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Endowment redistribution and Pareto improvements in GEI economies

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June 25, 2016

Abstract

With incomplete markets and numeraire assets, there are open sets of economies such that their equilibrium allocations can be improved upon by a reallocation of period zero endowments. This strengthens the classical results on constrained Pareto inefficiency of equilibria in GEI.

Keywords: GEI, constrained Pareto optimality.

1 Introduction

In the absence of completeness of financial markets, equilibrium allocations are typically Pareto inefficient. In fact, the set of equilibrium allocations itself may be Pareto ranked, completely, as in the Hart (1975) example, or partially, as in Pietra (2004) and Salto and Pietra (2013).[1] In economies with real assets, however, Pareto ranking of equilibria is the exception, and it becomes important to formulate an appropriate efficiency criterion. The canonical definition of constrained Pareto optimality (CPO) has been introduced by Stiglitz (1982) and developed by Geanakoplos and Polemarchakis (1986) and Citanna, Kajii and Villanacci (1998). It rests on the idea that the minimal efficiency requirement that an equilibrium allocation should satisfy is that it cannot be improved upon by a reallocation of asset holdings, and by the adjustment of prices required to restore the equilibrium in the commodity markets. Adopting the convenient fiction of a benevolent planner, this notion of CPO endows her with fairly limited instruments and, most important, it allows her to affect directly the intertemporal allocation of individual incomes using only the opportunities offered by the set of available assets. The possibility of improving upon the equilibrium allocation using portfolio reallocations rests on the welfare effects of the induced changes in equilibrium prices.

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[1] These last two papers deal with economies with nominal asset and, therefore, indeterminate equilibria. Under appropriate restrictions, generically each equilibrium allocation is Pareto inferior to some other equilibrium allocation.

Different notions of constrained efficiency can be developed, in much the same spirit, by choosing other policy instruments. Herings and Polemarchakis (2004) show that, under suitable regularity conditions, price regulation can attain a Pareto improvement over fix-price equilibria. Citanna, Polemarchakis and Tirelli (2006) show that taxation of asset trades may also induce Pareto improvements.

Here, we consider an alternative notion, which allows the planner to reallocate incomes just in the initial trading period, letting the agents choose their individually optimal portfolios and consumption bundles at the new equilibrium prices. Our notion shares the same basic idea behind the canonical Geanakoplos and Polemarchakis's criterion: the planner chooses the value of a policy instrument, allows people to choose their optimal behavior, and adjusts prices to restore market clearing. Evidently, to use period 0 endowment reallocations as policy tools is fully coherent with the absence of some assets.[2] In GEI models, the market failure is due to the distorted intertemporal allocation of incomes. Hence, it could appear harder to implement a Pareto improvement just by reallocating time 0 endowments. However, we show that there are open sets of economies such that this can be obtained, so that their equilibria are not CPO according to our criterion. Clearly, a key role is played by the choice of the specific vector of lump-sum taxes. Thus, the policy intervention cannot be anonymous (see Kajii (1994)).

There are several motivations for this paper. Its core issue - "can we improve upon a GEI equilibrium allocation by reallocating just period 0 endowments?" - has been around for a long time. It looks interesting to settle it. Our answer is only partially positive. We show that, first, given any specification of an economy in terms of numeraire asset structure and preferences, there are open sets of endowments such that the associated equilibria are CPO. Therefore, endowment reallocations are not generically sufficient to guarantee the possibility to attain some Pareto improvement. Secondly, we construct sets of economies where an appropriate period 0 endowment reallocation induces a Pareto improvement. These sets are open in the space of the economies. The same results hold even if we restrict the analysis to time-separable, or VNM, utility functions.

We believe that this second result is of interest for at least two additional reasons. First, this is, in a limited, but important, way, a counterexample to the claim that you need at least H independent policy instruments to implement H policy aims. As pointed out by Citanna et al. (1998), this viewpoint goes back to Tinbergen (1956). In our set up, there are H policy aims (the changes in the equilibrium utility of H agents) and $(H-1)$ independent instruments. Still, by properly exploiting the welfare effects of the induced price changes, we can attain a Pareto improvement for some open set of economies. Of course, the real issue is how we define a "policy aim." It is certainly true that, in general, at least H independent policy instruments are required to attain *each specific* vector of utility improvements, $du \equiv (du_1, ..., du_H)$, so that this cannot be ob-

[2] It is essential that the income transfers take place just in period zero. Otherwise, we would implicitly allow the planner to manufacture personalized assets, so that she could actually attain full Pareto optimality.

tained by reallocating endowments at time zero. However, if you just aim to implement *any* $du > 0$, less than H policy tools may be enough. The point is that the result established in Geanakoplos and Polemarchakis (1986) and in Citanna et al (1998) is much stronger than what is strictly required. Their CPO criterion simply requires that, by some policy intervention, we can attain *some* positive vector of changes in the equilibrium level of the utility of each agent. However, they provide conditions such that one can attain *each* positive vector. Not surprisingly, a stronger result requires stronger restrictions on the class of economies and of policy profiles than the ones minimally required. It must be stressed that our first result mentioned above implies that no generic constrained inefficiency result is possible if we just allow for endowment reallocations. Therefore, from this point of view, there is a compelling motivation for the adoption of a different, and, in a sense, larger, set of policy instruments as in Geanakoplos and Polemarchakis (1986) and in Citanna et al (1998).

Secondly, the kind of policy intervention considered in Geanakoplos and Polemarchakis (1986) could somehow be read as suggesting that the inefficiency associated with market incompleteness dictates Pareto improving measures related to interventions in the working of the financial markets - in their framework, to impose a portfolio to each agent - or, more generally, to intertemporal policies. Our result shows that efficiency can be improved using just time 0 lump-sum taxes.

The next section briefly presents the model. Section 3 formalizes our notion of constrained Pareto optimality and establishes our main results. Some final remarks follow.

2 The Model

We consider a standard GEI model with numeraire assets. There is a finite set of agents $(h = 1, ..., H)$ and a finite set of commodities $(c = 1, ..., C)$ at each of $(S+1)$ spots, $s = 0, ..., S$. A consumption plan is $x_h \equiv \left(x_h^0, x_h^1, ..., x_h^S\right) \in \mathbb{R}_+^{(S+1)C}$, a portfolio is $b_h \equiv \left(b_h^1, ..., b_h^J\right) \in \mathbb{R}_+^J$. Commodity prices are $p \equiv \left(p^0, p^1, ..., p^S\right) \in \mathbb{R}_{++}^{(S+1)C}$, asset prices are $q \equiv \left(q^1, ..., q^J\right) \in \mathbb{R}^J$. As usual, we normalize to 1 the price of good 1 in each spot. Asset trade takes place at spot 0.[3] Asset payoffs are defined in terms of the numeraire commodity and described by a $(S \times J)$ matrix R of full rank

$$R = \begin{bmatrix} r^{11} & & r^{1J} \\ \vdots & \ddots & \vdots \\ r^{S1} & & r^{SJ} \end{bmatrix}.$$

[3] A standard interpretation is that there are two periods and uncertainty on tomorrow state of the world. In view of the structure of some of the examples below, it is better to think of it as a multiperiod model, with or without uncertainty. The essential feature is that asset trade takes place just at time 0.

Finally, $u_h(x_h)$ is agent h's utility function, satisfying the standard assumptions for the differential analysis of equilibria: for each h, $u_h(x_h)$ is C^2, strictly monotone, differentiably strictly quasi-concave in x_h, and satisfies the boundary conditions: the closure of the set $\{x_h : u_h(x_h) \geq u_h(\overline{x}_h)\}$ is contained in $\mathbb{R}^{(S+1)C}_{++}$, for each $\overline{x}_h \gg 0$.

Consumers' behavior is described by the optimal solution to the problem: Given (p, q), choose

$$\begin{aligned} (x_h, b_h) &\in \arg\max u_h(x_h) \text{ subject to} \\ p^0\left(x_h^0 - \omega_h^0\right) &\equiv p^0 z_h^0 = -q b_h, \\ p^s\left(x_h^s - \omega_h^s\right) &\equiv p^s z_h^s = r^s b_h, \text{ for each } s > 0, \end{aligned} \tag{U}$$

where $\omega_h \equiv \left(\omega_h^0, \omega_h^1, ..., \omega_h^S\right) \in \mathbb{R}^{(S+1)C}_{++}$ is the initial endowment vector.

Given R, an economy is a profile $\{..., (u_h(.), \omega_h), ...\} \in \boldsymbol{E}$ where the space of endowments is endowed with the standard topology, and the one of utility functions with the C^2 compact-open topology, $\boldsymbol{E}$ with the product topology.

Let $\lambda_h \in \mathbb{R}^{S+1}_{++}$ be the vector of Lagrange multipliers associated with the optimal solution to optimization problem (U). We do not impose that preferences can be described by a Von Neumann-Morgenstern utility function. However, our main results hold as well for this more restricted class of economies.

Definition 1 *An equilibrium is a price vector $(\overline{p}, \overline{q})$, with associated allocation and portfolio profiles $\left\{\left(\overline{x}_1, \overline{b}_1\right), ..., \left(\overline{x}_H, \overline{b}_H\right)\right\}$, such that:*
a. for each h, $\left(\overline{z}_h, \overline{b}_h\right)$ solves problem (U) given $(\overline{p}, \overline{q})$,
b. $\sum_h \overline{z}_h = 0$ and $\sum_h \overline{b}_h = 0$.

