Where Porceddu is better than Pasta (DISCUSSION PAPER)

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Abstract. Preferences and contexts are fundamental aspects for deciding the best choices among possible options. We formalize the problem of *propagating preferences* from more generic to more specific contexts and study the key properties of propagation within an algebraic framework.

1 Introduction

Preferences and their influence on choices have been studied in a variety of fields, including psychology, sociology, economics, artificial intelligence, and data management. In all of the above fields, it is widely recognized that preferences highly depend on the *context*, as shown in the following example.

Example 1. We are ordering food at a restaurant in Italy: normally, then, we prefer pasta to beef. In Sardinia, though, we enjoy "porceddu" more than pasta. During summer, however, our preferred choice is just a fresh salad.

As in the example, we consider contexts as *states* (such as "Italy in summer") that can be compared via a *generalization hierarchy*, so that, say, "Sardinia" is more specific than "Italy". Preferences naturally *propagate* along the hierarchy, from the more generic to the more specific contexts (a preference defined for Italy normally holds in any Italian place). Things are not that simple, though.

Example 2. If we are in Sardinia in the summer, all the preferences given in Example 1 apply, since they refer to more generic contexts. Yet, the preferences defined in "Sardinia" and "Italy in summer" should take precedence over those given for "Italy", whereas the preference in "Sardinia" is on a par with the preference in "Italy in summer". Then, porceddu and salad are the best alternatives, since no other food is preferable to them in the given context.

Our research provides a principled approach to *context-aware preference propagation.* We develop a general framework whose only requirements are as follows:

– the contexts belong to a *poset*, i.e., a set C with a (strict) partial order relation $<_C$ on its elements: $c_1 <_C c_2$ means that the context c_1 is *more specific* than the context c_2 (and that c_2 is *more generic* than c_1);

– preferences define a strict partial order $\prec$ on the domain of objects O, where $o_1 \prec o_2$ means that the object o_2 *is preferable to* the object o_1;
– each stated preference is associated with one or more contexts.

The basic properties of the propagation process, implicitly at the basis of earlier approaches and naturally arising from the observations made for Example 2, are:

1. *coherence*: preferences only propagate from more generic contexts;
2. *fairness*: unordered contexts do not take precedence over each other;
3. *specificity*: a more specific context takes precedence over a more generic one.

Building on these properties, we focus on three main problems:

– (CFS) Determine whether there exist well-behaved (i.e., Coherent, Fair and Specific) methods for the propagation of preferences.
– (OrdRel) Efficiently determine the preference (order relation) between any two objects in a context according to a given propagation method.
– (Best) Establish the best objects in c wrt. the preferences propagated to c.

We tackle these problems through an axiomatization of the properties of propagation, and do so with an algebraic framework based on two abstract binary operators that express preference propagation by means of *Preference Composition* (PC) expressions: $+$ for unordered contexts, and $\rhd$ for ordered contexts.

Example 3. A possible PC-expression for propagating preferences to "summer in Sardinia" (c_4) is $c_4 \rhd ((c_2 + c_3) \rhd c_1)$. Here, first the preferences in "Sardinia" (c_2) and those in "summer in Italy" (c_3) are combined with $+$, since the two contexts are unordered. The result is then combined with the preferences in "Italy" (c_1) using $\rhd$, since this context is more generic than both c_2 and c_3. Finally, the result is combined with c_4 using $\rhd$, since this is the most specific context.

This paper summarizes some of the main findings reported in [3]. Here, we only focus on the axiomatization of $+$ and $\rhd$ and the study of their possible interpretations complying with the stated axioms. One of the main results is that, under mild assumptions, the well-known Pareto and Prioritized composition [1, 4] operators are the *only* possible interpretations of $+$ and $\rhd$, respectively.

Another notable achievement is a propagation method ($\mathcal{OC}$) that somehow provides the "ultimate" semantics for preference propagation. However, we refer to [3] for a full account of the solutions to Problems CFS and OrdRel. Here, we only hint at an algorithmic approach for propagating preferences according to the $\mathcal{OC}$ method, and characterize its asymptotic complexity, which, remarkably, is independent of the underlying domain size. Finally, we discuss how to determine the best objects according to the propagated preferences (Problem Best).

