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Discrete- and continuous-time random walks in 1D Lévy random medium

MARCO LENCI*

In memory of Francesca Romana Nardi
January 2022

Abstract

A Lévy random medium, in a given space, is a random point process where the distances between points, a.k.a. *targets*, are long-tailed. Random walks visiting the targets of a Lévy random medium have been used to model many (physical, ecological, social) phenomena that exhibit superdiffusion as the result of interactions between an agent and a sparse, complex environment. In this note we consider the simplest non-trivial Lévy random medium, a sequence of points in the real line with i.i.d. long-tailed distances between consecutive targets. A popular example of a continuous-time random walk in this medium is the so-called *Lévy-Lorentz gas*. We give an account of a number of recent theorems on generalizations and variations of such model, in discrete and continuous time.

Mathematics Subject Classification (2020): 60G50, 60G55, 60F17, 82C41, 60G51

Keywords: Lévy walk, Lévy flight, Lévy random medium, random walk on point process, stable processes, anomalous diffusion.

1 Introduction

In this note we give an overview of recent rigorous results on random walks (RWs) in random medium on the real line. The random medium is given by a point process $\omega = (\omega_k, k \in \mathbb{Z}) \subset \mathbb{R}$, where $\omega_0 = 0$ and the distances $\zeta_k := \omega_k - \omega_{k-1}$ between consecutive points are positive i.i.d. random variables with a long tail. By this

*Dipartimento di Matematica, Università di Bologna, Piazza di Porta San Donato 5, 40126 Bologna, Italy; and Istituto Nazionale di Fisica Nucleare, Sezione di Bologna, Via Irnerio 46, 40126 Bologna, Italy; E-mail: marco.lenci@unibo.it.

we mean that the variance of ζ_k is infinite. The points ω_k will be henceforth called *targets*. For reason that will be better clarified below we refer to ω as a *Lévy random medium*.

We consider two types of RWs on \mathbb{R} related to ω . To define them we introduce the auxiliary process $S = (S_n, n \in \mathbb{N})$, a \mathbb{Z} -valued RW with $S_0 = 0$ and independent increments. We postulate that S is independent of ω and call it the *underlying random walk*. The first process of interest is $Y \equiv Y^\omega := (Y_n, n \in \mathbb{N})$, where $Y_n := \omega_{S_n}$. This is the discrete-time RW (DTRW) that “jumps” on the targets of ω as determined by S . For example, if S produces the realization $(0, 2, -3, \dots)$, the walker Y starts at the origin, then jumps to the second target to the right of 0, then to the third target to the left of 0, etc. The second process of interest is $X \equiv X^\omega := (X(t), t \geq 0)$, the continuous-time RW (CTRW) defined as the unit-speed interpolation of Y . This means that the walker X visits all the points Y_n , ordered by n , but “walking” with unit speed rather than jumping. For instance, for S as in the above example, X starts at the origin and moves with velocity 1 until it reaches $Y_1 = \omega_2$, then it instantaneously turns its velocity to -1 and moves until it reaches $Y_2 = \omega_{-3}$, and so on.

In the case where the underlying RW S is simple and symmetric, the process X is generally referred to as the *Lévy-Lorentz gas*, after Barkai, Fleurov and Klafter introduced it in the physical literature in 2000 [1]. The Lévy-Lorentz gas has been used since as a simple model for a number of phenomena exhibiting superdiffusion, i.e., diffusion at a faster speed than square root of time. They include transport in porous media, disordered optical media (such as *Lévy glasses* [2]), nanowires, etc.; see [1, 3, 4, 5] and references therein.

In the physical literature, DTRWs, respectively CTRWs, whose distributions of jumps, respectively inertial stretches, are long-tailed, are often called *Lévy flights*, respectively *Lévy walks*. Lévy flights and walks, in regular or random media, have been employed as models for anomalous diffusion in a wide range of situations, from the physical to the biological and social sciences [6, 7]. A rigorous mathematical treatment of these systems has only been given for the simplest of them, mostly on regular media. In real-world applications, however, the anomalous behavior of a certain diffusing quantity is seldom due to a special law governing the diffusing agent *per se*, but rather to the interaction between the agent and an irregular medium (e.g., a photon in a Lévy glass, a signal in a small-world network, an animal foraging where food is scarce, etc.). Hence the interest in *Lévy media*, namely, media that induce superdiffusive behavior, such as the random point process ω defined earlier.

A fair amount of mathematical work on the processes X and Y has been done in recent years by different authors [8, 9, 10, 11, 12]. In particular, in a number of cases, limit theorems have been proved for suitable rescalings of either process. In some instances, the convergence of moments has been proved as well. The purpose of this note is to present these results in a concise, unified manner. For reasons of space and self-consistency, we will neglect interesting work by Artuso and collaborators on yet another type of RW related to the Lévy-Lorentz gas, a persistent RW in an

averaged medium [13, 14].

In Section 2 we present results on the DTRW Y and in Section 3 on the CTRW X . Understandably, the results depend on the assumptions on the random medium ω and the underlying RW S . Major differences occur depending on whether ζ_k , the distance between two consecutive targets, has infinite variance but finite mean, or infinite mean (and thus infinite variance), so we consider these cases in different subsections. No proofs are given, but references are placed throughout.