3 Constrained suboptimality of equilibria

Let's briefly discuss the standard approach to the analysis of constrained suboptimality in GEI economies. Consider the system of eqs.

$$\Xi(p, q, \xi) = [\Phi(.), (..., u_h(.) - \overline{u}_h, ...)],$$

where $\Phi(.) = 0$ defines the equilibrium, while ξ is some vector of policy instruments, for instance a vector describing the reallocation of portfolios or endowments. The key step in the classical proofs of constrained suboptimality is to show that $D_{(p,q,\xi)}\Xi(.)$ has, generically, full rank at each solution. This can be done using as equilibrium map the system of aggregate excess demand functions $\Phi(.)$, as in Geanakoplos and Polemarchakis (1986), or the entire system of equilibrium conditions (individual and aggregate), as in Citanna et al. (1998).[4] Anyhow, the basic idea is the same: add to the equilibrium conditions the system of equations $[..., u_h(.) - \overline{u}_h, ...]$ and show that the map so obtained has a

[4] There are other differences between the two papers. In particular, in the second, the authors consider the welfare effect of a policy profile defined in terms of both portfolio reallocation and period 0 endowment reallocation. This is irrelevant for the purposes of the current discussion.

full rank derivative. This implies that, by choosing appropriately the policy vector ξ, it is possible to implement every possible local variation of the equilibrium level of the utility of each agent. If equilibria are locally determined, this approach necessarily requires that the number of degrees of freedom in the selection of the policy vector is at least as large as the number of agents, H. Hence, it cannot be applied to study the possibility of Pareto improvements obtained by a reallocation of period 0 endowments, since this policy instrument has, in an essential way, dimension $(H-1)$.

However, what really matters is if it is possible to improve upon the equilibrium allocations, not the attainability of every possible Pareto improvement. From this perspective, the key issue is if the matrix $D_{(p,q,\xi)}\Xi(.)$ spans *some* non-trivial vector $[0,(...,du_h,...)] \geq 0$, not if it spans *all* vectors with this structure. Evidently, by adopting this weaker condition, we could be able to weaken the restriction on the minimal rank of $D_{(p,q,\xi)}\Xi(.)$. In fact, as we will see, one of the robust examples provided below can also be seen as an example of an economy with just one asset, so that the dimension of the policy profile "portfolio reallocation" is smaller than H. The unique equilibrium is not CPO, according to the Geanakoplos and Polemarchakis (1986) criterion.

Here, however, we will focus on the possibility of Pareto improvements obtainable through the reallocation of the initial endowments of good 1 in period zero. Thus, our policy vector is a profile $t \equiv [t_1,...,t_H] \in \mathbb{R}^H$ with $\sum_h t_h = 0$. Clearly, its dimension is $(H-1)$.[5] We now make precise our efficiency criterion. Let $(p(t),q(t))$ be the equilibrium prices associated with the vector t of endowments reallocation. Also, let $V_h\left(p(t),q(t),t_h\right) \equiv u_h\left(x_h(p(t),q(t),t_h)\right)$ be the associated maximal value of the utility.

Definition 2 *An equilibrium $(\overline{p},\overline{q})$ is $\omega-$Constrained Pareto Optimal ($\omega-$CPO) if there is no profile $t \in \mathbb{R}^H$ with $\sum_h t_h = 0$ such that, at one associated equilibrium $(p(t),q(t))$, $V_h\left(p(t),q(t),t_h\right) \geq V_h\left(\overline{p},\overline{q},t_h=0\right)$ for each h, with at least one strict inequality.*

We start establishing the negative part of our result: given any profile of utility functions, and any payoff matrix, there is an open set of economies, parameterized by endowment profiles, such that all equilibria are $\omega-$CPO (but not necessarily CPO according to the Geanakoplos and Polemarchakis (1986) criterion). The argument is straightforward. Still, it may be worthwhile to elaborate a little on its logic before getting into the details.

To begin, for completeness, we report a standard result, i.e., the generalization of Roy' s identity to financial economies.

Lemma 3 *Let $V_h(p,q,t_h)$ be the indirect utility function associated with opti-*

[5] We consider rellocations of good 1 endowments. Evidently, nothing would change by allowing for reallocation of the endowments of the other period 0 commodities.

mization problem (U), *and given a vector of endowment transfers* t_h. *Then,*

$$\begin{aligned}
\frac{\partial V_h}{\partial p^{sc}} &= -\lambda_h^s(p,q,t_h)z_h^{sc}(p,q,t_h), \text{ for each } sc, \\
\frac{\partial V_h}{\partial q^j} &= -\lambda_h^0(p,q,t_h)b_h^j(p,q,t_h), \text{ for each } j, \\
\frac{\partial V_h}{\partial t_h} &= \lambda_h^0(p,q,t_h).
\end{aligned}$$

Proof. In Appendix. ■

The effect of a change in portfolios and of the induced price changes on the utility of agent h can be decomposed into two parts: the direct effect of db_h on the indirect utility function, and the second order effect, due to the induced price changes. Let $\left[\overrightarrow{\frac{\partial G}{\partial t}x}\right]$ be the directional derivative of any function $G(.)$ in the direction $[x]$. By individual optimization, for each consumer, the direct impact on the utility of a marginal portfolio readjustment is nil. Hence, just the second order effects of the portfolios reallocation matter. Using Roy's identity, they can be written as $\frac{\partial V_h(.)}{\partial p^{sc}}\left[\overrightarrow{\frac{\partial p^{sc}}{\partial b}db}\right] = -\lambda_h^s(.)z_h^{sc}(.)\left[\overrightarrow{\frac{\partial p^{sc}}{\partial b}db}\right]$, and $\frac{\partial V_h(.)}{\partial q^j}\left[\overrightarrow{\frac{\partial q^j}{\partial b}db}\right] = -\lambda_h^0(.)b_h^j(.)\left[\overrightarrow{\frac{\partial q^j}{\partial b}db}\right]$. To establish lack of CPO, it suffices to show that the span of the collection of these directional derivatives with respect to prices contains at least one strictly positive vector.

With our notion of $\omega-$CPO, the first order effect is not trivial. Indeed, it is $\frac{\partial V_h(.)}{\partial t_h}t_h = \lambda_h^0(.)t_h \neq 0$ and, evidently, there must be at least one agent h with $\frac{\partial V_h(.)}{\partial t_h}t_h < 0$. To obtain a Pareto improvement, the second order effects must have the right sign and, additionally, they must be *sufficiently large*, so that they can compensate the, possibly negative, first order effects for each agent. Given the formulas for $\left(\frac{\partial V_h(.)}{\partial p^{sc}}, \frac{\partial V_h(.)}{\partial q^j}, \frac{\partial V_h(.)}{\partial t_h}\right)$ reported above, this can happen only if, at the initial equilibrium, the normalized vector of Lagrange multipliers are sufficiently different across agents, if net trades are sufficiently far away from 0, and/or if the directional derivatives of equilibrium prices are sufficiently large. Intuitively, this rules out the possibility to Pareto improve upon the equilibria of economies with initial endowments close to a PO allocation. Therefore, there are open sets of economies with $\omega-$CPO equilibria. This argument is formalized in Proposition 4. Its proof is in Appendix.

Proposition 4 *Given* R, *for each utility profile* $\{..., u_h, ...\}$, *there is an open set of endowments such that each equilibrium allocation is* $\omega-$*CPO.*

Given our aim, this is a negative result: lack of $\omega-$CPO cannot be a generic property. The "size" of the set of economies with $\omega-$CPO equilibria is still an open issue.

The main motivation for this paper is Prop. 7, showing that there are also open sets of economies with non $\omega-$CPO equilibria. We establish it by providing

two parametric examples, and then showing that the same result holds for some open set of economies. We are not claiming that a similar result holds for *some* set of endowment profiles given *any* specification of (u, R). It is fairly obvious that this cannot be true, for instance, for economies with identical, homothetic preferences.[6] Given the asset payoffs, our result holds for some open set of economies in the space defined by endowments and utilities. It is still an open issue if, given any (u, R) in some generic set, it holds for some appropriately chosen open set of endowments. We will come back to this issue in the conclusions.

The basic intuition for the possibility of a Pareto improvement can be most easily seen in an economy with just one asset, as in both examples: an endowment redistribution affects equilibrium prices for assets and commodities. Necessarily, within each spot, if some agent is made better off by the price change, some other agent must be made worst off, since market clear. However, due to market incompleteness, for each agent, these changes in spot utilities are aggregated over spots using a distinct vector of normalized Lagrange multipliers. Therefore, the total utility change due to the prices changes may be positive for each agent, as it is in fact true in Example 5. In addition, we need to take into account the direct effect of time 0 transfers. Clearly, to provide an example of an economy with $\omega-$CP inefficient equilibria, we need to balance carefully the three different effects. At the same time, we need to maintain a structure simple enough, so that the computational burden is not too heavy. In both our examples, we fix the class of economies and the endowment vectors. We then fix appropriately a price vector and an allocation and choose the values of some parameters of the utility functions in such a way that the assigned price and allocation hold as an equilibrium. As we will see, this leaves us with enough degrees of freedom to be able to pick these parameters so that the equilibrium is not $\omega - CPO$. We consider three-period economies with one numeraire asset. Both examples are highly non generic. Still, all the equilibrium variables are, locally, continuous functions of the parameters, so that, as we will see, the main result is robust to open perturbations. The first example, in the text, considers an economy with three agents, three spots and two consumption goods at each one of the future spots. The second (in Appendix) presents an economy with two agents, three periods and two goods just at one future spot. The two economies share many features. However, the first example requires computations which are somewhat more transparent, while the second shows that lack of CPO can hold even in two-agent economies.

