Alma Mater Studiorum Università di Bologna
Archivio istituzionale della ricerca

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(Article begins on next page)

26 June 2024

Effective metric outside bootstrapped Newtonian sources

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February 13, 2021

Abstract

We determine the complete space-time metric from the bootstrapped Newtonian potential generated by a static spherically symmetric source in the surrounding vacuum. This metric contains post-Newtonian parameters which can be further used to constrain the underlying dynamical theory and quantum state of gravity. For values of the post-Newtonian parameters within experimental bounds, the reconstructed metric appears very close to the Schwarzschild solution of General Relativity in the whole region outside the event horizon. The latter is however larger in size for the same value of the mass compared to the Schwarzschild case.

PACS - 04.70.Dy, 04.70.-s, 04.60.-m

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1 Introduction and motivation

General Relativity predicts that the gravitational collapse of any compact source will generate geodetically incomplete space-times whenever a trapping surface appears [1]. Moreover, eternal point-like sources are mathematically incompatible with the Einstein field equations [2]. A consistent quantum theory should fix this pathological classical picture of black hole formation, like quantum mechanics explains the stability of the hydrogen atom. Whether this can be achieved by modifications of the gravitational dynamics solely at the Planck scale or with sizeable implications for astrophysical compact objects remains open to debate, because it is intrinsically very difficult to describe quantum states of strongly interacting systems. Strong interactions imply large nonlinearities, so that the space of classical solutions does not admit a vector basis for the canonical variables which are usually lifted to quantum operators. Of course, this quantisation process can be introduced in a linearised version of any theory, but it becomes questionable that one can then effectively obtain a reliable approximation for the quantum state of what would classically be a strongly interacting configuration. For instance, the physical relevance of the quantum theory of linear perturbations around a given classical solution entirely relies on whether the chosen "background" is actually the one realised in nature.

In the Einstein theory of gravity, we know classical solutions, like the Schwarzschild metrics for the interior of a homogenous spherical star and the exterior of any spherical source, which cannot be obtained by perturbing the Minkowski vacuum. On the other hand, Deser [3] conjectured that it should be possible to reconstruct the full dynamics of General Relativity from the Fierz-Pauli action in Minkowski space-time by adding gravitational self-coupling terms consistent with diffeomorphisms invariance. On a closer inspection, this reconstruction of the Einstein-Hilbert action does not appear free of ambiguities since, for instance, it involves fixing the very important boundary terms in a specific way [4]. Generically, we know that any (modified) metric theory of gravity is invariant under changes of coordinates and must therefore be covariant under diffeomorphisms. Different choices of those boundary terms in the reconstruction proposed by Deser would therefore lead to different modified theories of gravity. What we do not know *a priori* is which (if any) of such theories describes the dynamics realised in nature and what the quantum state of the Universe really is. [1] Moreover, any reconstruction of the dynamics starting from the Minkowski vacuum can be practically effective only if the contribution of matter sources is perturbatively small, which introduces the further problem of reconstructing a large astrophysical source along with the ensuing gravitational field. Such considerations inspired a programme called *bootstrapped Newtonian gravity* [6,7], which consists in adding gravitational self-coupling terms to a Fierz-Pauli-type of action for the static Newtonian potential generated by an arbitrarily large matter source. Furthermore, the coupling constants for such additional terms are allowed to vary from their Einstein-Hilbert values in order to effectively accommodate for corrections arising from the underlying dynamics which, as mentioned above, we do not wish to restrict *a priori*. The direct outcome of this programme is a nonlinear equation, which determines the gravitational potential acting on test particles at rest, and which is generated by a static large source, including pressure effects and the gravitational self-interaction to next-to-leading order in the Newton constant. [2] It is important to remark that our main aim eventually is to investigate the actual quantum state of such systems and the resulting bootstrapped Newtonian potential must therefore be viewed as a mean-field result depending on effective coupling constants

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[1]We also remark that Lovelock's theorem [5] only holds in the vacuum, whereas our Universe is obviously a very different state and so are astrophysical compact objects.

[2]One could ideally iterate the process to any order, but the equations become quickly intractable analytically.

which entail properties of such a (otherwise unknown) state. Our approach is not meant to provide solutions of the linearised Einstein equations (or any modifications thereof), but to describe features of the proper quantum state of gravity. Compact objects were studied with this equation [8–10] and, at least for the simplest case of homogenous density, one can explicitly build a coherent quantum state (for a scalar field) which reproduces the classical gravitational potential [11,12]. Interestingly, these quantum states share some of the properties [13] found in the corpuscular model of black holes [14,15].

Accurate descriptions of the interior of matter sources, whether it is a black hole or a more regular, yet highly compact, distribution, should be given in terms of quantum physics, possibly resulting in an effective equation of state. The relevant observables would eventually be represented by the radius and the mass of stable configurations. Instead, the exterior region of any astrophysical compact object is phenomenologically characterised by the (geodesic) motion of test particles, including photon trajectories. Studying these trajectories, and comparing them with those predicted by General Relativity, is more directly done by means of a full (effective) metric tensor, rather than the bootstrapped Newtonian potential describing only forces which act on static particles.

The aim of this work is precisely to reconstruct a complete space-time metric from the boot-strapped Newtonian potential in the vacuum outside a spherically symmetric source. Of course, by employing an effective metric tensor we implicitly assume the effective dynamics is also invariant under changes of coordinates, which is compatible with the underlying fundamental theory of gravity being covariant under diffeomorphisms, although the particular metric we will find does not need to be a solution of the Einstein equations in the vacuum. Moreover, we will express this metric in terms of quantities which, if not directly observable, have at least an intrinsic geometric meaning. In particular, we will take advantage of the spherical symmetry and employ the usual angular coordinates on the spheres (as surfaces of symmetry of the system) of area $\mathcal{A} = 4\,\pi\,\bar{r}^2$, along with the areal radius $\bar{r}$. The latter differs from the radial coordinate r associated with the harmonic coordinates used to express the potential [16], which is a source of significant technical complication. Furthermore, starting from the potential acting on test particles at rest in a given (harmonic) reference frame does not fix the reconstructed spherically symmetric metric uniquely. For this reason, it will be useful to write the metric in the weak-field region in terms of post-Newtonian parameters, which allow for a direct comparison with experimental bounds. This procedure should, in principle, determine the entire metric in terms of the post-Newtonian parameters all the way into the strong coupling region, if we could solve all equations exactly. However, the post-Newtonian expansion fails near the horizon, so that an explicit calculation will require us to employ also a different near-horizon expansion. Since the potential is a smooth function of r, so must be the metric and the relation $\bar{r} = \bar{r}(r)$. The coefficients in the near-horizon expansion are therefore fully determined by the post-Newtonian parameters via matching conditions in a suitable intermediate region, but analytical expressions become rather involved very quickly. In the present work, we shall therefore just carry out the analysis by including the first few terms in each of the above two expansions.

The main result is that the bootstrapped metric at large distance from the source approaches the Schwarzschild form in a way that can make it compatible with bounds from Solar system tests and other measurements of the first post-Newtonian parameters. The bootstrapped metric is however necessarily different from the exact Schwarzschild form, and this can be interpreted from the point of view of General Relativity as indicating the presence of an effective fluid, filling the space around the source with a non-vanishing energy-momentum tensor which violates the classical energy conditions. The presence of an effective fluid in bootstrapped Newtonian gravity was already noted in Refs. [17]. Moreover, the near-horizon region differs from the General Relativistic prediction mostly in that

3

the horizon size is larger than the Schwarzschild radius for given black hole mass.

The paper is organised as follows: in Section 2, we review the derivation of the bootstrapped Newtonian potential acting on a static test particle generated by a static spherically symmetric source. In Section 3, we discuss the relation between the harmonic coordinates used to express the potential and the more common areal radius. This relation plays a crucial role in the reconstruction of the metric performed at large distance from the source in Section 4, where corrections to the perihelion precession, light deflection and time delay are also estimated. The geometry near the horizon is studied in Section 5 by matching with the weak-field expressions. We conclude with comments and an outlook in Section 6.

2   Potential in the vacuum

In General Relativity (and metric theories of gravity in general), the motion of test particles is determined by the entire metric tensor and there is no invariant notion of a gravitational potential. However, one can still introduce a potential for specific types of motion on specific metric space-times starting from the corresponding geodesic equation. For example, the geodesic equation in the weak field and non-relativistic limit reduces to the Newtonian equation of motion with the potential which solves the linearised Einstein equations in the vacuum provided one uses harmonic coordinates. In the following, we will reverse this argument and start from a bootstrapped Newtonian potential obtained in harmonic coordinates in order to reconstruct a compatible metric.

2.1   Potential for static test particles

We consider a massive particle moving along the trajectory $x^\mu = x^\mu(\tau)$ that satisfies the geodesic equation

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\,\dot{x}^\alpha\,\dot{x}^\beta = 0 \ , \tag{2.1}$$

where dots denote derivatives with respect to the particle's proper time τ and $\Gamma^\mu_{\alpha\beta}$ are the Christoffel symbols of the metric $g_{\mu\nu}$. If the space-time is static, one can choose a time coordinate x^0 in which the metric reads

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon\,h_{\mu\nu}(x^i) \ , \tag{2.2}$$

where ϵ is a parameter we introduce to keep track of deviations from flat space-time. We can now say that the particle is (initially) at rest if $\dot{x}^i = 0$ in this reference frame, which implies that $\dot{x}^0 \simeq 1$ and, as long as $|\dot{x}^i| \simeq \epsilon \ll 1$ (weak-field approximation), Eq. (2.1) to first order in ϵ reduces to

$$\dot{x}^i \simeq \frac{1}{2}\,\epsilon\,h_{00,i} \ , \tag{2.3}$$

which yields Newton's second law for a particle in the potential V if we set

$$g_{00} = -1 + \epsilon\,h_{00} = -(1 + 2\,V) \ , \tag{2.4}$$

and the spatial coordinates x^i in Eq. (2.3) are the analogue of Cartesian coordinates in Newtonian mechanics.

In fact, the explicit form of the potential V generated by a given source can be obtained from the linearised Einstein equations, which then reduce to the Poisson equation for the Newtonian potential in the *de Donder gauge*

$$\partial^\alpha h_{\alpha\mu} - \frac{1}{2}\,\partial_\mu h = 0 \ , \tag{2.5}$$

where $h \equiv \eta^{\alpha\beta}\,h_{\alpha\beta}$. We must correspondingly assume that the coordinates x^μ in which the components of the metric take the form in Eq. (2.2) are *harmonic* coordinates satisfying

$$\Box x^\mu = 0 \ . \tag{2.6}$$

Note that for a static metric with $|h_{ij}| \ll 1$, the condition (2.5) is always satisfied.

