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Abstract

We show that the bootstrapped Newtonian potential generated by a uniform and isotropic source does not depend on the one-loop correction for the matter coupling to gravity. The latter however affects the relation between the proper mass and the ADM mass and, consequently, the pressure needed to keep the configuration stable.

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1 Introduction and motivation

Black holes represent problematic predictions of general relativity, particularly in that they feature classical curvature singularities [1,2], which further seem to make hardly any sense in a quantum context. One therefore expects that a complete description of gravity will be modified by quantum physics. For this reason, an extension of Newtonian gravity that contains non-linear interaction terms was developed in Refs. [3–6], as a toy model to describe static, spherically symmetric sources in a quantum fashion. [1] As we shall review below, the non-linear term describing the gravitational self-interaction is in particular obtained by coupling the gravitational potential to the Newtonian gravitational potential energy density (1.5). [2] For this reason, this approach is termed *bootstrapped* Newtonian gravity. Solutions were then found corresponding to homogeneous matter distributions of radius R for which no Buchdahl limit [10] appears, but still require increasingly large pressure to counterbalance the gravitational pull for increasing compactness.

Indeed, the model naturally contains two mass parameters, one which appears in the potential outside the source and can be identified with the Arnowitt-Deser-Misner (ADM) mass [9], and a second mass term M_0 that is simply the volume integral of the proper density (from which the energy associated with the pressure is excluded). Since only M can be measured by studying orbits around the compact object, we shall define the compactness in terms of M as $G_{\rm N} M/R$ like in Ref. [6]. One then obtains a unique relation between M_0 and M. As a further

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[1] The issue of building a quantum description will be tackled elsewhere [7,8].

[2] The same term can also be obtained by expanding the Einstein-Hilbert action around flat space (for the details see Appendix B of Ref. [5]).

development of the model, we are here interested in analysing in more detail the effects of the couplings introduced in Ref. [6] on these two masses.

We recall from Ref. [5] that a non-linear equation for the potential $V = V(r)$ describing the gravitational pull on test particles generated by a matter density $\rho = \rho(r)$ can be obtained starting from the Newtonian Lagrangian [3]

$$L_{\rm N}[V] = -4\,\pi \int_0^\infty r^2\,{\rm d}r \left[\frac{(V')^2}{8\,\pi\,G_{\rm N}} + \rho\,V\right] \qquad (1.1)$$

and the corresponding Poisson equation of motion

$$r^{-2}\left(r^2\,V'\right)' \equiv \triangle V = 4\,\pi\,G_{\rm N}\,\rho \ . \qquad (1.2)$$

We can then include the effects of gravitational self-interaction by noting that the Hamiltonian

$$H_{\rm N}[V] = 4\,\pi \int_0^\infty r^2\,{\rm d}r \left(-\frac{V\,\triangle V}{8\,\pi\,G_{\rm N}} + \rho\,V\right) \ , \quad (1.3)$$

computed on-shell by means of Eq. (1.2), yields the total Newtonian potential energy

$$U_{\rm N}[V] = 2\,\pi \int_0^\infty r^2\,{\rm d}r\,\rho(r)\,V(r)$$
$$= -4\,\pi \int_0^\infty r^2\,{\rm d}r\,\frac{[V'(r)]^2}{8\,\pi\,G_{\rm N}} \ , \qquad (1.4)$$

where we assumed boundary terms vanish. Following Refs. [3–6], one can view $U_{\rm N}$ as given by the volume integral of the gravitational current

$$J_V = -\frac{[V'(r)]^2}{2\,\pi\,G_{\rm N}} \ . \qquad (1.5)$$

We can also include the source term

$$J_\rho = -2\,V^2 \ , \qquad (1.6)$$

which comes from the linearisation of the volume measure around the vacuum [5] and can be interpreted as a gravitational one-loop correction to the matter density. As we recalled above, in Ref. [3], no Buchdahl limit [10] was found but the pressure p becomes very large for compact sources with a size $R \lesssim R_{\rm H} \equiv 2\,G_{\rm N}\,M$, and one must therefore add a corresponding potential energy $U_{\rm B}$ such that

$$p = -\frac{{\rm d}U_{\rm B}}{{\rm d}\mathcal{V}} \ . \qquad (1.7)$$

This can be easily included by simply shifting $\rho \rightarrow \rho + p$ to yield [4]

