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An Analytical Valuation Framework for Financial Assets with Trading Suspensions*

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Abstract. In this paper we propose a derivative valuation framework based on Lévy processes which takes into account the possibility that the underlying asset is subject to information-related trading halts/suspensions. Since such assets are not traded at all times, we argue that the natural underlying for derivative risk-neutral valuation is not the asset itself but rather a contract that, when the asset is in trade suspensions upon maturity, cash settles the last quoted price plus the interest accrued since the last quote update. Combining some elements from semimartingale time changes and potential theory, we devise martingale dynamics and no-arbitrage relations for such a price process, provide Fourier transform–based pricing formulae for derivatives, and study the asymptotic behavior of the obtained formulae as a function of the halt parameters. The volatility surface analysis reveals that the short-term skew of our model is typically steeper than that of the underlying Lévy models, indicating that the presence of a trade suspension risk is consistent with the well-documented stylized fact of volatility skew persistence/explosion. A simple calibration example to market option prices is provided.

Key words. market halts and suspensions, time changes, Lévy subordinators, derivative pricing, Lévy processes

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1. Introduction.

Suspending or halting[1] of a stock from trading is a temporary emergency measure taking place in the event of abnormal market situations. Broadly speaking this action is generally triggered by two distinct types of circumstances. The first is the manifestation of severe market anomalies that may prevent the formation of a reliable price, e.g., crashes, order imbalances, and excessive bid ask spread/illiquidity holding back buyers. The second is the arrival of news that could have a potentially high impact on individual company quotes. We can thus distinguish between endogenous suspensions, generated by the market activity itself, and exogenous, news-related ones, typically independent from day-to-day trading. Trade-generated halts tend to be of fixed time and short lived, on the order of magnitude of minutes, whereas news-related suspensions might last up to hours or days, and their duration is typically discretionary. When impactful business news is expected, the firm might file for a trading suspension voluntarily, motivated by internal management decisions, or the action might be directly enforced by the market authority in the event of growing concerns regarding the ability

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[1]Depending on the stock markets, halts and suspensions might have slightly different meanings. In this article the two expressions are synonyms.

of the firm to meet the market's standards. In any case the purpose of a stock suspension is to give to all of the investors the opportunity to reassess their positions, facilitate the issuance of a better equilibrium price, and reduce market information asymmetries.

Trade suspensions can and do occur quite often. Engelen and Kabir [9] observe that in the years between 1992 and 2000 in the EuroNext stock market, there were 210 pure information-related suspensions, 30% of which lasted more than one trading day and involved a total of 112 companies whose 49% was halted more than once. Christie et al. [6] study a collective sample of 714 halts in the years 1997–1998 on the NASDAQ. Trading suspensions then appeared to be a marketwise repeatable process of possibly interdaily duration.

The financial literature surrounding market halts mainly focuses on whether market suspensions have the stabilizing effect on trade they are expected to deliver. The evidence is mixed to some extent. Greenwald and Stein [11] suggested that halts facilitate formation of an equilibrium price by reducing transactional risk, whereas statistical analyses in the New York Stock Exchange (NYSE) and other U.S. stock markets (Corwin and Lipson [7], Lee et al. [17]) point to an increase in both posthalt trade volume and volatility, at odds with what market suspensions are meant to achieve.

However, typically these analyses include suspensions caused by order imbalances or triggering of the so-called circuit breakers due to some financial variable (especially volatility) breaching a safety threshold, which are market-generated events. Indeed, once halts from order imbalances are removed from the sample or only interdaily suspensions are considered, the general findings (Christie et al. [6], Engelen and Kabir [9]) show that when the suspension lasts for more than one day, the volatility of a stock is not sensibly impacted by the halt.

To our knowledge, to date no research has been put forward to include the phenomenon of trade suspensions/halts in the valuation and risk management of derivatives. In this paper we aim at providing a no-arbitrage valuation framework in markets with news-related trading halts. Since for derivative pricing the minimum horizon is daily, we do not consider intradaily stoppages due to circuit breakers or transactional frictions, under the assumption that the changes in trading patterns these might determine are transient and do not extend to interdaily trading.

Of course, halts might not produce any significant effect on valuations if the expected suspensions are very short lived and maturity is long. However, the effect of a suspension lasting for several days to several hours cannot be ignored altogether in certain cases, e.g., for valuing weekly options, a product that has recently drawn much attention.

One difficulty in introducing suspensions in no-arbitrage valuation may be that a security that can be halted cannot be used as an underlying for martingale pricing. However paradoxical this might sound, by definition, suspendable assets are not traded at all times, and thus the replication/superreplication arguments establishing the equivalence between no-arbitrage and martingale dynamics of the underlying do not in principle apply.

The solution we propose in the present paper to such a problem is that of specifying the payoff as referencing the cash value at maturity of the last quoted market price, that is, the last quoted price accrued at the interest matured for the period in between. This is to be interpreted as a contractual fallback provision specified as part of the derivative contract: Considering interest accrual in such a clause is possible because derivative are usually cash settled.

In order to pursue this modeling idea, we first introduce a model for the fundamental value of the stock, recognizing that the evolution of the economic value might follow different dynamics when the asset can be traded or is suspended. Then we devise an observable last market quote price process Q_t by using a locally constant time change. What we argue is that the natural underlying for derivative valuation on a suspendable stock is a secondary "lookback" contract P_t that delivers at t the last observed quote Q_t plus the accrued interest since the last update of Q_t. This contract has all the characteristics we need: It can always be traded and exhibits martingale dynamics after an appropriate equivalent measure change. If we consider a legal stipulation of an over-the-counter (OTC) derivative whose payoff references the last market quote plus the interest rate payment, P_t effectively represents the real derivative underlying asset.

From a methodological viewpoint, the framework introduces the idea of using a locally constant time change in option prices, obtained by an *inverse* Lévy time change. Time changing in option pricing is a well-established technique (see, e.g., Geman et al. [10], but the literature is immense) that normally is used to capture the evolution of the business activity. Here the approach is rather different: Our time change is a continuous, piecewise linear process whose paths can be constant in random intervals. The constant sections of the time trajectories represent trade halts. However, this is not yet sufficient: In order to consistently model a market quote that undergoes halting, we must introduce a second time change capturing the last observable traded equity price, including the asset price jump when trade resumes after a halt. The process achieving this is the so-called *last passage process* of a subordinator.

We devise no-arbitrage relations for the model by identifying a set of martingale measures under which both the fundamental value S_t and the lookback price P_t are martingales. Adding a suspension process to a pricing Lévy model enriches its class of equivalent martingale measures. In other words, the intrinsic market incompleteness of these new models also accounts for an additional source of unhedgeable risk, the trading suspension risk, whose market price is embedded in the risk-neutral parameters of the Lévy subordinator generating the halts.

Remarkably, the whole framework produces closed-form formulae for the processes' characteristic functions. This means that the well-established machinery of Fourier pricing (e.g., Lewis [18]) is available, producing efficient valuation algorithms.

Finally, we consider the potential applications to the volatility surface modeling. Our numerical experiments show a volatility skew which is at the same time much steeper on the short-term section and declines more slowly than that of the underlying Lévy models, thus generating a volatility term structure better matching the one observed in the markets.

In section 2 we discuss equity derivatives with suspensions and outline the economic foundations of the framework. In section 3 we introduce the stochastic model for the fundamental value of the stock. In section 4 we define the market quote process and the traded underlying for derivative valuation. Section 5 deals with the equivalent martingale relations for the model. Section 6 is dedicated to the identification of a pricing formula, briefly discusses mean-variance hedging, and verifies convergence of option prices to those from Lévy models. Finally, in section 7 we perform some numerical tests for the pricing formula, analyze the arising volatility surfaces, and conduct a calibration test. Comparisons with the pure Lévy models are provided. Some concluding remarks are expressed in section 8.

2. Derivatives on suspendable assets.

The starting point for a valuation theory for stocks that are subject to halts is recognizing that the classic theory of no-arbitrage pricing cannot be directly applied. Indeed, the fundamental theorem of asset pricing (e.g., Delbaen and Schachermayer [8]) requires that the asset can be traded at any time in order to form hedge and superhedge portfolios, which is not the case when halts are present.

We are thus faced from the very beginning with the problem of manufacturing some form of synthetic underlying which can be traded at any time regardless of possible interruptions of the market activity, so that we can proceed in the usual vein within the theory of no-arbitrage pricing.

We denote by Q_t the stochastic process giving at all times $t > 0$ the last available market quote for the suspendable equity. Let τ_t be the last instant prior to t where the equity was last traded.[2] For any market asset X_t let

$$F^X(t,T) = e^{r(T-t)}X_t \tag{2.1}$$

be the forward price of X_t, contracted at t with delivery T. The prevailing risk-free rate $r > 0$ is assumed to be constant.

Let us consider the contract P that at all times entitles the holder to the cash payment of the forward price of a reference asset calculated from the time τ_t when the asset was last tradable (i.e., a live quote Q_t was available) to the present date. Since by definition $Q_t = Q_{\tau_t}$, for all $t > 0$ the value of such a contract is

$$P_t := F^Q(\tau_t, t) = e^{r(t-\tau_t)}Q_{\tau_t} = e^{r(t-\tau_t)}Q_t. \tag{2.2}$$

In this paper we propose using P as an underlying asset for derivative valuation on stocks whose trade can be interrupted. This mathematical modeling idea connects with the financial practice precisely because of (2.2). Indeed, a satisfactory legal definition of an OTC derivative on a suspendable market asset requires an explicit contractual specification of the actions to be undertaken when the market is closed at maturity because in such a case a current reference market value will not be available. The most natural choice and the one usually put in place for exchange-traded options is using as a reference value the last quoted price Q_t of the underlying. In such a case, the value to be used for calculation of the payoff would thus be the last market quote Q_{τ_T} recorded prior to the expiration time T. However, this seems to be unsatisfactory because it completely ignores the time value of money, i.e., the growth of the fundamental value of the asset during suspensions due to the interest rate component. However, in view of (2.2), including the risk-free interest rate accrual when determining the payoff corrects this issue, and as we shall show in this paper, it effectively reconnects discounting by the market numeraire with a halted discount factor which in turn allows for a fully analytical semimartingale valuation framework. On a side note, this choice is consistent with common practices in the collateralization of OTC derivatives, where counterparties pay interest on the collateral, which itself represents the (last determined) market value of the contract.

[2] Clearly, most of the time, $\tau_t = t$, but it is precisely when this does not happen that the discussion is significant.