2.2   Bootstrapped Newtonian vacuum

We just recalled that the interpretation of V in Eq. (2.4) as the gravitational potential for massive particles at rest is consistent with the fact that, in the same approximation, the linearised Einstein field equations reduce to the Poisson equation of Newton's theory,

$$\triangle V = 4\,\pi\,G_{\rm N}\,\rho \ , \tag{2.7}$$

where ρ is the energy density of the static source and $\triangle$ the flat space Laplacian. The de Donder gauge condition (2.5) implemented in the derivation of Eq. (2.7) was thus employed explicitly also in deriving the equation for the bootstrapped Newtonian potential V from the Einstein-Hilbert action in Ref. [12]. For the sake of brevity, we here review a more heuristic derivation of $V = V(r)$ outside static and spherically symmetric sources from a bootstrapped Newtonian effective action [6,8,10,12].

We start from the Newtonian Lagrangian for a source of density $\rho = \rho(r)$, to wit

$$L_{\rm N}[V] = -4\,\pi \int_0^\infty r^2\,{\rm d}r \left[\frac{(V')^2}{8\,\pi\,G_{\rm N}} + V\rho\right] \tag{2.8}$$

from which Eq. (2.7) can be derived, and stress that the radial coordinate r is the one obtained from harmonic coordinates x^i, as we shall see more in details in Section 3. To this action several interacting terms for the field potential V will be added for the motivation, stated in the introductory section, of describing mean-field deviations from General Relativity induced by quantum physics. First of all, we couple V to a gravitational current proportional to its own energy density,

$$J_V \simeq 4\frac{{\rm d}U_{\rm N}}{{\rm d}\mathcal{V}} = -\frac{[V'(r)]^2}{2\,\pi\,G_{\rm N}} \ , \tag{2.9}$$

where $\mathcal{V}$ is the spatial volume and $U_{\rm N}$ the Newtonian potential energy. Moreover, we add the "loop correction" $J_\rho \simeq -2V^2$, which couples to ρ and, since the pressure gravitates and becomes relevant for large compactness, we also add to the energy density the term [8] [3]

$$J_p \simeq -\frac{{\rm d}U_p}{{\rm d}\mathcal{V}} = p \ , \tag{2.10}$$

---

[3]We only consider isotropic fluids.

where U_p is the potential energy associated with the work done by the force responsible for the pressure. The total Lagrangian then reads

$$
\begin{aligned}
L[V] &= L_{\mathrm{N}}[V] - 4\pi \int_0^\infty r^2 \, \mathrm{d}r \left[q_V \, J_V \, V + 3 \, q_p \, J_p \, V + q_\rho \, J_\rho \, (\rho + 3 \, q_p \, p) \right] \\
&= -4\pi \int_0^\infty r^2 \, \mathrm{d}r \left[\frac{(V')^2}{8\pi \, G_{\mathrm{N}}} \left(1 - 4 \, q_V \, V \right) + (\rho + 3 \, q_p \, p) \, V \, (1 - 2 \, q_\rho \, V) \right] \,,
\end{aligned}
\tag{2.11}
$$

where the coupling constants q_V, q_p and q_ρ can be used to track the effects of the different contributions. As we mentioned previously, different values of these couplings would correspond to different quantum states and depend on the underlying microscopic quantum theory of gravity and matter. For instance, the case $q_V = q_p = q_\rho = 1$ reproduces the Einstein-Hilbert action at next-to-leading order in the expansion in ϵ in Eq. (2.2) and can be naturally used as a primary reference [12] (see also Refs. [8,10] for more details on the role of these coupling parameters). Eventually, their values should be fixed by experimental constraints. Finally, the field equation for V reads

$$
\triangle V = 4\pi \, G_{\mathrm{N}} \frac{1 - 4 \, q_\rho \, V}{1 - 4 \, q_V \, V} \, (\rho + 3 \, q_p \, p) + \frac{2 \, q_V \, (V')^2}{1 - 4 \, q_V \, V} \,,
\tag{2.12}
$$

which must be solved along with the conservation equation $p' = -V' \, (\rho + p)$.

In vacuum, where $\rho = p = 0$, Eq. (2.12) simplifies to

$$
\triangle V = \frac{2 \, q_V \, (V')^2}{1 - 4 \, q_V \, V} \,,
\tag{2.13}
$$

which allows for absorbing the coupling constant $V \to \tilde{V} = q_V \, V$. The exact solution was found in Ref. [6] and reads

$$
V_0 = \frac{1}{4 \, q_V} \left[1 - \left(1 + \frac{6 \, q_V \, G_{\mathrm{N}} \, M}{r} \right)^{2/3} \right] \,.
\tag{2.14}
$$

The asymptotic expansion away from the source yields

$$
V_0 \simeq -\frac{G_{\mathrm{N}} \, M}{r} + q_V \, \frac{G_{\mathrm{N}}^2 \, M^2}{r^2} - q_V^2 \, \frac{8 \, G_{\mathrm{N}}^3 \, M^3}{3 \, r^3} \,,
\tag{2.15}
$$

so that the Newtonian behaviour is always recovered and the post-Newtonian terms are seen to depend on the coupling q_V. The value of q_V can be constrained by experimental bounds once we compute trajectories to compare with.

3 Harmonic and areal coordinates for static spherical systems

The argument leading to the potential (2.14) starting from a general metric involves several approximations, which makes it impossible to determine the starting metric uniquely. In order to reconstruct a metric compatible with Eq. (2.14), we will therefore have to supply further conditions. Before we get to that point, however, we need to discuss in details the relation between the

radial coordinate r used to express the potential in the previous section and the areal coordinate $\bar{r}$ usually employed to write the general static spherically symmetric metric as

$$\mathrm{d}s^2 = -\bar{B}\,\mathrm{d}\bar{t}^2 + \bar{A}\,\mathrm{d}\bar{r}^2 + \bar{r}^2\,\mathrm{d}\Omega^2 \,, \tag{3.1}$$

where $\bar{A} = \bar{A}(\bar{r})$, $\bar{B} = \bar{B}(\bar{r})$, and $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$ is the usual solid angle on the unit sphere, with $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$.

Cartesian coordinates $x^i = (x, y, z)$ in flat space satisfy Eq. (2.6). This condition can be extended to general space-times by defining harmonic coordinates $x^\mu = (t, x, y, z) = (t, \boldsymbol{x})$ such that

$$\Box x^\mu = g^{\alpha\beta}\,\Gamma^\mu_{\alpha\beta} = 0 \,, \tag{3.2}$$

which coincides with the de Donder gauge condition (2.5). In particular, we are interested in spherically symmetric space-times with a metric of the form (3.1) and we therefore find it convenient to employ polar coordinates associated to the harmonic ones by

$$x = r(\bar{r})\,\sin\theta\,\cos\phi \,, \quad y = r(\bar{r})\,\sin\theta\,\sin\phi \,, \quad z = r(\bar{r})\,\cos\theta \,, \tag{3.3}$$

where we assume that the "harmonic" [4] r is an invertible smooth function of the areal coordinate $\bar{r}$. A straightforward calculation of Eq. (3.2) reveals that the function $r = r(\bar{r})$ must satisfy [16]

$$\frac{\mathrm{d}}{\mathrm{d}\bar{r}}\left(\bar{r}^2\,\sqrt{\frac{\bar{B}}{\bar{A}}}\,\frac{\mathrm{d}r}{\mathrm{d}\bar{r}}\right) = 2\,\sqrt{\bar{A}\,\bar{B}}\,r \,. \tag{3.4}$$

Expressing the metric (3.1) in terms of the the rotationally invariant forms $\mathrm{d}\boldsymbol{x}^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$ and $(\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x})^2 = r^2\,\mathrm{d}r^2$, we deduce that the line element in harmonic coordinates reads

$$\mathrm{d}s^2 = -B\,\mathrm{d}t^2 + \frac{\bar{r}^2}{r^2}\,\mathrm{d}\boldsymbol{x}^2 + \left[A\left(\frac{\mathrm{d}\bar{r}}{\mathrm{d}r}\right)^2 - \frac{\bar{r}^2}{r^2}\right]\frac{(\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x})^2}{r^2} \,, \tag{3.5}$$

where $\mathrm{d}t = \mathrm{d}\bar{t}$, $\bar{r} = \bar{r}(r)$, $A = \bar{A}(\bar{r}(r))$ and $B = \bar{B}(\bar{r}(r))$.

The unique Schwarzschild solution of the Einstein field equations in the vacuum outside a spherical source [5] is given by

$$\bar{B}_\mathrm{S} = \frac{1}{\bar{A}_\mathrm{S}} = 1 - \frac{R_\mathrm{H}}{\bar{r}} \,, \tag{3.6}$$

where

$$R_\mathrm{H} = 2\,G_\mathrm{N}\,M \tag{3.7}$$

is the gravitational radius. By solving Eq. (3.4), one finds that the harmonic radial coordinate for the Schwarzschild metric is simply given by

$$r = \bar{r} - \frac{R_\mathrm{H}}{2} \equiv \bar{r} - r_\mathrm{S} \,, \tag{3.8}$$

---

[4]Polar coordinates do not satisfy Eq. (3.2) even in Minkowski space-time, but we shall refer to r as the "harmonic" radial coordinate for the sake of brevity.

[5]Birkhoff's theorem ensures that uniqueness follows from spherical symmetry. In more general cases, other vacuum solutions can be obtained from the linearised solutions [18].
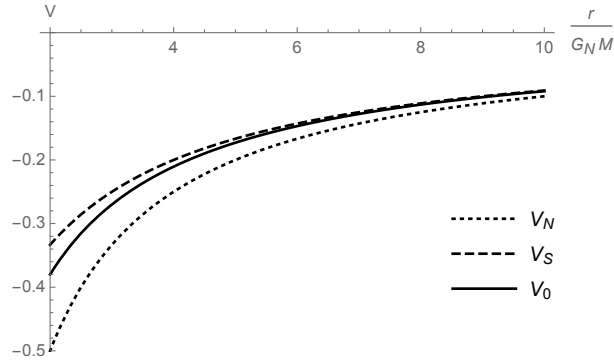
Figure 1: Comparison between the Newtonian V_{N}, Schwarzschild V_{S} and bootstrapped Newtonian V_0 (with $q_V = 1$).

which leads to the potential for the Schwarzschild metric in harmonic coordinates

$$V_{\mathrm{S}} = \frac{1}{2}\left[B_{\mathrm{S}} - 1\right] = -\frac{G_{\mathrm{N}}\,M}{r}\left(1 + \frac{G_{\mathrm{N}}\,M}{r}\right)^{-1} . \tag{3.9}$$

By comparing with the expansion of V_0 in Eq. (2.15), we then see that the unique prediction of General Relativity is recovered to first order in q_V if $q_V = 1$ (see Fig. 1).

