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Review

The Decreasing Hazard Rate Phenomenon: A Review of Different Models, with a Discussion of the Rationale behind Their Choice

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Abstract: It is well known that, especially in the field of electronic components reliability studies and applications, the Exponential reliability model is by far the most adopted, although the data fostering it are few. This appears to be due partly to its simplicity (also in view of estimation, since it is characterized by a unique parameter), and partly because most components seem to be well represented, at least in their "useful life" time interval, by the Exponential model. This adoption is basically due to its peculiar "memory-less" property, i.e., the fact that such model possesses a constant hazard rate function, meaning that stochastic "accidents" cause the failure of the component, independently of its service time. This theoretical reason behind the choice of the Exponential model is largely prevailing over the classical statistical "goodness of fit" tests, since the high-reliability values attained by such devices does not allow the availability of an adequate number of lifetime values to be observed and analyzed in a statistical data analysis procedure. A second model also widely adopted is the Weibull model, especially if characterized by a shape parameter greater than unity, so implying an increasing hazard rate function. However, there are many cases—which can be also justified on a theoretical basis, as reviewed in this paper—in which a decreasing hazard rate function (at least for relatively large mission times) may be the best suited to describe the true model behind a given failure mechanism. The afore-mentioned theoretical basis of these apparently peculiar models is the main core of the present review article, whose aim also includes the illustration of the basic features of the main reliability models featuring an hazard rate function diminishing with time. The paper also discusses, resorting to graphical and numerical case-studies relevant to both field and simulated data, the consequences of mistaken model identification in terms of the hazard rate function behaviour, which may imply wrong maintenance actions.

Keywords: electronics; hazard rate function; probability distributions; random variables; reliability; stress-strength models; wear processes

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1. Introduction

Reliability is more and more a key problem in engineering, especially in the field of electronics, as witnessed by a recent special issue on this topic, focused on Power electronic devices reliability [1–16]. Nowadays, indeed, these are becoming more and more critical devices, since—as highlighted in [16]—ower electronics plays a crucial role in renewable energy systems, lighting, electric mobility, and other systems that enable sustainable development.

It is well known that modern reliability theory, as witnessed by a growing series of authoritative books on the topic [17–44], is based upon consolidated theoretical probabilistic models [45] which find their justification in ad hoc statistical methods: these are however becoming difficult to apply, due to typical scarcity of failure data availability, as a consequence of the high and ever increasing reliability degree attained by modern components. Another aspect which undermines classical or traditional statistical methods is the randomness or unpredictability of environmental conditions in which the devices will operate, often very different from the laboratory test conditions. For instance, as discussed in [1] regarding again power electronics devices, these are expected to operate under challenging environmental conditions (e.g., high altitudes in more electric aircrafts or high temperatures on photovoltaic installations), compromising the effectiveness of traditional approaches, typically based on historical failure data and past observed scenarios. In fact, the rapid evolution of power electronic technologies and the ever more challenging operating frameworks pose severe limitations on the trustworthiness of available reliability data, as they are typically related to incoherent operating conditions [1–4].

Thus, new methods have been developed such that the reliability model identification and estimation of most of the modern components, as those above exemplified, may be better achieved, instead that using limited lifetime data, by the knowledge of the degradation mechanisms, and the consequent assessment of the wear process affecting the device's lifetime [46–73]. Such methods have developed the results initiated in 1973 by the seminal paper by Esary and Marshall [46] on shock models and wear processes, and successive basic papers as [47–53], and ultimately have taken advantage of a large series of theoretical results on the relationship between the theory of stochastic processes (or random functions of time) and reliability [74–84].

On the other hand, as discussed in [1], power electronic devices reliability is affected by thermal, electrical and mechanical stresses, and in the presence of such stresses it can be assessed through consolidated, traditional methods and techniques. Such methods, which often allow an efficient estimation of the state of the health of the device, can constitute a powerful basis for building the so-called "indirect" or "physical" reliability models assessment [72–76], even in the absence of many data. This is the basic purpose of the present review, aiming in particular at discussing the foundations of a particular sub-class of the Reliability models (RM), i.e., those possessing a decreasing *hazard rate function* (HRF)—at least for relatively large mission times. The definition of decreasing HRF is recalled afterwards, and it will be shown that the discussion of decreasing HRF may constitute a great help into a deeper understanding of the rationale behind choice of a given RM.