1.1 General notation

Throughout the paper we denote by \mathbb{P} the probability law that governs the whole system, both the random medium ω and the underlying RW S (that latter playing the role of the random dynamics, as it “drives” both X and Y , in a given ω). \mathbb{P} is called the *annealed law* and we denote by \mathbb{E} its expectation. For a fixed ω , the conditional probability $P_\omega := \mathbb{P}(\cdot|\omega)$ is called the *quenched law* relative to the realization ω of the medium. We denote by E_ω its expectation. A limit theorem, such as the CLT or the Invariance Principle, etc., relative to \mathbb{P} is referred to as an *annealed limit theorem*. One speaks instead of a *quenched limit theorem* if the result is proved w.r.t. P_ω , for \mathbb{P} -a.e. ω .¹ In what follows, for the most part, we present annealed and quenched results in different subsections.

We indicate with $\xi_n := S_n - S_{n-1}$ ($n \in \mathbb{Z}^+$) the i.i.d. increments of the underlying RW, whose drift is denoted $\nu := \mathbb{E}[\xi_n]$, if it exists in $\mathbb{R} \cup \{\pm\infty\}$. The mean distance between the targets of ω is denoted $\mu := \mathbb{E}[\zeta_k]$. Since $\zeta_k > 0$, μ always exists in $\mathbb{R} \cup \{+\infty\}$. Let us recall that all ξ_n ($n \in \mathbb{Z}^+$) and ζ_k ($k \in \mathbb{Z}$) are independent.

2 Discrete-time random walk

In this section we consider the asymptotic behavior of the DTRW $Y \equiv Y^\omega$, under a number of different assumptions on the distributions of ζ_1 and ξ_1 .

2.1 Finite mean distance between targets, quenched theorems

We start with the results of [8] on the quenched version of Y , which only require a very simple condition on the medium, $\mu = \mathbb{E}[\zeta_1] < \infty$, that is, the mean distance

¹Since we have neither introduced the measurable space (Ω, \mathcal{A}) where \mathbb{P} is defined, nor declared that ω are elements of Ω , mathematical formality requires that we define the phrase “ \mathbb{P} -almost every ω ”. The counterimages (equivalently, level sets) of the process ω form a partition of Ω . We assume this partition to be *measurable* in the sense of Rohlin [15]. Now, a property is said to hold for \mathbb{P} -a.e. ω if the values of ω which do not satisfy the property correspond to elements of the partition whose union has zero \mathbb{P} -measure. Incidentally, the existence of such a measurable partition is what guarantees that $\mathbb{P}(\cdot|\omega)$ is well-defined (for \mathbb{P} -a.e. ω).

between neighboring targets is finite. The assumptions on the underlying RW are instead as follows:

- the increment ξ_1 of S is symmetric, i.e., $\mathbb{P}(\xi_1 = j) = \mathbb{P}(\xi_1 = -j)$, for all $j \in \mathbb{N}$;
- its distribution is unimodal, i.e., $j \mapsto \mathbb{P}(\xi_1 = j)$ is non-increasing for $j \in \mathbb{N}$;
- it has finite variance: $V_\xi := \mathbb{E}[\xi_1^2] < \infty$.

The authors prove a quenched CLT for Y [8, Thm. 1]:

Theorem 2.1 *Assume the above conditions, most notably $\mu < \infty$. Then, as $n \rightarrow \infty$,*

$$\frac{Y_n}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, \mu^2 V_\xi),$$

w.r.t. P_ω , for \mathbb{P} -a.e. ω . Here $\mathcal{N}(0, \mu^2 V_\xi)$ is a Gaussian variable with mean 0 and variance $\mu^2 V_\xi$.

Obviously, a quenched distributional limit theorem with the same limit for a.e. quenched law implies the annealed version of the same theorem:

Corollary 2.2 *The limit in the statement of Theorem 2.1 holds w.r.t. \mathbb{P} as well.*

Convergence is known for the quenched moments of Y_n/\sqrt{n} as well, at least of lower order. Let $p := \sup\{q \geq 0 \mid \mathbb{E}[|\xi_1|^q] < \infty\}$. By the assumption on V_ξ , $p \geq 2$. For all $q \in \mathbb{R}^+$, denote by

$$\overline{m}_q := \sqrt{\frac{2^q}{\pi}} \Gamma\left(\frac{q+1}{2}\right) \quad (2.1)$$

the q -th absolute moment of the standard Gaussian $\mathcal{N}(0, 1)$ (here Γ is the usual Gamma function). It is not hard to show that, at least for all $q < p$,

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}[S_n^q]}{n^{q/2}} = V_\xi^{q/2} \overline{m}_q. \quad (2.2)$$

The following is a reformulation of Theorem 2 of [8].

Theorem 2.3 *Under the above assumptions and notation, fix $q \in (0, p)$. For a.a. ω ,*

$$\lim_{n \rightarrow \infty} \frac{E_\omega[Y_n^q]}{n^{q/2}} = \mu^q V_\xi^{q/2} \overline{m}_q.$$

Observe that $\mu^q V_\xi^{q/2} \overline{m}_q$ is the q -th absolute moment of $\mathcal{N}(0, \mu^2 V_\xi)$, so Theorem 2.3 is consistent with Theorem 2.1.

