Maz'ya-Shaposhnikova formula in magnetic fractional Orlicz–Sobolev spaces

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27 September 2024

MAZ'YA-SHAPOSHNIKOVA FORMULA IN MAGNETIC FRACTIONAL ORLICZ-SOBOLEV SPACES

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ABSTRACT. In this note we prove the validity of the Maz'ya-Shaposhnikova formula in magnetic fractional Orlicz-Sobolev spaces. This complements a previous asymptotic study of the limit as $s \uparrow 1$ performed by the second author in [20].

1. INTRODUCTION

The Hamiltonian formulation of classical mechanics leads to an expression for the energy H of a particle of mass m, given by

$$H(x,p) = \tfrac{p^2}{2m} + V(x), \quad (x,p) \in \mathbb{R}^n \times \mathbb{R}^n,$$

where p is the momentum, V is the potential energy and $T(p) := \tfrac{p^2}{2m}$ is the kinetic energy. Restricting our attention to the kinetic part of the energy H, if we are interested in the study of the motion of an electrically charged particle under the influence of a magnetic field B, the kinetic energy has to be modified as follows (see e.g. [31]):

$$T(x,p) := \tfrac{1}{2m}(p + \tfrac{e}{c}A(x))^2,$$

where e is the charge and c is the speed of light. The passage from classical mechanics to quantum mechanics requires the introduction of suitable operators. The quantization is performed by replacing the momentum p by the Euclidean gradient ∇, at least when we do not take care of the presence of an external magnetic field B, otherwise it is customary to replace the classical gradient ∇ by the so-called *magnetic gradient* $\nabla_A := \nabla - \mathrm{i}A$, where $A : \mathbb{R}^n \to \mathbb{R}^n$ is a vector potential and "i" denotes the imaginary unit. In particular, in the physical case of $\mathbb{R}^3$, if A is smooth enough, it holds that $\mathrm{curl}(A) = B$. The magnetic gradient can be used to define a second order differential operator usually called the *magnetic Laplacian*. See for instance [21, 25, 26, 28, 29, 46] and the references therein. From the mathematical point of view, this first order differential operator leads to the definition of the *magnetic Sobolev space* $H_A^1(\mathbb{R}^n; \mathbb{C})$ (see e.g. [30, Chapter 7, Section 19]), as the space of complex valued functions $u \in L^2(\mathbb{R}^n; \mathbb{C})$ such that $\nabla_A u \in (L^2(\mathbb{R}^n; \mathbb{C}))^n$. In order to have a proper definition it is necessary to ask for some regularity on the vector potential A, for example $L^2_{loc}(\mathbb{R}^n)$. With standard modifications one can define the magnetic Sobolev space $H_A^1(\Omega; \mathbb{C})$, where $\Omega \subset \mathbb{R}^n$ is an open set. The Hilbertian case (i.e. $p = 2$) is certainly the physically relevant one, but it is also possible to define magnetic Sobolev spaces $W_A^{1,p}(\mathbb{R}^n; \mathbb{C})$ (or $W_A^{1,p}(\Omega; \mathbb{C})$) for every $p \geq 1$, see e.g. [42].

In the last years, there has been a certain interest towards a characterization of

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magnetic Sobolev spaces (see [38, 39, 42, 44]), including the validity of a Maz'ya-Shaposhnikova formula in this setting, see [41]. In the classical diamagnetic case (i.e. when $A \equiv 0$), in [32] Maz'ya and Shaposhnikova showed that the fractional Sobolev seminorm recovers the L^p-norm. To be more precise, they proved that for every $u \in \bigcup_{s \in (0,1)} W_0^{s,p}(\mathbb{R}^n)$,

$$(1.1) \qquad \lim_{s \downarrow 0} s \iint_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy = 2 \frac{|S|^{n-1}}{p} \int_{\mathbb{R}^n} |u|^p \, dx,$$

where $|S|^{n-1}$ denotes the measure of the unit sphere. We point out that the function space $W_0^{s,p}(\mathbb{R}^n)$ stands for the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the fractional Gagliardo seminorm

$$[u]_{s,p} := \left(\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \right)^{1/p}.$$

We stress that (1.1) has been the object of other generalizations oriented to cover the case of ambient spaces different from ours, see e.g. [12, 35, 45]. We also recall that (1.1) can be considered as the natural counterpart of the limit as $s \uparrow 1$ studied by Bourgain, Brezis and Mironescu in [11].