2 Preliminaries

We consider the well-known *binary relation model* for expressing preferences over a domain of objects O $[1, 4, 8]$, where a *preference relation* over objects of a domain O is a strict partial order $\prec$ on O, associated with an unordered relation $\sim$. A refinement of $\sim$ allows some unordered objects to be considered as *indifferent*: an indifference relation $\approx$ is a reflexive, symmetric, and transitive subset of the unordered relation $\sim$ such that if $o_1 \approx o_2$ then, for all $o \in O$, $o_1 \prec o$ ($o \prec o_1$) entails $o_2 \prec o$ ($o \prec o_2$). If $o_1 \sim o_2$, but $o_1 \not\approx o_2$, then o_1 and o_2 are *incomparable*, denoted $o_1 \parallel o_2$.

Since $\parallel = \sim - \approx$, it suffices to consider the pair $\langle \prec, \approx \rangle$, collectively called a *preference structure*. Let θ be one of $\prec, \succ, \approx, \parallel$, where $\succ$ denotes the inverse of $\prec$; we say that the *order relation* between objects o_1 and o_2 is θ if $o_1 \theta o_2$.

Example 4. Let us consider the objects `pasta`, `beef`, `salad`, and `porceddu`. A possible preference structure over these objects is: `beef` $\prec$ `pasta`, `beef` $\prec$ `salad`, and `pasta` $\approx$ `salad`. In words, `pasta` and `salad` are both preferable to `beef`, whereas `pasta` and `salad` are indifferent. It follows that `porceddu` is incomparable with all other foods, i.e., `porceddu` $\parallel o$, for $o \in \{\texttt{pasta}, \texttt{beef}, \texttt{salad}\}$.

For a finite domain O, the best objects (i.e., maximal elements) in O according to $\prec$ can be selected by the *Best* operator $\beta_\prec(O) = \{o \in O \mid \nexists o' \in O, o \prec o'\}$. For instance, according to the preferences of Example 4, the best objects are $\beta_\prec(O) = \{\texttt{pasta}, \texttt{salad}, \texttt{porceddu}\}$. For any non-empty domain O, $\beta_\prec(O)$ is never empty [1]. In case O is infinite, $\beta_\prec$ is typically applied to a finite subset of O.

3 Preference Propagation

Throughout the paper we consider a context poset C and a domain O, and assume that each context $c \in C$ is associated with a preference structure $\langle \prec^c, \approx^c \rangle$ over O, called the *ground preference structure* in c. We also call $\langle \prec^C, \approx^C \rangle = \{\langle \prec^c, \approx^c \rangle \mid c \in C\}$ a *(preference structure) configuration* over C.

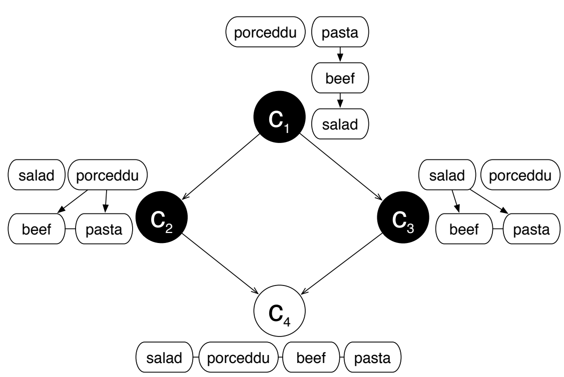

Fig. 1: A configuration for a context poset; each ground preference relation is shown next to its context (no ground preferences in c_4, shown as a white disk).

Example 5. The context poset in Figure 1 reports the ground preference structures discussed in Example 2. In context c_4 ("summer in Sardinia") all objects are indifferent. In context c_1 ("Italy"), `pasta` is preferable to `beef` and `beef` to `salad`, while, in context c_2 ("Sardinia"), `porceddu` is preferable to both `pasta` and `beef`, and `pasta` $\approx^{c_2}$ `beef`; similarly, `pasta` $\approx^{c_3}$ `beef` and `pasta` $\approx^{c_4}$ `beef`.