2.2 Finite mean distance between targets, annealed theorems

In the next two subsections we report the functional limit theorems of [11]. We refer the reader to [16] for background material on stable laws, Lévy processes, Skorokhod topologies, etc. All distributional convergences in these subsections are meant w.r.t. \mathbb{P} , that is, we are considering annealed functional limit theorems.

We assume that ζ_1 is in the normal basin of attraction of a α -stable distribution, with $\alpha \in (1, 2]$. This means that $\mu = \mathbb{E}[\zeta_1] < \infty$ and

$$\frac{1}{n^{1/\alpha}} \sum_{i=1}^n (\zeta_i - \mu) \xrightarrow{d} \tilde{Z}_1^{(\alpha)}, \quad (2.3)$$

as $n \rightarrow \infty$, for some α -stable variable $\tilde{Z}_1^{(\alpha)}$ (whose skewness index² must then be 0). As for the underlying RW, we assume ξ_1 is in the normal basin of attraction of an β -stable distribution, with $\beta \in (0, 1) \cup (1, 2]$. We must distinguish two cases, depending on whether $\nu = \mathbb{E}[\xi_1]$ exists and differs from 0, or otherwise.

- If $\beta \in (0, 1)$, or $\beta \in (1, 2]$ and $\nu = 0$, we assume that there exists a β -stable variable $W_1^{(\beta)}$ such that, as $n \rightarrow \infty$,

$$\frac{1}{n^{1/\beta}} \sum_{i=1}^n \xi_i \xrightarrow{d} W_1^{(\beta)}. \quad (2.4)$$

- If $\beta \in (1, 2]$ and $\nu \neq 0$, we assume that there exists a β -stable variable $\widetilde{W}_1^{(\beta)}$ such that

$$\frac{1}{n^{1/\beta}} \sum_{i=1}^n (\xi_i - \nu) \xrightarrow{d} \widetilde{W}_1^{(\beta)}. \quad (2.5)$$

To state the results of this section, we need two spaces of functions with jump discontinuities. In what follows, we denote by \mathcal{D}^+ the space of càdlàg³ functions $\mathbb{R}^+ \rightarrow \mathbb{R}$ and by \mathcal{D} the space of functions $\mathbb{R} \rightarrow \mathbb{R}$ whose restriction to $[0, +\infty)$, respectively $(-\infty, 0]$, is càdlàg, respectively càglàd.

Let $(\tilde{Z}_\pm^{(\alpha)}(s), s \geq 0)$ be two i.i.d. càdlàg α -stable Lévy processes such that $\tilde{Z}_\pm^{(\alpha)}(0) = 0$ and $\tilde{Z}_\pm^{(\alpha)}(1)$ is distributed like $\tilde{Z}_1^{(\alpha)}$, introduced in (2.3) (these conditions uniquely determine the distribution of the processes), and set

$$\tilde{Z}^{(\alpha)}(s) := \begin{cases} Z_+^{(\alpha)}(s), & s \geq 0; \\ -Z_-^{(\alpha)}(-s), & s < 0. \end{cases} \quad (2.6)$$

²This is the parameter that, in virtually all textbooks on stable variables (such as [16]) is denoted $\beta \in [-1, 1]$. In this paper β is used for the stability index of ξ_1 .

³I.e., right-continuous with left limits existing everywhere. Càglàd means left-continuous with right limits everywhere.

By construction, every realization $\tilde{Z}^{(\alpha)}$ belongs to \mathcal{D} , and so do the realizations

$$\bar{\omega}^{(n)}(s) := \frac{1}{n} \begin{cases} \omega_{\lfloor ns \rfloor}, & s \geq 0; \\ \omega_{\lceil ns \rceil}, & s < 0, \end{cases} \quad (2.7)$$

$$\tilde{\omega}^{(n)}(s) := \frac{1}{n^{1/\alpha}} \begin{cases} \sum_{i=1}^{\lfloor ns \rfloor} (\zeta_i - \mu), & s \geq 0; \\ -\sum_{i=\lceil (n-1)s \rceil}^0 (\zeta_i - \mu), & s < 0, \end{cases} \quad (2.8)$$

defining the processes $(\bar{\omega}^{(n)}(s), s \in \mathbb{R})$ and $(\tilde{\omega}^{(n)}(s), s \in \mathbb{R})$. The single-variable convergence (2.3) entails functional convergence of these processes: as $n \rightarrow \infty$, $\bar{\omega}^{(n)} \xrightarrow{\text{a.s.}} \mu \text{id}$ and $\tilde{\omega}^{(n)} \xrightarrow{\text{d}} \tilde{Z}^{(\alpha)}$, relative to the Skorokhod topology J_1 . From now on, we will write ‘in (\mathcal{D}, J_1) ’ for short.

We now introduce continuous-argument processes for the dynamics.