We can now replace V_{S} with the potential V_0 in Eq. (2.14), that is

$$B = 1 + 2\,V_0 , \tag{3.10}$$

and start to reconstruct the bootstrapped metric in the areal coordinate $\bar{r}$. In particular, we notice that the metric coefficient $\bar{B}$ is fully determined by the potential V_0 and the relation $r = r(\bar{r})$. Moreover, the Schwarzschild metric has the important property that $\bar{A}_{\mathrm{S}}\,\bar{B}_{\mathrm{S}} = 1$, which is related with the vanishing of the light-like component of the Ricci tensor, $R_{\mu\nu}\,k^\mu\,k^\nu = 0$ for any $k_\mu\,k^\mu = 0$ [19], and the validity of the Equivalence Principle. Using $\bar{C} \equiv \bar{A}\,\bar{B}$, it is also convenient to rewrite Eq. (3.4) as

$$\bar{r}\,r'' + \left(2 - \frac{\bar{r}\,\bar{C}'}{2\,\bar{C}} + \bar{r}\,\frac{\bar{B}'}{\bar{B}}\right)r' = 2\,\frac{\bar{C}\,r}{\bar{B}\,\bar{r}} , \tag{3.11}$$

where a prime denotes the derivative with respect to $\bar{r}$. This equation determines the relation between $\bar{A}$ and $\bar{r}$, but one equation is not sufficient to determine both $r = r(\bar{r})$ and $\bar{A} = \bar{A}(\bar{r})$ given $B = B(r)$, and we will have to resort to further conditions.

4 Effective space-time picture: weak field

We first analyse the region far from the source by Taylor expanding the metric coefficients and $r = r(\bar{r})$ in powers of the dimensionless ratio $R_{\mathrm{H}}/\bar{r} \sim M/\bar{r}$, that is

$$\bar{A} = 1 + \sum_{k=1} a_k \left(\frac{R_{\mathrm{H}}}{\bar{r}}\right)^k ,$$
$$\bar{B} = 1 + \sum_{k=1} b_k \left(\frac{R_{\mathrm{H}}}{\bar{r}}\right)^k \tag{4.1}$$

8

and

$$\frac{r}{\bar{r}} = 1 + \sum_{k=1} \sigma_k \left(\frac{R_{\mathrm{H}}}{\bar{r}}\right)^k . \tag{4.2}$$

We also introduce

$$\bar{C} = 1 + \sum_{k=1} c_k \left(\frac{R_{\mathrm{H}}}{\bar{r}}\right)^k , \tag{4.3}$$

in which the coefficients a_k's are fully determined by the c_k's and b_k's since $\bar{C} = \bar{A}\,\bar{B}$. The above expressions for $\bar{C}$, $\bar{B}$ and $r/\bar{r}$ solve Eq. (3.11) [equivalently, Eq. (3.4)] at zero order in $R_{\mathrm{H}}/\bar{r}$ and ensure asymptotic flatness for $r \sim \bar{r} \to \infty$. In the following, we will solve Eq. (3.11) in order to determine the metric up to third order in $R_{\mathrm{H}}/\bar{r}$.

At first and second order in $R_{\mathrm{H}}/\bar{r}$ we obtain

$$\begin{aligned}
\sigma_1 &= \frac{b_1}{2} - \frac{3}{4}\,c_1 \\
\sigma_2 &= \frac{c_1}{4}\,(2\,c_1 - b_1) - \frac{c_2}{2} ,
\end{aligned} \tag{4.4}$$

and the third-order equation yields

$$c_3 = \frac{5}{2}\,c_1\,c_2 - \frac{1}{2}\,b_1^2\,c_1 - b_1\,c_2 + b_2\,c_1 + \frac{5}{4}\,b_1\,c_1^2 - 2\,b_3 - \frac{3}{2}\,c_1^3 . \tag{4.5}$$

We can now fix the coefficients b_k to match Eq. (3.10), that is

$$\bar{B} \simeq 1 - \frac{R_{\mathrm{H}}}{r(\bar{r})} + \frac{q_V\,R_{\mathrm{H}}^2}{2\,[r(\bar{r})]^2} - \frac{2\,q_V^2\,R_{\mathrm{H}}^3}{3\,[r(\bar{r})]^3} , \tag{4.6}$$

which yields $b_1 = -1$ and

$$\begin{aligned}
b_2 &= \frac{q_V}{2} - \frac{3}{4}\,c_1 - \frac{1}{2} \\
b_3 &= \frac{q_V}{4}\,(2 + 3\,c_1) - \frac{2}{3}\,q_V^2 - \frac{c_1}{16}\,(8 + c_1) - \frac{c_2}{2} - \frac{1}{4} .
\end{aligned} \tag{4.7}$$

Upon replacing the above expressions in the expansion of $\bar{A}$, we obtain

$$\begin{aligned}
a_1 &= 1 + c_1 \\
a_2 &= \frac{3}{2} - \frac{q_V}{2} + \frac{7}{4}\,c_1 + c_2 \\
a_3 &= \frac{11}{4} + \left(2\,q_V - \frac{5}{2} - \frac{9}{4}\,c_1\right) q_V + \frac{7}{2}\,(c_1 + c_2) + \frac{c_1}{2}\left(5\,c_2 - \frac{17}{8}\,c_1 - 3\,c_1^2\right) .
\end{aligned} \tag{4.8}$$

In order to uniquely fix all of the coefficients in the above expansions from physical considerations, it is useful to introduce the Eddington-Robertson parameterised post-Newtonian (PPN) formalism, in which the metric reads [16]

$$\mathrm{d}s^2 \simeq - \left[1 - \alpha\,\frac{R_{\mathrm{H}}}{\bar{r}} + (\beta - \alpha\,\gamma)\,\frac{R_{\mathrm{H}}^2}{2\,\bar{r}^2} + (\zeta - 1)\,\frac{R_{\mathrm{H}}^3}{\bar{r}^3}\right] \mathrm{d}\bar{t}^2 + \left[1 + \gamma\,\frac{R_{\mathrm{H}}}{\bar{r}} + \xi\,\frac{R_{\mathrm{H}}^2}{\bar{r}^2}\right] \mathrm{d}\bar{r}^2 + \bar{r}^2\,\mathrm{d}\Omega^2 , \tag{4.9}$$

where one can set $\alpha = 1$ by the definition of the gravitational radius (3.7). This is in agreement with $b_1 = -\alpha = -1$ and allows us to identify the first order PPN parameters

$$c_1 = \gamma - 1 \quad \text{and} \quad q_V = \beta + \frac{\gamma - 1}{2} \ . \tag{4.10}$$

Finally, we obtain

$$\bar{B} \simeq 1 - \frac{R_{\mathrm{H}}}{\bar{r}} + (\beta - \gamma) \frac{R_{\mathrm{H}}^2}{2\,\bar{r}^2}$$
$$+ \left[7 + 4\,\beta\,(5 + \gamma) - 32\,\beta^2 - \gamma\,(26 - 7\,\gamma) - 24\,c_2 \right] \frac{R_{\mathrm{H}}^3}{48\,\bar{r}^3} \tag{4.11}$$

and

$$\bar{A} \simeq 1 + \gamma\,\frac{R_{\mathrm{H}}}{\bar{r}} - (\beta - 3\,\gamma - 2\,c_2) \frac{R_{\mathrm{H}}^2}{2\,\bar{r}^2}$$
$$+ \left[5 + 32\,\beta^2 - 4\,\beta\,(9 + \gamma) + 3\,\gamma\,(6 + 15\,\gamma - 8\,\gamma^2) + 8\,c_2\,(2 + 5\,\gamma) \right] \frac{R_{\mathrm{H}}^3}{16\,\bar{r}^3} \tag{4.12}$$

so that

$$\bar{C} \simeq 1 + (\gamma - 1) \frac{R_{\mathrm{H}}}{\bar{r}} + c_2 \frac{R_{\mathrm{H}}^2}{\bar{r}^2} \tag{4.13}$$
$$+ \left[11 + 32\,\beta^2 - 8\,\beta\,(4 - \gamma) - \gamma\,(22 - 59\,\gamma - 36\,\gamma^2) - 12\,c_2\,(1 - 5\,\gamma) \right] \frac{R_{\mathrm{H}}^3}{24\,\bar{r}^3} \ . \tag{4.14}$$

The harmonic radius is also given by

$$r \simeq \bar{r} + \frac{1 - 3\,\gamma}{4}\,R_{\mathrm{H}} + \left(1 - 3\,\gamma + 2\,\gamma^2 - 2\,c_2 \right) \frac{R_{\mathrm{H}}^2}{4\,\bar{r}} \ . \tag{4.15}$$

Experimental data strongly constrain $|\gamma - 1| \simeq |\beta - 1| \ll 1$. Upon assuming $\beta = \gamma = 1$, that is $c_1 = 0$ and $q_V = 1$, we find that the bootstrapped metric which describes the minimum deviation from the Schwarzschild form is given by

$$\bar{B} \simeq 1 - \frac{2\,G_{\mathrm{N}}\,M}{\bar{r}} - 2\,(5 + 6\,c_2) \frac{G_{\mathrm{N}}^3\,M^3}{3\,\bar{r}^3}$$
$$\simeq B_{\mathrm{S}}(\bar{r}) - 2\,(6\,\xi - 1) \frac{G_{\mathrm{N}}^3\,M^3}{3\,\bar{r}^3} \tag{4.16}$$
$$\bar{A} \simeq 1 + \frac{2\,G_{\mathrm{N}}\,M}{\bar{r}} + 4\,(1 + c_2) \frac{G_{\mathrm{N}}^2\,M^2}{\bar{r}^2} + 2\,(9 + 14\,c_2) \frac{G_{\mathrm{N}}^3\,M^3}{\bar{r}^3}$$
$$\simeq A_{\mathrm{S}}(\bar{r}) + (\xi - 1) \frac{G_{\mathrm{N}}^2\,M^2}{\bar{r}^2} + 2\,(14\,\xi - 9) \frac{G_{\mathrm{N}}^3\,M^3}{\bar{r}^3} \ , \tag{4.17}$$

and

$$r \simeq \bar{r} - G_{\mathrm{N}}\,M - 2\,(\xi - 1) \frac{G_{\mathrm{N}}^2\,M^2}{\bar{r}} \ . \tag{4.18}$$

For completeness, we also display the Ricci scalar obtained from the above metric coefficients to next-to-leading order in the $R_{\rm H}/r$ expansion,

$$
\begin{aligned}
\bar{R} \simeq & \left(1 - 4\,c_2 - 5\,\gamma + 4\,\gamma^2\right) \frac{R_{\rm H}^2}{2\,r^4} \\
& - \left(9 + 16\,c_2 - 40\,\beta + 32\,\beta^2 + 14\,\gamma + 16\,c_2\,\gamma + 16\,\beta\,\gamma + 5\,\gamma^2 - 16\,\gamma^3\right) \frac{R_{\rm H}^3}{8\,r^5} \\
\simeq & -2\,c_2 \frac{R_{\rm H}^2}{r^4} - \left(5 + 8\,c_2\right) \frac{R_{\rm H}^3}{2\,r^5} \;,
\end{aligned}
\tag{4.19}
$$

where we set $\beta = \gamma = 1$ in the second expression. Clearly, the above expression of $\bar{R}$ shows that the effective metric increasingly differs from Schwarzschild's $\bar{R}_{\rm S} = 0$ as one goes closer to the source.