Example 5 ***An economy with a unique, non*** $\omega - CPO$ ***equilibrium.*** *There are three periods and just one asset, inside money, paying one unit of the numeraire commodity at each future spot. To avoid unnecessary notation, we assume that there is just one commodity at time 0. This entails no loss of generality. Agents are endowed with strictly concave, time-separable utility*

[6] Since equilibrium prices are invariant with respect to the reallocation of spot income, our argument is bound to fail, as, in fact, does the one of Geanakoplos and Polemarchakis (1986).

functions:

$$u_h(.) = \alpha_h \ln x_h^0 + \beta_h \ln v_h^1\left(x_h^{11}, x_h^{12}\right) + (1-\alpha_h-\beta_h)\ln v_h^2\left(x_h^{21}, x_h^{22}\right).$$

There are three agents, with

$$\begin{aligned}
v_1^1\left(x_1^{s1}, x_1^{s2}\right) &= \left[(x_1^{s1})^{-2} + k^3(x_1^{s2})^{-2}\right]^{-\frac{1}{2}}, \text{ for } s = 1,2, \\
v_2^1\left(x_2^{11}, x_2^{12}\right) &= \left[k^3(x_2^{11})^{-2} + (x_2^{12})^{-2}\right]^{-\frac{1}{2}}, \quad v_2^2\left(x_2^{21}, x_2^{22}\right) = \left[(x_2^{21})^{-2} + k^3(x_2^{22})^{-2}\right]^{-\frac{1}{2}}, \\
v_3^1\left(x_3^{11}, x_3^{12}\right) &= \left[(x_3^{11})^{-2} + k^3(x_3^{12})^{-2}\right]^{-\frac{1}{2}}, \quad v_3^2\left(x_3^{21}, x_3^{22}\right) = \left[k^3(x_3^{21})^{-2} + (x_3^{22})^{-2}\right]^{-\frac{1}{2}}.
\end{aligned}$$

Endowments are $\omega_1 = (14, (2,0), (2,0))$, $\omega_2 = (0, (4,20), (14,0))$, *and* $\omega_3 = (0, (14,0), (4,20))$. *We will argue later on that our choice of a boundary endowment profile has no substantial effect on the results, which also hold for strictly positive endowment vectors.*
In this example, we fix $k = \frac{1}{3}$ *to simplify the computations. However, it is convenient to study some properties of this economy for an arbitrary value of the parameter* k, *because they will become handy in Example A1 in Appendix.*
Set $p^0 = p^{11} = p^{21} = 1$. *Given any* $\overline{b}_1$ *and each vector* p, *agent 1's associate spot* s *indirect utility functions are*

$$V_1^s\left(p^s, \overline{b}_1\right) \equiv \left(2+\overline{b}_1\right)\left[\left(\frac{p^{s2\frac{1}{3}}}{p^{s2\frac{1}{3}} + kp^{s2}}\right)^{-2} + k^3\left(\frac{k}{p^{s2\frac{1}{3}} + kp^{s2}}\right)^{-2}\right]^{-\frac{1}{2}} \equiv \left(2+\overline{b}_1\right) g_1^s(p^s).$$

The results for agents 2 and 3 are similar. A key property follows from our selection of the utility functions: spot commodity prices affect the choice of the optimal portfolio only because they determine the value of the spot endowments.[7]
Fix $\overline{b} \equiv \left(\overline{b}_1, \overline{b}_2, \overline{b}_3\right) = (8, -4, -4)$. *Given* $\overline{b}$, *at each spot, the equilibrium is obtained solving the market clearing condition for good 1, i.e.,*

$$\overline{Z}^{s1}(p^s, \overline{b}) \equiv \frac{10p^{s2\frac{1}{3}}}{kp^{s2} + p^{s2\frac{1}{3}}} + \frac{10p^{s2\frac{1}{3}}}{kp^{s2} + p^{s2\frac{1}{3}}} + \frac{(20p^{s2})kp^{s2\frac{1}{3}}}{p^{s2} + kp^{s2\frac{1}{3}}} - 20 = 0.$$

A convenient feature of this spot economy is that, for the given ω *and* $\overline{b}$, $\overline{p}^{s2} = 1$, *for each* s, *is an equilibrium for each* $k > 0$. *For* $k \geq \frac{1}{3}$, *it is the only equilibrium; for* $k < \frac{1}{3}$, *there are three equilibria, see Figure 1.*[8]

FIGURE 1 GOES HERE

At the equilibrium $\overline{p}^{s2} = 1$, *the derivative of the excess demand function is*

$$\frac{\partial \overline{Z}^{s1}(p^s, \overline{b})}{\partial p^{s2}}\Big|_{\overline{p}^{s2}=1} = \left(\frac{20}{3}\frac{3k^2 - k}{(k+1)(k+1)}\right),$$

[7] Hence, and due to the log utility functions, the functions $g_h^s(p^s)$ are irrelevant for the optimal portfolio choice.

[8] The spot 1 and 2 subeconomies are based on an example of a CES economy with multiple equilibria proposed by D. Blair.

which is zero at $k = \frac{1}{3}$, given the portfolio $\overline{b}$. Hence, the direct *effect of changes in spot equilibrium prices on the aggregate excess demand is zero.*[9]
Given $k = \frac{1}{3}$, $\overline{b}$ and $\left(\overline{p}^{12}, \overline{p}^{22}\right) = (1,1)$, the excess demands for good 2 at spot 1 and 2 are

$$\overline{z}_1^{s2} = \left[\frac{10}{4}, \frac{10}{4}\right], \quad \overline{z}_2^{s2} = \left[-\frac{20}{4}, \frac{10}{4}\right], \quad \overline{z}_3^{s2} = \left[\frac{10}{4}, -\frac{20}{4}\right].$$

Let t be the endowment reallocation, with $t_1 \equiv \tau > 0$ and $t_2 \equiv t_3 \equiv -\frac{\tau}{2}$. Consider now the portfolio optimization problems of the three agents. Using the previous observation, we can write them as

$$\begin{aligned}
\max_{b_1} V_1(.) &= \alpha_1 \ln(14 - qb_1 + \tau) + \beta_1 \ln(2 + b_1) + (1 - \alpha_1 - \beta_1) \ln(2 + b_1) + G_1(p^{12}, p^{22}), \\
\max_{b_2} V_2(.) &= \alpha_2 \ln\left(-qb_2 - \frac{\tau}{2}\right) + \beta_2 \ln\left(20p^{12} + 4 + b_2\right) + (1 - \alpha_2 - \beta_2) \ln(14 + b_2) + G_2(p^{12}, p^{22}), \\
\max_{b_3} V_3(.) &= \alpha_3 \ln\left(-qb_3 - \frac{\tau}{2}\right) + \beta_3 \ln(14 + b_3) + (1 - \alpha_3 - \beta_3) \ln\left(20p^{22} + 4 + b_3\right) + G_3(p^{12}, p^{22}).
\end{aligned}$$

It is easy to check that, at $\left(\overline{p}^{12}, \overline{p}^{22}, \overline{q}\right) = (1,1,1)$ and $\overline{t} = (0,0,0)$, $\left(\overline{b}_1, \overline{b}_2, \overline{b}_3\right) = (8, -4, -4)$ are the optimal portfolios if and only if

$$\alpha_1 = \frac{3}{8}, \ \beta_2 = (2 - 7\alpha_2), \text{ and } \beta_3 = (6\alpha_3 - 1).$$

This, and the positivity constraints on the parameters of the utility functions, imply that, for $h = 1, 2$, $\alpha_h \in \left(\frac{1}{6}, \frac{2}{7}\right)$.
Using these properties, and applying the implicit function thm. to the FOCs of the three optimization problems, we obtain

	$\frac{\partial b_h}{\partial p^{12}}$	$\frac{\partial b_h}{\partial p^{22}}$	$\frac{\partial b_h}{\partial q}$	$\frac{\partial b_h}{\partial \tau}$
$h = 1:$	0	0	$-\frac{70}{8}$	$\frac{5}{8}$
$h = 2:$	$\frac{280\alpha_2 - 80}{84\alpha_2 - 4}$	0	0	$\frac{25\alpha_2}{4 - 84\alpha_2}$
$h = 3:$	0	$\frac{280\alpha_3 - 80}{84\alpha_3 - 4}$.	0	$\frac{25\alpha_3}{4 - 84\alpha_3}$

The simple structure of these derivatives follows from our specific choice of the endowments and of the value of k.
We can now compute the values of $\left[\frac{\partial p^{12}}{\partial \tau}, \frac{\partial p^{22}}{\partial \tau}, \frac{\partial q}{\partial \tau}\right]$. Define the equilibrium map

$$\Phi(p, q, \tau) \equiv \left[\ \sum_h b_h(.), \sum_h z_h^{11}(.), \sum_h z_h^{21}(.)\ \right] = 0.$$

[9] Given portfolios, and for $k = \frac{1}{3}$, the spot equilibrium price $p^{s2} = 1$ is critical. For the entire intertemporal economy, the equilibrium associated with $p^{s2} = 1$ may, or may not, be critical, because changes in commodity prices affect the optimal porfolios and, consequently, the aggregate excess demand for the commodities. As we will show, in this example, the intertemporal equilibrium is, in fact, regular.