The extension of (1.1) to the magnetic setting, requires the introduction of appropriate *magnetic fractional Sobolev spaces* $W_A^{s,p}(\mathbb{R}^n; \mathbb{C})$. In particular, when $p = 2$, it is possible to define the fractional Sobolev space $H_A^s(\Omega; \mathbb{C})$ for a given open set $\Omega \subset \mathbb{R}^n$ as the space of complex-valued functions $u \in L^2(\Omega; \mathbb{C})$ for which the following magnetic Gagliardo seminorm $[u]_{H_A^s(\Omega;\mathbb{C})}$ is finite:

$$[u]_{H_A^s(\Omega;\mathbb{C})} := \left(\iint_{\Omega \times \Omega} \frac{|u(x) - e^{\mathrm{i}(x-y) \cdot A\left(\frac{x+y}{2}\right)} u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \right)^{1/2}.$$

This space boils down to the classical fractional Sobolev space when $A \equiv 0$ (and u is real-valued) and has been proved to be the correct space where to study nonlocal problems driven by the so-called *magnetic fractional Laplacian* $(-\Delta)_A^s$. Without any attempt of completeness, we refer to [3, 4, 5, 6, 9, 15, 22, 23, 27, 36, 37]. The nonlocal operator $(-\Delta)_A^s$ via *midpoint prescription* has been introduced in [15] for any $s \in (0,1)$ and it admits the following representation when acting on smooth complex-valued functions $u \in C_0^\infty(\mathbb{R}^n; \mathbb{C})$

$$(1.2) \quad (-\Delta)_A^s u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \frac{u(x) - e^{\mathrm{i}(x-y) \cdot A\left(\frac{x+y}{2}\right)} u(y)}{|x - y|^{n+2s}} \, dy, \qquad x \in \mathbb{R}^n,$$

where $B(x, \varepsilon)$ denotes the ball of center x and radius ε. It is easy to notice that when $A \equiv 0$ and u is real-valued, $(-\Delta)_A^s u$ coincides with the usual definition of fractional Laplacian. See Remark 2.4 and Remark 2.5 for further comments on the magnetic fractional Laplacian and magnetic fractional spaces.

The aim of this note is to continue the study of properties of magnetic spaces, focusing on the extension of (1.1) to the case of magnetic fractional Orlicz-Sobolev spaces, a class of function spaces which has been recently introduced in [20], where the authors proved the validity of a Bourgain-Brezis-Mironescu-type formula, so extending to the magnetic setting their previous result in [18]. In order to properly state our result, let us first give a basic glance to the objects in play. We consider a smooth enough magnetic potential $A \colon \mathbb{R}^n \to \mathbb{R}^n$ and a Young function G satisfying the following standard growth condition:

$$(\mathrm{L}) \qquad\qquad 1 \leq p^- \leq \frac{t \, G'(t)}{G(t)} \leq p^+ < \infty \quad \text{for every } t > 0.$$

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We also consider the Young function $\overline{G} \colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ naturally associated to G, which is defined as

$$(1.3) \qquad \overline{G}(t) := \int_0^t G(\tau) \frac{d\tau}{\tau}, \quad t \in \mathbb{R}_0^+.$$

For any $s \in (0,1)$ it is now possible to introduce a proper notion of *magnetic fractional Orlicz Sobolev spaces* $W_A^{s,G}$, see section 2 for the precise definition.

Our main result reads as follows.

Theorem 1.1. *Let G be a Young function satisfying* (L) *and $A \colon \mathbb{R}^n \to \mathbb{R}^n$ be a magnetic field. If $u \in \bigcup_{s \in (0,1)} W_{A,0}^{s,G}(\mathbb{R}^n; \mathbb{C})$, then*
(1.4)

$$\lim_{s \downarrow 0} s \iint_{\mathbb{R}^n \times \mathbb{R}^n} G\left(\left| \frac{u(x) - e^{\mathrm{i}(x-y)A\left(\frac{x+y}{2}\right)} u(y)}{|x-y|^s} \right| \right) \frac{dx\, dy}{|x-y|^n} = \frac{2\, \omega_n}{n} \int_{\mathbb{R}^n} \overline{G}(|u(x)|)\, dx.$$