In the above, the second order PPN parameters are both determined by the one parameter c_2 as

$$
\xi = 1 + c_2 \;, \quad \text{and} \quad \zeta = 1 - \frac{5 + 6\,c_2}{12} = \frac{13 - 6\,\xi}{12} \;,
\tag{4.20}
$$

so that the combination $\xi = \zeta = 1$ corresponding to the PPN expansion of the Schwarzschild metric is not allowed. We can see that the new contribution to $\bar{A}$ at second order in $R_{\rm H}/\bar{r}$ only vanishes for $c_2 = 0$, but higher-order corrections then cannot be eliminated. Correspondingly, for $\beta = \gamma = 1$, we have

$$
\bar{C} \simeq 1 + (\xi - 1)\frac{R_{\rm H}^2}{\bar{r}^2} + (12\,\xi - 7)\frac{R_{\rm H}^3}{6\,\bar{r}^3} \;,
\tag{4.21}
$$

and the Schwarzschild case $\bar{C} = 1$ cannot be reproduced. In the following, we shall analyse the effects of these second order terms in Eqs. (4.16) and (4.17).

4.1 Effective energy-momentum tensor

Since the effective metric with components (4.16) and (4.17) differs from the Schwarzschild geometry, the space-time must contain a non-vanishing effective spherically symmetric energy-momentum tensor

$$
T_{\mu\nu}^{\rm eff} = \rho^{\rm eff}\,u_\mu\,u_\nu + p_r^{\rm eff}\,r_\mu\,r_\nu + p_t^{\rm eff}\,\theta_\mu\,\theta_\nu + p_t^{\rm eff}\,\phi_\mu\,\phi_\nu \;,
\tag{4.22}
$$

where $\rho^{\rm eff} = \rho^{\rm eff}(\bar{r})$, $p_r^{\rm eff} = p_r^{\rm eff}(\bar{r})$ and $p_t^{\rm eff} = p_t^{\rm eff}(\bar{r})$ are respectively the energy density, the radial pressure and the surface tension of the static effective fluid. In the coordinates $\bar{x}^\mu = (\bar{t}, \bar{r}, \theta, \phi)$ of Eq. (3.1), we also have the tetrad components

$$
u^\mu = \frac{\delta_0^\mu}{\bar{B}^{1/2}} \;, \qquad r^\mu = \frac{\delta_1^\mu}{\bar{A}^{1/2}} \;, \qquad \theta^\mu = \frac{\delta_3^\mu}{\bar{r}} \;, \qquad \phi^\mu = \frac{\delta_4^\mu}{\bar{r}\,\sin\theta} \;.
\tag{4.23}
$$

We can compute the density and pressures from the Einstein tensor,

$$
\begin{aligned}
\rho^{\rm eff} &= T_{\mu\nu}^{\rm eff}\,u^\mu\,u^\nu = \frac{G_{00}}{8\,\pi\,G_{\rm N}\,\bar{B}} = \frac{(\bar{A} - 1)\,\bar{A} + \bar{r}\,\bar{A}'}{8\,\pi\,G_{\rm N}\,\bar{r}^2\,\bar{A}^2} \\
p_r^{\rm eff} &= T_{\mu\nu}^{\rm eff}\,r^\mu\,r^\nu = \frac{G_{11}}{8\,\pi\,G_{\rm N}\,\bar{A}} = \frac{\bar{B} - \bar{C} + \bar{r}\,\bar{B}'}{8\,\pi\,G_{\rm N}\,\bar{r}^2\,\bar{C}} \\
p_t^{\rm eff} &= T_{\mu\nu}^{\rm eff}\,\theta^\mu\,\theta^\nu = \frac{G_{22}}{8\,\pi\,G_{\rm N}\,\bar{r}^2} = \frac{2\,\bar{C}\left(2\,\bar{B}' + \bar{r}\,\bar{B}''\right) - \left(2\,\bar{B} + \bar{r}\,\bar{B}'\right)\bar{C}'}{32\,\pi\,G_{\rm N}\,\bar{r}\,\bar{C}^2} \;,
\end{aligned}
\tag{4.24}
$$

where a prime denotes differentiation with respect to $\bar{r}$. The above expressions of course vanish for the Schwarzschild metric, whereas we obtain [6]

$$\rho^{\text{eff}} \simeq \frac{G_{\text{N}} M^2}{2 \pi \bar{r}^4} \left[1 - \xi + (1 - 6\xi) \frac{G_{\text{N}} M}{\bar{r}} \right] \tag{4.25}$$

$$p_r^{\text{eff}} \simeq (1 - \xi) \frac{G_{\text{N}} M^2}{2 \pi \bar{r}^4} \left(1 + \frac{2 G_{\text{N}} M}{\bar{r}} \right)$$

$$\simeq -p_t^{\text{eff}} . \tag{4.26}$$

For $\xi = 1$ (that is $c_2 = 0$), the pressure and tension vanish, at this order of approximation, but one is still left with a negative energy density.

4.1.1 Energy conditions

One can now check if the effective source satisfies (some of) the energy conditions. Since $p_r \simeq -p_t$ the effective fluid is in general anisotropic. In particular, for anisotropic fluids, the null energy condition is implied by all other energy conditions and requires

$$0 \le \rho + p_r = \frac{\bar{B} \bar{C}'}{8 \pi G_{\text{N}} \bar{r} \bar{C}^2} \tag{4.27}$$

$$0 \le \rho + p_t = \frac{2 \bar{r}^2 \bar{C} \bar{B}'' - \bar{r}^2 \bar{B}' \bar{C}' + \bar{B} \left(2 \bar{r} \bar{C}' - 4 \bar{C} \right) + 4 \bar{C}^2}{32 \pi G_{\text{N}} \bar{r}^2 \bar{C}^2} , \tag{4.28}$$

where primes again denote differentiation with respect to $\bar{r}$.

For $\beta = \gamma = 1$, we have

$$\rho + p_r \simeq \frac{G_{\text{N}} M^2}{\pi \bar{r}^4} \left[1 - \xi + (3 - 8\xi) \frac{G_{\text{N}} M}{2 \bar{r}} \right] \tag{4.29}$$

$$\rho + p_t \simeq -(1 + 4\xi) \frac{G_{\text{N}}^2 M^3}{2 \pi \bar{r}^5} , \tag{4.30}$$

and, in order to enforce the above conditions (4.27) and (4.28) for $\bar{r} \gg R_{\text{H}}$, we would then need $\xi < -1/4$ (that is, $c_2 < -5/4$). The case $\xi = 1$ (or $c_2 = 0$) of minimal deviation from the Schwarzschild metric necessarily violates the classical energy conditions. In principle, this conclusion is in line with the original idea that the effective metric should incorporate corrections stemming from quantum physics. The fact that the effective energy-momentum tensor does not vanish at large distance from the source means that quantum effects associated with a localised source will affect the space-time even at much larger scales.

4.1.2 Misner-Sharp-Hernandez mass

It is also interesting to cast the above result in terms of the Misner-Sharp-Hernandez mass [20–22]

$$m(\bar{r}) = \frac{\bar{r}}{2 G_{\text{N}}} \left(1 - \frac{1}{A(\bar{r})} \right) , \tag{4.31}$$

---

[6]The general expressions in terms of Eddington-Robertson parameters is given in Appendix A.

which is known to play an important role in the study of the viability of quantum and quantum-corrected black hole solutions (see *e.g.* [23, 24] and references therein).[7] For $\beta = \gamma = 1$, we find

$$m(\bar{r}) \simeq M \left[1 + (\xi - 1) \frac{2 G_{\mathrm{N}} M}{\bar{r}} + (6\xi - 1) \frac{G_{\mathrm{N}}^2 M^2}{\bar{r}^2} \right] , \tag{4.32}$$

which equals

$$m(\bar{r}) = M_{\mathrm{s}} + 4\pi \int_{\bar{r}_{\mathrm{s}}}^{\bar{r}} \rho^{\mathrm{eff}}(x) x^2 \, \mathrm{d}x , \tag{4.33}$$

where $\bar{r}_{\mathrm{s}} \gg R_{\mathrm{H}}$ is the areal radius of the source of mass $M_{\mathrm{s}} = m(\bar{r}_{\mathrm{s}})$. For $\xi \geq 1$ (or $c_2 \geq 0$), one therefore finds that the asymptotic ADM [25] mass $m(\bar{r} \to \infty) = M < M_{\mathrm{s}}$ (the effective negative energy density screens gravity), whereas for $\xi < 1$ (or $c_2 < 0$) we have $M > M_{\mathrm{s}}$ (the positive effective energy density causes an anti-screening effect [26]).

4.2   Geodesics

Geodesics $\bar{x}^\mu = \bar{x}^\mu(\lambda)$ in a metric of the form in Eq. (3.1) can be obtained from the Lagrangian

$$2L = \bar{B} \dot{\bar{t}}^2 - \bar{A} \dot{\bar{r}}^2 - \bar{r}^2 \left(\dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2 \right) = k , \tag{4.34}$$

where a dot denotes differentiation with respect to λ. The constant $k = 1$ and $\lambda = \tau$ is the proper time for massive particles, whereas $k = 0$ and λ is an affine parameter for light signals. Staticity and spherical symmetry ensure the existence of the usual integrals of motion, namely

$$\mathcal{E} = \frac{\partial L}{\partial \dot{\bar{t}}} = \bar{B} \dot{\bar{t}} \tag{4.35}$$

and

$$\mathcal{J} = -\frac{\partial L}{\partial \dot{\phi}} = \bar{r}^2 \dot{\phi} , \tag{4.36}$$

which is proportional to the angular momentum around the axis that defines the angle ϕ having chosen the trajectory to lie on the plane $\theta = \pi/2$.

