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PRODUCTS OF ARITHMETIC MATROIDS AND QUASIPOLYNOMIAL INVARIANTS OF CW-COMPLEXES

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Abstract. In this note we prove that the product of two arithmetic multiplicity functions on a matroid is again an arithmetic multiplicity function. This allows us to answer a question by Bajo–Burdick–Chmutov [3], concerning the modified Tutte–Krushkal–Renhardy polynomials defined by these authors.

Furthermore, we show that the Tutte quasi-polynomial introduced by Brändén and Moci encompasses invariants defined by Beck–Breuer–Godkin–Martin [4] and Duval–Klivans–Martin [1] and can thus be considered as a dichromate for CW complexes.

1. Introduction

The enumeration of colorings, flows and spanning trees on graphs are classical topics, unified by a two-variable polynomial due to W. T. Tutte [23]. This polynomial specializes to both the coloring counting and the flow counting functions, and it evaluates to the number of spanning trees. H. Crapo extended Tutte's definition to arbitrary matroids and since then the *Tutte polynomial* went on to become one of the most studied matroid invariants, with great theoretical significance and a host of applications — e.g., in statistics and physics. Recently, this classical setup has been generalized in two ways.

On the one hand, the concepts of coloring and flow have been generalized from graphs to higher dimensional objects such as simplicial complexes by Beck and Kemper [5] and, more generally, to CW complexes by Beck–Breuer–Godkin–Martin [4] and Duval–Klivans–Martin [1]. These authors showed, among other things, that the functions counting the number of colorings and flows with q values on a CW complex is a quasi-polynomial in q. In a related vein, Bajo, Burdick and Chmutov [3] introduced a family of *modified TKR polynomials* that connects Kalai's enumeration of weighted cellular spanning trees of complexes [14] to a class of polynomials defined by Krushkal and Renhardy [16] in their study of graph embeddings and to a polynomial defined by Bott [6].

On the other hand, in collaboration with M. D'Adderio [9] and with P. Brändén [7] the second-named author developed a theory of arithmetic matroids as "matroids decorated with a multiplicity function", abstracting the arithmetic properties of lists of elements in finitely generated abelian groups. To each arithmetic matroid is naturally associated an *arithmetic Tutte polynomial*. These polynomials have been in the focus of recent and lively research, which brought to light manifold connections and a rich structure theory. For instance, arithmetic Tutte polynomials specialize to Poincaré polynomials of toric arrangements [20], to Ehrhart polynomials of zonotopes [8] and to the Hilbert series of some zonotopal spaces [18]. Moreover, they can be recovered from the Tutte polynomials for group actions on semimatroids [11], and they satisfy a convolution formula [2].

With a list of elements in a finitely generated abelian group is also associated a *Tutte quasi-polynomial* [7], which interpolates between the (ordinary) Tutte polynomial and the arithmetic Tutte polynomial. This quasi-polynomial does not depend only on the arithmetic matroid, but on a finer structure: *a matroid over* $\mathbb{Z}$ in the sense of [13]. As pointed out in [4], the enumerating functions of colorings and flows on a CW complex are not matroidal, and hence cannot be obtained from the ordinary Tutte polynomial. In this paper we show that, however, they are specializations of the Tutte quasi-polynomial, which hence can be viewed as the the *dichromate* for CW complexes, just as the Tutte polynomial is the dichromate for graphs. In the same spirit, one may look for a higher-dimensional analogue of the graphical arrangement associated to a graph. This is an arrangement of subgroups of the torus (see Definition 3.9) and, as such, a special case of a construction studied e.g. by Kamiya, Takemura and Terao [15] and Lawrence [17].

Moreover, we show that the set of arithmetic matroids over a fixed underlying matroid has a natural structure as a commutative monoid. This implies that the modified TKR polynomials are indeed arithmetic Tutte polynomials; in particular, their coefficients are positive.

In this way we address questions of the authors of [1, 3, 4], who ask whether and how the coloring and flow polynomials for CW complexes and the modified TKR polynomials are related to arithmetic matroids.

