \in S$.

Multiset union $\_ \oplus \_$ is defined as follows: $(m \oplus m')(s) = m(s) + m'(s)$; the operation $\oplus$ is commutative, associative and has θ as neutral element. *Multiset difference* $\_ \ominus \_$ is defined as follows: $(m_1 \ominus m_2)(s) = max\{m_1(s) - m_2(s), 0\}$. The *scalar product* of a number j with m is the multiset $j \cdot m$ defined as $(j \cdot m)(s) = j \cdot (m(s))$.

By s_i we also denote the multiset with s_i as its only element. Hence, a multiset m over $S = \{s_1, \dots, s_n\}$ can be represented as $k_1 \cdot s_1 \oplus k_2 \cdot s_2 \oplus \dots \oplus k_n \cdot s_n$, where $k_j = m(s_j) \geq 0$ for $j = 1, \dots, n$. □

Definition 2. (Finite-state machine) A labeled *finite-state machine* (FSM, for short) is a tuple $N = (S, A, T)$, where

- S is the finite set of *places*, ranged over by s (possibly indexed),
- A is the finite set of *labels*, ranged over by ℓ (possibly indexed), and
- $T \subseteq S \times A \times (S \cup \{\theta\})$ is the finite set of *transitions*, ranged over by t (possibly indexed).

Given a transition $t = (s, \ell, m)$, we use the notation: $^\bullet t$ to denote its *pre-set* s (which is a single place) of tokens to be consumed; $l(t)$ for its *label* ℓ, and $t^\bullet$ to denote its *post-set* m (which is a place or the empty multiset θ) of tokens to be produced. Hence, transition t can be also represented as $^\bullet t \xrightarrow{l(t)} t^\bullet$. □

Our definition of T as a set of triples ensures that the net is *transition simple*, i.e., for each $t_1, t_2 \in T$, if $^\bullet t_1 = {}^\bullet t_2$ and $t_1^\bullet = t_2^\bullet$ and $l(t_1) = l(t_2)$, then $t_1 = t_2$. Graphically, a place is represented by a little circle, a transition by a little box, which is connected by a directed arc from the place in its pre-set and to the place in its post-set, if any.

Definition 3. (Marking, FSM net system) A *marking* is a multiset over S. Given a marking m and a place s, we say that the place s contains $m(s)$ *tokens*, graphically represented by $m(s)$ bullets inside place s. An *FSM net system* $N(m_0)$ is a tuple (S, A, T, m_0), where (S, A, T) is an FSM and m_0 is a marking over S, called the *initial marking*. We also say that $N(m_0)$ is a *marked* net. An FSM net system $N(m_0) = (S, A, T, m_0)$ is *sequential* if m_0 is a singleton, i.e., $|m_0| = 1$; while it is *concurrent* if m_0 is arbitrary. □

Definition 4. (Token game, firing sequence, transition sequence, reachable markings) Given an FSM $N = (S, A, T)$, a transition t is *enabled* at marking m, denoted by $m[t\rangle$, if ${}^\bullet t \subseteq m$. The execution (or *firing*) of t enabled at m produces the marking $m' = (m \ominus {}^\bullet t) \oplus t^\bullet$. This is written as $m[t\rangle m'$. This procedure is called the *token game*. A *firing sequence* starting at m is defined inductively as follows:

- $m[\varepsilon\rangle m$ is a firing sequence (where ε denotes an empty transition sequence) and
- if $m[\sigma\rangle m'$ is a firing sequence and $m'[t\rangle m''$, then $m[\sigma t\rangle m''$ is a firing sequence.

If $\sigma = t_1 \dots t_n$ (for $n \geq 0$) and $m[\sigma\rangle m'$ is a firing sequence, then there exist $m_1, \dots, m_{n+1}$ such that $m = m_1[t_1\rangle m_2[t_2\rangle \dots m_n[t_n\rangle m_{n+1} = m'$, and $\sigma = t_1 \dots t_n$ is called a *transition sequence* starting at m and ending at m'. The set of *reachable markings* from m is $reach(m) = \{m' \mid \exists \sigma . m[\sigma\rangle m'\}$.

The labeling function can be extended to transition sequences by juxtaposition; formally: $l(\varepsilon) = \varepsilon$ (i.e., the label of an empty transition sequence is the empty word on A, both represented by the same symbol ε with abuse of notation) and $l(t\sigma) = l(t)l(\sigma)$.

The definition of pre-set and post-set can be extended to transition sequences as follows: ${}^\bullet\varepsilon = \theta$, ${}^\bullet(t\sigma) = {}^\bullet t \oplus ({}^\bullet\sigma \ominus t^\bullet)$, $\varepsilon^\bullet = \theta$, $(t\sigma)^\bullet = \sigma^\bullet \oplus (t^\bullet \ominus {}^\bullet\sigma)$. A transition sequence σ is *sequential* if $|{}^\bullet\sigma| \leq 1$. □

Definition 5. (Dynamically reduced) An FSM net system $N(m_0) = (S, A, T, m_0)$ is *dynamically reduced* if $\forall s \in S\, \exists m \in reach(m_0).m(s) \geq 1$ and $\forall t \in T\, \exists m, m' \in reach(m_0)$ such that $m[t\rangle m'$. □

Example 1. Figure 1 shows in (a) a sequential FSM, which performs a, possibly empty, sequence of a's and b's, until it performs one c and then stops *successfully* (the token disappears in the end). Note that a sequential FSM is *safe* (or 1-bounded): each place in any reachable marking can hold one token at most. In (b), a concurrent FSM is depicted: it can perform a forever, interleaved with two occurrences of b, only: the two tokens in s_4 will eventually reach s_5, which is a place representing unsuccessful termination (deadlock). Note that a concurrent FSM is *k-bounded*, where k is the size of the initial marking: each place in any reachable marking can hold k tokens at most. Finally, note that for each FSM and each place $s \in S$, the set $reach(s)$ is a subset of $S \cup \{\theta\}$. □

Definition 6. (Interleaving Bisimulation) Let $N = (S, A, T)$ be an FSM. An *interleaving bisimulation* is a relation $R \subseteq \mathscr{M}(S) \times \mathscr{M}(S)$ such that if $(m_1, m_2) \in R$ then

- $\forall t_1$ such that $m_1[t_1\rangle m_1'$, $\exists t_2$ such that $m_2[t_2\rangle m_2'$ with $l(t_1) = l(t_2)$ and $(m_1', m_2') \in R$,
- $\forall t_2$ such that $m_2[t_2\rangle m_2'$, $\exists t_1$ such that $m_1[t_1\rangle m_1'$ with $l(t_1) = l(t_2)$ and $(m_1', m_2') \in R$.

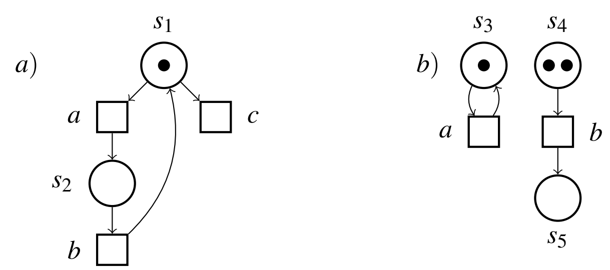

Figure 1: A sequential finite-state machine in (a), and a concurrent finite-state machine in (b)

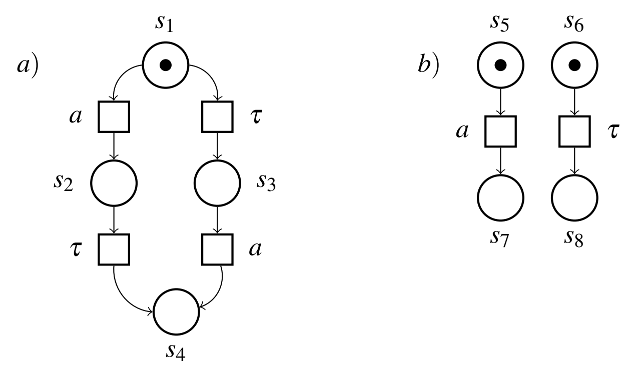

Figure 2: Two interleaving bisimilar FSMs

Two markings m_1 and m_2 are *interleaving bisimilar*, denoted by $m_1 \sim_{int} m_2$, if there exists an interleaving bisimulation R such that $(m_1, m_2) \in R$. □

Interleaving bisimilarity $\sim_{int}$, which is defined as the union of all the interleaving bisimulations, is the largest interleaving bisimulation and also an equivalence relation.

Example 2. Consider the net in Figure 2. It is not difficult to realize that $R = \{(s_1, s_5 \oplus s_6), (s_2, s_6 \oplus s_7), (s_3, s_5 \oplus s_8), (s_4, s_7 \oplus s_8)\}$ is an interleaving bisimulation. □

We now introduce some marking-based relations for FSMs with silent moves. Some auxiliary notation is necessary. Let $N = (S, A, T)$ be an FSM. By $A_\tau = A \setminus \{\tau\}$, where τ is the only invisible action, we denote the set of observable actions. Given a transition sequence σ, its *observable label* $o(\sigma)$ is computed inductively as follows.

$$o(\varepsilon) = \varepsilon$$

$$o(t\sigma) = \begin{cases} l(t)o(\sigma) & \text{if } l(t) \neq \tau \\ o(\sigma) & \text{otherwise.} \end{cases}$$

We also define the auxiliary function $o_\tau(\sigma)$ as follows: In case $o(\sigma) \neq \varepsilon$ or σ is empty, then $o_\tau(\sigma) = o(\sigma)$; in case $o(\sigma) = \varepsilon$ and σ is not empty, then $o_\tau(\sigma) = \tau$.

Definition 7. (Weak Interleaving Bisimulation) Let $N = (S, A, T)$ be an FSM with silent moves. A *weak* interleaving bisimulation is a relation $R \subseteq \mathscr{M}(S) \times \mathscr{M}(S)$ such that if $(m_1, m_2) \in R$ then

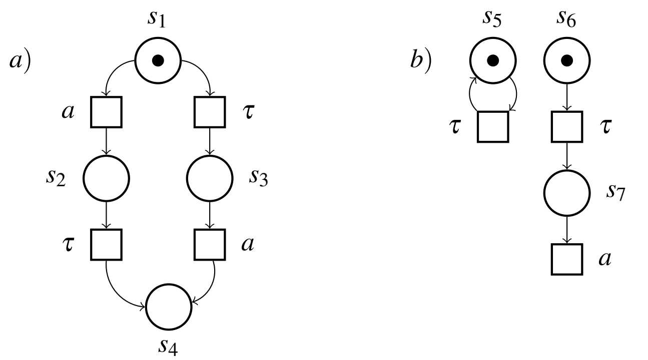

Figure 3: Two weak interleaving bisimilar FSMs

- $\forall t_1$ such that $l(t_1) \neq \tau$ and $m_1[t_1\rangle m_1'$, $\exists \sigma_2$ such that $m_2[\sigma_2\rangle m_2'$ with $l(t_1) = o(\sigma_2)$ and $(m_1', m_2') \in R$,
- $\forall t_1$ such that $l(t_1) = \tau$ and $m_1[t_1\rangle m_1'$, $\exists \sigma_2$ such that $m_2[\sigma_2\rangle m_2'$ with $o(\sigma_2) = \varepsilon$ and $(m_1', m_2') \in R$,

and symmetrically if m_2 moves first.

Two markings m_1 and m_2 are *weak* interleaving bisimilar, denoted by $m_1 \approx_{int} m_2$, if there exists a weak interleaving bisimulation R such that $(m_1, m_2) \in R$. □

Note that an invisible transition performed by one of the markings may be matched by the other one also by idling, i.e., by performing an empty transition sequence. Weak interleaving bisimilarity $\approx_{int}$, defined as the union of all the weak interleaving bisimulations, is the largest weak interleaving bisimulation and also an equivalence relation.

Example 3. Consider Figure 3. It is easy to realize that $R = \{(s_1, s_5 \oplus s_6), (s_2, s_5), (s_3, s_5 \oplus s_7), (s_4, s_5)\}$ is a weak interleaving bisimulation. And also $R \cup \{(s_3, s_5 \oplus s_6)\}$. □

Definition 8. (Rooted Weak Interleaving Bisimilarity) Let $N = (S, A, T)$ be an FSM with silent moves. Two markings m_1 and m_2 are *rooted weak* interleaving bisimilar, denoted by $m_1 \approx_{int}^c m_2$, if

- $\forall t_1$ such that $m_1[t_1\rangle m_1'$, $\exists \sigma_2$ s.t. $m_2[\sigma_2\rangle m_2'$ with $l(t_1) = o_\tau(\sigma_2)$ and $m_1' \approx_{int} m_2'$,
- $\forall t_2$ such that $m_2[t_2\rangle m_2'$, $\exists \sigma_1$ s.t. $m_1[\sigma_1\rangle m_1'$ with $l(t_2) = o_\tau(\sigma_1)$ and $m_1' \approx_{int} m_2'$. □

Therefore, if two markings are rooted weak interleaving bisimilar, in case one of the two initially performs an invisible transition (i.e., $l(t_1) = \tau$), then the other is able to respond with a nonempty sequence of invisible transitions (i.e., $o_\tau(\sigma_2) = \tau$); since the reached markings are simply weakly interleaving bisimilar (i.e., $m_1' \approx_{int} m_2'$), future invisible transitions performed by one of the two can be matched by the other one also by idling. Hence, $\approx_{int}^c$ is slightly finer than $\approx_{int}$.

Example 4. Consider again the net in Figure 3. It is easy to realize that $s_1 \approx_{int}^c s_5 \oplus s_6$, while $s_3 \not\approx_{int}^c s_5 \oplus s_6$, even if $s_3 \approx_{int} s_5 \oplus s_6$. □

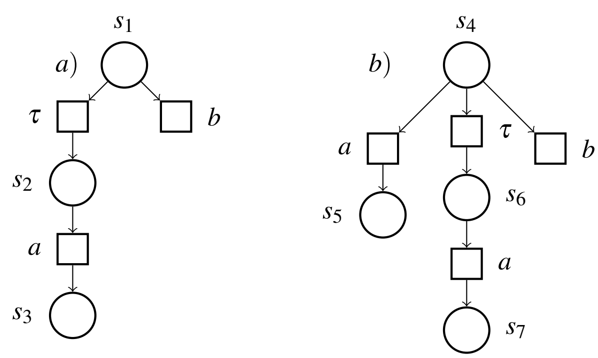

Figure 4: Other two non-branching bisimilar FSMs

Definition 9. (Branching interleaving bisimulation) Let $N = (S, A, T)$ be an FSM with τ-moves. A *branching* interleaving bisimulation is a relation $R \subseteq \mathscr{M}(S) \times \mathscr{M}(S)$ such that if $(m_1, m_2) \in R$ then

- $\forall t_1$ such that $m_1[t_1\rangle m_1'$,
  - either $l(t_1) = \tau$ and $\exists \sigma_2$ such that $o(\sigma_2) = \varepsilon$, $m_2[\sigma_2\rangle m_2'$ with $(m_1, m_2') \in R$ and $(m_1', m_2') \in R$,
  - or $\exists \sigma, t_2$ such that $o(\sigma) = \varepsilon$, $l(t_1) = l(t_2)$, $m_2[\sigma\rangle m[t_2\rangle m_2'$ with $(m_1, m) \in R$ and $(m_1', m_2') \in R$,
- and, symmetrically, $\forall t_2$ such that $m_2[t_2\rangle m_2'$.

Two markings m_1 and m_2 are *branching interleaving bisimilar* (or *branching interleaving bisimulation equivalent*), denoted $m_1 \approx_{bri} m_2$, if there exists a branching interleaving bisimulation R that relates them. □

Note that a silent transition performed by one of the two markings may be matched by the other one also by idling: this is due to the *either* case when $\sigma_2 = \varepsilon$ (or $\sigma_1 = \varepsilon$). Branching interleaving bisimilarity $\approx_{bri}$, which is defined as the union of all the branching interleaving bisimulations, is the largest branching interleaving bisimulation and also an equivalence relation. Note that the markings s_1 and $s_5 \oplus s_6$ in Figure 3 are branching interleaving bisimilar (but not *rooted* branching interleaving bisimilar, see below). Branching interleaving bisimilarity is finer than weak interleaving bisimilarity, because a branching interleaving bisimulation is also a weak interleaving bisimulation.

Example 5. Consider the nets in Figure 4. It is not difficult to see that $s_1 \approx_{int} s_4$. However, $s_1 \not\approx_{bri} s_4$, because to transition $s_4 \xrightarrow{a} s_5$, place s_1 can only try to respond with $s_1 \xrightarrow{\tau} s_2 \xrightarrow{a} s_3$, but not all the conditions required are satisfied; in particular, $s_2 \not\approx_{bri} s_4$, because only s_4 can do b. □

An important property hods for $\approx_{bri}$. Consider the *either* case: since $(m_1, m_2) \in \approx_{bri}$ by hypothesis, and $m_2[\sigma_2\rangle m_2'$ with $(m_1, m_2') \in \approx_{bri}$ and $(m_1', m_2') \in \approx_{bri}$, it follows that

$(m_2, m_2') \in \approx_{bri}$ because $\approx_{bri}$ is an equivalence relation. This implies that all the markings in the silent path from m_2 to m_2' are branching interleaving bisimilar (by the so-called *stuttering property*, cf. Remark 2). Similarly for the *or* case. This condition is not required by weak interleaving bisimilarity (cf. Example 5).

Definition 10. (Rooted branching interleaving bisimilarity) Let $N = (S, A, T)$ be an FSM with τ-moves. Two markings m_1 and m_2 are rooted branching interleaving bisimilar, denoted $m_1 \approx_{bri}^c m_2$, if

- $\forall t_1$ such that $m_1[t_1\rangle m_1'$, $\exists t_2$ such that $l(t_1) = l(t_2)$, $m_2[t_2\rangle m_2'$ and $m_1' \approx_{bri} m_2'$,
- $\forall t_2$ such that $m_2[t_2\rangle m_2'$, $\exists t_1$ such that $l(t_1) = l(t_2)$, $m_1[t_1\rangle m_1'$ and $m_1' \approx_{bri} m_2'$. □

Note that $\approx_{bri}^c$ is finer than $\approx_{bri}$: if in the net in Figure 4 we delete the b-labeled transitions, then $s_1 \approx_{bri} s_4$ but $s_1 \not\approx_{bri}^c s_4$, because only s_4 can perform a initially.

3. A Distributed Approach to Weak Equivalence Checking

We recall the definition of (strong) bisimulation on places for unmarked FSMs, originally introduced in [16], as it will be useful in the following. In this definition (and in the following ones), the markings m_1 and m_2 can only be either the empty marking θ or a single place, because of the shape of FSM transitions.

Definition 11. (Bisimulation on places) Let $N = (S, A, T)$ be an FSM. A *bisimulation on places* is a relation $R \subseteq S \times S$ such that if $(s_1, s_2) \in R$ then for all $\ell \in A$

- $\forall m_1$ such that $s_1 \xrightarrow{\ell} m_1$, $\exists m_2$ such that $s_2 \xrightarrow{\ell} m_2$ and either $m_1 = \theta = m_2$ or $(m_1, m_2) \in R$,
- $\forall m_2$ such that $s_2 \xrightarrow{\ell} m_2$, $\exists m_1$ such that $s_1 \xrightarrow{\ell} m_1$ and either $m_1 = \theta = m_2$ or $(m_1, m_2) \in R$.

Two places s and s' are *bisimilar* (or *bisimulation equivalent*), denoted $s \sim s'$, if there exists a bisimulation R such that $(s, s') \in R$. □

We argued in [16] that, if m is the number of net transitions and n is the number of places, checking whether two places of an FSM are (strong) bisimilar can be done in $O(m \cdot \log(n+1))$ time, by adapting the algorithm in [30] for ordinary bisimulation on LTSs. Moreover, bisimilarity on places enjoys the same properties of bisimulation on LTSs, i.e., it is coinductive and equipped with a fixed-point characterization.

3.1. Weak Bisimulation on Places

In order to adapt the definition of weak bisimulation on LTSs [25, 14] for unmarked FSMs, we need some auxiliary notation. We define relation $\overset{\varepsilon}{\Longrightarrow} \subseteq S \times (S \cup \{\theta\})$ as the reflexive and transitive closure of the silent transition relation; formally, $\forall s \in S$, $s \overset{\varepsilon}{\Longrightarrow} s$, denoting that each place can silently reach itself with zero steps; moreover, if $s \overset{\varepsilon}{\Longrightarrow} s'$ and $s' \xrightarrow{\tau} m$, then $s \overset{\varepsilon}{\Longrightarrow} m$. Note that $s \overset{\varepsilon}{\Longrightarrow} m$ if and only if there exists a *sequential* transition sequence σ such that $s[\sigma\rangle m$ and $o(\sigma) = \varepsilon$. If $s \stackrel{\varepsilon}{\Longrightarrow} m$ and the sequence of silent moves is not empty, we may also denote this by $s \stackrel{\tau}{\Longrightarrow} m$, i.e., there exists a *nonempty*, sequential transition sequence σ such that $s[\sigma\rangle m$ and $o_\tau(\sigma) = \tau$.

Finally, for any $\ell \in A_\tau$, we write $s \stackrel{\ell}{\Longrightarrow} m$ if there exist two places s' and s'' such that $s \stackrel{\varepsilon}{\Longrightarrow} s' \stackrel{\ell}{\longrightarrow} s'' \stackrel{\varepsilon}{\Longrightarrow} m$, or (in case $s'' = \theta = m$) if there exists one place s' such that $s \stackrel{\varepsilon}{\Longrightarrow} s' \stackrel{\ell}{\longrightarrow} \theta$. Note that $s \stackrel{\ell}{\Longrightarrow} m$ if and only if there exists a sequential transition sequence σ such that $s[\sigma\rangle m$ and $o(\sigma) = \ell$.

Definition 12. (Weak bisimulation on places) Let $N = (S, A, T)$ be an FSM with silent moves. A *weak bisimulation* is a relation $R \subseteq S \times S$ such that if $(s_1, s_2) \in R$ then for all $\ell \in A_\tau$

- $\forall m_1$ such that $s_1 \stackrel{\ell}{\longrightarrow} m_1$, $\exists m_2$ such that $s_2 \stackrel{\ell}{\Longrightarrow} m_2$ and either $m_1 = \theta = m_2$ or $(m_1, m_2) \in R$,

- $\forall m_1$ such that $s_1 \stackrel{\tau}{\longrightarrow} m_1$, $\exists m_2$ such that $s_2 \stackrel{\varepsilon}{\Longrightarrow} m_2$ and either $m_1 = \theta = m_2$ or $(m_1, m_2) \in R$,

and symmetrically if m_2 moves first.

Two places s and s' are *weakly bisimilar* (or *weak bisimulation equivalent*), denoted by $s \approx s'$, if there exists a weak bisimulation R such that $(s, s') \in R$. □

We now list some properties, whose proofs are rather similar to the corresponding ones for weak bisimulation on LTSs (see, e.g., [25, 34, 14]) and so they are omitted.

Lemma 1. Let $N = (S, A, T)$ be an FSM and let R be a weak bisimulation such that $(s, s') \in R$. Then, the following hold:

(i) For all m such that $s \stackrel{\varepsilon}{\Longrightarrow} m$, there exists m' such that $s' \stackrel{\varepsilon}{\Longrightarrow} m'$ and either $m = \theta = m'$ or $(m, m') \in R$;

(ii) For all m such that $s \stackrel{\ell}{\Longrightarrow} m$, there exists m' such that $s' \stackrel{\ell}{\Longrightarrow} m'$ and either $m = \theta = m'$ or $(m, m') \in R$;

and symmetrically if s' moves first.

PROOF. The proof is by induction on the length of the computation. □

Proposition 1. For each FSM $N = (S, A, T)$, the following hold:

1. the identity relation $\mathscr{I} = \{(s, s) \mid s \in S\}$ is a weak bisimulation;
2. the inverse relation $R^{-1} = \{(s', s) \mid (s, s') \in R\}$ of a weak bisimulation R is a weak bisimulation;
3. the relational composition $R_1 \circ R_2 = \{(s, s'') \mid \exists s'.(s, s') \in R_1 \wedge (s', s'') \in R_2\}$ of two weak bisimulations R_1 and R_2 is a weak bisimulation;
4. the union $\bigcup_{i \in I} R_i$ of weak bisimulations R_i is a weak bisimulation.

PROOF. Standard. For the proof of (3), it is necessary to use Lemma 1. □

Remember that $s \approx s'$ if there exists a weak bisimulation containing the pair (s, s'). This means that $\approx$ is the union of all weak bisimulations, i.e.,

$$\approx = \bigcup \{R \subseteq S \times S \mid R \text{ is a weak bisimulation}\}.$$

By Proposition 1(4), $\approx$ is also a weak bisimulation, hence the largest such relation.

Proposition 2. For each FSM $N = (S, A, T)$, relation $\approx \subseteq S \times S$ is the largest weak bisimulation relation. □

Directly from Proposition 1(1-3), we deduce the following.

Proposition 3. For each FSM $N = (S, A, T)$, $\approx \subseteq S \times S$ is an equivalence relation. □

Let $N = (S, A, T)$ be an FSM. Its *saturated* net is $N' = (S, A_\tau \cup \{\varepsilon\}, T')$, where $T' = \{(s, \delta, m) \mid \delta \in A_\tau \cup \{\varepsilon\} \text{ and } s \stackrel{\delta}{\Longrightarrow} m\}$. The transitions of N' are computed by means of the (partial) reflexive/transitive closure $\stackrel{\delta}{\Longrightarrow}$ of the transition relation $\rightarrow$ (with an algorithm that can be based on, e.g., the Floyd-Warshall algorithm [8]).

It is possible to offer an alternative, yet equivalent, definition of weak bisimulation on places over N as a strong bisimulation on places over its saturated net N'.

Proposition 4. Let $N = (S, A, T)$ be an FSM and let $R \subseteq S \times S$ be a weak bisimulation. It can be proved that if $(s_1, s_2) \in R$ then for all $\delta \in A_\tau \cup \{\varepsilon\}$

- $\forall m_1$ such that $s_1 \stackrel{\delta}{\Longrightarrow} m_1$, there exists m_2 such that $s_2 \stackrel{\delta}{\Longrightarrow} m_2$ and $m_1 = \theta = m_2$ or $(m_1, m_2) \in R$,
- $\forall m_2$ such that $s_2 \stackrel{\delta}{\Longrightarrow} m_2$, there exists m_1 such that $s_1 \stackrel{\delta}{\Longrightarrow} m_1$ and $m_1 = \theta = m_2$ or $(m_1, m_2) \in R$.

PROOF. Since R is a weak bisimulation, if $(s_1, s_2) \in R$ and $s_1 \stackrel{\delta}{\Longrightarrow} m_1$, then, by Lemma 1, $s_2 \stackrel{\delta}{\Longrightarrow} m_2$ and $m_1 = \theta = m_2$ or $(m_1, m_2) \in R$; symmetrically, if s_2 moves first. □

A consequence of this alternative characterization is that, from a complexity point of view, computing weak bisimulation equivalence is just a bit harder than computing (strong) bisimulation equivalence on places: one has first to derive the transitions of the saturated net, (which takes $O((n+1)^3)$, where n is the number of places, if the Floyd-Warshall algorithm [8] is used),[1] and then to check (strong) bisimulation equivalence on the saturated net, which is in $O(m \log (n+1))$ time (where m is the number of transitions of the saturated net). Another consequence is that it is also possible to characterize $\approx$ as the greatest fixed point of a suitable functional over binary relations, as done for (strong) bisimulation over LTSs in [25, 34, 14]. Finally, if we use this alternative definition of weak bisimulation (in strong style), the proof technique "strong bisimulation up to $\sim$" can be used also in this context with the use of $\approx$ in place of each occurrence of $\sim$ [35], as we will do in the proof of Proposition 27.

[1] By considering θ as an additional, dummy place, the total number of "places" is $n+1$. However, it is possible to compute weak bisimilarity with more performant algorithms with complexity $O(m \cdot n)$ [32].

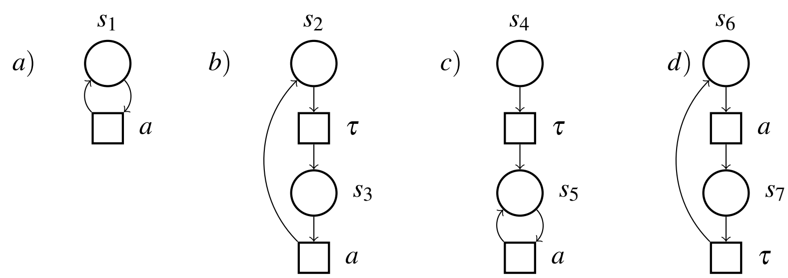

Figure 5: Some weakly bisimilar FSMs

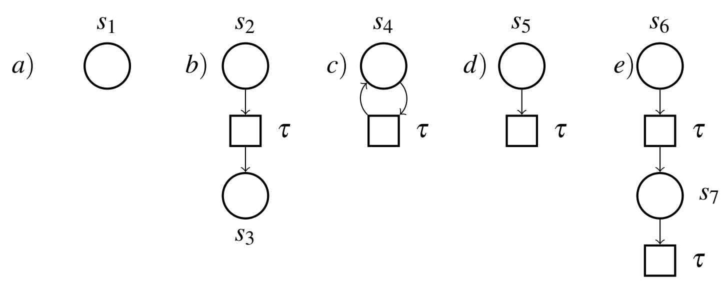

Figure 6: Some other (not) weakly bisimilar FSMs

Example 6. Consider Figure 5. Relation $R_1 = \{(s_1,s_2),\ (s_1,s_3)\}$ is a weak bisimulation. In fact, (s_1,s_2) is a weak bisimulation pair as if $s_1 \xrightarrow{a} s_1$, then $s_2 \stackrel{a}{\Longrightarrow} s_2$ and $(s_1,s_2) \in R_1$; conversely, if $s_2 \xrightarrow{\tau} s_3$, then $s_1 \stackrel{\varepsilon}{\Longrightarrow} s_1$ (i.e., it responds by idling) and $(s_1,s_3) \in R_1$. Similarly, we can prove that also (s_1,s_3) is a weak bisimulation pair. Moreover, also $R_2 = \{(s_1,s_4),\ (s_1,s_5)\}$ and $R_3 = \{(s_1,s_6),\ (s_1,s_7)\}$ are weak bisimulations. Actually, if $S = \{s_1,s_2,s_3,s_4,s_5,s_6,s_7\}$, then $\approx\ =\ S \times S$. □

Example 7. Consider Figure 6. It is easy to realize that $R_1 = \{(s_1,s_2),(s_1,s_3)\}$ is a weak bisimulation: if $s_2 \xrightarrow{\tau} s_3$, then s_1 responds by idling, and the reached places are still in R_1. Similarly, $R_2 = \{(s_1,s_4)\}$ is a weak bisimulation. This example shows that deadlock (i.e., place s_1) and divergence (i.e., place s_4) are not distinguished by $\approx$. More intriguing is the net in $d)$. Since the post-set of the τ-labeled transition is θ, this apparently silent transition is actually observable. In fact, $s_2 \not\approx s_5$: to transition $s_2 \xrightarrow{\tau} s_3$, place s_5 can only try to respond with $s_5 \xrightarrow{\tau} \theta$, but s_3 and θ are not weakly bisimilar, as a place cannot be related to the empty marking by definition. In fact, weak bisimulation equivalence is sensitive to the kind of termination of a process: even if s_3 is stuck, it is not equivalent to θ because the latter is the marking of a properly terminated process, while s_3 denotes a deadlock situation. However, s_5 and s_6 are weakly bisimilar, as relation $R_3 = \{(s_5,s_6),(s_5,s_7)\}$ is a weak bisimulation. In fact, if $s_5 \xrightarrow{\tau} \theta$, then $s_6 \stackrel{\varepsilon}{\Longrightarrow} \theta$; instead, if $s_6 \xrightarrow{\tau} s_7$, then s_5 responds by idling and the pair (s_5,s_7) is in R_3. In other words, a τ-labeled transition may be unobservable only if it does not change the number of tokens of the current marking, i.e., if its post-set is not empty. □

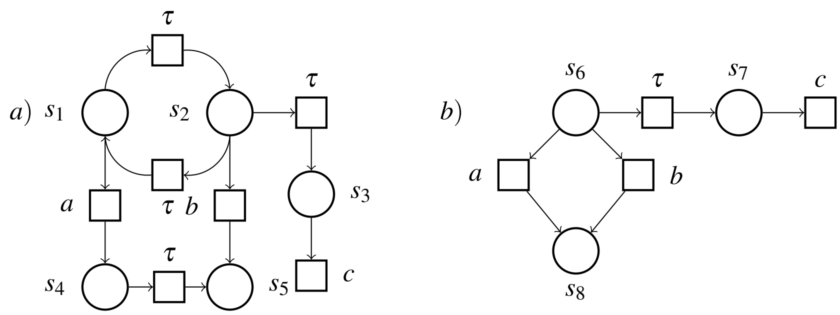

Figure 7: Two more complex weakly bisimilar FSMs

Example 8. Consider Figure 2. Note that $s_1 \approx s_5$ as $R = \{(s_1,s_5),(s_2,s_7),(s_3,s_5),(s_4,s_7)\}$ is a weak bisimulation. On the contrary, $s_1 \not\approx s_6$ as s_6 cannot perform a. Now, consider Figure 3. Note that $s_1 \not\approx s_6$ as if $s_1 \xrightarrow{a} s_2$, then s_6 can only try to respond with $s_6 \stackrel{a}{\Longrightarrow} \theta$, but $s_2 \not\approx \theta$, as a place is not weak bisimilar to the empty marking. □

Example 9. Consider Figure 7. It is easy to realize that relation $R = \{(s_1,s_6),(s_2,s_6),(s_3,s_7),(s_4,s_8),(s_5,s_8)\}$ is a weak bisimulation. □

Definition 13. (Rooted weak bisimulation on places) Let $N = (S,A,T)$ be an FSM. Two places s_1 and s_2 are *rooted weakly bisimilar*, denoted $s_1 \approx_c s_2$, if for all $\ell \in A$

- for all m_1 such that $s_1 \xrightarrow{\ell} m_1$, there exists m_2 such that $s_2 \stackrel{\ell}{\Longrightarrow} m_2$ and either $m_1 = \theta = m_2$ or $m_1 \approx m_2$,
- for all m_2 such that $s_2 \xrightarrow{\ell} m_2$, there exists m_1 such that $s_1 \stackrel{\ell}{\Longrightarrow} m_1$ and either $m_1 = \theta = m_2$ or $m_1 \approx m_2$. □

Note that if $s_1 \xrightarrow{\tau} m_1$, then s_2 must be able to respond with a nonempty sequence of silent moves: $s_2 \stackrel{\tau}{\Longrightarrow} m_2$. However, after this initial step, the reached places are to be related by weak bisimilarity, so that future silent moves of one of the two can be matched by the other one also by idling. Therefore, the following holds.