- In the case $\beta \in (0, 1)$, or $\beta \in (1, 2]$ and $\nu = 0$, we denote by $(W^{(\beta)}(t), t \geq 0)$ a càdlàg β -stable Lévy process whose distribution is uniquely determined by the conditions that $W^{(\beta)}(0) = 0$ and $W^{(\beta)}(1)$ be distributed like $W_1^{(\beta)}$; cf. (2.4). Also define $(\hat{S}^{(n)}(t), t \geq 0)$ via

$$\hat{S}^{(n)}(t) := \frac{S_{\lfloor nt \rfloor}}{n^{1/\beta}}. \quad (2.9)$$

It follows from (2.4) that $\hat{S}^{(n)} \xrightarrow{\text{d}} W^{(\beta)}$, in (\mathcal{D}^+, J_1) , as $n \rightarrow \infty$.

- In the case $\beta \in (1, 2]$ and $\nu \neq 0$, we consider $\widetilde{W}^{(\beta)}$, defined exactly as $W^{(\beta)}$ above but with $\widetilde{W}_1^{(\beta)}$ in place of $W_1^{(\beta)}$; cf. (2.5). In lieu of (2.9) we define two processes:

$$\bar{S}^{(n)}(t) := \frac{S_{\lfloor nt \rfloor}}{n}, \quad \tilde{S}^{(n)}(t) := \frac{\sum_{i=1}^{\lfloor nt \rfloor} (\xi_i - \nu)}{n^{1/\beta}}. \quad (2.10)$$

All these processes take values in \mathcal{D}^+ . By (2.5), $\bar{S}^{(n)} \xrightarrow{\text{a.s.}} \nu \text{id}$ and $\tilde{S}^{(n)} \xrightarrow{\text{d}} \widetilde{W}^{(\beta)}$, in (\mathcal{D}^+, J_1) , as $n \rightarrow \infty$.

The following result extends Theorem 2.3 of [11].

Theorem 2.4 *Under the above assumptions, in particular $\alpha \in (1, 2]$, the following convergences hold, w.r.t. \mathbb{P} :*

- (a) *If $\beta \in (0, 1)$, or $\beta \in (1, 2]$ with $\nu = 0$, let $\hat{Y}^{(n)}(t) := \frac{Y_{\lfloor nt \rfloor}}{n^{1/\beta}}$, for $t \geq 0$. As $n \rightarrow \infty$,*

$$\hat{Y}^{(n)} \xrightarrow{\text{d}} \mu W^{(\beta)} \quad \text{in } (\mathcal{D}^+, J_1).$$

- (b) *If $\beta \in (1, 2]$ with $\nu \neq 0$, let $\bar{Y}^{(n)}(t) := \frac{Y_{\lfloor nt \rfloor}}{n}$, for $t \geq 0$. As $n \rightarrow \infty$,*

$$\bar{Y}^{(n)} \xrightarrow{\text{d}} \mu \nu \text{id} \quad \text{in } (\mathcal{D}^+, J_1).$$

Remark 2.5 The statement of [11, Thm. 2.3] does not include the cases $\alpha = 2$ and/or $\beta = 2$, because the authors were mostly interested in *bona fide* Lévy media and Lévy flights in them. The proof of the theorem, however, works verbatim if the assumptions are generalized to include ζ_1 and/or ξ_1 in the normal domain of attraction of a 2-stable distribution, i.e., a Gaussian. In this case, of course, $\widetilde{Z}_\pm^{(2)}$, $W_1^{(2)}$ and $\widetilde{W}_1^{(2)}$ are Brownian motions. The same remark holds for Theorems 2.7 and 2.9 below.

Remark 2.6 Theorem 2.4 mostly supersedes Theorem 2.1 of [9] (which is stated for the case where S is simple and symmetric), but not quite, since the hypothesis on ζ_1 there is that $\mathbb{P}(\zeta_1 > x) \approx x^{-p}$, for $x \rightarrow +\infty$, with $p \geq 1$. This is weaker than asking that ζ_1 be in the *normal* domain of attraction of an α -stable distribution, with $\alpha \in (1, 2]$. For example it includes the case $\mathbb{P}(\zeta_1 > x) = Cx^{-2}$, where ζ_1 is in the domain, but not normal domain of attraction of a Gaussian [17, Ex. 5.10]. Also, the assertion of Theorem 2.4 is of course much stronger than that of Corollary 2.2, but the hypothesis on the medium for the latter is much weaker: simply $\mu < \infty$.

The case (b) of the above theorem is the case where Y has a drift. Understandably, the scaling rate of $Y_{[n\cdot]}$ is n (one says that the process is *ballistic*) and the convergence is to a deterministic function. It is therefore natural to study the fluctuations around the deterministic limit.

Theorem 2.7 *Under the same assumptions and notation as Theorem 2.4, if $\alpha, \beta \in (1, 2]$, the following convergences hold, w.r.t. \mathbb{P} :*

$$(a) \text{ If } \beta < \alpha, \text{ let } \widetilde{Y}^{(n)}(t) := \frac{n(\bar{Y}^{(n)}(t) - \mu\nu t)}{n^{1/\beta}} = \frac{Y_{[nt]} - n\mu\nu t}{n^{1/\beta}}. \text{ As } n \rightarrow \infty,$$

$$\widetilde{Y}^{(n)} \xrightarrow{d} \mu\widetilde{W}^{(\beta)} \quad \text{in } (\mathcal{D}^+, J_1).$$