Structure of the paper. In Section 2 we start off with some preliminaries on incidence algebras and arithmetic matroids. We prove a general theorem about products of integer functions on posets (Theorem 2.2) and specialize it to one about products of arithmetic multiplicity functions (Theorem 2.8).

In Section 3 we recall the definitions of flow and chromatic quasi-polynomials for CW complexes, and we show that they are indeed specializations of the Tutte quasi-polynomial (Theorem 3.6).

We close with Section 4 where we prove that the modified TKR polynomial is the arithmetic Tutte polynomial of an arithmetic matroid (Theorem 4.4).

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2. On arithmetic matroids

2.1. **Poset theory preliminaries.** The goal of this section is to prove a result on Möbius functions of posets (short for "partially ordered sets") which will serve as a stepping stone towards Theorem 2.8. We will assume familiarity with basic terminology of poset theory and refer the reader unfamiliar with it to [22].

Throughout, we will let P denote a finite poset.[1] An *interval* of P is any subset of P of the form $[x,y] := \{z \in P \mid x \leq z \leq y\}$ for some $x, y \in P$, $x \leq y$. The set of intervals of P is denoted $I(P)$.

The so-called *Möbius function* of P is the function

$$\mu : I(P) \to \mathbb{Z}$$

defined recursively as follows

$$\begin{cases} \mu(p,p) = 1 & \text{for all } p \in P, \\ \displaystyle\sum_{p_1 \leq q \leq p_2} \mu(p_1, q) = 0 & \text{for all } p_1 < p_2 \text{ in } P, \end{cases}$$

where for simplicity we write $\mu(x,y) := \mu([x,y])$.

[1] This will avoid unnecessary technicalities and will suffice for the applications later in the paper, even though most of what we will prove in this section holds in the generality of locally finite posets.

Let R be a commutative ring and let $m : P \to R$ be any function. The *(dual) Möbius transform*[2] of m is the function

$$\begin{aligned} m^\mu : P &\to R \\ p &\mapsto \sum_{q \geq p} \mu(p,q) m(q). \end{aligned}$$

It is characterized by $m(p) = \sum_{q\geq p} m^\mu(q)$.

Consider two elements $p, p' \in P$. If there is an element $x \in P$ with

$$\{q \in P \mid q \geq p,\ q \geq p'\} = \{q \in P \mid q \geq x\}$$

then x is unique, called the *meet* (or minimal upper bound) of p and p', and denoted by $p \vee p'$. If every pair $p, p' \in P$ admits a meet, the poset P is called a *meet semilattice*.

The following lemma should be folklore. We give here a proof for completeness, because we do not know of a reference for it.

Lemma 2.1. *Let P be a meet-semilattice, and $D : P \to \mathrm{Sets}$ be a function such that $D(p) \cap D(q) = D(p \vee q)$ for all $p, q \in P$. Then,*

$$\sum_{q \geq p} \mu(p,q)|D(q)| \geq 0$$

for all $p \in P$.

Proof. Define for all $p \in P$

$$G(p) := D(p) \setminus \bigcup_{q>p} D(q), \qquad f(p) := |G(p)| \geq 0.$$

We claim that

$$D(p) = \biguplus_{q \geq p} G(q).$$

The right-to left inclusion is clear: $q \geq p$ means $q = p \vee q$, hence $G(q) \subseteq D(q) = D(p) \cap D(q) \subseteq D(p)$. For the left-to-right inclusion consider $x \in D(p)$. The set $P_x = \{q \in P \mid x \in D(q)\}$ has a unique maximal element $\hat{p}$ (since $x \in D(q)$ and $x \in D(q')$ imply $x \in D(q \vee q')$ – hence, $q, q' \in P_x$ imply $q \vee q' \in P_x$). Now we see that $x \in D(\hat{p}) \setminus \bigcup_{q>\hat{p}} D(q) = G(\hat{p})$. Uniqueness of $\hat{p}$ implies that the union is indeed disjoint.