The proof of Theorem 1.1 is presented in section 4 and it is obtained combining careful estimates of both the liminf and the limsup. The technique adopted to achieve those estimates follows the lines of the original proof by Maz'ya and Shaposhnikova in [32]. We mention that the same scheme has been already successfully applied in providing extensions of (1.1) in different settings: we refer to [41] for the classical fractional magnetic setting, [1] for the case of fractional Orlicz-Sobolev spaces and [13] for the case of fractional Orlicz-Sobolev spaces in Carnot groups. We note that, coherently with [41], (1.4) shows that the limit as $s \downarrow 0$ kills the magnetic effect so recovering the same limit found in [1]. We also stress that assuming condition (L) on the Young function G is not restrictive at all, as showed in [1, Theorem 1.2]. It is noteworthy to mention that, in order to perform the estimates necessary to prove (1.4), we have to ensure the finiteness of certain integrals (see section 4 for the details). To this aim, in analogy with [1], we have to prove a magnetic Hardy-type inequality for Orlicz functions for *small* values of the fractional parameter s (we refer to Remark 3.4 for more details). This is the content of the following

Theorem 1.2. *Given a Young function G satisfying* (L) *and $s \in (0,1)$ such that $s < \frac{n}{p^+}$, there exists a constant $C = C(n, s, p^\pm)$ such that*

$$(1.5) \qquad \int_{\mathbb{R}^n} G\left(\frac{|u(x)|}{|x|^s} \right) dx \le C \iint_{\mathbb{R}^{2n}} G\left(\left| \frac{u(x) - e^{\mathrm{i}(x-y)A\left(\frac{x+y}{2}\right)} u(y)}{|x-y|^s} \right| \right) \frac{dx\, dy}{|x-y|^n}$$

for all $u \in W_{A,0}^{s,G}(\mathbb{R}^n; \mathbb{C})$.

This result recovers the fractional magnetic Hardy-inequality proved in [41] when $G(t) = t^p$ and [2, Theorem 5.1], if instead we consider the diamagnetic case, i.e. when $A \equiv 0$. The proof is an easy combination of the latter with a standard diamagnetic inequality, see section 3 for the details.

We want to stress that the interest towards fractional Orlicz-Sobolev spaces is recently moving to the study of nonlocal PDEs driven by the so-called G-Laplacian, which is a nonlocal operator naturally modeled on a given Young function G, see e.g. [7, 8, 16, 19, 43]. We refer also to [10, 33, 34] for some related problems concerning the local G-Laplacian.

The paper is organized as follows: in section 2 we provide the necessary definitions and basic results concerning magnetic fractional Orlicz-Sobolev spaces, in section 3 we prove Theorem 1.2 and finally in section 4 we prove Theorem 1.1.

2. Preliminaries

In this section we introduce definitions and notations and we recall few results needed in the next sections.

2.1. Young functions.
We recall now the basic notions concerning Young functions. We refer the interested reader to the books [17, 40] for a comprehensive introduction to the subject.

Definition 2.1. *Let $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a function satisfying the following conditions:*
- *(i) $\phi(0) = 0$, and $\phi(t) > 0$ for any $t > 0$;*
- *(ii) ϕ is non-decreasing;*
- *(iii) ϕ is right-continuous and $\lim_{t \to +\infty} \phi(t) = +\infty$.*

We call Young function the real valued function $G : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ given by

$$G(t) = \int_0^t \phi(s)\, ds.$$

We recall that properties (i) to (iii) imply that every Young function G is continuous, locally Lipschitz continuous, strictly increasing and convex on $\mathbb{R}_0^+$. It is also quite easy to see that $G(0) = 0$ and that G is superlinear both at zero and at $+\infty$. We can also assume without loss of generality that $G(1) = 1$. Finally, the inverse $G^{-1} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ of G is also well defined and inherits a bunch of properties: in particular, G^{-1} is continuous, concave, strictly increasing, $G^{-1}(0) = 0$ and $G^{-1}(1) = 1$.

As already stated in the introduction, we will work with Young functions satisfying (L). As a direct consequence, we notice that

(G_1) $$s^{\overline{p}} G(t) \leq G(st) \leq s^{\tilde{p}} G(t),$$

(G_2) $$G(s+t) \leq \frac{2^{p^+}}{2}(G(s) + G(t))$$

for any $s, t \in \mathbb{R}_0^+$, where $s^{\tilde{p}} := \max\{s^{p^-}, s^{p^+}\}$ and $s^{\overline{p}} := \min\{s^{p^-}, s^{p^+}\}$.

