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Target zones on firm's value☆

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ABSTRACT

In the present paper we extend the traditional literature on Dynamic Capital Structure initiated by Merton (1974) and Leland (1994) to a Target Zone à la Krugman (1991). The existing literature focuses on the effects caused by several type of restrictions in the bankruptcy state, without focusing on the evolution of the firm's value in good state. We show that the conditions excluding the explosive root of the differential equation solving debt and equity equation are in practice not sufficient to guarantee stability. The introduction of a Target Zone is in effect able to bound from above the value of the firm. We fully characterize the Target Zone of the firm's value and fully solve the dynamic system. The model delivers similar results to those obtained under Real Option approach. We extend our result to a mean reverting process leading revenue growth rate: this confirms all the conclusions from trade-off theory, as opposite to the case obtained with a simple arithmetic Brownian motion. The model is flexible enough to allow many further generalization.

1. Introduction

Merton's (1974) seminal model laid the foundation for the structural approach to credit risk by framing corporate debt as a contingent claim, extending the idea of valuing corporate claims as options originally presented in Black and Scholes (1973). From Merton (1974) seminal contribution, holding a defaultable bond is equivalent to possess a portfolio of risk-free bond and a short put on the value on the firm's assets. The price of defaultable debt includes a debt discount value that it is interpreted as the price of a plain-vanilla put option to default.

This insight allowed for the derivation of closed-form pricing formulas for risky debt under suitable assumptions on asset dynamics. Since then, two central questions have guided the evolution of this literature: (i) what explains the behavior and level of credit spreads? (ii) What determines the optimal capital structure of the firm?

Key extensions to Merton's framework include Black and Cox (1976), who incorporated safety covenants and default-triggering boundaries, and Leland (1994), who introduced tax and bankruptcy costs to endogenize optimal leverage and default timing. Later works – such as Longstaff and Schwartz (1995), Leland and Toft (1996), and Zhou (2001) – enhanced realism by incorporating stochastic interest rates, debt heterogeneity, and jump processes to better capture the empirical behavior of credit spreads and capital structures. Goldstein, Leland, and Ju (2001) shifted focus to pre-tax cash flows, highlighting the role of government in shaping firm value.

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Despite these advances, most of the literature has focused on lower-boundary behavior, particularly the default threshold. Far less attention has been paid to upper bounds in firm's value, even though such constraints may arise from regulatory ceilings, market discipline, or internal corporate policies.

In this paper, we fill this gap by proposing a structural model in which firm value is constrained within a target zone, bounded both below and above, a concept inspired by the real options literature on investment under uncertainty and the economic theory of exchange rate management.

Our framework draws on the intuition of real options with dual triggers (see Brennan & Schwartz, 1985; Dias & Shackleton, 2011; Dixit, 1989), where investment or disinvestment decisions are governed by crossing upper and lower thresholds. By imposing an upper barrier, we depart from the canonical assumption that firm value can grow indefinitely and address a neglected issue: the potential distortions introduced by contractual or market-imposed constraints on firm expansion. Technically, we demonstrate that excluding the positive root from the solution to the firm's value dynamics is a sufficient – but not necessary – condition to avoid explosive growth. Our analysis shows that the firm's behavior remains unstable even in favorable states if growth is unconstrained.

The innovation of this paper is to embed the firm's valuation within a Target Zone (TZ) structure, similar to the exchange rate literature pioneered by Krugman (1991), where central banks commit to defend a currency within a pre-announced band. In a similar vein, a firm's decision to restructure during distress and to restrain excessive expansion effectively creates a bounded state space, influencing both valuation and credit spreads across its entire operating spectrum, rather than solely in periods approaching default. A key result of our model is due to the different behavior conditional to the assumptions on the stochastic process driving growth rate of revenues: under a standard arithmetic Brownian motion, the trade-off theory as formulated by Miller and Modigliani (1963), Kraus and Litzenberger (1973) and Leland (1994) is not respected. However, generalizing the model to include a simple mean reverting process, implies a dynamic behavior of firm's value fully coherent with trade-off theory: a moderate debt (here considered as a debt with a constant coupon) translates into an higher equity value, via the benefits coming from interest tax shield, confirming the empirical evidence as in Graham (2000) and Fama and French (2002).

The main benefit of including an upper bound is due to the possibility to ease the constraint on the process coefficients, since it is no longer necessary to ensure convergence of a variable Y_t as $E_t e^{-rt} Y_t$ as t goes to infinity.

This dual-boundary approach aligns conceptually with recent developments in real options theory, particularly models with endogenous barriers and hysteresis. Our results resonate with recent work by Dias, Nunes, and da Silva (2024), who apply real options logic to credit risk with bounded investment regimes. However, our framework is distinct in that it is grounded in a dynamic capital structure model with perpetual debt and closed-form solutions, rather than discrete investment timing.

Das and Kim (2015) extend the classic Merton (1974) paper on structural credit risk model, by introducing dynamic debt policies, where leverage can be ratcheted up or written down based on firm's value threshold. Within the context proposed by Das and Kim (2015) debt adjustments are conceptualized via barrier options. Thanks to this, they can classify several variants of dynamic debt policy including standalone ratchets, swap downs, and combined regimes. Through barrier option mathematics, Das and Kim (2015) derive closed-form expression for ex-ante credit spreads under each covenant type. The inclusion of barrier option within this model help to capture features of credit spread-term structures often unexplained by static models. This setting helps firms and investors to interpret covenant-implied spread dynamics and optimize debt restructuring triggers with exogenous covenant A fertile area of research is represented by the definition of endogenous barriers, as it is considered in the present paper.

The idea of modeling investment or capital structure decisions using dual-threshold dynamics has deep roots in the real options literature. Traditional real options models, such as McDonald and Siegel (1986) or Dixit and Pindyck (1994), typically feature a single trigger threshold—e.g., the value above which it becomes optimal to invest. However, a growing strand of research incorporates bands or ranges defined by both upper and lower thresholds, reflecting irreversibility, adjustment costs, and regime-switching incentives.

For instance, Brennan and Schwartz (1985) analyze natural resource extraction using a model where the firm's value evolves between two boundaries, and decisions are made when thresholds are crossed. Dixit (1989) develops a framework for entry and exit decisions under uncertainty, explicitly characterizing the optimal hysteresis band where no action is taken. More recent contributions such as Dias and Shackleton (2011) and Guerra, Nunes, and Oliveira (2017) use free-boundary analysis to solve investment and disinvestment problems, capturing behavior within a corridor of inaction. These models highlight that two-sided thresholds more accurately describe real-world managerial decision-making under uncertainty, especially when the firm faces both downside risk and upside constraints (e.g., competition, regulatory ceilings, or saturation effects). Our approach draws on this logic by imposing a target zone on firm's value within a structural credit risk framework, bridging real options theory with capital structure modeling.

Among other contributions in the literature there is Arkin and Slastnikov (2016), where optimal investment timing are characterized via threshold strategies, derived through free-boundary analysis: the results are obtained by using general diffusion settings (with upper and lower settings). Guerra et al. (2017) provide analytical solutions for optimal stopping with free-boundary conditions defining action regions (with an implicit upper/lower thresholds). Detemple and Kitapbayev (2018) investigate the pricing of American options with upper and lower caps, yielding multiple exercise-region thresholds, parallel to real-option dual-boundary structures. Trigeorgis (1993, 1996) present an extensive review of entry/exit option theory with hysteresis bands. Smit and Trigeorgis (2004) develop managerial real option frameworks with multi-threshold switching capabilities. Carmona and Ludkovski (2010) study multiple switching problems (multi-unit capacity decisions), implementing multiple threshold boundaries.

The rest of the paper is organized as follows. Section 2 introduces the model setup and explores how bankruptcy-related constraints affect firm's behavior even in favorable states. Section 3 formalizes the Target Zone model and presents a full solution for bounded firm's value for a one-sided Target Zone. Section 4 formalizes a two-sided Target Zone model and Section 5 incorporates debt into the model and analyzes a levered firm issuing perpetual coupon-paying bonds and compares the dynamics of firm value under traditional one-sided models versus our bounded framework. Section 6 extends the model to a general mean reverting stochastic process for the firm's growth rate of output. Section 7 concludes the paper with additional considerations for further extensions. Some analytical results discussed in the text are collected in two appendices at the end of the paper.

2. The model

We assume frictionless capital markets and free of informational asymmetries. Agents are risk neutral and can borrow and lend freely at a constant risk free interest rate r. We assume the existence of no taxes. A firm is characterized by a growth rate of revenues (output) given by y_t for which we assume the following stochastic process:

$$dy_t = \mu dt + \sigma dH_t \tag{1}$$

where μ and σ are constant parameters and dH_t is a standard Brownian motion. The arithmetic Brownian motion under (1) can take also negative value: this is possible in our context, since y_t is here assumed to be the growth rate of output. We generalize the process (1) with a mean-reverting term in a specific later of the paper.

The firm incurs in a production costs which is set equal to k. Therefore, total earnings are defined as:

$$y_t - k \tag{2}$$

Under bankruptcy the firm is impaired in its efficiency, so that its earnings become:

$$\gamma_x \left(y_t - k\right) \tag{3}$$

with $\gamma_x < 1$, indicating a direct cost of bankruptcy. At liquidation, the scrapping value of the firm is given by θ. Indirect cost of bankruptcy is represented by the investment distortions induced by the cost of debt. Liquidation is a real investment decision: a consequence of the model is given by early liquidation investment. We model the firm under two possible states:

State 1: pure equity financing.

State 2: the firm starts with both equity and debt, it goes bankrupt and then is run under pure equity by former debt holders.