Thus, for all $p \in P$ we have

$$|D(p)| = \sum_{q \geq p} f(q)$$

and, by Möbius inversion,

$$\sum_{q \geq p} \mu(p,q)|D(q)| = f(p) \geq 0$$

as required. □

Theorem 2.2. *Let P be a meet-semilattice, and consider two functions $m_1, m_2 : P \mapsto \mathbb{Z}$. If $(m_1)^\mu(p) \geq 0$ and $(m_1)^\mu(p) \geq 0$ for all $p \in P$, then $(m_1 m_2)^\mu(p) \geq 0$ for all $p \in P$.*

[2] We will henceforth simply use the term *Möbius transform*. It is referred to as the "dual form" in [22, Proposition 3.7.2].

Proof. The positivity hypothesis allows us to define, for every $i = 1, 2$ and $p \in P$, a set

$$G_i(p) := \{X_1^{i,p}, \dots, X_{m_i^\mu(p)}^{i,p}\},$$

where the $X_j^{i,p}$ are pairwise distinct formal elements – i.e., $X_j^{i,p} = X_{j'}^{i',p'}$ if and only if $i = i'$, $p = p'$, $j = j'$. Then, set

$$A_i(p) := \biguplus_{q \geq p} G_i(q).$$

Then,

$$A_i(p') \cap A_i(p'') = A_i(p' \vee p'')$$

Notice also that, by definition of m_i^μ,

$$|A_i(p)| = \sum_{q \geq p} m_i^\mu(q) = m_i(p).$$

Consider now the family of sets $(A_{12}(p))_{p \in P}$ defined by

$$A_{12}(p) := A_1(p) \times A_2(p).$$

Since cartesian products commute with intersections, for $p', p'' \in P$ we have

$$A_{12}(p') \cap A_{12}(p'') = (A_1(p') \cap A_1(p'')) \times (A_2(p') \cap A_2(p'')) = A_{12}(p' \vee p'')$$

and thus, by Lemma 2.1,

$$\sum_{q \geq p} \mu(p, q)|A_{12}(q)| \geq 0.$$

The claim now follows because $|A_{12}(q)| = m_1(q)m_2(q)$ for all $q \in P$. □

2.2. **Arithmetic matroids.** In this section we recall basic definitions on matroids and arithmetic matroids in order to set some notation, and we prove Theorem 2.8. For background on matroid theory we refer, e.g., to Oxley's textbook [21], while our presentation of arithmetic matroids follows mostly [7].

Definition 2.3. A *matroid* is given by a pair (E, rk), where E is a finite set and $\mathrm{rk} : 2^E \to \mathbb{N}$ is a function such that, for all $X, Y \subseteq E$,

(R1) $\mathrm{rk}(X) \leq |X|$,
(R2) $X \subseteq Y$ implies $\mathrm{rk}(X) \leq \mathrm{rk}(Y)$,
(R3) $\mathrm{rk}(X \cup Y) + \mathrm{rk}(X \cap Y) \leq \mathrm{rk}(X) + \mathrm{rk}(Y)$.

A *molecule* in a matroid is a triple $\alpha := (R, F, T)$ of disjoint subsets of E such that, for every $A \subseteq E$ with $R \subseteq A \subseteq R \cup F \cup T$,

$$\mathrm{rk}(A) = \mathrm{rk}(R) + |A \cap F|.$$

Notice that if $\alpha = (R, F, T)$ is a molecule, then so is the triple (R', F', T') for every $F' \subseteq F$, every $T' \subseteq T$ and every R' with $R \subseteq R' \subseteq (R \cup F \cup T) \setminus (F' \cup T')$.

To any molecule α, following e.g. [11], we associate a poset

$$B_\alpha = \{(T', F') \mid T' \subseteq T, F' \subseteq F\}$$

ordered by $(T', F') \leq (T'', F'')$ if $T' \subseteq T''$, $F' \supseteq F''$.

Remark 2.4. The poset B_α is bounded, with unique minimal element $(\emptyset, F)$ and unique maximal element $(T, \emptyset)$. Moreover, every interval in B_α (say, $[(F', T'), (F'', T'')]$) is the poset $B_{\alpha'}$ for another molecule (i.e., $\alpha' = (R \cup F'' \cup T', F', T'' \setminus T')$).