Proposition 5. Let $N = (S,A,T)$ be an FSM. If $s_1 \approx_c s_2$, then $s_1 \approx s_2$. □

Proposition 6. Let $N = (S,A,T)$ be an FSM with silent moves. Relation $\approx_c$ is an equivalence.

PROOF. Trivial. Transitivity can be proved by exploting Lemma 1. □

Example 10. Consider the nets in Figure 5. Even if $s_1 \approx s_2$, we have that $s_1 \not\approx_c s_2$ because s_1 cannot reply to transition $s_2 \xrightarrow{\tau} s_3$; similarly, $s_1 \not\approx_c s_4$ and $s_1 \not\approx_c s_7$. On the contrary, $s_1 \approx_c s_3$, $s_1 \approx_c s_5$, $s_1 \approx_c s_6$, as well as $s_2 \approx_c s_4$ and $s_2 \approx_c s_7$. □

Example 11. Consider the nets in Figure 6. Even if $s_1 \approx s_2$, we have that $s_1 \not\approx_c s_2$ because s_1 cannot reply to transition $s_2 \xrightarrow{\tau} s_3$. On the contrary, $s_2 \approx_c s_4$. Note also that $s_5 \not\approx_c s_6$; in fact, if $s_6 \xrightarrow{\tau} s_7$, then s_5 cannot reply because $\theta \not\approx s_7$. □

3.2. Additive Closure

In order to define the various equivalences on markings we are presenting in the following, we need a technical, auxiliary definition.

Definition 14. (Additive closure) Given an FSM net $N = (S, A, T)$ and a *place relation* $R \subseteq S \times S$, we define a *marking relation* $R^{\oplus} \subseteq \mathscr{M}(S) \times \mathscr{M}(S)$, called the *additive closure* of R, as the least relation induced by the following axiom and rule.

$$\frac{}{(\theta, \theta) \in R^{\oplus}} \qquad \frac{(s_1, s_2) \in R \quad (m_1, m_2) \in R^{\oplus}}{(s_1 \oplus m_1, s_2 \oplus m_2) \in R^{\oplus}}$$

□

Note that, by definition, two markings are related by $R^{\oplus}$ only if they have the same size; in fact, the axiom states that the empty marking is related to itself, while the rule, assuming by induction that m_1 and m_2 have the same size, ensures that $s_1 \oplus m_1$ and $s_2 \oplus m_2$ have the same size. An alternative way to define that two markings m_1 and m_2 are related by $R^{\oplus}$ is to state that m_1 can be represented as $s_1 \oplus s_2 \oplus \ldots \oplus s_k$, m_2 can be represented as $s'_1 \oplus s'_2 \oplus \ldots \oplus s'_k$ and $(s_i, s'_i) \in R$ for $i = 1, \ldots, k$.

Proposition 7. For each FSM net $N = (S, A, T)$ and each place relation $R \subseteq S \times S$, if $(m_1, m_2) \in R^{\oplus}$, then $|m_1| = |m_2|$. □

Proposition 8. For each FSM net $N = (S, A, T)$ and each place relation $R \subseteq S \times S$, the following hold:

1. If R is an equivalence relation, then $R^{\oplus}$ is an equivalence relation.
2. If $R_1 \subseteq R_2$, then $R_1^{\oplus} \subseteq R_2^{\oplus}$, i.e., the additive closure is monotone. □

Another property of the additive closure $R^{\oplus}$ of a place relation R is that it is additive, indeed; moreover, it is also subtractive when R is an equivalence relation.

Proposition 9. (Additivity/Subtractivity) Given a BPP net $N = (S, A, T)$ and a place relation R, the following hold:

1. If $(m_1, m_2) \in R^{\oplus}$ and $(m'_1, m'_2) \in R^{\oplus}$, then $(m_1 \oplus m'_1, m_2 \oplus m'_2) \in R^{\oplus}$.
2. If R is an equivalence relation, $(m_1 \oplus m'_1, m_2 \oplus m'_2) \in R^{\oplus}$ and $(m_1, m_2) \in R^{\oplus}$, then $(m'_1, m'_2) \in R^{\oplus}$.

PROOF. By induction on the size of m_1. □

Note that the requirement that R is an equivalence relation is strictly necessary for Propositon 9(2). As a counterexample, consider $R = \{(s_1, s_3), (s_1, s_4), (s_2, s_4)\}$. We have that $(s_1 \oplus s_2, s_3 \oplus s_4) \in R^{\oplus}$ and $(s_1, s_4) \in R^{\oplus}$, but $(s_2, s_3) \notin R^{\oplus}$.

Remark 1. (Complexity of $R^{\oplus}$) Given an equivalence place relation R, the complexity of checking whether two markings m_1 and m_2 of equal size are related by $R^{\oplus}$ is very low. In fact, if R is implemented as an adjacency matrix, then the complexity of checking if two markings m_1 and m_2 (represented as an array of places with multiplicities) are related by $R^{\oplus}$ is $O(k^2)$, where k is the size of the markings, since the problem is essentially that of finding for each element s_1 of m_1 a matching, R-related element s_2 of m_2. The details of the algorithm, which is correct only if R is an *equivalence relation* (so that $R^{\oplus}$ is subtractive) are outlined in [16]. Moreover, if we want to check whether other two markings of the same net are related by $R^{\oplus}$, we can reuse R, so that the time complexity is again quadratic on the size of the two markings. However, note that the time spent in creating the adjacency matrix for the equivalence relation R has not been considered: since n is the number of places, $O(n^2)$ time is needed to implement this matrix, so that the time spent for the first check is $O(n^2)$, while for subsequent checks it is only $O(k^2)$, where k is the size of the markings.

The algorithm in [16] is not optimal. The algorithm in [23] simply scans the equivalence classes of R and, for each class, it checks whether the number of tokens in the places of m_1 belonging to this class equals the number of tokens in the places of m_2 in the same class; if this holds for all the equivalence classes, then $(m_1, m_2) \in R^{\oplus}$. Of course, the complexity of this algorithm is $O(n)$, even for the first check; hence, this algorithm is usually more performant, even if, from the second check onwards, it may be slower when applied to small markings; in fact, in case the number n of places is greater than k^2, then the original algorithm is better. □

3.3. Weak Team Equivalence

Starting from weak bisimilarity over an unmarked FSM, we can define *weak team equivalence* over its markings in a structural, distributed way, as the additive closure of $\approx$, i.e., $\approx^{\oplus}$. Hence, by Proposition 7 weak team equivalent markings have the same size: if $m_1 \approx^{\oplus} m_2$, then $|m_1| = |m_2|$.

Proposition 10. For each FSM $N = (S, A, T)$, relation $\approx^{\oplus} \subseteq \mathscr{M}(S) \times \mathscr{M}(S)$ is an equivalence relation.

PROOF. Since $\approx$ is an equivalence relation, by Proposition 8, $\approx^{\oplus}$ is an equivalence relation, too. □

Note that, once $\approx$ has been computed (e.g., in $O(m \cdot (n+1))$ by adapting the algorithm in [32]), checking whether two markings of size k are weak team equivalent takes only $O(k^2)$ time (or $O(n)$ time). This equivalence checking can be done for any pair of markings, hence reusing the already computed relation $\approx$.

The following theorem provides a characterization of $\approx^{\oplus}$ as a suitable bisimulation-like relation over markings, giving evidence of the fact that weak team equivalence does respect the global behavior of the net. It is interesting to observe that this characterization gives a dynamic interpretation of weak team equivalence as a relation on the global model of the system under scrutiny, while Definition 14 gives a structural definition of weak team equivalence $\approx^{\oplus}$ as the additive closure of the local relation $\approx$ on places.

Theorem 1. Let $N = (S, A, T)$ be an FSM. Two markings m_1 and m_2 are weak team equivalent, $m_1 \approx^{\oplus} m_2$, if and only if $|m_1| = |m_2|$ and

1. $\forall t_1$ such that $l(t_1) \neq \tau$ and $m_1[t_1\rangle m_1'$, $\exists \sigma_2$ such that σ_2 is sequential, ${}^{\bullet}t_1 \approx {}^{\bullet}\sigma_2$, $l(t_1) = o(\sigma_2)$, $t_1^{\bullet} \approx^{\oplus} \sigma_2^{\bullet}$, $m_2[\sigma_2\rangle m_2'$ and $m_1' \approx^{\oplus} m_2'$,

2. $\forall t_1$ such that $l(t_1) = \tau$ and $m_1[t_1\rangle m_1'$, *either* $\exists \sigma_2$ such that σ_2 is nonempty and sequential, $^\bullet t_1 \approx {}^\bullet\sigma_2$, $o(\sigma_2) = \varepsilon$, $t_1^\bullet \approx^\oplus \sigma_2^\bullet$, $m_2[\sigma_2\rangle m_2'$ and $m_1' \approx^\oplus m_2'$, *or* $\exists s_2 \in m_2$ such that $^\bullet t_1 \approx s_2$, $t_1^\bullet \approx s_2$ and $m_1' \approx^\oplus m_2$,
3. $\forall t_2$ such that $l(t_2) \neq \tau$ and $m_2[t_2\rangle m_2'$, $\exists \sigma_1$ such that σ_1 is sequential, $^\bullet\sigma_1 \approx {}^\bullet t_2$, $o(\sigma_1) = l(t_2)$, $\sigma_1^\bullet \approx^\oplus t_2^\bullet$, $m_1[\sigma_1\rangle m_1'$ and $m_1' \approx^\oplus m_2'$,
4. $\forall t_2$ such that $l(t_2) = \tau$ and $m_2[t_2\rangle m_2'$, *either* $\exists \sigma_1$ such that σ_1 is nonempty and sequential, $^\bullet\sigma_1 \approx {}^\bullet t_2$, $o(\sigma_1) = \varepsilon$, $\sigma_1^\bullet \approx^\oplus t_2^\bullet$, $m_1[\sigma_1\rangle m_1'$ and $m_1' \approx^\oplus m_2'$, *or* $\exists s_1 \in m_1$ such that $s_1 \approx {}^\bullet t_2$, $s_1 \approx t_2^\bullet$ and $m_1 \approx^\oplus m_2'$.

PROOF. ($\Rightarrow$) If $m_1 \approx^\oplus m_2$, then $|m_1| = |m_2|$ by Proposition 7. Moreover, for any t_1 such that $m_1[t_1\rangle m_1'$, we have that $m_1 = s_1 \oplus \overline{m}_1$, where $s_1 = {}^\bullet t_1$. As $m_1 \approx^\oplus m_2$, by Definition 14, it follows that there exist s_2 and $\overline{m}_2$ such that $m_2 = s_2 \oplus \overline{m}_2$, $s_1 \approx s_2$ and $\overline{m}_1 \approx^\oplus \overline{m}_2$. Since $s_1 \approx s_2$, by Definition 12, we have to consider two cases for the shape of t_1:

(*i*) if $t_1 = s_1 \xrightarrow{\ell} p_1$, then there exists p_2 such that $s_2 \stackrel{\ell}{\Longrightarrow} p_2$ and either $p_1 = \theta = p_2$ or $p_1 \approx p_2$. This means that for transition t_1, there exists a sequential transition sequence σ_2 such that $o(\sigma_2) = \ell = l(t_1)$, $^\bullet\sigma_2 = s_2$, $\sigma_2^\bullet = p_2$, hence with $^\bullet t_1 \approx {}^\bullet\sigma_2$ and $t_1^\bullet \approx^\oplus \sigma_2^\bullet$. We have to consider two subcases: either $t_1^\bullet = \theta = \sigma_2^\bullet$ or $t_1^\bullet \approx \sigma_2^\bullet$. In the former subcase, $m_1' = \overline{m}_1$ and $m_2' = \overline{m}_2$, and so $m_1' \approx^\oplus m_2'$ because $\overline{m}_1 \approx^\oplus \overline{m}_2$ by assumption. In the latter case, $m_1' = t_1^\bullet \oplus \overline{m}_1$ and $m_2' = \sigma_2^\bullet \oplus \overline{m}_2$, and so $m_1' \approx^\oplus m_2'$ by Definition 14. Hence, this corresponds to item 1 of the bisimulation conditions.

(*ii*) if $t_1 = s_1 \xrightarrow{\tau} p_1$, then there exists p_2 such that $s_2 \stackrel{\varepsilon}{\Longrightarrow} p_2$ and either $p_1 = \theta = p_2$ or $p_1 \approx p_2$. This means that for transition t_1, *either* there exists a nonempty sequential transition sequence σ_2 such that $o(\sigma_2) = \varepsilon$, $^\bullet\sigma_2 = s_2$ and $\sigma_2^\bullet = p_2$, hence with $^\bullet t_1 \approx {}^\bullet\sigma_2$ and $t_1^\bullet \approx^\oplus \sigma_2^\bullet$; *or* s_2 responds by idling, i.e., $^\bullet t_1 \approx s_2$ and $t_1^\bullet \approx s_2$. The *either* case is analogous to the previous one, and so omitted; this ensures the first part of item 2 of the bisimulation conditions. The *or* case, instead, accounts for the second part of item 2: since $m_1' = t_1^\bullet \oplus \overline{m}_1$, $m_2 = s_2 \oplus \overline{m}_2$, $\overline{m}_1 \approx^\oplus \overline{m}_2$ and $t_1^\bullet \approx s_2$, it follows that $m_1' \approx^\oplus m_2$.

The case when m_2 moves first is symmetric, hence omitted. These cases accounts for items 3 and 4 of the bisimulation conditions.

($\Leftarrow$) Let us assume that $|m_1| = |m_2|$ and that the four bisimulation-like conditions hold; then, we prove that $m_1 \approx^\oplus m_2$. First of all, assume that no transition t_1 is enabled at m_1; in such a case, no observable transition is enabled at m_2; in fact, if $m_2[t_2\rangle m_2'$ with $l(t_2) \neq \tau$, then, by the third condition, a nonempty transition sequence σ_1 must be executable at m_1, contradicting the assumption that no transition is enabled at m_1. However, m_2 may enable silent transitions: by the fourth condition, m_1 can reply by idling. This means that each place in m_1 is a deadlock, and similarly each place in m_2 is weakly bisimilar to a deadlock; therefore, all the places in m_1 and m_2 are pairwise weakly bisimilar; hence, the condition $|m_1| = |m_2|$ is enough to ensure that $m_1 \approx^\oplus m_2$.

Now, assume that $m_1[t_1\rangle m_1'$ for some t_1. If $l(t_1) \neq \tau$, then the first condition ensures that there exists a sequential transition sequence σ_2 such that $^\bullet t_1 \approx {}^\bullet\sigma_2$, $l(t_1) = o(\sigma_2)$, $t_1^\bullet \approx^\oplus \sigma_2^\bullet$, $m_2[\sigma_2\rangle m_2'$ and $m_1' \approx^\oplus m_2'$. Note that $t_1^\bullet \approx^\oplus \sigma_2^\bullet$ holds if and only if either $t_1^\bullet = \theta = \sigma_2^\bullet$ or $t_1^\bullet \approx \sigma_2^\bullet$. In the former subcase, we have that $m_1 = {}^\bullet t_1 \oplus m_1'$ and $m_2 = {}^\bullet\sigma_2 \oplus m_2'$, and so $m_1 \approx^\oplus m_2$ by Definition 14. In the latter subcase, we have that $m_1' = t_1^\bullet \oplus \overline{m}_1$, $m_2' = \sigma_2^\bullet \oplus \overline{m}_2$, $m_1 = {}^\bullet t_1 \oplus \overline{m}_1$, $m_2 = {}^\bullet\sigma_2 \oplus \overline{m}_2$. Since $m_1' \approx^\oplus m_2'$ and $t_1^\bullet \approx \sigma_2^\bullet$, it follows that $\overline{m}_1 \approx^\oplus \overline{m}_2$, and so $m_1 \approx^\oplus m_2$, because $^\bullet t_1 \approx {}^\bullet\sigma_2$. The second condition, accounting for the case when $l(t_1) = \tau$, is analogous, and so omitted.

Symmetrically, if we start from a transition t_2 enabled at m_2. □

By the theorem above, it is clear that $\approx^{\oplus}$ is a weak interleaving bisimulation; hence, the following corollary follows trivially.

Corollary 1. (Weak team equivalence is finer than weak interleaving bisimilarity) Let $N = (S, A, T)$ be an FSM. If $m_1 \approx^{\oplus} m_2$, then $m_1 \approx_{int} m_2$. □

Example 12. Weak team equivalence is a truly concurrent equivalence, strictly finer than weak interleaving bisimilarity. Consider the nets in Figure 2. Even if s_1 is weak bisimilar to s_5, s_1 is not weak team equivalent to $s_5 \oplus s_6$, because the size of the two markings is different. On the contrary, s_1 is (weak) interleaving bisimilar to $s_5 \oplus s_6$. As another striking example, consider the CFM terms $a.\mathbf{0} \,|\, b.\mathbf{0}$ and $a.b.\mathbf{0} + b.a.\mathbf{0}$. The nets of these two terms have initial markings $a.\mathbf{0} \oplus b.\mathbf{0}$ and $a.b.\mathbf{0} + b.a.\mathbf{0}$, according to the semantics in Section 5.2: as these two markings have different size, they cannot be (weak) team equivalent, even if they are (weak) interleaving equivalent. □

Example 13. If two markings m_1 and m_2 are weak interleaving bisimilar and have the same size, then they may be not weak team equivalent. For instance, suppose to add an isolated place s_8 to the net in Figure 3. In such a case, $s_1 \oplus s_8$ and $s_5 \oplus s_6$ have the same size, they are weak interleaving bisimilar, but they are not weak team equivalent. □

Example 14. Continuing Example 7 about the nets in Figure 6, it is not difficult to see that, e.g., $s_2 \oplus s_3 \approx^{\oplus} 2 \cdot s_4$ or that $s_1 \oplus s_5 \approx^{\oplus} s_2 \oplus s_6$. On the contrary, $s_1 \oplus s_2 \not\approx^{\oplus} s_4 \oplus s_6$, because, even if $s_1 \approx s_4$, the remaining tokens, s_2 and s_6, are not weakly bisimilar. □

The examples above make clear that two markings m_1 and m_2 are *not* weak team equivalent if either they have different size, or if we can single out a place s_i' in m_1 which has no matching weak bisimilar place in m_2, i.e., there is no weak-bisimulation-preserving bijection among the tokens of the two markings.

3.4. *Rooted Weak Team Equivalence*

We can also define *rooted weak team equivalence* as the additive closure of rooted weak bisimilarity, i.e., $\approx_c^{\oplus}$. Of course, by Proposition 7, rooted weak team equivalence relates markings of the same size only; moreover, $\approx_c^{\oplus}$ is an equivalence relation, by Proposition 8, as $\approx_c$ is an equivalence relation (by Proposition 6).

Proposition 11. (Rooted weak team equivalence is finer than weak team equivalence) Let $N = (S, A, T)$ be an FSM. If $m_1 \approx_c^{\oplus} m_2$, then $m_1 \approx^{\oplus} m_2$.

PROOF. By Proposition 5, we have that $\approx_c \subseteq \approx$. Since the additive closure is monotone (by Proposition 8(2)), the thesis follows trivially. □

The following theorem provides a characterization of rooted weak team equivalence as a suitable bisimulation-like relation over markings, i.e., over a global model of the overall behavior.

Theorem 2. Let $N = (S, A, T)$ be an FSM. If two markings m_1 and m_2 are rooted weak team equivalent, $m_1 \approx_c^{\oplus} m_2$, then $|m_1| = |m_2|$ and

1. $\forall t_1$ such that $m_1[t_1\rangle m_1'$, $\exists \sigma_2$ such that σ_2 is sequential, ${}^\bullet t_1 \approx_c {}^\bullet \sigma_2$, $l(t_1) = o_\tau(\sigma_2)$, $t_1^\bullet \approx^\oplus \sigma_2^\bullet$, $m_2[\sigma_2\rangle m_2'$ and $m_1' \approx^\oplus m_2'$,
2. $\forall t_2$ such that $m_2[t_2\rangle m_2'$, $\exists \sigma_1$ such that σ_1 is sequential, ${}^\bullet \sigma_1 \approx_c {}^\bullet t_2$, $o_\tau(\sigma_1) = l(t_2)$, $\sigma_1^\bullet \approx^\oplus t_2^\bullet$, $m_1[\sigma_1\rangle m_1'$ and $m_1' \approx^\oplus m_2'$.

PROOF. If $m_1 \approx_c^\oplus m_2$, then $|m_1| = |m_2|$ by Proposition 7. Moreover, for any t_1 such that $m_1[t_1\rangle m_1'$, we have that $m_1 = s_1 \oplus \overline{m}_1$, where $s_1 = {}^\bullet t_1$. As $m_1 \approx_c^\oplus m_2$, by Definition 14, it follows that there exist s_2 and $\overline{m}_2$ such that $m_2 = s_2 \oplus \overline{m}_2$, $s_1 \approx_c s_2$ and $\overline{m}_1 \approx_c^\oplus \overline{m}_2$. Since $s_1 \approx_c s_2$, by Definition 13, we have that for transition $t_1 = s_1 \xrightarrow{\ell} p_1$, there must exist p_2 such that $s_2 \stackrel{\ell}{\Longrightarrow} p_2$ and either $p_1 = \theta = p_2$ or $p_1 \approx p_2$. This means that for transition t_1, there exists a sequential transition sequence σ_2 such that $o_\tau(\sigma_2) = \ell = l(t_1)$, ${}^\bullet\sigma_2 = s_2$, $\sigma_2^\bullet = p_2$, hence with ${}^\bullet t_1 \approx_c {}^\bullet \sigma_2$ and $t_1^\bullet \approx^\oplus \sigma_2^\bullet$. We have to consider two subcases: either $t_1^\bullet = \theta = \sigma_2^\bullet$ or $t_1^\bullet \approx \sigma_2^\bullet$. In the former subcase, $m_1' = \overline{m}_1$ and $m_2' = \overline{m}_2$; since $\overline{m}_1 \approx_c^\oplus \overline{m}_2$ by assumption, we also have $\overline{m}_1 \approx^\oplus \overline{m}_2$ by Proposition 11, and so $m_1' \approx^\oplus m_2'$, as required. In the latter case, $m_1' = t_1^\bullet \oplus \overline{m}_1$ and $m_2' = \sigma_2^\bullet \oplus \overline{m}_2$, and so $m_1' \approx^\oplus m_2'$ by Definition 14.

The case when m_2 moves first is symmetric, hence omitted. □

Note that, contrary to Theorem 1, we do not have an *if-and-only-if* condition. As a matter of fact, it is not true that if two markings m_1 and m_2 of the same size are such that they satisfy the two bisimulation conditions of Theorem 2, then they are rooted weak team equivalent. As a counterexample, consider the net in Figure 5 and the two markings $2 \cdot s_1 \oplus s_2$ and $s_1 \oplus 2 \cdot s_2$, which are clearly weak team equivalent. However, since s_1 cannot perform any silent transition, we have that $s_1 \not\approx_c s_2$ and so $2 \cdot s_1 \oplus s_2 \not\approx_c^\oplus s_1 \oplus 2 \cdot s_2$. Nonetheless, the two bisimulation conditions are satisfied for these markings. In one direction, to transition $2 \cdot s_1 \oplus s_2 \xrightarrow{a} 2 \cdot s_1 \oplus s_2$, the other marking can reply with $s_1 \oplus 2 \cdot s_2 \xrightarrow{a} s_1 \oplus 2 \cdot s_2$ with $s_1 \approx_c s_1$; similarly, to transition $2 \cdot s_1 \oplus s_2 \xrightarrow{\tau} 2 \cdot s_1 \oplus s_3$, the other marking can reply with $s_1 \oplus 2 \cdot s_2 \xrightarrow{\tau} s_1 \oplus s_2 \oplus s_3$ with $s_2 \approx_c s_2$, $s_3 \approx s_3$ and $2 \cdot s_1 \oplus s_3 \approx^\oplus s_1 \oplus s_2 \oplus s_3$. Symmetrically, if $s_1 \oplus 2 \cdot s_2$ moves first.

Corollary 2. (Rooted weak team equivalence is finer than rooted weak interleaving bisimilarity) Let $N = (S, A, T)$ be an FSM. If $m_1 \approx_c^\oplus m_2$, then $m_1 \approx_{int}^c m_2$.

PROOF. We want to prove that if $m_1 \approx_c^\oplus m_2$, then

- $\forall t_1$ such that $m_1[t_1\rangle m_1'$, $\exists \sigma_2$ s.t. $m_2[\sigma_2\rangle m_2'$ with $l(t_1) = o_\tau(\sigma_2)$ and $m_1' \approx_{int} m_2'$,
- $\forall t_2$ such that $m_2[t_2\rangle m_2'$, $\exists \sigma_1$ s.t. $m_1[\sigma_1\rangle m_1'$ with $o_\tau(\sigma_1) = l(t_2)$ and $m_1' \approx_{int} m_2'$,

so that $m_1 \approx_{int}^c m_2$ follows directly by Definition 8. However, this implication is obvious, due to Theorem 2 and Corollary 1. □

Not surprisingly, rooted weak team equivalence is strictly finer than rooted weak interleaving bisimilarity. Consider Figure 2. Even if $s_1 \approx_{int}^c s_5 \oplus s_6$, s_1 is not rooted weak team equivalent to $s_5 \oplus s_6$, because the size of the two markings is different.

3.5. Minimizing Nets

In [16], we showed how to compute, for a given FSM N, its reduced net $N_\sim$, i.e., the minimized net according to bisimulation $\sim$ on places (cf. Definition 11), where the places of $N_\sim$ are equivalence classes of the places of N. We proved that this reduction is correct and we argued that $N_\sim$ is really the net with the least number of places exhibiting the same team behavior as N. By means of Proposition 4, we observed that weak bisimilarity on the places of N can be equivalently characterized as (strong) bisimulation on the places of the saturated net N'. Therefore, it is possible to minimize the net N w.r.t. the weak bisimulation equivalence $\approx$ over places by minimizing the saturated net N' w.r.t. $\sim$. Since N and its saturated net N' have the same set of places, the equivalence classes computed over N' w.r.t. $\sim$ are the same equivalence classes over N w.r.t. $\approx$.

Example 15. Consider the net in Figure 7(a). By saturating the net (this saturation is not described in the picture), the equivalence classes w.r.t. $\sim$ are $\{s_1, s_2\}$, $\{s_3\}$ and $\{s_4, s_5\}$. Hence, the reduced net has only three places and is actually isomorphic to (the saturated net originating from) the net in (b). If the initial marking of the original net is $s_1 \oplus s_2 \oplus s_3 \oplus s_4 \oplus s_5$, then the initial marking of the reduced net is $2 \cdot s_6 \oplus s_7 \oplus 2 \cdot s_8$. □

A direct construction of the reduced net w.r.t. $\approx$, which minimizes the number of places and transitions, can be also obtained by adapting, *mutatis mutandis*, the construction in Section 4.4 for the reduced net w.r.t. branching bisimilarity on places $\approx_{br}$.

4. A Distributed Approach to Branching Equivalence Checking

In [12], van Glabbeek and Weijland argued that weak bisimilarity $\approx$ is not completely respecting the timing of choices (the so-called *branching structure* of systems). For instance, consider the two nets in Figure 8. A weak bisimulation is $R = \{(s_1, s_4), (s_2, s_5), (s_3, s_6), (s_3, s_7)\}$, hence $s_1 \approx s_4$ (actually, $s_1 \approx_c s_4$). However, in the net in (a), in each computation the choice between b and c is made after the a-labeled transition, while in the net in (b) there is a computation where c is already discarded by performing a. Hence, it may be argued that the two nets should not be equivalent. A finer notion of equivalence that distinguishes between these two systems is as follows.

4.1. Branching Bisimulation on Places

Definition 15. (Branching bisimulation on places) Let $N = (S, A, T)$ be an FSM. A *branching bisimulation* is a relation $R \subseteq S \times S$ such that if $(s_1, s_2) \in R$ then for all $\ell \in A$

- $\forall m_1$ such that $s_1 \xrightarrow{\ell} m_1$,
  - either $\ell = \tau$ and $\exists m_2$ such that $s_2 \stackrel{\varepsilon}{\Longrightarrow} m_2$ with $(s_1, m_2) \in R$ and $(m_1, m_2) \in R$,
  - or $\exists m, m_2$ such that $s_2 \stackrel{\varepsilon}{\Longrightarrow} m \xrightarrow{\ell} m_2$ with $(s_1, m) \in R$ and either $m_1 = \theta = m_2$ or $(m_1, m_2) \in R$,
- and, symmetrically, $\forall m_2$ such that $s_2 \xrightarrow{\ell} m_2$.

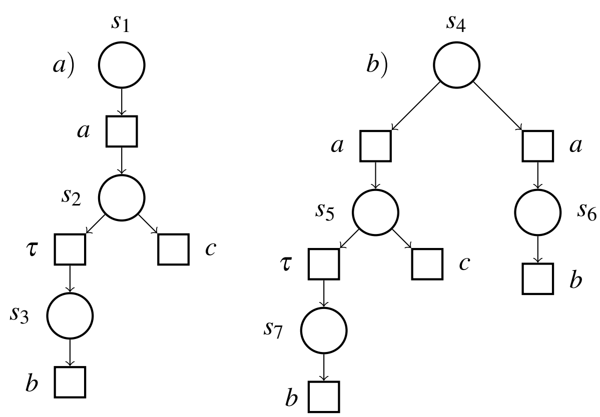

Figure 8: Two weakly bisimilar FSMs, which are not branching bisimilar

Two places s and s' are *branching bisimilar* (or *branching bisimulation equivalent*), denoted by $s \approx_{br} s'$, if there exists a branching bisimulation R such that $(s, s') \in R$. □

This definition is not a rephrasing of the original definition on LTS in [12], rather of a slight variant called *semi-branching bisimulation* [12, 4, 14], which gives rise to the same equivalence relation as the original definition but has better mathematical properties; in particular it ensures [4] that the relational composition of branching bisimulation on places is a branching bisimulation on places (see Proposition 13(3)).

Example 16. Considering Figure 8, note that $s_1 \not\approx_{br} s_4$. In fact, to transition $s_4 \xrightarrow{a} s_6$, place s_1 can only try to respond with $s_1 \xrightarrow{a} s_2$, but s_2 and s_6 are clearly not equivalent, because only s_2 can do c. □

Remark 2. (Stuttering Property) It is not difficult to prove that, given a τ-labeled path $s_1 \xrightarrow{\tau} s_2 \xrightarrow{\tau} \dots s_n \xrightarrow{\tau} s_{n+1}$, if $s_1 \approx_{br} s_{n+1}$, then $s_i \approx_{br} s_j$ for all $i, j = 1, \dots n+1$. This is sometimes called the *stuttering property* [12, 14].

This property justifies the following observation on the nature of branching bisimilarity. As $\approx_{br}$ is a branching bisimulation (by Proposition 14), it satisfies the conditions in Definition 15. Let us consider two branching bisimilar places $s_1 \approx_{br} s_2$. Then, suppose $s_1 \xrightarrow{\tau} m_1$ and that s_2 responds by performing $s_2 \stackrel{\varepsilon}{\Longrightarrow} m_2$ with $s_1 \approx_{br} m_2$ and $m_1 \approx_{br} m_2$. By transitivity of $\approx_{br}$ (by Proposition 15), we have that also $s_2 \approx_{br} m_2$. Hence, by the stuttering property, s_1 is branching bisimilar to each place in the path from s_2 to m_2, and so all the places traversed in the path from s_2 to m_2 are branching bisimilar. Similarly, assume $s_1 \xrightarrow{\ell} m_1$ (with ℓ that can be τ) and that s_2 responds by performing $s_2 \stackrel{\varepsilon}{\Longrightarrow} m \xrightarrow{\ell} m_2$ with $s_1 \approx_{br} m$ and $m_1 \approx_{br} m_2$. By transitivity, $s_2 \approx_{br} m$, hence, by the stuttering property, s_1 is branching bisimilar to each place in the path from s_2 to m. These constraints are not required by weak bisimilarity: given $s_1 \approx s_2$, when matching a transition $s_1 \xrightarrow{\ell} m_1$ with $s_2 \stackrel{\varepsilon}{\Longrightarrow} s_2' \xrightarrow{\mu} m_2 \stackrel{\varepsilon}{\Longrightarrow} m_2'$, weak bisimilarity only requires that $m_1 \approx m_2'$, but does not impose any condition on the intermediate states; in particular, it is not required that $s_1 \approx s_2'$, or that $m_1 \approx m_2$. □

Proposition 12. Let $N = (S, A, T)$ be an FSM. If $s_1 \approx_{br} s_2$, then $s_1 \approx s_2$. □

We now list some useful properties of branching bisimulations, whose proofs are slight adaptation of the original ones for (semi-)branching bisimulation on LTSs [4, 12, 14], and so they are omitted.

Lemma 2. Let $N = (S, A, T)$ be an FSM and let R be a branching bisimulation such that $(s_1, s_2) \in R$. Then, the following hold:

(i) For all m_1 such that $s_1 \stackrel{\varepsilon}{\Longrightarrow} m_1$, there exists m_2 such that $s_2 \stackrel{\varepsilon}{\Longrightarrow} m_2$ and either $m_1 = \theta = m_2$ or $(m_1, m_2) \in R$; and symmetrically,

(ii) For all m_2 such that $s_2 \stackrel{\varepsilon}{\Longrightarrow} m_2$, there exists m_1 such that $s_1 \stackrel{\varepsilon}{\Longrightarrow} m_1$ and either $m_1 = \theta = m_2$ or $(m_1, m_2) \in R$.

PROOF. The proof is by induction on the length of the computation. □

Proposition 13. For each FSM $N = (S, A, T)$, the following hold:

1. the identity relation $\mathscr{I} = \{(s, s) \mid s \in S\}$ is a branching bisimulation;
2. the inverse relation $R^{-1} = \{(s', s) \mid (s, s') \in R\}$ of a branching bisimulation R is a branching bisimulation;
3. the relational composition $R_1 \circ R_2 = \{(s, s'') \mid \exists s'.(s, s') \in R_1 \wedge (s', s'') \in R_2\}$ of two branching bisimulations R_1 and R_2 is a branching bisimulation;
4. the union $\bigcup_{i \in I} R_i$ of branching bisimulations R_i is a branching bisimulation.