We follow the same approach as in Mella-Barral and Perraudin (1997) to discuss the implications of these assumptions in terms of value of the firm, equity value and debt value.

2.1. The pure equity case

Under pure equity financing the value of the firm is V_t: this is the value of an asset with $y_t - k$ income. From previous assumptions, under risk neutrality, financial markets equilibrium requires that:

$$rV_t = y_t - k + E_t \left. \frac{d}{dt} V_{t+\Delta t} \right|_{\Delta t=0} = \tag{4}$$

$$= y_t - k + E_t \left(\frac{dV_t}{dt} \right) \tag{5}$$

Clearly, V_t depends on time only through its dependency on the state variable y_t. The left hand-side of both Eqs. (4)–(5) is the required return on investment, while in the right hand side $y_t - k$ is the dividend flow (net of costs) and $E_t\left(\frac{dV_t}{dt}\right)$ is the local change in the value function due to capital gains and long term payoffs.

Assume that $V_t = V(y_t)$ is a twice continuously differentiable function of y_t. By using Ito's Lemma we get:

$$dV_t = V'(y_t)dy_t + \frac{1}{2}V''(y_t)\left(dy_t\right)^2 \tag{6}$$

so that after using (1) into (6), we get:

$$dV_t = \left(V'(y_t)\mu + \frac{\sigma^2}{2}V''(y_t) \right) dt + V'(y_t)\sigma dH_t \tag{7}$$

which, after applying standard techniques, implies:

$$E_t \left(\frac{dV_t}{dt} \right) = \mu V'(y_t) + \frac{\sigma^2}{2}V''(y_t) \tag{8}$$

so that Eq. (5) becomes:

$$rV_t = y_t - k + \mu V'(y_t) + \frac{\sigma^2}{2}V''(y_t) \tag{9}$$

The second order differential equation (9) is solved subjected to boundary conditions.

Liquidation occurs when output hits a lower value of earnings y_b: the corresponding value of the firm at liquidation is θ. In other words, when $y_t = y_b$ the firm is liquidated and its residual value is θ. The trigger value y_b is obtained after imposing the smooth pasting condition $V'(y_b) = 0$. Clearly, θ represents a lower bound of the firm.

An important point we are making in the present paper is given by the existence of an upper bound of the firm's value. In the current literature, it is assumed that firm's value grows indefinitely without limits. To understand how this is realized, we know that the differential equation (9) has two roots: one positive and one negative. As showed by Leland (1994), the general solution of (9) has two parts: one is associated to the positive root, the other to the negative one. A common practice in the literature (see Leland, 1994 for all) is to exclude the part associated to the positive root from the analytical expression of general solution to (9).

Differently from the current literature, rather than excluding the positive root, we insert a maximum value reachable by firm. This can be due, for example, to the state of technology: the maximum capacity is forcefully limited by the existing state of technology or by the availability of a maximum level of natural resources to be deployed in the production process. In fact, when a firm reaches a given threshold needs to introduce some innovations in the organizational structure or in the production process. If nothing happens, the current organizational structure and the current plants become a constraint for the further development of the firm: this is the rationale behind the upper bound.

As we discussed in the introduction, the role of the upper bound is not new in the literature on real options as showed, among others, by Dias and Shackleton (2011) and other contributions already discussed in the literature review in the introduction. The novelty of this paper is the introduction of the upper bound via the usage of the Target Zone approach.

Thus, when $y_t = y_h$, the corresponding maximum firm's value is $V(y_h) = \Gamma$. Clearly, y_h is obtained by applying the smooth pasting condition $V'(y_h) = 0$, as necessary condition for a maximum.

2.2. Equity and debt

An alternative case is offered by a situation where the firm starts with both equity and debt, then it goes bankrupt and it is run via pure equity by former debt holders. Therefore, former debt holders face an option either to liquidate the firm now or to run it as a going concern: in this case the maximum amount of earning is given by (3). This is because bankruptcy impairs the firm. Assume that $Z_t = Z(y_t)$ is the value of the firm in the hands of new owners. Even in this case there is a value for y_t, given by y_x such that bankruptcy occurs. When $y_t = y_x$, the firm goes bankrupt and the residual value is still θ, so that: $Z(y_x) = \theta$. As in the pure equity case, the value of y_x is obtained after imposing the smooth pasting condition: $Z'(y_x) = 0$. The value of the firm $Z(y_t)$ is the solution of the following equation:

$$rZ_t = \gamma_x \left(y_t - k\right) + \mu Z'(y_t) + \frac{\sigma^2}{2} Z''(y_t) \tag{10}$$

As discussed earlier, the value of the firm can reach a maximum Φ, given the actual state of the technology. The maximum Φ can be bigger or equal to the pure equity case Γ, i.e.: $\Phi \geq \Gamma$.

3. The benchmark case

The value of the firm in the two states is governed by $V(y_t)$ and $Z(y_t)$ from Eqs. (9) and (10), respectively. We can now recast the problem by considering a Target Zones (TZ, henceforth) solution approach. In this section we focus on the one-sided solution. The result is condensed in the following Theorem:

Theorem 1. *The one-sided general solution of the pure equity firm is*

$$V(y_t) = \frac{y_t}{r} - \frac{kr - \mu}{kr^2} + \left[\theta - \frac{y_b}{r} + \frac{kr - \mu}{kr^2}\right] e^{\lambda_1 (y_t - y_b)} \tag{11}$$

where y_b is:

$$y_b = \frac{r}{\lambda_1 (r - \mu)} + \theta r + \frac{kr - \mu}{kr} \tag{12}$$

where λ_1 is the negative root of the quadratic equation associated to the homogeneous pde (9) whose expression is:

$$\lambda_1 = -\frac{\mu}{\sigma^2} - \frac{1}{\sigma^2}\sqrt{\mu^2 + 2\sigma^2} \tag{13}$$

Proof. See Appendix A. ∎

Analogously, we can state a similar result for the case where the firm starts with both equity and debt and is successively run by debtors as equity holders, after a bankruptcy episode. Recall that in this case the value of the firm is given by Eq. (10). The results are contained in the following Theorem:

Theorem 2. *The one-sided general solution of the mixed firm is given by the following equation:*

$$Z(y_t) = \frac{\gamma_x}{r} y_t - \frac{\gamma_x (rk - \mu)}{kr^2} + \left[\theta - \frac{\gamma_x}{r} y_x - \frac{\gamma_x (rk - \mu)}{kr^2}\right] e^{\lambda_1 (y_t - y_b)} \tag{14}$$

where the level of output at which bankruptcy occurs is defined as:

$$y_x = \frac{1}{\lambda_1} + \frac{r}{\gamma_x}\left[\theta - \frac{\gamma_x (rk - \mu)}{kr^2}\right] \tag{15}$$

with λ_1 as the root of the second order equation whose analytical expression is given by Eq. (13).

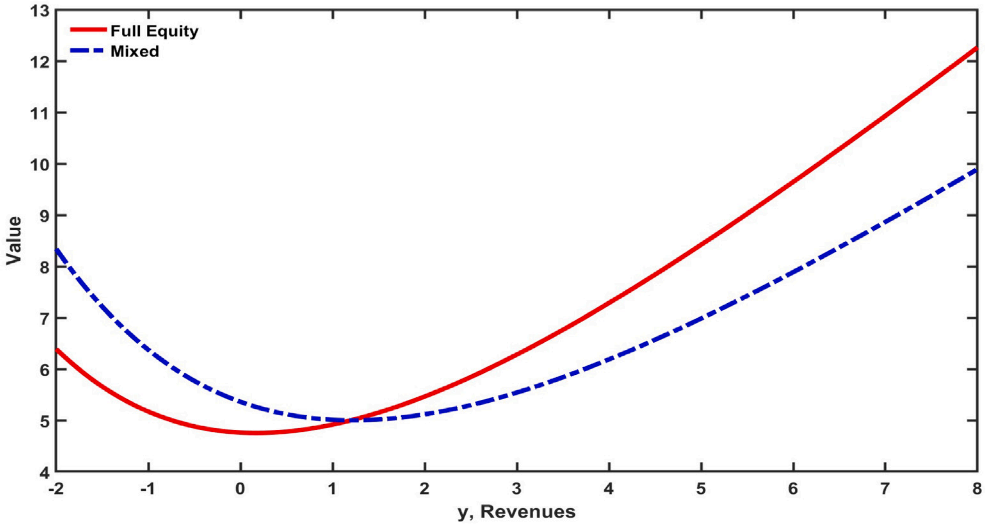

Fig. 1. Simulation of firm's value under full equity case and mixed (dashed). Case with $\sigma = 2$ per cent.

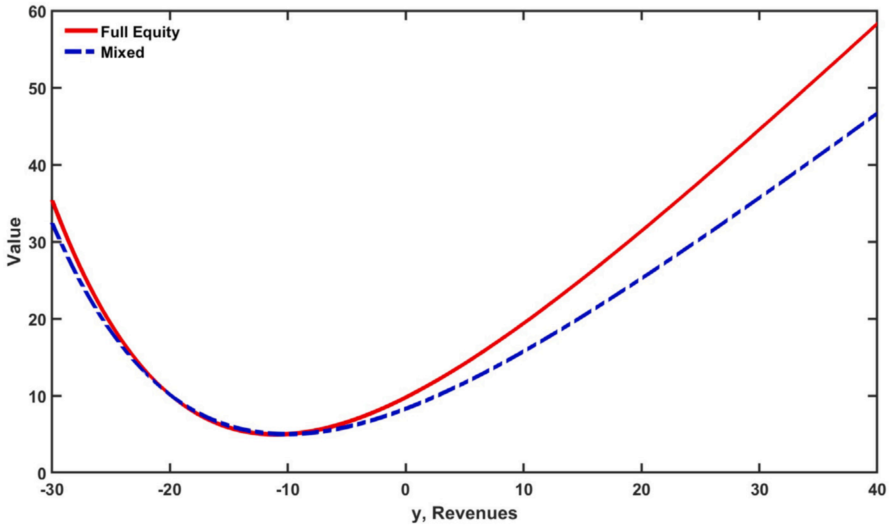

Fig. 2. Simulation of firm's value under full equity case and mixed (dashed). Case with $\sigma = 20$ per cent.