Given any function $m : 2^E \to \mathbb{Z}$ and a molecule $\alpha = (R, F, T)$ of a matroid over the ground set E, we define $m_\alpha : B_\alpha \to \mathbb{Z}$ as the function with

$$m_\alpha(F', T') := m(R \cup F' \cup T').$$

Definition 2.5. An *arithmetic matroid* is a triple (E, rk, m) where (E, rk) is a matroid, and $m : 2^E \to \mathbb{Z}$ is a function satisfying the following axioms.

(P) For every molecule $\alpha = (R, F, T)$ of (E, rk),
$$(-1)^T \sum_{R \subseteq A \subseteq R \cup F \cup T} (-1)^{|(R \cup F \cup T) \setminus A|} m(A) \geq 0$$

(Q) For every molecule $\alpha = (R, F, T)$ of (E, rk)
$$m(R)m(R \cup F \cup T) = m(R \cup F)m(R \cup T).$$

(A) For all $A \subseteq E$ and all $e \in E$,
if $\mathrm{rk}(A \cup e) > \mathrm{rk}(A)$, $m(A)$ divides $m(A \cup e)$;
if $\mathrm{rk}(A \cup e) = \mathrm{rk}(A)$, $m(A \cup e)$ divides $m(A)$.

Remark 2.6. Axiom (P) is equivalent to

(P') For every molecule α of (E, rk)
$$(m_\alpha)^\mu(\hat{0}) \geq 0.$$

In fact, for a molecule $\alpha = (R, F, T)$ we see that the poset B_α is boolean and the length of the interval $(B_\alpha)_{\leq (T', F')}$ is $|T'| + |F \setminus F'|$. Therefore, the Möbius function of B_α satisfies
$$\mu(\hat{0}, (T', F')) = (-1)^{|T'| + |F \setminus F'|}.$$
If we now expand axiom (P') we get
$$\begin{aligned}
(m_\alpha)^\mu(\hat{0}) &= \sum_{\substack{T' \subseteq T \\ F' \subseteq F}} \mu(\hat{0}, (T', F')) m(R \cup F' \cup T') \\
&= \sum_{\substack{T' \subseteq T \\ F' \subseteq F}} (-1)^{|T'| + |F \setminus F'|} m(R \cup F' \cup T') \\
&= (-1)^{|T|} \sum_{\substack{T' \subseteq T \\ F' \subseteq F}} (-1)^{|T \setminus T'| + |F \setminus F'|} m(R \cup F' \cup T') \\
&= (-1)^{|T|} \sum_{R \subseteq A \subseteq R \cup F \cup T} (-1)^{|(R \cup F \cup T) \setminus A|} m(A),
\end{aligned}$$
and we recover the formulation of axiom (P) in Definition 2.5.

2.3. **Product of multiplicity functions.** Consider now a fixed matroid (E, rk), two (possibly different) functions $m', m'' : 2^E \to \mathbb{Z}$ and their (pointwise) product $m := m'm''$

Lemma 2.7. *If both m' and m'' satisfy axiom (P), so does $m = m'm''$.*

Proof. Suppose m' and m'' both satisfy (P) and consider a molecule α. The poset B_α is boolean, hence in particular a (meet semi-)lattice. Since every interval of B_α defines a molecule, m'_α and m''_α satisfy the conditions of Theorem 2.2 on B_α. Hence, $((m'm'')_\alpha)^\mu(\hat{0}) \geq 0$.

□

Theorem 2.8. *If both (E, rk, m') and (E, rk, m'') are arithmetic matroids, then $(E, \mathrm{rk}, m'm'')$ is also an arithmetic matroid.*

Proof. The triple $(E, \mathrm{rk}, m'm'')$ satisfies (P) by Lemma 2.7, and (Q), (A) trivially.

□

Remark 2.9. This theorem endows the set of arithmetic matroids over a fixed underlying matroid with a natural product, which makes it into a commutative monoid. We leave the investigation of this algebraic structure as an open problem.