By the implicit function theorem,

$$\left[\frac{\partial p^{12}}{\partial \tau}, \frac{\partial p^{22}}{\partial \tau}, \frac{\partial q}{\partial \tau}\right]^T = -\left[D_{(p,q)}\Phi(.)\right]^{-1} D_\tau \Phi(.).$$

As shown in Appendix, because of the particular - trichotomous - structure of our economy,

$$\left[\frac{\partial p^{12}}{\partial \tau}, \frac{\partial p^{22}}{\partial \tau}, \frac{\partial q}{\partial \tau}\right]^T = -\left[\frac{\frac{\partial b_2}{\partial \tau}}{\frac{\partial b_2}{\partial p^{12}}}, \frac{\frac{\partial b_3}{\partial \tau}}{\frac{\partial b_3}{\partial p^{22}}}, \frac{\frac{\partial b_1}{\partial \tau}}{\frac{\partial b_1}{\partial q}}\right].$$

Using the previous results, this implies

$$\left[\frac{\partial p^{12}}{\partial \tau}, \frac{\partial p^{22}}{\partial \tau}, \frac{\partial q}{\partial \tau}\right]^T = \left[\frac{5\alpha_2}{56\alpha_2 - 16}, \frac{5\alpha_3}{56\alpha_3 - 16}, \frac{1}{14}\right].$$

Consider now the effect of a period 0 endowment reallocation on the equilibrium utilities:

$$\begin{aligned}
\frac{\partial V_1}{\partial \tau} &= \lambda_1^0 + \lambda_1^0\left(-\overline{b}_1\right)\frac{\partial q}{\partial \tau} + \lambda_1^1\left(-\overline{z}_1^{12}\right)\frac{\partial p^{12}}{\partial \tau} + \lambda_1^2\left(-\overline{z}_1^{22}\right)\frac{\partial p^{22}}{\partial \tau}, \\
\frac{\partial V_h}{\partial \tau} &= -\frac{\lambda_h^0}{2} + \lambda_h^0\left(-\overline{b}_h\right)\frac{\partial q}{\partial \tau} + \lambda_h^1\left(-\overline{z}_h^{12}\right)\frac{\partial p^{12}}{\partial \tau} + \lambda_h^2\left(-\overline{z}_h^{22}\right)\frac{\partial p^{22}}{\partial \tau}, \text{ for } h > 1.
\end{aligned}$$

Using $\lambda_1 = \left(\frac{3}{48}, \frac{\beta_1}{10}, \frac{\frac{5}{8}-\beta_1}{10}\right)$, $\lambda_2 = \left(\frac{\alpha_2}{4}, \frac{\beta_2}{20}, \frac{1-\alpha_2-\beta_2}{10}\right)$ *and* $\lambda_3 = \left(\frac{\alpha_3}{4}, \frac{\beta_3}{10}, \frac{1-\alpha_3-\beta_3}{20}\right)$, *and the values of the excess demands computed above, we obtain*

$$\begin{aligned}
\frac{\partial V_1}{\partial \tau} &= \frac{3}{112} - \frac{\beta_1}{4}\left(\frac{5\alpha_2}{56\alpha_2 - 16}\right) - \frac{\frac{5}{8} - \beta_1}{4}\left(\frac{5\alpha_3}{56\alpha_3 - 16}\right), \\
\frac{\partial V_2}{\partial \tau} &= -\frac{3}{56}\alpha_2 - \frac{5}{32}\alpha_2 - \frac{6\alpha_2 - 1}{4}\left(\frac{5\alpha_3}{56\alpha_3 - 16}\right), \\
\frac{\partial V_3}{\partial \tau} &= -\frac{3}{56}\alpha_3 - \frac{5}{32}\alpha_3 - \frac{6\alpha_3 - 1}{4}\left(\frac{5\alpha_2}{56\alpha_2 - 16}\right).
\end{aligned}$$

For $h = 2, 3$, $\alpha_h < \frac{2}{7}$, *so that* $(56\alpha_h - 16) < 0$. *Since* $\alpha_1 = \frac{3}{8}$, *it must be* $\beta_1 < \frac{5}{8}$. *Hence,* $\frac{\partial V_1}{\partial \tau}$ *is strictly positive and monotonically increasing in* (α_2, α_3). *Similarly,* $\frac{\partial V_2}{\partial \tau}$ *is monotonically increasing in* α_3 *and divergent for* $\alpha_3 \to \frac{2}{7}$ *(while* $\frac{\partial V_3}{\partial \tau}$ *has the same properties with respect to* α_2*). It follows that, for* α_2 *and* α_3 *close to* $\frac{2}{7}$, $\left(..., \frac{\partial V_h}{\partial \tau}, ...\right) >> 0$. *This is shown in Figure 2, which reports the values of* $\frac{\partial V_1}{\partial \tau}$ *(the thick curve) and* $\frac{\partial V_2}{\partial \tau} = \frac{\partial V_3}{\partial \tau}$ *(the thin one) for a range of values of* $\alpha_2 = \alpha_3 \in \left(\frac{1}{6}, \frac{2}{7}\right)$. *Evidently, for* $\alpha_2 = \alpha_3$ *sufficiently large, each equilibrium is not* $\omega - CPO$.

Our choice of the collection of endowments and utility functions is very specific, to simplify as much as possible the computations and to guarantee uniqueness. However, the substantive results just depend on the fact that the inequality

$\left(..., \frac{\partial V_h}{\partial \tau}, ...\right) >> 0$ *is satisfied for some* τ *sufficiently small.*
We conclude showing that the inefficiency result is robust to perturbations of the parameters, and that the (intertemporal) equilibrium is unique. Pick $\alpha_2 = \alpha_3 = \frac{1}{4}$, $\beta_2 = \frac{1}{4}$ *and* $\beta_3 = \frac{1}{4}$. *Then, at* $\left(\overline{p}^{12}, \overline{p}^{22}, \overline{q}\right) = (1,1,1)$,

$$\left(..., \frac{\partial V_h}{\partial \tau}, ...\right) = \left(\frac{223}{1792}, \frac{23}{896}, \frac{23}{896}\right) >> 0.$$

Moreover, by direct computation, the equilibrium map $\Phi(p, q, \tau = 0)$ *satisfies* $detD_{(p,q)}\Phi(.) = -\frac{875}{1156} \neq 0$. *Hence, such an equilibrium is regular. By continuity, the same properties are satisfied for each economy in some relatively open neighborhood of the given economy. In particular, they are also satisfied for some open set of economies with* $\omega \in \mathbb{R}^5_{++}$. *In Appendix, we also show that the intertemporal equilibrium* $\left(\overline{p}^{12}, \overline{p}^{22}, \overline{q}\right) = (1,1,1)$ *is the unique equilibrium of the economy described above. Its regularity implies that there is some (relatively) open neighborhood of economies with a unique equilibrium, that, as we have already shown, is not* $\omega-$*CPO.*

FIGURE 2 GOES HERE

In the example, we have chosen k so that the *spot* equilibria are critical. This simplifies a lot the computations, but nothing of relevance rests on it. We also propose another example, in Appendix, with two agents, obtaining a similar result. Some of the spot utility functions are CES as in Example 5.

Remark 6 *Example 5 can also be used to show that a pure portfolio reallocation of a single asset can be sufficient to guarantee a Pareto improvement: Figure 3 presents the values of the derivatives of the indirect utility functions of the three agents for an arbitrarily given change the portfolios which happens to be identical to the one induced by the endowment reallocation. Here,* $V_h^*(p(b), q(b), b_h)$ *captures the impact of the price change due to a reallocation of the portfolios. We just consider the effects of the changes in spot commodity prices in the future periods. There is an open range of values of the parameters such that the utility of each agent is increasing. Once again, the dimension of the policy profile is* $(H-1)$, *but we can Pareto improve upon the equilibrium allocation.*

FIGURE 3 GOES HERE

We can now state our second result in a somewhat more general form, showing that some of the features of the examples are not essential: they just allow for computational feasibility.

Proposition 7 *There are open set of economies* $\boldsymbol{E}^\circ \subset \boldsymbol{E}$ *with equilibria which are not* $\omega-$*CPO. This also holds if we restrict the class of economies to time-separable utility functions or to VNM utility functions.*

Proof. The first two results are established by Example 5 above. Since the equilibrium of the economy constructed in the example is regular and the payoff matrix has full rank, small changes of the parameters will not break down the results, so that it holds for an open set of economies. Evidently, this holds both if we consider the general space of economies, and if we restrict the analysis to economies with time-separable preferences, because preferences in Example 5 are in fact time-separable.

Therefore, we just need to argue that a similar result holds for VNM utility functions, since preferences in the example are not VNM. The easiest way to proceed is to consider a three-period economy, with a realization of uncertainty only at period 2. Consider an economy as the one described in Example A1. Add a second state in period 2. Endowments and preferences in the two states, each having probability $\frac{1}{2}$, are identical. It is easy to check that this new, sunspot-like, economy has the same market clearing conditions as the economy with three periods and no uncertainty analyzed in Example A1. It is also easy to check that, by the same argument reported in Appendix, for this VNM economy, there is a Pareto improving endowment reallocation. Using a, by now, standard argument, one can show that, modulo some arbitrarily small perturbation of the parameters, the regular equilibrium of the economy of Example A1 induces a regular equilibrium in the associated economy with (trivial) uncertainty in period 2. To conclude, perturb period 2 endowments in different directions in the two states, introducing intrinsic uncertainty. Regularity of the initial equilibrium guarantees that, provided that the perturbations are sufficiently small, $\omega-$CP suboptimality is preserved at the corresponding equilibrium for all the economies in some sufficiently small, but open, neighborhood of the economy we started with. ■

4 Conclusion

We have considered the canonical GEI model with numeraire assets, and we have shown that there are open sets of economies such that their equilibria can be improved upon by an appropriate reallocation of period zero initial endowments. We have also shown that, for each economy defined in terms of utility functions and asset payoffs, there are open sets of endowments such that it is impossible to attain any Pareto improvement by pure period zero endowment reallocation. Hence, our result is weaker than the generic one obtainable when the policy profile is the portfolio of each agent, as in Geanakoplos and Polemarchakis (1996) and Citanna et al. (1998). Still, we believe that it settles an open issue in the literature on constrained inefficiency in GEI and that it contributes to a better understanding of this phenomenon.