Proof. See Appendix B. ■

First of all, we admit that the bankruptcy may occur in correspondence of a level of output which can be different from the two cases. After a quick inspection of both theorems, we observe that in case of a full equity firm the level of bankruptcy can happen at y_b which is the level of bankruptcy associated to the equity and debt firm and is not necessarily equal to y_x

The model is calibrated by considering the following values: risk free rate $r = 4\%$, $\mu = 3.5\%$, $k = 2$, $\sigma = 2\%$, $\theta = 3$. This parametrization ensures a multiple around 4 between output (revenues) and firm's value. It is a conservative value, but it should be intended as a global average. These parameters are taken from the literature: Mella-Barral and Perraudin (1997) and Leland (1994).

Our results deliver $y_b = 1.19$ and $y_x = 1.21$. Even if the difference between the two values is not larger, the model correctly displays a key property evidencing the fact that the level of production leading to bankruptcy is hit earlier for the restructured firm ($y_x = 1.21$), rather than for the full equity firm ($y_b = 1.19$) showing that after restructuring agreement can impair the firm preventing it from a virtuous value. Our calibration captures quite well the equilibrium growth rate of output in both cases: these values are non-negative but certainly quite small if compared with fast growing firms. We perform a sensitivity analysis by setting the standard deviation to a value ten times larger than what has been assumed in Fig. 1. The results are represented in Fig. 2.

From Fig. 2 we still observe the larger value associated to the full equity case, given the assumptions previously considered. However, with a large level of volatility, we observe a different shape of the two functions: the lower bound occurs in correspondence of negative value for the growth rate of revenues. Moreover, when the growth rate of revenues is between −20% and 0, the two curves cross together, similarly to Fig. 1. In principle, negative values for y (revenues) are feasible, since they represent growth rate of revenue. On the other hand, after a strong increase in the variance, as in case for Fig. 1 the behavior of the model in the negative domain show a decreasing firm's value when the growth rate of revenues approaches higher values (still negative). To sum up, even if negative values for the growth rate of revenues can be, at least in principle, accepted, the model shows an easily interpretable behavior when the growth rate of revenues is positive.

In the next section we generalize this approach to the construction of a Target Zone.

4. The target zone

The key aspect related with the construction of a Target Zone is represented by the two-sided solution. In fact, we hypothesized that there is a value of output such that the firm reaches a maximum. The upper bound can be rationalized in many ways: it can be the maximum value of the firm reachable with the existing production methods and organizational structure. Or, alternatively, the upper bound can represent the maximum size of the market. Moreover, antitrust rules implicitly define a maximum market share for a firm which, implicitly implies a maximum enterprise and equity value, given the existing technological and organizational structure.

To be concrete, the Target Zone is qualified in the following definition:

Definition 3. The Target Zone for the all equity firm is characterized by the following conditions:

$$\text{(i)} \quad V(y_b) = \theta; \qquad \text{(iii)} V(y_h) = \Gamma; \tag{16}$$

$$\text{(ii)} \quad V'(y_b) = 0; \qquad \text{(iv)} V'(y_h) = 0; \tag{17}$$

and

$$\text{(i)} \quad Z(y_z) = \theta; \qquad \text{(iii)} Z(y_{zh}) = \Phi; \tag{18}$$

$$\text{(ii)} \quad Z'(y_z) = 0; \qquad \text{(iv)} Z'(y_{zh}) = 0; \tag{19}$$

for the other firm.

From Eqs. (16)–(17) and (18)–(19) we have that conditions (i) and (ii) are related to the lower bound, while (iii) and (iv) pertains to the upper bound. Recall that conditions (ii) and (iv) are the smooth pasting conditions. Clearly, y_h is the maximum level of output and Γ is the upper bound in the all equity firm, while y_{zh} is the maximum output for the firm run by equity holders and bond holders. The upper bound Γ is exogenously given. The solution techniques will deliver the level of the growth rate of revenues y associated to Γ.

Following Krugman (1991) and Delgado and Dumas (1991), the construction of a Target Zone implies the solution of a system of four equations as described by (16)–(17) and (18)–(19).

Given the general solution of $V(y_t)$ given in (11), the Target Zone is the result of the following system:

$$\frac{y_b}{r} - \frac{kr-\mu}{kr^2} + A_1 e^{\lambda_1 y_b} + A_2 e^{\lambda_2 y_b} = \theta \tag{20}$$

$$\frac{y_h}{r} - \frac{kr-\mu}{kr^2} + A_1 e^{\lambda_1 y_h} + A_2 e^{\lambda_2 y_h} = \Gamma \tag{21}$$

$$\frac{1}{r} + \lambda_1 A_1 e^{\lambda_1 y_b} + \lambda_2 A_2 e^{\lambda_2 y_b} = 0 \tag{22}$$

$$\frac{1}{r} + \lambda_1 A_1 e^{\lambda_1 y_h} + \lambda_2 A_2 e^{\lambda_2 y_h} = 0 \tag{23}$$

The system is solved for the four unknowns A_1, A_2, y_b y_h, either numerically or analytically. In what follows, we adopt a mixed approach, by solving for A_1 and A_2 from (20) and (21) an insert it into (22) and (23). Eqs. (20) and (21) can be cast in matrix form:

$$\begin{bmatrix} e^{\lambda_1 y_b} & e^{\lambda_2 y_b} \\ e^{\lambda_1 y_h} & e^{\lambda_2 y_h} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} \theta_b \\ \Gamma_b \end{bmatrix}$$

where $\theta_b = \theta - \frac{y_b}{r} + \frac{kr-\mu}{kr^2}$ and $\Gamma_b = \Gamma - \frac{y_h}{r} + \frac{kr-\mu}{kr^2}$.

The solutions are:

$$A_1 = \frac{\theta_b e^{\lambda_2 y_h} - \Gamma_b e^{\lambda_2 y_b}}{e^{\lambda_1 y_b + \lambda_2 y_h} - e^{\lambda_1 y_h + \lambda_2 y_b}} \tag{24}$$

$$A_2 = \frac{\Gamma_b e^{\lambda_1 y_b} - \theta_b e^{\lambda_2 y_h}}{e^{\lambda_1 y_b + \lambda_2 y_h} - e^{\lambda_1 y_h + \lambda_2 y_b}} \tag{25}$$

Substituting out (24)–(25) into the smooth pasting conditions (22) and (23) we obtain two equations to solve for y_b and y_h numerically. Another approach, equivalent to ours, to solve the system (20)–(23) has been proposed by Shackleton, Tsekrekos, and Wojakowski (2004). The two methods are perfectly equivalent.

Table 1
Solution to the systems (20)–(23) and (26)–(29).

	A_1	A_2	λ_1	λ_2
Full Equity	5.66	−7.56	−0.2	0.25
Mixed	5.97	−6.93	−0.18	0.11

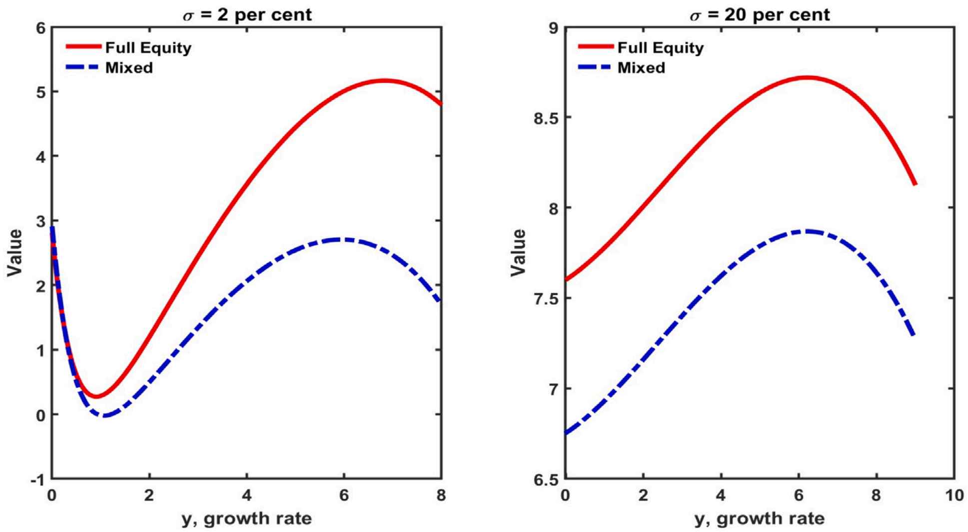

Fig. 3. Target Zone for both full equity firm and restructured (mixed).

A similar procedure can be followed to find a Target Zone for the firm managed by equity holders first and later by bondholders. In this case the system to be solved is:

$$\frac{\gamma_x}{r-\mu}y_x - \frac{\gamma_x k}{r} + A_1 e^{\lambda_1 y_x} + A_2 e^{\lambda_2 y_x} = \theta \tag{26}$$

$$\frac{\gamma_x}{r-\mu}y_x - \frac{\gamma_x k}{r} + A_1 e^{\lambda_1 y_{xh}} + A_2 e^{\lambda_2 y_{xh}} = \Gamma \tag{27}$$

$$\frac{\gamma_x}{r} + \lambda_1 A_1 e^{\lambda_1 y_x} + \lambda_2 A_2 e^{\lambda_2 y_x} = 0 \tag{28}$$

$$\frac{\gamma_x}{r} + \lambda_1 A_1 e^{\lambda_1 y_{xh}} + \lambda_2 A_2 e^{\lambda_2 y_{xh}} = 0 \tag{29}$$

Therefore, by following the same steps described before, we solve for B_1 and B_2 from (26) and (27) which can be substituted out into (28) and (29). The solution to systems (20)–(23) and (26)–(29) is provided in Table 1.