Preferences *propagate* along the poset C, and we call *complete preference relation* in c, denoted by ${}^{\mathcal{P}}\!\prec^c$, the result of combining $\prec^c$ with the ground preferences defined in the other contexts in C, according to a *propagation method* $\mathcal{P}$. Such a method also defines how indifference of objects is propagated to c, denoted by ${}^{\mathcal{P}}\!\approx^c$ (and thus also the unordered relation ${}^{\mathcal{P}}\!\sim^c$ and the incomparability relation ${}^{\mathcal{P}}\|^c$), thereby defining the *complete preference structure* $\langle {}^{\mathcal{P}}\!\prec^c, {}^{\mathcal{P}}\!\approx^c \rangle$.

In abstract terms, a propagation method $\mathcal{P}$ is a function that associates a context poset C, a configuration $\langle \prec^C, \approx^C \rangle$ over C, and a target context $c \in C$, with a complete preference structure $\langle {}^{\mathcal{P}}\!\prec^c, {}^{\mathcal{P}}\!\approx^c \rangle$. In this paper, we focus on propagation methods implemented through algebraic expressions.

We shall now capture the basic ideas underlying preference propagation.

Let us denote with $C\lfloor c \rfloor$ the *successor poset* of c, i.e., poset $C' = \{c' \in C \mid c \leqslant_C c'\}$, where $c_1 \leqslant_{C'} c_2$ iff $c_1 \leqslant_C c_2$. Context c' is *relevant* for c when a change in the ground preferences in c' may affect the propagated preferences in c.

Definition 1 (Coherence). *A propagation method $\mathcal{P}$ is* coherent wrt. C *if, for every context c in C, the relevant contexts for c according to $\mathcal{P}$ are exactly those in $C\lfloor c \rfloor$; $\mathcal{P}$ is* coherent *if it is coherent wrt. every context poset C.*

When combining preferences in unordered contexts, a cautious approach is to propagate only the preferences that hold in both contexts. The (more flexible) way we undertake just aims to avoid the propagation of conflicting preferences.

Definition 2 (Fairness). *$\mathcal{P}$ is* fair *for c in C if the following holds for all unordered contexts c_1 and c_2 in $C\lfloor c \rfloor$, all $o_1, o_2 \in O$ and all configuration $\langle \prec^C, \approx^C \rangle$:* ***if*** *i) $o_2 \prec^{c_1} o_1$, ii) $o_1 \prec^{c_2} o_2$, iii) $\forall c_i (c \leqslant_C c_i <_C c_1 \vee c \leqslant_C c_i <_C c_2) \Rightarrow o_1 \approx^{c_i} o_2$,* ***then*** *$o_1$ and o_2 are unordered in the complete preferences for c, i.e., $o_1 \, {}^{\mathcal{P}}\!\sim^c o_2$.*

A method $\mathcal{P}$ is fair *if it is fair for every context c in every poset C.*

So, if c_1 and c_2 disagree on the order of o_1 and o_2 while o_1 and o_2 are indifferent in all the more specific contexts, then o_1 and o_2 are not ordered in ${}^{\mathcal{P}}\!\prec^c$.

When combining ordered contexts, we give more importance to preferences that hold in contexts "closer" to the target context.

Definition 3 (Specificity). *$\mathcal{P}$ is* specific *for a context c in C if the following holds for every c_1 in $C\lfloor c \rfloor$, every $o_1, o_2 \in O$ and every configuration $\langle \prec^C, \approx^C \rangle$:* ***if*** *i) $o_1 \prec^{c_1} o_2$, ii) $o_1 \approx^{c_i} o_2$ for each $c \leqslant_C c_i <_C c_1$, and iii) it is either $o_1 \prec^{c_2} o_2$ or $o_1 \approx^{c_2} o_2$ for all $c_2 \in C\lfloor c \rfloor$ such that: 1) c_2 is unordered wrt. c_1, and 2) $o_1 \approx^{c_3} o_2$ $\forall c_3$ such that $c \leqslant_C c_3 <_C c_2$,* ***then*** *$o_1 \; {}^{\mathcal{P}}\!\prec^c \; o_2$.*

A method $\mathcal{P}$ is specific *if it is specific for every context c in every poset C.*

Thus if o_2 is preferable to o_1 in c_1 and such objects are indifferent in all the more specific contexts, then this preference does indeed propagate to context c (preferences from more generic contexts are overridden by those in more specific contexts). The preference must not propagate if this violates fairness.