Young functions aim at generalizing the power functions, hence $G(t) = t^p$, for a certain $p \geq 1$, is definitely a trivial example of a Young function for which $p^- = p = p^+$. There are of course less trivial instances of Young functions e.g. logarithmic perturbation of powers: given positive constants $a, b, c > 0$ one considers a function G such that $G'(t) = t^a \log(b + ct)$. In this case we have that $p^- = 1 + a$ and $p^+ = 2 + a$.

We want to recall that condition (L) is equivalent to another condition usually assumed when dealing with Young functions, namely the Δ_2-condition.

Definition 2.2. *Let G be a Young function. We say that G satisfies the Δ_2-condition if there exists a positive constant $\mathbf{C} > 2$ such that*

$$G(2t) \leq \mathbf{C}\, G(t) \quad \text{for every } t \in \mathbb{R}_0^+.$$

We close this brief recap noticing that $\overline{G}$ is a bounded Young function. Keeping in mind its definition (1.3), the boundedness follows from the monotonicity of both $G(t)$ and $\frac{G(t)}{t}$. We also stress that the functions G and $\overline{G}$ are *equivalent* Young functions, in the sense that

(2.1) $$G\left(\frac{t}{2}\right) \leq \overline{G}(t) \leq G(t) \quad \text{for every } t \in \mathbb{R}_0^+.$$

2.2. **Magnetic spaces.** We are now ready to introduce the relevant functional spaces appearing in Theorem 1.1.

To simplify the readability, we introduce the following compact notation: let $s \in (0,1)$, we denote the magnetic Hölder quotient of order s as

$$(2.2) \qquad D_s^A u(x,y) := \frac{u(x) - e^{\mathrm{i}(x-y)A\left(\frac{x+y}{2}\right)} u(y)}{|x-y|^s}.$$

We will also denote

$$d\mu(x,y) := \frac{dx\,dy}{|x-y|^n}.$$

We notice that when $A \equiv 0$, $D_s^0 u = D_s u := \dfrac{u(x) - u(y)}{|x-y|^s}$ is the usual s-Hölder quotient appearing in the definition of fractional Sobolev spaces.

Definition 2.3. *Let G be an Orlicz function, $s \in (0,1)$ be a fractional parameter and $A : \mathbb{R}^n \to \mathbb{R}^n$ a smooth enough vector potential. We define the spaces $L^G(\mathbb{R}^n; \mathbb{C})$ and $W_A^{s,G}(\mathbb{R}^n; \mathbb{C})$ as follows:*

$$L^G(\mathbb{R}^n; \mathbb{C}) := \left\{ u\colon \mathbb{R}^n \to \mathbb{C} \text{ measurable} \colon I_G(u) < \infty \right\},$$

$$W_A^{s,G}(\mathbb{R}^n; \mathbb{C}) := \left\{ u \in L^G(\mathbb{R}^n; \mathbb{C}) \colon I_{s,G}^A(u) < \infty \right\},$$

where I_G and $I_{s,G}^A$ are defined as

$$I_G(u) := \int_{\mathbb{R}^n} G(|u(x)|)\,dx$$

and

$$(2.3) \qquad I_{s,G}^A(u) := \iint_{\mathbb{R}^{2n}} G\left(|D_s^A u(x,y)| \right)\,d\mu.$$

These spaces become Banach spaces when endowed with the so-called Luxemburg norms defined through $I_{s,G}^A$, namely

$$\|u\|_{s,G}^A = \|u\|_G + |u|_{s,G}^A,$$

where

$$\|u\|_G := \inf\{\lambda > 0 \colon I_G(\tfrac{u}{\lambda}) \le 1\}$$

is the usual (Luxemburg) norm on $L^G(\mathbb{R}^n; \mathbb{C})$ and

$$|u|_{s,G}^A := \inf\{\lambda > 0 \colon I_{s,G}^A(\tfrac{u}{\lambda}) \le 1\}.$$

Finally, we define the fractional Magnetic Orlicz-Sobolev space $W_{A,0}^{s,G}(\mathbb{R}^n; \mathbb{C})$ as the closure of $C_0^\infty(\mathbb{R}^n; \mathbb{C})$ with respect to the magnetic fractional Orlicz seminorm $|u|_{s,G}^A$. We note that when $G(t) = t^p$ we recover the magnetic fractional Sobolev spaces defined in [41, 42, 44]. At the same time, if we assume $A \equiv 0$ the above definitions lead to the fractional Orlicz-Sobolev spaces considered in [1, 2]. Combining the last observations, it is also obvious that for $G(t) = t^p$ and $A \equiv 0$ we recover the classical fractional Sobolev spaces.