$$L[V] = -4\,\pi \int_0^\infty r^2\,{\rm d}r \left[\frac{(V')^2}{8\,\pi\,G_{\rm N}} + (V + q_\rho\,J_\rho)\,(\rho + p)\,q_V\,J_V\,V\right]$$
$$= -4\,\pi \int_0^\infty r^2\,{\rm d}r \left[\frac{(1 - 4\,q_V\,V)\,(V')^2}{8\,\pi\,G_{\rm N}} + V\,(1 - 2\,q_\rho\,V)\,(\rho + p)\right] \ , \qquad (1.8)$$

where the non-negative coefficients q_V and q_ρ play the role of coupling constants for the graviton currents J_V and J_ρ [5]. The associated effective Hamiltonian is simply given by

$$H[V] = -L[V] \ , \qquad (1.9)$$

and the Euler-Lagrange equation for V is given by the modified Poisson equation

$$\triangle V = 4\,\pi\,G_{\rm N}\,\frac{1 - 4\,q_\rho\,V}{1 - 4\,q_V\,V}\,(\rho + p) + \frac{2\,q_V\,(V')^2}{1 - 4\,q_V\,V} \ . (1.10)$$

We can therefore see that in this simplified bootstrapped picture, there appears an "effective Newton constant"

$$\tilde{G}_{\rm eff} = \frac{1 - 4\,q_\rho\,V}{1 - 4\,q_V\,V}\,G_{\rm N} \ , \qquad (1.11)$$

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[3]Since all functions only depend on the radial coordinate r, we use the notation $f' \equiv {\rm d}f/{\rm d}r$.

[4]This way of including the pressure is in analogy with the definition of the Tolman mass [11].

[5]Different values of q_V can be implemented in order to obtain the approximate potential for different motions of the test particles in general relativity.

as well as an "effective self-coupling"

$$q_{\text{eff}} = \frac{q_V}{1 - 4\,q_V\,V} \ .$$ (1.12)

It is interesting to note that both effective couplings decrease when the field V is negative and large if $q_\rho < q_V$, something one would expect, *e.g.* in the asymptotic safety scenario [12].

The conservation equation that determines the pressure reads

$$p' = -V'\,(\rho + p) \ .$$ (1.13)

In the vacuum (where $\rho = p = 0$), Eq. (1.13) is trivially satisfied and Eq. (1.10) is exactly solved by [3]

$$V = \frac{1}{4\,q_V} \left[1 - \left(1 + \frac{6\,q_V\,G_N\,M}{r} \right)^{2/3} \right] \ ,$$ (1.14)

where the integration constants were fixed in order to recover the Newtonian behaviour at large distance,

$$V_N = -\frac{G_N\,M}{r} \ .$$ (1.15)

Note that we can now take the limit $q_V \to 0$ and precisely recover the Newtonian potential (1.15), as one would expect by first considering this limit in Eq. (1.10). We also note that the large r expansion of the solution (1.14) reads

$$V \simeq -\frac{G_N\,M}{r} + q_V\,\frac{G_N^2\,M^2}{r^2} \ ,$$ (1.16)

so that q_V always affects the post-Newtonian order.

In the following analysis, we are specifically interested in the effect of the one-loop coupling q_ρ on the relation between the mass M and the proper mass M_0 of the source (which we will introduce shortly), hence we set $q_V = 1$ and consider the range $q_\rho \geq 0$.

2 Interior solutions

In order to derive the interior potential, we proceed as in the previous Refs. [3–6], in which the source is simply modelled as a spherically-uniform proper density distribution of matter with radius R,

$$\rho = \rho_0 \equiv \frac{3\,M_0}{4\,\pi\,R^3}\,\Theta(R - r) \ ,$$ (2.1)

where Θ is the Heaviside step function and the total mass M_0 is defined as

$$M_0 = 4\,\pi \int_0^R r^2\,\mathrm{d}r\,\rho(r) \ .$$ (2.2)

We use Eq. (1.13) to express the pressure in terms of the potential itself like in Ref. [6] as

$$p = \rho_0 \left(e^{V_R - V} - 1 \right)$$ (2.3)

and obtain

$$\triangle V = \frac{3\,G_N\,M_0}{R^3} \left(\frac{1 - 4\,q_\rho\,V}{1 - 4\,V} \right) e^{V_R - V} + \frac{2\,(V')^2}{1 - 4\,V} \ .$$ (2.4)

Regularity conditions in the centre are required to be met by the solutions, specifically

$$V'_{\text{in}}(0) = 0 \ ,$$ (2.5)

where $V_{\text{in}} = V(0 \leq r \leq R)$, and they must also satisfy matching conditions with the exterior solution at the surface,

$$V_{\text{in}}(R) = V_{\text{out}}(R) \equiv V_R = \frac{1}{4} \left[1 - (1 + 6X)^{2/3} \right]$$ (2.6)

$$V'_{\text{in}}(R) = V'_{\text{out}}(R) \equiv V'_R = \frac{X}{R\,(1 + 6\,X)^{1/3}} \ ,$$ (2.7)

where $V_{\text{out}} = V(R \leq r)$. We also introduced the "outer" compactness

$$X = \frac{G_N\,M}{R} \ ,$$ (2.8)

where it is important to keep in mind that the ADM mass $M \neq M_0$ in general.