We are now left with just the equations of motion for $\phi = \phi(\tau)$ and $r = r(\tau)$, for which it is easier to use the mass-shell condition (4.34), which we write as

$$\dot{\bar{r}}^2 + \mathcal{V}_{\mathrm{eff}} = \frac{\mathcal{E}^2}{\bar{C}} , \tag{4.37}$$

where the effective potential

$$\mathcal{V}_{\mathrm{eff}} = \frac{1}{\bar{A}} \left(k + \frac{\mathcal{J}^2}{\bar{r}^2} \right) . \tag{4.38}$$

---

[7]It is also worth mentioning that the Misner-Sharp-Hernandez has a role in determining the location of horizons for static spherically symmetric spacetimes, thus providing a straightforward method for the characterization of the causal structure of such spaces (see *e.g.* Ref. [22]).

An interesting feature is that $\bar{C} = \bar{A}\bar{B} \neq 1$ in general, see Eq (4.14), and one therefore expects an energy-dependent term in the acceleration experienced by a particle, in apparent violation of the equivalence principle [27], as predicted by some quantum models of gravity [28].

For the purpose of studying orbits with $\mathcal{J} \neq 0$, it is more useful to parameterise the trajectories with the angle ϕ, and therefore solve

$$\left(\frac{d\bar{r}}{d\phi}\right)^2 = \left(\frac{\dot{\bar{r}}}{\dot{\phi}}\right)^2 . \tag{4.39}$$

We next analyse massive ($k = 1$) and massless ($k = 0$) cases separately.

4.2.1 Perihelion precession

The precession of almost Newtonian orbits of planets and stars ($k = 1$) with *semilatus rectum* ℓ and eccentricity ε can be easily expressed in terms of the PPN parameters. In particular, at first order in R_H/ℓ, one finds [16]

$$\Delta\phi^{(1)} = 2\pi(2 - \beta + 2\gamma)\frac{G_N M}{\ell} , \tag{4.40}$$

which reproduces the General Relativistic result

$$\Delta\phi_S^{(1)} = 6\pi\frac{G_N M}{\ell} \tag{4.41}$$

for $\beta = \gamma = 1$ of the Schwarzschild metric. The second order correction depends on both ξ and ζ and, for $\beta = \gamma = 1$, is given by

$$\begin{aligned}
\Delta\phi^{(2)} &= \pi\left[(41 + 10\xi - 24\zeta) + (16\xi - 13)\frac{\varepsilon^2}{2}\right]\frac{G_N^2 M^2}{\ell^2} \\
&\simeq \pi\left[(37 + 22c_2) + (3 + 16c_2)\frac{\varepsilon^2}{2}\right]\frac{G_N^2 M^2}{\ell^2} \\
&\simeq \Delta\phi_S^{(2)} + 2\pi\left[11\xi - 7 + 4(\xi - 1)\varepsilon^2\right]\frac{G_N^2 M^2}{\ell^2} ,
\end{aligned} \tag{4.42}$$

where we used Eq. (4.20), and the General Relativistic result $\Delta\phi_S^{(2)}$ corresponds to $\xi = \zeta = 1$. We see that, in the minimal case with $c_2 = 0$, we have $\xi = 1$ but $\zeta \neq 1$, and a correction remains which is independent of the eccentricity. Binary systems could therefore be employed in order to test the effective bootstrapped Newtonian metric at the second PPN order.

4.2.2 Light deflection

For light signals ($k = 0$), one can likewise express the weak lensing angle for a trajectory reaching the minimum areal radius $\bar{r}_0$ from infinity in terms of the PPN parameters. At first order in $R_H/\bar{r}_0$, we have [16]

$$\Delta\phi^{(1)} = (1 + \gamma)\frac{2G_N M}{\bar{r}_0} , \tag{4.43}$$

14

which reproduces the result from the Schwarzschild geometry for $\gamma = 1$ by construction. The second order correction for $\beta = \gamma = 1$, however, only depends on ξ and is given by

$$
\begin{aligned}
\Delta\phi^{(2)} &= \left[\left(\frac{11}{4} + \xi\right)\pi - 4\right]\frac{G_N^2\,M^2}{\bar{r}_0^2} \\
&\simeq \left[(15 + 4\,c_2)\,\pi - 16\right]\frac{G_N^2\,M^2}{4\,\bar{r}_0^2} \\
&\simeq \Delta\phi_S^{(2)} + (\xi - 1)\,\pi\,\frac{G_N^2\,M^2}{\bar{r}_0^2}\ ,
\end{aligned}
\tag{4.44}
$$

which equals the General Relativistic result in the minimal case $\xi = 1$ (or $c_2 = 0$). This shows that light is not significantly affected and weak gravitational lensing cannot be efficiently used to test the bootstrapped Newtonian metric.

4.2.3   Time delay

The radial equation (4.37) for $\beta = \gamma = 1$ reads

$$
\begin{aligned}
\dot{\bar{r}}^2 &\simeq -k\left\{1 - \frac{2\,G_N\,M}{\bar{r}}\left[1 + \frac{2\,c_2\,G_N\,M}{\bar{r}} + (5 + 6\,c_2)\frac{G_N^2\,M^2}{\bar{r}^2}\right]\right\} - \frac{\mathcal{J}^2}{\bar{r}^2}\left(1 - \frac{2\,G_N\,M}{\bar{r}}\right) \\
&\quad + \mathcal{E}^2\left\{1 - \frac{4\,G_N^2\,M^2}{\bar{r}^2}\left[c_2 + (5 + 12\,c_2)\frac{G_N\,M}{3\,\bar{r}}\right]\right\}\ ,
\end{aligned}
\tag{4.45}
$$

which, even for the minimal deviation with $c_2 = 0$, contains an additional term proportional to $\mathcal{E}^2$. This terms will give rise to an additional acceleration

$$
\ddot{\bar{r}} \sim \frac{G_N^3\,M^3}{\bar{r}^4}\,\mathcal{E}^2\ ,
\tag{4.46}
$$

which will affect the time of flight of both massive and light signals compared to the General Relativistic expectation.

Let us consider, in particular, a trajectory with $\mathcal{J} = 0$ between $\bar{r}_1 = \bar{r}(\lambda_1)$ and $\bar{r}_2 = \bar{r}(\lambda_2) > \bar{r}_1$. Eq. (4.37) with $c_2 = 0$ then reads

$$
\dot{\bar{r}}^2 \simeq -k\left[1 - \frac{2\,G_N\,M}{\bar{r}}\left(1 + \frac{5\,G_N^2\,M^2}{\bar{r}^2}\right)\right] + \mathcal{E}^2\left(1 - \frac{20\,G_N^3\,M^3}{3\,\bar{r}^3}\right)\ ,
\tag{4.47}
$$

For light signals, since $k = 0$,

$$
\left(1 + \frac{10\,G_N^3\,M^3}{3\,\bar{r}^3}\right)\dot{\bar{r}} \simeq \mathcal{E}\ ,
\tag{4.48}
$$

which yields

$$
\begin{aligned}
\lambda_2 - \lambda_1 &\simeq \frac{\bar{r}_2 - \bar{r}_1}{\mathcal{E}}\left[1 + \frac{5\,G_N^3\,M^3}{3\,\bar{r}_1^2\,\bar{r}_2^2}(\bar{r}_1 + \bar{r}_2)\right] \\
&\equiv \Delta\lambda\left(1 + \frac{\delta\lambda}{\Delta\lambda}\right)\ .
\end{aligned}
\tag{4.49}
$$

The expected relative time delay $\delta\lambda/\Delta\lambda$ for light signals is therefore independent of $\mathcal{E}$.

5 Effective space-time picture: near horizon

The task of reconstructing a metric from the potential (2.14) is more challenging near the horizon, as we have far less experimental constraints to rely upon. Moreover, we need to first discuss how the horizon would be determined by the potential in harmonic coordinates. For the Schwarzschild solution (3.6), the horizon areal radius is given by $\bar{r} = R_{\mathrm{H}}$, which corresponds to the harmonic radius $r_{\mathrm{S}} = R_{\mathrm{H}}/2 = G_{\mathrm{N}} M$, according to Eq. (3.8). The potential (3.9) then takes the value

$$V_{\mathrm{S}}(r_{\mathrm{S}}) = -1/2 \ , \tag{5.1}$$

in agreement with the Newtonian concept of escape velocity being equal to the speed of light.

In Refs. [6,8,9,13], we relied on this result and likewise defined the horizon as the radius where the escape velocity equals the speed of light for the bootstrapped Newtonian potential, that is

$$2\, V_0(r_{\mathrm{H}}) = -1 \ , \tag{5.2}$$

which yields

$$r_{\mathrm{H}} = \frac{6\, q_V\, G_{\mathrm{N}}\, M}{(1 + 2\, q_V)^{3/2} - 1} \ , \tag{5.3}$$

provided $q_V > 0$. Note also that

$$\lim_{q_V \to 0} r_{\mathrm{H}} = R_{\mathrm{H}} \ , \tag{5.4}$$

which is twice the Schwarzschild value $r_{\mathrm{S}} = R_{\mathrm{H}}/2$. Considering Eq. (4.10) and the constraints on the PPN parameters γ and β from the weak-field regime, we must have $q_V \simeq 1$. In particular, for the minimal deviation from Schwarzschild given by $q_V = 1$, we have

$$r_{\mathrm{H}} = \frac{6\, G_{\mathrm{N}}\, M}{3\sqrt{3} - 1} \simeq 1.43\, G_{\mathrm{N}}\, M \ , \tag{5.5}$$

which is also significantly larger than the corresponding harmonic Schwarzschild radius r_{S}.

Since $R_{\mathrm{H}}/r_{\mathrm{H}} \sim 1$, the perturbative PPN expansion (4.1) cannot be effectively extended into the near-horizon region. We instead have

$$B = 1 + 2\, V_0 = \left(1 - \frac{r_{\mathrm{H}}}{r}\right) \mathcal{B} \ , \tag{5.6}$$

where $\mathcal{B} = \mathcal{B}(r)$ is a regular and strictly positive function for $r \geq r_{\mathrm{H}}$, which can be Taylor expanded as

$$\mathcal{B} = \sum_{k=0}^{\infty} \beta_k \left(\frac{r - r_{\mathrm{H}}}{r_{\mathrm{H}}}\right)^k \ . \tag{5.7}$$

Of course, the coefficients β_k are fully determined from the explicit form of V_0, although their expressions are rather cumbersome. The first few ones, for instance, are given by

$$\begin{aligned}
\beta_0 &= \frac{(1 + 2\, q_V)^{3/2} - 1}{3\, q_V\, \sqrt{1 + 2\, q_V}} \simeq 0.81 \\
\beta_1 &= \frac{q_V\left(3 + 6\, q_V + 4\, q_V^2\right) - (1 + 2\, q_V)^{3/2}}{9\, q_V\, (1 + 2\, q_V)^2} \simeq 0.11 \\
\beta_2 &\simeq -0.07 \ ,
\end{aligned} \tag{5.8}$$

16

where the numerical estimates are obtained by setting $q_V = 1$.