From Table 1 we see that one root is positive and the other is negative. Symmetrically A_1 and A_2 are one positive and the other negative crossed with their respective root in order to maintain the system within the bounds.

We can represent the solution in the following graphs where we adopted the same parametrization assumed for the one-sided case in Fig. 3, where in the left panel we represented the evolution of the model with the standard parametrization, with $\sigma = 2\%$. In the right panel we show the same model conditional to a level of standard deviation set equal to 20 per cent.

The left panel in Fig. 3 shows the evolution of the value of the full equity firm (continuous line) together with the firm with both equity and debt (dashed-dotted line), with $\sigma = 2$ per cent, while the right panel shows the evolution when $\sigma = 20$ per cent. From the left panel, we show the role of the upper and lower bound for both type of firms, fully reflecting the logic of the target zone model: clearly, the restructured firm show an evolution of its value within a smaller range.

A similar pattern is observed also in the right panel, for a firm characterized by a volatility ten times larger: the full equity firm ranges with higher values with respect to the both equity and debt firm, for the same growth rate of revenues. On other hand, for a model with endogenous target zone, we have that, an increasing volatility implies an increase of the firm's value, as it is easily observed by comparing the peaks and the trough of the firms' value under all cases reported. However, the increase of variance, keeping all other parameters given, shows a smaller difference between the value of a full equity firm and that of mixed one, as it is proved in the right panel of Fig. 3, a sign of the fact that higher volatility raises firm's value for high growth rate of revenues.

This example shows that the imposition of both upper and lower bound implies an evolution of the value of the firm within a range. This result is similar to Dias and Shackleton (2011). It is important to emphasize that the upper and lower level of y is endogenous to the model, given the upper and lower value of the firm's value. This endogeneity emerges naturally given the assumptions of the model about the link existing between revenues growth rate and firm's value and as the result of the smooth pasting condition.

Table 2
Sensitivity analysis for the all equity case and for the 'mixed' case, represented by full equity after bankruptcy.

		$\sigma = 2\%$	$\sigma = 20\%$	
All Equity	y_h	6.83	8.71	6.25
	y_b	0.91	7.6	0.03
Mixed	y_{xh}	6.07	7.86	6.24
	y_x	1.07	6.75	0.0

In Table 2 we report the value associated to each case under study obtained as a solution of the system (20)–(23) for both cases: the full equity firm and the mixed firm under bankruptcy. In both cases we report also the solution conditional to two different value for the standard deviation: $\sigma = 2\%$ and $\sigma = 20\%$. From Table 1 we clearly observe that increasing the volatility implies and increase in the value of the firm under all different cases. This result is aligned with the impact of volatility on the option value.

5. The levered firm

In what follows we extend previous results to the case characterized by a levered firm. We solve both debt and equity for a firm issuing a perpetual debt with principal c/r and a coupon flow c per period of time.

Assume $D(y_t)$ is the value of the levered firm's debt and $S(y_t)$ is the value of equity. Following a similar approach as that presented before, financial markets equilibrium requires that:

$$rD(y_t) = c + \mu D'(y_t) + \frac{\sigma^2}{2} D''(y_t) \tag{30}$$

The equity value is by:

$$rS(y_t) = y_t - k - c + \mu S'(y_t) + \frac{\sigma^2}{2} S''(y_t) \tag{31}$$

Equity holders are ready to cover firm's operating losses by inserting new capital. Therefore, this implies that equity holders decide the timing of bankruptcy: when they stop to insert new capital, the firm will go bankrupt. When a low level of the growth rate of sales y_b is reached, bankruptcy occurs. As we proved earlier, y_b is chosen via smooth pasting condition. Without arbitrage, the value of the firm in case of bankruptcy is:

$$D(y_b) = \min\left\{ Z(y_b), \frac{c}{r} \right\} \tag{32}$$

and

$$S(y_b) = \max\left\{ 0, V(y_b) - \frac{c}{r} \right\} \tag{33}$$

Thus, the value of debt and equity is delivered according to the following Theorem:

Theorem 4. *The value of debt is given by:*

$$D(y_t) = \frac{c}{r} + \left[Z(y_b) - \frac{c}{r} \right] e^{\lambda_1 (y_t - y_b)} \tag{34}$$

and the value of equity is

$$S(y_t) = \frac{y_t}{r} - \frac{(c+k+\mu)}{r} - \left[\frac{y_b}{r} - \frac{(c+k+\mu)}{r} \right] e^{\lambda_1 (y_t - y_b)} \tag{35}$$

where

$$y_b = \frac{1}{\lambda_1} + c + k + \mu \tag{36}$$

Proof. See Appendix ■

We can represent the evolution of the value of the firm given by $W(y_t) = D(y_t) + S(y_t)$ in Fig. 4.

Fig. 4 represents the one-sided solution as obtained from Eqs. (34)–(35). The curve representing debt value is increasing with respect to growth rate of revenues and concave. At the same time, the curve representing the equity value is increasing and convex. The curve describing the total value of the firm is obtained as the sum of both equity and debt value. The example shows that the one-sided solution is not enough to bound the evolution of firm's value (enterprise value considered as the sum of debt and equity). This is clearly observed by the fact that equity, debt and firm's value grow indefinitely, notwithstanding the upper bound.

Focusing our attention on the behavior of equity, we observe that for revenues set equal to 1 per cent, the equity value is close to zero: paradoxically, equity holders seem to be better off, when revenues are zero, rather than when they are 1. However, this can be interpreted along the traditional pattern of corporate value evolution: at the start, when revenues are not materialized yet, the value of equity is positive: it is the starting capital. After revenues start to raise, also debt increases as well. When revenues are

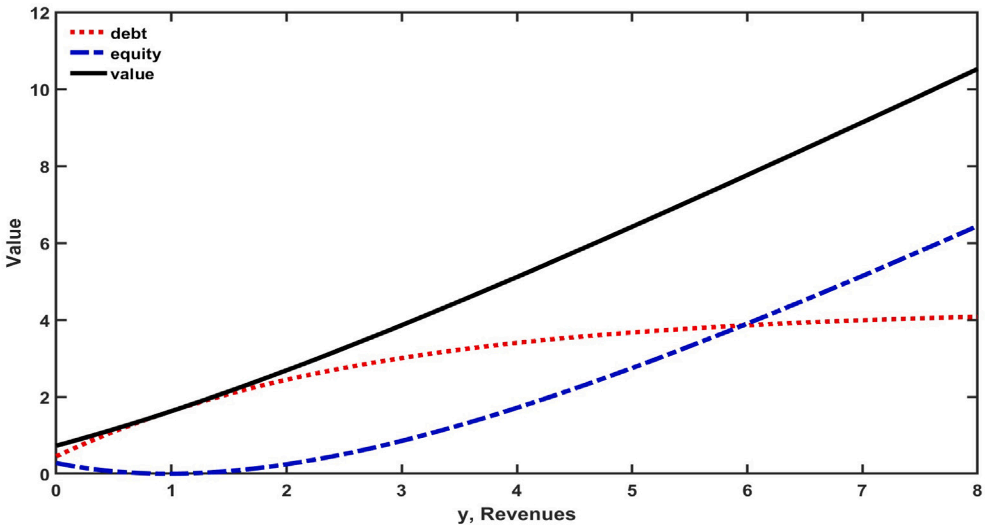

Fig. 4. Value of debt (dotted line), value of equity (dashed line), total value of firm (continuous line), for a levered firm, $\sigma = 2\%$.

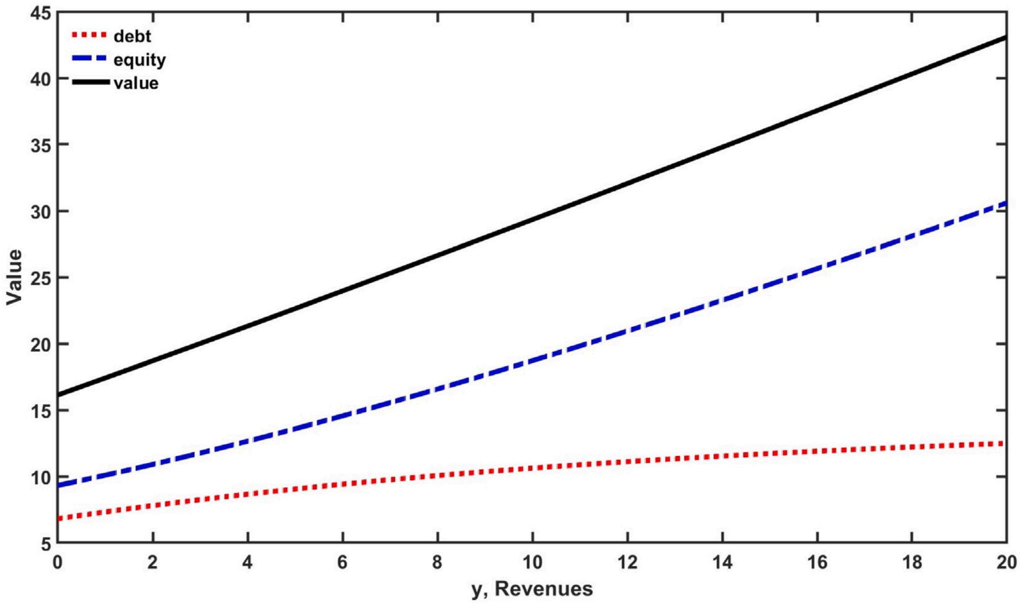

Fig. 5. Value of debt (dotted line), value of equity (dashed line), total value of firm (continuous line), for a levered firm, $\sigma = 20\%$.

too low (say $y \leq 1.3$), they are not sufficiently high to ensure enough cash flows: equity value is compressed until when the value of revenues increase up to a point above 1.3, when their level starts to be sufficiently high to payback debt and guarantee equity increase.