Remark 2.4. *In the definition of the Hölder quotient of order s, and hence in the definition of the fractional Orlicz-Sobolev spaces, we actually chose the so-called midpoint prescription*

$$(x,y) \mapsto A\left(\frac{x+y}{2}\right),$$

which is closely related to the magnetic fractional Laplacian defined in (1.2). It is noteworthy to mention that actually, for $s = \frac{1}{2}$, the definition of the fractional

operator $(-\Delta)^s_A$ dates back to the '80s, and it is closely related to the proper definition of a quantized operator corresponding to the symbol of the classical relativistic Hamiltonian, namely

$$\sqrt{(\xi - A(x))^2 + m^2} + V(x), \quad (\xi, x) \in \mathbb{R}^n \times \mathbb{R}^n.$$

In particular, it is related to the kinetic part of the above symbol. In the survey [24] it is explained that there are at least three definitions for such a quantized operator appearing in the literature: two of them are given in terms of pseudo-differential operators, while the third one as the square root of a suitable non-negative operator. In [24], Ichinose showed that these three non–local operators are in general not the same, but they do coincide whenever dealing with a linear vector potential A. A well studied example of a linear potential A is the so-called Ahronov-Bohm potential. We finally notice that, in the physically relevant case of $\mathbb{R}^3$, the linearity of A is actually equivalent to require a constant magnetic field.

Remark 2.5. As mentioned before, one may then replace midpoint prescription with other prescriptions, e.g. the averaged one

$$(x, y) \mapsto \int_0^1 A\left((1-\vartheta)x + \vartheta y\right) d\vartheta =: A_\sharp(x, y).$$

From the physical point of view, the latter has the advantage that the magnetic fractional Laplacian associated to it, i.e. $(-\Delta)^s_{A_\sharp}$, turns out to be Gauge covariant (see e.g. [24, Proposition 2.8]), namely

$$(-\Delta)^s_{(A+\nabla\phi)_\sharp} = e^{i\phi}(-\Delta)^s_{A_\sharp}e^{-i\phi}.$$

It is clear that one can define fractional Orlicz-Sobolev spaces according to the averaged prescription. All the results presented in this note remain true even for these spaces, once the obvious modifications are made.

3. A FRACTIONAL MAGNETIC HARDY-TYPE INEQUALITY

In this section we prove Theorem 1.2. The proof comes out as a combination of the Hardy-type inequality proved in [2] with the fractional diamagnetic inequality, which in turn heavily relies upon the so-called *diamagnetic inequality*. The latter is well-known in the classical setting, see e.g. [30, Theorem 7.21], and it reads as follows:

Proposition 3.1. Let $A: \Omega \to \mathbb{R}^n$ be a measurable magnetic potential such that $|A| < \infty$ a.e. in Ω and let $u \in W^{1,1}_{loc}(\mathbb{R}^n; \mathbb{C})$. Then

$$(3.1) \qquad |\nabla|u|(x)| \le |\nabla u(x) - iA(x)u(x)|,$$

for a.e. $x \in \Omega$.

The fractional analogue of (3.1) was provided in [15, Lemma 3.1 and Remark 3.2]:

Proposition 3.2. Let $A: \mathbb{R}^n \to \mathbb{R}^n$ be a measurable magnetic potential such that $|A| < \infty$ a.e. in $\mathbb{R}^n$ and let $u: \mathbb{R}^n \to \mathbb{C}$ be a measurable function such that $|u| < \infty$ a.e. in $\mathbb{R}^n$. Then

$$(3.2) \qquad ||u(x)| - |u(y)|| \le \left| u(x) - e^{i(x-y)A\left(\frac{x+y}{2}\right)} u(y) \right|,$$

for a.e. $x, y \in \mathbb{R}^n$.

Remark 3.3. *Observe that, using the compact notation introduced before, the fractional diamagnetic inequality (3.2) can be re-stated as*

$$(3.3) \qquad\qquad |D_s|u|(x,y)| \leq |D_s^A u(x,y)|,$$

a.e. $x, y \in \mathbb{R}^n$, *where* $D_s v(x,y) = \frac{v(x)-v(y)}{|x-y|^s}$.