PROOF. Standard. For proving case (3), it is necessary to use Lemma 2. □

Remember that $s \approx_{br} s'$ if there exists a branching bisimulation containing the pair (s, s'). This means that $\approx_{br}$ is the union of all branching bisimulations, i.e.,

$$\approx_{br} = \bigcup \{R \subseteq S \times S \mid R \text{ is a branching bisimulation}\}.$$

By Proposition 13(4), $\approx_{br}$ is also a branching bisimulation, hence the largest such relation.

Proposition 14. For each FSM $N = (S, A, T)$, relation $\approx_{br} \subseteq S \times S$ is the largest branching bisimulation relation. □

By Proposition 13(1-3) we deduce that $\approx_{br}$ is an equivalence relation.

Proposition 15. For each FSM $N = (S, A, T)$ with silent moves, relation $\approx_{br} \subseteq S \times S$ is an equivalence relation. □

From a complexity point of view, branching bisimilarity is the easiest bisimulation-based equivalence to decide. According to [18, 12], it can be checked on finite-state LTSs with time complexity $O(l + nm)$ and space complexity $O(n + m)$, where l is the number of labels, n the number of states and m the number of transitions. Recently, an apparently optimal algorithm has been proposed in [19, 21], whose time complexity is $O(m \cdot log\, n)$. Therefore, essentially the same complexity is necessary to compute branching bisimilarity on places of an FSM, with the usual adaptation of counting the empty marking as an additional, dummy place.

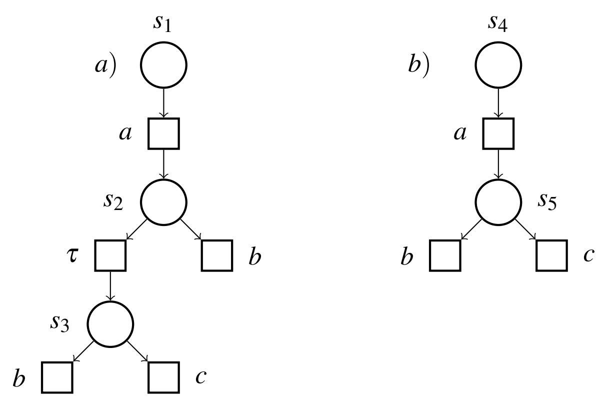

Figure 9: Two branching bisimilar FSMs

Example 17. Consider Figure 4. It is easy to see that $s_1 \approx s_4$. However, $s_1 \not\approx_{br} s_4$, as to transition $s_4 \xrightarrow{a} s_5$, place s_1 can only try to respond with $s_1 \xrightarrow{\tau} s_2 \xrightarrow{a} s_3$, but not all the conditions required are satisfied; in particular, $s_2 \not\approx_{br} s_4$, as only s_4 can do b. □

Example 18. Consider Figure 9. It is easy to see that $R = \{(s_1,s_4),(s_2,s_5),(s_3,s_5)\}$ is a branching bisimulation. To move $s_2 \xrightarrow{\tau} s_3$, place s_5 responds by idling. Note that to move $s_5 \xrightarrow{c} \theta$, place s_2 responds with $s_2 \xrightarrow{\tau} s_3 \xrightarrow{c} \theta$ and, indeed, by performing the τ move, the system passes through branching bisimilar states only, i.e., $s_2 \approx_{br} s_3$. □

Definition 16. (Rooted branching bisimulation on places) Let $N = (S,A,T)$ be an FSM. Two places s_1 and s_2 are *rooted branching bisimilar*, denoted $s_1 \approx_{brc} s_2$, if $\forall \ell \in A$

- for all m_1 such that $s_1 \xrightarrow{\ell} m_1$, there exists m_2 such that $s_2 \xrightarrow{\ell} m_2$ and either $m_1 = \theta = m_2$ or $m_1 \approx_{br} m_2$,
- for all m_2 such that $s_2 \xrightarrow{\ell} m_2$, there exists m_1 such that $s_1 \xrightarrow{\ell} m_1$ and either $m_1 = \theta = m_2$ or $m_1 \approx_{br} m_2$. □

The peculiar feature of $\approx_{brc}$ is that initial moves are matched as in strong bisimulation, while subsequent moves are matched as for branching bisimilarity. Therefore, rooted branching bisimilarity is a slightly finer variant of branching bisimilarity.

Proposition 16. (Rooted branching bisimilarity is finer than branching bisimilarity) Let $N = (S,A,T)$ be an FSM. If $s_1 \approx_{brc} s_2$, then $s_1 \approx_{br} s_2$. □

Proposition 17. (Rooted branching bisimilarity is finer than rooted weak bisimilarity) Let $N = (S,A,T)$ be an FSM. If $s_1 \approx_{brc} s_2$, then $s_1 \approx_c s_2$.

PROOF. It follows directly by Proposition 12. □

Proposition 18. Let $N = (S,A,T)$ be an FSM. Relation $\approx_{brc}$ is an equivalence.

PROOF. Standard. It follows by the fact that $\approx_{br}$ is an equivalence relation. □

Example 19. Considering again Figure 9, we have that $s_1 \approx_{brc} s_4$ because $s_2 \approx_{br} s_5$; however, note that $s_2 \not\approx_{brc} s_5$. □

4.2. Branching Team Equivalence

We can also define *branching team equivalence* as the additive closure of branching bisimilarity, i.e., $\approx_{br}^{\oplus}$. Of course, by Proposition 7, branching team equivalence relates markings of the same size only; moreover, $\approx_{br}^{\oplus}$ is an equivalence relation, by Proposition 8, as $\approx_{br}$ is an equivalence relation (by Proposition 15).

The following theorem provides a characterization of $\approx_{br}^{\oplus}$ as a suitable bisimulation-like relation over markings, i.e., over a global model of the overall behavior.

Theorem 3. Let $N = (S, A, T)$ be an FSM. Two markings m_1 and m_2 are branching team equivalent, $m_1 \approx_{br}^{\oplus} m_2$, if and only if $|m_1| = |m_2|$ and

1. $\forall t_1$ such that $m_1[t_1\rangle m_1'$,
   - either $l(t_1) = \tau$ and
     - (i) either $\exists \sigma_2$ nonempty and sequential, such that $^\bullet t_1 \approx_{br} {}^\bullet\sigma_2$, $o(\sigma_2) = \varepsilon$, $t_1^\bullet \approx_{br} \sigma_2^\bullet$, $^\bullet t_1 \approx_{br} \sigma_2^\bullet$, $m_2[\sigma_2\rangle m_2'$ with $m_1 \approx_{br}^{\oplus} m_2'$ and $m_1' \approx_{br}^{\oplus} m_2'$,
     - (ii) or $\exists s_2 \in m_2$ such that $^\bullet t_1 \approx_{br} s_2$, $t_1^\bullet \approx_{br} s_2$, with $m_1' \approx_{br}^{\oplus} m_2$,
   - or $\exists \sigma, t_2$ such that σt_2 is sequential, $o(\sigma) = \varepsilon$, $l(t_1) = l(t_2)$, $^\bullet t_1 \approx_{br} {}^\bullet\sigma t_2$, $^\bullet t_1 \approx_{br} {}^\bullet t_2$, $t_1^\bullet \approx_{br}^{\oplus} t_2^\bullet$, $m_2[\sigma\rangle m[t_2\rangle m_2'$ with $m_1 \approx_{br}^{\oplus} m$ and $m_1' \approx_{br}^{\oplus} m_2'$;
2. and, symmetrically, $\forall t_2$ such that $m_2[t_2\rangle m_2'$,
   - either $l(t_2) = \tau$ and
     - (i) either $\exists \sigma_1$ nonempty and sequential, such that $^\bullet\sigma_1 \approx_{br} {}^\bullet t_2$, $o(\sigma_1) = \varepsilon$, $\sigma_1^\bullet \approx_{br} t_2^\bullet$, $\sigma_1^\bullet \approx_{br} {}^\bullet t_2$, $m_1[\sigma_1\rangle m_1'$ with $m_1' \approx_{br}^{\oplus} m_2$ and $m_1' \approx_{br}^{\oplus} m_2'$,
     - (ii) or $\exists s_1 \in m_1$ such that $s_1 \approx_{br} {}^\bullet t_2$, $s_1 \approx_{br} t_2^\bullet$, with $m_1 \approx_{br}^{\oplus} m_2'$,
   - or $\exists \sigma, t_1$ such that σt_1 is sequential, $o(\sigma) = \varepsilon$, $l(t_1) = l(t_2)$, $^\bullet\sigma t_1 \approx_{br} {}^\bullet t_2$, $^\bullet t_1 \approx_{br} {}^\bullet t_2$, $t_1^\bullet \approx_{br}^{\oplus} t_2^\bullet$, $m_1[\sigma\rangle m[t_1\rangle m_1'$ with $m \approx_{br}^{\oplus} m_2$ and $m_1' \approx_{br}^{\oplus} m_2'$.

PROOF. ($\Rightarrow$) If $m_1 \approx_{br}^{\oplus} m_2$, then $|m_1| = |m_2|$ by Proposition 7. Moreover, for any t_1 such that $m_1[t_1\rangle m_1'$, we have that $m_1 = s_1 \oplus \overline{m}_1$, where $s_1 = {}^\bullet t_1$. As $m_1 \approx_{br}^{\oplus} m_2$, by Definition 14, it follows that there exist s_2 and $\overline{m}_2$ such that $m_2 = s_2 \oplus \overline{m}_2$, $s_1 \approx_{br} s_2$ and $\overline{m}_1 \approx_{br}^{\oplus} \overline{m}_2$. Since $s_1 \approx_{br} s_2$, by Definition 15, if $t_1 = s_1 \xrightarrow{\ell} p_1$, we have to consider two cases:

(*i*) either $\ell = \tau$ and $\exists p_2$ such that $s_2 \overset{\varepsilon}{\Longrightarrow} p_2$ with $s_1 \approx_{br} p_2$ and $p_1 \approx_{br} p_2$,

(*ii*) or $\exists p, p_2$ such that $s_2 \overset{\varepsilon}{\Longrightarrow} p \xrightarrow{\ell} p_2$, with $s_1 \approx_{br} p$ and either $p_1 \approx_{br} p_2$ or $p_1 = \theta = p_2$.

Case (i): We have to consider two subcases: (a) Either there exists a nonempty sequential transition sequence σ_2 such that $o(\sigma_2) = \varepsilon$, ${}^\bullet\sigma_2 = s_2$, $\sigma_2^\bullet = p_2$, hence with ${}^\bullet t_1 \approx_{br} {}^\bullet\sigma_2$, ${}^\bullet t_1 \approx_{br} \sigma_2^\bullet$ and $t_1^\bullet \approx_{br} \sigma_2^\bullet$. (b) Or s_2 replies by idling, i.e., $p_2 = s_2$; in such a case, ${}^\bullet t_1 \approx_{br} s_2$ and $t_1^\bullet \approx_{br}^{\oplus} s_2$.

In subcase (a), $m_1' = t_1^\bullet \oplus \overline{m}_1$ and $m_2' = \sigma_2^\bullet \oplus \overline{m}_2$, and so $m_1' \approx_{br}^{\oplus} m_2'$ by Definition 14. For the same reason, $m_1 \approx_{br}^{\oplus} m_2'$, as ${}^\bullet t_1 \approx_{br} \sigma_2^\bullet$.

Similarly, in subcase (b), $m_1' = t_1^\bullet \oplus \overline{m}_1$, $m_2 = s_2 \oplus \overline{m}_2$ and so $m_1' \approx_{br}^{\oplus} m_2$.

Case (ii): This means that for transition t_1, there exists a (possibly empty) sequential transition sequence σ and a transition t_2 such that ${}^\bullet\sigma t_2 = s_2$, $o(\sigma) = \varepsilon$, $l(t_1) = l(t_2)$, ${}^\bullet t_2 = p = \sigma^\bullet$, $t_2^\bullet = p_2$, and so ${}^\bullet t_1 \approx_{br} {}^\bullet t_2 = \sigma^\bullet$ and either $t_1^\bullet = \theta = t_2^\bullet$ or $t_1^\bullet \approx_{br} t_2^\bullet$ (hence, $t_1^\bullet \approx_{br}^{\oplus} t_2^\bullet$). Now, $m = p \oplus \overline{m}_2$ and so $m_1 \approx_{br}^{\oplus} m$ by Definition 14. If $t_1^\bullet = \theta = t_2^\bullet$, then $m_1' = \overline{m}_1$ and $m_2' = \overline{m}_2$, hence $m_1' \approx_{br}^{\oplus} m_2'$ as we already know that $\overline{m}_1 \approx_{br}^{\oplus} \overline{m}_2$. Instead, if $t_1^\bullet \approx_{br} t_2^\bullet$, then $m_1' = t_1^\bullet \oplus \overline{m}_1$ and $m_2' = t_2^\bullet \oplus \overline{m}_2$, and so $m_1' \approx_{br}^{\oplus} m_2'$ by Definition 14.

The case when m_2 moves first is symmetric, hence omitted.

$(\Leftarrow)$ Let us assume that $|m_1| = |m_2|$ and that the bisimulation-like conditions hold; then, we prove that $m_1 \approx_{br}^{\oplus} m_2$. First of all, assume that no transition t_1 is enabled at m_1; in such a case, no observable transition is enabled at m_2; in fact, if $m_2[t_2\rangle m_2'$ with $l(t_2) \neq \tau$, then, by the (2-or) condition, a nonempty, sequential transition sequence σt_1 must be executable at m_1, contradicting the assumption that no transition is enabled at m_1. However, m_2 may enable silent transitions: by the (2-either-(ii)) condition, m_1 can reply by idling. This means that each place in m_1 is a deadlock, and similarly each place in m_2 is branching bisimilar to a deadlock; therefore, all the places in m_1 and m_2 are pairwise branching bisimilar; hence, the condition $|m_1| = |m_2|$ is enough to ensure that $m_1 \approx_{br}^{\oplus} m_2$.

Now, assume that $m_1[t_1\rangle m_1'$ for some t_1. Let us consider first the (1-either) condition, i.e., with $l(t_1) = \tau$. This case is actually composed of two subcases.

In subcase (i), we know that there exists a nonempty sequential transition sequence σ_2 such that ${}^\bullet t_1 \approx_{br} {}^\bullet\sigma_2$, $o(\sigma_2) = \varepsilon$, $t_1^\bullet \approx_{br} \sigma_2^\bullet$, $m_2[\sigma_2\rangle m_2'$ and $m_1' \approx_{br}^{\oplus} m_2'$. Therefore, we have that $m_1' = t_1^\bullet \oplus \overline{m}_1$, $m_2' = \sigma_2^\bullet \oplus \overline{m}_2$, $m_1 = {}^\bullet t_1 \oplus \overline{m}_1$, $m_2 = {}^\bullet\sigma_2 \oplus \overline{m}_2$. Since $m_1' \approx_{br}^{\oplus} m_2'$ and $t_1^\bullet \approx_{br} \sigma_2^\bullet$, it follows that $\overline{m}_1 \approx_{br}^{\oplus} \overline{m}_2$, and so $m_1 \approx_{br}^{\oplus} m_2$, because ${}^\bullet t_1 \approx_{br} {}^\bullet\sigma_2$.

In subcase (ii), we have that $\exists s_2 \in m_2$ such that ${}^\bullet t_1 \approx_{br} s_2$, $t_1^\bullet \approx_{br} s_2$, with $m_1' \approx_{br}^{\oplus} m_2$. Note that $m_1' = t_1^\bullet \oplus \overline{m}_1$, $m_1 = {}^\bullet t_1 \oplus \overline{m}_1$ and $m_2 = s_2 \oplus \overline{m}_2$. Since $m_1' \approx_{br}^{\oplus} m_2$ and $t_1^\bullet \approx_{br} s_2$, it follows that $\overline{m}_1 \approx_{br}^{\oplus} \overline{m}_2$, and so $m_1 \approx_{br}^{\oplus} m_2$, because ${}^\bullet t_1 \approx_{br} s_2$.

Let us now consider the (1-or) condition. This means that $\exists \sigma, t_2$ such that σt_2 is sequential, $o(\sigma) = \varepsilon$, $l(t_1) = l(t_2)$, ${}^\bullet t_1 \approx_{br} {}^\bullet\sigma t_2$, ${}^\bullet t_1 \approx_{br} {}^\bullet t_2$, $t_1^\bullet \approx_{br}^{\oplus} t_2^\bullet$, $m_2[\sigma\rangle m[t_2\rangle m_2'$ with $m_1 \approx_{br}^{\oplus} m$ and $m_1' \approx_{br}^{\oplus} m_2'$. Note that $m_1 = {}^\bullet t_1 \oplus \overline{m}_1$, $m_1' = t_1^\bullet \oplus \overline{m}_1$, $m_2 = {}^\bullet\sigma t_2 \oplus \overline{m}_2$ and $m_2' = t_2^\bullet \oplus \overline{m}_2$. We have to consider two subcases: $t_1^\bullet \approx_{br}^{\oplus} t_2^\bullet$ holds if either $t_1^\bullet = \theta = t_2^\bullet$ or $t_1^\bullet \approx_{br} t_2^\bullet$. In the former subcase, $m_1' = \overline{m}_1$ and $m_2' = \overline{m}_2$, and so $m_1 \approx_{br}^{\oplus} m_2$ because ${}^\bullet t_1 \approx_{br} {}^\bullet t_2$. In the latter subcase, since $m_1' \approx_{br}^{\oplus} m_2'$ and $t_1^\bullet \approx_{br} t_2^\bullet$, it follows that $\overline{m}_1 \approx_{br}^{\oplus} \overline{m}_2$; moreover, since ${}^\bullet t_1 \approx_{br} {}^\bullet\sigma t_2$, it follows that $m_1 \approx_{br}^{\oplus} m_2$.

Symmetrically, if we start from a transition t_2 enabled at m_2. □

By the theorem above, it is clear that $\approx_{br}^{\oplus}$ is a branching interleaving bisimulation. As a matter of fact, let us just consider the first *either* case: since ${}^\bullet\sigma_2 \approx_{br} \sigma_2^\bullet$, then not only $m_2 \approx_{br} m_2'$, but also all the intermediate markings in the path $m_2[\sigma_2\rangle m_2'$ must be branching team equivalent. Hence, the following corollary follows trivially.

Corollary 3. (Branching team equivalence is finer than branching interleaving bisimilarity) Let $N = (S, A, T)$ be an FSM. If $m_1 \approx_{br}^{\oplus} m_2$, then $m_1 \approx_{bri} m_2$. □

Example 20. The containment in the above corollary is strict. Consider the nets in Figure 3. Clearly, the markings s_1 and $s_5 \oplus s_6$ are branching interleaving bisimilar, but not branching team equivalent, as the two markings have different size. □

4.3. Rooted Branching Team Equivalence

We can also define *rooted branching team equivalence* as the additive closure of rooted branching bisimilarity, i.e., $\approx_{brc}^{\oplus}$. Of course, rooted branching team equivalence relates markings of the same size only; moreover, $\approx_{brc}^{\oplus}$ is an equivalence relation, by Proposition 8, as $\approx_{brc}$ is an equivalence relation (by Proposition 18).

Proposition 19. (Rooted branching team equivalence is finer than branching team equivalence) Let $N = (S, A, T)$ be an FSM. If $m_1 \approx_{brc}^{\oplus} m_2$, then $m_1 \approx_{br}^{\oplus} m_2$.

PROOF. By Proposition 16, we have that $\approx_{brc} \subseteq \approx_{br}$. Since the additive closure is monotone (by Proposition 8(4)), the thesis follows trivially. □

The following theorem provides a characterization of rooted branching team equivalence as a suitable bisimulation-like relation over markings, i.e., over a global model of the overall behavior.

Theorem 4. Let $N = (S, A, T)$ be an FSM. If two markings m_1 and m_2 are rooted branching team equivalent, $m_1 \approx_{brc}^{\oplus} m_2$, then $|m_1| = |m_2|$ and

1. $\forall t_1$ such that $m_1[t_1\rangle m_1'$, $\exists t_2$ such that $^\bullet t_1 \approx_{brc} {}^\bullet t_2$, $l(t_1) = l(t_2)$, $t_1^\bullet \approx_{br}^{\oplus} t_2^\bullet$, $m_2[t_2\rangle m_2'$ and $m_1' \approx_{br}^{\oplus} m_2'$,
2. $\forall t_2$ such that $m_2[t_2\rangle m_2'$, $\exists t_1$ such that $^\bullet t_1 \approx_{brc} {}^\bullet t_2$, $l(t_1) = l(t_2)$, $t_1^\bullet \approx_{br}^{\oplus} t_2^\bullet$, $m_1[t_1\rangle m_1'$ and $m_1' \approx_{br}^{\oplus} m_2'$.

PROOF. If $m_1 \approx_{brc}^{\oplus} m_2$, then $|m_1| = |m_2|$ by Proposition 7. Moreover, for any t_1 such that $m_1[t_1\rangle m_1'$, we have that $m_1 = s_1 \oplus \overline{m}_1$, where $s_1 = {}^\bullet t_1$. As $m_1 \approx_{brc}^{\oplus} m_2$, by Definition 14, it follows that there exist s_2 and $\overline{m}_2$ such that $m_2 = s_2 \oplus \overline{m}_2$, $s_1 \approx_{brc} s_2$ and $\overline{m}_1 \approx_{brc}^{\oplus} \overline{m}_2$. Since $s_1 \approx_{brc} s_2$, by Definition 16, we have that for transition $t_1 = s_1 \xrightarrow{\ell} p_1$, there must exist p_2 such that $s_2 \xrightarrow{\ell} p_2$ and either $p_1 = \theta = p_2$ or $p_1 \approx_{br} p_2$. This means that for transition t_1, there exists a transition t_2 such that $l(t_2) = \ell = l(t_1)$, ${}^\bullet t_2 = s_2$, $t_2^\bullet = p_2$, hence with ${}^\bullet t_1 \approx_{brc} {}^\bullet t_2$ and $t_1^\bullet \approx_{br}^{\oplus} t_2^\bullet$.

We have to consider two subcases: either $t_1^\bullet = \theta = t_2^\bullet$ or $t_1^\bullet \approx_{br} t_2^\bullet$. In the former subcase, $m_1' = \overline{m}_1$ and $m_2' = \overline{m}_2$; since $\overline{m}_1 \approx_{brc}^{\oplus} \overline{m}_2$ by assumption, we have $\overline{m}_1 \approx_{br}^{\oplus} \overline{m}_2$ by Proposition 19, and so $m_1' \approx_{br}^{\oplus} m_2'$, as required. In the latter case, $m_1' = t_1^\bullet \oplus \overline{m}_1$ and $m_2' = t_2^\bullet \oplus \overline{m}_2$, and so $m_1' \approx_{br}^{\oplus} m_2'$ by Definition 14.

The case when m_2 moves first is symmetric, hence omitted. □

Note that, contrary to Theorem 3, we do not have an *if-and-only-if* condition: the same example discussed after Theorem 2 explains, *mutatis mutandis*, this fact.

Corollary 4. (Rooted branching team equivalence is finer than rooted branching interleaving bisimilarity) Let $N = (S, A, T)$ be a finite-state machine. If $m_1 \approx^{\oplus}_{brc} m_2$, then $m_1 \approx^{c}_{bri} m_2$.

PROOF. We want to prove that if $m_1 \approx^{\oplus}_{brc} m_2$, then

- $\forall t_1$ such that $m_1[t_1\rangle m_1'$, $\exists t_2$ s.t. $m_2[t_2\rangle m_2'$ with $l(t_1) = l(t_2)$ and $m_1' \approx_{bri} m_2'$,
- $\forall t_2$ such that $m_2[t_2\rangle m_2'$, $\exists t_1$ s.t. $m_1[t_1\rangle m_1'$ with $l(t_1) = l(t_2)$ and $m_1' \approx_{bri} m_2'$,

so that $m_1 \approx^{c}_{bri} m_2$ follows directly by Definition 8. However, this implication is obvious, due to Theorem 4 and Corollary 3. □

Not surprisingly, $\approx^{\oplus}_{brc}$ is strictly finer than rooted branching interleaving bisimilarity. Consider the nets in Figure 2. Even if $s_1 \approx^{c}_{bri} s_5 \oplus s_6$, s_1 is not rooted branching team equivalent to $s_5 \oplus s_6$, because the size of the two markings is different.

4.4. Minimization

Definition 17. (Reduced net) Let $N = (S, A, T)$ be an FSM and let $\approx_{br}$ be the branching bisimulation equivalence relation over its places. The *reduced* net $N_{br} = (S_{br}, A, T_{br})$ is defined as follows:

- $S_{br} = \{[s] \mid s \in S\}$, where $[s] = \{s' \in S \mid s \approx_{br} s'\}$;
- $T_{br} = \{([s], \ell, [m]) \mid (s, \ell, m) \in T, \ell \neq \tau\} \cup \{([s], \tau, [m]) \mid (s, \tau, m) \in T, [s] \neq [m]\}$,

where $[m]$ is defined as: $[\theta] = \theta$ and $[m_1 \oplus m_2] = [m_1] \oplus [m_2]$. If N has initial marking $m_0 = k_1 \cdot s_1 \oplus \ldots \oplus k_n \cdot s_n$, then N_{br} has initial marking $[m_0] = k_1 \cdot [s_1] \oplus \ldots \oplus k_n \cdot [s_n]$. □

Lemma 3. Let $N = (S, A, T)$ be an FSM and let $N_{br} = (S_{br}, A, T_{br})$ be its reduced net w.r.t. $\approx_{br}$. Relation $R = \{(s, [s]) \mid s \in S\}$ is a branching bisimulation.

PROOF. If $s \xrightarrow{\ell} m$ with $\ell \neq \tau$, then also $[s] \xrightarrow{\ell} [m]$ by definition of T_{br}; if $m = \theta$, then also $[\theta] = \theta$ and so the branching bisimulation condition is satisfied; otherwise, if $m = s'$, then $(s', [s']) \in R$, as required. If $s \xrightarrow{\tau} m$ and $[s] = [m]$, then $[s]$ replies by idling, and $(m, [s]) \in R$, because $[s] = [m]$. Finally, if $s \xrightarrow{\tau} m$ and $[s] \neq [m]$, then $[s] \xrightarrow{\tau} [m]$ by definition of T_{br}; if $m = \theta$, then also $[\theta] = \theta$ and so the branching bisimulation condition is satisfied; otherwise, if $m = s'$, then $(s', [s']) \in R$, as required.

The case when $[s]$ moves first is slightly more complex for the freedom in choosing the representative in an equivalence class. Transition $[s] \xrightarrow{\ell} [m]$ is possible, by Definition of T_{br}, if there exist $s' \in [s]$ and $m' \in [m]$ such that $s' \xrightarrow{\ell} m'$; as $s \approx_{br} s'$ and $s' \xrightarrow{\ell} m'$, then

- either $\ell = \tau$ and $\exists p_1$ such that $s \stackrel{\varepsilon}{\Longrightarrow} p_1$ with $p_1 \approx_{br} s'$ and $p_1 \approx_{br} m'$,
- or there exist $\overline{s}, m_1$ such that $s \stackrel{\varepsilon}{\Longrightarrow} \overline{s} \xrightarrow{\ell} m_1$ with $\overline{s} \approx_{br} s'$ and either $m_1 = \theta = m'$ or $m_1 \approx_{br} m'$.

Summing up, if $[s] \xrightarrow{\ell} [m]$, then

- either $\ell = \tau$ and $\exists p_1$ such that $s \stackrel{\varepsilon}{\Longrightarrow} p_1$ with $p_1 \in [s']$ and $p_1 \in [m']$ (i.e., with $(p_1, [s]) \in R$ and $(p_1, [m]) \in R$, because $[s] = [s'] = [p_1] = [m'] = [m]$);
- or $\exists \overline{s}, m_1$ such that $s \stackrel{\varepsilon}{\Longrightarrow} \overline{s} \xrightarrow{\ell} m_1$ with $\overline{s} \in [s']$ (i.e., with $(\overline{s}, [s]) \in R$, as $[s] = [s'] = [\overline{s}]$) and $m_1 \in [m']$ (i.e., with either $m_1 = \theta = [m]$ or $(m_1, [m]) \in R$, as $[m_1] = [m'] = [m]$).

Hence, $R = \{(s, [s]) \mid s \in S\}$ is a branching bisimulation. □

Theorem 5. Let $N = (S, A, T)$ be an FSM and let $N_{br} = (S_{br}, A, T_{br})$ be its reduced net w.r.t. $\approx_{br}$. For any $m \in \mathscr{M}(S)$, we have that $m \approx_{br}^{\oplus} [m]$.

PROOF. By induction on the size of m. If $m = \theta$, then $[m] = \theta$ and the thesis follows trivially. If $m = s \oplus m'$, then $[m] = [s] \oplus [m']$; by Lemma 3, $s \approx_{br} [s]$ and, by induction, $m' \approx_{br}^{\oplus} [m']$; therefore, by the rule in Definition 14, $m \approx_{br}^{\oplus} [m]$. □

As a consequence of this theorem, we would like to point out that the reduced net w.r.t. $\approx_{br}$ is indeed the *least* net offering the same branching team behavior as the original net: no further fusion of places can be done, as there are not two places in the reduced net which are branching bisimilar. Moreover, silent transitions relating branching bisimilar places in the original net do not generate any silent transition in the reduced net, so that the number of transitions is minimized, too.

As an example, consider the net in Figure 7(a). The equivalence classes w.r.t. $\approx_{br}$ are $\{s_1, s_2\}$, $\{s_3\}$ and $\{s_4, s_5\}$. Hence, the reduced net has only three places and is actually isomorphic to the net in (b). Note that the transitions $s_1 \xrightarrow{\tau} s_2$, $s_2 \xrightarrow{\tau} s_1$ and $s_4 \xrightarrow{\tau} s_5$, which connect branching bisimilar places, do not originate any silent transition in the reduced net.

5. CFM: Syntax and Net Semantics

Now we define the process algebra CFM [15] (where CFM is the acronym of *Concurrent Finite-state Machines*) that truly represents FSMs.

5.1. Syntax

Let $\mathscr{A}$ be a finite set of *observable* actions, ranged over by $a, b, c \ldots$. Let τ be the invisible action. Let $Act = \mathscr{A} \cup \{\tau\}$ be the finite set of actions, ranged over by μ, and let $\mathscr{C}$ be a finite set of constants, disjoint from Act, ranged over by $A, B, C, \ldots$. The size of the sets Act and $\mathscr{C}$ is not important: we assume that they can be chosen are as large as needed. The CFM *terms* are generated from actions and constants by the following abstract syntax (using three syntactic categories):

$$
\begin{array}{lclllll}
s & ::= & \mathbf{0} & \mid & \mu.q & \mid & s+s \qquad \textit{guarded processes} \\
q & ::= & s & \mid & C & & \qquad \textit{sequential processes} \\
p & ::= & q & \mid & p \mid p & & \qquad \textit{parallel processes}
\end{array}
$$

$dec(\mathbf{0})$	$=$	θ	$dec(\mu.p)$	$=$	$\{\mu.p\}$
$dec(p+p')$	$=$	$\{p+p'\}$	$dec(C)$	$=$	$\{C\}$
$dec(p \mid p')$	$=$	$dec(p) \oplus dec(p')$			

Table 1: Decomposition function

where $\mathbf{0}$ is the empty process, $\mu.q$ is a process where action μ prefixes the residual q ($\mu.-$ is the *action prefixing* operator), $s_1 + s_2$ denotes the alternative composition of s_1 and s_2 ($- + -$ is the *choice* operator), $p_1 \mid p_2$ denotes the asynchronous parallel composition of p_1 and p_2 and C is a constant. A constant C may be equipped with a definition, but this must be a guarded process, i.e., $C \doteq s$. A term p is a CFM *process* if each constant in $Const(p)$ (the set of constants used by p; see below for details) is equipped with a defining equation (in category s). The set of CFM processes is denoted by $\mathscr{P}_{CFM}$, the set of its sequential processes, i.e., those in syntactic category q, by $\mathscr{P}^{seq}_{CFM}$ and the set of its guarded processes, i.e., those in syntactic category s, by $\mathscr{P}^{grd}_{CFM}$.

By $Const(p)$ we denote the set of process constants $\delta(p, \emptyset)$, where the auxiliary function δ, which has, as an additional parameter, a set I of already known constants, is defined as follows:

$$\begin{array}{rcl}
\delta(\mathbf{0}, I) & = & \emptyset, \\
\delta(\mu.p, I) & = & \delta(p, I), \\
\delta(p_1 + p_2, I) & = & \delta(p_1, I) \cup \delta(p_2, I), \\
\delta(p_1 \mid p_2, I) & = & \delta(p_1, I) \cup \delta(p_2, I), \\
\delta(C, I) & = & \begin{cases} \emptyset & C \in I, \\ \{C\} & C \notin I \wedge C \text{ undefined}, \\ \{C\} \cup \delta(p, I \cup \{C\}) & C \notin I \wedge C \doteq p. \end{cases}
\end{array}$$

5.2. Net Semantics

The net for CFM, originally described in [15], has the set of the sequential CFM processes (without $\mathbf{0}$) as the set S_{CFM} of places, i.e., $S_{CFM} = \mathscr{P}^{seq}_{CFM} \setminus \{\mathbf{0}\}$. The decomposition function, mapping process terms to markings, is $dec : \mathscr{P}_{CFM} \to \mathscr{M}(S_{CFM})$, defined in Table 1. An easy induction proves that for any $p \in \mathscr{P}_{CFM}$, $dec(p)$ is a finite multiset of sequential processes. Note that, if $C \doteq \mathbf{0}$, then $\theta = dec(\mathbf{0}) \neq dec(C) = \{C\}$.

Now we provide a construction of the net system $[\![p]\!]_\emptyset$ associated with process p, which is compositional and denotational in style. The details of the construction are outlined in Table 2. The mapping is parametrized by a set of constants that have already been found while scanning p; such a set is initially empty and it is used to avoid looping on recursive constants. The definition is syntax driven and also the places of the constructed net are syntactic objects, i.e., CFM sequential process terms. For instance, the net system $[\![a.\mathbf{0}]\!]_\emptyset$ is a net composed of one single marked place, namely term $a.\mathbf{0}$, and one single transition $(\{a.\mathbf{0}\}, a, \theta)$. A bit of care is needed in the rule for choice: in order to include only strictly necessary places and transitions, the initial place p_1 (or p_2) of the subnet $[\![p_1]\!]_I$ (or $[\![p_2]\!]_I$) is to be kept in the net for $p_1 + p_2$ only if there exists a transition reaching place p_1 (or p_2) in $[\![p_1]\!]_I$ (or $[\![p_2]\!]_I$), otherwise p_1 (or p_2) can be safely removed in the new net. Similarly, for the rule for constants.