In order to change to the standard coordinates, we similarly expand the harmonic coordinate r around $\bar{r}_H \equiv \bar{r}(r_H)$ as

$$r = \rho_0 \, \bar{r}_H + \bar{r}_H \sum_{k=1} \rho_k \left(\frac{\bar{r} - \bar{r}_H}{\bar{r}_H} \right)^k , \tag{5.9}$$

where $r_H = \rho_0 \, \bar{r}_H$ is again the harmonic horizon radius in Eq. (5.3). By inserting Eq. (5.9) into Eq. (5.6), one can write

$$\bar{B} = \left(1 - \frac{\bar{r}_H}{\bar{r}} \right) \bar{\mathcal{B}} , \tag{5.10}$$

with

$$\bar{\mathcal{B}} = \sum_{k=0} \mathcal{B}_k \left(\frac{\bar{r} - \bar{r}_H}{\bar{r}_H} \right)^k , \tag{5.11}$$

where the coefficients $\mathcal{B}_k$ are determined by the known β_j's in Eq. (5.7) and the still undetermined ρ_j's in Eq. (5.9). We notice in particular that $\bar{B}(\bar{r} > \bar{r}_H) > 0$ implies that $\mathcal{B}_0 > 0$ and each $\mathcal{B}_k$ depends on the $\rho_{j \leq k+1}$'s, which quickly makes all expressions very cumbersome.

In order to have a proper event horizon, we must require that both $\bar{B}$ and $\bar{A}$ become negative for $\bar{r} < \bar{r}_H$. We thus assume [8]

$$\bar{A} = \left(1 - \frac{\bar{r}_H}{\bar{r}} \right)^{-1} \bar{\mathcal{A}} , \tag{5.12}$$

where the function $\bar{\mathcal{A}}$ is also regular and strictly positive for $\bar{r} \geq \bar{r}_H$ and can be expanded as

$$\bar{\mathcal{A}} = \sum_{k=0} \mathcal{A}_k \left(\frac{\bar{r} - \bar{r}_H}{\bar{r}_H} \right)^k , \tag{5.13}$$

where $\mathcal{A}_0 > 0$. It follows that $\bar{C} = \bar{\mathcal{A}} \, \bar{\mathcal{B}}$ and, upon replacing into Eq. (3.11), we obtain

$$\bar{r} \, r'' - \frac{2 \, \bar{\mathcal{A}} \, r}{\bar{r} - \bar{r}_H} + \left(2 + \frac{\bar{r}_H}{\bar{r} - \bar{r}_H} + \frac{\bar{r} \, \bar{\mathcal{B}}'}{2 \, \bar{\mathcal{B}}} - \frac{\bar{r} \, \bar{\mathcal{A}}'}{2 \, \bar{\mathcal{A}}} \right) r' = 0 , \tag{5.14}$$

where primes again denote derivatives with respect to $\bar{r}$. In principle, this equation can be solved order by order in $(\bar{r} - \bar{r}_H)$, thus relating the coefficients $\mathcal{A}_k$ to the $\mathcal{B}_k$'s and ρ_k's (equivalently, to the β_k's and ρ_k's).

At leading order, for $\bar{r} \simeq \bar{r}_H$, we have

$$\frac{\bar{r}_H}{\bar{r} - \bar{r}_H} \left(\rho_1 - 2 \, \rho_0 \, \mathcal{A}_0 \right) \simeq 0 , \tag{5.15}$$

which implies

$$\mathcal{A}_0 = \frac{\rho_1}{2 \, \rho_0} . \tag{5.16}$$

---

[8]Note that we require that the determinant of the metric $g \sim \bar{A} \, \bar{B}$ is regular everywhere for $\bar{r} \geq \bar{r}_H$.

At next to leading order, we then have

$$
\begin{aligned}
\rho_2 &= \rho_0 \, \mathcal{A}_1 + \rho_1 \, \mathcal{A}_0 - \frac{\rho_1}{4} \left(4 + \frac{\mathcal{B}_1}{\mathcal{B}_0} - \frac{\mathcal{A}_1}{\mathcal{A}_0} \right) \\
&= \frac{\rho_0}{2} \left[3 \, \mathcal{A}_1 - \mathcal{A}_0 \left(4 - 4 \, \mathcal{A}_0 + \frac{\mathcal{B}_1}{\mathcal{B}_0} \right) \right] ,
\end{aligned}
\tag{5.17}
$$

where we recall that $\mathcal{A}_0$ and $\mathcal{B}_0$ must be positive. In particular, if $\rho_1 \simeq 1$ and $|\rho_2| \ll 1$, [9] we must have

$$
\mathcal{A}_0 \simeq \frac{1}{2 \, \rho_0} , \qquad \mathcal{A}_1 \simeq \frac{2}{3 \, \rho_0} \left(1 - \frac{1}{2 \, \rho_0} - \frac{\mathcal{B}_1}{4 \, \mathcal{B}_0} \right) ,
\tag{5.18}
$$

where the known and exact coefficient

$$
\mathcal{B}_0 = \frac{\beta_0}{\rho_0} = \frac{(1 + 2 \, q_V)^{3/2} - 1}{3 \, \rho_0 \, q_V \, \sqrt{1 + 2 \, q_V}} ,
\tag{5.19}
$$

and, since $\mathcal{B}_1$ depends also on ρ_2, we do not show its rather long expression here.

It is important to remark that the unknown coefficients $\mathcal{A}_k$'s depend on the coefficients ρ_k's, both through Eq. (5.14) and because the $\mathcal{B}_k$'s depend on the ρ_k's. The only way to fix this ambiguity, related with the expression of the harmonic $r = r(\bar{r})$, on physical grounds is to match the near-horizon expressions of the metric components $\bar{A}$ and $\bar{B}$ with their analogue in the weak-field regime. The latter was obtained previously by imposing observational constraints to partly fix $r = r(\bar{r})$ therein. The matching between the two regimes will therefore leave unspecified only those parameters which do not conflict with the experimental bounds at large distance from the source.

5.1 Matching with weak field

Let us start from noting that the Taylor expansion for the near-horizon regime is comparable with the one for weak field when

$$
\frac{\bar{r} - \bar{r}_{\mathrm{H}}}{\bar{r}_{\mathrm{H}}} \simeq \frac{R_{\mathrm{H}}}{\bar{r}} ,
\tag{5.20}
$$

or $\bar{r} \simeq \bar{r}_{\mathrm{m}}$, with

$$
\begin{aligned}
\bar{r}_{\mathrm{m}} &= \frac{\bar{r}_{\mathrm{H}}}{2} \left(1 + \sqrt{1 + 4 \, \frac{R_{\mathrm{H}}}{\bar{r}_{\mathrm{H}}}} \right) \\
&= \frac{r_{\mathrm{H}}}{2 \, \rho_0} \left(1 + \sqrt{1 + 4 \, \rho_0 \, \frac{R_{\mathrm{H}}}{r_{\mathrm{H}}}} \right) ,
\end{aligned}
\tag{5.21}
$$

where we recall that $\rho_0 > 0$ and the harmonic r_{H} is given in Eq. (5.3). Moreover, the first few terms in the two expansions still provide a reliable approximation at $\bar{r} = \bar{r}_{\mathrm{m}}$ if

$$
\frac{R_{\mathrm{H}}}{\bar{r}_{\mathrm{m}}} = 2 \, \rho_0 \, \frac{R_{\mathrm{H}}}{r_{\mathrm{H}}} \left(1 + \sqrt{1 + 4 \, \rho_0 \, \frac{R_{\mathrm{H}}}{r_{\mathrm{H}}}} \right)^{-1} \lesssim 1 .
\tag{5.22}
$$

---

[9]We will see next that this is a rather accurate estimate.

The above condition is satisfied for

$$\rho_0 \lesssim \rho_c \equiv 2 \frac{r_H}{R_H} = \frac{6 \, q_V}{(1 + 2 \, q_V)^{3/2} - 1} \ . \tag{5.23}$$

In particular, by matching the expressions of the harmonic coordinates (4.2) and (5.9) at $\bar{r} = \bar{r}_m$, that is

$$
\begin{aligned}
\bar{r}_m - \frac{R_H}{2} + \bar{r}_m \sum_{k=2} \sigma_k \left(\frac{R_H}{\bar{r}_m} \right)^k &= \rho_0 \, \bar{r}_H + \bar{r}_H \sum_{k=1} \rho_k \left(\frac{\bar{r}_m - \bar{r}_H}{\bar{r}_H} \right)^k \\
&= \rho_0 \, \bar{r}_H + \rho_1 \, (\bar{r}_m - \bar{r}_H) + \bar{r}_H \sum_{k=2} \rho_k \left(\frac{R_H}{\bar{r}_m} \right)^k \ ,
\end{aligned}
\tag{5.24}
$$

we obtain

$$\rho_0 = (1 - \rho_1) \frac{\bar{r}_m}{\bar{r}_H} + \rho_1 - \frac{R_H}{2 \, \bar{r}_H} - \sum_{k=2} \left(\rho_k - \frac{\bar{r}_m}{\bar{r}_H} \sigma_k \right) \left(\frac{R_H}{\bar{r}_m} \right)^k \ . \tag{5.25}$$

At leading order, we thus find

$$\rho_0 \simeq (1 - \rho_1) \frac{\bar{r}_m}{\bar{r}_H} + \rho_1 - \frac{R_H}{2 \, \bar{r}_H} \ . \tag{5.26}$$

This estimate can be further improved by considering yet another expansion about $\bar{r} = \bar{r}_m$ and determining the corresponding Taylor coefficients from the matching with the weak-field expansion for $\bar{r} \gtrsim \bar{r}_m$ and with the near-horizon expansion for $\bar{r} \lesssim \bar{r}_m$. This is equivalent to imposing continuity of the function $r = r(\bar{r})$ and its derivatives across $\bar{r}_m$ (see Appendix B). We remark here that the result for $|c_2| = |\xi - 1| \lesssim 1$ is consistent with the above expressions for $\rho_1 \simeq 1$ and $|\rho_2| \ll 1$.

5.2 Near-horizon geometry

The better estimate of ρ_0 in Eq. (B.11) yields for the areal radius of the bootstrapped Newtonian horizon

$$\bar{r}_H = \frac{r_H}{\rho_0} \simeq (1.21 + 0.27 \, c_2) R_H \ . \tag{5.27}$$

The value of $c_2 = c_2^S$ which would give $\bar{r}_H = R_H = 2 \, G_N \, M$ according to this equation is outside our range of approximation (namely, $|c_2| \ll 1$). In fact, resorting to Eq. (B.6), we obtain $c_2^S \approx -0.696$, corresponding to $\xi = c_2^S + 1 \simeq 0.3$.