With this at hand, we are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Given a Young function G, by [2, Theorem 5.1] there exist a Young function $\hat{G}$ and a positive constant $C = C(n,s) > 0$ such that

$$(3.4) \qquad \int_{\mathbb{R}^n} \hat{G}\left(\frac{|u(x)|}{|x|^s}\right) dx \leq (1-s) \iint_{\mathbb{R}^{2n}} G(C|D_s u(x,y)|)\, d\mu.$$

Moreover, in light of [14, Propositions 5.1 and 5.2], since $s < \frac{n}{p^+}$, $\hat{G}$ and G are equivalent Young functions, i.e., there exist two positive constants $c_1 < c_2$ such that $G(c_1 t) \leq \hat{G}(t) \leq G(c_2 t)$ for all $t \in \mathbb{R}_0^+$. Therefore, by using inequality (3.4) on $|u|$ and the Diamagnetic inequality (3.2), it holds that

$$\int_{\mathbb{R}^n} G\left(c_1 \frac{|u(x)|}{|x|^s}\right) dx \leq (1-s) \iint_{\mathbb{R}^{2n}} G\left(C|D_s|u|(x,y)|\right) d\mu$$
$$\leq (1-s) \iint_{\mathbb{R}^{2n}} G\left(C|D_s^A u(x,y)|\right) d\mu.$$

Thus, the result follows by (G_1). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.4. *Let us notice that, accordingly to the Hardy-type inequality proved by Maz'ya-Shaposhnikova, see [32, Theorem 2], and to the analogous result holding in the case of Orlicz spaces, see [2], the inequality (1.5) holds just for small values of the fractional parameter s, meaning that $s < \frac{n}{p^+}$. Here, the term p^+, which is defined in (L), is bigger or equal to the upper Matuszewska-Orlicz index $I(G)$, taken into account in [1]. We stress that, if one is interested in proving a Hardy-type inequality for all $s \in (0,1)$ one has to reinforce the assumptions on the Orlicz function, see e.g. [2, Section 5]. On the other hand, as explained in [1], the validity of condition (L), ensures that for small enough s such assumptions are satisfied by all Orlicz functions. Finally, we notice that, as a consequence of Theorem 1.2 and (2.1), it holds that*

$$(3.5) \qquad \int_{\mathbb{R}^n} \overline{G}\left(\alpha \frac{|u(x)|}{|x|^s}\right) dx \leq \int_{\mathbb{R}^n} G\left(\alpha \frac{|u(x)|}{|x|^s}\right) dx < \infty$$

for any $u \in W_{A,0}^{s,G}(\mathbb{R}^n; \mathbb{C})$ *and for any* $\alpha > 0$.

4. Proof of Theorem 1.1

In this section we prove Theorem 1.1 combining two estimates for the liminf and the limsup respectively.

Lemma 4.1 (Liminf estimate). *Let* $u \in \cup_{s \in (0,1)} W_{A,0}^{s,G}(\mathbb{R}^n; \mathbb{C})$. *Then*

$$\liminf_{s \downarrow 0} s I_{s,G}^A(u) \geq \frac{2\omega_n}{n} I_{\overline{G}}(|u|).$$

Proof. If $\liminf_{s \downarrow 0} s I_{s,G}^A(u) = +\infty$ the result follows. Otherwise, there exists a sequence $\{s_k\}_{k \in \mathbb{N}} \subset (0,1)$ such that $s_k \downarrow 0$ and

$$\liminf_{s \downarrow 0} s I_{s,G}^A(u) = \lim_{k \to \infty} s_k I_{s_k,G}^A(u).$$

By using Remark 3.3 and the monotonicity of G we get

$$s_k \iint_{\mathbb{R}^{2n}} G(|D_{s_k}|u||) \, d\mu \leq s_k \iint_{\mathbb{R}^{2n}} G(|D_{s_k}^A u|) \, d\mu$$

that is, for any $k \in \mathbb{N}$,

$$s_k \, I_{s_k, G}(|u|) \leq s_k \, I_{s_k, G}^A(u).$$

Thus, taking the limit as $k \to \infty$ in the last inequality and applying [1, Theorem 1.1] to $|u|$, we get

$$\frac{2\omega_n}{n} I_{\overline{G}}(|u|) \leq \lim_{s_k \to \infty} s_k \, I_{s_k, G}^A(u),$$

which concludes the proof. $\qquad\square$

We can now move to the upper estimate for the limsup.