$[\![\mathbf{0}]\!]_I = (\emptyset,\emptyset,\emptyset,\theta)$	
$[\![\mu.p]\!]_I = (S,A,T,\{\mu.p\})$	given $[\![p]\!]_I = (S',A',T',dec(p))$ and where $S = \{\mu.p\} \cup S'$, $A = \{\mu\} \cup A'$, $T = \{(\{\mu.p\},\mu,dec(p))\} \cup T'$
$[\![p_1 + p_2]\!]_I = (S,A,T,\{p_1+p_2\})$	given $[\![p_i]\!]_I = (S_i,A_i,T_i,dec(p_i))$ for $i = 1,2$, $S = \{p_1+p_2\} \cup S'_1 \cup S'_2$, with, for $i = 1,2$, $S'_i = \begin{cases} S_i & \exists t \in T_i.\ t^\bullet(p_i) > 0 \\ S_i \setminus \{p_i\} & \text{otherwise} \end{cases}$ $A = A_1 \cup A_2$, $T = T' \cup T'_1 \cup T'_2$, with, for $i = 1,2$, $T'_i = \begin{cases} T_i & \text{if } \exists t \in T_i\,.t^\bullet(p_i) > 0 \\ T_i \setminus \{t \in T_i \mid {}^\bullet t(p_i) > 0\} & \text{otherwise} \end{cases}$ $T' = \{(\{p_1+p_2\},\mu,m) \mid (\{p_i\},\mu,m) \in T_i, i = 1,2\}$
$[\![C]\!]_I = (\{C\},\emptyset,\emptyset,\{C\})$	if $C \in I$
$[\![C]\!]_I = (S,A,T,\{C\})$	if $C \notin I$, given $C \doteq p$ and $[\![p]\!]_{I\cup\{C\}} = (S',A',T',dec(p))$ $A = A', S = \{C\} \cup S''$, where $S'' = \begin{cases} S' & \exists t \in T'\,.t^\bullet(p) > 0 \\ S' \setminus \{p\} & \text{otherwise} \end{cases}$ $T = \{(\{C\},\mu,m) \mid (\{p\},\mu,m) \in T'\} \cup T''$ where $T'' = \begin{cases} T' & \text{if } \exists t \in T'\,.t^\bullet(p) > 0 \\ T' \setminus \{t \in T' \mid {}^\bullet t(p) > 0\} & \text{otherwise} \end{cases}$
$[\![p_1 \mid p_2]\!]_I = (S,A,T,m_0)$	given $[\![p_i]\!]_I = (S_i,A_i,T_i,m_i)$ for $i = 1,2$, and where $S = S_1 \cup S_2$, $A = A_1 \cup A_2$, $T = T_1 \cup T_2$, $m_0 = m_1 \oplus m_2$

Table 2: Denotational net semantics

Example 21. Consider constant $B \doteq b.A$, where $A \doteq a.b.A$. By using the definitions in Table 2, $[\![A]\!]_{\{A,B\}} = (\{A\},\emptyset,\emptyset,\{A\})$. Then, by action prefixing,

$[\![b.A]\!]_{\{A,B\}} = (\{b.A,A\},\{b\},\{(\{b.A\},b,\{A\})\},\{b.A\})$. Again, by action prefixing,

$[\![a.b.A]\!]_{\{A,B\}} = (\{a.b.A,b.A,A\},\{a,b\},\{(\{a.b.A\},a,\{b.A\}),(\{b.A\},b,\{A\})\},\{a.b.A\})$.

Now, the rule for constants ensures that

$[\![A]\!]_{\{B\}} = (\{b.A,A\},\{a,b\},\{(\{A\},a,\{b.A\}),(\{b.A\},b,\{A\})\},\{A\})$.

Note that place $a.b.A$ has been removed, as no transition in $[\![a.b.A]\!]_{\{A,B\}}$ reaches that place. By action prefixing,

$[\![b.A]\!]_{\{B\}} = (\{b.A,A\},\{a,b\},\{(\{A\},a,\{b.A\}),(\{b.A\},b,\{A\})\},\{b.A\})$,

i.e., this operation changes only the initial marking, but does not affect the underlying net! Finally,

$[\![B]\!]_\emptyset = (\{B,b.A,A\},\{a,b\},\{(\{B\},b,\{A\}),(\{A\},a,\{b.A\}),(\{b.A\},b,\{A\})\},\{B\})$.

Note that place $b.A$ has been kept, because there is a transition in the net $[\![b.A]\!]_{\{B\}}$ that reaches that place. □

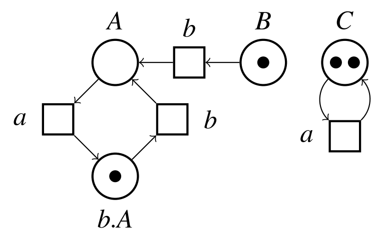

Figure 10: The concurrent finite-state machine for $B\,|\,b.A\,|\,C\,|\,C$ of Example 22

Example 22. Consider the CFM process $B\,|\,b.A\,|\,C\,|\,C$, where $B \doteq b.A$, $A \doteq a.b.A$ and $C \doteq a.C$. The nets for the processes $b.A$ and B are described in Example 21. The concurrent FSM associated with $B\,|\,b.A$ is
$[\![B\,|\,b.A]\!]_\emptyset =$
$= (\{B, b.A, A\}, \{a, b\}, \{(\{B\}, b, \{A\}), (\{A\}, a, \{b.A\}), (\{b.A\}, b, \{A\})\}, \{B, b.A\})$,
where the only addition to the net for B is one token in place $b.A$. The sequential FSM for C is $[\![C]\!]_\emptyset = (\{C\}, \{a\}, \{(\{C\}, a, \{C\})\}, \{C\})$.
The concurrent FSM for $C\,|\,C$ is $[\![C\,|\,C]\!]_\emptyset = (\{C\}, \{a\}, \{(\{C\}, a, \{C\})\}, \{C, C\})$.
And the whole net for $B\,|\,b.A\,|\,C\,|\,C$ is $[\![B\,|\,b.A\,|\,C\,|\,C]\!]_\emptyset = (S, A, T, m_0)$, where

$$\begin{array}{rcl} S & = & \{B, b.A, A, C\}, \\ A & = & \{a, b\}, \\ T & = & \{(\{B\}, b, \{A\}), (\{A\}, a, \{b.A\}), (\{b.A\}, b, \{A\}), (\{C\}, a, \{C\})\}, \\ m_0 & = & \{B, b.A, C, C\}. \end{array}$$

The resulting net is depicted in Figure 10. □

We now list some properties of the semantics, whose proofs are in [15], which state that CFM really represents the class of FSMs.

Theorem 6. (Only Concurrent FSMs) For each CFM process p, $[\![p]\!]_\emptyset$ is a concurrent finite-state machine. □

Definition 18. (Translating Concurrent FSMs into CFM Process Terms) Let $N(m_0)$ $= (S, A, T, m_0)$ — with $S = \{s_1, \ldots, s_n\}$, $A \subseteq Act$, $T = \{t_1, \ldots, t_k\}$, and $l(t_j) = \mu_j$ — be a concurrent finite-state machine. Function $\mathscr{T}_{CFM}(-)$, from concurrent finite-state machines to CFM processes, is defined as

$$\mathscr{T}_{CFM}(N(m_0)) = \underbrace{C_1|\cdots|C_1}_{m_0(s_1)}|\cdots|\underbrace{C_n|\cdots|C_n}_{m_0(s_n)}$$

where each C_i is equipped with a defining equation $C_i \doteq c_i^1 + \cdots + c_i^k$ (with $C_i \doteq \mathbf{0}$ if $k = 0$), and each summand c_i^j, for $j = 1, \ldots, k$, is equal to

- $\mathbf{0}$, if $s_i \not\in {}^\bullet t_j$;
- $\mu_j.\mathbf{0}$, if ${}^\bullet t_j = \{s_i\}$ and $t_j^\bullet = \emptyset$;
- $\mu_j.C_h$, if ${}^\bullet t_j = \{s_i\}$ and $t_j^\bullet = \{s_h\}$. □

Theorem 7. **(All Concurrent FSMs)** Let $N(m_0) = (S, A, T, m_0)$ be a dynamically reduced, concurrent finite-state machine such that $A \subseteq Act$, and let $p = \mathscr{T}_{CFM}(N(m_0))$. Then, $[\![p]\!]_\emptyset$ is isomorphic to $N(m_0)$. □

6. Congruence and Algebraic Properties

Thanks to the theorems of the previous section, we can transfer the definitions of the various team-based bisimulation equivalences from FSM nets to CFM process terms in a simple way.

Definition 19. Two CFM processes p and q are (strong) team bisimilar, denoted by $p \sim^\oplus q$, if, by considering the (union of the) nets $[\![p]\!]_\emptyset$ and $[\![q]\!]_\emptyset$, $dec(p) \sim^\oplus dec(q)$ holds. In the same way, we can define all the other team equivalences; for instance, $p \approx_c^\oplus q$ if $dec(p) \approx_c^\oplus dec(q)$ and $p \approx_{brc}^\oplus q$ if $dec(p) \approx_{brc}^\oplus dec(q)$. □

Of course, for sequential CFM processes, strong team equivalence $\sim^\oplus$ coincides with strong bisimilarity on places $\sim$. The same is true for rooted weak team equivalence $\approx_c^\oplus$ (rooted branching team equivalence $\approx_{brc}^\oplus$), which coincides with rooted weak bisimilarity $\approx_c$ (rooted branching bisimilarity $\approx_{brc}$) for sequential terms.

Thanks to Definition 19, we can now perform the usual process algebraic study of a behavioral equivalence: to prove that it is a congruence for the operators of the CFM process algebra and to study its algebraic properties. These will be the subject of the next subsections.

6.1. Congruence

We first prove that strong/weak/branching bisimilarities are congruences for action prefixing.

Proposition 20. 1) For each $p, q \in \mathscr{P}_{CFM}^{seq}$, if $p \sim q$ (or $p = q = \mathbf{0}$), then $\mu.p \sim \mu.q$ for all $\mu \in Act$.
2) For each $p, q \in \mathscr{P}_{CFM}^{seq}$, if $p \approx q$ (or $p = q = \mathbf{0}$), then $\mu.p \approx \mu.q$ for all $\mu \in Act$.
3) For each $p, q \in \mathscr{P}_{CFM}^{seq}$, if $p \approx_{br} q$ (or $p = q = \mathbf{0}$), then $\mu.p \approx_{br} \mu.q$ for all $\mu \in Act$.
4) For each $p, q \in \mathscr{P}_{CFM}^{seq}$, if $p \approx_c q$ (or $p = q = \mathbf{0}$), then $\mu.p \approx_c \mu.q$ for all $\mu \in Act$.
5) For each $p, q \in \mathscr{P}_{CFM}^{seq}$, if $p \approx_{brc} q$ (or $p = q = \mathbf{0}$), then $\mu.p \approx_{brc} \mu.q$ for all $\mu \in Act$.

PROOF. For cases 1-3, assume R is a strong/weak/branching bisimulation such that $(p, q) \in R$ (or $R = \emptyset$ in case $p = q = \mathbf{0}$). Consider, for each $\mu \in Act$, relation $R_\mu = \{(\mu.p, \mu.q)\} \cup R$. It is very easy to check that R_μ is a strong/weak/branching bisimulation on places. Case 4 derives from the following exercise: $p \approx q$ if and only if $\mu.p \approx_c \mu.q$ for each $\mu \in Act$. With a similar exercise, also case 5 can be proved. □

Now we prove that strong/rooted-weak/rooted-branching bisimilarities are congruences for summation. As expected, weak/branching bisimilarities are not congruences in this case. For instance, $\tau.a.\mathbf{0} \approx a.\mathbf{0}$ but $\tau.a.\mathbf{0} + b.\mathbf{0} \not\approx a.\mathbf{0} + b.\mathbf{0}$.

Proposition 21. 1) For each $p,q \in \mathscr{P}_{CFM}^{grd}$, if $p \sim q$ (or $p = q = \mathbf{0}$), then $p+r \sim q+r$ for all $r \in \mathscr{P}_{CFM}^{grd}$.
2) For each $p,q \in \mathscr{P}_{CFM}^{grd}$, if $p \approx_c q$ (or $p = q = \mathbf{0}$), then $p+r \approx_c q+r$ for all $r \in \mathscr{P}_{CFM}^{grd}$.
3) For each $p,q \in \mathscr{P}_{CFM}^{grd}$, if $p \approx_{brc} q$ (or $p = q = \mathbf{0}$), then $p+r \approx_{brc} q+r$ for all $r \in \mathscr{P}_{CFM}^{grd}$.

PROOF. For case 1, assume R is a bisimulation such that $(p,q) \in R$ (or $R = \emptyset$ in case $p = q = \mathbf{0}$). It is easy to check that, for each $r \in \mathscr{P}_{CFM}^{grd}$, the relation $R_r = \{(p+r, q+r)\} \cup R \cup \mathscr{I}_r$ is a strong bisimulation, where $\mathscr{I}_r = \{(r', r') \mid r' \in reach(r), r' \neq \theta\}$ if $r \neq \mathbf{0}$, otherwise $\mathscr{I}_r = \emptyset$.

For case 2, if $p+r \xrightarrow{\mu} s$, this can be due to either $p \xrightarrow{\mu} s$ or $r \xrightarrow{\mu} s$, according to the net semantics in Table 2. In the former case, since $p \approx_c q$, we have that $q \stackrel{\mu}{\Longrightarrow} q'$, with $s \approx q'$. Hence, by lifting the first transition on that path with source place $q+r$ (cf. set T' in Table 2), we can derive $q+r \stackrel{\mu}{\Longrightarrow} q'$ with $s \approx q'$, as required. In the latter case, by the definition of the net semantics, we can derive also $q+r \xrightarrow{\mu} s$, hence $q+r \stackrel{\mu}{\Longrightarrow} s$ with $s \approx s$, as required. The symmetric case when $q+r$ moves first is omitted.

For case 3, if $p+r \xrightarrow{\mu} s$, this can be due to either $p \xrightarrow{\mu} s$ or $r \xrightarrow{\mu} s$. In the former case, since $p \approx_{brc} q$, we have that $q \xrightarrow{\mu} q'$, with $s \approx_{br} q'$. Hence, by the definition of the net semantics, we can also derive $q+r \xrightarrow{\mu} q'$ with $s \approx_{br} q'$, as required. In the latter case, by the definition of the net semantics, we can also derive $q+r \xrightarrow{\mu} s$, with $s \approx_{br} s$. The symmetric case when $q+r$ moves first is omitted. □

Now we show that all the bisimulations on places are congruences for parallel composition.

Proposition 22. 1) For every $p,q,r \in \mathscr{P}_{CFM}$, if $p \sim^{\oplus} q$, then $p \,|\, r \sim^{\oplus} q \,|\, r$.
2) For every $p,q,r \in \mathscr{P}_{CFM}$, if $p \approx^{\oplus} q$, then $p \,|\, r \approx^{\oplus} q \,|\, r$.
3) For every $p,q,r \in \mathscr{P}_{CFM}$, if $p \approx_{br}^{\oplus} q$, then $p \,|\, r \approx_{br}^{\oplus} q \,|\, r$.
4) For every $p,q,r \in \mathscr{P}_{CFM}$, if $p \approx_{c}^{\oplus} q$, then $p \,|\, r \approx_{c}^{\oplus} q \,|\, r$.
5) For every $p,q,r \in \mathscr{P}_{CFM}$, if $p \approx_{brc}^{\oplus} q$, then $p \,|\, r \approx_{brc}^{\oplus} q \,|\, r$.

PROOF. By induction on the size of $dec(p)$. The proof is identical in all the five cases. So, we show only the first one.

If $|dec(p)| = 0$, then $dec(p) = \theta$; as $p \sim^{\oplus} q$, necessarily also $dec(q) = \theta$. Hence, $dec(p \,|\, r) = dec(r) = dec(q \,|\, r)$ and the thesis follows trivially, because $\sim^{\oplus}$ is reflexive. Since $dec(p) \sim^{\oplus} dec(q)$, if $|dec(p)| = k+1$ for some $k \geq 0$, then by Definition 14, there exist p_1, p_2, q_1, q_2 such that $p_1 \sim q_1$, $dec(p_2) \sim^{\oplus} dec(q_2)$, $dec(p) = p_1 \oplus dec(p_2)$ and $dec(q) = q_1 \oplus dec(q_2)$. Since $|dec(p_2)| = k = |dec(q_2)|$ and $p_2 \sim^{\oplus} q_2$, by induction, we have that $p_2 \,|\, r \sim^{\oplus} q_2 \,|\, r$. Since $p_1 \sim q_1$, by Definition 14, we have that $dec(p \,|\, r) = p_1 \oplus dec(p_2 \,|\, r) \sim^{\oplus} q_1 \oplus dec(q_2 \,|\, r) = dec(q \,|\, r)$. Hence, $p \,|\, r \sim^{\oplus} q \,|\, r$. □

Still there is one construct missing: recursion, defined over guarded terms only. Here we simply sketch the issue. Consider an extension of CFM where terms can be constructed using variables, such as $x, y, \dots$ (which are in syntactic category q): this defines an "open" CFM, where terms may be not given a complete semantics. For instance, $p_1(x) = a.(b.\mathbf{0} + c.x)$ and $p_2(x) = a.(c.x + b.\mathbf{0})$ are open guarded CFM terms.

Definition 20. (Open CFM) Let $Var = \{x, y, z, \ldots\}$ be a finite set of variables. The CFM *open terms* are generated from actions, constants and variables by the following abstract syntax:

s	$::=$	$\mathbf{0}$	$\mid$	$\mu.q$	$\mid$	$s+s$		*guarded open processes*
q	$::=$	s	$\mid$	C	$\mid$	x		*sequential open processes*
p	$::=$	q	$\mid$	$p \mid p$				*parallel open processes*

where x is any variable taken from Var. The *open* net semantics for open CFM extends the net semantics in Table 2 with $[\![x]\!]_I = (\{x\}, \emptyset, \emptyset, \{x\})$, so that, e.g., the semantics of $a.x$ is $(\{a.x, x\}, \{a\}, \{(a.x, a, x)\}, a.x)$. □

However, a place x is not equivalent to $\mathbf{0}+\mathbf{0}$, even if both are stuck, because x is intended to be a placeholder for a sequential CFM term. All the behavioral equivalences we have defined on (closed) process terms can be extended to open terms, by considering the variables in a proper way. An open term $p(x_1, \ldots, x_n)$ can be *closed* by means of a substitution as follows:

$$p(x_1, \ldots, x_n)\{r_1/x_1, \ldots, r_n/x_n\}$$

with the effect that each occurrence of the variable x_i (within p and the body of each constant in $Const(p)$) is replaced by the *closed* CFM sequential process r_i, for $i = 1, \ldots, n$. E.g, if $p_1(x) = a.(b.\mathbf{0}+c.x)$ and $p_2(x) = a.(c.x+b.\mathbf{0})$, then $p_1(x)\{d.\mathbf{0}/x\} = a.(b.\mathbf{0}+c.d.\mathbf{0})$ and $p_2(x)\{d.\mathbf{0}/x\} = a.(c.d.\mathbf{0}+b.\mathbf{0})$.

A natural extension of strong bisimulation equivalence $\sim$ over open *guarded* terms is as follows: $p(x_1, \ldots, x_n) \sim q(x_1, \ldots, x_n)$ if for all tuples $(r_1, \ldots, r_n)$ of (closed) CFM sequential terms, $p(x_1, \ldots, x_n)\{r_1/x_1, \ldots, r_n/x_n\} \sim q(x_1, \ldots, x_n)\{r_1/x_1, \ldots, r_n/x_n\}$. For instance, it is easy to see that $p_1(x) \sim p_2(x)$. As a matter of fact, for all r, $p_1(x)\{r/x\} = a.(b.\mathbf{0}+c.r) \sim a.(c.r+b.\mathbf{0}) = p_2(x)\{r/x\}$, which can be easily proved by means of the algebraic properties (discussed in the next subsection) and the congruence ones of $\sim$.

In the same way, we can define weak bisimilarity $\approx$ (as well as branching bisimilarity $\approx_{br}$, rooted weak bisimilarity $\approx_c$ and rooted branching bisimilarity $\approx_{brc}$) over *open guarded* CFM terms.

For simplicity's sake, let us now restrict our attention to open guarded terms using a single undefined variable. We can *recursively close* an open term $p(x)$ by means of a recursively defined constant. For instance, $A \doteq p(x)\{A/x\}$. The resulting process constant A is a closed CFM sequential process. By saying that strong bisimilarity is a congruence for recursion we mean the following: If $p(x) \sim q(x)$ and $A \doteq p(x)\{A/x\}$ and $B \doteq q(x)\{B/x\}$, then $A \sim B$. Similarly, we can state that (rooted) weak/branching bisimilarity is congruence for recursion. The following theorem states these facts.

Theorem 8. Let p and q be two open guarded CFM terms, with one variable x at most. Let $A \doteq p\{A/x\}$, $B \doteq q\{B/x\}$.
1) If $p \sim q$, then $A \sim B$;
2) if $p \approx q$, then $A \approx B$;
3) if $p \approx_{br} q$, then $A \approx_{br} B$;
4) if $p \approx_c q$, then $A \approx_c B$;
5) if $p \approx_{brc} q$, then $A \approx_{brc} B$;

PROOF. Consider the relation $R = \{(r\{A/x\}, r\{B/x\}) \mid r \in reach(p) \cup reach(q), r \neq \theta\}$. Note that when r is x, we get $(A,B) \in R$.

For case 1, it is enough to prove that R is a strong bisimulation up to $\sim$ [25, 14]. For case 2, it is enough to prove that R is a weak bisimulation up to $\approx$ [25, 14]. For case 3, it is enough to prove that R is a branching bisimulation up to $\approx_{br}$ [14]. These three cases are very similar and their proofs differ only when $r = x$. For cases 4 and 5, it is enough to elaborate a bit on cases 2 and 3, respectively. By symmetry, for case 1 it is enough to prove that if $r\{A/x\} \xrightarrow{\mu} p'$, then $r\{B/x\} \xrightarrow{\mu} q'$ with $p' \sim R \sim q'$ (or $p' = \theta = q'$); instead, for case 2, we have to prove that if $r\{A/x\} \xrightarrow{a} p'$ (or $r\{A/x\} \xrightarrow{\tau} p'$), then $r\{B/x\} \stackrel{a}{\Longrightarrow} q'$ (or $r\{B/x\} \stackrel{\varepsilon}{\Longrightarrow} q'$) with $p' \sim R \approx q'$ (or $p' = \theta = q'$); finally, for case 3 we have to prove that if $r\{A/x\} \xrightarrow{\mu} p'$, then *either* $\mu = \tau$ and $r\{B/x\} \stackrel{\varepsilon}{\Longrightarrow} q'$ with $p' \sim R \approx_{br} r\{B/x\}$ and $p' \sim R \approx_{br} q'$, *or* $r\{B/x\} \stackrel{\varepsilon}{\Longrightarrow} \overline{q} \xrightarrow{\mu} q'$ with $r\{A/x\} \sim R \approx_{br} \overline{q}$ and $p' \sim R \approx_{br} q'$.

The proof proceeds by induction on the definition of the net for $r\{A/x\}$. We examine the possible shapes of r, which is an open sequential process.

(a) $r = \mu.r'$. In this case, $r\{A/x\} = \mu.r'\{A/x\} \xrightarrow{\mu} r'\{A/x\}$ (in case $r' \neq \mathbf{0}$) or $r\{A/x\} = \mu.r'\{A/x\} \xrightarrow{\mu} \theta$ (in case $r' = \mathbf{0}$). Similarly, $r\{B/x\} = \mu.r'\{B/x\} \xrightarrow{\mu} r'\{B/x\}$ (or $r\{B/x\} = \mu.r'\{B/x\} \xrightarrow{\mu} \theta$) is derivable, with $(r'\{A/x\}, r'\{B/x\}) \in R$ (or the reached markings are both θ).

(b) $r = r_1 + r_2$. In this case, $r\{A/x\} = r_1\{A/x\} + r_2\{A/x\}$. A transition from $r\{A/x\}$, e.g., $r_1\{A/x\} + r_2\{A/x\} \xrightarrow{\mu} p'$, is derivable only if $r_i\{A/x\} \xrightarrow{\mu} p'$ for some $i = 1,2$. Without loss of generality, assume the transition is due to $r_1\{A/x\} \xrightarrow{\mu} p'$. Since r_1 is guarded, $r_1\{A/x\} \xrightarrow{\mu} p'$ is derivable only if $r_1 \xrightarrow{\mu} \overline{r}$, with $p' = \overline{r}\{A/x\}$. Therefore, we can derive $r_1\{B/x\} \xrightarrow{\mu} \overline{r}\{B/x\}$ and so also $r\{B/x\} = r_1\{B/x\} + r_2\{B/x\} \xrightarrow{\mu} \overline{r}\{B/x\}$, with $(\overline{r}\{A/x\}, \overline{r}\{B/x\}) \in R$ (or $\overline{r} = \theta$).

(c) $r = D$, with $D \doteq s$. So, $r\{A/x\} \doteq s\{A/x\}$ and $r\{B/x\} \doteq s\{B/x\}$. If $r\{A/x\} \xrightarrow{\mu} p'$, then this is possible only if $s\{A/x\} \xrightarrow{\mu} p'$. Since s is guarded, $s\{A/x\} \xrightarrow{\mu} p'$ is derivable only if $s \xrightarrow{\mu} \overline{s}$, with $p' = \overline{s}\{A/x\}$. So, we can derive $s\{B/x\} \xrightarrow{\mu} \overline{s}\{B/x\}$ and so, by the net semantics, also $r\{B/x\} \xrightarrow{\mu} \overline{s}\{B/x\}$, with $(\overline{s}\{A/x\}, \overline{s}\{B/x\}) \in R$ (or $\overline{s} = \theta$).

(d) $r = x$. Then, we have $r\{A/x\} = A$ and $r\{B/x\} = B$. By hypothesis, $A \doteq p\{A/x\}$, hence, if $A \xrightarrow{\mu} p'$, then also $p\{A/x\} \xrightarrow{\mu} p'$ is a transition in the net for $p\{A/x\}$. Since p is guarded, $p\{A/x\} \xrightarrow{\mu} p'$ is derivable only if $p \xrightarrow{\mu} \overline{p}$ with $p' = \overline{p}\{A/x\}$. Hence, $p\{B/x\} \xrightarrow{\mu} \overline{p}\{B/x\}$ is derivable, too. Now we have the following cases:

1. If $p \sim q$, then $q \xrightarrow{\mu} \overline{q}$ with $\overline{p} \sim \overline{q}$ (or $\overline{p} = \theta = \overline{q}$). Hence, $q\{B/x\} \xrightarrow{\mu} \overline{q}\{B/x\}$ is derivable, too, with $\overline{p}\{B/x\} \sim \overline{q}\{B/x\}$ (or $\overline{p} = \theta = \overline{q}$). Since $B \doteq q\{B/x\}$, $B \xrightarrow{\mu} \overline{q}\{B/x\}$ is derivable, too, with $\overline{p}\{A/x\} \sim \overline{p}\{A/x\} R\, \overline{p}\{B/x\} \sim \overline{q}\{B/x\}$ (or $\overline{p} = \theta = \overline{q}$), as required. This concludes the proof that R is a strong bisimulation up to $\sim$.

2. If $p \approx q$, then $q \stackrel{a}{\Longrightarrow} \overline{q}$ (if $\mu = a$; or $q \stackrel{\varepsilon}{\Longrightarrow} \overline{q}$ if $\mu = \tau$), with $\overline{p} \approx \overline{q}$ (or $\overline{p} = \theta = \overline{q}$). Hence, $q\{B/x\} \stackrel{a}{\Longrightarrow} \overline{q}\{B/x\}$ (or $q\{B/x\} \stackrel{\varepsilon}{\Longrightarrow} \overline{q}\{B/x\}$) is derivable, too, with $\overline{p}\{B/x\} \approx \overline{q}\{B/x\}$ (or $\overline{p} = \theta = \overline{q}$). Since $B \doteq q\{B/x\}$, $B \stackrel{a}{\Longrightarrow} \overline{q}\{B/x\}$ (or $B \stackrel{\varepsilon}{\Longrightarrow} \overline{q}\{B/x\}$) is derivable, too, with $\overline{p}\{A/x\} \sim \overline{p}\{A/x\} R \overline{p}\{B/x\} \approx \overline{q}\{B/x\}$ (or $\overline{p} = \theta = \overline{q}$), as required. This concludes the proof that R is a weak bisimulation up to $\approx$.
3. If $p \approx_{br} q$, then *either* $\mu = \tau$ and $q \stackrel{\varepsilon}{\Longrightarrow} \overline{q}$, such that $p \approx_{br} \overline{q}$ and $\overline{p} \approx_{br} \overline{q}$, *or* $q \stackrel{\varepsilon}{\Longrightarrow} s \stackrel{\mu}{\longrightarrow} \overline{q}$ such that $p \approx_{br} s$ and $\overline{p} \approx_{br} \overline{q}$. Hence, *either* $\mu = \tau$ and $q\{B/x\} \stackrel{\varepsilon}{\Longrightarrow} \overline{q}\{B/x\}$, such that $p\{B/x\} \approx_{br} \overline{q}\{B/x\}$ and $\overline{p}\{B/x\} \approx_{br} \overline{q}\{B/x\}$, *or* $q\{B/x\} \stackrel{\varepsilon}{\Longrightarrow} s\{B/x\} \stackrel{\mu}{\longrightarrow} \overline{q}\{B/x\}$ such that $p\{B/x\} \approx_{br} s\{B/x\}$ and $\overline{p}\{B/x\} \approx_{br} \overline{q}\{B/x\}$. Since $B \doteq q\{B/x\}$, in the either case $B \stackrel{\varepsilon}{\Longrightarrow} \overline{q}\{B/x\}$ is derivable, too, with $p\{A/x\} \sim p\{A/x\} R\, p\{B/x\} \approx_{br} \overline{q}\{B/x\}$ and $\overline{p}\{A/x\} \sim \overline{p}\{A/x\} R \overline{p}\{B/x\} \approx_{br} \overline{q}\{B/x\}$ (or $\overline{p} = \theta = \overline{q}$), as required; in the or case $B \stackrel{\varepsilon}{\Longrightarrow} s\{B/x\} \stackrel{\mu}{\longrightarrow} \overline{q}\{B/x\}$ is derivable, too, with
$p\{A/x\} \sim p\{A/x\} R\, p\{B/x\} \approx_{br} s\{B/x\}$ and
$\overline{p}\{A/x\} \sim \overline{p}\{A/x\} R \overline{p}\{B/x\} \approx_{br} \overline{q}\{B/x\}$ (or $\overline{p} = \theta = \overline{q}$), as required. This concludes the proof that R is a branching bisimulation up to $\approx_{br}$.
4. If $p \approx_c q$, then $q \stackrel{\mu}{\Longrightarrow} \overline{q}$, with $\overline{p} \approx \overline{q}$ (or $\overline{p} = \theta = \overline{q}$). So, $q\{B/x\} \stackrel{\mu}{\Longrightarrow} \overline{q}\{B/x\}$ is derivable, too, with $\overline{p}\{B/x\} \approx \overline{q}\{B/x\}$. Since $B \doteq q\{B/x\}$, $B \stackrel{\mu}{\Longrightarrow} \overline{q}\{B/x\}$ is derivable, too. Since from item 2 above (which holds because $p \approx_c q$ implies $p \approx q$), we know that R is a weak bisimulation up to $\approx$, we have that $\overline{p}\{A/x\} \approx \overline{p}\{B/x\}$, and so we can conclude that if $A \stackrel{\mu}{\longrightarrow} \overline{p}\{A/x\}$, then $B \stackrel{\mu}{\Longrightarrow} \overline{q}\{B/x\}$ with $\overline{p}\{A/x\} \approx \overline{q}\{B/x\}$, as required.
5. If $p \approx_{brc} q$, then $q \stackrel{\mu}{\longrightarrow} \overline{q}$, with $\overline{p} \approx_{br} \overline{q}$ (or $\overline{p} = \theta = \overline{q}$). So, $q\{B/x\} \stackrel{\mu}{\longrightarrow} \overline{q}\{B/x\}$ is derivable, too, with $\overline{p}\{B/x\} \approx_{br} \overline{q}\{B/x\}$. As $B \doteq q\{B/x\}$, the transition $B \stackrel{\mu}{\longrightarrow} \overline{q}\{B/x\}$ is derivable, too. As from item 3 above (which holds because $p \approx_{brc} q$ implies $p \approx_{br} q$), we know that relation R is a branching bisimulation up to $\approx_{br}$, we have that $\overline{p}\{A/x\} \approx_{br} \overline{p}\{B/x\}$, and so we can conclude that if $A \stackrel{\mu}{\longrightarrow} \overline{p}\{A/x\}$, then $B \stackrel{\mu}{\longrightarrow} \overline{q}\{B/x\}$ with $\overline{p}\{A/x\} \approx_{br} \overline{q}\{B/x\}$, as required. □

The extension to the case of open terms with multiple undefined variables, e.g., $p(x_1, \ldots, x_n)$ can be obtained in a standard way [25, 14].

6.2. Algebraic Properties

The algebraic properties we list in this section are a slight variation of those for strong/weak/branching bisimilarities over finite-state CCS, studied in [24, 20, 25, 26, 11, 12, 3, 14]. Interestingly enough, they can be adapted to the various team equivalences we propose for CFM.

6.2.1. Strong Team Equivalence

Now we list the algebraic properties of strong team equivalence, whose proof is outlined in [17]. On sequential processes we have the following algebraic laws.

Proposition 23. (Laws of the choice operator for $\sim$) For each $p,q,r \in \mathscr{P}_{CFM}^{grd}$, the following hold:

$$\begin{array}{rcllr} p+(q+r) & \sim & (p+q)+r & & \text{(associativity)} \\ p+q & \sim & q+p & & \text{(commutativity)} \\ p+\mathbf{0} & \sim & p & \text{if } p \neq \mathbf{0} & \text{(identity)} \\ p+p & \sim & p & \text{if } p \neq \mathbf{0} & \text{(idempotency)} \end{array}$$

□

Proposition 24. (Laws of the constant for $\sim$) For each $p \in \mathscr{P}_{CFM}^{grd}$, and each $C \in \mathscr{C}$, the following hold:

$$\begin{array}{lll} \text{if } C \doteq \mathbf{0}, \text{ then} & C \sim \mathbf{0}+\mathbf{0} & \text{(stuck)} \\ \text{if } C \doteq p \text{ and } p \neq \mathbf{0}, \text{ then} & C \sim p & \text{(unfolding)} \\ \text{if } C \doteq p\{C/x\} \text{ and } q \sim p\{q/x\} \text{ then} & C \sim q & \text{(folding)} \end{array}$$

where in the last law p is actually also open on x (while q is closed). □

Proposition 25. (Laws of the parallel operator for $\sim^{\oplus}$) For each $p,q,r \in \mathscr{P}_{CFM}$, the following hold:

$$\begin{array}{rcll} p\,|\,(q\,|\,r) & \sim^{\oplus} & (p\,|\,q)\,|\,r & \text{(associativity)} \\ p\,|\,q & \sim^{\oplus} & q\,|\,p & \text{(commutativity)} \\ p\,|\,\mathbf{0} & \sim^{\oplus} & p & \text{(identity)} \end{array}$$

□

As strong team equivalence is finer than rooted weak/branching team equivalences, all these algebraic properties (except for the folding law, which is to be proved again) hold also for these coarser equivalences.