3. On Tutte quasi-polynomials associated to cell complexes

3.1. **The Tutte quasi-polynomial.** Let G be a finitely generated abelian group, E be a finite set, and $\mathcal{L} = \{\{g_e : e \in E\}\}$ be a list (multiset) of elements in G. For every $A \subseteq E$ we denote by $\mathcal{L}_A$ the sublist $\{\{g_e : e \in A\}\}$, by $\langle \mathcal{L}_A \rangle$ the subgroup that it generates, and by $G_A := \operatorname{tor}(G/\langle \mathcal{L}_A \rangle)$ the torsion subgroup of the quotient $G/\langle \mathcal{L}_A \rangle$. In [7, Section 7], the *Tutte quasi-polynomial* of $\mathcal{L}$ is defined as follows.

$$Q_{\mathcal{L}}(x, y) := \sum_{A \subseteq E} \frac{|G_A|}{|(x-1)(y-1)G_A|} (x-1)^{\operatorname{rk} E - \operatorname{rk} A} (y-1)^{|A| - \operatorname{rk} A}.$$

Remark 3.1. If for every $A \subseteq E$ the integer $k = (x-1)(y-1)$ is coprime with $|G_A|$, then $kG_A := \{kg | g \in G_A\}$ equals G_A and we get the ordinary *Tutte polynomial* of the matroid of linear dependencies among elements of $\mathcal{L}$:

$$T_{\mathcal{L}}(x, y) := \sum_{A \subseteq E} (x-1)^{\operatorname{rk} E - \operatorname{rk} A} (y-1)^{|A| - \operatorname{rk} A}.$$

On the other hand, when for every $A \subseteq E$ the integer k is a multiple of $|G_A|$ we have that kG_A is trivial and we obtain the *arithmetic Tutte polynomial*:

$$M_{\mathcal{L}}(x, y) := \sum_{A \subseteq E} |G_A| (x-1)^{\operatorname{rk} E - \operatorname{rk} A} (y-1)^{|A| - \operatorname{rk} A}.$$

Therefore $Q_{\mathcal{L}}(x, y)$ is a quasi-polynomial function that in some sense interpolates between these two polynomials. It appeared as a specialization of a multivariate "Fortuin–Kasteleyn quasi-polynomial".

Now recall the following definitions.

Definition 3.2 ([7, Section 7])**.** Let G, E and $\mathcal{L}$ be as above.

(1) A *proper q-coloring* is an element $c \in \operatorname{Hom}(G, \mathbb{Z}_q)$ such that $c(g_e) \neq 0$ for all $e \in E$.
(2) A *nowhere zero q-flow* is a function $\phi : E \longrightarrow \mathbb{Z}_q \setminus \{0\}$ such that $\sum_{e \in E} \phi(e) g_e = 0$ in G/qG.

The number of proper q-colorings and the number of nowhere zero q-flows are denoted by $\chi_{\mathcal{L}}(q)$ and $\chi^*_{\mathcal{L}}(q)$ respectively.

The following statement generalizes a result of [10].

Lemma 3.3 ([7, Theorem 9.1])**.**

$$\chi_{\mathcal{L}}(q) = (-1)^{\operatorname{rk} E} q^{\operatorname{rk} G - \operatorname{rk} E} Q_{\mathcal{L}}(1-q, 0)$$

$$\chi^*_{\mathcal{L}}(q) = (-1)^{|E| - \operatorname{rk} E} |\operatorname{tor}(G)|^{-1} Q_{\mathcal{L}}(0, 1-q).$$

In particular, $\chi_{\mathcal{L}}(q)$ and $\chi^*_{\mathcal{L}}(q)$ are quasi-polynomial functions of q, called the *chromatic quasi-polynomial* and the *flow quasi-polynomial* respectively.

3.2. **On flows and colorings on CW complexes.** We start by recalling some definitions by Beck–Breuer–Godkin–Martin [4] and Duval–Klivans–Martin [1]. Let C be a CW complex of dimension d and, for every $i = 0, 1, \dots, d$, let C_i be the set of the i-dimensional cells of C. The top-dimensional boundary map $\partial : \mathbb{Z}^{C_d} \to \mathbb{Z}^{C_{d-1}}$ is represented by a matrix with integer entries, that (by a slight abuse of notation) we denote again by ∂. By reducing modulo q, we get a map $\overline{\partial} : \mathbb{Z}_q^{C_d} \to \mathbb{Z}_q^{C_{d-1}}$, that we can view as a matrix with coefficients in $\mathbb{Z}_q$.