It remains an open issue under which general conditions existence of an open set of endowments with $\omega-$CP suboptimal equilibria generically holds in the space of the economies defined by asset structure and preferences. Our analysis shows that the key ingredients are the matrix Λ with typical element $[-\lambda_h^s z_h^{sc}]$ (or $\left[-\lambda_h^0 b_h^j\right]$), and the matrices $D_{(p,q)}\Phi(.)$ and $D_t\Phi(.)$. Generically in

(u,ω) −space, Λ has full row rank H. This essentially requires some degree of heterogeneity across agents. Next, we need that there is some vector t such that $\left[\Lambda_h \left[D_{(p,q)}\Phi(.)\right]^{-1} D_t\Phi(.)t\right] t_h > 0$ for each h such that $t_h < 0$, i.e., the second order effect must increase the utility of the agents with $t_h < 0$. Finally, we need that these second order effects are sufficiently strong, so that they can overcome the (possibly) negative first order effects. This may be guaranteed if we are sufficiently close to a critical equilibrium. This motivates our conjecture: provided that an economy (defined by utilities and asset structure), has a critical equilibrium, then, with sufficient heterogeneity, there is some open set of endowments such that at least one equilibrium is not $\omega - CPO$.[10]

5 Appendix

Proof of Lemma 3. Consider any commodity $\overline{sc}$. Then,

$$\begin{aligned}\frac{\partial V_h}{\partial p^{\overline{sc}}} &= \sum_{sc} \frac{\partial u_h}{\partial x_h^{sc}} \frac{\partial x_h^{sc}}{\partial p^{\overline{sc}}} = \sum_s \lambda_h^s \sum_c p^{sc} \frac{\partial x_h^{sc}}{\partial p^{\overline{sc}}} \\ &= -\lambda_h^{\overline{s}} z_h^{\overline{sc}} + \sum_j \left[-\lambda_h^0 q^j \frac{\partial b_h^j}{\partial p^{\overline{sc}}} + \sum_s \lambda_h^s r^{sj} \frac{\partial b_h^j}{\partial p^{\overline{sc}}} \right] = -\lambda_h^{\overline{s}} z_h^{\overline{sc}}\end{aligned}$$

The last two equalities are obtained taking the derivative of the budget constraint in spot s,

$$\begin{aligned}\sum_c p^{sc} \frac{\partial x_h^{sc}}{\partial p^{\overline{sc}}} &= \sum_j r^{sj} \frac{\partial b_h^j}{\partial p^{\overline{sc}}}, \text{ if } s \neq \overline{s}, \\ \sum_c p^{sc} \frac{\partial x_h^{sc}}{\partial p^{\overline{sc}}} &= -z_h^{\overline{sc}} + \sum_j r^{sj} \frac{\partial b_h^j}{\partial p^{\overline{sc}}}, \text{ if } s = \overline{s},\end{aligned}$$

and taking into account the noarbitrage conditions.

Similarly,

$$\begin{aligned}\frac{\partial V_h}{\partial q^{\overline{j}}} &= \sum_{sc} \frac{\partial u_h}{\partial x_h^{sc}} \frac{\partial x_h^{sc}}{\partial q^{\overline{j}}} = \sum_s \lambda_h^s \sum_c p^{sc} \frac{\partial x_h^{sc}}{\partial q^{\overline{j}}} = -\lambda_h^0 b_h^{\overline{j}} + \sum_j \left[-\lambda_h^0 q^j \frac{\partial b_h^j}{\partial q^{\overline{j}}} + \sum_s \lambda_h^s r^{sj} \frac{\partial b_h^j}{\partial q^{\overline{j}}} \right] \\ &= -\lambda_h^0 b_h^{\overline{j}}.\end{aligned}$$

Finally, $\frac{\partial V_h}{\partial t_h} = \lambda_h^0$ is obvious. ■

Proof of Proposition 4. Given (u, R), pick a Pareto optimal endowment profile, $\overline{\omega}$. By a standard argument, the equilibrium is locally unique and

[10] This conjecture should remind of Safra (1981), concerning the transfer paradox.

regular. Moreover, for each agent, excess demand and portfolio are identically zero, while, for each $\widehat{t}$,

$$\frac{1}{\overline{\lambda}_h^0}\left[\overrightarrow{\frac{\partial V_h}{\partial t}}\widehat{t}\right]=\widehat{t}_h-\sum_j\overline{b}_h\left[\overrightarrow{\frac{\partial q^j}{\partial t}}\widehat{t}\right]-\sum_{sc}\frac{\overline{\lambda}_h^s}{\overline{\lambda}_h^0}\overline{z}_h^{sc}\left[\overrightarrow{\frac{\partial p^{sc}}{\partial t}}\widehat{t}\right]=\widehat{t}_h.$$

Since the original equilibrium is regular, the vector $\left[...,\left[\overrightarrow{\frac{\partial q^j}{\partial t}}\widehat{t}\right],...,\left[\overrightarrow{\frac{\partial p^{sc}}{\partial t}}\widehat{t}\right],...\right]$ is uniformly bounded above for each $(t,\omega)\in S\times V(\overline{\omega})$, any sufficiently small open neighborhood of $(0,\overline{\omega})$.

We break the argument into two parts and we start considering "small" vectors t. Let S be the set of possible profiles $\{t_1,...,t_H\}$, with $\sum_h t_h=0$ and $||t||$ sufficiently small. Given $\overline{\omega}$, assume that, for each open neighborhood $B_n(\overline{\omega})$ there is an endowment profile, $\omega^n\in B_n(\overline{\omega})$, such that the unique associated equilibrium allocation, $x(\omega^n)$, is not $\omega-$CPO. Then, there is a profile $t^n\in S$ such that the associated (unique) equilibrium $x(\omega^n,t^n)$ Pareto dominates $x(\omega^n,t=0)$. Bear in mind that, for t^n sufficiently small, this implies $\left[\overrightarrow{\frac{\partial V_h}{\partial t^n}}t^n\right]\geq 0$.

Define accordingly the sequence $\{(\omega^n,t^n)\}_{n=1}^{n=\infty}$. Evidently, for each n, $t^n\neq 0$. Pick any subsequence, without loss of generality the sequence itself, such that, for each n, the same agent, say agent 1, has $t_1^n\leq t_h^n$ for each h. Clearly, for each n, it must be $t_1^n<0$, and, by construction, $\left[\frac{t^n}{|t_1^n|}\right]\subset[-1,H-1]^H$, for each n, so that the sequence is bounded. Finally, along this sequence,

$$\begin{aligned}\frac{1}{\overline{\lambda}_1^{0n}|t_1^n|}\left[\overrightarrow{\frac{\partial V_1}{\partial t^n}}t^n\right] &= \frac{t_1^n}{|t_1^n|}\left[1-\sum_j\overline{b}_1^n\left[\frac{\partial q^{jn}}{\partial t_1^n}\right]-\sum_{sc}\frac{\overline{\lambda}_1^{sn}}{\overline{\lambda}_1^{0n}}\overline{z}_1^{scn}\left[\frac{\partial p^{scn}}{\partial t_1^n}\right]\right]\\ &\quad-\sum_{h>1}\frac{t_h^n}{|t_1^n|}\left[\sum_j\overline{b}_1^n\left[\frac{\partial q^{jn}}{\partial t_h^n}\right]+\sum_{sc}\frac{\overline{\lambda}_1^{sn}}{\overline{\lambda}_1^{0n}}\overline{z}_1^{scn}\left[\frac{\partial p^{scn}}{\partial t_h^n}\right]\right].\end{aligned}$$

Since $\left(\overline{b}_1^n,\overline{z}_1^{scn}\right)\to 0$, while, locally, $\left(\frac{\partial q^{jn}}{\partial t_h^n},\frac{\partial p^{scn}}{\partial t_h^n}\right)$ can be taken to be uniformly bounded for each h, it must be

$$\lim_{n\to\infty}\frac{1}{\overline{\lambda}_1^{0n}|t_1^n|}\left[\overrightarrow{\frac{\partial V_1}{\partial t^n}}t^n\right]=\lim_{n\to\infty}\frac{t_1^n}{|t_1^n|}=-1.$$

By continuity, this contradicts the initial claim that, for each n, $\left[\overrightarrow{\frac{\partial V_h}{\partial t^n}}t^n\right]\geq 0$ for each h.

We now consider arbitrary vectors t. By contradiction, assume that there is no open neighborhood of $\overline{\omega}$, $B(\overline{\omega})$, such that the (unique) equilibrium associated

with $\omega \in B(\overline{\omega})$ is $\omega-$CPO. We proceed as above. The only difference is that, in view of the previous result, we can consider only sequences $\{\omega^n\}_{n=1}^{n=\infty} \subset B(\overline{\omega})$, $\omega^n \to \overline{\omega}$, such that the associated sequence $\{t^n\}_{n=1}^{n=\infty}$ satisfies $t^n \to \overline{t} \neq 0$.