6.2.2. Rooted Weak Team Equivalence

A simple law that holds for both weak and branching bisimilarities is the following:

$$\text{if } q \neq \mathbf{0}, \text{ then } \tau.q \approx q \text{ and } \tau.q \approx_{br} q.$$

In fact, relation $R = \{(\tau.q,q)\} \cup \mathscr{I}$ is a weak/branching bisimulation, where $\mathscr{I} = \{(r,r) \mid r \in \mathit{reach}(q), r \neq \theta\}$. Note that the semantics of $\tau.\mathbf{0}$ is the net in Figure 6(d), while the semantics of $\mathbf{0}$ is the empty marking θ; hence $\tau.\mathbf{0} \not\approx \mathbf{0}$; nonetheless, $\tau.\tau.\mathbf{0} \approx \tau.\mathbf{0}$. Of course, this simple law is invalid for rooted weak/branching bisimilarity.

Now we list some laws for rooted weak bisimilarity.

Proposition 26. (τ-laws for rooted weak bisimilarity) For each $p \in \mathscr{P}_{CFM}^{grd}$, for each $q \in \mathscr{P}_{CFM}^{seq}$ and for each $\mu \in Act$, the following hold:

$$\begin{array}{lrcll} (i) & \mu.\tau.q & \approx_c & \mu.q & \text{if } q \neq \mathbf{0} \\ (ii) & p+\tau.p & \approx_c & \tau.p & \\ (iii) & \mu.(p+\tau.q) & \approx_c & \mu.(p+\tau.q)+\mu.q & \end{array}$$

PROOF. The first law follows directly by the fact that $\tau.q \approx q$ if $q \neq \mathbf{0}$.

For the second law, observe that the only move from $\tau.p$, namely $\tau.p \xrightarrow{\tau} p$ (or $\tau.p \xrightarrow{\tau} \theta$ if $p = \mathbf{0}$), can be easily matched by $p+\tau.p$, with $p+\tau.p \xrightarrow{\tau} p$ and $p \approx p$ (or $p+\tau.p \xrightarrow{\tau} \theta$). Conversely, if $p+\tau.p \xrightarrow{\mu} p'$, then $\tau.p \stackrel{\mu}{\Longrightarrow} p'$ with $p' \approx p'$ (or $p' =$

$$\frac{}{og(\mathbf{0})} \qquad \frac{}{og(\alpha.p)} \qquad \frac{og(p)}{og(\tau.p)} \qquad \frac{og(p) \quad C \doteq p\{C/x\}}{og(C)} \qquad \frac{og(p) \quad og(q)}{og(p+q)}$$

Table 3: Observationally guarded predicate

θ). Indeed, $p+\tau.p \xrightarrow{\mu} p'$ is derivable if either $p \xrightarrow{\mu} p'$ or $\tau.p \xrightarrow{\tau} p$ (and $\mu = \tau$ and $p' = p$). In the former case $\tau.p \xrightarrow{\tau} p \xrightarrow{\mu} p'$ and $p' \approx p'$ (or $p' = \theta$); in the latter case, $\tau.p \xrightarrow{\tau} p$ and $p \approx p$ (or $p = \theta$).

For the third law, the only nontrivial case is when $\mu.(p+\tau.q)+\mu.q \xrightarrow{\mu} q$ (or $\mu.(p+\tau.q)+\mu.q \xrightarrow{\mu} \theta$, if $q = \mathbf{0}$); in such a case $\mu.(p+\tau.q) \stackrel{\mu}{\Longrightarrow} q$ with $q \approx q$ (or $\mu.(p+\tau.q) \stackrel{\mu}{\Longrightarrow} \theta$). □

In order to define the properties concerning the constants, we need an auxiliary definition.

Definition 21. (Observationally guarded) A sequential (possibly open) CFM process p is *observationally guarded* if $og(p)$ holds, where $og(-)$ is defined in Table 3. □

In the definition above (and throughout the paper), it is assumed that whenever a constant C is defined by $C \doteq p\{C/x\}$, then the open guarded (but it may be not observationally guarded) term p does not contain occurrences of C, i.e., $C \notin Const(p)$, so that all the instances of C in $p\{C/x\}$ are due to substitution of C for x. Moreover, it assumed that for each constant definition we use a different variable, e.g., $D \doteq q\{D/y\}$.

For instance, if $C \doteq \tau.C$, then C is not observationally guarded because $\tau.x$ is not so. On the contrary, let us consider

$$C_1 \doteq a.C_2 + \tau.a.C_1$$
$$C_2 \doteq b.C_2 + \tau.C_1$$

Note that $og(C_1)$ holds; in fact, $og(a.C_2\{x/C_1\} + \tau.a.x)$[2] holds, as $og(a.C_2\{x/C_1\})$ hods and $og(\tau.a.x)$ holds as $og(a.x)$ holds. Similarly, $og(C_2)$ holds, as $og(b.y)$ holds and $og(\tau.C_1\{y/C_2\})$ holds; in fact $og(C_1\{y/C_2\})$ holds, because $og(a.y+\tau.a.x)$ holds. Note that $C \doteq p\{C/x\}$ is observationally guarded if and only if $C \stackrel{\tau}{\not\Rightarrow} C$.

Lemma 4. Let p be a sequential process, open on x, $p \neq x$. If $p\{q/x\} \xrightarrow{\mu} r$, then there exists a process p' such that $p \xrightarrow{\mu} p'$, $r = p'\{q/x\}$ and for any sequential process t, $p\{t/x\} \xrightarrow{\mu} p'\{t/x\}$. Moreover, if $\mu = \tau$ and p is observationally guarded, then p' is observationally guarded.

PROOF. By induction on the structure of p. If p is a constant $D \doteq s$, then $p\{q/x\} \doteq s\{q/x\}$ and $p\{q/x\} \xrightarrow{\mu} r$ only if $s\{q/x\} \xrightarrow{\mu} r$; since s is guarded, then $s \xrightarrow{\mu} \bar{s}$, with $r =$

[2]Note that we have used the inverse substitution $\{x/C_1\}$ to denote that each occurrence of constant C_1 in the body of C_2 is replaced by x.

$\overline{s}\{q/x\}$, and so $p' = \overline{s}$. If $p = p_1 + p_2$, then the result follows by easy induction. If $p = \mu.p'$, clearly $r = p'\{q/x\}$ and so the result is immediate. Note that, in each case of the induction, if p is observationally guarded and $\mu = \tau$, then p' must be observationally guarded, as required. □

Proposition 27. (Weak folding) For each $p \in \mathscr{P}_{CFM}^{grd}$ (open on x), and each $C \in \mathscr{C}$, if $C \doteq p\{C/x\}$, $og(p)$ and $q \approx_c p\{q/x\}$, then $C \approx_c q$.

PROOF. This statement can be proven by showing that the relation
$R = \{(r\{C/x\}, r\{q/x\}) \mid r \in reach(p), r \neq \theta\}$
is a weak bisimulation up to $\approx$ [25, 35, 14] satisfying the rootedness conditions. Here, we consider the alternative definition of weak bisimulation as a strong bisimulation of Proposition 4. This has the nice consequence that the proof technique that we use is actually a strong bisimulation up to, where each occurrence of $\sim$ is actually replaced by $\approx$ [35]. To ease notation, we write $r(t)$ for $r\{t/x\}$.

Clearly, when $r = x$, we have that $(C,q) \in R$. So, it remains to prove the weak bisimulation (up to) conditions: for each $\delta \in \mathscr{A} \cup \{\varepsilon\}$

- if $r(C) \stackrel{\delta}{\Longrightarrow} p'$, then $r(q) \stackrel{\delta}{\Longrightarrow} q'$ and $p' \approx R \approx q'$ (or $p' = \theta = q'$), and
- if $r(q) \stackrel{\delta}{\Longrightarrow} q'$, then $r(C) \stackrel{\delta}{\Longrightarrow} p'$ and $p' \approx R \approx q'$ (or $p' = \theta = q'$).

If R is a weak bisimulation up to, then $R \subseteq \approx$ and so $C \approx q$; since we will show that the rootedness conditions are satisfied for the pair (C,q), we get the thesis $C \approx_c q$.

If $r(C) \stackrel{\delta}{\Longrightarrow} p'$, then we have three cases: (i) either $p' = \overline{r}(C)$ and $r(C) \stackrel{\delta}{\Longrightarrow} \overline{r}(C)$ because $r \stackrel{\delta}{\Longrightarrow} \overline{r}$ (with $\overline{r} \neq x$); (ii) or $r(C) \stackrel{\varepsilon}{\Longrightarrow} x(C) \stackrel{\delta}{\Longrightarrow} p'$; (iii) or $r(C) \stackrel{\delta}{\Longrightarrow} x(C) \stackrel{\varepsilon}{\Longrightarrow} p'$.

(i) If $r(C) \stackrel{\delta}{\Longrightarrow} \overline{r}(C)$ because $r \stackrel{\delta}{\Longrightarrow} \overline{r}$ (with $\overline{r} \neq x$), then also $r(q) \stackrel{\delta}{\Longrightarrow} \overline{r}(q)$ is derivable by (repeated applications of) Lemma 4, and $(\overline{r}(C), \overline{r}(q)) \in R$. If $\overline{r}(C) = \theta$, then also $\overline{r}(q) = \theta$, so the thesis follows trivially.

(ii) To derivation $r(C) \stackrel{\varepsilon}{\Longrightarrow} x(C)$ (due to $r \stackrel{\varepsilon}{\Longrightarrow} x$), $r(q)$ replies by mimicking each move, by Lemma 4, with $r(q) \stackrel{\varepsilon}{\Longrightarrow} x(q)$ and $(C,q) \in R$. Now, derivation $x(C) \stackrel{\delta}{\Longrightarrow} p'$ ensures that $p(C) \stackrel{\delta}{\Longrightarrow} p'$ because $C \doteq p(C)$. Now we have only these two subcases:[3]

(a) $p(C) \stackrel{\delta}{\Longrightarrow} \overline{p}(C)$ because $p \stackrel{\delta}{\Longrightarrow} \overline{p}$ and $p' = \overline{p}(C)$. In such a case, also $p(q) \stackrel{\delta}{\Longrightarrow} \overline{p}(q)$ is derivable by (repeated applications of) Lemma 4. As $q \approx_c p(q)$, by Lemma 1 there exists q' such that $x(q) \stackrel{\delta}{\Longrightarrow} q'$ with $\overline{p}(q) \approx q'$ (assuming $\overline{p} \neq \theta$). Summing up, if $r(C) \stackrel{\varepsilon}{\Longrightarrow} x(C) \stackrel{\delta}{\Longrightarrow} p' = \overline{p}(C)$, then $r(q)$ replies with $r(q) \stackrel{\varepsilon}{\Longrightarrow} x(q) \stackrel{\delta}{\Longrightarrow} q'$ with $\overline{p}(C) \, R \, \overline{p}(q) \approx q'$. If $\overline{p} = \theta$, then also $q' = \theta$ and the thesis follows trivially.

(b) $p(C) \stackrel{\delta}{\Longrightarrow} x(C) \stackrel{\varepsilon}{\Longrightarrow} p'$ because $p \stackrel{\delta}{\Longrightarrow} x$. In such a case, also $p(q) \stackrel{\delta}{\Longrightarrow} x(q)$ is derivable by (repeated applications of) Lemma 4, with $(C,q) \in R$. As $q \approx_c p(q)$, by

[3] Note that the case $p(C) \stackrel{\varepsilon}{\Longrightarrow} x(C) \stackrel{\delta}{\Longrightarrow} p'$ is impossible because p is observationally guarded.

Lemma 1 there exists $\overline{q}$ such that $q \stackrel{\delta}{\Longrightarrow} \overline{q}$ with $x(q) = q \approx \overline{q}$. Then, $x(C) \stackrel{\varepsilon}{\Longrightarrow} p'$ ensures that $p(C) \stackrel{\varepsilon}{\Longrightarrow} p'$ because $C \doteq p(C)$. Since p is observationally guarded, by Lemma 4 it follows that $p' = \overline{p}(C)$ and $p \stackrel{\varepsilon}{\Longrightarrow} \overline{p}$. Therefore, also $p(q) \stackrel{\varepsilon}{\Longrightarrow} \overline{p}(q)$ is derivable. Since $\overline{q} \approx_c q \approx_c p(q)$, there exists q' such that $\overline{q} \stackrel{\varepsilon}{\Longrightarrow} q'$ with $\overline{p}(q) \approx q'$ (assuming $\overline{p} \neq \theta$). Summing up, if $r(C) \stackrel{\delta}{\Longrightarrow} x(C) \stackrel{\varepsilon}{\Longrightarrow} p' = \overline{p}(C)$, then $r(q)$ replies with $r(q) \stackrel{\delta}{\Longrightarrow} \overline{q} \stackrel{\varepsilon}{\Longrightarrow} q'$ with $\overline{p}(C) \, R \, \overline{p}(q) \approx q'$. If $\overline{p} = \theta$, then also $q' = \theta$ and the thesis follows trivially.

(iii) This case is analogous to the previous one, and so omitted.

The case when $r(q)$ moves first is analogous, and so omitted. Therefore, R is a weak bisimulation up to $\approx$, so that we can conclude that $C \approx q$, since $(C,q) \in R$.

Actually, we can prove that $C \approx_c q$. Reconsidering the proof above, we have actually shown that if $x(C) \stackrel{\mu}{\Longrightarrow} \overline{p}(C)$, then $x(q) \stackrel{\mu}{\Longrightarrow} q'$ with $\overline{p}(C) \, R \, \overline{p}(q) \approx q'$; since R is a weak bisimulation up to $\approx$, this is the same as $\overline{p}(C) \approx q'$, as required. □

Observe that the requirement that p is observationally guarded is crucial for the correctness of the weak folding property. For instance, consider $p = \tau.x + a.\mathbf{0}$, which is not observationally guarded. Then, $C \doteq \tau.C + a.\mathbf{0}$ defines a process that can perform only a, possibly after some internal activity, so it is rooted weak bisimilar to $\tau.a.\mathbf{0}$. However, $q = \tau.(a.\mathbf{0} + b.\mathbf{0})$ is such that $q \approx_c \tau.q + a.\mathbf{0}$, but $C \not\approx_c q$.

Proposition 28. (Other laws of the constant for $\approx_c$) For each $p, r \in \mathscr{P}_{CFM}^{grd}$, and each $C, D \in \mathscr{C}$, the following holds:

if $C \doteq (\tau.x + p)\{C/x\}$ and $D \doteq (\tau.(p + \mathbf{0}))\{D/x\}$, then $C \approx_c D$ (w-excision)
if $C \doteq (\tau.(\tau.x + p) + r)\{C/x\}$ and $D \doteq \tau.(p + r)\{D/x\}$, then $C \approx_c D$ (w-out)

PROOF. (W-excision) Relation $R = \{(C,D), (C, (p+\mathbf{0})\{D/x\})\} \cup \{(p'\{C/x\}, p'\{D/x\}) \mid p' \in reach(p)\}$ is a weak bisimulation. Note that if $p = \mathbf{0}$, it would be incorrect to state that $C \approx p\{D/x\}$, because p is not a place: this is reason why we add the summand $\mathbf{0}$ to get $p + \mathbf{0}$. Now, the proof that $C \approx_c D$ follows easily. If $C \stackrel{\tau}{\longrightarrow} C$, then $D \stackrel{\tau}{\longrightarrow} (p + \mathbf{0})\{D/x\}$ and $C \approx (p + \mathbf{0})\{D/x\}$; symmetrically if D moves first. If $C \stackrel{\mu}{\longrightarrow} p'\{D/x\}$ because $p \stackrel{\mu}{\longrightarrow} p'$, then $D \stackrel{\tau}{\longrightarrow} (p + \mathbf{0})\{D/x\} \stackrel{\mu}{\longrightarrow} p'\{D/x\}$ with $p'\{C/x\} \approx p'\{D/x\}$ (or $p' = \theta$).

(W-out) Relation $R = \{(C,D), (C, (p+r)\{D/x\}), ((\tau.x+p)\{C/x\}, (p+r)\{D/x\})\} \cup \{(q'\{C/x\}, q'\{D/x\}) \mid r' \in reach(p+r)\}$ is a weak bisimulation. Now, the proof that $C \approx_c D$ follows easily. If $C \stackrel{\tau}{\longrightarrow} (\tau.x + p)\{C/x\}$, then $D \stackrel{\tau}{\longrightarrow} (p+r)\{D/x\}$ with $(\tau.x + p)\{C/x\} \approx (p+r)\{D/x\}$; symmetrically if D moves first. If $C \stackrel{\mu}{\longrightarrow} r'\{C/x\}$ because $r \stackrel{\mu}{\longrightarrow} r'$, then $D \stackrel{\tau}{\longrightarrow} (p+r)\{D/x\} \stackrel{\mu}{\longrightarrow} r'\{D/x\}$ with $r'\{C/x\} \approx r'\{D/x\}$ (or $r' = \theta$). □

6.2.3. Rooted Branching Team Equivalence

As $\approx_{brc}$ is strictly finer than $\approx_c$, it may be not a surprise that some of the τ-laws in Proposition 26 do not hold for it. In particular, to get convinced that the third τ-law $\mu.(p + \tau.q) \approx_{brc} \mu.(p + \tau.q) + \mu.q$ is invalid, assume that $\mu = a$, $p = c.\mathbf{0}$ and $q = b.\mathbf{0}$. Then, the transition $a.(c.\mathbf{0} + \tau.b.\mathbf{0}) + a.b.\mathbf{0} \stackrel{a}{\longrightarrow} b.\mathbf{0}$ is matched (strongly, as required by the definition of rooted branching bisimilarity) by $a.(c.\mathbf{0}+\tau.b.\mathbf{0}) \xrightarrow{a} c.\mathbf{0}+\tau.b.\mathbf{0}$, but of course $b.\mathbf{0} \not\approx_{br} c.\mathbf{0}+\tau.b.\mathbf{0}$. Similarly, the second τ-law $p+\tau.p \approx_{brc} \tau.p$ is invalid. E.g., take $p = \tau.a.\mathbf{0}+b.\mathbf{0}$; then, to transition $p+\tau.p \xrightarrow{\tau} a.\mathbf{0}$ (which is due to transition $p \xrightarrow{\tau} a.\mathbf{0}$), $\tau.p$ can react only with $\tau.p \xrightarrow{\tau} p$, but $a.\mathbf{0} \not\approx_{br} p$.

Proposition 29. (τ-law for rooted branching bisimilarity) For each $p,r \in \mathscr{P}_{CFM}^{grd}$ and for each $\mu \in Act$, the following holds:

$$\mu.(\tau.(p+r)+p) \quad \approx_{brc} \quad \mu.(p+r)$$

PROOF. Observe that the two terms match their initial μ-labeled transition as in strong bisimulation, as required, and so it remains to prove that $\tau.(p+r)+p \approx_{br} p+r$. To achieve this, we will prove that the relation

$$R = \{(\tau.(p+r)+p, p+r)\} \cup \mathscr{I}$$

is a branching bisimulation. If $\tau.(p+r)+p \xrightarrow{\tau} p+r$, then $p+r \stackrel{\varepsilon}{\Longrightarrow} p+r$ and $(\tau.(p+r)+p, p+r) \in R$ as well as $(p+r, p+r) \in R$. Instead, if $\tau.(p+r)+p \xrightarrow{\mu} p'$ (or $\tau.(p+r)+p \xrightarrow{\mu} \theta$), because $p \xrightarrow{\mu} p'$ (or $p \xrightarrow{\mu} \theta$), then $p+r \stackrel{\varepsilon}{\Longrightarrow} p+r \xrightarrow{\mu} p'$ and $(p', p') \in R$ (or $p+r \stackrel{\varepsilon}{\Longrightarrow} p+r \xrightarrow{\mu} \theta$) as well as $(\tau.(p+r)+p, p+r) \in R$.

Conversely, if $p+r$ moves first, we have the following two cases: (i) $p+r \xrightarrow{\mu} r'$ (or $p+r \xrightarrow{\mu} \theta$) because $r \xrightarrow{\mu} r'$ (or $r \xrightarrow{\mu} \theta$); in such a case, $\tau.(p+r)+p \xrightarrow{\tau} p+r \xrightarrow{\mu} r'$ with $(r', r') \in R$ (or $\tau.(p+r)+p \xrightarrow{\tau} p+r \xrightarrow{\mu} \theta$) and $(p+r, p+r) \in R$; (ii) $p+r \xrightarrow{\mu} p'$ (or $p+r \xrightarrow{\mu} \theta$) because $p \xrightarrow{\mu} p'$ (or $p \xrightarrow{\mu} \theta$); in such a case, we have that $\tau.(p+r)+p \xrightarrow{\tau} p+r \xrightarrow{\mu} p'$ with $(p', p') \in R$ (or $\tau.(p+r)+p \xrightarrow{\tau} p+r \xrightarrow{\mu} \theta$) and $(p+r, p+r) \in R$.

In any case, the branching bisimulation conditions are respected, hence R is a branching bisimulation proving that $\tau.(p+r)+p \approx_{br} p+r$, and consequently, $\mu.(\tau.(p+r)+p) \approx_{brc} \mu.(p+r)$ holds. □

The first τ-law for rooted weak bisimilarity is derivable from the τ-law above, i.e., $\mu.\tau.r \approx_{brc} \mu.r$ if $r \neq \mathbf{0}$. In fact, we can first instantiate p to $\mathbf{0}$, yielding $\mu.(\tau.(\mathbf{0}+r)+\mathbf{0}) \approx_{brc} \mu.(\mathbf{0}+r)$, and then, by absorbing $\mathbf{0}$ with the identity law in Proposition 23, we would get $\mu.\tau.r \approx_{brc} \mu.r$. However, we have to be a bit careful because of a typing problem: in the first τ-law ($\mu.\tau.q \approx_{brc} \mu.q$ if $q \neq \mathbf{0}$) the process q is any sequential term (hence, also a constant C), while in the τ-law above the process r is a guarded process (hence, it cannot be a constant C). However, we can overcome this problem by using the unfolding property as follows. If we have a constant $C \doteq r$, then $C \approx_{brc} r$ by unfolding, where r is guarded;[4] so, $\mu.\tau.C$ is rooted branching equivalent to $\mu.\tau.r$, which is then equivalent to $\mu.(\tau.(\mathbf{0}+r)+\mathbf{0})$, so that we can apply the τ-law above to get $\mu.(\tau.(\mathbf{0}+r)+\mathbf{0}) \approx_{brc} \mu.(\mathbf{0}+r)$, and then $\mu.(\mathbf{0}+r) \approx_{brc} \mu.r \approx_{brc} \mu.C$. Summing up, $\mu.\tau.C \approx_{brc} \mu.C$, so that the first τ-law is indeed derivable in general.

[4]If $r = \mathbf{0}$, then we use the stuck property to get $C \approx_{br} \mathbf{0}+\mathbf{0}$ and the argument follows in the same way.

In order to prove that the folding property holds for rooted branching bisimilarity, too, we need to introduce an auxiliary proof technique.

Definition 22. **(Branching bisimulation up to $\approx_{br}$)** Given an FSM $N = (S, A, T)$, a branching bisimulation up to $\approx_{br}$ is a relation $R \subseteq S \times S$ such that if $(p,q) \in R$ then, for all $\mu \in A$

- if $p \stackrel{\varepsilon}{\Longrightarrow} p' \stackrel{\mu}{\longrightarrow} p''$, then
  - either $\mu = \tau$ and there exist $p'_1, p''_1, q'_1, q''_1, q'$ such that $q \stackrel{\varepsilon}{\Longrightarrow} q'$ with $p' \approx_{br} p'_1 R q'_1 \approx_{br} q'$ and $p'' \approx_{br} p''_1 R q''_1 \approx_{br} q'$;
  - or there exist $p'_1, p''_1, q'_1, q''_1, q', q''$ such that $q \stackrel{\varepsilon}{\Longrightarrow} q' \stackrel{\mu}{\longrightarrow} q''$ with $p' \approx_{br} p'_1 R q'_1 \approx_{br} q'$ and either $p'' \approx_{br} p''_1 R q''_1 \approx_{br} q''$ or $p'' = \theta = q''$.
- and symmetrically, if q moves first. □

Lemma 5. If R is a branching bisimulation up to $\approx_{br}$, then $\approx_{br} R \approx_{br}$ is a branching bisimulation.

PROOF. If $(p,q) \in \approx_{br} R \approx_{br}$, then there exist p', q' such that $p \approx_{br} p' R q' \approx_{br} q$. Since $p \approx_{br} p'$, if $p \stackrel{\mu}{\longrightarrow} \overline{p}$, then one of the following two cases are possible:

- Either $\mu = \tau$ and $p' \stackrel{\varepsilon}{\Longrightarrow} \overline{p}'$ with $p \approx_{br} \overline{p}'$ and $\overline{p} \approx_{br} \overline{p}'$. This means that either $\overline{p}' = p'$ (and this case is empty, because q can reply by idling and all the conditions are satisfied) or there exists r such that $p' \stackrel{\varepsilon}{\Longrightarrow} r \stackrel{\tau}{\longrightarrow} \overline{p}'$ with $p \approx_{br} r$. As $(p', q') \in R$, the derivation $p' \stackrel{\varepsilon}{\Longrightarrow} r \stackrel{\tau}{\longrightarrow} \overline{p}'$ is matched
  - (a) either by $q' \stackrel{\varepsilon}{\Longrightarrow} \overline{q}'$ such that $r \approx_{br} R \approx_{br} \overline{q}'$ and $\overline{p}' \approx_{br} R \approx_{br} \overline{q}'$. Since $q \approx_{br} q'$, by Lemma 2, from $q' \stackrel{\varepsilon}{\Longrightarrow} \overline{q}'$ it follows that $q \stackrel{\varepsilon}{\Longrightarrow} \overline{q}$ with $\overline{q}' \approx_{br} \overline{q}$. Summing up, if $p \stackrel{\tau}{\longrightarrow} \overline{p}$, then $q \stackrel{\varepsilon}{\Longrightarrow} \overline{q}$ such that $p \approx_{br} R \approx_{br} \overline{q}$ (because $p \approx_{br} r \approx_{br} R \approx_{br} \overline{q}' \approx_{br} \overline{q}$) and $\overline{p} \approx_{br} R \approx_{br} \overline{q}$ (because $\overline{p} \approx_{br} \overline{p}' \approx_{br} R \approx_{br} \overline{q}' \approx_{br} \overline{q}$).
  - (b) or by $q' \stackrel{\varepsilon}{\Longrightarrow} s \stackrel{\tau}{\longrightarrow} \overline{q}'$ such that $r \approx_{br} R \approx_{br} s$ and $\overline{p}' \approx_{br} R \approx_{br} \overline{q}'$. Since $q \approx_{br} q'$, by Lemma 2, from $q' \stackrel{\varepsilon}{\Longrightarrow} s$ it follows that $q \stackrel{\varepsilon}{\Longrightarrow} \overline{s}$ with $s \approx_{br} \overline{s}$. Now, to transition $s \stackrel{\tau}{\longrightarrow} \overline{q}'$, place $\overline{s}$ can reply
    - Either with $\overline{s} \stackrel{\varepsilon}{\Longrightarrow} \overline{q}$, such that $s \approx_{br} \overline{q}$ and $\overline{q}' \approx_{br} \overline{q}$. Summing up, to transition $p \stackrel{\tau}{\longrightarrow} \overline{p}$, q replies with $q \stackrel{\varepsilon}{\Longrightarrow} \overline{q}$ so that $p \approx_{br} R \approx_{br} \overline{q}$ (because $p \approx_{br} r \approx_{br} R \approx_{br} s \approx_{br} \overline{q}$) and $\overline{p} \approx_{br} R \approx_{br} \overline{q}$ (as $\overline{p} \approx_{br} \overline{p}' \approx_{br} R \approx_{br} \overline{q}' \approx_{br} \overline{q}$).
    - Or with $\overline{s} \stackrel{\varepsilon}{\Longrightarrow} \overline{s}' \stackrel{\tau}{\longrightarrow} \overline{q}$, such that $s \approx_{br} \overline{s}'$ and $\overline{q}' \approx_{br} \overline{q}$. Summing up, to transition $p \stackrel{\tau}{\longrightarrow} \overline{p}$, q replies with $q \stackrel{\varepsilon}{\Longrightarrow} \overline{s}' \stackrel{\tau}{\longrightarrow} \overline{q}$ so that $p \approx_{br} R \approx_{br} \overline{s}'$ (because $p \approx_{br} r \approx_{br} R \approx_{br} s \approx_{br} \overline{s}'$) and $\overline{p} \approx_{br} R \approx_{br} \overline{q}$ (because $\overline{p} \approx_{br} \overline{p}' \approx_{br} R \approx_{br} \overline{q}' \approx_{br} \overline{q}$).
- Or $p' \stackrel{\varepsilon}{\Longrightarrow} r \stackrel{\mu}{\longrightarrow} \overline{p}'$ with $p \approx_{br} r$ and $\overline{p} \approx_{br} \overline{p}'$ (or $\overline{p} = \theta = \overline{p}'$). Since $(p', q') \in R$, the derivation $p' \stackrel{\varepsilon}{\Longrightarrow} r \stackrel{\mu}{\longrightarrow} \overline{p}'$ is matched

(a) either by $q' \stackrel{\varepsilon}{\Longrightarrow} \overline{q}'$ (if $\mu = \tau$) such that $r \approx_{br} R \approx_{br} \overline{q}'$ and $\overline{p}' \approx_{br} R \approx_{br} \overline{q}'$. Since $q \approx_{br} q'$, by Lemma 2, from $q' \stackrel{\varepsilon}{\Longrightarrow} \overline{q}'$ it follows that $q \stackrel{\varepsilon}{\Longrightarrow} \overline{q}$ with $\overline{q'} \approx_{br} \overline{q}$. Summing up, if $p \stackrel{\tau}{\longrightarrow} \overline{p}$, then $q \stackrel{\varepsilon}{\Longrightarrow} \overline{q}$ such that $p \approx_{br} R \approx_{br} \overline{q}$ (because $p \approx_{br} r \approx_{br} R \approx_{br} \overline{q}' \approx_{br} \overline{q}$) and $\overline{p} \approx_{br} R \approx_{br} \overline{q}$ (because $\overline{p} \approx_{br} \overline{p}' \approx_{br} R \approx_{br} \overline{q}' \approx_{br} \overline{q}$).

(b) or by $q' \stackrel{\varepsilon}{\Longrightarrow} s \stackrel{\mu}{\longrightarrow} \overline{q}'$ such that $r \approx_{br} R \approx_{br} s$ and $\overline{p}' \approx_{br} R \approx_{br} \overline{q}'$ (or $\overline{p}' = \theta = \overline{q}'$). Since $q \approx_{br} q'$, to derivation $q' \stackrel{\varepsilon}{\Longrightarrow} s$, by Lemma 2, q can reply with $q \stackrel{\varepsilon}{\Longrightarrow} \overline{s}$ with $s \approx_{br} \overline{s}$. Hence, to transition $s \stackrel{\mu}{\longrightarrow} \overline{q}'$, place $\overline{s}$ can reply

- Either with $\overline{s} \stackrel{\varepsilon}{\Longrightarrow} \overline{q}$ (if $\mu = \tau$), such that $s \approx_{br} \overline{q}$ and $\overline{q}' \approx_{br} \overline{q}$. Summing up, to transition $p \stackrel{\tau}{\longrightarrow} \overline{p}$, q replies with $q \stackrel{\varepsilon}{\Longrightarrow} \overline{q}$ so that $p \approx_{br} R \approx_{br} \overline{q}$ (as $p \approx_{br} r \approx_{br} R \approx_{br} s \approx_{br} \overline{q}$) and $\overline{p} \approx_{br} R \approx_{br} \overline{q}$ (as $\overline{p} \approx_{br} \overline{p}' \approx_{br} R \approx_{br} \overline{q}' \approx_{br} \overline{q}$).
- Or with $\overline{s} \stackrel{\varepsilon}{\Longrightarrow} \overline{s}' \stackrel{\mu}{\longrightarrow} \overline{q}$, such that $s \approx_{br} \overline{s}'$ and $\overline{q}' \approx_{br} \overline{q}$ (or $\overline{q}' = \theta = \overline{q}$). Summing up, if $p \stackrel{\mu}{\longrightarrow} \overline{p}$, then $q \stackrel{\varepsilon}{\Longrightarrow} \overline{s}' \stackrel{\mu}{\longrightarrow} \overline{q}$ such that $p \approx_{br} R \approx_{br} \overline{s}'$ (because $p \approx_{br} r \approx_{br} R \approx_{br} s \approx_{br} \overline{s}'$) and $\overline{p} \approx_{br} R \approx_{br} \overline{q}$ (as $\overline{p} \approx_{br} \overline{p}' \approx_{br} R \approx_{br} \overline{q}' \approx_{br} \overline{q}$) or $\overline{p} = \theta = \overline{q}$.

The symmetric case when q moves first is analogous, hence omitted. Therefore, if R is a branching bisimulation up to $\approx_{br}$, then $\approx_{br} R \approx_{br}$ is a branching bisimulation. □

Proposition 30. If R is a branching bisimulation up to $\approx_{br}$, then $R \subseteq \approx_{br}$.

PROOF. Note that $R = \mathscr{I} R \mathscr{I} \subseteq \approx_{br} R \approx_{br} \subseteq \approx_{br}$, the last inclusion due to Lemma 5. Hence, $R \subseteq \approx_{br}$. □

Proposition 31. (Folding law for $\approx_{brc}$) For each $p \in \mathscr{P}_{CFM}^{grd}$ (open on x), and each $C \in \mathscr{C}$, if $C \doteq p\{C/x\}$, $og(p)$ and $q \approx_{brc} p\{q/x\}$, then $C \approx_{brc} q$.