Example 6. Consider Figure 1. With a fair propagation method $\mathcal{P}$, `pasta` and `porceddu` must be unordered in context c_4, since `pasta` $\prec^{c_2}$ `porceddu` whereas `porceddu` $\prec^{c_3}$ `pasta`, and similarly for `beef` and `porceddu`, i.e., `pasta` ${}^{\mathcal{P}}\!\!\sim^{c_4}$`porceddu` and `beef` ${}^{\mathcal{P}}\!\!\sim^{c_4}$`porceddu`. If $\mathcal{P}$ is specific, then `pasta` is preferable to `beef` in contexts c_2, c_3, and c_4, i.e., `beef` ${}^{\mathcal{P}}\!\!\prec^{c_2}$`pasta`, `beef` ${}^{\mathcal{P}}\!\!\prec^{c_3}$`pasta`, and `beef` ${}^{\mathcal{P}}\!\!\prec^{c_4}$`pasta`.

4 Preference Composition Expressions

We now characterize two binary operators, $+$ (for unordered contexts) and $\rhd$ (for ordered contexts), combining two ground preference structures $\langle \prec^{c_1}, \approx^{c_1} \rangle$ and $\langle \prec^{c_2}, \approx^{c_2} \rangle$ into a new preference structure, denoted $\langle \prec^{c_1} + \prec^{c_2}, \approx^{c_1} + \approx^{c_2} \rangle$ and, respectively, $\langle \prec^{c_1} \rhd \prec^{c_2}, \approx^{c_1} \rhd \approx^{c_2} \rangle$. Clearly, $+$ is commutative but $\rhd$ is not. Both operators are associative and idempotent and both have the *full indifference structure* $\emptyset_{\approx} = \langle \emptyset, O \times O \rangle$ as identity element, i.e., contexts in which all elements are indifferent should not influence the result. Axiomatically:

Definition 4 (+ operator). *A + operator satisfies the following axioms, for all objects* $o_1, o_2 \in O$ *and all preference structures* $\langle \prec_1, \approx_1 \rangle, \langle \prec_2, \approx_2 \rangle, \langle \prec_3, \approx_3 \rangle$*:*

i) $\langle \prec_1, \approx_1 \rangle + \langle \prec_2, \approx_2 \rangle = \langle \prec_2, \approx_2 \rangle + \langle \prec_1, \approx_1 \rangle$ *(commutativity)*
ii) $(\langle \prec_1, \approx_1 \rangle + \langle \prec_2, \approx_2 \rangle) + \langle \prec_3, \approx_3 \rangle = \langle \prec_1, \approx_1 \rangle + (\langle \prec_2, \approx_2 \rangle + \langle \prec_3, \approx_3 \rangle)$ *(ass.)*
iii) $\langle \prec_1, \approx_1 \rangle + \langle \prec_1, \approx_1 \rangle = \langle \prec_1, \approx_1 \rangle$ *(idempotence)*
iv) $\langle \prec_1, \approx_1 \rangle + \emptyset_{\approx} = \langle \prec_1, \approx_1 \rangle$ *(identity element)*
v) $o_1 \prec_1 o_2, o_2 \prec_2 o_1 \Rightarrow \neg(o_1 \prec_1 + \prec_2 o_2) \wedge \neg(o_2 \prec_1 + \prec_2 o_1)$ *(fairness)*

Definition 5 ($\rhd$ operator). *A $\rhd$ operator satisfies the following axioms, for all objects* $o_1, o_2 \in O$ *and all preference structures* $\langle \prec_1, \approx_1 \rangle, \langle \prec_2, \approx_2 \rangle, \langle \prec_3, \approx_3 \rangle$*:*

i) $(\langle \prec_1, \approx_1 \rangle \rhd \langle \prec_2, \approx_2 \rangle) \rhd \langle \prec_3, \approx_3 \rangle = \langle \prec_1, \approx_1 \rangle \rhd (\langle \prec_2, \approx_2 \rangle \rhd \langle \prec_3, \approx_3 \rangle)$ *(ass.)*
ii) $\langle \prec_1, \approx_1 \rangle \rhd \langle \prec_1, \approx_1 \rangle = \langle \prec_1, \approx_1 \rangle$ *(idempotence)*
iii) $\langle \prec_1, \approx_1 \rangle \rhd \emptyset_{\approx} = \emptyset_{\approx} \rhd \langle \prec_1, \approx_1 \rangle = \langle \prec_1, \approx_1 \rangle$ *(identity element)*
iv) $o_1 \prec_1 o_2 \Rightarrow o_1 \prec_1 \rhd \prec_2 o_2$ *(specificity)*

Preference structures can be combined with $+$ and $\rhd$ to form a *PC-expression*.