In Fig. 5, after an increase of volatility up to 20 per cent, we observe a dramatic change in the shape of equity, debt and enterprise value as a function of revenues growth rate for the same range of values for revenues as adopted in Fig. 4: the curve representing equity value is proportionally increasing with respect to y. Interestingly, the curve representing the evolution of debt is increasing in a less than proportional way with respect to revenues. Debt curve is still mildly concave and the convexity of the equity curve is preserved.

We now extend the Target Zone framework to the case of a levered firm. Following the same sort of argument described in the previous section, the Target Zone model is defined as follows:

$$\text{(i)} \quad S(y_b) = \theta_e; \qquad \text{(iii)} \ S(y_h) = \Gamma_e; \tag{37}$$

$$\text{(ii)} \quad S'(y_b) = 0; \qquad \text{(iv)} \ S'(y_h) = 0; \tag{38}$$

Table 3
Solution to the systems (39)–(42).

	A_1	A_2	λ_1	λ_2
$S(y)$	5.71	−0.26	−0.36	0.34

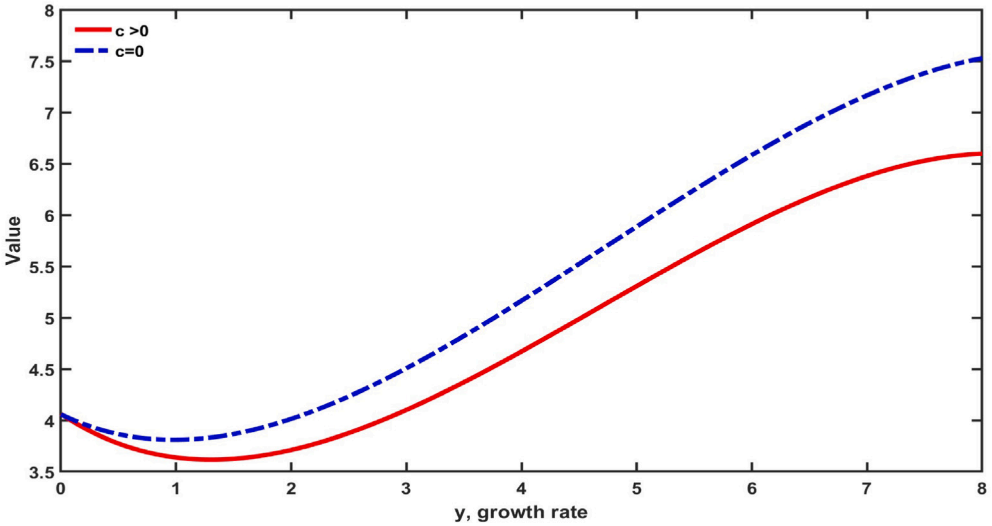

Fig. 6. Target Zone for equity value of a levered firm with different coupon: positive and zero. Simulation of the system (39)–(42), with $c > 0$ and $c = 0$.

The conditions (37)–(38) originate the following system:

$$\frac{y_b}{r} - \frac{c+\mu+k}{r} + A_1 e^{\lambda_1 y_b} + A_2 e^{\lambda_2 y_b} = \theta_e \tag{39}$$

$$\frac{y_h}{r} - \frac{c+\mu+k}{r} + A_1 e^{\lambda_1 y_h} + A_2 e^{\lambda_2 y_h} = \Gamma \tag{40}$$

$$\frac{1}{r} + \lambda_1 A_1 e^{\lambda_1 y_b} + \lambda_2 A_2 e^{\lambda_2 y_b} = 0 \tag{41}$$

$$\frac{1}{r} + \lambda_1 A_1 e^{\lambda_1 y_h} + \lambda_2 A_2 e^{\lambda_2 y_h} = 0 \tag{42}$$

where, as in the previous section, θ_e is the lower value of equity and Γ_e is the upper value of equity. The system of Eqs. (39)–(42) can be solved exactly by following the same steps described in previous section. In Table 3 the solution to system (39)–(42) is reported.

A rapid look at Table 3 evidences how the solution obtained respects the cross positivity/negativity associated between roots and constant. We can represent the evolution of the Target Zone for equity in Fig. 6.

The evolution of the TZ represented in Fig. 6. shows the impact on the equity value for different value coupon values. Larger is the coupon lower is the equity value, since the TZ curve for the levered firm lies below that obtained with zero coupon. The model shows again the role of upper and lower bound. This result contrasts with the *trade-off* theory firstly proposed by Miller and Modigliani (1963), formalized, among others, by Kraus and Litzenberger (1973), where it is showed that a moderate debt increases firm value due to tax shields. Excessive debt, at the contrary, reduces the firm's value due to expected bankruptcy costs.

The evolution of TZ here represented shows the internal consistency of the model, which can be further generalized to the case with a large σ, by setting $\sigma = 20$ per cent. The relationship between revenues growth rate and firm's value is represented in Fig. 7.

A quick inspection of Fig. 7 reveals the same conclusion already discussed previously: the level of debt reduces the equity value. In the following section we want to test if this conclusion is still verified for a different stochastic process.

6. A more general stochastic process

A drawback of the arithmetic Brownian motion assumed in (1) is represented by the possibility to express negative values for the growth rate of firm's output y. We can generalize the stochastic process for the growth rate of revenues by assuming that firm's output y_t is governed by:

$$dy_t = \eta\left(\mu - y_t\right)dt + \sigma dZ_t \tag{43}$$

Eq. (43) is a general formulation for the stochastic process for y_t, as discussed by Tristani (1994) and Dias and Shackleton (2011). The long run mean is identified by $\mu > 0$, and $\eta > 0$ identifies the intensity with which the growth rate of the firm is reverted back

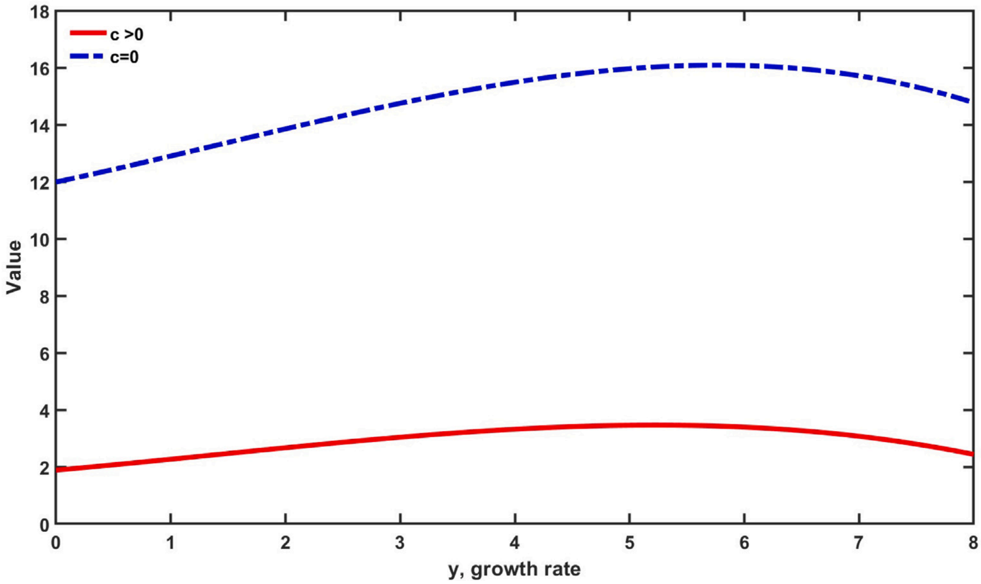

Fig. 7. Target Zone for equity value of a levered firm with different coupon: positive and zero. Simulation of the system (39)–(42), with $c > 0$ and $c = 0$, with $\sigma = 20$.

towards its long run mean. Moreover, σ is the volatility of the stochastic process (43). In what follows we derive again the Target Zone conditions, under the assumption that the stochastic process driving the growth rate of revenues is given by (43).