Clearly, when interest rates are zero, $Q_t = P_t$, and in this special case the quote process can indeed be used as a derivative underlying. However, what we will show in this paper is that for positive rates and under some economic assumptions, it is P_t and not Q_t to possess martingale dynamics under some pricing measure. This result, together with the previous remarks, seems to implicate that a market practice of using the last available quote Q_t for payoff calculation might be questionable from the theoretical perspective, at least whenever rates are high or suspensions are long lived, that is, when the difference in valuation between calculating or not the interest during the final suspension may be significant.

The starting element of our framework is an observable process S_t modeling the fundamental (or intrinsic, or economic) stock value, which is distinct from its market quote Q_t. This can be understood as some form of "shadow price" available to the investor, possibly outside the stock market. However, we emphasize again that neither of these two processes are traded assets. The main idea is that the two must coincide when the asset is tradable and may (will) differ otherwise. When the asset is not traded, Q_t is constant, but S_t still evolves to keep track of the economic activity surrounding the real asset. This assumption is naturally rooted in the efficiency principle: When an asset can be traded, all the available information is reflected in its market quote. The process S_t is modeled by a two-factor process: one factor representing the component purely due to trade and the other the impact on price of business and market news and expert valuations. Only the first component is halted during the market suspensions. Our model thus captures the existence of a background noise of business-related information whose contribution to price formation is distinct to that generated purely by the trading activity and which persists also during trading halts.

The next two sections are devoted to the identification of a rigorous mathematical model for Q_t and S_t. Once this is done, introducing the process P_t as indicated in this section will pave the way for a valuation theory for derivatives on suspendable stocks.

3. Fundamental value dynamics.

In this section we begin structuring the fundamental value of the market stock S_t. We consider a market filtration $(\Omega, \mathcal{F}_t, \mathcal{F}_\infty, \mathbb{P})$ satisfying the usual conditions and supporting Lévy processes and a money market account process paying a constant rate $r > 0$.

For a càdlàg one-dimensional Lévy process Y_t with Lévy triplet $(\mu_Y, \sigma_Y, \nu_Y(dx))$ and $z \in U \subseteq \mathbb{C}$, for the characteristic function of Y_t we use the nonstandard notation

$$\mathbb{E}[e^{-izY_t}] = e^{-t\psi_Y(z)}, \tag{3.1}$$

where

$$\psi_Y(z) = iz\mu_Y + \frac{z^2\sigma_Y^2}{2} - \int_{\mathbb{R}} (e^{-izx} - 1 + izx\mathbb{1}_{|x|<1})\nu_Y(dx) \tag{3.2}$$

is the Fourier characteristic exponent of Y_t. We denote the process of the left limits (the "predictable projection") of Y_t with Y_{t-}. By stochastic continuity, for all fixed $t > 0$ we have $Y_t = Y_{t-}$ almost surely. We write $\Delta Y_t := Y_t - Y_{t-}$ for the process of the jumps of Y_t.

As basic building blocks of our model we consider two one-dimensional independent Lévy processes X_t and R_t, with corresponding Lévy triplets $(\mu_X, \sigma_X, \nu_X(dx))$ and $(\mu_R, \sigma_R, \nu_R(dx))$

and characteristic exponents ψ_X and ψ_R. We assume that $\mathcal{F}_t$ is the filtration generated by X_t and R_t. We also hasten to add the standard conditions

$$\int_{|x|>1} e^{2x}\nu_X(dx) < \infty, \quad \int_{|x|>1} e^{2x}\nu_R(dx) < \infty, \tag{3.3}$$

which are necessary for exponential Lévy models to be square integrable.

The choice of Lévy dynamics is motivated by the analytical tractability of the pricing formulae we attain (see section 6), but the theoretical framework exposed can be extended to general semimartingales X_t and R_t, thereby allowing us to integrate suspensions into many popular pricing models (e.g., stochastic volatility).

The processes X_t and R_t retain the following financial interpretations. The evolution of X_t represents the component of the log-asset price coming purely from the execution of trades. The process R_t (the "rumor" process) instead models all of the other external factors that may impact the price, mostly the dissemination of external news, both financial and nonfinancial. Normally, X_t is expected to dominate R_t, but this does not always need to be the case. When trading in the asset is allowed, the stock returns are defined to be the independent sum of these two factors. However, as explained in the previous section, we shall require that as a trade halt occurs, X_t does not evolve, while R_t still contributes to the fundamental value formation.

Let us introduce the generator of the market suspensions as a compound Poisson process G_t independent of (X_t, R_t) of the following form:

$$G_t = t + \sum_{i=0}^{N_t} \xi_i \tag{3.4}$$

where the variables ξ_i are independent identically exponentially[3] distributed of common rate parameter β and N_t is a Poisson process of intensity λ independent of the ξ_is and all the remaining processes. For $s \geq 0$ the Laplace characteristic exponent $\phi_G(s)$ of G_t satisfies

$$\mathbb{E}[e^{-sG_t}] = e^{-t\phi_G(s)} \tag{3.5}$$

and is given by

$$\phi_G(s) = s + \int_{\mathbb{R}^+} \left(1 - e^{-su}\right) \nu_G(du) = \frac{\lambda s}{s+\beta} + s. \tag{3.6}$$

Finally, we introduce the market suspensions process H_t as the "inverse" of G_t. More precisely, for all t we define H_t as the first exit time of the level t of G_t, that is,

$$H_t = \inf\left\{s > 0 \,|\, G_s > t\right\}. \tag{3.7}$$

When G_t jumps, H_t has a flat spot, and a market suspension occurs. Furthermore, the duration of the suspension is exactly given by the size of the jump. In the instants between the jumps of G_t, H_t is just the linear calendar time. In other words, we have the following definition.

[3]For the sake of this section alone, an unspecified positive law could be equally assumed for ξ_i. However, most of the analytic results of the paper hinge upon the exponential distribution, so we directly make this choice.

Definition 3.1. *Let $\mathcal{R}$ be the image of the process G_t, i.e., the random set*

$$\mathcal{R}(\omega) = \{G_t(\omega),\, t \geq 0\}. \tag{3.8}$$

We say that S_t is suspended, halted, or nontradable at $t > 0$ if $t \in \mathcal{R}^c$. If $s > 0$ is such that $G_{s-} \neq G_s$, then ΔG_s is the duration of the halt.

It is important to notice that $H_t \leq G_t$ and the equivalence $\{H_t \leq s\} = \{G_s \geq t\}$, which in particular yields $\{H_t \leq t\} = \Omega$. Since G_t is strictly increasing, H_t is continuous, and it is a stopping time for all fixed t. Also, H_t is almost surely increasing, bounded almost surely, and $\lim_{t\to\infty} H_t = \infty$ almost surely, and thus it is a valid time change (Jacod [12, Chapter 10]).

We are now in the position of describing the *fundamental value*, or economic value, of our suspendable asset S_t, which is given by

$$S_t := S_0 \exp(X_{H_t} + R_t), \quad S_0 > 0. \tag{3.9}$$

The writing X_{H_t} indicates the time-changed process in the sense of Jacod [12]. Since H_t is continuous, X_t is H_t-continuous.[4] This means that X_{H_t} retains many of the good properties of X_t (Jacod [12, Chapter 10]); in particular, it is an $\mathcal{F}_{H_t}$-adapted semimartingale. Finally, recalling that $A \in \mathcal{F}_{H_t}$ if and only if $A \cap \{H_t \leq s\} \in \mathcal{F}_s$ for all s, choosing $s = t$ and observing $\{H_t \leq t\} = \Omega$ shows $\mathcal{F}_{H_t} \subset \mathcal{F}_t$, so that S_t is also an $\mathcal{F}_t$-adapted semimartingale.

The fundamental value evolution S_t has the property we were striving for. Conditionally on the asset being tradable, i.e., G_t not jumping, we have that $X_{H_t} + R_t = X_t + R_t$, and the price process is jointly determined by the economic reaction to trade and the external news flow. When G_t jumps, the stochastic time H_t and thus the value component X_{H_t} remain constant, and the fundamental value is driven only by the news dissemination process R_t.

Finally, we associate to S_t the corresponding Lévy exponential model S^0_t without halts, whose dynamics are given by

$$S^0_t := S_0 \exp(X_t + R_t). \tag{3.10}$$

Further on, we shall be interested in comparing financial valuations relying on S_t with the analogous on its pure Lévy counterpart S^0_t in order to assess the impact of the introduction of trading halt periods in derivative pricing.

4. The asset quote process and the traded underlying.

We must at this point rigorously define the quote process Q_t recording the last available market quote of S_t at time t. Recall that $\mathcal{R}$ is the range of G_t. The *last passage process* of G_t for the level $[0, t]$ is defined as

$$\tau_t = \sup\{s < t, s \in \mathcal{R}\}. \tag{4.1}$$

Observe the important relation $\tau_t \leq t$. This process keeps track of the last position of G_t in all the level sets, and for all fixed t we will interpret it as a random time. Indeed, as t varies, this process can be regarded as a time change. We have the following result (see also Bertoin [2, section 1.4]):

[4] A process Y_t is said to be continuous with respect to a time change T_t if it is almost surely constant on the sets $[T_{t-}, T_t]$.

Lemma 4.1. *The process τ_t satisfies*

$$\tau_t = G_{H_t -}, \tag{4.2}$$

and there exists a right-continuous modification of τ_t which is an $\mathcal{F}_t$-time change.

Proof. As G_{t-} is adapted to $\mathcal{F}_t$ the process, G_{H_t-} is adapted to $\mathcal{F}_{H_t}$, but since $\mathcal{F}_{H_t} \subset \mathcal{F}_t$, it is also $\mathcal{F}_t$-adapted, and thus G_{H_t-} is an $\mathcal{F}_t$-stopping time for all t.

To show (4.2), observe that for all s we have

$$\{G_{H_t-} > G_s\} = \{H_t > s\} = \{G_s < t\}, \tag{4.3}$$

so that $G_{H_t-} \geq \tau_t$ almost surely. But since $G_{H_t-} \leq t$ surely, then the equality holds. Therefore, being that the process τ_t is increasing and almost surely finite, it is a time change if a right-continuous modification exists. Consider the process τ_t^+ obtained by replacing τ_t with τ_{t+} in the set where τ_t is discontinuous, that is, the set of points in $\mathcal{R}$ isolated on their right. Denote by $\partial\mathcal{R}^l$ such a set and by $\mathcal{R}^{\Delta G}$ the image of the Lévy process ΔG. Because of the regenerative property of $\mathcal{R}^{\Delta G}$ (see Bertoin [2, Chapter 2]), $\partial\mathcal{R}^l$ is distributed as $\mathcal{R}^{\Delta G}$, and hence

$$\mathbb{P}(\tau_t^+ \neq \tau_t) = \mathbb{P}(t \in \partial\mathcal{R}^l) = \mathbb{P}(t \in \mathcal{R}^{\Delta G}) = 0, \tag{4.4}$$

where the last equality follows by, e.g., Bertoin [2, Proposition 1.9.i]. ■

From now on we will make use of the right-continuous version of τ_t. It is crucial to observe that $\tau_t < t$ if and only if $t \in \overline{\mathcal{R}}^c$, i.e., using Definition 3.1, if and only if the asset is suspended *strictly before* time t. Conversely, $\tau_t = t$ if and only if the asset is tradable in t or has been halted *exactly* in t. We therefore denominate τ_t the *last market quote time* process. Observe again that τ_t has a jump discontinuity exactly at the market reopening times given by $t = G_s$, $\Delta G_s \neq 0$, i.e., the points in $\mathcal{R}$ isolated on their left.