Let $\{y(\omega^n, t^n)\}_{n=1}^{n=\infty}$ be the sequence of the equilibrium allocations with, for each n, $y_h(\omega^n, t^n) \succsim_h x_h(\omega^n, t=0)$, for each h, and $y_h(\omega^n, t^n) \succ_h x_h(\omega^n, t=0)$, for some h. Given that $\{y(\omega^n, t^n)\}_{n=1}^{n=\infty}$ is a sequence of feasible allocations (and $\omega^v \to \overline{\omega}$), it lies in some compact set, so that, without loss of generality, we can assume that $y(\omega^n, t^n) \to \overline{y}$. Since $\overline{\omega}$ is Pareto optimal, it cannot be $\overline{y}_h \succsim_h \overline{\omega}$, for each h. Otherwise, there would be a feasible allocation $(\phi\overline{y} + (1-\phi)\overline{\omega})$, $\phi \in (0,1)$, such that $(\phi\overline{y}_h + (1-\phi)\, x_h(\overline{\omega})) \succ_h \overline{\omega}_h$, for each h, contradicting PO of $\overline{\omega}$. Then, regularity of equilibria and continuity of preferences imply that, for sufficiently small neighborhoods of $\overline{y}$ and $\overline{\omega}$, $B_\varepsilon(\overline{y})$ and $B_\varepsilon(\overline{\omega})$, there are no feasible allocations $\widetilde{y} \in B_\varepsilon(\overline{y})$, $\widetilde{x} \in B_\varepsilon(\overline{\omega})$ such that $\widetilde{y}_h \succsim_h \widetilde{x}_h$ for each h. This contradicts our original claim.

Hence, for all the economies in some open neighborhood of $\overline{\omega}$, each equilibrium is $\omega-$CPO. ■

Example 5: a. Computation of $\left[\frac{\partial p^{12}}{\partial \tau}, \frac{\partial p^{22}}{\partial \tau}, \frac{\partial q}{\partial \tau}\right]$.

Because of the endowment profile, $\frac{\partial FOC_1}{\partial p^{12}} = \frac{\partial FOC_1}{\partial p^{22}} = 0$, $\frac{\partial FOC_2}{\partial q} = \frac{\partial FOC_2}{\partial p^{22}} = 0$, and $\frac{\partial FOC_3}{\partial q} = \frac{\partial FOC_3}{\partial p^{12}} = 0$. Hence,

$$D_{(p^{12},p^{22},q,\tau)}\Phi(p,q,\tau) = \begin{bmatrix} \frac{\partial b_2}{\partial p^{12}} & \frac{\partial b_3}{\partial p^{22}} & \frac{\partial b_1}{\partial q} & \sum_h \frac{\partial b_h}{\partial \tau} \\ \frac{\partial x_2^{11}}{\partial b_2}\frac{\partial b_2}{\partial p^{12}} + \frac{\partial \overline{Z}^{11}}{\partial p^{12}} & 0 & \frac{\partial x_1^{11}}{\partial b_1}\frac{\partial b_1}{\partial q} & \sum_h \frac{\partial x_h^{11}}{\partial b_h}\frac{\partial b_h}{\partial \tau} \\ 0 & \frac{\partial x_3^{12}}{\partial b_3}\frac{\partial b_3}{\partial p^{22}} + \frac{\partial \overline{Z}^{21}}{\partial p^{22}} & \frac{\partial x_1^{21}}{\partial b_1}\frac{\partial b_1}{\partial q} & \sum_h \frac{\partial x_h^{21}}{\partial b_h}\frac{\partial b_h}{\partial \tau} \end{bmatrix},$$

where $\frac{\partial \overline{Z}^{s1}}{\partial p^{s2}}$ is computed for a given portfolio $\overline{b}$, $\frac{\partial \overline{Z}^{s1}}{\partial p^{s2}}|_{\overline{p}=1} = \left[\frac{20}{3}\ \frac{3k^2-k}{(k+1)(k+1)}\right]$.

Evidently,

$$\begin{aligned} \frac{\partial x_1^{s1}}{\partial b_1}|_{\overline{p}^{s2}=1} &= \frac{1}{k+1}, \left[\frac{\partial x_2^{11}}{\partial b_2}|_{\overline{p}^{12}=1}, \frac{\partial x_2^{21}}{\partial b_2}|_{\overline{p}^{22}=1}\right] = \left[\frac{k}{k+1}, \frac{1}{1+k}\right], \text{ and} \\ \left[\frac{\partial x_3^{11}}{\partial b_3}|_{\overline{p}^{12}=1}, \frac{\partial x_3^{21}}{\partial b_3}|_{\overline{p}^{22}=1}\right] &= \left[\frac{1}{k+1}, \frac{k}{1+k}\right] \end{aligned}$$

Set $k = \frac{1}{3}$, so that $\frac{\partial \overline{Z}^{s1}}{\partial p^{s2}} = 0$. Then, we can write

$$D_{(p,q,\tau)}\Phi(p,q,\tau) = \begin{bmatrix} 1 & 1 & 1 \\ \frac{k}{1+k} & \frac{1}{k+1} & \frac{1}{k+1} \\ \frac{1}{k+1} & \frac{k}{1+k} & \frac{1}{k+1} \end{bmatrix} \begin{bmatrix} \frac{\partial b_2}{\partial p^{12}} & 0 & 0 & \frac{\partial b_2}{\partial \tau} \\ 0 & \frac{\partial b_3}{\partial p^{22}} & 0 & \frac{\partial b_3}{\partial \tau} \\ 0 & 0 & \frac{\partial b_1}{\partial q} & \frac{\partial b_1}{\partial \tau} \end{bmatrix}.$$

Therefore, by the implicit function theorem,

$$\begin{aligned}
\begin{bmatrix} \frac{\partial p^{12}}{\partial \tau} \\ \frac{\partial p^{22}}{\partial \tau} \\ \frac{\partial q}{\partial \tau} \end{bmatrix} &= -\begin{bmatrix} 1/\frac{\partial b_2}{\partial p^{12}} & 0 & 0 \\ 0 & 1/\frac{\partial b_3}{\partial p^{22}} & 0 \\ 0 & 0 & 1/\frac{\partial b_1}{\partial q} \end{bmatrix} \begin{bmatrix} \frac{\partial b_2}{\partial \tau} \\ \frac{\partial b_3}{\partial \tau} \\ \frac{\partial b_1}{\partial \tau} \end{bmatrix} \\
&= -\left[\frac{\frac{\partial b_2}{\partial \tau}}{\frac{\partial b_2}{\partial p^{12}}}, \frac{\frac{\partial b_3}{\partial \tau}}{\frac{\partial b_3}{\partial p^{22}}}, \frac{\frac{\partial b_1}{\partial \tau}}{\frac{\partial b_1}{\partial q}} \right]^T .
\end{aligned}$$

b. Global uniqueness of the intertemporal equilibrium.

Fix $\alpha_2 = \alpha_3 = \frac{1}{4}$, $\beta_2 = \frac{1}{4}$ and $\beta_3 = \frac{1}{4}$. To simplify notation, let $\sqrt[3]{p^{12}} \equiv \phi_1$ and $\sqrt[3]{p^{22}} \equiv \phi_2$. The equilibrium map is

$$\begin{aligned}
\sum_h b_h(\phi_1, \phi_2, q) &= \frac{35}{4q} - \frac{3}{4} - 10 + \sum_{s=1,2} \left(\frac{1}{2}\sqrt{225\phi_s^6 + 20\phi_s^3 + 44} - \frac{15}{2}\phi_s^3 \right) = 0, \\
\sum_h z_h^{11}(\phi_1, \phi_2, q) &= \left(\frac{3\left(16 + b_1(.) + b_3(.)\right)\phi_1}{\phi_1^3 + 3\phi_1} + \frac{\left(4 + 20\phi_1^3 + b_2(.)\right)\phi_1}{3\phi_1^3 + \phi_1^{\frac{1}{3}}} - 20 \right) = 0, \\
\sum_h z_h^{21}(\phi_1, \phi_2, q) &= \left(\frac{3\left(16 + b_1(.) + b_2(.)\right)\phi_2}{\phi_2^3 + 3\phi_2} + \frac{\left(4 + 20\phi_2^3 + b_3(.)\right)\phi_2}{3\phi_2^3 + \phi_2^{\frac{1}{3}}} - 20 \right) = 0.
\end{aligned}$$

Since $(b_2(.), b_3(.))$ do not depend upon q, for any given pair (ϕ_1, ϕ_2), while $b_1(.)$ just depends upon q, we can always find q solving the first eq. Hence, we can exploit market clearing on the asset market to replace $(b_1(.) + b_3(.))$ with $-b_2(.)$ in the second eq. (and $(b_1(.) + b_2(.))$ with $-b_3(.)$ in the third). Now, the second and third eqs. are independent and can be solved and analyzed separately. Consider, for instance, the last eq. (by symmetry the same argument holds for the second) and replace $b_2(.)$ with its explicit formula. Then, the eq. reduces to

$$F(\phi_1) \equiv \frac{N(\phi_1)}{D(\phi_1)} \equiv \frac{15\phi_1^2 - 15\phi_1 - 20\phi_1^3 + \sqrt{225\phi_1^6 + 20\phi_1^3 + 44} + 3}{3\phi_1^4 + 10\phi_1^2 + 3} = 0$$

At each equilibrium, $\frac{dF}{d\phi_1} = \frac{\frac{\partial N(\phi_1)}{\partial \phi_1}}{D(\phi_1)}$. At $\overline{\phi}_1 = 1$, $\frac{dF}{d\phi_1} = -\frac{60}{17}$. Evidently, if at each possible equilibrium $\frac{dF}{d\phi_1} < 0$, then the equilibrium must be unique. By direct computation, at each equilibrium,

$$sign \frac{dF}{d\phi_1} = sign \left(2\phi_1^2 + 45\phi_1^5 - \left(4\phi_1^2 + 1 - 2\phi_1\right)\sqrt{225\phi_1^6 + 20\phi_1^3 + 44} \right).$$

Simple computations show that, for each $\phi_1 > 0$,

$$2\phi_1^2 + 45\phi_1^5 - \left(4\phi_1^2 + 1 - 2\phi_1\right)\sqrt{225\phi_1^6 + 20\phi_1^3 + 44} < 0.$$