Recall the financial market equilibrium equation, given by:

$$rV_t = y_t - k + E_t\left(\frac{dV_t}{dt}\right) \tag{44}$$

Using the Ito's Lemma together with (43) and taking the expectation, on the same guise of Eq. (9), we have:

$$dV_t = \eta\left(\mu - y_t\right)V'(y_t)dt + V'(y_t)\sigma dZ_t + \frac{\sigma^2}{2}V''(y_t)dt \tag{45}$$

which, after taking expectations, together with (9) becomes after setting $\gamma = 1/r$:

$$V_t = \gamma\left(y_t - k\right) + \eta\left(\mu - y_t\right)V'(y_t) + \frac{\sigma^2}{2}V''(y_t) \tag{46}$$

Set the variable:

$$x = \frac{\eta}{\sigma^2}(\mu - y)^2 \tag{47}$$

where we drop time index. After the change of variable, we can rewrite derivatives $V'(y_t)$, $V''(y_t)$:

$$V'(y) = -\frac{2\eta}{\sigma^2}(\mu - y)V'(x(y)) \tag{48}$$

$$V''(y) = \frac{2\eta}{\sigma^2}V'(x(y)) + \frac{4\eta^2}{\left(\sigma^2\right)^2}(\mu - y)^2 V''(x(y)) \tag{49}$$

Using (48) and (49) into (46) and rearranging after taking into account the definition x in (47):

$$V(x(y)) = \gamma\left(y_t - k\right) + 2\gamma\eta\left(\frac{1}{2} - x\right)V'(x(y)) + 2\gamma\eta x V''(x(y)) \tag{50}$$

Divide both sides by $2\gamma\eta$ to get:

$$\frac{1}{2\gamma\eta}V(x) = \frac{\left(y_t - k\right)}{2\eta} + \left(\frac{1}{2} - x\right)V'(x) + xV''(x) \tag{51}$$

If we focus on the homogeneous part of second order differential equation (50) and rearrange, we get:

$$xV''(x) + \left(\frac{1}{2} - x\right)V'(x) - \frac{1}{2\gamma\eta}V(x) = 0 \tag{52}$$

Eq. (52) is a Kummer equation whose solution is:

$$V(x) = AM(a, b, x) + Bx^{1-b}M(1 + a - b, 2 - b, x) \tag{53}$$

Table 4
Solution to the systems equations (58), (66).

	A_1	A_2	λ_1	λ_2
$S(y)$	−0.041	−0.086	0.45	8.96
$Z(y)$	−0.01	−0.015	0.96	9

where $M(a,b,x)$ and $M(1+a-b,2-b,x)$ are the Kummer confluent hypergeometric functions, with:

$$a = \frac{1}{2\gamma\eta} \tag{54}$$

$$b = \frac{1}{2} \tag{55}$$

and A and B are constant to be determined. A full characterization of function $M(a,b,x), x^{1-b}M(1+a-b,2-b,x)$, is given in Slater (1960, eq. 1.1.8), Abramowitz and Stegun (1972, eq. 13.1.2).

To obtain the general solution, we guess the following solution for the homogeneous part $f(y) = m_0 + m_1 y$, with $f'(y) = m_1$ to be included into (45). After matching coefficients, we find:

$$m_0 = \frac{\gamma^2\eta\mu - \gamma k(1+\gamma\eta)}{1+\gamma\eta} \tag{56}$$

$$m_1 = \frac{\gamma}{1+\gamma\eta} \tag{57}$$

Therefore, all solutions are defined by:

$$V(y) = m_0 + m_1 y + AM\left(\frac{1}{2\gamma\eta}, \frac{1}{2}, \frac{\eta}{\sigma^2}(\mu-y)^2\right) + B\left(\frac{\sqrt{\eta}}{\sigma}\right)(\mu-y)\,M\left(\frac{1+\gamma\eta}{2\gamma\eta}, \frac{3}{2}, \frac{\eta}{\sigma^2}(\mu-y)^2\right) \tag{58}$$

where m_0 and m_1 are given by:

$$m_0 = \frac{r-\omega-k\mu}{rk(r-\omega)} \tag{59}$$

$$m_1 = \frac{1}{r-\omega} \tag{60}$$

Thus, from Definition 3, the Target Zone for the all equity firm is the solution of the system:

$$V(y_b) = \theta \tag{61}$$

$$V'(y_b) = 0 \tag{62}$$

$$V(y_h) = \Gamma \tag{63}$$

$$V'(y_h) = 0 \tag{64}$$

with respect to four unknowns: y_b, y_h, A, B, where $V(y_b)$ and $V(y_h)$ are determined by Eq. (58) computed conditionally to y_b, y_h and where the first derivative of $V(y)$ is (see Abramowitz & Stegun, 1972; Slater, 1960):

$$M'(a_1,b,z_1) = \frac{a_1}{b}M(a_1+1,b+1,z_1) \tag{65}$$

Thus, by using (58) together with (65) and conditions (61)–(64), we can solve a four equation system with respect to y_h, y_b, A, B along the same lines discussed in the case with an arithmetic Brownian motion examined in the previous section. The 'mixed' case represented by debt and equity and only equity post bankruptcy, implies a slight modification of Eq. (58) along the same lines already discussed in Eq. (10), to get the value of the firm Z_t:

$$Z_t(y_t) = \gamma_x\gamma(y_t-k) + \gamma\eta(\mu-y_t)Z'(y_t) + \gamma\frac{\sigma^2}{2}Z''(y_t)$$

where $\gamma_x < 1$ is the cost of bankruptcy as discussed in (10). Following the same steps as discussed before, we obtain the following solution:

$$Z(y) = m_{0x} + m_{1x}y + AM\left(\frac{1}{2\gamma\eta}, \frac{1}{2}, \frac{\eta}{\sigma^2}(\mu-y)^2\right) + B\left(\frac{\sqrt{\eta}}{\sigma}\right)(\mu-y)\,M\left(\frac{1+\gamma\eta}{2\gamma\eta}, \frac{3}{2}, \frac{\eta}{\sigma^2}(\mu-y)^2\right) \tag{66}$$

where A_x, B_x are constant to be determined and m_{0x}, m_{1x} are given by:

$$m_{0x} = \gamma_x\gamma\left(\frac{\eta\mu\gamma_x\gamma}{1+\eta\gamma_x\gamma} - k\right) \tag{67}$$

$$m_{1x} = \frac{\gamma_x\gamma}{1+\eta\gamma_x\gamma} \tag{68}$$

The solution is reported in the Table 4.

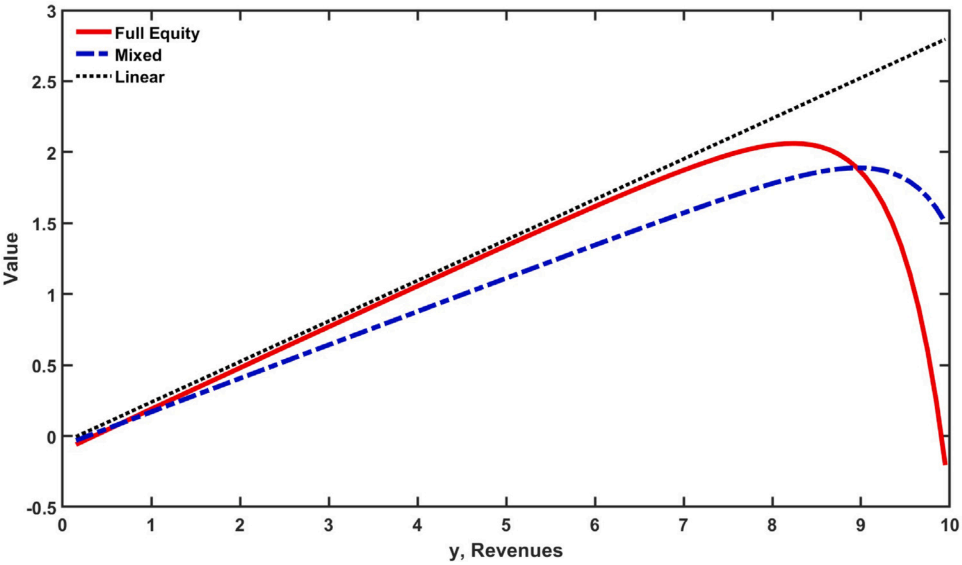

Fig. 8. Firm's value under a mean reverting process. Full equity case (continuous line), debt and equity post bankruptcy (dashed/dotted) with $\gamma_x = 0.7$. The dotted line is the deterministic linear relation between revenues and value.

Table 5
Sensitivity analysis for $\sigma = 2\%$, 20% for the full equity (58), 'mixed' (66) and levered firm (70).

		$\sigma = 2\%$	$\sigma = 20\%$
Full Equity	y_h	2.06	18.93
	y_b	0.02	0.01
Mixed, $\gamma_x = 0.7$	y_h	1.88	15.58
	y_b	0.015	0.009
Levered $c > 0$	y_h	3.08	79.5
	y_b	0.0057	0.013

The solution, conditional to standard parameter values: can be simulated conditional to the evolution of the growth rate of revenues, to obtain the pictorial representation in Fig. 8 for the all equity case.

Fig. 8 reports the evolution of firm's value derived from a mean reverting process like (43): the continuous line represents the equity only case, while the dashed/dotted line is the value of the firm under the hypothesis that the firm starts with both equity ('Mixed') and debt and it is run via pure equity by former debt holders. The dotted line represents the linear relationship emerging from the model, absent any stochastic source, and it is obtained from the solution (58) by setting the value of the constant A, B equal to zero. The picture shows clearly how the target zone constrains the value of the firm, given the existing capital structure in both cases.

Thus, after a continuous growth rate of revenues, the existing set of constraint in terms of both technology and capital structure impose a non-linear evolution of firm value reaching the upper limit showed above. The lower limit is reached for negative values of the growth rate of revenues and is not showed in the picture in order to preserve the proportions of the graph.

We consider in Fig. 9 a variant represented by a change in the volatility of revenues up to 20 per cent.

From Fig. 9 we observe a similar behavior for the curves of full equity and mixed: the main difference existing between the case with $\sigma = 2$ represented in Fig. 8 is given by the fact that the maximum firm value is reached when the growth rate of revenues is very high. This clearly shows a large impact on firm's value delivered by the increase of volatility of the underlying process.

In Table 5 we represent the maximum and minimum values of the growth rate of revenues conditional to different values for the various parameters of the model.