Very often, indeed, the largely most widespread RM, i.e., the Exponential model (the unique model possessing a constant HRF) and the Weibull model (especially if characterized by a shape parameter greater than unity, so implying an increasing HRF), are adopted without any reason apart from—in practice—their simplicity, also in view of consolidated estimation procedure [33].

The Exponential model in particular is sometimes justified on a statistical basis, i.e., by conducting adequate statistical tests on the available failure data, but it is mostly motivated relying on the theoretical basis that such model is the only one characterized by a constant hazard rate function, meaning that stochastic "accidents" cause the failure of the component, independently of its service time. This second, theoretical reason, behind the choice of the Exponential model is largely the prevailing one nowadays, since the high-reliability values attained by such devices does not allow the availability of an adequate number of lifetime values to be observed and analyzed in a statistical data analysis procedure.

As above hinted at, this is the fundamental reason behind the development of "indirect" or "physical" reliability models assessment, based on the study of wear or degradation processes acting on the device. A peculiar consequence of such kind of analysis, surprisingly rarely emphasized in literature, is that such models imply a fact that is maybe contrary to intuition, namely a hazard rate function which is decreasing over time—at least for relatively large mission times. More precisely, most of such theoretical models imply an HRF which is, alternatively:

- always decreasing in time (the class of such models is denoted as "*decreasing hazard rate*", DHR);

- first increasing, then decreasing in time (the class of such models is denoted as "*first increasing, then decreasing hazard rate*", IDHR).

Sometimes, the IDHR class is also denoted as "unimodal HRF", since in this class the HRF possesses a unique maximum at a certain time instant t*, while it is increasing from t = 0 to t*, then decreasing after t*. Such instant t* is also denoted, for obvious reasons, a "change point" of the HRF.

The previously mentioned characteristics may be connected to the features of wear or degradation processes. Moreover, new insights have been gained on the theory from extensive studies on maintenance strategies [85–93], since they are in some way related to the HRF behavior, as will be clear when we introduce the meaning of the HRF later.

Indeed, in view of the purpose of the present paper, in the following a brief review is presented for illustrating the basic relations existing among the fundamental Reliability measures, the most important of which are:

- the Reliability function (RF);
- the above introduced hazard rate function (HRF);
- the Mean Time to Failure (MTTF);
- the "residual reliability function" (RRF)

More refined details on such relations can be found in the above cited references [17–45].

Indicating as T the non-negative RV "time to failure", or "lifetime" (LT), of the device and as $F(t)$ its cumulative distribution function (CDF), the RF can be stated as here below:

$$R(t) = P(T > t) = 1 - F(t) \tag{1}$$

being $P(A)$ the probability of the generic random event A. The above RF is sometimes denoted also as "*survival function*" in literature. The RV T is typically assumed, of course being a time variable, as a continuous RV, distributed according to a probability density function (PDF) $f(t)$ such that:

$$f(t) = dF/dt = -dR/dt \tag{2}$$

and

$$R(t) = \int_t^{\infty} f(u)du \tag{3}$$

It is remarked that the RF $R(t)$ represents the probability that the component works satisfactorily over all the period (0,t). MTTF indicates the expected LT and can be obtained, provided that the integral exists, by:

$$E[T] = \int_0^{\infty} R(t)dt \tag{4}$$

As pointed out in the above cited references [17–45], the Reliability theory can be viewed mathematically as an applied probability theory focusing on positive RVs. Nevertheless, it features parameters and functions established precisely to illustrate time-related RVs (especially–but non exclusively–Lifetimes). In practice these parameters and functions miss counterparts in different fields of probability theory.