2.1 Small and medium compactness

We can approach the problem in a similar way as in Ref. [6] for the case when the radius of the source R is much larger or of the order of $G_N\,M$. An analytic approximation V_s for V_{in} can be obtained by expanding around $r = 0$, and thus the expression for the potential in (2.4) can be written

$$V_s \simeq V_0 + \frac{G_N\,M_0}{2\,R^2} \left(\frac{1 - 4\,q_\rho\,V_0}{1 - 4\,V_0} \right) e^{V_R - V_0}\,r^2 \ ,$$ (2.9)

where $V_0 \equiv V_{\text{in}}(0) < 0$. We also used the regularity condition (2.5), which constrains all odd order terms in r from the Taylor expansion about $r = 0$ to vanish.

After imposing the boundary conditions (2.6) and (2.7), we find that the potential has the same expression for any values of q_ρ,

$$V_s \simeq \frac{R^2 \left[(1 + 6\,X)^{1/3} - 1 \right] + 2\,X\,(r^2 - 4\,R^2)}{4\,R^2\,(1 + 6\,X)^{1/3}} \ ,$$ (2.10)
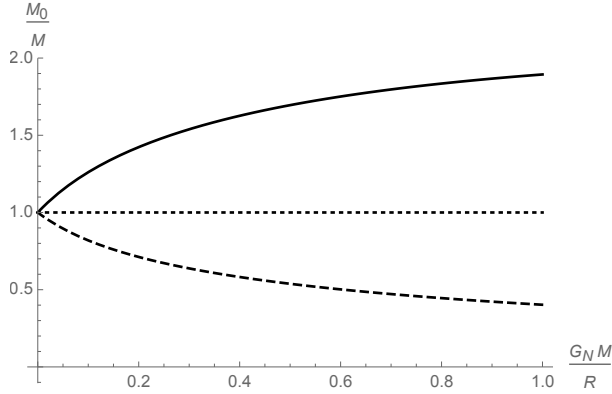
Figure 1: Ratio M_0/M for small and medium compactness for $q_\rho = 1$ (dashed line), and $q_\rho = 0$ (solid line). In these two cases M_0 is always different from M (dotted line).

but the relation between M_0 and M does depend on q_ρ,

$$\frac{M_0}{M} \simeq \frac{e^{-\frac{X}{2(1+6\,X)^{1/3}}}\,(1+8\,X)}{(1+6\,X)^{2/3}\left[1-q_\rho+\frac{(1+8\,X)}{(1+6\,X)^{1/3}}q_\rho\right]} \ , \quad (2.11)$$

which is plotted for the two cases $q_\rho = 1$, respectively $q_\rho = 0$ in Fig. 1. Different values of q_ρ interpolate between these cases and a critical value of $q_\rho = q_s$ can be found such that $M_0 = M$ (see Fig. 2),

$$q_s \simeq \frac{(1+8\,X)\,e^{-\frac{X}{2\,(1+6\,X)^{1/3}}} - (1+6\,X)^{1/3}}{(1+6\,X)^{1/3}\left[1+8\,X-(1+6\,X)^{1/3}\right]} \ . \quad (2.12)$$

For $q_s \lesssim q_\rho$ the mass $M_0 < M$ as in Ref. [6], whereas $M_0 > M$ for $0 \le q_\rho \lesssim q_s$. It is also worth noting that the pressure p in Eq. (2.3) grows faster with the compactness for $0 \le q_\rho \lesssim q_s$ than it does for $q_s \lesssim q_\rho$ (see Fig. 3).