$$(b) \text{ If } \beta > \alpha, \text{ let } \widetilde{Y}^{(n)}(t) := \frac{n(\bar{Y}^{(n)}(t) - \mu\nu t)}{n^{1/\alpha}} = \frac{Y_{[nt]} - n\mu\nu t}{n^{1/\alpha}}. \text{ As } n \rightarrow \infty,$$

$$\widetilde{Y}^{(n)} \xrightarrow{d} \text{sgn}(\nu) |\nu|^{1/\alpha} \widetilde{Z}_+^{(\alpha)} \quad \text{in } (\mathcal{D}^+, J_2).$$

$$(c) \text{ If } \beta = \alpha, \text{ let } \widetilde{Y}^{(n)}(t) := \frac{n(\bar{Y}^{(n)}(t) - \mu\nu t)}{n^{1/\beta}} = \frac{Y_{[nt]} - n\mu\nu t}{n^{1/\beta}}. \text{ As } n \rightarrow \infty,$$

$$\widetilde{Y}^{(n)} \xrightarrow{d} \mu\widetilde{W}^{(\beta)} + \text{sgn}(\nu) |\nu|^{1/\alpha} \widetilde{Z}_+^{(\alpha)} \quad \text{in } (\mathcal{D}^+, J_2).$$

where $\widetilde{W}^{(\beta)}$ and $\widetilde{Z}_+^{(\alpha)}$ are two independent processes, defined as in (2.5) ff.

Remark 2.8 The limits in (b) and (c) involve the unusual Skorokhod topology J_2 [16, §11.5]. But this is the strongest amongst the classical Skorokhod topologies relative to which such limits hold, cf. Remark 2.11 of [11]. However, see Remark A.2 in the same paper.

2.3 Infinite mean distance between targets, annealed theorems

In this subsection we assume that ζ_1 is in the normal basin of attraction of a α -stable distribution with $\alpha \in (0, 1)$. Since $\mathbb{E}[\zeta_1] = \infty$, this means that, for $n \rightarrow \infty$,

$$\frac{1}{n^{1/\alpha}} \sum_{i=1}^n \zeta_i \xrightarrow{d} Z_1^{(\alpha)}, \quad (2.11)$$

for some α -stable variable $Z_1^{(\alpha)}$ (whose skewness index is 1, since $\zeta_i > 0$). Out of $Z_1^{(\alpha)}$, we construct continuous-argument processes $(Z_{\pm}^{(\alpha)}(s), s \geq 0)$ and $(Z^{(\alpha)}(s), s \in \mathbb{R})$ in complete analogy with the previous case; cf. (2.6). $Z^{(\alpha)}$ takes values in \mathcal{D} , and the same is true for

$$\hat{\omega}^{(n)}(s) := \frac{1}{n^{1/\alpha}} \begin{cases} \omega_{\lfloor ns \rfloor}, & s \geq 0, \\ \omega_{\lceil ns \rceil}, & s < 0. \end{cases} \quad (2.12)$$

It is a basic fact that, as $n \rightarrow \infty$, $\hat{\omega}^{(n)} \xrightarrow{d} Z^{(\alpha)}$, in (\mathcal{D}, J_1) .

As the underlying RW, we maintain the same assumptions and notation as in §2.2, recalling that the fundamental assumption is that ξ_1 is in the normal basin of attraction of an β -stable distribution, with $\beta \in (0, 1) \cup (1, 2]$. Once again, ν denotes the expectation of ξ_1 , when defined.

The following theorem comprises and extends Theorems 2.1 and 2.2 of [11]:

Theorem 2.9 *Under the above assumptions, in particular $\alpha \in (0, 1)$, the following convergences hold, w.r.t. \mathbb{P} :*

(a) *If $\beta \in (0, 1)$, or $\beta \in (1, 2]$ with $\nu = 0$, let $\hat{Y}^{(n)}(t) := \frac{Y_{\lfloor nt \rfloor}}{n^{1/\alpha\beta}}$, for $t \geq 0$.*

As $n \rightarrow \infty$, the finite-dimensional distributions of $\hat{Y}^{(n)}$ converge to those of $Z^{(\alpha)} \circ W^{(\beta)}$. This means that, for all $m \in \mathbb{Z}^+$ and $t_1, \dots, t_m \in \mathbb{R}^+$,

$$(\hat{Y}^{(n)}(t_1), \dots, \hat{Y}^{(n)}(t_m)) \xrightarrow{d} (Z^{(\alpha)}(W^{(\beta)}(t_1)), \dots, Z^{(\alpha)}(W^{(\beta)}(t_m))).$$

(b) *If $\beta \in (1, 2]$ with $\nu \neq 0$, let $\hat{Y}^{(n)}(t) := \frac{Y_{\lfloor nt \rfloor}}{n^{1/\alpha}}$, for $t \geq 0$. As $n \rightarrow \infty$,*

$$\hat{Y}^{(n)} \xrightarrow{d} \operatorname{sgn}(\nu) |\nu|^{1/\alpha} Z_+^{(\alpha)} \quad \text{in } (\mathcal{D}^+, J_2).$$

Remark 2.10 The convergence of the finite-dimensional distributions in assertion (a) is certainly a weak form of convergence, but it is morally the best one can do, given that $Z^{(\alpha)} \circ W^{(\beta)}$ is not càdlàg with positive probability; see the comments after Theorem 2.2 of [10]. As for assertion (b), the considerations of Remark 2.8 apply here too.