Definition 6 (PC-expression). *A* PC-expression *over* C *is any expression* E *of the form:* $E ::= c \mid (E + E) \mid (E \rhd E) \mid \bot$*, where* c *is a context in* C*.*

The base case is the name of some context c, denoting the preference structure $\langle \prec^c, \approx^c \rangle$; one can also compose PC-expressions (i.e., preference structures) via $+$ and $\rhd$, and denote the full indifference structure $\emptyset_{\approx}$ via the $\bot$ symbol.

PC-expressions can be used to compute complete preference structures (and thus to implement propagation methods). Fairness and specificity can be extended to PC-expressions in a straightforward way.

5 Interpreting the Propagation Operators

In this section we investigate on possible interpretations of the operators $+$ and $\rhd$. We start with a general result about *idempotent semirings.* A semiring is an algebraic structure with an associative and commutative additive operator (like $+$) and an associative multiplicative operator (like $\rhd$), which is both left- and right-distributive over addition. Let $(\mathcal{PR}_O, +, \rhd)$ be an algebraic structure, where $\mathcal{PR}_O$ denotes the set of all preference structures over a domain O. Then, with the additional hypothesis that $\rhd$ distributes over $+$, $(\mathcal{PR}_O, +, \rhd)$ would be a semiring in which both operators are idempotent, i.e., an idempotent semiring.

However, the following major result rules out the possibility of using idempotent semirings for providing an interpretation to the propagation operators, since distributivity of $\rhd$ over $+$ turns out to be incompatible with the axioms of specificity and fairness of the operators.

Theorem 1. *No $(\mathcal{PR}_O, +, \rhd)$ structure is an idempotent semiring.*

The cause of incompatibility of distributivity with the axioms of the operators lies only in assuming that $\rhd$ *right-distributes* over $+$. For this reason, here we consider the case in which $(\mathcal{PR}_O, +, \rhd)$ is an *idempotent left near-semiring*, that is, an algebraic structure satisfying all requirements of idempotent semirings except for right-distributivity of $\rhd$. We still assume that $\rhd$ left-distributes over $+$.

Consider Pareto and Prioritized composition as interpretations of $+$ and $\rhd$.

Definition 7 (Pareto and Prioritized comp.). *Let $\langle \prec_1, \approx_1 \rangle$ and $\langle \prec_2, \approx_2 \rangle$ be two preference structures over a domain O. The* Prioritized composition *of $\langle \prec_1, \approx_1 \rangle$ and $\langle \prec_2, \approx_2 \rangle$, written $\langle \prec_1, \approx_1 \rangle \circledcirc \langle \prec_2, \approx_2 \rangle$, is defined as*
$o_1 \prec_1 \circledcirc \prec_2 o_2 \Leftrightarrow (o_1 \prec_1 o_2) \vee (o_1 \prec_2 o_2 \wedge o_1 \approx_1 o_2)$
$o_1 \approx_1 \circledcirc \approx_2 o_2 \Leftrightarrow o_1 \approx_1 o_2 \wedge o_1 \approx_2 o_2$.
and their Pareto composition*, written $\langle \prec_1, \approx_1 \rangle \oplus \langle \prec_2, \approx_2 \rangle$, is:*
$o_1 \prec_1 \oplus \prec_2 o_2 \Leftrightarrow (o_1 \prec_1 o_2 \wedge o_1 \prec_2 o_2) \vee (o_1 \prec_1 o_2 \wedge o_1 \approx_2 o_2) \vee (o_1 \approx_1 o_2 \wedge o_1 \prec_2 o_2)$
$o_1 \approx_1 \oplus \approx_2 o_2 \Leftrightarrow o_1 \approx_1 o_2 \wedge o_1 \approx_2 o_2$.
where o_1 and o_2 are any two objects in O.