Lemma 4.2 (Limsup estimate). *Let $u \in \cup_{s \in (0,1)} W_{A,0}^{s,G}(\mathbb{R}^n; \mathbb{C})$. Then*

$$\limsup_{s \downarrow 0} s \, I_{s,G}^A(u) \leq \frac{2\omega_n}{n} I_{\overline{G}}(|u|).$$

Proof. First observe that by Fubini's Theorem and a change of variables

$$\int_{\mathbb{R}^n} \left(\int_{\{|y|<|x|\}} G\left(\left| \frac{u(x) - e^{i(x-y)A\left(\frac{x+y}{2}\right)} u(y)}{|x-y|^s} \right| \right) \frac{dy}{|x-y|^n} \right) dx$$

$$= \int_{\mathbb{R}^n} \left(\int_{\{|x|>|y|\}} G\left(\left| \frac{u(x) - e^{i(x-y)A\left(\frac{x+y}{2}\right)} u(y)}{|x-y|^s} \right| \right) \frac{dx}{|x-y|^n} \right) dy$$

$$= \int_{\mathbb{R}^n} \left(\int_{\{|y|>|x|\}} G\left(\left| \frac{u(y) - e^{-i(x-y)A\left(\frac{x+y}{2}\right)} u(x)}{|x-y|^s} \right| \right) \frac{dy}{|x-y|^n} \right) dx.$$

Since

$$\left| \left(\frac{u(y) - e^{-i(x-y)A\left(\frac{x+y}{2}\right)} u(x)}{|x-y|^s} \right) \frac{e^{i(x-y)A\left(\frac{x+y}{2}\right)}}{e^{i(x-y)A\left(\frac{x+y}{2}\right)}} \right| = \left| \frac{u(x) - e^{i(x-y)A\left(\frac{x+y}{2}\right)} u(y)}{|x-y|^s} \right|,$$

then

$$s \, I_{s,G}^A(u) = s \int_{\mathbb{R}^n} \left(\int_{\{|y| \geq |x|\}} G(|D_s^A u(x,y)|) \frac{dy}{|x-y|^n} \right) dx$$

$$+ s \int_{\mathbb{R}^n} \left(\int_{\{|y| < |x|\}} G(|D_s^A u(x,y)|) \frac{dy}{|x-y|^n} \right) dx$$

$$= 2s \int_{\mathbb{R}^n} \left(\int_{\{|y| \geq |x|\}} G(|D_s^A u(x,y)|) \frac{dy}{|x-y|^n} \right) dx.$$

Let us now fix $\varepsilon > 0$. By the monotonicity and convexity of G, we can split the previous integral as

$$s \, I_{s,G}^A(u) = 2s \int_{\mathbb{R}^n} \left(\int_{\{|y| \geq 2|x|\}} G(|D_s^A u(x,y)|) \frac{dy}{|x-y|^n} \right) dx$$

(4.1)

$$+ 2s \int_{\mathbb{R}^n} \left(\int_{\{|x| \leq |y| < 2|x|\}} G(|D_s^A u(x,y)|) \frac{dy}{|x-y|^n} \right) dx$$

$$\leq \frac{2s}{1+\varepsilon} J_1 + \frac{2s\varepsilon}{1+\varepsilon} J_2 + 2s J_3$$

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where

$$J_1 := \int_{\mathbb{R}^n} \left(\int_{\{|y| \geq 2|x|\}} G\left((1+\varepsilon) \frac{|u(x)|}{|x-y|^s} \right) \frac{dy}{|x-y|^n} \right) dx$$

$$J_2 := \int_{\mathbb{R}^n} \left(\int_{\{|y| \geq 2|x|\}} G\left(\frac{1+\varepsilon}{\varepsilon} \frac{|u(y)|}{|x-y|^s} \right) \frac{dy}{|x-y|^n} \right) dx$$

$$J_3 := \int_{\mathbb{R}^n} \left(\int_{\{|x| \leq |y| < 2|x|\}} G(|D_s^A u(x,y)|) \frac{dy}{|x-y|^n} \right) dx.$$

Taking into account [1, (2.22) and (2.23)], it holds that

$$(4.2) \qquad J_1 \leq \frac{w_n}{ns} \int_{\mathbb{R}^n} \overline{G}\left((1+\varepsilon) \frac{|u(x)|}{|x|^s} \right) dx$$

and

$$(4.3) \qquad J_2 \leq w_n \int_{\mathbb{R}^n} G\left(\frac{1+\varepsilon}{\varepsilon} 2^s \frac{|u(y)|}{|y|^s} \right) dy.$$

To conclude the proof, we only need to provide a suitable upper estimate from above for J_3. We claim that, fixed $N > 3$, there exists $\overline{s} \in (0,1)$ such that

$$(4.4) \qquad J_3 \leq \iint_{\mathbb{R}^n \times E} G\left(N^{\overline{s}-s} |D_{\overline{s}}^A u(x,y)| \right) d\mu + \frac{\varepsilon}{s}$$

for any $s \in (0, \overline{s})$, where $E := \{|x| \leq |y| < 2|x|, |x-y| \leq N\}$.