2.2 Large compactness

In the large compactness case, $G_N\,M \gg R$, we can employ the linear approximation [6]

$$V_c \simeq V_R + V_R'\,(r-R) \ , \quad (2.13)$$

which obviously does not depend on q_ρ (see Appendix A for more details). The matching conditions (2.6) and (2.7) at $r = R$ are now satisfied by construction and we can hence determine the relation between M and M_0 by imposing the field equa-
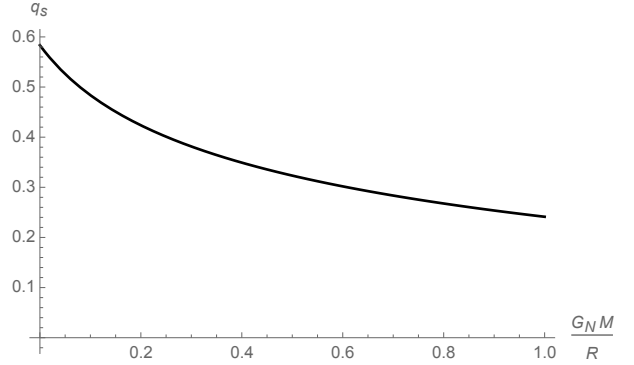

Figure 2: Critical value q_s of q_ρ for which $M = M_0$ for small and medium compactness.
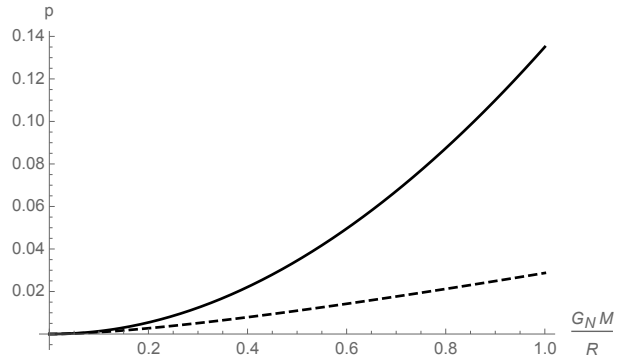

Figure 3: Pressure p for small and medium compactness for $q_\rho = 1$ (dashed line), and $q_\rho = 0$ (solid line).

tion (2.4), yielding

$$\frac{M_0}{M} \simeq \frac{2\,(1+5\,X)}{3\,(1+6X)^{2/3}\left\{1-q_\rho\left[1-(1+6X)^{2/3}\right]\right\}} , (2.14)$$

which is plotted for the two cases $q_\rho = 1$, respectively $q_\rho = 0$, in Fig. 4 and the critical value of $q_\rho = q_c$ such that $M_0 = M$,

$$q_c \simeq \frac{2\,(1+5\,X)-3\,(1+6\,X)^{2/3}}{3\,(1+6\,X)^{2/3}\left[(1+6\,X)^{2/3}-1\right]} \ , \quad (2.15)$$

is plotted in Fig. 5. It is easy to see from Eq. (2.15) that $q_c \sim X^{-1/3} \to 0$ for $X \to \infty$. As with smaller values of the compactness, the mass $M_0 < M$ for $q_c \lesssim q_\rho$, whereas $M_0 > M$ for $0 \le q_\rho \lesssim q_c$, and the pressure again grows with the compactness much faster when $M_0 > M$ (see Fig. 6). Finally, one should keep in mind that the linear approximation becomes rather accurate only for values of the compactness $X \gg 1$, which explains why the ratios M_0/M and
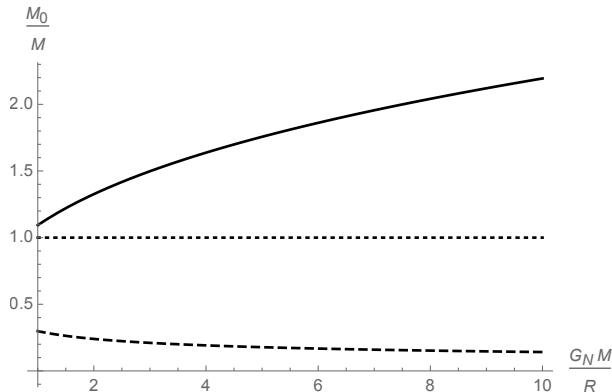
Figure 4: Ratio M_0/M for large compactness for $q_\rho = 1$ (dashed line), and $q_\rho = 0$ (solid line). In these two cases M_0 is always different from M (dotted line).