PROOF. This statement can be proven by showing that the relation

$$R = \{(r\{C/x\}, r\{q/x\}) \;\big|\; r \in reach(p), r \neq \theta\}$$

is a branching bisimulation up to $\approx_{br}$ satisfying the rootedness conditions. Clearly, when $r = x$, we have that $(C,q) \in R$. To ease notation, we write $r(t)$ for $r\{t/x\}$. So, it remains to prove the branching bisimulation (up to) conditions: for each $\mu \in Act$

- if $r(C) \stackrel{\varepsilon}{\Longrightarrow} \overline{p} \stackrel{\mu}{\longrightarrow} \overline{p}'$, then
  - either $\mu = \tau$ and $r(q) \stackrel{\varepsilon}{\Longrightarrow} \overline{q}'$ such that $\overline{p} \approx_{br} R \approx_{br} \overline{q}'$ and $\overline{p}' \approx_{br} R \approx_{br} \overline{q}'$;
  - or $r(q) \stackrel{\varepsilon}{\Longrightarrow} \overline{q} \stackrel{\mu}{\longrightarrow} \overline{q}'$ such that $\overline{p} \approx_{br} R \approx_{br} \overline{q}$ and $\overline{p}' \approx_{br} R \approx_{br} \overline{q}'$ (or $\overline{p}' = \theta = \overline{q}'$).
- symmetrically if $r(q)$ moves first.

If R is a branching bisimulation up to, then $R \subseteq \approx_{br}$ and so $C \approx_{br} q$; since we will show that the rootedness conditions are satisfied for the pair (C,q), we get the thesis $C \approx_{brc} q$.

If $r(C) \stackrel{\varepsilon}{\Longrightarrow} \overline{p} \stackrel{\mu}{\longrightarrow} \overline{p}'$ with $\mu \in Act$, then we have three cases: (i) either $\overline{p} = \overline{r}(C)$, $\overline{p}' = \overline{r}'(C)$ because $r \stackrel{\varepsilon}{\Longrightarrow} \overline{r} \stackrel{\mu}{\longrightarrow} \overline{r}'$ (with $\overline{r} \neq x$); (ii) or $r(C) \stackrel{\varepsilon}{\Longrightarrow} \overline{r}(C) \stackrel{\mu}{\longrightarrow} x(C)$, (iii) or even $r(C) \stackrel{\varepsilon}{\Longrightarrow} x(C) \stackrel{\varepsilon}{\Longrightarrow} p' \stackrel{\mu}{\longrightarrow} p''$.

(i) If $r(C) \stackrel{\varepsilon}{\Longrightarrow} \overline{p} \stackrel{\mu}{\longrightarrow} \overline{p}'$ (with $\overline{p}' \neq \theta$) because $r \stackrel{\varepsilon}{\Longrightarrow} \overline{r} \stackrel{\mu}{\longrightarrow} \overline{r}'$ (with $\overline{r} \neq x$) and $\overline{p} = \overline{r}(C)$, $\overline{p}' = \overline{r}'(C)$, then by (repeated use of) Lemma 4, also $r(q) \stackrel{\varepsilon}{\Longrightarrow} \overline{r}(q) \stackrel{\mu}{\longrightarrow} \overline{r}'(q)$ with $(\overline{r}(C), \overline{r}(q)) \in R$ and $(\overline{r}'(C), \overline{r}'(q)) \in R$. If $\overline{p}' = \theta$, then also $\overline{r}' = \theta$, so the thesis follows trivially.

(ii) If $r(C) \stackrel{\varepsilon}{\Longrightarrow} \overline{r}(C) \stackrel{\mu}{\longrightarrow} x(C)$, then $r(q)$ replies by mimicking each move, by Lemma 4, with $r(q) \stackrel{\varepsilon}{\Longrightarrow} \overline{r}(q) \stackrel{\mu}{\longrightarrow} x(q)$, and we have that $(\overline{r}(C), \overline{r}(q)) \in R$ and $(C, q) \in R$.

(iii) If $r(C) \stackrel{\varepsilon}{\Longrightarrow} x(C) \stackrel{\varepsilon}{\Longrightarrow} p' \stackrel{\mu}{\longrightarrow} p''$, then derivation $r(C) \stackrel{\varepsilon}{\Longrightarrow} x(C)$ (due to $r \stackrel{\varepsilon}{\Longrightarrow} x$), is matched by $r(q)$ by mimicking each move, by Lemma 4, with $r(q) \stackrel{\varepsilon}{\Longrightarrow} x(q)$ and $(C, q) \in R$. Now, derivation $x(C) \stackrel{\varepsilon}{\Longrightarrow} p' \stackrel{\mu}{\longrightarrow} p''$ ensures that $p(C) \stackrel{\varepsilon}{\Longrightarrow} p' \stackrel{\mu}{\longrightarrow} p''$ because $C \doteq p(C)$. Since p is observationally guarded, $p' = t'(C)$ and $p'' = t''(C)$ by (repeated use of) Lemma 4, so that $p \stackrel{\varepsilon}{\Longrightarrow} t' \stackrel{\mu}{\longrightarrow} t''$ is derivable, too. This ensures that also derivation $p(q) \stackrel{\varepsilon}{\Longrightarrow} t'(q) \stackrel{\mu}{\longrightarrow} t''(q)$ is derivable. Since $q \approx_{br} p(q)$, by Lemma 2 there exists q' such that $x(q) \stackrel{\varepsilon}{\Longrightarrow} q'$ with $t'(q) \approx_{br} q'$. Now, transition $t'(q) \stackrel{\mu}{\longrightarrow} t''(q)$ can be matched by q' with:

- Either $\mu = \tau$ and $q' \stackrel{\varepsilon}{\Longrightarrow} q''$ with $t'(q) \approx_{br} q''$ and $t''(q) \approx_{br} q''$. Summing up, if $r(C) \stackrel{\varepsilon}{\Longrightarrow} t'(C) \stackrel{\tau}{\longrightarrow} t''(C)$, then $r(q) \stackrel{\varepsilon}{\Longrightarrow} q''$ such that $t'(C) R t'(q) \approx_{br} q''$ and $t''(C) R t''(q) \approx_{br} q''$, as required.

- Or $q' \stackrel{\varepsilon}{\Longrightarrow} \overline{q} \stackrel{\mu}{\longrightarrow} q''$ with $t'(q) \approx_{br} \overline{q}$ and $t''(q) \approx_{br} q''$. Summing up, if $r(C)$ moves as $r(C) \stackrel{\varepsilon}{\Longrightarrow} t'(C) \stackrel{\mu}{\longrightarrow} t''(C)$, then $r(q) \stackrel{\varepsilon}{\Longrightarrow} \overline{q} \stackrel{\mu}{\longrightarrow} q''$ such that $t'(C) R t'(q) \approx_{br} \overline{q}$ and, moreover, $t''(C) R t''(q) \approx_{br} q''$, as required.

If $p'' = t''(C) = \theta$, then also $t''(q) = \theta = q''$ and the thesis follows trivially. The case when $r(q)$ moves first is analogous, so omitted. Therefore, R is a branching bisimulation up to $\approx_{br}$, so that we can conclude that $C \approx_{br} q$, since $(C, q) \in R$.

Actually, we can prove that $C \approx_{brc} q$. Since $C \doteq p(C)$, if $C \stackrel{\mu}{\longrightarrow} \overline{p}$, then $p(C) \stackrel{\mu}{\longrightarrow} \overline{p}$; since p is guarded, transition $p \stackrel{\mu}{\longrightarrow} p'$ is derivable, with $\overline{p} = p'(C)$. So, $p(q) \stackrel{\mu}{\longrightarrow} p'(q)$ is derivable, too. Since $q \approx_{brc} p(q)$, transition $p(q) \stackrel{\mu}{\longrightarrow} p'(q)$ must be matched by $q \stackrel{\mu}{\longrightarrow} q'$ with $p'(q) \approx_{br} q'$. Summing up, if $C \stackrel{\mu}{\longrightarrow} p'(C)$, then $q \stackrel{\mu}{\longrightarrow} q'$ with $p'(C) R p'(q) \approx_{br} q'$. As we have proved that R is a branching bisimulation up to $\approx_{br}$, then the condition $p'(C) R p'(q) \approx_{br} q'$ is equivalent to $p'(C) \approx_{br} q'$, as required by the definition of rooted branching bisimilarity. The case when q moves first is symmetric, hence omitted. □

Observe that the condition that p is observationally guarded is crucial for the correctness of the branching folding property. The same example discussed after Proposition 27 applies also in this case.

Proposition 32. (Other laws of the constant for $\approx_{brc}$) For each $p, r \in \mathscr{P}_{CFM}^{grd}$, and each $C, D \in \mathscr{C}$, the following hold:

1. If $C \doteq (\tau.x + p)\{C/x\}$ and $D \doteq (\tau.p + \mathbf{0})\{D/x\}$, then
   $C \approx_{brc} D$ (b-excision)
2. If $C \doteq (\tau.(\tau.x + p) + r)\{C/x\}$ and $D \doteq (\tau.(p + r) + r)\{D/x\}$, then
   $C \approx_{brc} D$ (b-out)
3. If $C \doteq (\tau.(\tau.q + p) + r)\{C/x\}$, x unguarded in $q \in \mathscr{P}_{CFM}^{grd}$
   and $D \doteq (\tau.(q + p) + r)\{D/x\}$, then
   $C \approx_{brc} D$ (gen-out)
4. If $C \doteq (\tau.(\tau.x + p) + \tau.(\tau.x + q) + r)\{C/x\}$,
   and $D \doteq (\tau.(\tau.x + p + q) + r)\{D/x\}$, then
   $C \approx_{brc} D$ (fuse)

PROOF. (B-excision) Relation $R = \{(C,D), (C, (p+\mathbf{0})\{D/x\})\} \cup \{(p'\{C/x\}, p'\{D/x\}) \mid p' \in reach(p)\}$ is a branching bisimulation. Now, the proof that $C \approx_{brc} D$ follows easily. If $C \xrightarrow{\tau} C$, then $D \xrightarrow{\tau} (p + \mathbf{0})\{D/x\}$ and $C \approx_{br} (p + \mathbf{0})\{D/x\}$. If $C \xrightarrow{\mu} p'\{C/x\}$ because $p \xrightarrow{\mu} p'$, then D replies with $D \xrightarrow{\mu} p'\{C/x\}$ and $p'\{C/x\} \approx_{br} p'\{C/x\}$ (or $p' = \theta$). Symmetrically, if D moves first.

(B-out) Relation $R = \{(C,D), ((\tau.x+p)\{C/x\}, (p+r)\{D/x\}), (C, (p+r)\{D/x\})\} \cup \{(t'\{C/x\}, t'\{D/x\}) \mid t \in reach(p + r)\}$ is a branching bisimulation. Now, the proof that $C \approx_{brc} D$ follows easily. If $C \xrightarrow{\tau} (\tau.x + p)\{C/x\}$, then $D \xrightarrow{\tau} (p + r)\{D/x\}$ and $(\tau.x+p)\{C/x\} \approx_{br} (p+r)\{D/x\}$. If $C \xrightarrow{\mu} r'\{C/x\}$ because $r \xrightarrow{\mu} r'$, then $D \xrightarrow{\mu} r'\{D/x\}$ and $r'\{C/x\} \approx_{br} r'\{D/x\}$. Symmetrically if D moves first.

(Gen-out) Relation $R = \{(C,D), (\tau.(q+p)\{C/x\}, (q+p)\{D/x\}), ((q+p)\{C/x\}, (q+p)\{D/x\})\} \cup \{(t'\{C/x\}, t'\{D/x\}) \mid t \in reach(p+q+r)\}$ is a branching bisimulation. Now, the proof that $C \approx_{brc} D$ follows easily. If $C \xrightarrow{\tau} \tau.(q + p)\{C/x\}$, then $D \xrightarrow{\tau} (q + p)\{D/x\}$ and $(q + p)\{C/x\} \approx_{br} (q + p)\{D/x\}$. If $C \xrightarrow{\mu} r'\{C/x\}$ because $r \xrightarrow{\mu} r'$, then $D \xrightarrow{\mu} r'\{D/x\}$ and $r'\{C/x\} \approx_{br} r'\{D/x\}$. Symmetrically if D moves first.

(Fuse) Relation $R = \{(C,D), ((\tau.x+p)\{C/x\}, (\tau.x+p+q)\{D/x\}), ((\tau.x+q)\{C/x\}, (\tau.x + p + q)\{D/x\})\} \cup \{(t'\{C/x\}, t'\{D/x\}) \mid t \in reach(p + q + r)\}$ is a branching bisimulation. Now, the proof that $C \approx_{brc} D$ follows easily. If $C \xrightarrow{\tau} (\tau.x+p)\{C/x\}$, then $D \xrightarrow{\tau} (\tau.x + p + q)\{C/x\}$ and $(\tau.x + p)\{C/x\} \approx_{br} (\tau.x + p + q)\{D/x\}$. If $C \xrightarrow{\tau} (\tau.x + q)\{C/x\}$, then $D \xrightarrow{\tau} (\tau.x + p + q)\{C/x\}$ and $(\tau.x + q)\{C/x\} \approx_{br} (\tau.x + p + q)\{D/x\}$. If $C \xrightarrow{\mu} r'\{C/x\}$ because $r \xrightarrow{\mu} r'$, then $D \xrightarrow{\mu} r'\{D/x\}$ and $r'\{C/x\} \approx_{br} r'\{D/x\}$. Symmetrically if D moves first. □

Since rooted branching bisimilarity is finer than rooted weak bisimilarity, the laws above, that we proved correct for the former, are sound also for the latter.

7. Axiomatizations

Now we provide a sound and (ground-)complete, finite axiomatization of rooted weak/branching team equivalence over observationally guarded CFM processes. For simplicity's sake, the syntactic definition of open CFM is given with only one syntactic category, but each ground instantiation of an axiom must respect the syntactic definition

A1	Associativity	$x+(y+z) = (x+y)+z$	
A2	Commutativity	$x+y = y+x$	
A3	Identity	$x+\mathbf{0} = x$	if $x \neq \mathbf{0}$
A4	Idempotence	$x+x = x$	if $x \neq \mathbf{0}$
W1		$\mu.\tau.x = \mu.x$	if $x \neq \mathbf{0}$
W2		$x+\tau.x = \tau.x$	
W3		$\mu.(x+\tau.y) = \mu.(x+\tau.y)+\mu.y$	
B		$\mu.(\tau.(x+y)+x) = \mu.(x+y)$	
R1	Stuck	if $C \doteq \mathbf{0}$,	then $C = \mathbf{0}+\mathbf{0}$
R2	Unfolding	if $C \doteq p \wedge p \neq \mathbf{0}$,	then $C = p$
R3	Folding	if $C \doteq p\{C/x\} \wedge q = p\{q/x\}$,	then $C = q$
R3′	BW-Folding	if $C \doteq p\{C/x\} \wedge og(p) \wedge q = p\{q/x\}$,	then $C = q$
WU1		if $C \doteq (\tau.x+p)\{C/x\} \wedge D \doteq (\tau.(p+\mathbf{0}))\{D/x\}$	then $C = D$
BU1		if $C \doteq (\tau.x+p)\{C/x\} \wedge D \doteq (\tau.(p+\mathbf{0})+p)\{D/x\}$	then $C = D$
WU2		if $C \doteq (\tau.(\tau.x+p)+r)\{C/x\} \wedge D \doteq (\tau.(p+r))\{D/x\}$	then $C = D$
BU2		if $C \doteq (\tau.(\tau.x+p)+r)\{C/x\} \wedge D \doteq (\tau.(p+r)+r)\{D/x\}$	then $C = D$
U3		if $C \doteq (\tau.(\tau.q+p)+r)\{C/x\} \wedge D \doteq (\tau.(q+p)+r)\{D/x\}$, x unguarded in $q \in \mathscr{P}_{CFM}^{grd}$,	then $C = D$
U4		if $C \doteq (\tau.(\tau.x+p)+\tau.(\tau.x+q)+r)\{C/x\}$ $\wedge D \doteq (\tau.(\tau.x+p+q)+r)\{D/x\}$	then $C = D$
P1	Associativity	$x\|(y\|z) = (x\|y)\|z$	
P2	Commutativity	$x\|y = y\|x$	
P3	Identity	$x\|\mathbf{0} = x$	

Table 4: Axioms for rooted weak/branching team equivalence

of CFM given in Section 5; this means that we can write the axiom $x+(y+z) = (x+y)+z$, but it is invalid to instantiate it to $C+(a.\mathbf{0}+b.\mathbf{0}) = (C+a.\mathbf{0})+b.\mathbf{0}$ because these are not legal CFM processes (the constant C cannot be used as a summand).

The set of axioms is outlined in Table 4. By the notation $E \vdash p = q$ we mean that there exists an equational deduction proof of the equality $p = q$, by using the axioms in the set E. Besides the usual equational deduction rules of reflexivity, symmetry, transitivity, substitutivity and instantiation (see, e.g., [14]), in order to deal with constants we need also the following recursion congruence rule:

$$\frac{p = q \ \wedge \ A \doteq p\{A/x\} \ \wedge \ B \doteq q\{B/x\}}{A = B}$$

SB is the set $\{$**A1, A2, A3, A4, R1, R2, R3, P1, P2, P3**$\}$ that constitutes a sound and complete, finite axiomatization of strong team equivalence [17]. The axioms **A1-A4** are the usual axioms for choice (originally in [24]) where, however, **A3-A4** have the side condition $x \neq \mathbf{0}$; hence, it is not possible to prove $SB \vdash \mathbf{0}+\mathbf{0} = \mathbf{0}$, as expected, because these two terms have a completely different semantics. The conditional axioms **R1-R3** are about process constants. Note that axiom **R2** requires that p is not $\mathbf{0}$. Note also that these conditional axioms are actually a finite collection of axioms, one for each constant definition: since the set $\mathscr{C}$ of process constants is finite, the instances of **R1-R3** are finitely many. Finally, we have axioms **P1-P3** for parallel composition.

We call WB_g the set SB of axioms, where **R3** is replaced by its more restrictive variant **R3′**, with the addition of the axioms $\{$**W1, W2, W3**$\}$ (originally in [20]). These axioms are the analogous of the three τ-laws for rooted weak bisimilarity. We call BB_g the set of axioms WB_g, where $\{$**W1, W2, W3**$\}$ are replaced by $\{$**B**$\}$ (originally in [12]). We will prove that WB_g (BB_g, respectively) is a sound and complete, finite axiomatization of rooted weak (rooted branching, resp.) team bisimilarity over observationally guarded CFM processes.

Finally, WB is obtained by adding to WB_g also the axioms $\{$**WU1, WU2, U3**$\}$, while BB is obtained by adding to BB_g also the axioms $\{$**BU1, BU2, U3, U4**$\}$. We will prove that these are sound and complete, finite axiomatization for rooted weak/rooted branching team equivalence for the whole of CFM, hence including observationally unguarded processes.

Theorem 9. (Soundness) For every $p, q \in \mathscr{P}_{CFM}$:
1) If $SB \vdash p = q$, then $p \sim^{\oplus} q$.
2) If $WB \vdash p = q$, then $p \approx_c^{\oplus} q$.
3) If $BB \vdash p = q$, then $p \approx_{brc}^{\oplus} q$.

PROOF. The proof is by induction on the proof of $E \vdash p = q$, where E can be SB, or WB or BB.

1) The thesis follows by observing that all the (closed instantiations of the) axioms in SB are sound by Propositions 23, 24 and 25.

2) All the axioms in WB which are also in SB are sound because they are sound also for strong team equivalence. Moreover, axiom **R3′** is sound by Proposition 27; axioms **W1, W2, W3** are sound by Proposition 26; axioms **WU1, WU2** and **U3** are sound for Propositions 28 and 32, respectively.

3) Similarly, all the axioms in BB are sound; axiom **R3′** is sound by Proposition 31, while axiom **B** is sound by Proposition 29, and, finally, axioms **BU1, BU2, U3, U4** are sound by Proposition 32. □

7.1. Normal Forms, Unique Solutions and Saturated Normal Forms

A CFM sequential process p is a *normal form* if the predicate $nf(p)$ holds. This predicate stands for $nf(p, \emptyset)$, whose inductive definition is displayed in Table 5. Examples of terms which are not in normal form are $a.b.\mathbf{0}$ and $C \doteq a.b.C$.

Note that if C is a normal form, then its body (ignoring possible empty summands $\mathbf{0}$, that can be absorbed) is of the form $\sum_{i=1}^{n} \mu_i.C_i + \sum_{j=1}^{m} \mu'_j.\mathbf{0}$, (assuming that this term is $\mathbf{0}$ if $n = m = 0$) where, in turn, each C_i is a normal form.

We will show that, for each sequential CFM process p, there exists a normal form q such that $SB \vdash p = q$.

Proposition 33. (Reduction to normal form) Given a sequential CFM process p, there exists a normal form q such that $SB \vdash p = q$.

$$\frac{}{nf(\mathbf{0},I)} \qquad \frac{C \in I}{nf(C,I)} \qquad \frac{nf(p,I\cup\{C\}) \quad C \doteq p \quad C \notin I}{nf(C,I)}$$

$$\frac{}{nf(\mu.\mathbf{0},I)} \qquad \frac{nf(C,I)}{nf(\mu.C,I)} \qquad \frac{nf(p,I) \quad nf(q,I)}{nf(p+q,I)}$$

Table 5: Normal form predicate

PROOF. The proof is by induction on the structure of p, with the proviso to use a set I of already scanned constants, in order to avoid looping on recursively defined constants, where I is initially empty. We prove that for (p,I) there exists a term q such that $nf(q,I)$ holds and $(SB,I) \vdash p = q$, where this means that the equality $p = q$ can be derived by the axioms in SB when each constant $C \in I$ is assumed to be equated to itself. The thesis then follows by considering $(p,\emptyset)$.

The base case is $(\mathbf{0},I)$; in such a case, $q = \mathbf{0}$, because $nf(\mathbf{0},I)$ holds, and the thesis $(SB,I) \vdash \mathbf{0} = \mathbf{0}$ follows by reflexivity.

Case $(\mu.p,I)$: by induction, (p,I) has an associated normal form $nf(q,I)$ such that $(SB,I) \vdash p = q$; hence, $(SB,I) \vdash \mu.p = \mu.q$ by substitutivity. If q is a constant or $q = \mathbf{0}$, then $\mu.q$ is already a normal form. Otherwise, take a new constant $C \doteq q$, so that $nf(C,I)$ holds because $nf(q,I\cup\{C\})$ holds (note that C does not occur in q, so that this is the same as stating $nf(q,I)$, which holds by induction). The required normal form is $\mu.C$. Indeed, $nf(\mu.C,I)$ holds because $nf(C,I)$ holds; moreover, since $(SB,I) \vdash p = q$ by induction and $(SB,I) \vdash C = q$ by axiom **R2**, we have that $(SB,I) \vdash p = C$ by transitivity, so that $(SB,I) \vdash \mu.p = \mu.C$ by substitutivity.

Case $(p_1 + p_2,I)$: by induction, we can assume that for (p_i,I) there exists a normal form $nf(q_i,I)$ such that $(SB,I) \vdash p_i = q_i$ for $i = 1,2$. Then, $nf(q_1 + q_2,I)$ holds and $(SB,I) \vdash p_1 + p_2 = q_1 + q_2$ by substitutivity.

Case (C,I), where $C \doteq r\{C/x\}$. If $C \in I$, then we can stop induction, by returning C itself: $nf(C,I)$ holds and $(SB,I) \vdash C = C$ by reflexivity. Otherwise (i.e., if $C \notin I$), if $r = \mathbf{0}$, then by axiom **R1**, $(SB,I) \vdash C = \mathbf{0}+\mathbf{0}$, where $\mathbf{0}+\mathbf{0}$ is a normal form. If $r \neq \mathbf{0}$, then, by induction on $(r\{C/x\},I\cup\{C\})$, we can assume that there exists a normal form $q\{C/x\}$ such that $nf(q\{C/x\},I\cup\{C\})$ holds and $(SB,I\cup\{C\}) \vdash r\{C/x\} = q\{C/x\}$. Note that, by construction, if $(SB,I\cup\{C\}) \vdash r\{C/x\} = q\{C/x\}$, then also $(SB,I) \vdash r\{C/x\} = q\{C/x\}$. Hence, as $(SB,I) \vdash C = r\{C/x\}$ by axiom **R2**, it follows that $(SB,I) \vdash C = q\{C/x\}$ by transitivity. Then, we take a new constant $D \doteq q\{D/x\}$ such that $nf(D,I)$ because $nf(q\{D/x\},I\cup\{D\})$ holds. Hence, $(SB,I) \vdash C = D$ by axiom **R3**, where D is a normal form. □

Remark 3. (Normal forms as systems of equations) As a matter of fact, we can restrict our attention only to normal forms defined by constants: if p is a normal form, $p \neq \mathbf{0}$ and p is not a constant, then take a new constant $D \doteq p$, which is a normal form such that $SB \vdash D = p$ by axiom **R2**. For this reason, in the following we often restrict our attention to normal forms that can be defined by means of a *system of equations*: a set $\widetilde{C} = \{C_1, C_2, \ldots, C_n\}$ of defined constants in normal form, such that $Const(C_1) = \widetilde{C}$ and $Const(C_i) \subseteq \widetilde{C}$ for $i = 1, \ldots, n$; the equations of the system $E(\widetilde{C})$ are of the following form (where μ and ν stands for any kind of action and function $f(i, j)$ returns a value k such that $1 \leq k \leq n$):

$$\begin{array}{lcl} C_1 & \doteq & \sum_{i=1}^{m(1)} \mu_{1i}.C_{f(1,i)} + \sum_{j=1}^{\overline{m}(1)} \nu_{1j}.\mathbf{0} \\ C_2 & \doteq & \sum_{i=1}^{m(2)} \mu_{2i}.C_{f(2,i)} + \sum_{j=1}^{\overline{m}(2)} \nu_{2j}.\mathbf{0} \\ \ldots & & \\ C_n & \doteq & \sum_{i=1}^{m(n)} \mu_{ni}.C_{f(n,i)} + \sum_{j=1}^{\overline{m}(n)} \nu_{nj}.\mathbf{0} \end{array}$$

where $C_i \doteq \mathbf{0}$ in case $m(i) = 0 = \overline{m}(i)$. We sometimes use the notation $body(C_h)$ to denote the sumform $\sum_{i=1}^{m(h)} \mu_{hi}.C_{f(h,i)} + \sum_{j=1}^{\overline{m}(h)} \nu_{hj}.\mathbf{0}$. □

Remark 4. (Observationally guarded system of equations) We say that a system of equations $E(\widetilde{C})$, not necessarily in normal form like the one above, is *observationally guarded* if, for each $C_i \in \widetilde{C}$, there is no silent cycle $C_i \stackrel{\tau}{\Longrightarrow} C_i$. This definition is coherent with Definition 21 For instance, consider again the following example:

$$\begin{array}{lcl} C_1 & \doteq & a.C_2 + \tau.a.C_1 \\ C_2 & \doteq & b.C_2 + \tau.C_1 \end{array}$$

Of course, $C_i \stackrel{\tau}{\not\Rightarrow} C_i$ for $i = 1, 2$, so that the system is observationally guarded. And indeed we proved that $og(C_1)$ and $og(C_2)$ hold. □

The next theorem (and the following corollary) shows that every observationally guarded system of equations (not necessarily in normal form) has a unique solution up to provably equality.

Theorem 10. (Unique solution for WB_g) Let $\widetilde{X} = (x_1, x_2, \ldots, x_n)$ be a tuple of variables and let $\widetilde{p} = (p_1, p_2, \ldots, p_n)$ be a tuple of open guarded CFM terms, using the variables in $\widetilde{X}$. Let $\widetilde{C} = (C_1, C_2, \ldots, C_n)$ be a tuple of constants (not occurring in $\widetilde{p}$) such that the system of equations

$$\begin{array}{lcl} C_1 & \doteq & p_1\{\widetilde{C}/\widetilde{X}\} \\ C_2 & \doteq & p_2\{\widetilde{C}/\widetilde{X}\} \\ \ldots & & \\ C_n & \doteq & p_n\{\widetilde{C}/\widetilde{X}\} \end{array}$$

is observationally guarded. Then, for $i = 1, \ldots, n$, $WB_g \vdash C_i = p_i\{\widetilde{C}/\widetilde{X}\}$ if $p_i \neq \mathbf{0}$, while $WB_g \vdash C_i = \mathbf{0} + \mathbf{0}$ if $p_i = \mathbf{0}$.

Moreover, if the same property holds for $\widetilde{q} = (q_1, q_2, \ldots, q_n)$, i.e., for $i = 1, \ldots, n$ $WB_g \vdash q_i = p_i\{\widetilde{q}/\widetilde{X}\}$ (or $WB_g \vdash q_i = \mathbf{0} + \mathbf{0}$ if $p_i = \mathbf{0}$), then $WB_g \vdash C_i = q_i$.

PROOF. By induction on n. For $n = 1$, if $p_1 \neq \mathbf{0}$, we have $C_1 \doteq p_1\{C_1/x_1\}$, and so the result $WB_g \vdash C_1 = p_1\{C_1/x_1\}$ follows immediately using axiom **R2**. This solution is unique because p_1 is observationally guarded (cf. Remark 4): if $WB_g \vdash q_1 = p_1\{q_1/x_1\}$, by axiom **R3′** we get $WB_g \vdash C_1 = q_1$. In case $p_1 = \mathbf{0}$, then, by axiom **R1**, we have $WB_g \vdash C_1 = \mathbf{0} + \mathbf{0}$. Since q_1 is such that $WB_g \vdash q_1 = \mathbf{0} + \mathbf{0}$, the thesis $WB_g \vdash C_1 = q_1$ follows by transitivity.

Now assume a tuple $\widetilde{p} = (p_1, p_2, \dots, p_n)$ and the term p_{n+1}, so that they are all open on $\widetilde{X} = (x_1, x_2, \dots, x_n)$ and the additional x_{n+1}. Assume, w.l.o.g., that x_{n+1} occurs in p_{n+1}. First, define $C_{n+1} \doteq p_{n+1}\{C_{n+1}/x_{n+1}\}$, so that C_{n+1} is now open on $\widetilde{X}$. Therefore, also for $i = 1, \dots, n$, each $p_i\{C_{n+1}/x_{n+1}\}$ is now open on $\widetilde{X}$. The resulting observationally guarded system of equations is:

$$\begin{array}{lcl} C_1 & \doteq & p_1\{C_{n+1}/x_{n+1}\}\{\widetilde{C}/\widetilde{X}\} \\ C_2 & \doteq & p_2\{C_{n+1}/x_{n+1}\}\{\widetilde{C}/\widetilde{X}\} \\ \dots & & \\ C_n & \doteq & p_n\{C_{n+1}/x_{n+1}\}\{\widetilde{C}/\widetilde{X}\} \end{array}$$

so that now we can use induction on $\widetilde{X}$ and $(p_1\{C_{n+1}/x_{n+1}\}, \dots, p_n\{C_{n+1}/x_{n+1}\})$, to conclude that the tuple $\widetilde{C} = (C_1, C_2, \dots, C_n)$ of closed constants is such that, for $i = 1, \dots, n$, if $p_i \neq \mathbf{0}$, then:

$$WB_g \vdash C_i = (p_i\{C_{n+1}/x_{n+1}\})\{\widetilde{C}/\widetilde{X}\} = p_i\{\widetilde{C}/\widetilde{X}, C_{n+1}\{\widetilde{C}/\widetilde{X}\}/x_{n+1}\},$$

while, in case $p_i = \mathbf{0}$, $WB_g \vdash C_i = \mathbf{0} + \mathbf{0}$. Note that above by $C_{n+1}\{\widetilde{C}/\widetilde{X}\}$ we have implicitly closed the definition of C_{n+1} as

$$C_{n+1} \doteq p_{n+1}\{C_{n+1}/x_{n+1}\}\{\widetilde{C}/\widetilde{X}\} = p_{n+1}\{\widetilde{C}/\widetilde{X}\}\{C_{n+1}/x_{n+1}\},$$

so that $WB_g \vdash C_{n+1} = p_{n+1}\{\widetilde{C}/\widetilde{X}\}\{C_{n+1}/x_{n+1}\}$ by axiom **R2**.

Unicity of the tuple $(\widetilde{C}, C_{n+1})$ can be proved by using axiom **R3′**. Assume to have another solution tuple $(\widetilde{q}, q_{n+1})$. This means that, for $i = 1, \dots, n+1$, if $p_i \neq \mathbf{0}$, then

$$WB_g \vdash q_i = p_i\{\widetilde{q}/\widetilde{X}, q_{n+1}/x_{n+1}\},$$

while, in case $p_i = \mathbf{0}$, $WB_g \vdash q_i = \mathbf{0} + \mathbf{0}$. By induction, we can assume, for $i = 1, \dots, n$, that $WB_g \vdash C_i = q_i$.

Since $WB_g \vdash C_{n+1} = p_{n+1}\{\widetilde{C}/\widetilde{X}\}\{C_{n+1}/x_{n+1}\}$ by axiom **R2**, by substitutivity we get $WB_g \vdash C_{n+1} = p_{n+1}\{\widetilde{q}/\widetilde{X}\}\{C_{n+1}/x_{n+1}\}$. Note that $p_{n+1}\{\widetilde{q}/\widetilde{X}\}$ is term open on x_{n+1} which is observationally guarded. Let F be a constant defined as follows: $F \doteq p_{n+1}\{\widetilde{q}/\widetilde{X}\}\{F/x_{n+1}\}$. Then, by axiom **R3′**, $C_{n+1} = F$. Hence, since

$$WB_g \vdash q_{n+1} = p_{n+1}\{\widetilde{q}/\widetilde{X}\}\{q_{n+1}/x_{n+1}\}$$

by axiom **R3′**, we get $WB_g \vdash F = q_{n+1}$; and so the thesis $WB_g \vdash C_{n+1} = q_{n+1}$ follows by transitivity. □

Corollary 5. (Unique solution for BB_g) Let $\widetilde{X} = (x_1, x_2, \dots, x_n)$ be a tuple of variables and let $\widetilde{p} = (p_1, p_2, \dots, p_n)$ be a tuple of open guarded CFM terms, using the variables in $\widetilde{X}$. Let $\widetilde{C} = \{C_1, C_2, \dots, C_n\}$ be a set of constants such that the system of equations

$$\begin{array}{lcl} C_1 & \doteq & p_1\{\widetilde{C}/\widetilde{X}\} \\ C_2 & \doteq & p_2\{\widetilde{C}/\widetilde{X}\} \\ \dots & & \\ C_n & \doteq & p_n\{\widetilde{C}/\widetilde{X}\} \end{array}$$

is observationally guarded. Then, for $i = 1, \dots, n$, $BB_g \vdash C_i = p_i\{\widetilde{C}/\widetilde{X}\}$ if $p_i \neq \mathbf{0}$, while $BB_g \vdash C_i = \mathbf{0} + \mathbf{0}$ if $p_i = \mathbf{0}$.