On using Eqs. (B.11), (5.19) and (5.16), we obtain

$$
\begin{aligned}
\mathcal{B}_0 &\simeq 1.37 + 0.50 \, c_2 \\
\mathcal{A}_0 &\simeq 0.85 + 0.31 \, c_2 \ .
\end{aligned}
\tag{5.28}
$$

In particular, for $c_2 = 0$, we find $\rho_1 \simeq 1$ and Eq. (5.26) yields the same relation between the harmonic and the areal horizon radii which holds for the Schwarzschild solution, that is

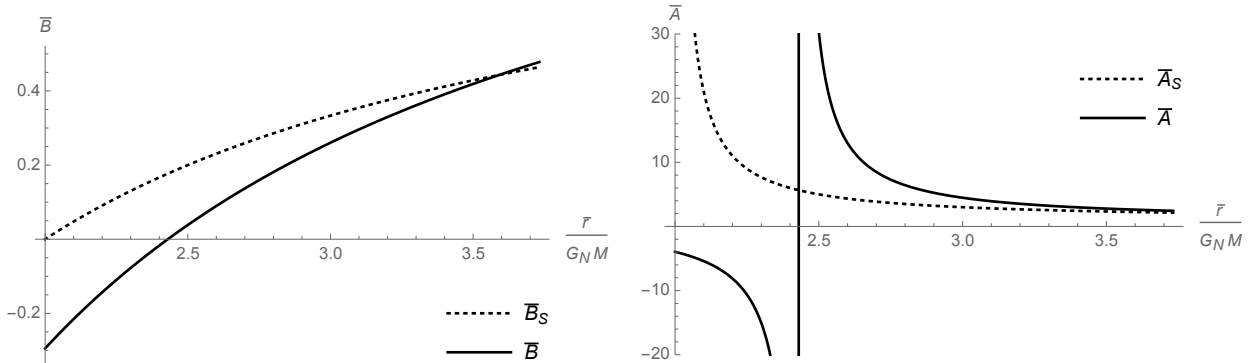$$\bar{r}_H \simeq r_H + \frac{R_H}{2} \ . \tag{5.29}$$

Figure 2: Comparison between the Schwarzschild and bootstrapped Newtonian metric components for $R_{\rm H} < \bar{r} < \bar{r}_{\rm m}$. The vertical line in the right panel is the location of the bootstrapped horizon $\bar{r} = \bar{r}_{\rm H}$.

From the bootstrapped potential we thus obtain

$$\bar{r}_{\rm H} \simeq \frac{(1 + 2\,q_V)^{3/2} - 1 + 6\,q_V}{(1 + 2\,q_V)^{3/2} - 1}\, G_{\rm N}\, M \simeq 2.43\, G_{\rm N}\, M \ , \tag{5.30}$$

where the last value is for $q_V = 1$. The corresponding metric coefficients $\bar{B}$ and $\bar{A}$ at leading order in the near-horizon expansion are shown in Fig. 2, where they are compared with their Schwarzschild analogues. The only relevant difference is given by the areal radius of the bootstrapped horizon. For this reason we plot $\bar{r}_{\rm H}$ in units of $R_{\rm H}$ in Fig. 3, and note that $\bar{r}_{\rm H} = R_{\rm H}$ for

$$q_V = \frac{3 + 2\sqrt{3}}{2} \simeq 3.23 \ . \tag{5.31}$$

Clearly, this much stronger self-coupling would not be compatible with the weak-field bounds, further supporting the result that the bootstrapped Newtonian metric contains an horizon $\bar{r}_{\rm H}$ larger than Schwarzschild's $R_{\rm H}$.

Since the matching radius $\bar{r}_{\rm m} \simeq 3.73\, G_{\rm N}\, M$, one can expect a correction for the radius $\bar{r}_{\rm ph}$ of the photon orbit, whose value is $3\, G_{\rm N}\, M$ in General Relativity. Using $\bar{C} \simeq \mathcal{A}_0\, \mathcal{B}_0 \simeq 1.17$ and constant,
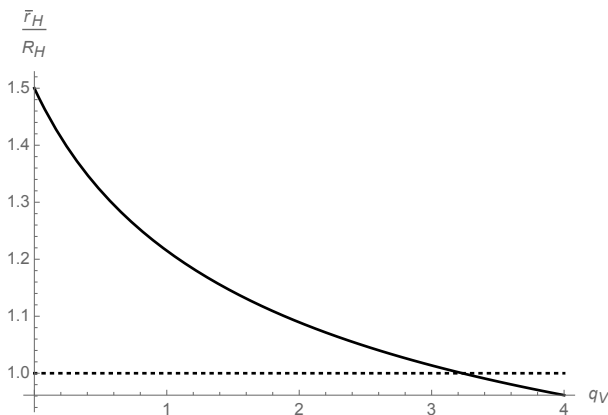

Figure 3: Bootstrapped horizon $\bar{r}_{\rm H}$ in units of $R_{\rm H}$. The horizontal dotted line is unity.

20

the latter can be estimated from Eq. (4.37) as

$$0 = \mathcal{V}'_{\text{eff}} \sim 3\,\bar{r}_{\text{H}} - 2\,\bar{r}_{\text{ph}} \ , \tag{5.32}$$

where $\mathcal{V}^{\text{eff}}$ is the potential in Eq. (4.38) with $k = 0$ for null trajectories. The result $\bar{r}_{\text{ph}} \simeq 3.64\,G_{\text{N}}\,M$ is just short of $\bar{r}_{\text{m}}$, and a better reconstruction of the near-horizon metric including a few higher order terms $\mathcal{A}_k$ and $\mathcal{B}_k$ is therefore likely to modify this value. In fact, we note that $\bar{C}$ must approach the General Relativistic value $\bar{C} = 1$ rather fast in the weak-field regime, according to Eq. (4.21), and $\bar{C}'$ cannot therefore be neglected near the horizon. For example, if we simply employ a linear approximation for $\bar{\mathcal{B}}$, and take $\rho_1 = 1$ and $\rho_2 = 0$, we get

$$\bar{r}_{\text{ph}} \simeq \frac{3\,\mathcal{B}_0 - 2\,\mathcal{B}_1}{2\,\mathcal{B}_0 - 2\,\mathcal{B}_1}\,\bar{r}_{\text{H}} \simeq 3.26\,G_{\text{N}}\,M \ , \tag{5.33}$$

which is closer to the prediction of General Relativity.

On the other hand, the innermost stable circular orbit of General Relativity is located at $\bar{r}_{\text{ISCO}} = 6\,G_{\text{N}}\,M$, and its location in the bootstrapped Newtonian metric should instead be recovered rather accurately from the weak-field approximation. From Eq. (4.16) and (4.17) evaluated at $\bar{r} = \bar{r}_{\text{ISCO}}$ we indeed obtain for the deviation of the bootstrapped metric from Schwarzschild's

$$\begin{aligned}
\frac{\bar{B} - \bar{B}_{\text{S}}}{\bar{B}_{\text{S}}} &\simeq \frac{5 + c_2}{216} \simeq 0.02 \\
\frac{\bar{A} - \bar{A}_{\text{S}}}{\bar{A}_{\text{S}}} &\simeq \frac{5 + 17\,c_2}{72} \simeq 0.07 \ ,
\end{aligned} \tag{5.34}$$

where we expect $|c_2| = |\xi - 1| \ll 1$.

5.3 Harmonic and areal compactness

For a source of harmonic radius R in the Schwarzschild space-time, one can introduce the compactness in the harmonic coordinate as

$$X_{\text{S}} \equiv \frac{G_{\text{N}}\,M}{R} \ , \tag{5.35}$$

or in the areal coordinate as

$$\bar{X}_{\text{S}} \equiv \frac{2\,G_{\text{N}}\,M}{\bar{R}} = \frac{2\,X_{\text{S}}}{1 + X_{\text{S}}} \ , \tag{5.36}$$

where we used Eq. (3.8). In particular, $\bar{X}_{\text{S}}(R_{\text{H}}) = X_{\text{S}}(r_{\text{S}}) = 1$ for a Schwarzschild black hole.

For the bootstrapped metric, we could likewise introduce the harmonic compactness

$$X \equiv \frac{r_{\text{H}}}{R} \tag{5.37}$$

and the areal compactness

$$\bar{X} \equiv \frac{\bar{r}_{\text{H}}}{\bar{R}} = \frac{X}{\rho_0 + (1 - \rho_0)\,X} \ , \tag{5.38}$$

in which we employed the leading order transformation (5.9) with $\rho_1 \simeq 1$,

$$r \simeq \rho_0\,\bar{r}_{\text{H}} + \bar{r} - \bar{r}_{\text{H}} \ , \tag{5.39}$$

21

so that $\bar{X}(\bar{r}_{\rm H}) = X(r_{\rm H}) = 1$.

For the purpose of comparing with General Relativity, it is however more convenient to use the Schwarzschild quantities and note that, for a bootstrapped Newtonian black hole

$$X_{\rm S}(r_{\rm H}) = \frac{G_{\rm N}\, M}{r_{\rm H}} = \frac{6\, q_V}{(1 + 2\, q_V)^{3/2} - 1} \simeq 0.70 \qquad (5.40)$$

and

$$\bar{X}_{\rm S}(\bar{r}_{\rm H}) = 2\, \rho_0\, X_{\rm S} = \frac{2\, X_{\rm S}}{1 + X_{\rm S}} \simeq 0.83 \ , \qquad (5.41)$$

where the numerical values are those for $q_V = 1$, as usual. We notice incidentally that this value is just slightly smaller than the Buchdahl limit $\bar{X}_{\rm B} = 8/9 \simeq 0.89$.

6   Conclusions and outlook

The bootstrapped Newtonian approach is devised to capture quantum effects which induce large (mean-field) deviations from classical General Relativity when large matter sources are involved. Such effects would be completely determined if we knew the proper quantum state describing specific self-gravitating systems. What we know for certain is that the strong field regime of gravity governed by the Einstein field equations is not linear. Determining the relevant quantum state therefore requires that one solves nonlinear quantum dynamics, which seems hardly a tenable task for large and very compact sources. The bootstrapped Newtonian approach considers a simplified form of nonlinear dynamics for gravity, compared to General Relativity, but aims at including quantum deviations from classicality in a form that is sufficiently general to confront observations. This generality is manifested in the coupling constants appearing in the action (2.11).

The potential experienced by test particle at rest is however not sufficient to determine all deviations from the classical solutions of General Relativity. Starting from the bootstrapped Newtonian potential outside a static and spherically symmetric source, we here obtained a complete metric by supplying further conditions of compatibility with observations in the weak-field regime. The main difference with respect to the unique Schwarzschild solution of General Relativity. is given by the larger horizon radius estimated in Eq. (5.30). This prediction makes the bootstrapped Newtonian programme experimentally testable, for instance, by measurements of light trajectories reaching the photon orbit. A more detailed analysis of these trajectories in terms of the parameters of the effective metric is the natural continuation of the work presented here.