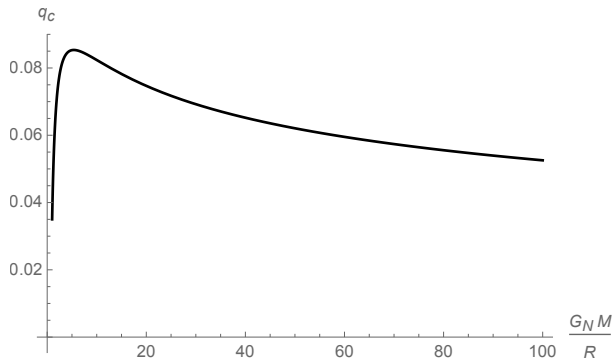the values of q_ρ for which $M_0 = M$ do not match around $X = 1$ [6].

Figure 5: Critical value q_c of q_ρ for which $M = M_0$ for large compactness.

We have therefore shown that, not only is the outer potential insensitive to the matter coupling q_ρ, but so is the interior potential (within our approximations). Since the outer potential only depends on the "total ADM energy" M, the fact that the value of q_ρ does not change it is expected. The value of q_ρ, however, can affect the relation between M_0 and M very significantly.

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[6]We find that the critical couplings q_s and q_c are numerically very close for values of $X \sim 4$, and that the masses $M_0 = M_0(X)$ estimated in the two regimes are also rather close for the same compactness.
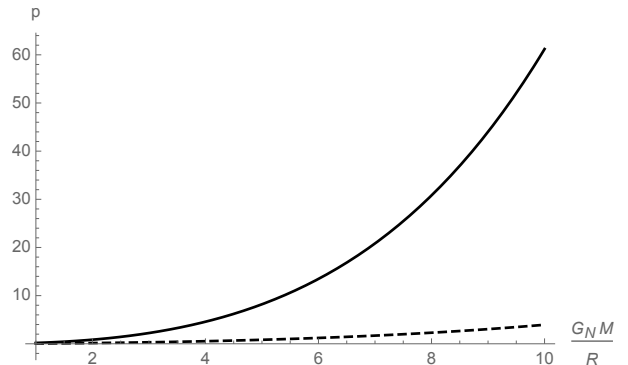

Figure 6: Pressure p for large compactness for $q_\rho = 1$ (dashed line), and $q_\rho = 0$ (solid line).

3 Discussion and conclusions

In this work, we focused on the effects induced by the strength of the one-loop coupling q_ρ in the Lagrangian (1.8) on the potential V generated by a static compact source of uniform density. For this analysis, we set $q_V = 1$ and values of q_ρ therefore measure the relative strength of this contribution with respect to the gravitational self-interaction proportional to q_V.

The main conclusions are that a) the potential V is totally insensitive to the value of $q_\rho \geq 0$ but b) the relation between the ADM mass M and the proper mass M_0 does depend on q_ρ. In particular, $M_0 > M$ and the pressure necessary to keep the system in equilibrium is much larger when $q_\rho < q_{cr}$, where $q_{cr} \simeq q_s$ in Eq. (2.12) for small compactness $G_N M \lesssim R$ and $q_{cr} \simeq q_c$ in Eq. (2.15) for large compactness $G_N M > R$. Since $q_{cr} < 1 = q_V$, this case was not covered in Ref. [6], where we assumed $q_\rho = q_V$ and we always had $M_0 < M$ accordingly. We also remark that $q_c \ll 1$ for very large compactness $G_N M \gg R$ and that it asymptotes to zero, which makes this case somewhat less likely to play a relevant role in modelling (quantum) black holes than the case studied in Ref. [6].

We conclude by noting that the fact the potential V for static configurations does not change with q_ρ, and is therefore insensitive to M_0, but only depends on the total mass M and radius R of the source appears as a form of Birkhoff's theorem in the bootstrapped Newtonian picture.

5

Acknowledgments

A Comparison method for large compactness

Using the comparison method for non-linear differential equations, it was shown in Ref. [6] that the linear potential (2.13) is a good approximation in the large compactness regime for $q_\rho = 1$, except in a (very) small region near $r = 0$, where it does not satisfy the boundary condition (2.5). We briefly show here that this still holds for $q_\rho \geq 0$.