The model can be immediately extended to the case with both equity and debt, generalizing the model represented by Eqs. (30) and (31) with the stochastic process under (43). The presence of the coupon rate in the equity value given by Eq. (31), allows us to study the equity value of a firm with also debt financing represented by a constant coupon. The value of the levered firm evolves through the following equation:

$$S(y_t) = \gamma\left(y_t - k - c\right) + \gamma\eta\left(\mu - y_t\right) S'(y_t) + \frac{\gamma\sigma^2}{2} S''(y_t) \tag{69}$$

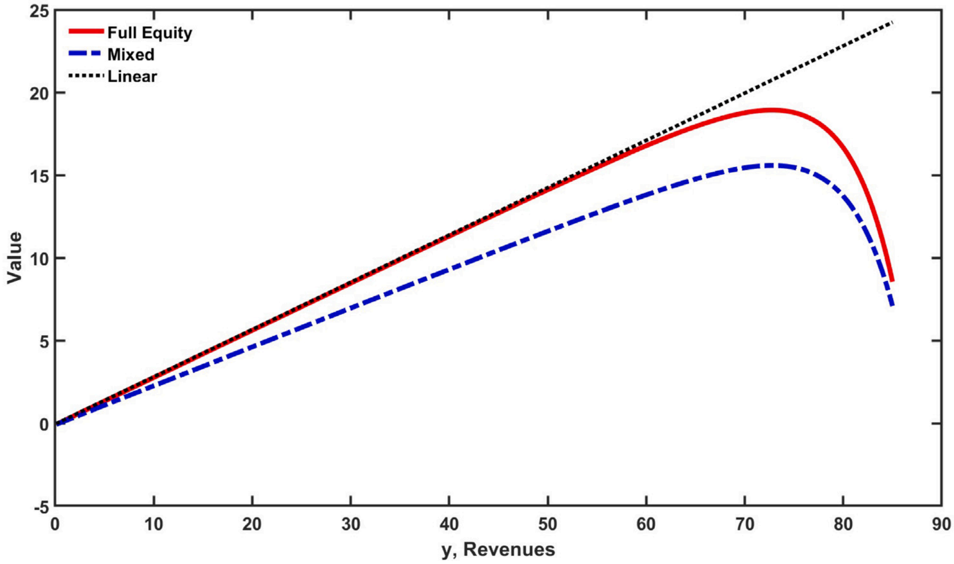

Fig. 9. Simulation for the Full equity model (58), continuous line and 'mixed' (66), dashed/dotted, for different volatility values: $\sigma = 2\%$, $\sigma = 20\%$. The dotted line is the deterministic model without bounds.

Following the same approach described before, we obtain the solution for the equity value of a levered firm:

$$S(y_t) = m_{0s} + m_{1s}y + A_s M\left(\frac{1}{2\gamma\eta}, \frac{1}{2}, \frac{\eta}{\sigma^2}(\mu - y)^2\right) + B_s\left(\frac{\sqrt{\eta}}{\sigma}\right)(\mu - y)\, M\left(\frac{1+\gamma\eta}{2\gamma\eta}, \frac{3}{2}, \frac{\eta}{\sigma^2}(\mu - y)^2\right) \tag{70}$$

where A_s, B_s are constant to be determined and m_{0s}, m_{1s} are defined according to:

$$m_{0s} = \frac{\eta\mu\gamma^2 - \gamma(k+c)(1+\gamma\eta)}{1+\eta\gamma} \tag{71}$$

$$m_{1x} = \frac{\gamma}{1+\eta\gamma} \tag{72}$$

The benchmark values for the coupon is 2 per cent. The pictorial representation of the equity value given in (70) (see Fig. 10).

From Fig. 10 we observe how a small level of leverage represented by the coupon makes the value of the levered company similar to the case with $c = 0$. The value of the levered company becomes larger when the growth rate of revenues increases. This result contrasts with the case represented with a simple stochastic process as (1) and emerges here only after considering a more general process like (43) with a mean reverting process: the presence of a moderate coupon makes equity value higher. This result is in line with the well known *trade-off* theory: along the lines of Miller and Modigliani (1963) and Kraus and Litzenberger (1973), moderate debt enhances firm value because of tax shields, which, indirectly increases also the equity value. In this model the role of taxes is not explicit, but our results emerge even without an explicit formulation of firm's tax structure. The introduction of a moderate level of debt allows the firm to reduce its overall tax burden, increasing the total firm value and equity value, as long as the risk of financial distress is low. This conclusion is also supported by empirical evidence collected by Graham (2000) and Fama and French (2002). Our model proves the existence of trade off theory within the context of target zone model. A possible immediate extension is to consider explicitly the role of taxes, as in Goldstein et al. (2001).

Let us consider the same case with an higher level of variance $\sigma = 20$. The results are reported in Fig. 11, where we observe two important phenomena: first of all the maximum is reached at very high values for both cases $c > 0$ and $c = 0$. Secondly, the line for $c = 0$ show low equity value. This is another example where we observe that a small level of debt works as an incentive and creates an increase in equity value as well.

In Table 6 we report a set of sensitivity exercises for various parameters of the model. In particular, we considered a sensitivity on the stochastic process parameter η, for which we take two values $\eta = 0.1$ and $\eta = 10$. Moreover, we also present a sensitivity exercise relative to the mean of the stochastic process: we examine two case, one for $\mu = 0.1$ and another one for $\mu = 10$. The results are reported in Table 6.

From the numbers reported in Table 6, we observe the highly non-linear nature of the relationship explored in the paper: increasing η makes the model more concentrated. Overall, it is confirmed that the levered firm is characterized by an higher value, for any values for growth rate of revenues.

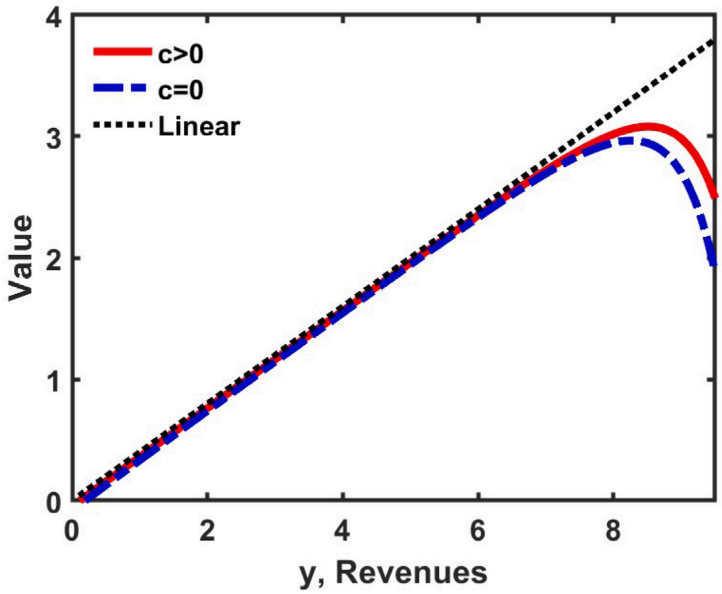

Fig. 10. The picture shows the evolution of the value for the levered equity for $c > 0$ (continuous line) and for $c = 0$ (dashed/dotted line). The dotted line is the linear deterministic relation between revenues growth and value.

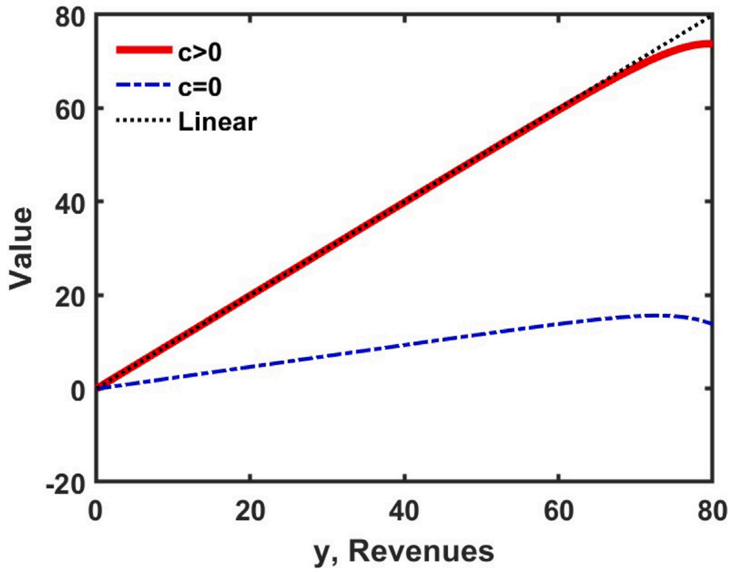

Fig. 11. The picture shows the evolution of the value for the levered equity for $c > 0$ (continuous line) and for $c = 0$ (dashed/dotted line) when $\sigma = 20\%$. The dotted line is the linear deterministic relation between revenues growth and value.

7. Concluding remarks

In this paper, we proposed a novel framework to reinterpret and extend the dynamic capital structure theory initiated by Leland (1994) and further developed in subsequent literature. We take the main framework from Mella-Barral and Perraudin (1997) and we extend it to include a Target Zone model on the same lines as proposed by Krugman (1991) and Delgado and Dumas (1991). Our main contribution lies in introducing a Target Zone structure for firm value, characterized by both lower and upper boundaries. While the lower boundary reflects the familiar bankruptcy threshold, the upper boundary represents a conceptual and practical innovation: it captures the limits imposed by investor protection mechanisms and other institutional or contractual constraints that become relevant when the firm performs exceptionally well.

Table 6
Sensitivity analysis for the model with mean reverting stochastic process. Full Equity case (58), 'mixed' (66), and levered firm (70), for $\eta = 0.1$, 1.5 and $\mu = 0.1$, 10.