3 Continuous-time random walk

In this section we deal with the CTRW $X \equiv X^\omega$, again under various assumptions, depending on the papers we report on.

3.1 Finite mean distance between targets, quenched theorems

Once more, we start by presenting a result by [8], namely the quenched CLT for X [8, Thm. 1]. The assumptions are the same as in §2.1 above: the mean distance μ between targets is finite and the underlying RW has symmetric, unimodal, finite-variance increments ξ_n . Let us recall in particular the notation $V_\xi := \mathbb{E}[\xi_1^2]$.

Theorem 3.1 *Under the above assumptions, most notably $\mu < \infty$, let $M_\xi := \mathbb{E}[|\xi_1|]$ denote the first absolute moment of the underlying RW. Then, as $t \rightarrow \infty$ and w.r.t. P_ω , for a.a. ω ,*

$$\frac{X(t)}{\sqrt{t}} \xrightarrow{d} \mathcal{N}\left(0, \mu \frac{V_\xi}{M_\xi}\right).$$

Here, once again, $\mathcal{N}(0, \cdot)$ is a centered Gaussian variable with the specified variance.

The annealed CLT follows immediately:

Corollary 3.2 *The limit in the statement of Theorem 3.1 holds w.r.t. \mathbb{P} as well.*

A recent preprint of Zamparo [12] claims the convergence of *all* quenched moments of $X(t)/\sqrt{t}$, under the additional assumption that the underlying RW is simple and symmetric (implying that X is the *bona fide* Lévy-Lorentz gas). Recall the notation \overline{m}_q for the q -th absolute moment of the standard Gaussian, cf. (2.1).

Theorem 3.3 *Assume that $\mu < \infty$ and S is a simple symmetric RW. Then, for a.a. ω ,*

$$\lim_{t \rightarrow \infty} \frac{E_\omega[|X(t)|^q]}{t^{q/2}} = \mu^{q/2} \overline{m}_q.$$

Remark 3.4 When S is simple and symmetric, $V_\xi = M_\xi = 1$, so Theorem 3.3 is consistent with Theorem 3.1, showing that X is completely diffusive in this case.

Theorem 3.3 descends from another result of independent interest, concerning the *large deviations* of X , namely, events of the type $\{|X(t)| > at\}$, for $a > 0$. Since $X(t)$ is centered and scales like \sqrt{t} , the probability of such “ballistic events” is expected to be exceedingly small. In [12, Thm. 2.3] it is proved that this probability vanishes like a stretched exponential. We report such result here:

Theorem 3.5 *Under the same assumptions as in Theorem 3.3, there exists $\kappa > 0$ such that, for all $a \in (0, 1]$, the limit*

$$\limsup_{t \rightarrow \infty} \frac{1}{\sqrt{at}} \log P_\omega(|X(t)| > at) \leq -\kappa$$

holds for a.e. ω .

3.2 Finite mean distance between targets, annealed theorems

Apart from recalling Corollary 3.2, which establishes the annealed CLT for X under the assumptions of [8] ($\mu < \infty$ and S has symmetric, unimodal, finite-variance increments, cf. §2.1), in this section we present the results of [12] on the moments and large deviations of the annealed version of X .

The assumptions for this part are stronger than for Theorems 3.3 and 3.5. Like before, S must be a simple symmetric RW, but now we also posit that the tail of the distribution of ζ_1 is *regularly varying* with index $-p \leq -1$. This means that

$$\tau_\zeta(x) := \mathbb{P}(\zeta_1 > x) = \frac{\ell(x)}{x^p}, \quad (3.1)$$

where ℓ is a *slowly varying* function at $+\infty$, namely, for all $c > 0$,

$$\lim_{x \rightarrow +\infty} \frac{\ell(cx)}{\ell(x)} = 1.^4 \quad (3.2)$$

In order to describe the upcoming theorems in their full power, we need more notation. For $0 < r < 1$, set

$$f_p(r) := \sum_{j=0}^{\lceil (1-r)/2r \rceil - 1} \left(\left(\frac{2j+2}{1+r} \right)^p - \left(\frac{2j}{1-r} \right)^p \right). \quad (3.3)$$

It can be seen [12, §2.1] that $0 < f_p(r) \leq r^{-p}$ and, as $r \rightarrow 0^+$,

$$f_p(r) \sim \frac{1}{(p+1)r^p}. \quad (3.4)$$

Here and in the rest of the paper \sim denotes *exact* asymptotic equivalence. This limit shows in particular that $\int_0^1 r^{q-1} f_p(r) dr$ converges for $q = 2p - 1$ and $p > 1$, or $q > 2p - 1$ and $p \geq 1$.