Hence, the equilibrium is globally unique. ■

Example A1: There is just one commodity at time 0 and at time 2. This entails no essential loss of generality. Preferences of agent 1 and 2 at spot 1 are as in the previous example. The utility functions are

$$u_h(.) = \alpha_h \ln x_h^0 + \beta_h \ln v_h^1\left(x_h^{11}, x_h^{12}\right) + (1-\alpha_h-\beta_h)\ln x_h^2.$$

Endowments are $\omega_1 = (10,(6,0),0)$, and $\omega_2 = (0,(4,10),10)$. As in Example 5, the boundary nature of the endowment profile does not imply any substantive loss of generality.
Essentially as above, and omitting the redundant superscript for the price of commodity 2 at spot 1,

$$x_1^1 = \left(\frac{\left(6+\overline{b}_1\right)p^{\frac{1}{3}}}{kp+p^{\frac{1}{3}}}, \frac{\left(6+\overline{b}_1\right)k}{kp+p^{\frac{1}{3}}}\right), \qquad x_2^1 = \left(\frac{\left(10p+4+\overline{b}_2\right)kp^{\frac{1}{3}}}{kp^{\frac{1}{3}}+p}, \frac{10p+4+\overline{b}_2}{kp^{\frac{1}{3}}+p}\right).$$

and

$$V_1^1\left(p,\overline{b}_1\right) \equiv \left(6+\overline{b}_1\right)\left[\left(\frac{p^{\frac{1}{3}}}{kp+p^{\frac{1}{3}}}\right)^{-2} + k^3\left(\frac{kp^{-\frac{1}{3}}}{kp+p^{\frac{1}{3}}}\right)^{-2}\right]^{-\frac{1}{2}} \equiv \left(6+\overline{b}_1\right)g_1^1(p).$$

The result for agents 2 is similar. For the given ω and $\overline{b}$, $\overline{p}=1$ is an equilibrium for each $k>0$. The derivative of the excess demand function depends upon k. It is given by $\frac{\partial \overline{Z}^{11}}{\partial p}|_{\overline{p}=1} = \left(\frac{10}{3}\frac{3k^2-k}{(k+1)(k+1)}\right)$.

Evidently,

$$D_{(p,q,t)}\Phi(p,q,t) = \begin{bmatrix} \frac{\partial b_2}{\partial p} & \frac{\partial b_1}{\partial q} & \sum_h \frac{\partial b_h}{\partial t} \\ \frac{\partial x_2^{11}}{\partial b_2}\frac{\partial b_2}{\partial p} + \frac{\partial \overline{Z}^{11}}{\partial p} & \frac{\partial x_1^{11}}{\partial b_1}\frac{\partial b_1}{\partial q} & \sum_h \frac{\partial x_h^{11}}{\partial b_h}\frac{\partial b_h}{\partial t} \end{bmatrix},$$

and we can rewrite the two blocks as

$$D_{(p,q)}\Phi(p,q) = \begin{bmatrix} 1 & 1 \\ \frac{\partial x_2^{11}}{\partial b_2} + \frac{\frac{\partial \overline{Z}^{11}}{\partial p}}{\frac{\partial b_2}{\partial p}} & \frac{\partial x_1^{11}}{\partial b_1} \end{bmatrix} \begin{bmatrix} \frac{\partial b_2}{\partial p} & 0 \\ 0 & \frac{\partial b_1}{\partial q} \end{bmatrix}$$

$$D_t\Phi(p,q) = \begin{bmatrix} 1 & 1 \\ \frac{\partial x_1^{11}}{\partial b_1} & \frac{\partial x_2^{11}}{\partial b_2} \end{bmatrix} \begin{bmatrix} \frac{\partial b_1}{\partial t} \\ \frac{\partial b_2}{\partial t} \end{bmatrix}$$

Let $\det \equiv \left[\frac{\partial x_1^{11}}{\partial b_1} - \frac{\partial x_2^{11}}{\partial b_2} - \frac{\frac{\partial \hat{Z}^{11}}{\partial p}|_{\overline{p}=1}}{\frac{\partial b_2}{\partial p}}\right] = \left[\frac{1-k}{1+k} - \frac{\frac{\partial \hat{Z}^{11}}{\partial p}|_{\overline{p}=1}}{\frac{\partial b_2}{\partial p}}\right]$. Then,

$$\begin{bmatrix} 1 & 1 \\ \frac{\partial x_2^{11}}{\partial b_2} + \frac{\frac{\partial \overline{Z}^{11}}{\partial p}}{\frac{\partial b_2}{\partial p}} & \frac{\partial x_1^{11}}{\partial b_1} \end{bmatrix}^{-1} = \frac{1}{\det}\begin{bmatrix} \frac{1}{1+k} & -1 \\ -\left(\frac{k}{1+k} + \frac{\frac{\partial \overline{Z}^{11}}{\partial p}}{\frac{\partial b_2}{\partial p}}\right) & 1 \end{bmatrix},$$

and

$$\left[\frac{\partial p}{\partial t},\frac{\partial q}{\partial t}\right]^T = -D_{(p,q)}\Phi(.)^{-1}D_t\Phi(.)$$

$$= \frac{-1}{\det}\begin{bmatrix} \frac{1}{\frac{\partial b_2}{\partial p}} & 0 \\ 0 & \frac{1}{\frac{\partial b_1}{\partial q}} \end{bmatrix} \times \begin{bmatrix} \frac{1}{1+k} & -1 \\ -\left(\frac{k}{1+k}+\frac{\frac{\partial \overline{Z}^{11}}{\partial p}}{\frac{\partial b_2}{\partial p}}\right) & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{1}{1+k} & \frac{k}{1+k} \end{bmatrix} \begin{bmatrix} \frac{\partial b_1}{\partial t} \\ \frac{\partial b_2}{\partial t} \end{bmatrix}$$

$$= \left[\frac{\frac{1-k}{1+k}\frac{\frac{\partial b_2}{\partial t}}{\frac{\partial b_2}{\partial p}}}{\left[\frac{\frac{\partial \widehat{Z}^{11}}{\partial p}|_{\overline{p}=1}}{\frac{\partial b_2}{\partial p}}+\frac{k-1}{1+k}\right]}, \quad -\frac{\frac{\partial b_1}{\partial t}}{\frac{\partial b_1}{\partial q}} - \frac{\frac{\frac{\partial \widehat{Z}^{11}}{\partial p}|_{\overline{p}=1}}{\frac{\partial b_1}{\partial q}}\frac{\frac{\partial b_2}{\partial t}}{\frac{\partial b_2}{\partial p}}}{\left[\frac{\frac{\partial \widehat{Z}^{11}}{\partial p}|_{\overline{p}=1}}{\frac{\partial b_2}{\partial p}}+\frac{k-1}{1+k}\right]}\right]^T$$

To conclude, we need to compute $\left(\frac{\partial b_1}{\partial q},\frac{\partial b_1}{\partial t}\right)$ and $\left(\frac{\partial b_2}{\partial p},\frac{\partial b_2}{\partial t}\right)$ using the implicit function thm. applied to the first order conditions of the portfolio optimization problem. First, observe that optimality of $\left(\overline{b}_1,\overline{b}_2\right)=(4,-4)$ at $(\overline{p},\overline{q})=(1,1)$ and the nonnegativity constraint on the values of $(\alpha_h,\beta_h,1-\alpha_h-\beta_h)$ require that $\beta_1(\alpha_1)=\left(\frac{5}{3}-\frac{25}{9}\alpha_1\right)$ and $\alpha_1\in\left(\frac{3}{8},\frac{3}{5}\right)$, while $\beta_2(\alpha_2)=\left(\frac{5}{2}-\frac{25}{4}\alpha_2\right)$ and $\alpha_2\in\left(\frac{2}{7},\frac{2}{5}\right)$.

Consider a negative transfer for agent 1. By direct computation, at $t=0$, $\overline{p}=\overline{q}=1$, $\overline{b}_1=-\overline{b}_2=4$, and using $\beta(\alpha),\delta(\gamma)$:

$$\frac{\partial FOC_1}{\partial t} = -\frac{20}{3}\alpha_1,\quad \frac{\partial FOC_1}{\partial q}=-\frac{200}{3}\alpha_1,\quad \frac{\partial FOC_1}{\partial q}=0,\quad \frac{\partial FOC_1}{\partial b_1}=6-\frac{80}{3}\alpha_1,$$

$$\frac{\partial FOC_2}{\partial t} = 15\alpha_2,\quad \frac{\partial FOC_2}{\partial q}=0,\quad \frac{\partial FOC_2}{\partial p}=(150\alpha_2-60),\quad \frac{\partial FOC_2}{\partial b_2}=(4-35\alpha_2),$$

so that

$$\frac{\partial b_1}{\partial t} = \frac{10\alpha_1}{9-40\alpha_1},\quad \frac{\partial b_1}{\partial q}=\frac{100\alpha_1}{9-40\alpha_1},\quad \frac{\partial b_1}{\partial p}=0$$

and

$$\frac{\partial b_2}{\partial t} = \frac{-15\alpha_2}{4-35\alpha_2},\quad \frac{\partial b_2}{\partial q}=0,\quad \frac{\partial b_2}{\partial p}=-\frac{150\alpha_2-60}{4-35\alpha_2}.$$