To see the importance of τ_t we begin by showing that using this process we can calculate the probability of the asset being tradable at any given time t.

Proposition 4.2. *We have that*

$$\mathbb{P}(\tau_t = t) = \frac{\beta + \lambda e^{-(\lambda+\beta)t}}{\beta + \lambda} \tag{4.5}$$

for all $t > 0$.

Proof. We recall that the qth potential measure $U^q(dx)$ of a Lévy process L_t is defined as the occupation measure

$$U^q(dx) = \int_0^\infty e^{-qt}\mathbb{P}(L_t \in dx)dt. \tag{4.6}$$

If $U^q(dx)$ is absolutely continuous, its Radon derivative $u^q(x)$ is called the potential density. When $q = 0$ and L_t is a subordinator, $U^0(dx)$ and $u^0(x)$ also go under the name of renewal measure (resp., density). In this case we drop the superscript and write $U(dx)$ and $u(x)$.

By Theorem 5 in Bertoin [3, Chapter 3], we have that since G_t has drift $d = 1$, its renewal density exists, can be chosen continuous, and satisfies $\mathbb{P}(\mathcal{T}_t = t) = u(t)$, where $\mathcal{T}_t = G_{H_t}$ is the first passage of G_t of $[t, \infty)$. Now observe that the well-known relationship (e.g., Bertoin [2, section 1.3])

$$\mathcal{L}(U(dt), s) = \int_0^\infty e^{-st} u(t) dt = \frac{1}{\phi_G(s)} \tag{4.7}$$

yields

$$\mathcal{L}(U(dt), s) = \frac{1}{s} \frac{\beta + s}{\beta + \lambda + s}. \tag{4.8}$$

On the other and, by a direct calculation

$$\int_0^\infty e^{-st} \frac{\beta + \lambda e^{-(\lambda+\beta)t}}{\lambda + \beta} dt = \frac{1}{s} \frac{\beta}{\beta + \lambda} + \frac{\lambda}{\beta + \lambda} \frac{1}{\lambda + \beta + s} = \frac{1}{s} \frac{\beta + s}{\beta + \lambda + s}. \tag{4.9}$$

The uniqueness of the Laplace transform for continuous functions then yields

$$\mathbb{P}(\mathcal{T}_t = t) = \frac{\beta + \lambda e^{-(\lambda+\beta)t}}{\lambda + \beta} \tag{4.10}$$

To conclude, observe that by Bertoin [3, Chapter 3, Proposition 2.*ii*], we know that $\mathbb{P}(\{\tau_t < t, \mathcal{T}_t = t\}) = 0$ for all fixed t entailing $\mathbb{P}(\tau_t = t | \mathcal{T}_t = t) = 1$. Also, $\mathbb{P}(\tau_t = t, \mathcal{T}_t > t) = \mathbb{P}(\Delta G_t \neq 0) = 0$ because a Lévy process is stochastically continuous, so that also $\mathbb{P}(\mathcal{T}_t = t | \tau_t = t) = 1$. By conditioning the event $\{\tau_t = t, \mathcal{T}_t = t\}$ one then sees that $\mathbb{P}(\mathcal{T}_t = t) = \mathbb{P}(\tau_t = t)$, so that (4.5) follows in view of (4.10). ∎

The last available market quote Q_t of S_t at time t is then nothing else than the value of S_t time-changed with τ_t:

$$Q_t := S_{\tau_t} = S_0 \exp\left(X_{H_t} + R_{\tau_t}\right). \tag{4.11}$$

The quote process Q_t is therefore an $\mathcal{F}_{\tau_t}$-semimartingale. The second equality follows from the obvious identity $H_{\tau_t} = H_t$. Moreover, since $\{\tau_t \leq t\} = \Omega$, $\mathcal{F}_{\tau_t} \subset \mathcal{F}_t$, but since we know $\mathcal{F}_{H_t} \subset \mathcal{F}_t$, we conclude that Q_t is $\mathcal{F}_t$-adapted.

The process Q_t acts as required. Whenever the time runs in a suspension interval, τ_t does not affect the trade component X_{H_t}, which is already halted during such intervals. However, such a time change *does* halt the evolution of R_t at the last value R_{τ_t} before the suspension, thus achieving the desired interpretation of S_{τ_t} as the last available market quote. Although R_t is stopped during the suspensions, its background evolution at those time spans still plays an important role in the dynamics of Q_t, as it combines with the discontinuities of τ_t to determine the "jump" in the reopening price.[5] Outside the suspension intervals we have the plain relation $\tau_t = t$.

Now, according to section 2, the traded underlying to be used for derivative valuation is the process P_t in (2.2). Observe that this process is $\mathcal{F}_t$-adapted. Let f be a square-integrable

[5]Strictly speaking, also X_t determines the new opening price, but R_t accumulates variation during the suspension intervals, whereas X_t just affects the price through X_{H_t} which in turn, by construction, only releases instantaneous variability at reopening. Thus, its contribution to the variance is much inferior.

contingent claim maturing at T. Since P_t is a marketable asset, by the fundamental theorem of asset pricing, the value V_0 of f written on P_t is given by

$$V_0 = \mathbb{E}^{\mathbb{Q}}[e^{-rT} f(P_T)], \tag{4.12}$$

where $\mathbb{Q}$ is a $\mathbb{P}$-equivalent martingale measure under which the discounted process $e^{-rt}P_t$ is a martingale. As we will detail later, such measure will not be unique, entailing market incompleteness for the model under inspection. Now observe that

$$e^{-rt}P_t = e^{-r\tau_t}Q_t. \tag{4.13}$$

Therefore, we have the derived following no-arbitrage principle for suspendable stocks.

No-arbitrage principle for securities with market suspensions. *The martingale property of the lookback value P_t discounted using the risk-free rate is equivalent to the martingale property of the quote process Q_t discounted with the stochastic discount factor $e^{-r\tau_t}$.*

In the next section we explore the implications of this principle for the determination of equivalent martingale measures/no-arbitrage relations for option pricing on financial securities with market halts. Before moving on, let us briefly summarize the framework.

Construction of an underlying traded security when securities can be suspended.

1. Select processes X_t and R_t for the market trade and rumor value components as well as a specification of the market halt generating process G_t.
2. Introduce the fundamental value S_t of an asset with market halts as in (3.9).
3. Determine the quote process Q_t by time-changing S_t with the last price observation time τ_t prior to t.
4. Define the traded contract P_t delivering the cash amount Q_{τ_t} in t, whose value in t is $P_t = e^{r(t-\tau_t)}Q_t$.

Using a common drawing from X_t, R_t, and G_t, we visualize the processes G_t, H_t, S_t, τ_t, Q_t, and P_t in Figures 1–4. We have used $S_0 = 100$, $\mu_X = 0.3$, $\sigma_X = 0.5$, $\mu_R = -0.2$, $\sigma_R = 0.2$,

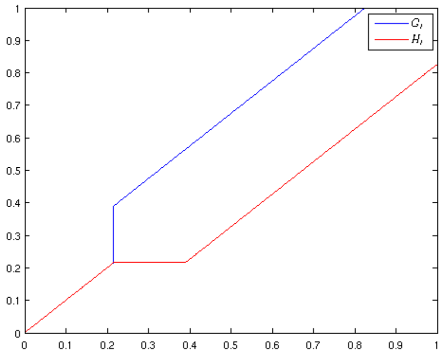

Figure 1. *Halts generator G_t and halts process H_t.*

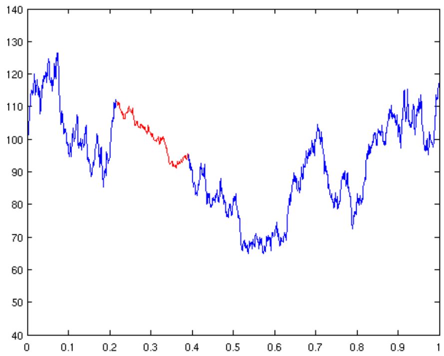

Figure 2. *Fundamental asset price value S_t.*

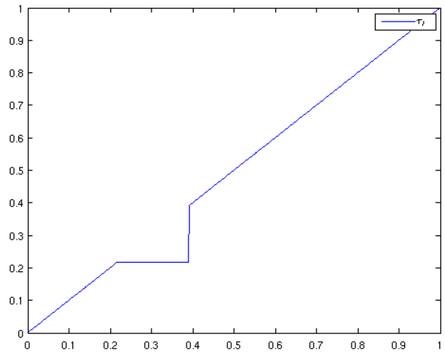

Figure 3. *Last quote time processes τ_t. The process is equal to t on the sets where G_t does not jump.*

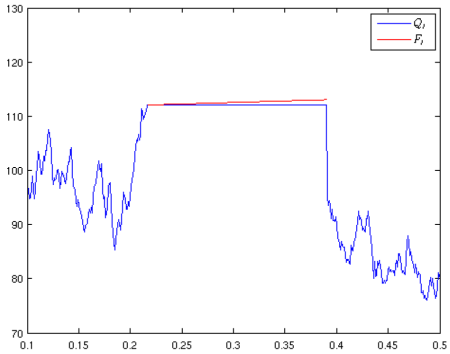

Figure 4. *Quote process Q_t and lookback price P_t (close-up of Figure 2). The processes coincide when S_t is tradable.*

and $\nu_X = \nu_R = 0$, so that conditional on being traded the asset follows the Black–Scholes–Samuelson model with drift $\mu = 0.1$ and volatility coefficient $\sigma = 0.5385$. The asset halt parameters are $\lambda = 1$ and $\beta = 7$.

5. Risk-neutral dynamics and price of trade suspension risk. We now proceed to investigate the martingale dynamics of the discounted asset $e^{-rt}P_t$. In view of the no-arbitrage principle, this is equivalent to determining the no-arbitrage dynamics of the stochastically discounted quote price $e^{-r\tau_t}Q_t$. As we shall see, because of the boundedness of τ_t, to achieve this it is sufficient to determine martingale relations on the fundamental value S_t.