Among them, the HRF is the most widely employed to illustrate component aging. Contrary to the CDF and the RF—both related to a time period, see above—the HRF $h(s)$ is related to the instant $s \geq 0$, to be meant as the component "age". Indeed, it is also known as the "*instantaneous failure rate*" at a certain time instant. From a formal viewpoint, if $f(t)$ is the PDF of LT, T, the HRF is defined as:

$$h(t) = \frac{f(t)}{R(t)} = -\frac{d}{dt}[\log(R(t))] \tag{5}$$

at each time $t \geq 0$ for which $R(t) \neq 0$.

An equivalent definition, which also explains the origin of its name, as well as its meaning, is the following:

$$h(x) = \lim_{\Delta x \to 0^+} \frac{P\{(x < T \leq x \leq x + \Delta x)(T > x)\}}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{F(x + \Delta x) - F(x)}{\Delta x \, R(x)} \quad (6)$$

Therefore, for $\Delta x \to 0^+$, the quantity $h(x)\Delta x$ is equal to the *conditional* probability that the failure takes place within the interval (x, $x + \Delta x$), provided that the component has reached an age x (i.e., it did not fail—or survived—until x); namely $h(x)Dx$ can be viewed as the *instantaneous* failure (conditional) probability of a component having age x.

Once imposed the trivial condition $R(0^+) = 1$, an integral relationship can be derived that lets the RF, $R(t)$, be explained as a function of the instantaneous HRF:

$$R(t) = exp\left(-\int_0^t h(\xi)d\xi\right), \; t > 0 \quad (7)$$

As a consequence, by fixing either the PDF, or the RF or the HRF of a reliability model, the model itself is completely defined, as every one of such three functions can provide straightforwardly the other two: for instance, from (7) $h(t)$—whose functional expression can be deduced in some cases relying on its physical grounds—provides directly the RF. Thereafter, the PDF can be obtained via the well-known equation:

$$f(t) = h(t)\, R(t) \quad (8)$$

In a similar way, the MTTF, CDF, etc. can be simply attained, too.

As a further illustration, in such a way the 2-parameter Weibull model (the most popular one when investigating reliability in engineering applications) can be easily derived, too, which features the following expressions for the dependency of RF, HRF, PDF vs. time $t > 0$:

$$\mathrm{R(t)} = \exp(-\alpha t^{\beta}) \quad (9)$$

$$\mathrm{h(t)} = \alpha\beta t^{\beta-1} \quad (10)$$

$$\mathrm{f(t)} = \mathrm{h(t)\,R(t)} = \alpha\beta t^{\beta-1}\exp(-\alpha t^{\beta}) \quad (11)$$

where positive-defined quantities α and β are the scale and shape parameters respectively.

In the sequel, the Weibull model is indicated as $W(\alpha,\beta)$ (an alternative form is introduced in Section 6 (Equation (54)). When β = 1, the Weibull model reduces to the above-mentioned Exponential reliability model. It is an increasing hazard rate (IHR) model when β = 1, and a decreasing hazard rate model when $\beta < 1$. As illustrated in many textbooks [17–45], the time trend of the HRF may suggest the reasons behind a failure: a decreasing HRF indicates "infant mortality" or "wear-in": namely, poorly-manufactured devices broke soon due to brittleness, flaws, etc.; on the whole the HRF drops over time with the removal of failed devices from the population. A steady HRF (Exponential reliability model only, see above) indicates that the component breaks, independently of its service time, due to stochastic "accidents". A rising HRF indicates that the component undergoes "wear-out", thus its failure probability increases with time.

The possible behaviors of the HRF are perhaps better understood, following [73], if the "*residual reliability function*" (RRF) is introduced, which is a function of two time variables (s and t in the sequel) defined as follows:

$$R(t\,|\,s) = P\{T > s + t\,|\,T > s\} = R(t + s)/R(s) \quad (t, s \geq 0) \quad (12)$$