Moreover, if the same property holds for $\widetilde{q} = (q_1, q_2, \dots, q_n)$, i.e., for $i = 1, \dots, n$ $BB_g \vdash q_i = p_i\{\widetilde{q}/\widetilde{X}\}$ (or $BB_g \vdash q_i = \mathbf{0} + \mathbf{0}$ if $p_i = \mathbf{0}$), then $BB_g \vdash C_i = q_i$.

PROOF. It is the same as the one for Theorem 10. □

A normal form p is *saturated* if whenever $p \stackrel{\mu}{\Longrightarrow} p'$, then p' is saturated and, if p is not a constant, then $\mu.p'$ is a summand of p, instead if $p = C$ (with $C \doteq q$), then $\mu.p'$ is a summand of q. As an example, consider the following normal form (i.e., system of equations):

$$C_1 \doteq a.C_1 + \tau.C_2$$
$$C_2 \doteq \tau.C_3 + b.C_2 + c.\mathbf{0}$$
$$C_3 \doteq d.C_1 + \tau.\mathbf{0}$$

This is not saturated because, e.g., $C_1 \stackrel{b}{\Longrightarrow} C_2$, but $b.C_2$ is not a summand of the body of C_1. However, by using the axioms in WB_g, we can equate this to another normal form which is saturated:

$$D_1 \doteq a.D_1 + \tau.D_2 + \tau.D_3 + b.D_2 + c.\mathbf{0} + d.D_1 + \tau.\mathbf{0} + d.D_2 + d.D_3$$
$$D_2 \doteq \tau.D_3 + b.D_2 + c.\mathbf{0} + d.D_1 + \tau.\mathbf{0} + d.D_2 + d.D_3$$
$$D_3 \doteq d.D_1 + \tau.\mathbf{0} + d.D_2 + d.D_3$$

Let us now prove that a normal form can always be saturated.

Lemma 6. (Saturation Lemma)
If p is a normal form and $p \stackrel{\mu}{\Longrightarrow} p'$, then $WB_g \vdash p = p + \mu.p'$.

PROOF. By induction on the length of $p \stackrel{\varepsilon}{\Longrightarrow} \stackrel{\mu}{\longrightarrow} \stackrel{\varepsilon}{\Longrightarrow} p'$. The proof is very similar to the original one [25] (also in [14]), that makes use of axioms **W2** and **W3**. □

Proposition 34. (Reduction to saturated normal form) For each normal form p, which is observationally guarded, there exists an observationally guarded, saturated normal form q such that $WB_g \vdash p = q$.

PROOF. By Remark 3, we may restrict our attention to an observationally guarded normal form of the form $E(\widetilde{C})$:

$$C_1 \doteq \sum_{i=1}^{m(1)} \mu_{1i}.C_{f(1,i)} + \sum_{j=1}^{\overline{m}(1)} \nu_{1j}.\mathbf{0}$$
$$C_2 \doteq \sum_{i=1}^{m(2)} \mu_{2i}.C_{f(2,i)} + \sum_{j=1}^{\overline{m}(2)} \nu_{2j}.\mathbf{0}$$
$$\dots$$
$$C_n \doteq \sum_{i=1}^{m(n)} \mu_{ni}.C_{f(n,i)} + \sum_{j=1}^{\overline{m}(n)} \nu_{nj}.\mathbf{0}$$

Then, for each C_h, we have two cases: Either $C_h \doteq \mathbf{0}$ and in such a case, by axiom **R1**, $WB_g \vdash C_h = \mathbf{0} + \mathbf{0}$ with $\mathbf{0} + \mathbf{0}$ a saturated normal form. Or take the set

$$R_h = \{(\mu_k^h, C_{g(h,k)}) \mid C_h \stackrel{\mu_k^h}{\Longrightarrow} C_{g(h,k)}, C_h \stackrel{\mu_k^h}{\not\longrightarrow} C_{g(h,k)}\} \cup \{(\nu_k^h, \mathbf{0}) \mid C_h \stackrel{\nu_k^h}{\Longrightarrow} \mathbf{0}, C_h \stackrel{\mu_k^h}{\not\longrightarrow} \mathbf{0}\},$$

where g is a function returning a value in the range $\{1, \dots, n\}$. By the saturation Lemma 6, we have that

$$WB_g \vdash C_h = \sum_{i=1}^{m(h)} \mu_{hi}.C_{f(h,i)} + \sum_{j=1}^{\overline{m}(h)} \nu_{hj}.\mathbf{0} + \sum_{(\mu_k^h, C_{g(h,k)}) \in R_h} \mu_k^h.C_{g(h,k)} + \sum_{(\nu_k^h, \mathbf{0}) \in R_h} \nu_k^h.\mathbf{0}.$$

Therefore, if we saturate each C_h in this way, we get

$WB_g \vdash$

$$C_1 = \sum_{i=1}^{m(1)} \mu_{1i}.C_{f(1,i)} + \sum_{j=1}^{\overline{m}(1)} \nu_{1j}.\mathbf{0} + \sum_{(\mu_k^1, C_{g(1,k)}) \in R_1} \mu_k^1.C_{g(1,k)} + \sum_{(\nu_k^1, \mathbf{0}) \in R_1} \nu_k^1.\mathbf{0}$$
$$C_2 = \sum_{i=1}^{m(2)} \mu_{2i}.C_{f(2,i)} + \sum_{j=1}^{\overline{m}(2)} \nu_{2j}.\mathbf{0} + \sum_{(\mu_k^2, C_{g(2,k)}) \in R_2} \mu_k^2.C_{g(2,k)} + \sum_{(\nu_k^2, \mathbf{0}) \in R_2} \nu_k^2.\mathbf{0}$$
$$\dots$$
$$C_n = \sum_{i=1}^{m(n)} \mu_{ni}.C_{f(n,i)} + \sum_{j=1}^{\overline{m}(n)} \nu_{nj}.\mathbf{0} + \sum_{(\mu_k^n, C_{g(n,k)}) \in R_n} \mu_k^n.C_{g(n,k)} + \sum_{(\nu_k^n, \mathbf{0}) \in R_n} \nu_k^n.\mathbf{0}$$

where, by the "either" case above, some of the C_h may be actually equated to $\mathbf{0}+\mathbf{0}$. Hence, it is enough to take a tuple $\widetilde{D} = (D_1, D_2, \dots, D_n)$ of new constants, defined as the following system of equations $F(\widetilde{D})$:

$$D_1 \doteq \sum_{i=1}^{m(1)} \mu_{1i}.D_{f(1,i)} + \sum_{j=1}^{\overline{m}(1)} \nu_{1j}.\mathbf{0} + \sum_{(\mu_k^1, C_{g(1,k)}) \in R_1} \mu_k^1.D_{g(1,k)} + \sum_{(\nu_k^1, \mathbf{0}) \in R_1} \nu_k^1.\mathbf{0}$$
$$D_2 \doteq \sum_{i=1}^{m(2)} \mu_{2i}.D_{f(2,i)} + \sum_{j=1}^{\overline{m}(2)} \nu_{2j}.\mathbf{0} + \sum_{(\mu_k^2, C_{g(2,k)}) \in R_2} \mu_k^2.D_{g(2,k)} + \sum_{(\nu_k^2, \mathbf{0}) \in R_2} \nu_k^2.\mathbf{0}$$
$$\dots$$
$$D_n \doteq \sum_{i=1}^{m(n)} \mu_{ni}.D_{f(n,i)} + \sum_{j=1}^{\overline{m}(n)} \nu_{nj}.\mathbf{0} + \sum_{(\mu_k^n, C_{g(n,k)}) \in R_n} \mu_k^n.D_{g(n,k)} + \sum_{(\nu_k^n, \mathbf{0}) \in R_n} \nu_k^n.\mathbf{0}$$

where actually $D_h \doteq \mathbf{0}$ in case $WB_g \vdash C_h = \mathbf{0}+\mathbf{0}$. Clearly, if $E(\widetilde{C})$ is observationally guarded, then also $F(\widetilde{D})$ is so. Moreover, $F(\widetilde{D})$ is clearly saturated. Finally, by Theorem 10 (unique solution), $WB_g \vdash C_h = D_h$ for $h = 1, \dots, n$, as required. □

A consequence of Proposition 33 and of the proposition above is that each observationally guarded, sequential CFM process can be equated, by means of the axioms in WB_g, to an observationally guarded, saturated normal form. Therefore, in the following proof of completeness of WB_g, we can focus our attention to observationally guarded, saturated normal forms.

Remark 5. (Weak bisimulation on saturated normal forms) Let p and q be two observationally guarded, saturated normal forms such that $p \approx q$. If $p \xrightarrow{\mu} p'$, then

- $q \xrightarrow{\mu} q'$ with $p' \approx q'$ (or $p' = \theta = q'$); or
- $\mu = \tau$ and $p' \approx q$;

and symmetrically, if $q \xrightarrow{\mu} q'$ then

- $p \xrightarrow{\mu} p'$ with $p' \approx q'$ (or $p' = \theta = q'$); or
- $\mu = \tau$ and $p \approx q'$.

As a matter of fact, if $p \xrightarrow{\mu} p'$, then, as $p \approx q$, either $q \stackrel{\mu}{\Longrightarrow} q'$ with $p' \approx q'$ (or $p' = \theta = q'$), or $\mu = \tau$ and $q \stackrel{\varepsilon}{\Longrightarrow} q'$ with $p' \approx q'$ (or $p' = \theta = q'$). In the former case, as q is saturated, also $q \xrightarrow{\mu} q'$ is a transition, so that the first condition is satisfied. In the latter case, if $q' \neq q$, then $q \stackrel{\tau}{\Longrightarrow} q'$, and we are in the same case as above: $q \xrightarrow{\tau} q'$ with $p' \approx q'$ (or $p' = \theta = q'$); instead, if $q' = q$, then the second condition is satisfied: $p' \approx q$. Symmetrically, if q moves first. □

7.2. Completeness of WB_g

Lemma 7. (Completeness for saturated normal forms) For every p, p' observationally guarded, saturated normal forms, if $p \approx_c p'$ (or $p = \mathbf{0} = p'$), then $WB_g \vdash p = p'$.

PROOF. If $p = \mathbf{0} = p'$, then $WB_g \vdash p = p'$ by reflexivity. Otherwise, by Remark 3, we can assume that p is (or is equated to) the saturated system of equations $E(\widetilde{C})$:

$$\begin{array}{lcl} C_1 & \doteq & \sum_{h=1}^{m(1)} \mu_{1h}.C_{f(1,h)} + \sum_{j=1}^{\overline{m}(1)} \nu_{1j}.\mathbf{0} \\ C_2 & \doteq & \sum_{h=1}^{m(2)} \mu_{2h}.C_{f(2,h)} + \sum_{j=1}^{\overline{m}(2)} \nu_{2j}.\mathbf{0} \\ \ldots & & \\ C_n & \doteq & \sum_{h=1}^{m(n)} \mu_{nh}.C_{f(n,h)} + \sum_{j=1}^{\overline{m}(n)} \nu_{nj}.\mathbf{0} \end{array}$$

where actually $C_i \doteq \mathbf{0}$ in case $m(i) = 0 = \overline{m}(i)$. For each $i = 1, \ldots, n$, in case $C_i \doteq \mathbf{0}$, by axiom **R1**, we get $WB_g \vdash C_i = \mathbf{0} + \mathbf{0}$; otherwise, we get $WB_g \vdash C_i = body(C_i)$, by axiom **R2**, where by $body(C_i)$ we denote the saturated normal form $\sum_{h=1}^{m(i)} \mu_{ih}.C_{f(i,h)} + \sum_{j=1}^{\overline{m}(i)} \nu_{ij}.\mathbf{0}$.
Similarly, by Remark 3 we can assume that p' is (or is equated to) the saturated system of equations $F(\widetilde{C'})$:

$$\begin{array}{lcl} C'_1 & \doteq & \sum_{h=1}^{m'(1)} \mu'_{1h}.C'_{f'(1,h)} + \sum_{j=1}^{\overline{m}'(1)} \nu'_{1j}.\mathbf{0} \\ C'_2 & \doteq & \sum_{h=1}^{m'(2)} \mu'_{2h}.C'_{f'(2,h)} + \sum_{j=1}^{\overline{m}'(2)} \nu'_{2j}.\mathbf{0} \\ \ldots & & \\ C'_{n'} & \doteq & \sum_{h=1}^{m'(n')} \mu'_{n'h}.C'_{f'(n',h)} + \sum_{j=1}^{\overline{m}'(n')} \nu'_{n'j}.\mathbf{0} \end{array}$$

For each $i = 1, \ldots, n'$, in case $C'_i \doteq \mathbf{0}$, by axiom **R1**, we get $WB_g \vdash C'_i = \mathbf{0} + \mathbf{0}$; otherwise, we get $WB_g \vdash C'_i = body(C'_i)$, by axiom **R2**, where by $body(C'_i)$ we denote the saturated normal form $\sum_{h=1}^{m'(i)} \mu'_{ih}.C'_{f'(i,h)} + \sum_{j=1}^{\overline{m}'(i)} \nu'_{ij}.\mathbf{0}$. Moreover, as $p \approx_c p'$, we have $C_1 \approx_c C'_1$.

Now, let $I = \{(i,i') \mid C_i \approx C'_{i'}\}$.

Note that if $C_i \approx C'_{i'}$, then $C_i \xrightarrow{\nu} \theta$ if and only if $C'_{i'} \xrightarrow{\nu} \theta$ for all ν. Therefore, the summands that ends immediately successfully are the same, up to reordering (axioms **A1-A2**) and the presence of possible duplicates that can be absorbed (axiom **A4**):

$WB_g \vdash \sum_{j=1}^{\overline{m}(i)} \nu_{ij}.\mathbf{0} = \sum_{j=1}^{\overline{m}'(i')} \nu'_{i'j}.\mathbf{0}$ for each $(i,i') \in I$.

Hence, we can equate these summands in the following.

Clearly, since $C_1 \approx_c C'_1$, we have that $(1,1) \in I$. Moreover, since C_1 and C'_1 are rooted weakly bisimilar and saturated, the following hold: for $(1,1) \in I$, there exists a total surjective relation J_{11} between $\{1,2,\ldots m(1)\}$ and $\{1,2,\ldots m'(1)\}$ given by

$J_{11} = \{(j,j') \mid \mu_{1j} = \mu'_{1j'} \wedge (f(1,j), f'(1,j')) \in I\}$.

For any other $(i,i') \in I$, since C_i and $C'_{i'}$ are only weakly bisimilar (and saturated! - see Remark 5), there exists a total surjective relation $J_{ii'}$ between $\{1,2,\ldots m(i)\}$ and $\{1,2,\ldots m'(i')\}$ given by $J_{ii'} = J^1_{ii'} \cup J^2_{ii'} \cup J^3_{ii'}$, where

$$\begin{array}{lcl} J^1_{ii'} & = & \{(j,j') \mid \mu_{ij} = \mu'_{i'j'} \wedge (f(i,j), f'(i',j')) \in I\} \\ J^2_{ii'} & = & \{(j,i') \mid \mu_{ij} = \tau \wedge (f(i,j), i') \in I\} \\ J^3_{ii'} & = & \{(i,j') \mid \mu'_{i'j'} = \tau \wedge (i, f'(i',j')) \in I\}. \end{array}$$

Now, for $(1,1) \in I$, let us consider the set of variables $\widetilde{X} = \{x_{ii'} \mid (i,i') \in I\}$ and the open term

$$t_{11} = \sum_{(j,j') \in J_{11}} \mu_{1j}.x_{f(1,j)f'(1,j')} + \sum_{j=1}^{\overline{m}(1)} \nu_{1j}.\mathbf{0}$$

and, for each $(i,i') \in I$, the open terms

$$t_{ii'} = \sum_{(j,j') \in J^1_{ii'}} \mu_{ij}.x_{f(i,j)f'(i',j')} + \sum_{(j,i') \in J^2_{ii'}} \tau.x_{f(i,j)i'} + \sum_{(i,j') \in J^3_{ii'}} \tau.x_{if'(i',j')} + \sum_{j=1}^{\overline{m}(i)} \nu_{ij}.\mathbf{0}$$

This gives rise to a system of equations $G(\widetilde{D})$, where $\widetilde{D} = \{D_{ii'} \mid (i,i') \in I\}$, of the following form:

$$\begin{array}{lcl} D_{11} & \doteq & t_{11}\{\widetilde{D}/\widetilde{X}\} \\ \ldots & & \\ D_{ii'} & \doteq & t_{ii'}\{\widetilde{D}/\widetilde{X}\} \\ \ldots & & \end{array}$$

The system $G(\widetilde{D})$ is observationally guarded and saturated by construction. For each $(i,i') \in I$, by axiom **R2**, we get $WB_g \vdash D_{ii'} = t_{ii'}\{\widetilde{D}/\widetilde{X}\}$ provided that $t_{ii'} \neq \mathbf{0}$. In case $t_{ii'} = \mathbf{0}$, then $WB_g \vdash D_{ii'} = \mathbf{0} + \mathbf{0}$ by axiom **R1**.

Now, let us consider the terms $q_i = C_i$ if $J^3_{ii'} = \emptyset$, $q_i = \tau.C_i$ otherwise, which are cumulatively represented as $\widetilde{q}$. If we close each $t_{ii'}$ by replacing $x_{f(i,j)f'(i',j')}$ with $C_{f(i,j)}$ in the first summation, $x_{f(i,j)i'}$ with $C_{f(i,j)}$ in the second summation and $x_{if'(i',j')}$ with $\tau.C_i$ in the third summation, we get the term $t_{ii'}\{\widetilde{q}/\widetilde{X}\} =$

$$\sum_{(j,j') \in J^1_{ii'}} \mu_{ij}.C_{f(i,j)} + \sum_{(j,i') \in J^2_{ii'}} \tau.C_{f(i,j)} + \sum_{(i,j') \in J^3_{ii'}} \tau.\tau.C_i + \sum_{j=1}^{\overline{m}(i)} \nu_{ij}.\mathbf{0}$$

which is provably equal to C_i in case $J^3_{ii'} = \emptyset$, or to $\tau.C_i$ otherwise. (Of course, the special case $t_{ii'}\{\widetilde{q}/\widetilde{X}\} = \mathbf{0}$ is trivial: this means that $C_i \doteq \mathbf{0}$ and so $WB_g \vdash C_i = \mathbf{0} + \mathbf{0}$.)

In fact, if $J^3_{ii'} = \emptyset$, then the summation $t_{ii'}\{\widetilde{q}/\widetilde{X}\}$ contains (with possible repetitions) exactly the same terms $\mu.C_k$ for which $C_i \xrightarrow{\mu} C_k$, and the thesis $WB_g \vdash C_i = t_{ii'}\{\widetilde{q}/\widetilde{X}\}$ follows from $WB_g \vdash C_i = \sum_{h=1}^{m(i)} \mu_{ih}.C_{f(i,h)} + \sum_{j=1}^{\overline{m}(i)} \nu_{ij}.\mathbf{0}$.

On the other hand, if $J^3_{ii'} \neq \emptyset$, then $t_{ii'}\{\widetilde{q}/\widetilde{X}\}$ contains, in addition, the terms $\tau.\tau.C_i$. In this case, we can first use axiom **W1** (instance $\tau.\tau.C_i = \tau.C_i$), then possibly axiom **A4** (instance $\tau.C_i + \tau.C_i = \tau.C_i$, for removing possible duplicates of $\tau.C_i$), then **R2** $\vdash$ $C_i = body(C_i)$ (where $body(C_i) = \sum_{h=1}^{m(i)} \mu_{ih}.C_{f(i,h)} + \sum_{j=1}^{\overline{m}(i)} \nu_{ij}.\mathbf{0}$), so that $WB_g \vdash \tau.C_i = \tau.body(C_i)$, then axiom **W2** (instance $\tau.body(C_i) = \tau.body(C_i) + body(C_i)$), in order to prove that $t_{ii'}\{\widetilde{q}/\widetilde{X}\}$ is provably equal to $\tau.C_i$, as follows:

$$\begin{array}{lll} WB_g & \vdash & \tau.C_i = \tau.C_i + body(C_i) = \\ & & = \tau.C_i + \sum_{(j,j') \in J^1_{ii'}} \mu_{ij}.C_{f(i,j)} + \sum_{(j,i') \in J^2_{ii'}} \tau.C_{f(i,j)} + \sum_{j=1}^{\overline{m}(i)} \nu_{ij}.\mathbf{0} = \\ & & = t_{ii'}\{\widetilde{q}/\widetilde{X}\}. \end{array}$$

By Theorem 10, since $WB_g \vdash C_1 = q_1 = t_{11}\{\widetilde{q}/\widetilde{X}\}$ and $D_{11} \doteq t_{11}\{\widetilde{D}/\widetilde{X}\}$, we have that $WB_g \vdash D_{11} = C_1$.

In exactly the same way, we can define the terms $q'_{i'} = C'_{i'}$ if $J^2_{ii'} = \emptyset$, $q'_{i'} = \tau.C'_{i'}$ otherwise, which are cumulatively represented as $\widetilde{q'}$. If we close each $t_{ii'}$ by replacing $x_{f(i,j)f'(i',j')}$ with $C'_{f'(i',j')}$ in the first summation, $x_{f(i,j)i'}$ with $\tau.C'_{i'}$ in the second summation and $x_{if'(i',j')}$ with $C_{f'(i',j')}$ in the third summation, we get the term $t_{ii'}\{\widetilde{q'}/\widetilde{X}\} =$

$$\sum_{(j,j') \in J^1_{ii'}} \mu_{ij}.C'_{f'(i',j')} + \sum_{(j,i') \in J^2_{ii'}} \tau.\tau.C'_{i'} + \sum_{(i,j') \in J^3_{ii'}} \tau.C'_{f'(i',j')} + \sum_{j=1}^{\overline{m}(i)} \nu_{ij}.\mathbf{0}$$

which, by substitutivity, is provably equal to:

$\sum_{(j,j')\in J^1_{ii'}} \mu_{ij}.C'_{f'(i',j')} + \sum_{(j,i')\in J^2_{ii'}} \tau.\tau.C'_{i'} + \sum_{(i,j')\in J^3_{ii'}} \tau.C'_{f'(i',j')} + \sum_{j=1}^{\overline{m}'(i')} \nu'_{i'j}.\mathbf{0}$
which, in turn, can be proved equal to $C'_{i'}$, in case $J^2_{ii'} = \emptyset$, or to $\tau.C'_{i'}$ otherwise, by the same argument as above. By Theorem 10, since $WB_g \vdash C'_1 = t_{11}\{\widetilde{q'}/\widetilde{X}\}$ and $D_{11} \doteq t_{11}\{\widetilde{D}/\widetilde{X}\}$, we have that $WB_g \vdash D_{11} = C'_1$. Therefore, the thesis $WB_g \vdash C_1 = C'_1$ follows by transitivity. □

Proposition 35. (Completeness for sequential processes) For every p, p' observationally guarded, sequential CFM processes, if $p \approx_c p'$ (or $p = \mathbf{0} = p'$), then $WB_g \vdash p = p'$.

PROOF. By Propositions 33 and 34, there exist saturated normal forms q and q' such that $WB_g \vdash p = q$ and $WB_g \vdash p' = q'$. By Theorem 9, we have $p \approx_c q$ and $p' \approx_c q'$, so that $q \approx_c q'$ by transitivity. By Lemma 7, we have that $WB_g \vdash q = q'$, so that the thesis $WB_g \vdash p = p'$ follows by transitivity. □

We can extend the definition of the observationally guarded predicate $og(p)$ to any CFM process, by adding to the rules in Table 3, also the following:

$$\frac{og(p,I) \wedge og(q,I)}{og(p\,|\,q,I)}$$

With this extension, we can state the following theorem of completeness for observationally guarded CFM processes.

Theorem 11. (Completeness of WB_g) For every $p, q \in \mathscr{P}_{CFM}$ observationally guarded, if $p \approx_c^{\oplus} q$, then $WB_g \vdash p = q$.

PROOF. The proof is by induction on the size of $dec(p)$. If $|dec(p)| = 0$, then $dec(p) = \theta$; as $p \approx_c^{\oplus} q$, necessarily also $dec(q) = \theta$. By observing the definition of the decomposition function in Table 1, this is possible only if p and q are either $\mathbf{0}$ or a parallel composition of $\mathbf{0}$'s, e.g., $\mathbf{0}\,|\,\mathbf{0}$; hence, $E \vdash p = \mathbf{0}$ and $E \vdash q = \mathbf{0}$, possibly using axioms **P1-P3**; hence, by transitivity we get $E \vdash p = q$. If $|dec(p)| = k+1$, then there exist p_1, p_2, q_1, q_2, which are all observationally guarded, such that $dec(p) = p_1 \oplus dec(p_2)$, $dec(q) = q_1 \oplus dec(q_2)$, $p_1 \approx_c q_1$ and $dec(p_2) \approx_c^{\oplus} dec(q_2)$. By the definition of the decomposition function and by axioms **P1-P3**, this means that $WB_g \vdash p = p_1\,|\,p_2$ and $WB_g \vdash q = q_1\,|\,q_2$. By Proposition 35 we have that $WB_g \vdash p_1 = q_1$. By induction, we have that $WB_g \vdash p_2 = q_2$. By substitutivity we get $WB_g \vdash p_1\,|\,p_2 = q_1\,|\,q_2$ and so the thesis follows by transitivity. □

7.3. Completeness of BB_g

Remark 6. (Branching bisimulation and normal forms) Let p and q be two observationally guarded, normal forms such that $p \approx_{br} q$. If $p \xrightarrow{\mu} p'$, then

- $q \xrightarrow{\mu} q'$ with $p' \approx_{br} q'$ (or $p' = \theta = q'$); or
- $\mu = \tau$ and $p' \approx_{br} q$; or

- $q \xrightarrow{\tau} q'$ with $p \approx_{br} q'$,

and symmetrically if q moves first. As a matter of fact, if $p \xrightarrow{\mu} p'$, then, as $p \approx_{br} q$, (*i*) either $\mu = \tau$ and $q \stackrel{\varepsilon}{\Longrightarrow} q'$ with $p \approx_{br} q'$ and $p' \approx_{br} q'$, (*ii*) or there exists $\overline{q}$ such that $q \stackrel{\varepsilon}{\Longrightarrow} \overline{q} \xrightarrow{\mu} q'$ with $p \approx_{br} \overline{q}$ and $p' \approx_{br} q'$ (or $p' = \theta = q'$). In the former case, we end up in the second condition: $p' \approx_{br} q$ by transitivity. In the latter case, we have two subcases. Either $\overline{q} = q$, so that we end up in the first condition: $q \xrightarrow{\mu} q'$ with $p' \approx_{br} q'$ (or $p' = \theta = q'$). Or $q \xrightarrow{\tau} q'' \stackrel{\varepsilon}{\Longrightarrow} \overline{q} \xrightarrow{\mu} q'$ so that we end up in the third condition: $q \xrightarrow{\tau} q''$ with $p \approx_{br} q''$, because $p \approx_{br} \overline{q}$ and, by stuttering property (see Remark 2), we have that $q'' \approx_{br} \overline{q}$. □

Lemma 8. (Completeness for normal forms) For every p, p' observationally guarded, normal forms, if $p \approx_{brc} p'$ (or $p = \mathbf{0} = p'$), then $BB_g \vdash p = p'$.

PROOF. If $p = \mathbf{0} = p'$, then $BB_g \vdash p = p'$ by reflexivity. Otherwise, by Remark 3, we can assume that p is (or is equated to) the system of equations $E(\widetilde{C})$:

$$
\begin{array}{lcl}
C_1 & \doteq & \sum_{h=1}^{m(1)} \mu_{1h}.C_{f(1,h)} + \sum_{j=1}^{\overline{m}(1)} \nu_{1j}.\mathbf{0} \\
C_2 & \doteq & \sum_{h=1}^{m(2)} \mu_{2h}.C_{f(2,h)} + \sum_{j=1}^{\overline{m}(2)} \nu_{2j}.\mathbf{0} \\
\dots & & \\
C_n & \doteq & \sum_{h=1}^{m(n)} \mu_{nh}.C_{f(n,h)} + \sum_{j=1}^{\overline{m}(n)} \nu_{nj}.\mathbf{0}
\end{array}
$$

where actually $C_i \doteq \mathbf{0}$ in case $m(i) = 0 = \overline{m}(i)$. For each $i = 1, \dots, n$, in case $C_i \doteq \mathbf{0}$, by axiom **R1**, we get $BB_g \vdash C_i = \mathbf{0} + \mathbf{0}$; otherwise, we get $BB_g \vdash C_i = body(C_i)$, by axiom **R2**, where by $body(C_i)$ we denote the normal form $\sum_{h=1}^{m(i)} \mu_{ih}.C_{f(i,h)} + \sum_{j=1}^{\overline{m}(i)} \nu_{ij}.\mathbf{0}$.
Similarly, by Remark 3 we can assume that p' is (or is equated to) the system of equations $F(\widetilde{C'})$:

$$
\begin{array}{lcl}
C'_1 & \doteq & \sum_{h=1}^{m'(1)} \mu'_{1h}.C'_{f'(1,h)} + \sum_{j=1}^{\overline{m}'(1)} \nu'_{1j}.\mathbf{0} \\
C'_2 & \doteq & \sum_{h=1}^{m'(2)} \mu'_{2h}.C'_{f'(2,h)} + \sum_{j=1}^{\overline{m}'(2)} \nu'_{2j}.\mathbf{0} \\
\dots & & \\
C'_{n'} & \doteq & \sum_{h=1}^{m'(n')} \mu'_{n'h}.C'_{f'(n',h)} + \sum_{j=1}^{\overline{m}'(n')} \nu'_{n'j}.\mathbf{0}
\end{array}
$$

For each $i = 1, \dots, n'$, in case $C'_i \doteq \mathbf{0}$, by axiom **R1**, we get $BB_g \vdash C'_i = \mathbf{0} + \mathbf{0}$; otherwise, we get $BB_g \vdash C'_i = body(C'_i)$, by axiom **R2**, where by $body(C'_i)$ we denote the normal form $\sum_{h=1}^{m'(i)} \mu'_{ih}.C'_{f'(i,h)} + \sum_{j=1}^{\overline{m}'(i)} \nu'_{ij}.\mathbf{0}$. Moreover, as $p \approx_{brc} p'$, we have $C_1 \approx_{brc} C'_1$.

Now, let $I = \{(i, i') \mid C_i \approx_{br} C'_{i'}\}$.

Clearly, since $C_1 \approx_{brc} C'_1$, we have that $(1,1) \in I$. Since $C_1 \approx_{brc} C'_1$, it follows that $C_1 \xrightarrow{\nu} \theta$ if and only if $C'_1 \xrightarrow{\nu} \theta$ for all ν. Therefore, the summands that ends immediately successfully are the same for these two terms, up to reordering (axioms **A1-A2**) and the presence of possible duplicates that can be absorbed (axiom **A4**):

$$BB_g \vdash \sum_{j=1}^{\overline{m}(1)} \nu_{1j}.\mathbf{0} = \sum_{j=1}^{\overline{m}'(1)} \nu'_{1j}.\mathbf{0} \quad \text{for } (1,1) \in I.$$

Hence, we can equate these summands in the following. Moreover, since C_1 and C'_1 are rooted branching bisimilar, the following holds: for $(1,1) \in I$, there exists a total surjective relation J_{11} between $\{1, 2, \dots m(1)\}$ and $\{1, 2, \dots m'(1)\}$ given by

$$J_{11} = \{(j, j') \mid \mu_{1j} = \mu'_{1j'} \wedge (f(1,j), f'(1,j')) \in I\}.$$

For any other $(i,i') \in I$, since C_i and $C'_{i'}$ are only branching bisimilar, by Remark 6, there exists a total surjective relation $J_{ii'}$ between $\{1,2,\dots m(i)\}$ and $\{1,2,\dots m'(i')\}$ given by $J_{ii'} = J^1_{ii'} \cup J^2_{ii'} \cup J^3_{ii'}$, where

$$\begin{array}{lcl} J^1_{ii'} & = & \{(j,j') \mid \mu_{ij} = \mu'_{i'j'} \wedge (f(i,j), f'(i',j')) \in I\} \\ J^2_{ii'} & = & \{(j,i') \mid \mu_{ij} = \tau \wedge (f(i,j),i')) \in I\} \\ J^3_{ii'} & = & \{(i,j') \mid \mu'_{i'j'} = \tau \wedge (i, f'(i',j')) \in I\}. \end{array} .$$

Now, for $(1,1) \in I$, let us consider the set of variables $\widetilde{X} = \{x_{ii'} \mid (i,i') \in I\}$ and the open term

$$t_{11} = \sum_{(j,j') \in J_{11}} \mu_{1j}.x_{f(1,j)f'(1,j')} + \sum_{j=1}^{\overline{m}(1)} \nu_{1j}.\mathbf{0}$$

and, for each $(i,i') \in I$, the open terms

$$t_{ii'} = \sum_{(j,j') \in J^1_{ii'}} \mu_{ij}.x_{f(i,j)f'(i',j')} + \sum_{(j,i') \in J^2_{ii'}} \tau.x_{f(i,j)i'} + \sum_{(i,j') \in J^3_{ii'}} \tau.x_{if'(i',j')} + K_{ii'}$$

where $K_{ii'}$ is the sumform composed of all the termination summands that occur in both $\sum_{j=1}^{\overline{m}(i)} \nu_{ij}.\mathbf{0}$ (i.e., in $body(C_i)$) and $\sum_{j=1}^{\overline{m}'(i')} \nu'_{i'j}.\mathbf{0}$ (i.e., in $body(C'_{i'})$.

This gives rise to a system of equations $G(\widetilde{D})$, where $\widetilde{D} = \{D_{ii'} \mid (i,i') \in I\}$, of the following form:

$$\begin{array}{lcl} D_{11} & \doteq & t_{11}\{\widetilde{D}/\widetilde{X}\} \\ \dots & & \\ D_{ii'} & \doteq & t_{ii'}\{\widetilde{D}/\widetilde{X}\} \\ \dots & & \end{array}$$

The system $G(\widetilde{D})$ is observationally guarded by construction. For each $(i,i') \in I$, by axiom **R2**, we get $BB_g \vdash D_{ii'} = t_{ii'}\{\widetilde{D}/\widetilde{X}\}$ provided that $t_{ii'} \neq \mathbf{0}$. In case $t_{ii'} = \mathbf{0}$, then $BB_g \vdash D_{ii'} = \mathbf{0} + \mathbf{0}$ by axiom **R1**.