Intuitively, Prioritized composition gives precedence to preferences in $\prec_1$, while preferences in $\prec_2$ are used only if two objects are indifferent according to $\prec_1$. Conversely, Pareto considers the two preference relations equally important.

Example 7. Consider the objects: *i)* o_1: a comedy with Favino, *ii)* o_2: a comedy without Favino, *iii)* o_3: a drama with Favino, *iv)* o_4: a drama without Favino. Let $\prec_1$ be the preference relation corresponding to "I prefer comedies to all other genres" and $\prec_2$ to "I prefer movies with Favino to all other movies". If we consider $\prec_1$, then o_1 and o_2 are both preferable to o_3 and o_4; also, $o_1 \approx_1 o_2$ and $o_3 \approx_1 o_4$. With $\prec_2$, we have instead that o_1 and o_3 are both preferable to o_2 and o_4, with $o_1 \approx_2 o_3$ and $o_2 \approx_2 o_4$. Let $\prec_{\texttt{Par}} = \prec_1 \oplus \prec_2$; then $o_2 \prec_{\texttt{Par}} o_1$, $o_3 \prec_{\texttt{Par}} o_1$, $o_4 \prec_{\texttt{Par}} o_2$, $o_4 \prec_{\texttt{Par}} o_3$, with $o_2 \parallel_{\texttt{Par}} o_3$. Let now $\prec_{\texttt{Pri}} = \prec_1 \circledcirc \prec_2$; then $o_4 \prec_{\texttt{Pri}} o_3$, $o_3 \prec_{\texttt{Pri}} o_2$, $o_2 \prec_{\texttt{Pri}} o_1$. The intuition is that o_1 is always the best, since it satisfies both preferences, while o_4 is the worst one; o_2 and o_3 remain unordered if no priority is assumed between $\prec_1$ and $\prec_2$ (if $\oplus$ is used), whereas o_2 is preferred to o_3 when $\prec_1$ has priority over $\prec_2$ (via $\circledcirc$).

Both Prioritized and Pareto composition preserve strict partial orders (both yield a strict partial order when applied to two strict partial orders), whereas this is not guaranteed by replacing in their definition $\approx$ with $\sim$ [1]. It is known that both are associative and $\oplus$ is also commutative. Both operators are idempotent and have $\emptyset_{\approx}$ as the identity. Finally, we can show that $\circledtriangleright$ left-distributes over $\oplus$.

Theorem 2. *$(\mathcal{PR}_O, \oplus, \circledtriangleright)$ is an idempotent left near-semiring.*

One may wonder whether other interpretations, besides the one based on $\oplus$ and $\circledtriangleright$, exist for the $+$ and $\triangleright$ operators. Our answer is negative for an important class of operators, which we call *independent of irrelevant objects*.

Definition 8 (IIO operator). *An operator $\diamond$ is* independent of irrelevant objects *(IIO) if, for any two objects o and o' in O, the order relation between o and o' according to the combined preference structure $\langle <^1, \approx^1 \rangle \diamond \langle <^2, \approx^2 \rangle$ only depends on the order relation between o and o' according to $\langle <^1, \approx^1 \rangle$ and $\langle <^2, \approx^2 \rangle$.*

Thus, to determine the order relation between o_1 and o_2, an IIO operator does not need to consider any other objects. Both $\oplus$ and $\circledtriangleright$ are IIO. The following theorems show that $\oplus$ and $\circledtriangleright$ are the only possible IIO interpretations of $+$ and $\triangleright$: there is no other IIO interpretation of $+$ and $\triangleright$ that satisfies all the axioms.

Theorem 3. *Operator $\oplus$ is the only IIO $+$ operator.*

Theorem 4. *Operator $\circledtriangleright$ is the only IIO $\triangleright$ operator.*

6 Solutions and complexities

Problem CFS is solved by exhibiting a propagation method, called *Object-specific Cover* ($\mathcal{OC}$) propagation, based on PC-expressions using $\oplus$ and $\circledtriangleright$. It can be shown that no straightforward combination of the ground preferences in a context poset attains both fairness and specificity (each pair of objects seems to require a dedicated PC-expression). However, as a major result, $\mathcal{OC}$ implements propagation through a *single* PC-expression, denoted $\mathsf{RG}^C(c)$, for all pairs of objects; this is obtained, algebraically, by maximally "grouping on the right".