Theorem 3.6 *Assume that S is a simple symmetric RW, $\mu < \infty$ and τ_ζ is regularly varying with index $-p \leq -1$, cf. (3.1). For all $q > 0$, recall the notation \bar{m}_q for the q -th moment of the standard Gaussian, cf. (2.1), and set*

$$d_{\mu,p,q} := \sqrt{\frac{2}{\mu}} \frac{\Gamma(q-p+1)}{\Gamma(q-p+3/2)} \int_0^1 r^{q-1} f_p(r) dr.$$

Then, as $t \rightarrow \infty$,

$$\mathbb{E}[|X(t)|^q] \sim \begin{cases} \bar{m}_q \mu^{q/2} t^{q/2}, & q < 2p - 1, \text{ or } q = p = 1; \\ \bar{m}_q \mu^{q/2} t^{q/2} + d_{\mu,p,q} t^{q+1/2} \tau_\zeta(t), & q = 2p - 1, \text{ and } p > 1; \\ d_{\mu,p,q} t^{q+1/2} \tau_\zeta(t), & q > 2p - 1. \end{cases}$$

⁴See [18] for a treatise on regularly varying functions.

A few words of comment: In the subject of anomalous diffusion, an important quantity to investigate is the *scaling exponent* of the moments,

$$\gamma(q) := \lim_{t \rightarrow \infty} \frac{\log \mathbb{E}[|X(t)|^q]}{\log t}, \quad (3.5)$$

assuming this limit exists at least for a.e. $q > 0$. In many relevant models one observes that $q \mapsto \gamma(q)$ is piecewise linear with two branches, a left one with slope $1/2$ and a right one with slope 1 . Researchers named this situation *strong anomalous diffusion*,⁵ cf. [19, 6]. Theorem 3.6 shows that this is precisely what happens for the annealed Lévy-Lorentz gas, under the above assumptions. The corner between the two branches occurs at the moment of order $2p - 1 \geq 2$, so the behavior of the second moment is still normal, at least in terms of the leading exponent. Even more interestingly, this picture is very different from that of the corresponding quenched Lévy-Lorentz gas, which is fully diffusive, as seen in Theorem 3.3.⁶

As for the quenched case, Theorem 3.6 is based on a large deviation result, which, however, is very different from Theorem 3.5:

Theorem 3.7 *Under the same assumptions as in Theorem 3.6, for $a \in (0, 1]$, let*

$$F_{\mu,p,a} := \frac{1}{\sqrt{2\pi\mu}} \int_a^1 f_p\left(\frac{a}{\eta}\right) \frac{\eta^{-p}}{\sqrt{1-\eta}} d\eta.$$

Then, as $t \rightarrow \infty$,

$$\mathbb{P}(X(t) > at) = \mathbb{P}(X(t) < -at) \sim F_{\mu,p,a} \sqrt{t} \tau_\zeta(t).$$

Moreover, for any $\delta \in (0, 1)$, the lower order terms are uniformly bounded for $a \in [\delta, 1]$.

Remark 3.8 The first equality of the above assertion is obvious because, by the symmetry of the distributions of ω and S , the annealed distribution of $X(t)$ is the same as that of $-X(t)$.

3.3 Infinite mean distance between targets, annealed theorems

The last and hardest case is that of the CTRW X in a medium ω with $\mu = \infty$. At least to this author's knowledge, only a limit theorem seems to be available, that

⁵Though different authors use different terminologies, not always compatible with each other, or even fully self-consistent.

⁶This does not mean that Theorems 3.3 and 3.6 are incompatible: what is happening here is that $t^{-q/2} E_\omega[|X(t)|^q]$ converges to the suitable limit for a.a. ω , but, at least for large q , the convergence rate depends heavily on ω . Moreover, the convergence is not monotonic in t . Mathematically speaking, the convergence is neither dominated nor monotonic, so one cannot interchange the limit in t and the integration on ω , to obtain the limit of the annealed moments from that of the quenched moments.

of Bianchi *et al*, recently appeared in [10]. We present it after some preparatory material.

First off, the assumption on the medium is that ζ_1 is in the normal domain of attraction of an α -stable positive variable, with $\alpha \in (0, 1)$. As far ω is concerned, this is the same assumption as in §2.3, so we use the same notation introduced there, in particular for the processes $(Z_{\pm}^{(\alpha)}(s), s \geq 0)$ and $(Z^{(\alpha)}(s), s \in \mathbb{R})$. The underlying random walk S is assumed to be centered and such that $\mathbb{E}[|\xi_1|^q] < \infty$, for some $q > 2/\alpha$. This implies in particular that $V_{\xi} = \mathbb{E}[|\xi_1|^2] < \infty$. So this is a special case of the assumptions on S of §2.3 (which were the same as in §2.2).

All the preliminary results seen earlier then apply, in particular, for $n \rightarrow \infty$, $\hat{\omega}^{(n)} \xrightarrow{d} Z^{(\alpha)}$, in (\mathcal{D}, J_1) , cf. (2.12), and $\hat{S}^{(n)} := n^{-1/2} S_{\lfloor n \cdot \rfloor} \xrightarrow{d} W^{(2)}$, in (\mathcal{D}^+, J_1) . Here $W^{(2)}$ is a Brownian motion such that $W^{(2)}(t)$ has mean 0 and variance $V_{\xi}t$. As clarified in §2.3, the processes $Z_{\pm}^{(\alpha)}$ and $W^{(2)}$ are independent. Recalling the notation $M_{\xi} := \mathbb{E}[|\xi_1|]$, let $\Delta := (\Delta(t), t \geq 0)$ be defined by

$$\Delta(t) := M_{\xi} \left(\int_0^{\infty} L_t(x) dZ_+^{(\alpha)}(x) + \int_0^{\infty} L_t(-x) dZ_-^{(\alpha)}(x) \right), \quad (3.6)$$

where, for all $x \in \mathbb{R}$, $L_t(x) := \#(W^{(2)}|_{[0,t]})^{-1}(x)$. In other words, $L_t(x)$ is the local time of the Brownian motion $W^{(2)}$ in x , up to time t . As a function of x , L_t is compactly supported and almost surely continuous, thus the above r.h.s. is well-defined. Since L_t is also strictly increasing in t , Δ is almost surely continuous and strictly increasing. Processes like Δ are called *Kesten-Spitzer processes* and arise in the context of *RW in random scenery* [20], which is one of the technical ingredients of Theorem 2.1 of [10] which we now present.