To make the discussion more transparent, we introduce the following general proposition, stating that under certain conditions time change and measure change commute. For general background, see Jacod and Shiryaev [13], Jacod [12], and Kallsen and Shiryaev [14].

Lemma 5.1. *Let X_t be a semimartingale on a filtered space $(\Omega, \mathbb{P}, \mathcal{F}, \mathcal{F}_t)$ which is continuous with respect to a time change T_t and Z_t a martingale density process having the stochastic exponential representation*

$$Z_t = \mathcal{E}\left(\int_0^t H_u dX_u^c + \int_0^t (W(u,x) - 1)(\mu^X - \nu^X)(dx \times du)\right), \tag{5.1}$$

where X_t^c is the continuous martingale part of X_t; μ^X and ν^X are, respectively, its jump measure and jump compensator; H_t is some square-integrable process integrable with respect to X_t^c; and $W(t,x)$ is a random function such that the second integral in (5.1) exists. The symbol $\mathcal{E}(\cdot)$ stands for the stochastic exponential.

Assume further that Z_{T_t} is a true martingale, and denote by $\mathbb{Q}$ and $\mathbb{Q}^T$ the $\mathbb{P}$-equivalent measures associated, respectively, with Z_t and Z_{T_t}. We have, up to evanescence,

$$(X_{T_t})^{\mathbb{Q}^T} = X_{T_t}^{\mathbb{Q}}. \tag{5.2}$$

Proof. Let $(\mu_t, \sigma_t, \nu(dt \times dx))$ be the $\mathbb{P}$-characteristics of X_t. By the Girsanov theorem for semimartingales (Jacod and Shiryaev [13, Chapter III, Theorem 3.24]), their $\mathbb{Q}$ counterparts, in the "disintegrated" form (Jacod and Shiryaev [13, Chapter II, Proposition 2.9]), are

$$\mu_t^{\mathbb{Q}} = \mu_t + \int_0^t H_u \sigma_u dA_u + \int_{|x|<1} (W(t,x) - 1) K_t(dx), \tag{5.3}$$

$$\sigma_t^{\mathbb{Q}} = \sigma_t, \tag{5.4}$$

$$\nu^{\mathbb{Q}}(dt \times dx) = dA_t W(t,x) K_t(dx) \tag{5.5}$$

for some predictable process A_t and random measure $K_t(dx)$. Furthermore, according to Jacod and Shiryaev [13, Lemma 2.7], by the adaptedness of X_t to T_t, the characteristics of X_{T_t} under $\mathbb{Q}$ are $(\mu_{T_t}, \sigma_{T_t}, \nu(dT_t \times dx))$. Now by Jacod [12, Theorems 10.19, 10.27], we have that

$$Z_{T_t} = \mathcal{E}\left(\int_0^t H_{T_u} dX_{T_u}^c + \int_0^t W(T_u, x)(\mu^X - \nu^X)(dx \times dT_u) \right). \tag{5.6}$$

Therefore, by applying the Girsanov's theorem to X_{T_t} with respect to the density Z_{T_t}, we obtain, taking into account $\langle \int_0^{\cdot} H_{T_u} dX_{T_u}^c \rangle_t = \int_0^{T_t} H_u \sigma_u dA_u$ (because of Jacod [12, Theorem 10.17]), the following characteristics:

$$\mu_t^{\mathbb{Q}^T} = \mu_{T_t} + \int_0^{T_t} H_u \sigma_u dA_u + \int_{|x|<1} (W(T_t, x) - 1) K_{T_t}(dx), \tag{5.7}$$

$$\sigma_t^{\mathbb{Q}^T} = \sigma_{T_t}, \tag{5.8}$$

$$\nu^{\mathbb{Q}^T}(dt \times dx) = dA_{T_t} W(T_t, x) K_{T_t}(dx), \tag{5.9}$$

which match $(\mu_{T_t}^{\mathbb{Q}}, \sigma_{T_t}^{\mathbb{Q}}, \nu^{\mathbb{Q}}(dT_t \times dx))$, proving the claim. ∎

We can then directly state the main result of this section on the martingale relations for the fundamental asset price process.

Theorem 5.2. *Let S_t be given by (3.9). If neither X_t nor R_t is an increasing or decreasing process, then there exists an equivalent martingale measure $\mathbb{Q}^{X,R,H} \sim \mathbb{P}$ for S_t with density given by*

$$\frac{d\mathbb{Q}^{X,R,H}}{d\mathbb{P}} = \mathcal{X}_t \mathcal{R}_t \mathcal{H}_t \tag{5.10}$$

for some martingale density processes $\mathcal{X}_t$ and $\mathcal{R}_t$ and

$$\mathcal{H}_t = \exp\left((\lambda - \lambda^*)t + \sum_{s \le t} \log\left(h\left(\Delta G_s\right)\right) \right), \tag{5.11}$$

where $\lambda^* > 0$ and $h(x) = \beta^{-1} e^{\beta x} p^*(x) \mathbb{1}_{\{x>0\}}$ for some positively supported probability density function $p^*(x)$.

Moreover, under $\mathbb{Q}^{X,R,H}$ the discounted fundamental value process $e^{-rt}S_t$ is of the form $S_0 \exp(X_{H_t^*}^* + R_t^* - rt)$ for some Lévy processes X_t^* and R_t^*, and H_t^* is the inverse of a *compound Poisson process of drift one, intensity* λ^**, and i.i.d. jumps of probability density* $p^*(x)$*. In particular,* $\mathbb{Q}^{X,R,H}$ *is never unique.*

Proof. Under the equivalent martingale measure induced by $\mathcal{H}_t$, we have that the dynamics of G_t are those of a compound Poisson process of unit drift with intensity λ^* and jump probability density p^* (e.g., Sato [21, Theorem 33.1]). We denote by H_t^* the dynamics of H_t under the $\mathbb{P}$-equivalent measure $\mathbb{Q}^H$ induced by $d\mathbb{Q}^H/d\mathbb{P} = \mathcal{H}_t$. Clearly, H_t^* remains a time change.

The first step is to isolate a change of measure under which $\exp(X_t)$ and $\exp(R_t)$ are individually martingales. By a classic construction (Cherny and Shiryaev [5, Theorem 4.6]), under the given assumptions on R_t and X_t, it is possible to find martingale density processes $\mathcal{R}_t$ and $\mathcal{X}_t^0$ determining equivalent martingale measures for $\exp(X_t)$ and $\exp(R_t)$ such that under the respective measures change, R_t and X_t are Lévy processes of triplets, respectively, $(\mu_X^0, \sigma_X, \nu_X^0(dx))$ and $(\mu_R^*, \sigma_R^*, \nu_R^*(dx))$, where

$$\mu_X^0 = -(\sigma_X^0)^2/2 - \int_{\mathbb{R}} (e^x - 1 - x\mathbb{1}_{|x|<1})\nu_X^0(dx), \tag{5.12}$$

$$\sigma_X^0 = \sigma_X, \tag{5.13}$$

$$\nu_X^0 \sim \nu_X, \qquad \int_{\{|x|>a_X\}} e^x \nu_X^0(dx) < \infty, \tag{5.14}$$

and

$$\mu_R^* = -(\sigma_R^*)^2/2 - \int_{\mathbb{R}} (e^x - 1 - x\mathbb{1}_{|x|<a_X})\nu_R^*(dx) + r, \tag{5.15}$$

$$\sigma_R^* = \sigma_R, \tag{5.16}$$

$$\nu_R^* \sim \nu_R, \qquad \int_{\{|x|>a_R\}} e^x \nu_R^*(dx) < \infty, \tag{5.17}$$

for some constants $a_X, a_R > 0$. The integrability conditions in (5.14) and (5.17) are equivalent to state that the corresponding exponential moments of X_t and R_t exist.

Now under $\mathbb{Q}^H$ consider $\mathcal{X}_t = \mathcal{X}^0_{H_t^*}$; clearly, $\mathcal{X}_t^0$ and $\mathcal{R}_t$ remain $\mathbb{Q}^H$-martingales. Furthermore, since H_t^* a bounded stopping time, by Doob's optional sampling theorem, $\mathcal{X}_t$ is also a martingale. Thus, by independence, $\mathcal{X}_t\mathcal{R}_t$ is a $\mathbb{Q}^H$-martingale, and $d\mathbb{Q}^{X,R,H}/d\mathbb{Q}^H = \mathcal{X}_t\mathcal{R}_t$ is well defined.

We can thus proceed to calculate the characteristics of R_t^* and $X^*_{H_t^*}$. Again by independence, the characteristics of R_t^* coincide with (5.15)–(5.17). Furthermore, since $\mathcal{X}_t^0$ has the exponential representation (5.1), we can use Lemma 5.1, and the characteristics of $X^*_{H_t^*}$ are obtained by simply time-changing (5.12)–(5.14):

$$\mu_X^*(t) = -H_t^*(\sigma_X^0)^2/2 - H_t^* \int_{\mathbb{R}} (e^x - 1 - x\mathbb{1}_{|x|<1})\nu_X^0(dx), \tag{5.18}$$

$$\sigma_X^*(t) = \sigma_X H_t^*, \tag{5.19}$$

$$\nu_X^*(dx \times dt) = dH_t^* \nu_X^0(dx). \tag{5.20}$$

Recall now that for $\theta \in \mathbb{C}$, the (Fourier) cumulant process $K_t^X(\theta)$ of a quasi-left-continuous semimartingale X_t is the almost surely uniquely determined process $K_t^X(\theta)$ such that in the appropriate domains of integrability, $\exp(i\theta X_t - K_t^X(\theta))$ is a local martingale. In the case of a Lévy process, in our notation $K_t^X(\theta) = -t\psi_X(-\theta)$. By Kallsen and Shiryaev [14, Lemma 2.7] one has that if T_t is a time change and X_t is T_t-continuous, $K_t^{X_T}(\theta) = K_{T_t}^X(\theta)$.

In our case, by virtue of (5.14) and (5.17) the relevant cumulant processes $K_t^X(-i)$ and $K_t^R(-i)$ are well defined. Moreover, we have

$$X^*_{H^*_t} = \tilde{X}_{H^*_t} - K^{\tilde{X}}_{H^*_t}(-i), \tag{5.21}$$

$$R^*_t = \tilde{R}^*_t - K^{\tilde{R}}_t(-i) + rt \tag{5.22}$$

with $\tilde{X}_{H^*_t}$ and $\tilde{R}^*_t$ being the driftless processes with characteristics given, respectively, by $(0, \sigma^*_X(t), \nu^*_X(dx \times dt))$ and $(0, t\sigma^*_R, \nu^*_R(dx)dt)$. Therefore, by independence, $\exp(X^*_{H^*_t} + R^*_t - rt)$ is a local martingale under $\mathbb{Q}^{X,R,H}$, with $\exp(R^*_t - rt)$ being a true martingale because $\tilde{R}^*_t$ is a Lévy process with finite first exponential moment. Finally, conditioning and using the independence of H^*_t and X^*_t yields that for all t, $\mathbb{E}[\exp(X^*_{H^*_t})] = 1$ again by the martingale property of X^*_t following from finiteness of the exponential moment. Therefore, $\exp(X^*_{H^*_t} + R^*_t - rt)$ is indeed a martingale, and this terminates the proof. ■

It is then a simple consequence of the discussion in section 3 and Theorem 5.2 that P_t is also a martingale under the measures $\mathbb{Q}^{X,R,H}$.