Definition 3.4 (cf. [5] and [4])**.** Let C and ∂ be as above.

(1') a *proper q-coloring* of C is an element $c \in \mathbb{Z}_q^{C_{d-1}}$ such that all the entries of the vector $c\overline{\partial}$ are nonzero.

(2') a *nowhere zero q-flow* on C is an element $\phi \in \ker \overline{\partial}$ such that the coordinate $\phi(e)$ is nonzero for every $e \in C_d$.

The authors of [5] and [4] prove that the number of proper q-colorings and the number of nowhere zero q-flows are quasi-polynomial functions, that we will denote by $\chi_C(q)$ and $\chi_C^*(q)$.

In fact, to the (integer) matrix ∂ one can associate a Tutte quasi-polynomial, an arithmetic matroid and an arithmetic Tutte polynomial. With the following lemma we address Remark 3.15 of Beck-Breuer-Godkin-Martin [4] by showing that the coloring- and flow-counting quasi-polynomials of [4] and [1] are instances of the coloring- and flow- quasi-polynomials associated to the matrix ∂.

Lemma 3.5. *Definitions (1') and (2') agree with definitions (1) and (2), when* $G = \mathbb{Z}^{C_{d-1}}$, $E = \{1, 2, \dots |C_d|\}$, *and* $\mathcal{L} = \{$*columns of* $\partial\}$.

Proof. Every $c \in \mathbb{Z}_q^{C_{d-1}}$ uniquely extends to a homomorphism $\tilde{c} \in \mathrm{Hom}(\mathbb{Z}^{C_{d-1}}, \mathbb{Z}_q)$. Then since $\tilde{c}(g_e) = c\partial$, (1) specializes to (1'). On the other hand, definition (2') is equivalent to saying that ϕ is a function $C_d \to \mathbb{Z}_q \setminus \{0\}$ such that $\partial\phi = 0$. This is precisely the specialization of definition (2). □

The following statement, which now follows immediately from Lemmas 3.3 and 3.5, can be interpreted as saying that the Tutte quasi-polynomial is indeed a "higher-dimensional analogue" of Tutte's dichromate for graphs [23] :

Theorem 3.6. *With the notations above, we have:*

$$\chi_C(q) = (-1)^{\mathrm{rk}\,\partial} q^{|C_{d-1}| - \mathrm{rk}\,\partial} Q_\partial(1-q, 0)$$

$$\chi_C^*(q) = (-1)^{|C_d| - \mathrm{rk}\,\partial} Q_\partial(0, 1-q).$$

Remark 3.7. As pointed out in [7], the Tutte quasi-polynomial is not an invariant of the arithmetic matroid, but is an invariant of the *matroid over* $\mathbb{Z}$ associated to the matrix ∂. We call this matroid the *cellular matroid* over $\mathbb{Z}$ of C.

Remark 3.8. Underlying matroids of cellular matroids over $\mathbb{Z}$ (i.e., the matroids defined by the matrices ∂) have been studied in their own right. Allowing different generality for the complex C one obtains different interesting classes of matroids. Already in the case where C is a simplicial complex, the matroids obtained this way are strictly more general than graphical matroids [19].

The following subgroup arrangement has been studied by Kamiya, Takemura and Terao [15] and Lawrence [17] for an arbitrary list of vectors in a lattice.

Definition 3.9. Let $v_1, \dots v_m$ denote the columns of the top-degree boundary matrix ∂ of a CW-complex C (so $m = |C_{d-1}|$). For every $i = 1, \dots, m$ let

$$\phi_i : (\mathbb{Z}_q)^{|C_d|} \to \mathbb{Z}_q, \quad x \mapsto \langle x, v_i \rangle.$$

Define then $\mathcal{A}(C, q)$ to be the arrangement of the subgroups $\{\ker \phi_1, \dots, \ker \phi_m\}$ in $(\mathbb{Z}_q)^{|C_d|}$. (This is, in fact, a family of subgroup arrangements parametrized by q).