Now, let us consider the terms $q_i = C_i$ if $J^3_{ii'} = \emptyset$, $q_i = \tau.C_i + H_i$ otherwise, which are cumulatively represented as $\widetilde{q}$, where H_i is defined below. If we close each $t_{ii'}$ by replacing $x_{f(i,j)f'(i',j')}$ with $C_{f(i,j)}$ in the first summation, $x_{f(i,j)i'}$ with $C_{f(i,j)}$ in the second summation and $x_{if'(i',j')}$ with $\tau.C_i + H_i$ in the third summation, we get the term

$$t_{ii'}\{\widetilde{q}/\widetilde{X}\} = \textstyle\sum_{(j,j') \in J^1_{ii'}} \mu_{ij}.C_{f(i,j)} + \sum_{(j,i') \in J^2_{ii'}} \tau.C_{f(i,j)} + \sum_{(i,j') \in J^3_{ii'}} \tau.(\tau.C_i + H_i) + K_{ii'}$$

which is provably equal to C_i in case $J^3_{ii'} = \emptyset$, or to $\tau.C_i + H_i$ otherwise. (Of course, the special case $t_{ii'}\{\widetilde{q}/\widetilde{X}\} = \mathbf{0}$ is trivial: this means that $C_i \doteq \mathbf{0}$ and so $BB_g \vdash C_i = \mathbf{0} + \mathbf{0}$.) First of all, let us define H_i: it is the abbreviation for the sumform

$$\textstyle\sum_{(j,j') \in J^1_{ii'}} \mu_{ij}.C_{f(i,j)} + \sum_{(j,i') \in J^2_{ii'}} \tau.C_{f(i,j)} + K_{ii'},$$

so that $t_{ii'}\{\widetilde{q}/\widetilde{X}\} = H_i$ in case $J^3_{ii'} = \emptyset$, otherwise $BB_g \vdash t_{ii'}\{\widetilde{q}/\widetilde{X}\} = \tau.(\tau.C_i + H_i) + H_i$.

If $J^3_{ii'} = \emptyset$, then $BB_g \vdash K_{ii'} = \sum_{j=1}^{\overline{m}(i)} \nu_{ij}.\mathbf{0}$ and, moreover, H_i contains (with possible repetitions) exactly the same terms $\mu.C_k$ for which $C_i \xrightarrow{\mu} C_k$, and the thesis $BB_g \vdash C_i = t_{ii'}\{\widetilde{q}/\widetilde{X}\}$ follows, by axioms **A1-A4**, from $BB_g \vdash C_i = \sum_{h=1}^{m(i)} \mu_{ih}.C_{f(i,h)} + \sum_{j=1}^{\overline{m}(i)} \nu_{ij}.\mathbf{0}$.

On the other hand, if $J^3_{ii'} \neq \emptyset$, then $BB_g \vdash t_{ii'}\{\widetilde{q}/\widetilde{X}\} = \tau.(\tau.C_i + H_i) + H_i$. By axiom **R2**, $C_i = body(C_i)$, so that $BB_g \vdash \tau.(\tau.C_i + H_i) + H_i = \tau.(\tau.body(C_i) + H_i) + H_i$. Note that H_i is a sub-sumform of $body(C_i)$, so that, by axiom **B**, we can derive that

$BB_g \vdash \tau.(\tau.body(C_i)+H_i)+H_i = \tau.body(C_i)+H_i$,
and so the thesis $BB_g \vdash t_{ii'}\{\widetilde{q}/\widetilde{X}\} = \tau.C_i + H_i$ follows by transitivity. By Corollary 5, since $BB_g \vdash C_1 = q_1 = t_{11}\{\widetilde{q}/\widetilde{X}\}$ and $D_{11} \doteq t_{11}\{\widetilde{D}/\widetilde{X}\}$, we have that $BB_g \vdash D_{11} = C_1$.

In exactly the same way, we can define the terms $q'_{i'} = C'_{i'}$ if $J^2_{ii'} = \emptyset$, $q'_{i'} = \tau.C'_{i'} + H'_{i'}$ otherwise, which are cumulatively represented as $\widetilde{q'}$, where $H'_{i'}$ is defined below. If we close each $t_{ii'}$ by replacing $x_{f(i,j)f'(i',j')}$ with $C'_{f'(i',j')}$ in the first summation, $x_{f(i,j)i'}$ with $\tau.C'_{i'} + H'_{i'}$ in the second summation and $x_{if'(i',j')}$ with $C'_{f'(i',j')}$ in the third summation, we get the term $t_{ii'}\{\widetilde{q'}/\widetilde{X}\} =$

$\sum_{(j,j')\in J^1_{ii'}} \mu_{ij}.C'_{f'(i',j')} + \sum_{(j,i')\in J^2_{ii'}} \tau.(\tau.C'_{i'} + H'_{i'}) + \sum_{(i,j')\in J^3_{ii'}} \tau.C'_{f'(i',j')} + K_{ii'}$

(where $H'_{i'} = \sum_{(j,j')\in J^1_{ii'}} \mu_{ij}.C'_{f'(i',j')} + \sum_{(i,j')\in J^3_{ii'}} \tau.C'_{f'(i',j')} + K_{ii'}$) which can be proved equal to $C'_{i'}$ in case $J^2_{ii'} = \emptyset$, or to $\tau.C'_{i'} + H'_{i'}$ otherwise, by the same argument as above. By Corollary 5, since $BB_g \vdash C'_1 = t_{11}\{\widetilde{q'}/\widetilde{X}\}$ and $D_{11} \doteq t_{11}\{\widetilde{D}/\widetilde{X}\}$, we have that $BB_g \vdash D_{11} = C'_1$. Therefore, the thesis $BB_g \vdash C_1 = C'_1$ follows by transitivity. □

Proposition 36. (Completeness for sequential processes) For every p, p' observationally guarded, sequential CFM processes, if $p \approx_{brc} p'$ (or $p = \mathbf{0} = p'$), then $BB_g \vdash p = p'$.

PROOF. By Proposition 33, there exist normal forms q and q' such that $BB_g \vdash p = q$ and $BB_g \vdash p' = q'$. By Theorem 9, we have $p \approx_{brc} q$ and $p' \approx_{brc} q'$, so that $q \approx_{brc} q'$ by transitivity. By Lemma 8, we have that $BB_g \vdash q = q'$, so that the thesis $BB_g \vdash p = p'$ follows by transitivity. □

Theorem 12. (Completeness of BB_g) For every $p, q \in \mathscr{P}_{CFM}$ observationally guarded, if $p \approx^{\oplus}_{brc} q$, then $BB_g \vdash p = q$.

PROOF. The proof, by induction on the size of $dec(p)$, is analogous to that of Theorem 11 and so omitted. □

7.4. Completeness of WB and BB over the whole of CFM

We now prove that the addition of the three axioms **WU1, WU2, U3** to WB_g is enough to equate any CFM process p to an observationally guarded CFM process q. Similarly, we will prove that the addition of the four axioms **BU1, BU2, U3, U4** to BB_g is enough to equate any CFM process p to an observationally guarded CFM process q. In order to prove these results, we need some auxiliary notation.

First of all, we denote by $n_x(p)$ the number of unguarded occurrences of x in p (including the bodies of the constants occurring in p). E.g., if $C \doteq p\{C/x\}$ and $p = \tau.x + a.x + \tau.D$, with $D \doteq \tau.D + \tau.x$, then $n_x(p) = 2$; moreover, we set $n_x(C) = n_x(p)$.

Then, we need to introduce a measure of the length of all the silent computations leading to an occurrence of an unguarded variable x. This value $len(p)$ for the sequential CFM term p (which may be open on x, but the definition applies also to closed terms) is computed by the length function defined in Table 6. Note that if an observable action prefixes a process, then the returned value is 0, while the variable x returns 1. Moreover, in case of composition with the choice operator, $p + p'$, we take the sum of $len(p)$ and $len(p')$. The crucial rule is that for the constant: if $C \doteq p\{C/x\}$, then, even if

$len(\mathbf{0})$	$=$	0	$len(p+p')$	$=$	$len(p)+len(p')$
$len(x)$	$=$	1	$len(\tau.p)$	$=$	$\begin{cases} 1+len(p) & \text{if } len(p)\neq 0 \\ 0 & \text{otherwise} \end{cases}$
$len(a.p)$	$=$	0	$len(C)$	$=$	$len(p)$ if $C \doteq p\{C/x\}$

Table 6: Length function

C is closed, its length is computed on the open term p. It is an easy exercise to show that the open sequential term p is observationally guarded (cf. Definition 21) if and only if $len(p) = n_x(p) = 0$. Moreover, if p is closed (actually a closed system of equations), then p is observationally guarded (cf. Remark 4) if and only if $len(p) = n_x(p) = 0$.

Lemma 9. Let p be an open sequential CFM term. Then, the following holds:

- $og(p)$ if and only if $len(p) = n_x(p) = 0$, and
- $len(p) > 0$ if and only if $n_x(p) > 0$.

Moreover, p is closed and observationally guarded iff $len(p) = n_x(p) = 0$. □

In order to illustrate the procedure for converting an observationally unguarded term p into an observationally guarded term q, we provide an example, which explains the crucial steps in the proof that follows.

Example 23. Let us consider the constant definitions:

$$C \doteq \tau.C+\tau.D$$
$$D \doteq \tau.C+b.D+\tau.D$$

Clearly, both C and D are not observationally guarded. We start the procedure from C. Define $C_1 \doteq \tau.(\tau.D+\mathbf{0})$: by axiom **WU1**, we have $WB \vdash C_1 = C$ so that the silent self-loop is removed. Now, by axioms **A3** and **W1**, we get $WB \vdash C_1 = \tau.D$. Define a new constant $C_2 \doteq \tau.D$, so that by recursion congruence, $WB \vdash C_1 = C_2$. Note that no occurrence of C_2 occurs in its body. (In general, after these preliminary steps, no instance of the processed constant occurs observationally unguarded.)

Now we start the procedure for D. First of all, the occurrence of C in the body of D is replaced by the body of C_2, i.e., by $\tau.D$, so that $WB \vdash D = \tau.(\tau.D)+b.D+\tau.D$. Then, by reordering the summands,

$$WB \vdash D = \tau.D+\tau.(\tau.D)+b.D$$

Now, we define a new constant $D_1 \doteq \tau.D_1+\tau.(\tau.D_1)+b.D_1$ so that, by recursion congruence, $WB \vdash D = D_1$. Then define $D_2 \doteq \tau.(\tau.(\tau.D_2)+b.D_2)$: by axiom **WU1**, $WB \vdash D_1 = D_2$. Now we define a new constant $D_3 \doteq \tau.((\tau.D_3)+b.D_3)$: by axiom **U3**, $WB \vdash D_2 = D_3$. Now we define a new constant $D_4 \doteq \tau.(b.D_4)$: by axiom **WU2**, $WB \vdash D_3 = D_4$. And we are done because D_4 is observationally guarded.

Now, by transitivity, substitutivity and axiom **R2**, we get $WB \vdash C = \tau.D$ and $WB \vdash D = \tau.b.D$. Therefore, the observationally guarded system of equations

$$A \doteq \tau.B$$
$$B \doteq \tau.b.B$$

is such that $WB \vdash C = A$, $WB \vdash D = B$ by Theorem 10. □

Proposition 37. **(Reduction to observationally guarded process for** WB**)** Let $\widetilde{X} = (x_1, x_2, \ldots, x_n)$ be a tuple of variables and let $\widetilde{p} = (p_1, p_2, \ldots, p_n)$ be a tuple of open guarded CFM terms, using the variables in $\widetilde{X}$. Let $\widetilde{C} = (C_1, C_2, \ldots, C_n)$ be a tuple of constants (not occurring in $\widetilde{p}$) such that the system of equations

$$\begin{array}{lcl} C_1 & \doteq & p_1\{\widetilde{C}/\widetilde{X}\} \\ C_2 & \doteq & p_2\{\widetilde{C}/\widetilde{X}\} \\ \ldots & & \\ C_n & \doteq & p_n\{\widetilde{C}/\widetilde{X}\} \end{array}$$

is observationally unguarded. Then, there exist a tuple $\widetilde{q} = (q_1, q_2, \ldots, q_n)$ of open guarded CFM terms, using the variables in $\widetilde{X}$, and a tuple of constants $\widetilde{D} = (D_1, D_2, \ldots, D_n)$ (not occurring in $\widetilde{q}$) such that the system of equations

$$\begin{array}{lcl} D_1 & \doteq & q_1\{\widetilde{D}/\widetilde{X}\} \\ D_2 & \doteq & q_2\{\widetilde{D}/\widetilde{X}\} \\ \ldots & & \\ D_n & \doteq & q_n\{\widetilde{C}/\widetilde{X}\} \end{array}$$

is observationally guarded and, for $i = 1, \ldots, n$, $WB \vdash C_i = D_i$.

PROOF. The proof is by double induction: first on n, and then on the pair $(n_x(p), len(p))$ (for the considered open guarded term p), where we assume that $(n_1, k_2) < (n_2, k_2)$ if $n_1 < n_2$, or $n_1 = n_2$ and $k_1 < k_2$.

For $n = 1$, if $(n_x(p_1), len(p_1)) = (0, 0)$, then we are done, as $C_1 \doteq p_1\{C_1/x\}$ is actually observationally guarded. Instead, if p_1 is observationally unguarded, then it must be of the form $\tau.r + q$, with r observationally unguarded. We now proceed by case analysis:

- $r = x$: In this case, $WB \vdash C_1 = (\tau.x + q)\{C_1/x\}$ by axiom **R2** (and possibly also **A1-A3**). Now define $C_2 \doteq (\tau.x + q)\{C_2/x\}$ so that $WB \vdash C_1 = C_2$ by recursion congruence. Define $C_3 \doteq \tau.(q + \mathbf{0})\{C_3/x\}$: by axiom **WU1** we have $WB \vdash C_2 = C_3$. Note that $n_x(\tau.(q + \mathbf{0})) = n_x(\tau.x + q) - 1$, so that induction can be invoked to conclude that there exist an observationally guarded process q_1 and a constant D_1 such that $D_1 \doteq q_1\{D_1/x\}$ and $WB \vdash C_3 = D_1$, so that $WB \vdash C_1 = D_1$ follows by transitivity.

- $r = \tau.p_1 + p_2$, with p_1 observationally unguarded: We have two subcases:

  - $p_1 = x$: Thus, $WB \vdash C_1 = (\tau.(\tau.x + p_2) + q)\{C_1/x\}$ by axiom **R2** (and possibly also **A1-A3**). Now define $C_2 \doteq (\tau.(\tau.x + p_2) + q)\{C_2/x\}$ so that $WB \vdash C_1 = C_2$ by recursion congruence. Define $C_3 \doteq (\tau.(p_2 + q))\{C_3/x\}$: by axiom **WU2**, we have $WB \vdash C_2 = C_3$. Note that $n_x(\tau.(p_2 + q)) = n_x(\tau.(\tau.x + p_2) + q) - 1$, so that induction can be invoked to conclude that there exist an observationally guarded process q_1 and a constant D_1 such that $D_1 \doteq q_1\{D_1/x\}$ and $WB \vdash C_3 = D_1$, so that $WB \vdash C_1 = D_1$ follows by transitivity.
  - p_1 is a guarded term, observationally unguarded in x: In this case, $WB \vdash C_1 = (\tau.(\tau.p_1 + p_2) + q)\{C_1/x\}$ by axiom **R2** (and possibly also **A1-A3**). Now define $C_2 \doteq (\tau.(\tau.p_1 + p_2) + q)\{C_2/x\}$ so that $WB \vdash C_1 = C_2$ by

recursion congruence. Define $C_3 \doteq (\tau.(p_1 + p_2) + q)\{C_3/x\}$: by axiom **U3**, $WB \vdash C_2 = C_3$. Note that $len((\tau.(p_1 + p_2) + q)) = len((\tau.(\tau.p_1 + p_2) + q)) - 1$, so that induction can be invoked to conclude that there exist an observationally guarded process q_1 and a constant D_1 such that $D_1 \doteq q_1\{D_1/x\}$ and $WB \vdash C_3 = D_1$, so that $WB \vdash C_1 = D_1$ follows by transitivity.

As no other cases are possible, the proof of the base case for $n = 1$ is complete.

Now assume a tuple $\widetilde{p} = (p_1, p_2, \ldots, p_n)$ and the term p_{n+1}, so that they are all open on $\widetilde{X} = (x_1, x_2, \ldots, x_n)$ and the additional x_{n+1}. Assume, w.l.o.g., that x_{n+1} occurs in p_{n+1}. First, define

$C_{n+1} \doteq p_{n+1}\{C_{n+1}/x_{n+1}\}$,

so that C_{n+1} is now open on $\widetilde{X}$. Then, we define $p_i\{C_{n+1}/x_{n+1}\}$, which, for $i = 1, \ldots, n$, is the term p_i where each occurrence of variable x_{n+1} has been replaced by C_{n+1}; note that each term $p_i\{C_{n+1}/x_{n+1}\}$ is open on $\widetilde{X}$. Hence

$$\begin{array}{lcl} C_1 & \doteq & p_1\{C_{n+1}/x_{n+1}\}\{\widetilde{C}/\widetilde{X}\} \\ C_2 & \doteq & p_2\{C_{n+1}/x_{n+1}\}\{\widetilde{C}/\widetilde{X}\} \\ \ldots & & \\ C_n & \doteq & p_n\{C_{n+1}/x_{n+1}\}\{\widetilde{C}/\widetilde{X}\} \end{array}$$

is a system of equation of size n, so that, by induction, we can conclude that there exist a tuple $\widetilde{q} = (q_1, q_2, \ldots, q_n)$ of terms and a tuple of constants $\widetilde{D} = (D_1, D_2, \ldots, D_n)$ such that

$$\begin{array}{lcl} D_1 & \doteq & q_1\{C_{n+1}/x_{n+1}\}\{\widetilde{D}/\widetilde{X}\} \\ D_2 & \doteq & q_2\{C_{n+1}/x_{n+1}\}\{\widetilde{D}/\widetilde{X}\} \\ \ldots & & \\ D_n & \doteq & q_n\{C_{n+1}/x_{n+1}\}\{\widetilde{D}/\widetilde{X}\} \end{array}$$

is observationally guarded and $WB \vdash C_i = D_i$ for $i = 1, \ldots, n$. Note that

$q_i\{C_{n+1}/x_{n+1}\}\{\widetilde{D}/\widetilde{X}\} = q_i\{\widetilde{D}/\widetilde{X}, C_{n+1}\{\widetilde{D}/\widetilde{X}\}/x_{n+1}\}$

so that by $C_{n+1}\{\widetilde{D}/\widetilde{X}\}$ we have implicitly closed the definition of C_{n+1} as

$C_{n+1} \doteq p_{n+1}\{C_{n+1}/x_{n+1}\}\{\widetilde{D}/\widetilde{X}\} = p_{n+1}\{\widetilde{D}/\widetilde{X}\}\{C_{n+1}/x_{n+1}\}$.

Now we unfold each constant D_i inside the term $p_{n+1}\{\widetilde{D}/\widetilde{X}\}\{C_{n+1}/x_{n+1}\}$ so that possible further occurrences of C_{n+1} are now exposed. Note that, by axiom **R2**,

$WB \vdash p_{n+1}\{\widetilde{D}/\widetilde{X}\} = p_{n+1}\{\widetilde{body(D)}/\widetilde{X}\}$.

More compactly, we can write

$p_{n+1}\{\widetilde{body(D)}/\widetilde{X}\} = r_{n+1}\{\widetilde{D}/\widetilde{X}\}$.

and define a new constant $C'_{n+1} \doteq r_{n+1}\{\widetilde{D}/\widetilde{X}\}\{C'_{n+1}/x_{n+1}\}$ so that $WB \vdash C_{n+1} = C'_{n+1}$ by recursion congruence. Note that we are now in a situation similar to the base case for $n = 1$: if $(n_x(r_{n+1}\{\widetilde{D}/\widetilde{X}\}), len(r_{n+1}\{\widetilde{D}/\widetilde{X}\})) = (0,0)$, then we are done, as $C'_{n+1} \doteq r_{n+1}\{\widetilde{D}/\widetilde{X}\}\{C'_{n+1}/x_{n+1}\}$ is actually observationally guarded. Instead, if $r_{n+1}\{\widetilde{D}/\widetilde{X}\}$ is observationally unguarded, then it must be of the form $\tau.r + q$, with r observationally unguarded. We can now proceed by case analysis as done for the base case; as this is indeed very similar, we omit this part of the proof. So, at the end, we get a new observationally guarded term q_{n+1} and a new constant D_{n+1} such that $D_{n+1} \doteq$

$q_{n+1}\{D_{n+1}/x_{n+1}\}\{\widetilde{D}/\widetilde{X}\}$ and $WB \vdash C'_{n+1} = D_{n+1}$. Assuming $q_{n+1} \neq \mathbf{0}$, by axiom **R2** (the case $q_{n+1} = \mathbf{0}$, using axiom **R1**, is obvious), we have that

$WB \vdash D_{n+1} = q_{n+1}\{D_{n+1}/x_{n+1}\}\{\widetilde{D}/\widetilde{X}\}$.

Similarly, also that $WB \vdash D_i \doteq q_i\{C_{n+1}/x_{n+1}\}\{\widetilde{D}/\widetilde{X}\}$ for $i = 1,\dots,n$. By substitutivity, we also get $WB \vdash D_i \doteq q_i\{D_{n+1}/x_{n+1}\}\{\widetilde{D}/\widetilde{X}\}$ for $i = 1,\dots,n$. Finally, we define a new observationally guarded system of equations

$$\begin{aligned} E_1 &\doteq q_1\{E_{n+1}/x_{n+1}\}\{\widetilde{E}/\widetilde{X}\} \\ E_2 &\doteq q_2\{E_{n+1}/x_{n+1}\}\{\widetilde{E}/\widetilde{X}\} \\ &\dots \\ E_n &\doteq q_n\{E_{n+1}/x_{n+1}\}\{\widetilde{E}/\widetilde{X}\} \\ E_{n+1} &\doteq q_{n+1}\{E_{n+1}/x_{n+1}\}\{\widetilde{E}/\widetilde{X}\} \end{aligned}$$

such that $WB \vdash D_i = E_i$ for $i = 1,\dots,n+1$ by Theorem 10. The thesis, $WB \vdash C_i = E_i$ for $i = 1,\dots,n+1$, follows by transitivity. □

Theorem 13. For any closed CFM process p, there exists an observationally guarded, closed, CFM process q such that $WB \vdash p = q$.

PROOF. By induction on the size of $dec(p)$. If $|dec(p)| = 0$, then $dec(p) = \theta$. By observing the definition of the decomposition function in Table 1, this is possible only if p is either $\mathbf{0}$ or a parallel composition of $\mathbf{0}$'s, e.g., $\mathbf{0}\,|\,\mathbf{0}$; hence, $E \vdash p = \mathbf{0}$, possibly using axioms **P1-P3**, where $\mathbf{0}$ is observationally guarded. If $|dec(p)| = k+1$, then there exist p_1 and p_2 such that $dec(p) = p_1 \oplus dec(p_2)$. By Proposition 33 and Remark 3, p_1 can be equated, via SB, to a constant C_1 in normal form. By Proposition 37, there exists an observationally guarded constant D_1 such that $WB \vdash C_1 = D_1$. By induction, there exists an observationally guarded process q_2 such that $WB \vdash p_2 = q_2$. Note that $D_1\,|\,q_2$ is observationally guarded. By substitutivity we have $WB \vdash p_1\,|\,p_2 = D_1\,|\,q_2$. By axioms **P1-P3** we have that $WB \vdash p = p_1\,|\,p_2$. So the thesis $WB \vdash p = D_1\,|\,q_2$ follows easily. □

Corollary 6. (Completeness of WB for the whole of CFM) For every $p,q \in \mathscr{P}_{CFM}$, if $p \approx_c^{\oplus} q$, then $WB \vdash p = q$.

PROOF. By Theorem 13, there exists p' and q', observationally guarded, such that $WB \vdash p = p'$ and $WB \vdash q = q'$. By Theorem 9, we have that $p \approx_c^{\oplus} p'$ as well as $q \approx_c^{\oplus} q'$, so that, by transitivity, we also have $p' \approx_c^{\oplus} q'$. By Theorem 11, we have $WB \vdash p' = q'$, so that the thesis $WB \vdash p = q$ follows by transitivity. □

Now we sketch how to adapt the previous proofs for the case of rooted branching bisimilarity.

Proposition 38. (Reduction to observationally guarded process for BB) Let $\widetilde{X} = (x_1,x_2,\dots,x_n)$ be a tuple of variables and let $\widetilde{p} = (p_1,p_2,\dots,p_n)$ be a tuple of open guarded CFM terms, using the variables in $\widetilde{X}$. Let $\widetilde{C} = (C_1,C_2,\dots,C_n)$ be a tuple of constants (not occurring in $\widetilde{p}$) such that the system of equations

$$\begin{aligned} C_1 &\doteq p_1\{\widetilde{C}/\widetilde{X}\} \\ C_2 &\doteq p_2\{\widetilde{C}/\widetilde{X}\} \\ &\dots \\ C_n &\doteq p_n\{\widetilde{C}/\widetilde{X}\} \end{aligned}$$

is observationally unguarded. Then, there exist a tuple $\widetilde{q} = (q_1, q_2, \dots, q_n)$ of open guarded CFM terms, using the variables in $\widetilde{X}$, and a tuple of constants $\widetilde{D} = (D_1, D_2, \dots, D_n)$ (not occurring in $\widetilde{q}$) such that the system of equations

$$\begin{array}{lll} D_1 & \doteq & q_1\{\widetilde{D}/\widetilde{X}\} \\ D_2 & \doteq & q_2\{\widetilde{D}/\widetilde{X}\} \\ \dots & & \\ D_n & \doteq & q_n\{\widetilde{C}/\widetilde{X}\} \end{array}$$

is observationally guarded and, for $i = 1, \dots, n$, $BB \vdash C_i = D_i$.

PROOF. (Sketch) The proof follows the same induction pattern of the proof of Proposition 37. We sketch only the base case for $n = 1$, following the idea in [11]. Let $C_1 \doteq p_1\{C_1/x\}$. If p_1 is observationally unguarded, then it may, or may not, contain summands of the form $\tau.x$.

In the former case, $BB \vdash p_1 = \tau.x + q$ where q does not contain any summand $\tau.x$ (as possible replicas are absorbed via axiom **A4**). Now, let us define $B \doteq (\tau.x + q)\{B/x\}$, so that, by recursion congruence, $BB \vdash C_1 = B$. The constant $B' \doteq \tau.(q\{B'/x\} + \mathbf{0}) + q\{B'/x\}$ is such that $BB \vdash B = B'$ by axiom **BU1**. Note that $\tau.(q + \mathbf{0}) + q$ does not contain any summand of the form $\tau.x$, so that all these silent self-loops have been removed. If q is observationally guarded, then we are done. Otherwise, q must be of the form $\tau.q_1 + q_2$ (with q_1 observationally unguarded), so that $\mathbf{0}$ can be absorbed, and we have

$$BB \vdash B' = (\tau.(\tau.q_1 + q_2) + \tau.q_1 + q_2)\{B'/x\}.$$

Let $D \doteq (\tau.(\tau.q_1 + q_2) + \tau.q_1 + q_2)\{D/x\}$, so that $BB \vdash B' = D$. Now, define $D' \doteq (\tau.(q_1 + q_2) + \tau.q_1 + q_2)\{D'/x\}$: by axiom **U3**, we have $BB \vdash D' = D$. Now by repeated use of axiom **U3**, it is possible to shorten the silent computations leading to x, so that, at the end, each summand of D' will be of the form $\tau.(\tau.x + r_1)$, $\tau.(\tau.x + r_2)$, ..., where r_i does not contain any observationally unguarded occurrence of x, because possible replicas of $\tau.x$ can be absorbed via axiom **A4**. In other words, D' can be equated to a constant

$$F \doteq (\tau.(\tau.x + r_1) + \tau.(\tau.x + r_2) + \dots \tau.(\tau.x + r_k) + r)\{F/x\}$$

where $r_1, r_2, \dots, r_k$ and r are all without any observationally unguarded occurrence of variable x. Then, by repeated application of axiom **U4**, this term can be equated to

$$\tau.(\tau.x + r_1 + r_2 + \dots + r_k) + r,$$

i.e., a term of the form $\tau.(\tau.x + p) + r$, so that by axiom **BU2**, it can be equated to $\tau.(p + r) + r$, which is an observationally guarded term. Then, $D_1 \doteq (\tau.(p + r) + r)\{D_1/x\}$ and $BB \vdash C_1 = D_1$.

In the latter case (i.e., when p_1 does not contain summands of the form $\tau.x$), $BB \vdash p_1 = \tau.q_1 + q_2$ where $q_1 \neq x$ is observationally unguarded and q_2 does not contain summands of the form $\tau.x$. Term q_1 can only be of the form $\tau.r_1 + r_2$, with r_1 observationally unguarded, so that $BB \vdash p_1 = \tau.(\tau.r_1 + r_2) + q_2$. Now define $B \doteq (\tau.(r_1 + r_2) + q_2)\{B/x\}$: by axiom **U3**, we have $BB \vdash C_1 = B$. Now the proof proceeds as in the previous case: by repeated use of axiom **U3**, it is possible to shorten the silent computations leading to x in r_1 (and possibly in r_2 and q_2), so that, at the end, each summand of B will be of the form $\tau.(\tau.x + t_1)$, $\tau.(\tau.x + t_2)$, ..., where t_i does not contain any observationally unguarded occurrence of x, because possible replicas of

$\tau.x$ can be absorbed via axiom **A4**. In other words, B can be equated to a constant
$G \doteq (\tau.(\tau.x+t_1)+\tau.(\tau.x+t_2)+\dots\tau.(\tau.x+t_h)+t)\{G/x\}$
where $t_1, t_2, \dots, t_h$ and t are all without any observationally unguarded occurrence of variable x. Then, by repeated application of axiom **U4**, this term can be equated to
$\tau.(\tau.x+t_1+t_2+\dots+t_k)+t,$
i.e., a term of the form $\tau.(\tau.x+p)+r$, so that by axiom **BU2**, it can be equated to $\tau.(p+r)+r$, which is an observationally guarded term. Then, $D_1 \doteq (\tau.(p+r)+r)\{D_1/x\}$ and $BB \vdash C_1 = D_1$ also in this case. □

Theorem 14. For any closed CFM process p, there exists an observationally guarded, closed, CFM process q such that $BB \vdash p = q$.

PROOF. As for Theorem 13. □

Corollary 7. (Completeness of ***BB*** **for the whole of CFM)** For every $p, q \in \mathscr{P}_{CFM}$, if $p \approx^{\oplus}_{brc} q$, then $BB \vdash p = q$.

PROOF. Similar to the proof of Corollary 6, and so omitted. □

8. Conclusion

Finite-state machines with silent moves have been equipped with simple, efficiently decidable, truly-concurrent behavioral semantics. Indeed, weak (or branching) team equivalence seems the most natural, intuitive and simple extension of LTS weak (or branching) bisimulation equivalence to FSMs with silent moves; it also has a very low complexity, actually the lowest one for FMSs with silent moves. More precisely, weak bisimilarity on places can be checked in $O(m \cdot (n+1))$ time (where n is the number of places and m the number of transitions), by adapting the algorithm in [32], and then weak team equivalence on markings can be checked in $O(k^2)$ time, where k is the size of the involved markings (or in $O(n)$, *cf.* Remark 1). Moreover, branching bisimilarity on places can be checked with time complexity $O(m \cdot (log\, n+1))$, where n is the number of places and m the number of transitions (adapting the algorithms proposed in [19, 21]), and branching team equivalence on markings can be checked in $O(k^2)$, where k is the size of the involved markings (or in $O(n)$, *cf.* Remark 1). Note that these results are in striking contrast with interleaving equivalences, which are all checkable in exponential time w.r.t. the size of the initial marking. As, in order to perform team equivalence checking, there is no need to compute the LTSs of the global behavior of the systems under scrutiny, our proposal seems a natural solution to solving the state-space explosion problem for FSMs with silent moves.

Furthermore, on FSMs *without* silent moves, they coincide with (strong) team equivalence, i.e., the additive closure of (strong) bisimilarity $\sim$ on places; this strong equivalence, studied in [16], was proved to coincide with strong place bisimilarity, proposed by Autant, Belmesk and Schnoebelen in [1, 2], an equivalence relation refining an earlier proposal by Olderog [27], under the name of strong bisimilarity. Team equivalence, which also coincides with *structure preserving bisimilarity* [13], is coarser than the branching-time semantics of *isomorphism of (nondeterministic) occurrence nets* (or unfoldings) [9] and finer than the linear-time semantics of *isomorphism of causal* *(or deterministic occurrence) nets* [5, 27]; moreover, it is finer than *history-preserving bisimilarity* [31, 7, 10], which on nets takes the form of so-called *fully concurrent bisimilarity* [6]. Hence, strong team equivalence does respect the causal behavior of nets. Weak (or branching) team equivalence seems the most natural extension of this causality-based equivalence in a setting with silent moves.

It is possible to prove that weak team equivalence is finer than *weak* fully concurrent bisimilarity (whose precise, one-step definition can be obtained by adapting the definitions in [10, 6, 36]). The formal proof of this fact, which is rather technical, is left for future research. The inclusion is strict: for instance, the places s_1 and s_5 in Figure 6 are weakly fully concurrent bisimilar, but they are not weakly team equivalent, because these two places are not even weakly bisimilar, as discussed in Example 7. It is also possible to define *branching* fully concurrent bisimilarity and to prove that branching team equivalence is finer than that; also this fact is left for future research.

The axiomatizations we have provided for rooted weak/branching team bisimilarity are based on previous work by Milner [24, 26] and van Glabbeek [11, 12] over finite-state CCS (and also on [3]). However, our calculus CFM uses guarded constants for recursion, rather than a recursive construct (e.g., $\mu X(a.E+X)$) with possibly unguarded variables; moreover, our semantics is sensitive to the kind of termination: a stuck place (unsuccessful termination) and the empty marking (successful termination) are never equated (cf. Example 7). Hence, these axioms have been adapted to our case. Our axiomatizations are the first *finite*, sound and (ground-)complete, axiomatization of truly-concurrent equivalences *in the presence of silent moves* for a process algebra admitting recursive behavior.

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