Corollary 5.3. *Let $\mathcal{F}^S_t$ be the filtration generated by S_t. Under the measures $\mathbb{Q}^{X,R,H}$ of Theorem 5.2, P_t is an $\mathcal{F}^S_{\tau_t}$-martingale.*

Proof. By Proposition 5.2 we have that $e^{-rt}S_t$ is an $\mathcal{F}^S_t$-martingale under $\mathbb{Q}^{X,H,R}$. But since τ_t is a bounded stopping time, we can apply Doob's optimal sampling theorem, from which follows that $e^{-r\tau_t}S_{\tau_t} = e^{-r\tau_t}Q_t = e^{-rt}P_t$ is an $\mathcal{F}^S_{\tau_t}$-martingale. ■

In the process of isolating the martingale density process corresponding to the change of measure to the equivalent risk-neutral ones, we can see that the model is intrinsically incomplete, *even* if the underlying model S^0_t in (4.11) is complete. In the most general situation of an incomplete underlying Lévy model, pricing in our framework incorporates two sources of unhedgeable risk. First, as in the unhalted model, the presence of jumps in the drivers X_t and R_t bears a source of systematic risk which cannot be completely hedged by trading in a set of fundamental securities. Second, (5.11) characterizes the class of the equivalent measures under which the halts generator G_t remains a (necessarily increasing) Lévy process. In particular, under any such measure change, the jumps of G_t and hence the intervals on which H_t is constant cannot disappear. Therefore, modeling the asset halts by a random time change inverse to a Lévy subordinator introduces an additional source of market risk which is equally unhedgeable in terms of replication. This risk corresponds to the "totally inaccessible" events of a suspension taking place. A suspension can happen at any time without notice: Suspension times are not predictable times. Hence, derivatives on assets with halts cannot be perfectly replicated using a predictable trading strategy.

Consequently, the introduction of a "horizontal jumpiness" of the securities brings about the concept of *market price of suspension risk* embedded in the risk-neutral parameter λ^* and probability law $p^*(x)dx$. These new parameters encode the premium that the investors should demand for holding an investment which is subject to suspensions.

6. Contingent claim valuation.

In this section we show how the model leads to analytical pricing formulae under the assumption that the pricing measure $\mathbb{Q}$ preserves the exponential distribution of the jumps in G_t. Furthermore, we illustrate how the underlying Lévy model can be recovered as a limiting case of the suspended one.

Let us begin from the derivation of the Laplace–Laplace transform of the joint density of an inverse subordinator H_t and its last passage process G_{H_t-}.

Proposition 6.1. *Let G_t be any strictly increasing subordinator and H_t be inverse as defined in (3.7), denote by $P_t(x,y)$ the joint law of (H_t, G_{H_t-}), and let $\hat{P}_t(q,k) = \mathbb{E}[e^{-qH_t - kG_{H_t-}}]$. The Laplace transform in the variable t of $\hat{P}_t(q,k)$ satisfies*

$$\mathcal{L}(\hat{P}_t(q,k), s) = \int_0^\infty e^{-st}\hat{P}_t(q,k)dt = \frac{1}{s}\frac{\phi_G(s)}{q+\phi_G(k+s)}. \tag{6.1}$$

Proof. Because of the possibility of an atom at t in G_{H_t-}, we must divide

$$\mathcal{L}(\hat{P}_t(q,k), s) = \int_0^\infty e^{-st}\mathbb{E}[e^{-qH_t-kG_{H_t-}}\mathbb{1}_{\{G_{H_t-}=t\}}] + \int_0^\infty e^{-st}\mathbb{E}[e^{-qH_t-kG_{H_t-}}\mathbb{1}_{\{G_{H_t-}<t\}}]. \tag{6.2}$$

It is shown in Kyprianou [15, Chapter 5], that if the drift d of G_t is positive,

$$\mathbb{E}[e^{-qH_t}\mathbb{1}_{\{G_{H_t-}=t\}}] = d\, u^q(x), \tag{6.3}$$

where $u^q(x)$ is the qth potential measure of G_t (and such quantity is zero otherwise), whence

$$\begin{aligned}\int_0^\infty e^{-st}\mathbb{E}[e^{-qH_t-kG_{H_t-}}\mathbb{1}_{\{G_{H_t-}=t\}}]dt &= d\int_0^\infty e^{-(s+k)t}u^q(x)dx \\ &= d\int_0^\infty e^{-(q+\phi_G(k+s))t}dx = \frac{d}{q+\phi_G(s+k)}.\end{aligned} \tag{6.4}$$

Let us turn to study the transform of (H_t, G_{H_t-}) on the set $\{G_{H_t-} < t\}$. By conditioning on $\{H_t = x\}$, applying Fubini's theorem, and writing $f(x,t)$ for the density of H_t (which exists by Meerschaert and Scheffler [20, Theorem 3.1]),

$$\begin{aligned}\mathbb{E}[e^{-qH_t-kG_{H_t-}}\mathbb{1}_{\{G_{H_t-}<t\}}] &= \int_0^\infty e^{-qx}f(x,t)dx\int_0^t e^{-ky}\mathbb{P}(G_{H_t-}\in dy|H_t=x) \\ &= \int_0^t e^{-ky}dx\int_0^\infty e^{-qx}\mathbb{P}(G_{H_t-}\in dy, H_t = x).\end{aligned} \tag{6.5}$$

The crucial remark is now that

$$\{G_{H_t-} < y, H_t = x\} = \{G_{x-} < y, G_x \geq t\} = \{G_{x-} < y, \Delta G_x > t - G_{x-}\}. \tag{6.6}$$

Hence, define the point process

$$\gamma_x^t = \sum_{s\leq x\leq t}\mathbb{1}_{\{\Delta G_s > t - G_{s-}\}}, \tag{6.7}$$

and denote the tail density $\overline{\nu}_G(u) = \nu_G(u,\infty)$. Since G_x has Lévy measure ν_G, for a Borel random set A the point process $\sum_{s\le x} \mathbb{1}_{\{\Delta G_s \in A\}}$ has compensating measure $\nu_G(A)dx$, and thus γ_x^t has compensating measure $\overline{\nu}_G(t-G_{x-})dx$. Therefore, as $\mathbb{1}_{\{G_{x-}<y\}}$ is predictable, by virtue of (6.6), the compensation formula (e.g., Last and Brandt [16, Proposition 4.1.6]), Fubini's theorem, and stochastic continuity of G_x, we calculate

$$\begin{aligned}\int_0^\infty e^{-qx}\mathbb{P}(G_{H_t-}\le y, H_t = x)dx &= \mathbb{E}\left[\int_0^\infty e^{-qx}\mathbb{1}_{\{G_{x-}<y\}}d\gamma_x^t\right]\\ &= \mathbb{E}\left[\int_0^\infty e^{-qx}\mathbb{1}_{\{G_{x-}<y\}}\overline{\nu}_G(t-G_{x-})dx\right]\\ &= \int_0^\infty e^{-qx}dx\int_0^y \mathbb{P}(G_x\in dz)\overline{\nu}_G((t-z)-)\\ &= \int_0^y U^q(dz)\overline{\nu}_G(t-z). \qquad (6.8)\end{aligned}$$

The last equality follows because U^q has no atoms and the set of discontinuities of a Lévy measure has Lebesgue measure zero. Thus,

$$\mathbb{E}[e^{-qH_t-kG_{H_t-}}\mathbb{1}_{\{G_{H_t-}<t\}}] = \int_0^t e^{-ky}U^q(dy)\overline{\nu}_G(t-y)dy, \qquad (6.9)$$

so that, again applying Fubini's theorem,

$$\begin{aligned}\int_0^\infty e^{-st}\mathbb{E}[e^{-qH_t-kG_{H_t-}}\mathbb{1}_{\{G_{H_t-}<t\}}]dt &= \int_0^\infty e^{-st}dt\int_0^t e^{-ky}U^q(dy)\overline{\nu}_G(t-y)dy\\ &= \int_0^\infty\int_0^t e^{-s(t-y)}e^{-(k+s)y}U^q(dy)\overline{\nu}_G(t-y)dtdy\\ &= \mathcal{L}(\overline{\nu}_G(t),s)\mathcal{L}(U^q(dy),k+s). \qquad (6.10)\end{aligned}$$

Using the formula (e.g., Bertoin [3, section III.1])

$$\mathcal{L}(\overline{\nu}_G(t),s) = \frac{\phi_G(s)}{s} - d \qquad (6.11)$$

and the second and third equalities in (6.4) again (with $d=1$), we obtain

$$\int_0^\infty e^{-st}\mathbb{E}[e^{-qH_t-kG_{H_t-}}\mathbb{1}_{\{G_{H_t-}<t\}}] = \frac{1}{q+\phi_G(s+k)}\left(\frac{\phi_G(s)}{s}-d\right), \qquad (6.12)$$

and by substituting (6.12) and (6.4) in (6.2), the proof is complete. ∎

This proposition extends Bertoin [2, Lemma 1.11], and is somewhat reminiscent of the Wiener–Hopf factorization formulae and analogous identities in Lévy potential theory (see Bertoin [3], Kyprianou [15]).

Crucially, since X_t and R_t are independent of G_t, they are of both τ_t and H_t. By combining this property with the proposition above, we can obtain pricing formulae applying the well-known Fourier techniques. Further, there exists risk-neutral martingale densities preserving the exponential distribution of the suspensions and under which the risk-neutral characteristic function can be computed analytically.