The comparison theorems [13–15] (see also Appendix C in Ref. [6]) ensure that the solution to Eq. (2.4) must lie in between any two bounding functions,

$$V_- < V_{\rm in} < V_+ \ . \tag{A.1}$$

which satisfy (suitably generalised) boundary conditions and are such that $E_+(r) < 0$ and $E_-(r) > 0$ for $0 \leq r \leq R$, where

$$E_\pm \equiv \triangle V_\pm - \frac{3\,G_{\rm N}\,M_0^\pm(M)}{R^3}\left(\frac{1-4\,q_\rho\,V_\pm}{1-4\,V_\pm}\right)e^{V_R-V_\pm} - \frac{2\left(V'_\pm\right)^2}{1-4\,V_\pm} \ . \tag{A.2}$$

For $X \equiv G_{\rm N}\,M/R \gg 1$, we consider the simpler equation

$$\psi'' = \frac{3\,G_{\rm N}\,M_0}{R^3}\,e^{V_R-\psi} \ , \tag{A.3}$$

which is solved by

$$\psi(r; A, B) = -A\left(B + \frac{r}{R}\right)$$
$$+ 2\ln\left[1 + \frac{3\,G_{\rm N}\,M_0}{2\,A^2\,R}\,e^{A\,(B+r/R)+V_R}\right], \tag{A.4}$$

where the constants A, B and M_0 are determined by the boundary conditions (2.5), (2.6) and (2.7). Regularity at $r = 0$ in particular yields

$$M_0 = \frac{2\,A^2\,R}{3\,G_{\rm N}}\,e^{-A\,B-V_R} \ . \tag{A.5}$$

Eq. (2.7) for the continuity of the derivative across $r = R$ then reads

$$A \tanh(A/2) \simeq A = R\,V'_R \ , \tag{A.6}$$

and he continuity Eq. (2.6) for the potential,

$$2\ln\left(1 + e^{R\,V'_R}\right) - R\,V'_R\,(1+B) = V_R \ , \tag{A.7}$$

can be used to express B in terms of M and R. Putting everything together, we obtain [6]

$$\psi(r; X, R) \simeq \frac{1}{2}\left(\frac{X}{\sqrt{6}}\right)^{2/3}\left(\frac{2\,r}{R} - 5\right) \ . \tag{A.8}$$

Bounding functions for Eq. (2.4) can then be obtained as

$$V_\pm = C_\pm\,\psi(r; A_\pm, B_\pm) \ , \tag{A.9}$$

where $A_\pm$, $B_\pm$ and $C_\pm$ are constants computed by imposing the boundary conditions (2.5), (2.6) and (2.7). One first determines a function $V_C = C\,\psi(r; A, B)$ and corresponding mass M_0 which satisfy the three boundary conditions for any constant C and, for fixed values of R, X and q_ρ, one can then numerically determine a constant C_+ such that $E_+ < 0$ and a constant $C_- < C_+$ such that $E_- > 0$. For example, for the limiting case $q_\rho = 0$ and $X = 10^3$, we obtain $C_+ \simeq 1.73$ and $C_- \simeq 1.05$. The two bounding functions are then plotted in Fig. 7 along with the linear approximation (2.13). For a comparison, we recall that $C_+ \simeq 1.6$ and $C_- \simeq 1$ for $q_\rho = 1$ and $X = 10^3$ from Ref. [6].
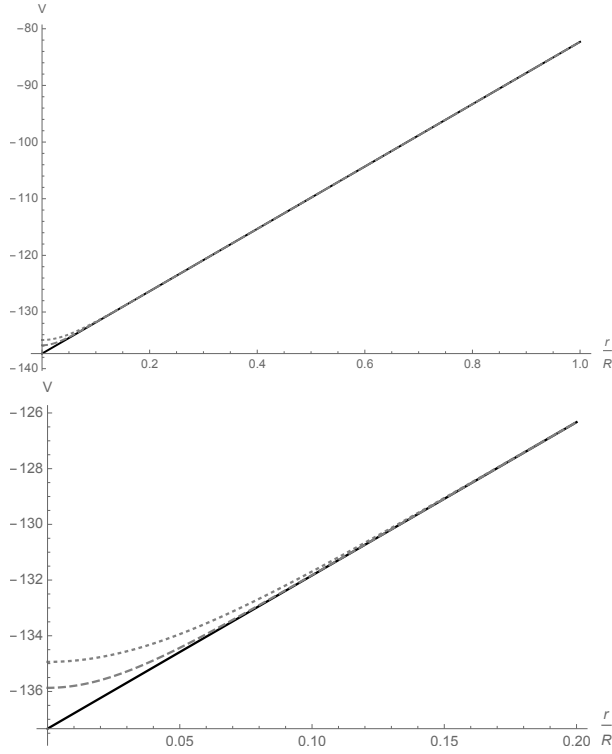
Figure 7: Bounding functions V_- (dashed line) and V_+ (dotted line) *vs* linear approximation (solid line) for $q_\rho = 0$ and $x = 10^3$. Bottom panel is a close up view near $r = 0$.

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