The above RRF, denoted as the *residual reliability function* for a mission time t of a component of age s, is equal to the conditional probability that, after the component has survived until age s, it will survive at least until time ($s + t$), i.e., its age will increase of at least t time units in addition to s. Of course the RRF shall behave as a RF vs. time t, for example it shall satisfy: $R(0\,|\,s) = 1$, $R(\infty\,|\,s) = 0$, and be decreasing with t. Its behavior with age s might appear less obvious, as discussed in [73]. E.g., in principle one might guess that, as aging is expected to make all components weaker, $R(t\,|\,s)$ drops with s, but this does not necessarily hold, as this guess neglects that the RRF is a *conditional* probability, and conditioning might remarkably modify our information. For instance, in some cases, the information that a component's age has reached a value s might make engineers more confident in its "future" survival than in the case that such information is missing; therefore, the component *seems* to "get stronger" with age, and the RRF may rise with s. In any case, it should be clear that (*un- conditional*) RF must always decrease with time: this is a basic property originating from the very definition of a RF, and has no connection with the time variation of the HRF and/or the RRF.

Instead, there is a close relationship between the HRF and the RRF behavior [17,73]: i.e., the HRF $h(s)$ increases (decreases) with age if and only if the RRF $R(t\,|\,s)$ decreases (increases) with age s; the HRF $h(s)$ is constant with age s if and only if the RRF $R(t\,|\,s)$ constant with s, what happens only—as already pointed out many times above—with the Exponential model, by far the most adopted in the field of electronics, thereby proving that such model is "memoryless".

However, as witnessed by all well-known standards on reliability [94,95], the methodology of the Exponential reliability model has been indeed criticized in the last years [96,97]. Also to the popular "*bathtub curve*" of the HRF [98] has been object of very recent criticisms in an authoritative paper by Pecht in 2021 [99]. Indeed, according such model—which is based upon the above interpretation of the RRF, but also relies upon experimental evidence originated by studies on human lifetimes-implies a HRF which first decreases, then becomes approximately constant with age (the "useful life" period, characterized only by accidental failures), finally increases with time due to wear out.

Nonetheless, as can be derived from all the models which will be analyzed in the following (and which are of course not exhaustive of the class of the RM), often the HRF behaviour in time is quite the opposite of the above model. It is also noticeable that most, if not all, of such DHR or IDHR models can be deduced, as it will be shown, in the framework of adequate so-called "stress-strength" (SS) models [100–105], to be recalled in the next section.

By discussing such aging models, this review is also aimed at highlighting the rationale behind a proper and accurate selection of a reliability model for the above devices, which is a very useful approach in view of the above discussed scarce availability of lifetime values. It also discusses the consequences of mistaken model identification in terms of the HRF behaviour. Indeed, an insight into the models shows that, although some of them appear to possess similar shape of their PDF or RF, and possess very close values of the mean and the SD, or of their "central" quantiles (e.g., the median), as clearly shown in the application of the final section—their HRF may be rather (or very) different, and the same holds for their "extreme" quantiles. This is apparent when examining various couples of RM, e.g., the Weibull and the Lognormal models (as also illustrated in the numerical application), as well the Weibull and the Log-Logistic models, or the Birnbaum-Saunders and Inverse Gaussian model, and more). This theoretical aspect may have serious consequences, e.g., in terms of the maintenance actions to be taken on such devices [85].

The rest of this paper is organized as follows. Section 2 presents a brief account of the SS model and then, a list of the basic DHR and IDHR models deducible from the above SS model, to be discussed in the successive sections. Such DHR and IDHR models (*Birnbaum-Saunders, Inverse Gaussian, Inverse Weibull, Log-Logistic, Lognormal model)* are indeed discussed in Sections 3–8. Also the *Exponential* model is discussed (in Section 4) for its importance, for its being at same time a DHR and IHR model, and also for illustrating a peculiar method for its deduction from an SS model.

Also, in Section 9, some DHR or IDHR models generated by mixtures of Exponential or Weibull RV are presented, in order to illustrate another completely different mechanism involved in the same above-discussed family of models. This kind of mechanism appears to be a fundamental one for electronics devices, because of the aforementioned randomness of environmental conditions in which these devices typically operate [1].

Finally, in Section 10, the aforementioned application related to real electronics lifetime data is illustrated, in order to show the possible consequences of model mis-specifications which may become very likely in the presence of a limited amount of data.