Theorem 6.2. *Let $f(x)$ be a contingent claim on P_t maturing at time T, assume that the Fourier transform $\hat{w}(z)$ of $w(x) = f(e^x)$ exists, and let $\mathcal{S}_w$ be its domain of regularity. Choose among all the equivalent martingale measures $\mathbb{Q}$ from Theorem 5.2 those such that in equation (5.11) we have*

$$\mathcal{H}_t = \exp\left((\lambda - \lambda^*)t + \sum_{s \le t} \log\left(\frac{\beta^*}{\beta}\right)(\beta^* - \beta)\Delta G_s\right) \tag{6.13}$$

for some $\beta^* > 0$. Denote with $\mathcal{S}_F$ the domain of regularity of $\mathbb{E}^{\mathbb{Q}}[e^{-iz\log P_T}]$, and assume $\mathcal{S}_w \cap \mathcal{S}_F \neq \emptyset$. The price V_0 of the derivative is given by

$$V_0 = \mathbb{E}^{\mathbb{Q}}[e^{-rT} f(P_T)] = \frac{e^{-rT}}{2\pi} \int_{i\gamma-\infty}^{i\gamma+\infty} S_0^{-iz} e^{-izrT} \hat{w}(z) \Phi_T\big(\psi_X^*(z), \psi_R^*(z), \lambda^*, \beta^*\big) dz \tag{6.14}$$

with

$$\begin{aligned} &\Phi_t\big(z_1, z_2, \lambda, \beta\big) = (De^{b_t})^{-1} \\ &\cdot \Big(\beta^2 c_t(d_t - 1)(\lambda + z_2) - z_2\big(2e^{b_t}\lambda a + e(c_t d_t(a - \lambda - z_1) + c_t(\lambda + z_1 + a))\big) + \beta c((d_t - 1)\lambda^2 \\ &+ \lambda(z_1 - d_t z_1 + a + d_t(a - z_2) + z_2) + z_2(-2(d_t - 1)z_1 + a + d_t(a - z_2) + z_2))\Big), \end{aligned} \tag{6.15}$$

where

$$\begin{aligned} a &= \sqrt{\beta^2 + 2\beta(\lambda - z_1) + (\lambda - z_1)^2}, \\ b_t &= \exp\left(\tfrac{t}{2}(z_1 + 2z_2 + \lambda + a)\right), \\ c_t &= e^{\beta t/2}, \\ d_t &= e^{at}, \\ e &= z_1 + z_2, \\ D &= 2a(\beta(\lambda + z_2) - z_2(\lambda + e)) \end{aligned} \tag{6.16}$$

and γ is chosen such that the integration contour lies in $\mathcal{S}_w \cap \mathcal{S}_F$.

Proof. Clearly, (6.13) is obtained in Theorem 5.2 by choosing for $p^*(x)$ an exponential density of parameter β^*. Since $\mathcal{S}_w \cap \mathcal{S}_F \neq \emptyset$, by the discussion in Lewis [18], and conditioning under independence, we have that the value V_0 of a derivative can be represented as the Parseval-type convolution

$$\begin{aligned} V_0 = e^{-rt}\mathbb{E}^{\mathbb{Q}}[w(\log(P_T))] &= \frac{e^{-rt}}{2\pi} \int_{i\gamma-\infty}^{i\gamma+\infty} \mathbb{E}^{\mathbb{Q}}[e^{-iz\log P_T}]\hat{w}(z) dz \\ &= \frac{e^{-rt}}{2\pi} \int_{i\gamma-\infty}^{i\gamma+\infty} S_0^{-iz} e^{-izrT} \mathbb{E}^{\mathbb{Q}}[e^{-\psi_X^*(z)H_T - \psi_R^*(z)G_{H_T-}}]\hat{w}(z) dz \end{aligned} \tag{6.17}$$

for some γ chosen in $\mathcal{S}_w \cap \mathcal{S}_F$. But then using Proposition 6.1 and taking the analytic continuation on the convergence domain of the Laplace transform, we have for some $z_1, z_2 \in \mathbb{C}$

$$\begin{aligned}\int_0^\infty e^{-sT}\mathbb{E}^{\mathbb{Q}}[e^{-z_1 H_T - z_2 G_{H_T -}}]dt &= \frac{1}{s}\frac{\phi_G^*(s)}{z_1 + \phi_G^*(z_2+s)}\\ &= \frac{1}{\beta^*+s}\frac{(\beta^*+\lambda^*+s)(\beta^*+z_2+s)}{(z_1+z_2+s)(\beta^*+z_2+s)+\lambda^*(z_2+s)}.\end{aligned} \tag{6.18}$$

Explicitly calculating the inverse Laplace transform of the last line of (6.18) with MATHEMATICA, we obtain (6.15)–(6.16). ■

Note that if the risk-neutral distribution of the jumps of G_t is not exponential in the pricing measure, pricing would still have been possible, but at the cost of a double integral inversion.

6.1. Hedging. As is often the case in incomplete models, hedging in a Lévy framework with halts can be operated in the mean-variance sense. That is to say, the hedging problem reduces to the identification of a trading strategy in the underlying market asset which minimizes the variance of the difference between the option value and the hedge position at maturity (or at any given time). To this end, essential tools are the martingale decomposition theorems of local martingales on the subspaces generated by stochastic integrals with respect to a martingale, specifically, the Galtchouk–Kunita–Watanabe decomposition and the Föllmer–Schweizer decomposition (for a full account, see, e.g., Schweizer [22]).

However, in the presented model the martingale underlying in a risk-neutral measure is the discounted price $\tilde{P}_t = e^{-rt}P_t$ and not the traded equity with price $\tilde{Q}_t = e^{-rt}Q_t$. On the other hand, the price P_t is a contracted reference cash flow agreement rather than a genuine marketed asset and as such might be unavailable for trading and hedging. Therefore, the only option available to market participants to hedge a derivative is trading (outside the halts) in the physical equity. It is then not immediately obvious in the present setup how to reconcile the mathematics with the financial aspects. However, as we illustrate below, an approximate mean-variance hedge can still be devised.

Let $\tilde{V}_t = e^{-rt}V_s = e^{-rt}\mathbb{E}_t[f(P_T)|\mathcal{F}^S_{\tau_t}]$ be the time $t < T$ value of a derivative f on P expiring at T and

$$\mathcal{S} = \left\{\psi_t \text{ predictable, } \mathbb{Q}\text{-square integrable on } [0,T] \text{ with respect to } \tilde{Q}_t\right\}. \tag{6.19}$$

Since $\tilde{P}_t$ and $\tilde{V}_t$ are martingales, by the aforementioned Galtchouk–Kunita–Watanabe decomposition (see, e.g., Jacod and Shiryaev [13]), for any risk-neutral measure $\mathbb{Q} \sim \mathbb{P}$ (thus, in particular, if it exists for the minimal martingale measure of $\tilde{P}_t$) there exists a local martingale N_t such that $\langle N, \tilde{P}\rangle_t = 0$ and an $\mathcal{F}^S_{\tau_t}$-predictable process ϕ_t such that

$$\tilde{V}_t = V_0 + \int_0^t \phi_s d\tilde{P}_s + N_t, \tag{6.20}$$

and it can be verified that

$$\phi_t = \arg\min_{\psi_t \in \mathcal{S}} \mathbb{E}^{\mathbb{Q}}\left[\left(\tilde{V}_t - \int_0^t \psi_s d\tilde{P}_s - V_0\right)^2\right]. \tag{6.21}$$

Remembering the discounting principle, since the random set $\{\tau_s < s\}$ is a finite union of disjoint intervals where $e^{-r\tau}Q$ is constant and τ_t has discontinuities in all the points of $\mathcal{R}$ isolated on their left, so that

$$\{s \leq t : \tau_s = s, \Delta\tau_s \neq 0\} = \{s \leq t,\, s = G_u,\, \Delta G_u \neq 0\}, \tag{6.22}$$

we can write

$$\begin{aligned}\int_0^t \phi_s d\tilde{P}_s = \int_0^t \phi_s d(e^{-r\tau_s}Q_s) = \int_{\{s\leq t:\tau_s=s,\Delta\tau_s=0\}} \phi_{\tau_s} d(e^{-rs}Q_s)\\ + \sum_{s\leq t,\, s=G_u,\, \Delta G_u\neq 0} e^{-r\Delta G_u}(Q_s - Q_{s-})\phi_s = \int_0^t \phi_{\tau_s} d\tilde{Q}_s - \epsilon_t.\end{aligned} \tag{6.23}$$

Here ϵ_t is the contribution of the integral with respect to $d\tilde{Q}_t$ on the set $\{\tau_s < s\}$; i.e., it is the value of the position held in Q when the halts occur, discounted throughout all the halts duration. More precisely,

$$\epsilon_t = \sum_{s=G_{u-},\, G_u<t,\, \Delta G_u\neq 0} \phi_s Q_s e^{-\Delta G_u} + \phi_s Q_s e^{-r(t-s)} \mathbb{1}_{\{s=G_{u-}<t,\, G_u>t\}}. \tag{6.24}$$

The first term is the accrued error due to the past halts, while the second refers to a possible present halt (it is nonzero if and only if the asset is halted at t). From (6.24), we deduce then that if ϵ_t is small, the trading strategy $\phi_t^* := \phi_{\tau_t}$ in $\tilde{Q}_t$ approximates well the minimum variance attained instead by using the strategy ϕ_t in $\tilde{P}_t$.

The stochastic error term ϵ_t does admit a more explicit point process representation; a detailed analysis is not in the scope of this paper. Here we will limit ourselves to the heuristic remarks that, everything else being equal, when the halt intensity λ is small, ϵ_t has a lesser dispersion, as it will on average consist of fewer terms. Also, when β is small, ΔG_t is on average larger, and therefore ϵ_t will be on average smaller. This last fact can be interpreted in the sense that when long halts are expected, it does not make much difference whether you hedge using P_t or Q_t: In both cases the hedge value will be dominated by the growth of the bond position in the hedging portfolio and the hedge performance essentially dictated by the relationship between the growth rate of the equity fundamental value and the risk-free rate accrual.

6.2. The Lévy price asymptotics. We conclude this section by a natural result that guarantees, in line with the intuition, that in the setup of Theorem 6.2 prices of claims written on P_t should converge to those from the benchmark Lévy model without halts S_t^0, as the halt frequency and average duration tend to zero.

Proposition 6.3. *Let f be a bounded contingent claim maturing at T. With the notation and under the assumptions of Theorem 6.2 we have the following asymptotic relations for V_0:*

(i) *If V_0^0 is the value of the claim f written on S_t^0, then*

$$\lim_{\lambda^*\to 0} V_0 = \lim_{\beta^*\to\infty} V_0 = V_0^0. \tag{6.25}$$

(ii) *Let ξ be an exponential independent time of parameter λ^*. For a stochastic process Y_t, define*

$$Y_t^\xi = \begin{cases} Y_t & \text{if } t < \xi \\ Y_\xi & \text{if } t \geq \xi. \end{cases} \tag{6.26}$$

Then

$$\lim_{\beta^* \to 0} V_0 = V_0^\xi, \tag{6.27}$$

where V_0^ξ is the discounted expectation of f taken with respect to the distribution of the terminal random variable $S_T^\xi = S_0 \exp((X^*)_T^\xi + (R^*)_T^\xi)$.