		$\eta = 0.1$	$\eta = 1.5$	$\mu = 0.1$	$\mu = 10$
Full Equity	y_h	7.45	5.25	9.95	9
	y_b	0.15	0.03	0.15	0.1
Mixed, $\gamma_x = 0.7$	y_h	7.45	5.05	9.9	9.9
	y_b	0.15	0.032	0.1	0.12
Levered $c > 0$	y_h	9	4	6.3	10
	y_b	0.08	0.01	0.01	0.012

The introduction of an upper boundary resolves one of the long-standing paradoxes in structural models of capital structure—namely, the implausible implication that firm value can grow without bound as EBIT or sales tend toward infinity. By establishing a finite target zone, our model restores economic realism and aligns more closely with observed corporate behavior, where growth is often tempered by frictions such as agency costs, payout constraints, and policy interventions. Furthermore, this bounded framework naturally embeds a real options perspective, wherein management optimally navigates between investment, financing, and restructuring decisions within a bounded opportunity set. We also solve the target zone for a general mean-reverting problem, to avoid eventual negative values of the domain: we end up by solving a Kummer equation, as it is customary in the Target Zone literature.

A key result of the present paper is that under a mean reverting process for the growth rate of revenues the trade-off theory of Miller and Modigliani (1963), Kraus and Litzenberger (1973) and Leland (1994) is confirmed within a Target Zone framework. A natural extension to be considered consists now to include an explicit role of taxes, extending the model by Goldstein et al. (2001) with a Target Zone similar to what has been considered in the present paper.

The Target Zones approach not only enriches the interpretation of optimal capital structure under uncertainty but also lays the groundwork for meaningful extensions. Future research could incorporate the role of corporate and investor-level taxes, more complex and realistic stochastic processes governing firm output (e.g., mean-reverting, jump-diffusion, or regime-switching dynamics), and alternative debt designs that differentiate between performance-contingent instruments and state-contingent covenants. Moreover, the model offers fertile ground for investigating capital structure choices in heterogeneous firms environments, particularly in scenarios where firm value is path-dependent or subject to macroeconomic shocks.

We believe that this framework can offer valuable insights into the strategic financial behavior of firms, especially in periods of strong performance, which are often neglected in traditional models that emphasize distress and default. By conditioning on adverse state assumptions while modeling behavior in favorable states, our approach bridges the gap between risk management and value creation. Ultimately, the Target Zones perspective provides a unifying lens through which capital structure decisions can be re-evaluated in both theoretical and applied corporate finance settings.

Appendix A. Proof of Theorem 1

To solve Eq. (9), we guess the following function $V(y_t) = e^{\lambda y_t}$. Taking derivatives and plugging into Eq. (9) we obtain:

$$rV_t = y_t - k + \mu\lambda e^{\lambda y_t} + \frac{\lambda^2\sigma^2}{2}e^{\lambda y_t} \tag{73}$$

If we focus on the homogeneous part, the root λ is obtained as solution of the following second order equation:

$$\frac{\lambda^2\sigma^2}{2} + \mu\lambda - r = 0$$

whose roots are:

$$\lambda_{1,2} = -\frac{\mu}{\sigma^2} \pm \frac{1}{\sigma^2}\sqrt{\mu^2 + 2\sigma^2} \tag{74}$$

A general solution takes the following general functional form:

$$V(y_t) = f(y_t) + A_1 e^{\lambda_1 y_t} + A_2 e^{\lambda_2 y_t} \tag{75}$$

where $\lambda_1 < 0$, $\lambda_2 > 0$ are the roots of (74) and A_1, A_2 are constant to be determined. To complete the solution we need an expression for the non-homogeneous part $f(y_t)$. Given the linear structure of Eq. (9) we guess the following function for $f(y_t)$:

$$f(y_t) = \alpha_1 y_t - \alpha_0 k \tag{76}$$

with α_1, α_0 constant to be determined. Thus, using $f'(y_t) = \alpha_1$ and (76) into equation we obtain:

$$\alpha_1 r y_t - \alpha_0 r k = y_t - k + \alpha_1 \mu$$

Thus equating coefficients of y_t and the constant terms, we find that:

$$\alpha_1 = \frac{1}{r}; \qquad \alpha_0 = \frac{rk - \mu}{r^2 k} \tag{77}$$

Therefore, the general solution is:

$$V(y_t) = \frac{y_t}{r} - \frac{(rk-\mu)}{kr^2} + A_1 e^{\lambda_1 y_t} + A_2 e^{\lambda_2 y_t} \tag{78}$$

If we exclude explosive part of the solution by setting $A_2 = 0$, we obtain the one sided solution. To get the value of constant A_1 and the value of y_b, we need to solve the following system of equations:

$$\frac{y_b}{r} - \frac{(rk-\mu)}{kr^2} + A_1 e^{\lambda_1 y_b} = \theta \tag{79}$$

$$\frac{1}{r} + \lambda_1 A_1 e^{\lambda_1 y_b} = 0 \tag{80}$$

Eq. (79) is the condition $V(y_b) = \theta$, i.e. the value of the firm when output hits y_b is equal to the bankruptcy case. Eq. (80) is the smooth pasting condition computed conditional to $y_t = y_b$. From (79) we immediately get:

$$A_1 = \left[\theta - \frac{y_b}{r} + \frac{(rk-\mu)}{kr^2}\right] e^{-\lambda_1 y_b} \tag{81}$$

Plugging (81) into (80) we obtain the following expression for y_b:

$$y_b = \frac{1}{\lambda_1} + \theta r + \frac{(rk-\mu)}{kr} \tag{82a}$$

as stated in the text. □

Appendix B. Proof of Theorem 2

The proof goes along with the same steps already considered for Therefore, we provide here a brief sketch. The general solution to Eq. (10) is given by the following equation:

$$Z(y_t) = f(y_t) + B_1 e^{\lambda_1 y_t} + B_2 e^{\lambda_2 y_t} \tag{83}$$

As in case of Theorem 1, we guess $Z(y_t) = e^{\lambda y_t}$ which, after taking derivatives and plugged into Eq. (10) delivers a second order equation identical to Eq. (74) determining the roots $\lambda_1 < 0$, $\lambda_2 > 0$. To determine the function $f(y_t)$, we guess the same Eq. (76) to be inserted into (10). After equating the coefficient of similar terms, as we proceeded in Theorem 1, we find the following results for constant coefficients α_0, α_1:

$$\alpha_0 = \frac{\gamma_x(rk-\mu)}{kr^2}; \qquad \alpha_1 = \frac{\gamma_x}{r}$$

So that the general solution will be:

$$Z(y_t) = \frac{\gamma_x}{r} y_t - \frac{\gamma_x(rk-\mu)}{kr^2} + B_1 e^{\lambda_1 y_t} + B_2 e^{\lambda_2 y_t} \tag{84}$$

Concentrate again on the one-sided case, by excluding the explosive solution, setting $B_2 = 0$. To determine B_1 and the level of output associated to bankruptcy y_x we need to solve the following system:

$$\frac{\gamma_x}{r} y_x - \frac{\gamma_x(rk-\mu)}{kr^2} + B_1 e^{\lambda_1 y_x} = \theta \tag{85}$$

$$\frac{\gamma_x}{r} + \lambda_1 B_1 e^{\lambda_1 y_x} = 0 \tag{86}$$

Clearly, Eq. (85) is the value of the firm conditional to the bankruptcy case if $y_t = y_x$ and Eq. (86) is the smooth pasting condition. Therefore, solving for B_1 from (85) and substituting into (86), after rearranging, we directly obtain the following expression for y_x:

$$y_x = \frac{1}{\lambda_1} + \frac{r}{\gamma_x}\left[\theta + \frac{\gamma_x(rk-\mu)}{kr^2}\right] = \frac{1}{\lambda_1} + \frac{r\theta}{\gamma_x} + \frac{(rk-\mu)}{kr} \tag{87}$$

As stated in the Theorem. □

Appendix C. Proof of Theorem 4

Start with the proof of Eq. (34). Using the guess $D(y_t) = e^{\lambda y_t}$, we obtain a quadratic function whose roots are still given by:

$$\lambda_{1,2} = -\frac{\mu}{\sigma^2} \pm \frac{1}{\sigma^2}\sqrt{\mu^2 + 2\sigma^2} \tag{88}$$

The general solution of (34) is then:

$$D(y_t) = \frac{c}{r} + A_1 e^{\lambda_1 y_t} + A_2 e^{\lambda_2 y_t} \tag{89}$$

setting $A_2 = 0$, when $y_t = y_b$, from (32), we have that $D(y_b) = Z(y_b)$. Therefore, if

$$Z(y_b) = \frac{c}{r} + A_1 e^{\lambda_1 y_b}$$

it is immediate to get:

$$A_1 = \left[Z(y_b) - \frac{c}{r} \right] e^{-\lambda_1 y_b}$$

which implies:

$$D(y_t) = \frac{c}{r} + \left[Z(y_b) - \frac{c}{r} \right] e^{\lambda_1 (y_t - y_b)} \tag{90}$$

which proves the result.

In the same fashion, we can solve (35). Following the same approach already considered, we find that the general solution of (35) is:

$$S(y_t) = \frac{y_t}{r} - \frac{(c + k + \mu)}{r} + A_1 e^{\lambda_1 y_t} + A_2 e^{\lambda_2 y_t} \tag{91}$$

Under bankruptcy the value of equity is zero, i.e. $S(y_b) = 0$. Next, setting $A_2 = 0$, from (91) with $y_t = y_b$, we can recover A_1, so that the general solution is:

$$S(y_t) = \frac{y_t}{r} - \frac{(c + k + \mu)}{r} - \left[\frac{y_b}{r} - \frac{(c + k + \mu)}{r} \right] e^{\lambda_1 (y_t - y_b)} \tag{92}$$

The value of y_b is next obtained from smooth pasting condition $S'(y_b) = 0$. □

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