Theorem 3.9 *Under the above assumptions, in particular $\alpha \in (0, 1)$, $\nu = 0$, and ξ_1 has a finite absolute moment of order $q > 2/\alpha$, let $\hat{X}^{(n)}(t) := \frac{X(nt)}{n^{1/(\alpha+1)}}$, for $t \geq 0$. Then the annealed finite-dimensional distributions of $\hat{X}^{(n)}$ converge to those of $Z^{(\alpha)} \circ W^{(2)} \circ \Delta^{-1}$. This means that, for all $m \in \mathbb{Z}^+$ and $t_1, \dots, t_m \in \mathbb{R}^+$,*

$$(\hat{X}^{(n)}(t_1), \dots, \hat{X}^{(n)}(t_m)) \xrightarrow{d} (Z^{(\alpha)}(W^{(2)}(\Delta^{-1}(t_1))), \dots, Z^{(\alpha)}(W^{(2)}(\Delta^{-1}(t_m))))),$$

as $n \rightarrow \infty$, relative to \mathbb{P} .

Remark 3.10 The process $Z^{(\alpha)} \circ W^{(2)} \circ \Delta^{-1}$ is not a.s. càdlàg, so the same considerations and reference as in Remark 2.10 apply here.

4 A brief discussion on perspectives

Just by looking at the titles of the previous subsections, one notices that no quenched theorems were given for the case of infinite mean distance between targets. This is

the main shortcoming of the current mathematical description of the processes X and Y . Technically speaking, the problem is that, without the condition $\mu < \infty$, one does not have a strong law of large numbers for the variables ζ_k . This is after all the simplest form of a quenched result and provides the scaling of $k \mapsto \omega_k$, as $|k| \rightarrow \infty$, for *each* realization ω of the medium, apart from a negligible set of exceptions. How to prove quenched limit theorems without this basic ingredient is not clear to me at the moment.

An open question of a different nature is that of devising a good model of Lévy-Lorentz gas in dimension $d \geq 2$. Here ‘good’ means that it should have the following features, in one form or another:

- The random medium should be *homogeneous*, in the sense that the distribution of the relative positions of two or more targets should not depend on the absolute position of any of the targets involved.⁷
- The distances between targets should be heavy-tailed.⁸
- The law of the random medium should be rotation-invariant, at least for a subgroup of rotations, e.g., the coordinate directions. In other words, the model should be *isotropic*, unless it has a clear reason not to be.
- The transition probabilities from one target to the next should not depend on their distance, only on the “degree of accessibility” of the new target. For example, the next target might always be the nearest one along a random (isotropically selected) direction. The meaning of this condition is that it should be the medium, not the walker, to decide how long the next inertial stretch will be.

Even with all these features, a model might not be very interesting. Here is an example of a feasible, yet not very instructive model. Let $(\omega'_{k_1}, k_1 \in \mathbb{Z})$ and $(\omega''_{k_2}, k_2 \in \mathbb{Z})$ be two i.i.d. point processes in \mathbb{R} , as introduced in Section 1. For $k = (k_1, k_2) \in \mathbb{Z}^2$, set $\omega_k := (\omega'_{k_1}, \omega''_{k_2})$. This defines a random medium $\omega = (\omega_k, k \in \mathbb{Z}^2)$ in \mathbb{R}^2 . An independent, \mathbb{Z}^2 -valued, underlying RW $(S_n, n \in \mathbb{N})$ is given, whereby we introduce the DTRW $Y := (Y_n := \omega_{S_n}, n \in \mathbb{N})$. The CTRW $X := (X(t), t \geq 0)$ is then defined as the unit-speed interpolation of Y .

Obviously, the process X is simply the direct sum, in a very natural sense, of two independent, orthogonal, 1D CTRWs X' and X'' . Its properties are thus (for the most part) easily derived from those of X' and X'' , as presented in Section 3.

⁷The reader who feels this condition is not well-defined is right, see footnote below.

⁸This condition is ill-defined in the same way as the previous condition was. It would be well-defined if the points of the random medium were labeled in a consistent way, so that it would make sense to consider, say, the distribution of the distance between ω_k and ω_ℓ (here k and ℓ are generic indices, not necessarily in \mathbb{Z}). But no labeling is assumed on the random medium, as it is not easy to think of a general, physically relevant way to label the points of a d -dimensional point process, for $d \geq 2$.

Introducing and investigating more relevant, and truly d -dimensional, flights and walks in Lévy random medium will be the subject of future work.

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