Definition 9 (PC-expression for $\mathcal{OC}$ propagation). *Let $c' \in C\lfloor c \rfloor$ and let $\{c_1, \ldots, c_k\}$ be immediately more specific than c'. The expression $\mathsf{RG}^C(c, c')$ is defined as $\mathsf{RG}^C(c, c) = c$; and $\mathsf{RG}^C(c, c') = (\mathsf{RG}^C(c, c_1) \oplus \ldots \oplus \mathsf{RG}^C(c, c_k)) \circledtriangleright c'$ if $c <_C c'$. Let $\hat{c}_1, \ldots, \hat{c}_n$ be maximal in $C\lfloor c \rfloor$. Then: $\mathsf{RG}^C(c) = \mathsf{RG}^C(c, \hat{c}_1) \oplus \ldots \oplus \mathsf{RG}^C(c, \hat{c}_n)$.*

For the poset in Figure 1, $\mathsf{RG}^C(c_4) = ((c_4 \circledtriangleright c_2) \oplus (c_4 \circledtriangleright c_3)) \circledtriangleright c_1 \equiv c_4 \circledtriangleright (c_2 \oplus c_3) \circledtriangleright c_1$.

Ultimately, $\mathcal{OC}$ is the only method guaranteeing both fairness and specificity.

Theorem 5. *For any method $\mathcal{P}$, if ${}^{\mathcal{P}}{<^c} \subset {}^{\mathcal{OC}}{<^c}$ then $\mathcal{P}$ is not specific.*

Theorem 6. *For any coherent method $\mathcal{P}$ based on $\oplus$ and $\circledtriangleright$, whose PC-expression only depends on c and C, if ${}^{\mathcal{OC}}{<^c} \subset {}^{\mathcal{P}}{<^c}$ then $\mathcal{P}$ is not both fair and specific.*

So, the $\mathcal{OC}$ semantics is "final" and therefore used to study the other problems.

For both OrdRel and Best, the exact complexity depends on the underlying context model as well as on the language used for expressing preferences. We remain parametric wrt. these aspects and assume that two context descriptions can be compared in $\mathcal{O}(\delta)$ time, and that $\mathcal{O}(\gamma)$ is the complexity of determining the order relation of any two objects according to ground preferences in a context.

Examples are the context model in [5–7], for which $\mathcal{O}(\delta) = \mathcal{O}(d)$, where d is the number of contextual dimensions and the language of [1], where, if preferences are expressed via a CNF with n conjuncts, we have $\mathcal{O}(\gamma) = \mathcal{O}(n^2)$.

Problem OrdRel can be solved by first computing the PC-expression $\mathsf{RG}^C(c)$ given by $\mathcal{OC}$ and then using it against the configuration to compute the complete preferences. However, materializing $\mathsf{RG}^C(c)$ may be inefficient due to repeated sub-expressions, leading, in the worst case, to a size of the PC-expression that is *exponential* in the number of contexts in $C\lfloor c \rfloor$. Yet, via efficient bookkeeping structures, we can solve Problem OrdRel in $\mathcal{O}(|A| \cdot (\delta + w(A) + \gamma))$ time, where A is the subset of C with only the contexts with some ground preferences, and $w(A)$ is its width. Problem Best requires at most $\mathcal{O}(N^2)$ comparisons for a set of N objects, leading to a complexity of $\mathcal{O}(|A| \cdot (\delta + N^2 \cdot (w(A) + \gamma)))$.

7 Conclusion and future work

In this paper we have discussed how preferences propagate when they depend on the context. We have proposed an algebraic model for expressing preference propagation, based on two abstract operators, and have shown that Pareto and Prioritized composition are the only natural possible interpretations satisfying all the required algebraic properties. Finally, we have studied the problem of efficiently computing the best objects according to the preferences propagated to a target context, and have shown that this can be done with polynomial complexity in all the main involved parameters.

Further research might analyze the application of the principles of preference propagation to numerical preferences. An example of the possibilities opened by this line of research is discussed in [2], where qualitative preferences expressed through constraints are imposed over numeric (quantitative) attributes.

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