Remark 3.10. The chromatic quasi-polynomial of the cellular complex coincides with the *characteristic quasi-polynomial* of the arrangement $\mathcal{A}(C, q)$ studied in [15], just like the chromatic polynomial of a graph coincides with the characteristic polynomial of the corresponding graphical arrangement. In particular, the complement of this arrangement has cardinality $\chi_C(q) = (-1)^{\mathrm{rk}\,\partial} q^{|C_{d-1}| - \mathrm{rk}\,\partial} Q_\partial(1-q, 0)$.

...vero? (l'ha detto il referee 1)

Remark 3.11. Given a d-dimensional CW-complex C, for every $j = 0, 1, \dots, d-1$ the j-skeleton of C is itself a j-dimensional CW-complex C^j for which we can carry out all considerations of this section. Thus, C in fact gives rise to a class of arithmetic quasi-polynomials and arithmetic matroids. In the following section we will consider properties of this class as a whole.

4. On the modified Tutte-Krushkal-Renhardy polynomial

When considering cell complexes as higher dimensional generalizations of graphs, besides flows and colorings it is natural to enumerate the analogue of spanning trees. Following Kalai [14], this enumeration is weighted by the square of the cardinality of the torsion of the subcomplexes that are enumerated. This line of thought inspired [3], where the authors introduced a class of polynomials arising as a modification of Krushkal and Renhardy's polynomial invariants of triangulations. This last section is devoted to answering a question of [3] which we will state after reviewing some definitions (following [3, 16]).

Definition 4.1. We denote by $\mathcal{S}^j$ the family of all *spanning subcomplexes of dimension* j, i.e., of all the subcomplexes S such that $C^{j-1} \subseteq S \subseteq C^j$. These are naturally identified with the subsets of $(C^j)_j$, the set of j-dimensional cells of the j-skeleton of C. Let $b_i(S)$ be the i-th Betti number of S (i.e., the rank of the homology $H_i(S, \mathbb{Z})$), and let $t_i(S)$ be the cardinality of its torsion, $t_i(S) := |\operatorname{tor}(H_i(S, \mathbb{Z}))|$.

Remark 4.2. As has been pointed out e.g. in [4], the function $t_i(S)$ is the multiplicity function of the arithmetic matroid defined by the matrix ∂^j.

Definition 4.3 ([3, Definition 3.1])**.** The j-th Tutte–Krushkal–Renhardy (TKR for short) polynomial of C is defined in [16] as

$$T_C^j(x, y) = \sum_{S \in \mathcal{S}^j} (x-1)^{b_{j-1}(S) - b_{j-1}(C)} (y-1)^{b_j(S)}.$$

The "modified j-th Tutte–Krushkal–Renhardy polynomial" of C is

$$M_C^j(x, y) = \sum_{S \in \mathcal{S}^j} t_j^2(S)(x-1)^{b_{j-1}(S) - b_{j-1}(C)} (y-1)^{b_j(S)}.$$

As noted in [12, Section 5.4], $T_C^j(1,1)$ is the number of "cellular j-spanning trees" of C (according to Definition 2.1 of [3]), while $M_C^j(1,1)$ is an invariant introduced by G. Kalai in [14]: the number of cellular j-spanning trees S of C, each counted with multiplicity $t_j^2(S)$. In [3, Remark 3.3] the authors ask whether the multiplicity t_j^2 defines an arithmetic matroid. The results established in Section 2 allow us to give a positive answer to this question.

Theorem 4.4. *Let C be a CW-complex of dimension d and, for every $j = 1, \dots, d$ let $\mathcal{M}_j(C)$ denote the cellular matroid of the j-skeleton of C (see Remark 3.8). Then, for every j the pair $(\mathcal{M}_j(C), t_j^2)$ is an arithmetic matroid, and the modified j-th Tutte–Krushkal–Renhardy polynomial $M_C^j(x, y)$ is the associated arithmetic Tutte polynomial. In particular, the coefficients of $M_C^j(x, y)$ are nonnnegative.*

Proof. We first notice that the pair $(\mathcal{M}_j(C), t_j)$ is an arithmetic matroid (see Remark 4.2), then apply Theorem 2.8 with $m' = m'' = t_j$. □

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