Since $\lambda_1=\left(\frac{\alpha_1}{6},\frac{\beta_1}{10},\frac{1-\alpha_1-\beta_1}{4}\right)$ and $\lambda_2=\left(\frac{\alpha_2}{4},\frac{\beta_2}{10},\frac{1-\alpha_2-\beta_2}{6}\right)$, while $\overline{z}_1^{12}=\left(\frac{10k}{1+k}\right)$, and $\overline{z}_2^{12}=\left(-\frac{10k}{1+k}\right)$, replacing (β_1,β_2) with $(\beta_1(\alpha_1),\beta_2(\alpha_2))$, we obtain

$$\frac{\partial V_1}{\partial t} = \left(-1-4\frac{\partial q}{\partial t}\right)\frac{\alpha_1}{6}-\left(\frac{5}{3}-\frac{25}{9}\alpha_1\right)\left(\frac{k}{1+k}\right)\frac{\partial p}{\partial t},$$

$$\frac{\partial V_2}{\partial t} = \left(1+4\frac{\partial q}{\partial t}\right)\frac{\alpha_2}{4}+\left(\frac{5}{2}-\frac{25}{4}\alpha_2\right)\left(\frac{k}{1+k}\right)\frac{\partial p}{\partial t}.$$

Set $\alpha_1 = \frac{38}{100}$ and $\alpha_2 = \frac{39}{100}$. Then,

$$\begin{aligned}
\frac{\partial q}{\partial t} &= \frac{1}{190}\left[\frac{40k + 51k^2 - 171}{570k^2 - 193k + 9}\right], \\
\frac{\partial p}{\partial t} &= -\frac{351}{10}\left[\frac{(k+1)(k-1)}{570k^2 - 193k + 9}\right],
\end{aligned}$$

and $(\overline{q}, \overline{p}) = (1, 1)$ is a critical equilibrium for $k = \left(\frac{193}{1140} \pm \frac{1}{1140}\sqrt{16\,729}\right)$. The rates of change of the indirect utilities are

$$\begin{aligned}
\frac{\partial V_1}{\partial t} &= \left(-1 - \frac{4}{190}\left(\frac{40k + 51k^2 - 171}{570k^2 - 193k + 9}\right)\right)\frac{38}{600} + \frac{3861}{180}\left(\frac{k}{1+k}\right)\left(\frac{(k+1)(k-1)}{570k^2 - 193k + 9}\right) \\
\frac{\partial V_2}{\partial t} &= \left(1 + \frac{4}{190}\left(\frac{40k + 51k^2 - 171}{570k^2 - 193k + 9}\right)\right)\frac{39}{400} - \frac{351}{160}\left(\frac{k}{1+k}\right)\left(\frac{(k+1)(k-1)}{570k^2 - 193k + 9}\right)
\end{aligned}$$

Figure 4 shows their values[11] for $k \in \left(\frac{7}{100}, \frac{27}{100}\right)$, an interval contained in one of the connected components of the equilibrium manifold, defined with respect to k. For values of k in this range, the equilibrium with $(\overline{p}, \overline{q}) = (1, 1)$ is clearly not $\omega-$CPO, since it can be improved upon by a small reallocation of period 0 endowment with $t_1 < 0$.

An argument similar to the one used for Example 5 shows that a similar result holds for a, relatively, open set of economies.

FIGURE 4 GOES HERE

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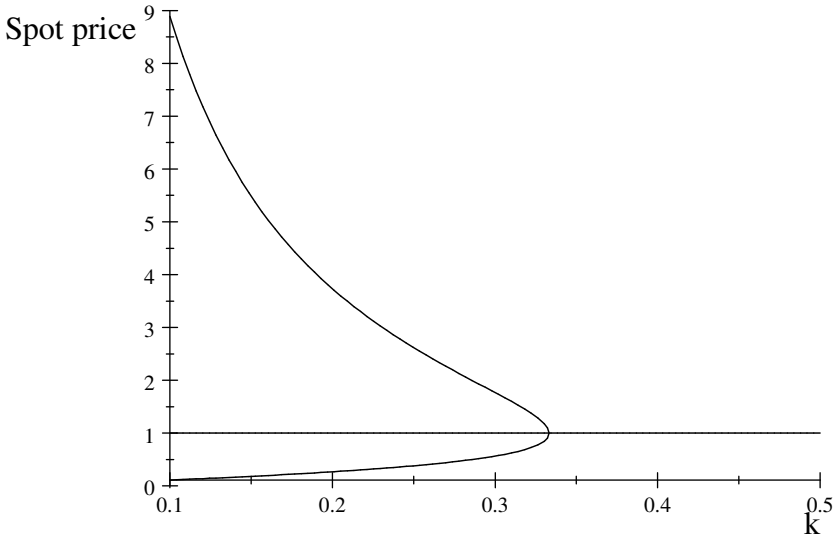

Figure 1: Equilibrium prices $\sqrt[3]{p^{s2}}$ as a function of k

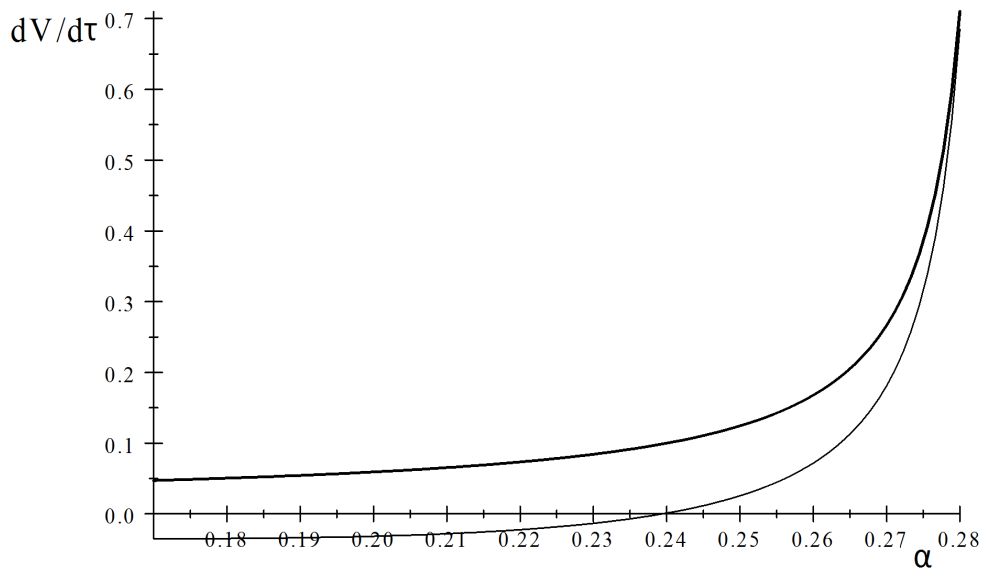

Figure 2: Values of $\frac{\partial V_1}{\partial \tau}, \frac{\partial V_2}{\partial \tau}$ and $\frac{\partial V_3}{\partial \tau}$ as functions of $\alpha_2 = \alpha_3$

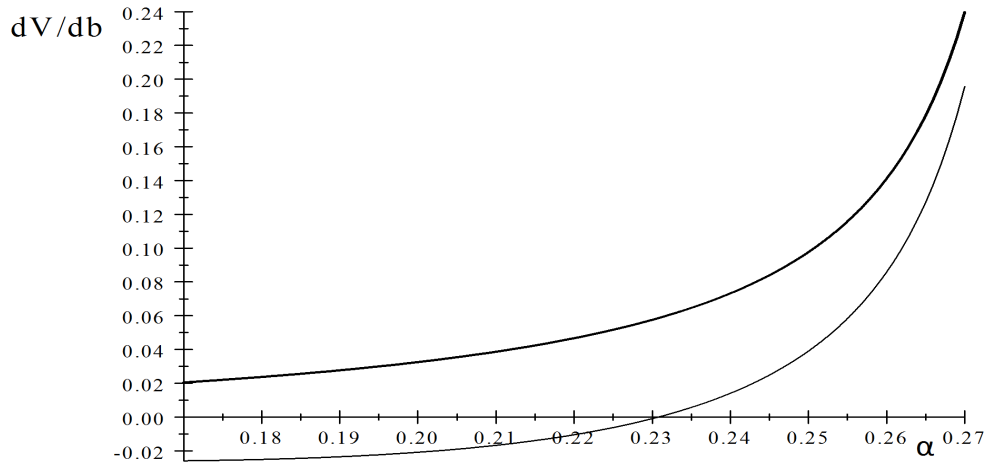

Figure 3: $\frac{\partial V_b}{\partial b}$ due to changes in commodity prices

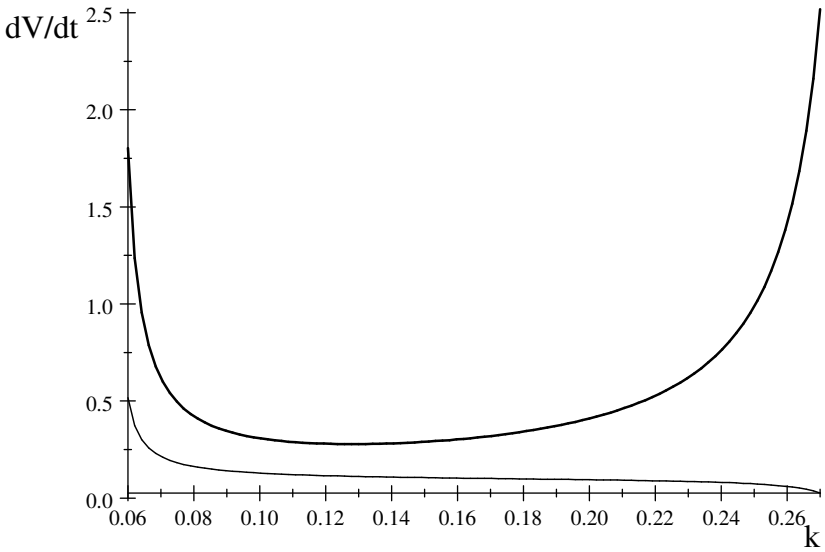

Figure 4: $\frac{\partial V_1}{\partial t}$ and $\frac{\partial V_2}{\partial t}$ at different values of k