Proof. From Proposition 6.1 and using independence, we see that in the risk-neutral measure

$$\begin{aligned} \lim_{\lambda^* \to 0} \hat{P}(z,s) = \lim_{\beta^* \to \infty} \hat{P}(z,s) &= \frac{1}{\psi_X^*(z) + \psi_R^*(z) + s} \\ &= \mathcal{L}(e^{-T(\psi_X^*(z)+\psi_R^*(z))}, s) = \mathcal{L}(\mathbb{E}[e^{-iz(X_T^*+R_T^*)}], s). \end{aligned} \tag{6.28}$$

Taking the limits inside the Laplace integral by dominated convergence and inverting the transform, we see by the Lévy continuity theorem that S_T tends in distribution to S_T^0 for the given parameter asymptotics. This completes the proof of (i).

For ξ as in (ii), define the killed linear drift

$$\lambda_t^\infty = \begin{cases} t & \text{if } t < \xi, \\ \infty & \text{if } t \geq \xi, \end{cases} \tag{6.29}$$

whose De Finetti–Lévy–Kincthine exponent is $\phi_\lambda(s) = \lambda^* + s$, and consider its first exit time process

$$\lambda_t = \inf\{s > 0 |\, \lambda_s^\infty > t\} = \begin{cases} t & \text{if } t < \xi, \\ \xi & \text{if } t \geq \xi. \end{cases} \tag{6.30}$$

Evidently, $\lambda_{\lambda_t -}^\infty = \lambda_t$, so we can apply Lemma 1.11 in Bertoin [2] for a subordinator with killing directly to the process λ_t; note also that $S_t^\xi = S_{\lambda_t}$. Taking the limit in Proposition 6.1 and using independence shows that

$$\begin{aligned} \lim_{\beta^* \to 0} \hat{P}(z,s) &= \frac{1}{s} \frac{\lambda^* + s}{\psi_X^*(z) + \psi_R^*(z) + \lambda^* + s} \\ &= \mathcal{L}(\mathbb{E}[e^{-(\psi_X^*(z)+\psi_R^*(z))\lambda_T}], s) = \mathcal{L}(\mathbb{E}[e^{-iz((X^*)_T^\xi+(R^*)_T^\xi)}], s), \end{aligned} \tag{6.31}$$

and the result again follows again by interchanging integration and limit, inverting the transform, and applying the Lévy continuity theorem. ■

The first part of this proposition guarantees convergence of put prices on P_t to those of the associated unhalted model S_t^0. Convergence of call prices then also holds by call-put parity.

The second part has the interpretation that as the expected length of jumps tends to infinity, the implied asset process tends to a price distribution with only one halt that freezes the asset value at the last recorded price until maturity, and that for every possible maturity. The risk-neutral distribution of the waiting time of such an halt is that of a single Poisson event in G_t, that is, an exponential independent time of parameter λ^*.

In combination with Proposition 4.2, this result can be helpful to assess the relative impact of the halts on prices, something we will pursue in the next section.

7. Numerical experiments.

For the numerical tests of the next two subsections, we use the halts exponential risk-neutral specification of Theorem 6.2; we take for X_t the CGMY model of Carr et al. [4] with one set of calibrated parameters found therein, namely,

$$C = 6.51, \quad G = 18.75, \quad M = 32.95, \quad Y = 0.57. \tag{7.1}$$

We choose R_t as a Brownian motion with $\mu_R = 0$ and $\sigma_R = 0.2$ and fix a risk-free rate $r = 0.02$ and a spot equity price $S_0 = 100$. This parameter set provides the baseline scenario.

7.1. Dependency of prices on suspension parameters.

We begin by visualizing (4.5) to get a better idea of how the probabilities of the asset being tradable and thus option prices depend on λ, β.

In Figure 5, for a given time horizon $t = 0.5$ we plot the probabilities as a function of λ and β. As $\lambda \to 0$, this probability tends to 1, and the model converges to S_t^0. In Figure 6, instead we fixed $\lambda = 1$ and $\beta = 10$, and as time increases, the probability of t falling in a suspension decreases to its asymptotic value $\beta/(\beta + \lambda)$. This means that, everything else being equal, we expect the absolute differences of prices compared to S_t^0 to be higher for longer maturities.

In Figure 7 we represent the prices corresponding to an at-the-money call option on P_t with same maturity and parameters λ^* and β^* as in Figure 5. We can see that Figure 7 closely mirrors Figure 5. As $\lambda^* \to 0$ and $\beta^* \to \infty$, in accordance to Proposition 6.3, the prices converge to the line 11.883 given by the price in the associated Lévy model S_t^0. Also, note that this convergence is naturally increasing in both λ and β since halting the asset has the effect

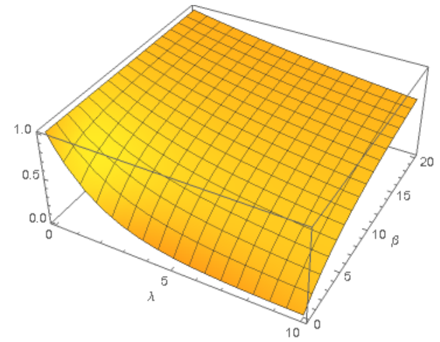

Figure 5. *Probability of S_t being tradable at time $t = 0.5$.*

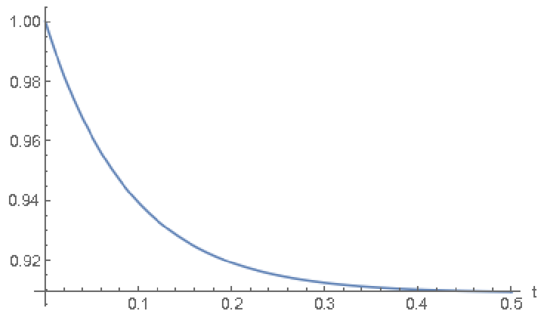

Figure 6. *Probability of S_t being tradable as a function of t. $\lambda = 1$, $\beta = 10$.*

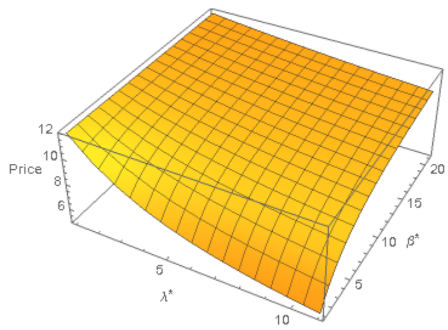

Figure 7. *ATM call option prices at* $t = 0.5$.

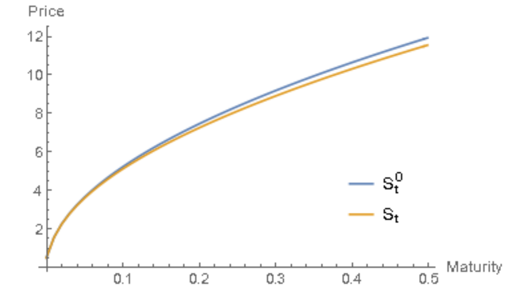

Figure 8. *Effect of halts on option prices time growth; ATM option,* $\lambda^* = 1, \beta^* = 10$.

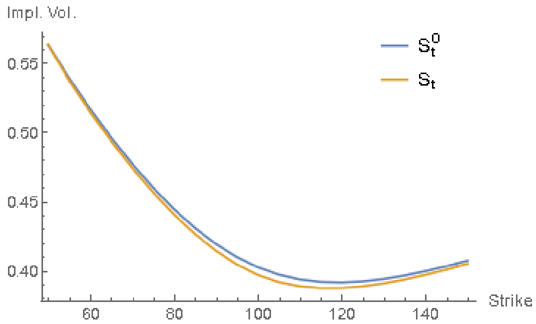

Figure 9. *Baseline,* $T = 1/12$. *Excess skew observed.*

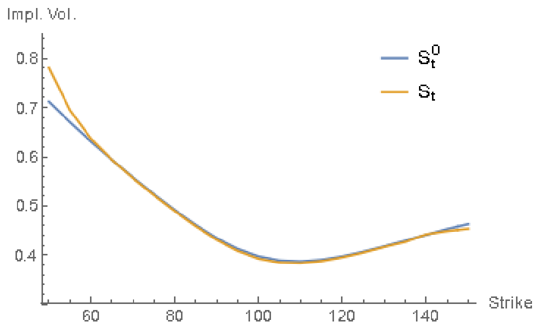

Figure 10. *Baseline,* $T = 1/24$. *At closer maturity the impact of halts is immaterial.*

of compressing the volatility and thus lowering the price. As the number of expected halts and their average duration go to zero, the variability of S_t^0 is restored and price convergence attained. In Figure 8 we represent the effect of this lowering on theta. As one would expect, also in view of Figure 6, the option prices grow slower as time to maturity increases.

7.2. Impact of suspensions on the volatility skew. In Figures 9–16 we compare some volatility skews extracted from options on S_t^0 and S_t. We want to show how acting on the halt parameters λ^*, β^*, and σ_R^*, dictating, respectively, the (risk-neutral) frequency and average duration of the halts and the variance of the price quote jump at reopening, fundamentally alters the skew structure of the benchmark model S_t^0. We initially set as baseline $\lambda^* = 2$ and $\beta^* = 50$, corresponding to a biyearly suspension frequency with an average length of five days.

Figure 9 shows the baseline scenario with monthly maturity. The excess at-the-money steepness of the halted model compared to the Lévy one can be noticed, while the two skews retain the same structure in and out of the money. As we shorten the maturity to biweekly, this difference gets lost, as can be seen in Figure 10: The likelihood of a halt $\lambda^* t$ is too small for the given parameters β^* and σ_R^* to generate any noticeable difference of the implied price distributions from those of S_t^0.