In order to prove (4.4), we first notice that J_3 can be written as

$$J_3 = (i) + (ii),$$

where, denoting $F := \{|x| \leq |y| < 2|x|, |x-y| > N\}$, we write

$$(i) := \iint_{\mathbb{R}^n \times E} G(|D_s^A u(x,y)|) \, d\mu, \qquad (ii) := \iint_{\mathbb{R}^n \times F} G(|D_s^A u(x,y)|) \, d\mu.$$

Since $u \in \bigcup_{s \in (0,1)} W_{A,0}^{s,G}(\mathbb{R}^n; \mathbb{C})$, there exists $\overline{s} \in (0,1)$ such that $u \in W_{A,0}^{\overline{s},G}(\mathbb{R}^n; \mathbb{C})$. Let now $s < \overline{s}$. Then

(4.5)
$$(i) = \iint_{\mathbb{R}^n \times E} G\left(|x-y|^{\overline{s}-s} |D_{\overline{s}}^A u(x,y)| \right) d\mu \leq \iint_{\mathbb{R}^n \times E} G\left(N^{\overline{s}-s} |D_{\overline{s}}^A u(x,y)| \right) d\mu.$$

Moreover, since $|x| \leq |y| < 2|x|$ and $|x-y| > N$ imply that

$$(4.6) \qquad |x| > \frac{N}{3} \text{ and } |y| > \frac{N}{3},$$

then, by (4.6), the monotonicity and the convexity of G, by a change of variable and taking into account that $\left| e^{i(x-y)A\left(\frac{x+y}{2} \right)} \right| = 1$, we get

$$(ii) \leq \frac{1}{2} \iint_{\mathbb{R}^n \times F} G\left(\frac{2|u(x)|}{|x-y|^s} \right) d\mu + \frac{1}{2} \iint_{\mathbb{R}^n \times F} G\left(\frac{2|u(y)|}{|x-y|^s} \right) d\mu$$

$$\leq \frac{1}{2} \int_{\{|x| > \frac{N}{3}\}} \left(\int_{\{|x-y| > N\}} G\left(\frac{2|u(x)|}{|x-y|^s} \right) \frac{dy}{|x-y|^n} \right) dx$$

$$+ \frac{1}{2} \int_{\{|y| > \frac{N}{3}\}} \left(\int_{\{|x-y| > N\}} G\left(\frac{2|u(y)|}{|x-y|^s} \right) \frac{dx}{|x-y|^n} \right) dy$$

$$= \int_{\{|x| > \frac{N}{3}\}} \left(\int_{\{|x-y| > N\}} G\left(\frac{2|u(x)|}{|x-y|^s} \right) \frac{dy}{|x-y|^n} \right) dx.$$

Finally, the last inequality and a change of variables gives that

$$(ii) \leq \frac{w_n}{n} \int_{\{|x| > \frac{N}{3}\}} \left(\int_N^{+\infty} G\left(\frac{2|u(x)|}{r^s} \right) \frac{dr}{r} \right) dx$$

$$= \frac{w_n}{ns} \int_{\{|x| > \frac{N}{3}\}} \left(\int_0^{\frac{2|u(x)|}{N^s}} G(\tau) \frac{d\tau}{\tau} \right) dx$$

$$= \frac{w_n}{ns} \int_{\{|x| > \frac{N}{3}\}} \overline{G}\left(\frac{2|u(x)|}{N^s} \right) dx$$

$$< \frac{w_n}{ns} \int_{\{|x| > \frac{N}{3}\}} \overline{G}(2|u(x)|) \, dx < \frac{\varepsilon}{s}$$

for N sufficiently large. From this, (4.4) easily follows.

In order to justify the passage to the limsup, we can argue as in [1] once again. In this way, by the arbitrariness of ε, by gathering (4.1), (4.2), (4.3) and (4.4), by (3.5) and by using Fatou's Lemma, we close the proof. $\qquad \square$

Proof of Theorem 1.1. Combining the lower bounds obtained in Lemma 4.1 with the upper bounds provided by Lemma 4.2, we easily get (1.4). $\qquad \square$

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