A possible conclusion of such a phenomenological analysis could be that a consistent description of the near-horizon region of black holes requires more than the first few nonlinear terms included in the bootstrapped Newtonian Lagrangian (2.11). This possibility will be investigated in the future, but, in this respect, it is important to recall that the entire programme about bootstrap Newtonian gravity is motivated by the idea that black holes and similarly compact sources might require a fully quantum, rather than semiclassical, description. It is therefore not *a priori* clear to what extent the effective metric we obtained is meaningful at such short distances from the (would-be classical) horizon. More precisely, one expects that the interaction of matter and light falling towards the black hole should be described in terms of scattering processes, for which classical geodesic lines will become an unreliable approximation if black holes are indeed extended quantum objects (for a non-exhaustive list, see Refs. [13, 14, 29–34]). This viewpoint will also require a more detailed quantum description of the matter source itself, which is left completely out here.

Finally, we would like to mention that the weak-field regime is also worthy of further study. First of all, there is the possibility that deviations from the Schwarzschild geometry reproduce the kind of effective dark fluid responsible for Dark Matter phenomenology as explored in Refs. [17]. Moreover, propagation of gravitational waves and other signals would also be affected by the non-trivial background corresponding to the effective metric. All of these developments are left for future investigations.

Acknowledgments

A   Weak-field effective energy-momentum tensor

The effective fluid density for general values of the Robertson-Eddington parameters is given by

$$
\rho^{\mathrm{eff}} \simeq \frac{G_{\mathrm{N}}\,M^2}{4\,\pi\,\bar{r}^4} \left\{ \left(\beta - 3\,\gamma + 2\,\gamma^2 - 2\,c_2\right) \right.
$$
$$
\left. - \left[5 + 32\,\beta^2 - 12\,\beta\,(3 - \gamma) + \gamma\,(18 - 3\,\gamma - 8\,\gamma^2) + 8\,c_2\,(2 + \gamma)\right] \frac{G_{\mathrm{N}}\,M}{2\,\bar{r}} \right\} \ , \quad (\mathrm{A}.1)
$$

the pressure by

$$
p_r^{\mathrm{eff}} \simeq \frac{M}{4\,\pi\,\bar{r}^3} \left\{ (1 - \gamma) + \left(2 - \beta - 3\,\gamma + 2\,\gamma^2 - 2\,c_2\right) \frac{G_{\mathrm{N}}\,M}{\bar{r}} \right.
$$
$$
\left. + (1 + \gamma)\left(1 - 3\,\gamma + 2\,\gamma^2 - 2\,c_2\right) \frac{G_{\mathrm{N}}^2\,M^2}{\bar{r}^2} \right\} \ , \quad (\mathrm{A}.2)
$$

and the tension by

$$
p_t^{\mathrm{eff}} \simeq \frac{M}{8\,\pi\,\bar{r}^3} \left\{ (\gamma - 1) + \left(2\,\beta - 3 + 5\,\gamma - 4\,\gamma^2 + 4\,c_2\right) \frac{G_{\mathrm{N}}\,M}{\bar{r}} \right.
$$
$$
\left. + \left[(1 + \gamma)\left(1 - 2\,\beta + \gamma + 6\,\gamma^2\right) + c_2\,(2 + 6\,\gamma)\right] \frac{G_{\mathrm{N}}^2\,M^2}{\bar{r}^2} \right\} \ . \quad (\mathrm{A}.3)
$$

The anisotropy $\Pi \equiv p_r - p_t$ therefore is

$$
\Pi \simeq \frac{M}{8\,\pi\,\bar{r}^3} \left\{ 3\,(1 - \gamma) + \left(7 - 4\,\beta - 11\,\gamma + 8\,\gamma^2 - 8\,c_2\right) \frac{G_{\mathrm{N}}\,M}{\bar{r}} \right.
$$
$$
\left. + \left[(1 - \gamma)\left(1 + 2\,\beta - 3\,\gamma + 10\,\gamma^2\right) - 2\,c_2\,(3 + 5\,\gamma)\right] \frac{G_{\mathrm{N}}^2\,M^2}{\bar{r}^2} \right\} \ . \quad (\mathrm{A}.4)
$$

The Misner-Sharp-Hernandez mass reads

$$
m(\bar{r}) \simeq M \left\{ \gamma - \left(\beta - 3\,\gamma + 2\,\gamma^2 - 2\,c_2\right) \frac{G_{\mathrm{N}}\,M}{\bar{r}} \right.
$$
$$
\left. + \left[5 - 32\,\beta^2 - 12\,\beta\,(3 - \gamma) + 18\,\gamma - 3\,\gamma^2 - 8\,\gamma^3 + 8\,c_2\,(2 + \gamma)\right] \frac{G_{\mathrm{N}}^2\,M^2}{4\,\bar{r}^2} \right\} \ . \quad (\mathrm{A}.5)
$$

For $\beta = \gamma = 1$, the above expressions reduce to those shown in the main text.

B Intermediate range expansion

Let us start by expanding the Schwarzschild metric in harmonic coordinates around the horizon $r_S = R_H/2$. From the general form (3.5) and Eq. (3.8) we obtain

$$B_S = \frac{1}{A_S} \simeq \frac{r - G_N M}{2 G_N M} \ .$$
(B.1)

The analogous expansion for the coefficient B of the bootstrapped Newtonian metric can be derived from Eq. (5.6). To be more specific, we shall only consider the case $q_V = \gamma = 1$ here, which yields

$$B \simeq \frac{r - 1.43 G_N M}{1.77 G_N M} \ .$$
(B.2)

However, we need both ρ_0 and ρ_1 in Eq. (5.9) in order to obtain the leading order expression for the coefficient A.

For this purpose, we expand $r = r(\bar{r})$ around $\bar{r}_m$ defined in Eq. (5.21) as the radius at which the weak-field expansion becomes comparable to the near-horizon one. This intermediate expansion of $r = r(\bar{r})$ can then be constrained by using the weak-field expansion to the right of $\bar{r}_m$ and the near-horizon expansion to the left of $\bar{r}_m$. In particular, continuity of $r = r(\bar{r})$ and of its first few derivatives around $\bar{r}_m$ implies

$$\bar{r}_H \left[\rho_0 + \rho_1 \frac{R_H}{\bar{r}_m} + \rho_2 \left(\frac{R_H}{\bar{r}_m} \right)^2 + \mathcal{O}(3) \right] = \bar{r}_m \left[1 - \frac{R_H}{2\bar{r}_m} - \frac{c_2}{2} \left(\frac{R_H}{\bar{r}_m} \right)^2 + \mathcal{O}(3) \right]$$
(B.3)

$$\rho_1 + 2\rho_2 \frac{R_H}{\bar{r}_m} + \mathcal{O}(2) = 1 + \frac{c_2}{2} \left(\frac{R_H}{\bar{r}_m} \right)^2 + \mathcal{O}(3)$$
(B.4)

$$\frac{1}{\bar{r}_H} [2\rho_2 + \mathcal{O}(1)] = \frac{1}{\bar{r}_m} \left[-c_2 \left(\frac{R_H}{\bar{r}_m} \right)^2 + \mathcal{O}(3) \right] \ ,$$
(B.5)

where $\mathcal{O}(k)$ denotes a quantity proportional to $(R_H/\bar{r}_m)^k$. Solving these equations gives

$$\rho_0 = 1 - \frac{R_H}{2\bar{r}_m} - \frac{c_2 + 1}{2} \left(\frac{R_H}{\bar{r}_m} \right)^2 - \frac{c_2}{2} \left(\frac{R_H}{\bar{r}_m} \right)^3 + \mathcal{O}(3)$$
(B.6)

$$\rho_1 = 1 + \frac{c_2}{2} \left(\frac{R_H}{\bar{r}_m} \right)^2 + \mathcal{O}(2)$$
(B.7)

$$\rho_2 = \mathcal{O}(1) \ .$$
(B.8)

We next note that

$$\rho_0 = \frac{r_H}{\bar{r}_H} = \frac{r_H}{R_H} \frac{R_H}{\bar{r}_m} \frac{\bar{r}_m}{\bar{r}_H} = \frac{r_H}{R_H} \frac{R_H}{\bar{r}_m} \left(\frac{R_H}{\bar{r}_m} + 1 \right) \ ,$$
(B.9)

where the bootstrapped Newtonian horizon r_H is given in Eq. (5.3) and the last equality follows from the definition (5.21). With this expression for ρ_0, Eq. (B.6) can be used to relate $R_H/\bar{r}_m$ to the weak-field coefficient $c_2 = \xi - 1$, and one can then obtain explicit estimates for ρ_0, ρ_1 and ρ_2. Using again $q_V = 1$, we get

$$\frac{R_H}{\bar{r}_m} \simeq 0.54 - 0.09 c_2 + \mathcal{O}(3) \ ,$$
(B.10)

which is smaller than one, as required for the validity of the truncation. Correspondingly, we have

$$
\begin{aligned}
\rho_0 &\simeq 0.59 - 0.13\, c_2 + \mathcal{O}(3) \\
\rho_1 &\simeq 1 + 0.14\, c_2 + \mathcal{O}(2) \ .
\end{aligned}
\tag{B.11}
$$

From Eq. (B.10), we can further estimate the orders of magnitude of neglected quantities, namely $\mathcal{O}(1) \approx 0.54$, $\mathcal{O}(2) \approx 0.29$ and $\mathcal{O}(3) \approx 0.15$, assuming proportionality constants of order one as well.

Moreover, using the above estimates in Eq. (5.26) yields $\rho_0 \simeq 0.59$ to leading order. This confirms that the direct matching between the weak-field and near-horizon expansions is already rather accurate. In fact, the ratio between the first term that we neglected and the last we included in the direct matching in Eq. (5.24), that is

$$
\left| \frac{\rho_2\, \bar{r}_{\mathrm{H}} - \sigma_2\, \bar{r}_{\mathrm{m}}}{\rho_1\, \bar{r}_{\mathrm{H}} - \sigma_1\, \bar{r}_{\mathrm{m}}} \cdot \frac{R_{\mathrm{H}}}{\bar{r}_{\mathrm{m}}} \right| \simeq 0.08 + 0.19\, c_2 \ ,
\tag{B.12}
$$

is reasonably small in the expected range of values of $c_2 = \xi - 1$.

Finally, Eq. (5.16) with the above estimates yields

$$
A \simeq \frac{(2.06 + 1.50\, c_2)\, G_{\mathrm{N}}\, M}{r - 1.43\, G_{\mathrm{N}}\, M} \ ,
\tag{B.13}
$$

where we used the approximation of small c_2 and neglected all $\mathcal{O}(k)$ terms.

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