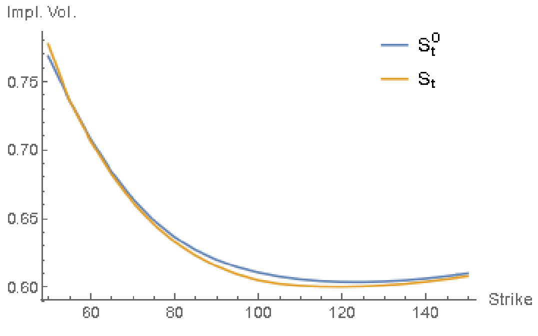

Figure 11. $\sigma_R^* = 0.5$, $T = 1/24$. *Increasing σ_R^* re-creates skew.*

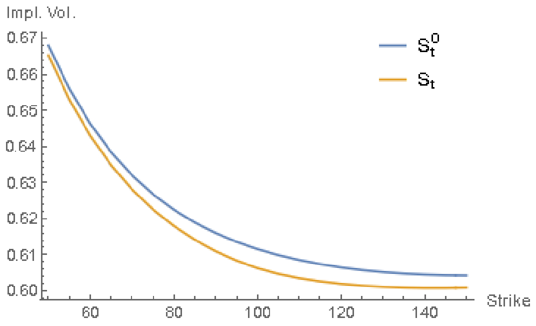

Figure 12. *Even more so at monthly level.*

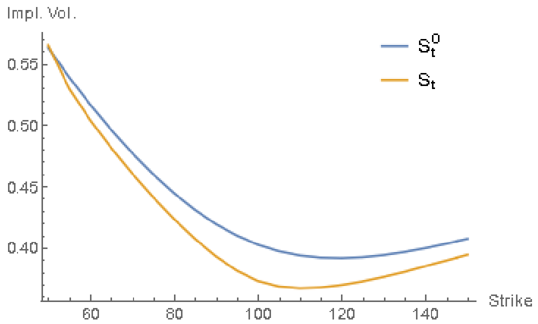

Figure 13. $\lambda^* = 12$, $T = 1/12$. *Increasing λ^* steepens and lowers the skew.*

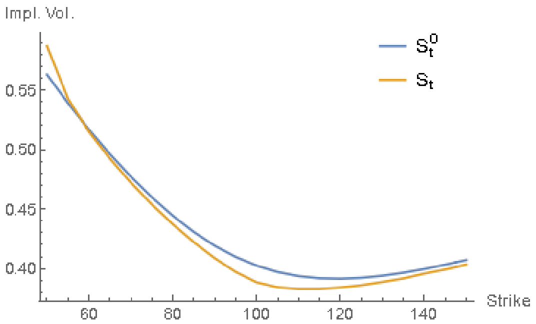

Figure 14. $\beta^* = 12$, $T = 1/12$. *Reducing β^* also decreases the level and increases convexity.*

Therefore, in the biweekly maturity case we change σ_R^* to $\sigma_R^* = 0.5$ and hold the other parameters constant. We can see in Figure 11 that the resulting increase in the variance of the reopening price shocks is enough to re-create the excess at-the-money skew already observed in Figure 9. Of course, with this modification, the one-month skew difference is exacerbated (Figure 12).

Analogously, we proceed to alter λ^* and β^*. Fixing the maturity to monthly and all the remaining parameters to the baseline case, we first change $\lambda^* = 12$ (suspensions expected with monthly frequency) and then $\beta^* = 12$ (monthly expected suspension length). The resulting Figures 13 and 14 show similar effects on the skew that the one attained in Figure 11 by changing σ_R^*. Note also that associated to this parameter change is a minimal lowering of the level of the skew, consistent with the discussed effect that a decrease in β^* and an increase in λ^* determine a global reduction of the option prices.

Finally, we find the effect on the skew for λ^* and β^* to be persistent in time. In Figures 15 and 16 the same situation of Figures 13 and 14 is reproduced, but this time with maturity six months. The halted model skew decay is evidently slower than that of S_t^0. This effect is in line with the real market volatility skew shapes. Of course, the lowering of the

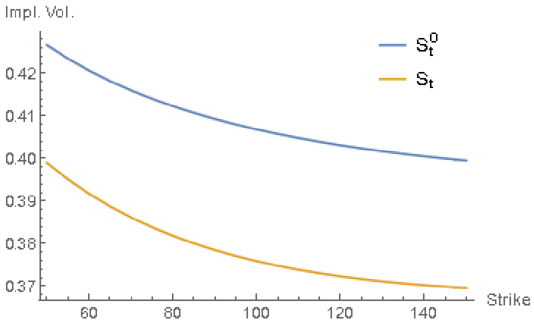

Figure 15. $\lambda^* = 12$, $T = 0.5$. *Skew increase and level lowering still visible at longer maturity.*

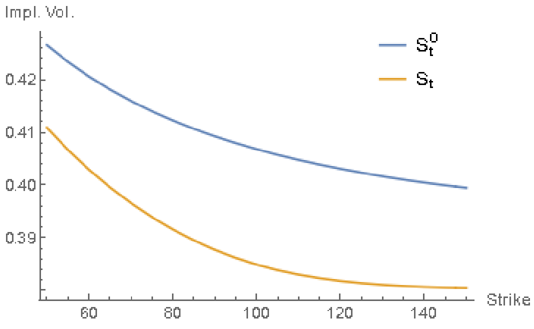

Figure 16. $\beta^* = 12$, $T = 0.5$. *Effect even more pronounced for* β^*.

implied volatilities in these examples is even stronger, again following the pattern of Figure 8.

In conclusion, we can see that the risk-neutral suspension parameters λ^*, β^*, and σ_R^* act as further steepening the surface. Like the jump parameters, these are able to create implicit distribution kurtosis and skewness by a combination of price movement delays and reopening jumps. However, unlike the skewness generated by jumps, which dissipates quickly with maturity because the jump asymmetries "even out" after temporal aggregation, halting generates genuine skew persistence since long-run returns processes driven by inverse-subordinated Lévy processes do not generally possess normality features, as explained, e.g., in Meerschaert and Scheffler [20] and references therein.

7.3. Calibration test. To trace evidence of market halts in traded option prices, we have calibrated our model to the NYSE-traded call options on State Street Corporation, as observed on November 27, 2019. Trading on the State Street stock was halted less than one month before, on November 6, for a full trading day, following a large company buyback of redeemable shares. The halt was operated by the exchange and assigned the news-related halt code T3.[6] We use as an input the quoted option prices instead of the implied volatilities, since the latter are typically pre-processed and smoothed by market-makers according to standard models, and an implied trading halt premium might be removed after this process.

The calibration consists of a comparison between a halted model and its pure Lévy benchmark (3.10). We chose a variance gamma [19] specification for the trade value process X_t and a driftless Brownian motion with volatility σ_R for the rumor process R_t. The variance gamma process is characterized in its time-changed representations by an absolute volatility σ, a kurtosis parameter κ, and a skew parameter ρ. The error function chosen is the mean absolute percentage error between market and model prices, i.e.,

$$\text{ERROR} = \frac{1}{n \times m} \sum_{i,j=1}^{n,m} \frac{|C^{Market}(K_i, T_j) - C^{Model}(K_i, T_j)|}{C^{Market}(K_i, T_j)}. \tag{7.2}$$

[6] https://www.nasdaqtrader.com/Trader.aspx?id=TradeHaltCodes.

Table 1

Calibration to STT call option prices as of November 27, 2019, of the pure Lévy model S_t^0 in (3.10). The process X_t is variance gamma (κ, σ, ρ), and R_t is a Brownian motion of diffusion σ_R.

Time to maturity	23 days	86 days	170 days	415 days
κ	7.9622	47.8487	48.1460	0.7521
σ	0.1973	0.2606	0.1969	0.5654
ρ	−0.0375	−0.0131	0.0014	1.8959
σ_R	0.0297	0.0341	0.0275	0.0001
Error	5.17%	8.87%	11.63%	9.39%

Table 2

Halted model calibration based on the same data and with same specifications for X_t and R_t as in Table 1. We used the historical halt parameters $\lambda = 1$ and $\beta = 250$.

Time to maturity	23 days	86 days	170 days	415 days
κ	0.0718	0.0077	1.0857	0.0103
σ	0.2335	0.1484	1.4538	0.2478
ρ	−0.3580	−1.7095	3.9839	−0.6679
σ_R	0.0410	0.1049	0.0001	0.1130
Error	3.52%	7.03%	8.80%	10.94%
Improvement	31.90%	20.73%	24.31%	−16.42%

The calibration has been performed using a differential evolution algorithm. Rather than proceeding with an unconstrained minimization on all parameters, we set λ and β as the market-observed values and then calibrate on the remaining ones and compare to the Lévy model calibration. This constrained approach allows us to compare solutions to two minimization problems with same dimensionality (four) so that a better fit of one model over the other cannot be merely attributed to the difference in the number of the target parameters. In other words, fixing the halt parameters ensures that the halted model does not adapt better to the market-implied distributions simply because it has more degrees of freedom. Obviously, the historical estimates of the halt parameter need not to coincide with the risk-neutral ones; however, for the purpose of showing an impact of halts on market prices, it is not necessary to find the real risk-neutral parameters, and it suffices to show that already for a suboptimal specific set of λ, β, the halted model improves on the pure Lévy one. Therefore, we use the historically observed daily halt duration $\beta = 250$ and use $\lambda = 1$ according to our availability of only a one-year history of NYSE market halts, during which trading on the State Street stock was halted only in the mentioned instance.

We ran four calibrations of both models, each to a set of liquid options from a single maturity cross section. The results are summarized in Tables 1 and 2. We report the pricing errors with the market prices for the Lévy and the halted model calibrations and in Table 2 also the improvement on the Lévy calibration, expressed as the percentage error decrease. We notice that the halted model sensibly improves on the Lévy one for the approximate one-, three-, and 5-month maturities, whereas its calibration quality for the 14-month expiry is inferior. This leads us to conclude that for options with short time to maturity, there is good evidence of a presence of a market price of suspension risk in the option premia. At longer expiries, either of the following scenarios are possible: The halt parameter assumption we operated is very far from the actual market-implied ones, or a market price of suspension risk is absent altogether.

8. Conclusions.

In this paper we presented a martingale derivative valuation framework for stocks with suspensions exogenous to the trading activity. We did so by observing that the natural underlying of a derivative on a suspendable asset is neither the asset itself nor its last market quote price but rather a contract of cash delivery of the last stock quote plus interest, which can always be traded and can be made into an asset earning the risk-free rate after an equivalent measure change.

In order to mathematically formulate one such a framework, we resorted to a Lévy setup comprised of two independent price factors, one modeling the trading and the other the news effects on price, together with a finite activity subordinator whose jumps generate the market halts. The economic value of the asset is then recovered by halting the component with the time-change obtained by the first exit time of the halts generator. The last available market quote is then attained by further time-changing the asset value to the last passage process τ_t of G_t.

Martingale relations pose no difficulties, and a class of equivalent martingale measures has been identified. In this context, the concept of market price of suspension risk emerges as the fraction of the option risk premium borne by the risk-neutral parameters of halt intensity and duration.

Furthermore, we have been able to produce an option pricing formula through the popular technique of Fourier integral pricing by deriving the joint Laplace–Laplace transform of the time changes and then inverting it in time to obtain the characteristic function of the log value of P_t. Finally, mean-variance hedging has been discussed.

Analyses of the volatility surfaces show that the short time skew of a model with suspension is much steeper than that of the corresponding Lévy model without halts. In addition, the smile decays slowly over time, a pattern consistent with real market volatility term structures which is not normally captured by Lévy models.

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