2. A Premise to the Review of the Basic DHR or IDHR Lifetime Models on the Basis of Stress-Strength Models

In the following Sections 3–9, a review of the basic DHR or IDHR lifetime models is discussed; the models are theoretically deduced by generative mechanisms originated by wear or stress processes, as explained below. For this purpose, a brief reminder of "stress-strength" (SS) models [100–105], arising from shock or wear processes is presented here, while a list of the above models, which will be illustrated in the following sections, is presented at the end of this section.

The first part of this section presents a brief account of the stress-strength model, which is probably the most straightforward method among the aforementioned "indirect reliability assessment" methods. In its simplest form, such model is introduced as a method for evaluating the reliability a device (or a system) that is subjected to an external stress, modeled by a RV *X*, against which the unit sets its own strength, modeled by another RV Y. So, the device survives if and only if its strength Y is greater than its stress *X*.

This allows to evaluate the reliability R of the unit as the probability that the unit withstands the stress:

$$R = P(X < Y) \tag{13}$$

Once the PDF and/or the CDF of the RV (X,Y) are known, the evaluation of the reliability R can be computed as follows

Let f(y) and F(y) be respectively the PDF and the CDF of the RV Y;

Let g(x) and G(x) be respectively the PDF the PDF and the CDF of the RV X.

Then under the hypothesis that the RV X and Y are statistically independent—the reliability R of the device is given by:

$$R = \int_0^\infty g(x)P(X < YX = x)dx = \int_0^\infty g(x)(1 - F(x))dx \tag{14}$$

In the above equations, time doesn't appears (at least in explicit form), i.e., the PDF of *X* and *Y* are assumed as time-independent, or the mission time pre-determined, as in the particular case of the so-called "one-shot devices" (such as automobile air-bags, munitions, rockets), or electrical generation systems which, in a given mission time, must supply a constant load, and so on. Instead, in its most general form, accounting for time variation of the basic RV, the SS model regards *X* and/or *Y* as stochastic processes in time. For instance, let the stress be a wear (or degradation) process, here denoted by $X(t)$, i.e., a stress which varies with time accumulating progressively on the device until the "threshold" y is exceeded (i.e., y is the device's strength, which is here considered as a fixed constant for sake of illustration; the hypothesis will be relaxed afterwards). Then, the reliability of the device is expressed as a time function, i.e., as the RF $R(t)$, as follows:

$$R(t) = P\{X(s) < y, \forall s \in (0,t)\} \tag{15}$$

The failure time can be defined in such case as the first time instant in which the stress process $X(t)$ exceeds the failure threshold y:

$$T = \inf\{s: s > 0, X(s) > y\} \tag{16}$$

The above equations may be easily extended to the case in which also the strength *Y* is a stochastic process, *Y(t).*

In some cases, the analytical expression of stress *X(t)* and/or the strength *Y(t)* is known or may be expressed in terms of RV with known distributions, so—once the parameters of such RV are estimated—the distribution of lifetime T may be deduced by the stress and strength distributions.

For instance, one of the most adopted analytical expressions for stress X(t) is the one of a linear process:

$$X(t) = \beta t \tag{17}$$

in which β is a (positive) RV. This model assumes that the stress at time zero is zero: X(0) = 0, but can be easily extended to a more general linear model (X(t) = α + βt). In [21] these models are derived relying on physical grounds, providing exemplifications in addition to the ones given here. In many cases, the model in [17] may be a reasonable approximation (e.g., as obtained by the 1st term of a Taylor expansion) over some time interval, of a more complex random function. Returning to eqn. (17), in case the device's strength, is a constant y, it is easy to deduce that the failure time T is expressed by the RV A and the threshold y as:

$$T = y/\beta \tag{18}$$

In case also the device's strength is a RV, Y, then the failure time T is expressed by the ratio of the two RV Y and β as:

$$T = Y/\beta \tag{19}$$

Thus, as above hinted at in the general case, the distribution of lifetime T may be deduced once the stress and strength distributions are known, i.e., the distributions of the RV β and Y are known.

Another model, used for instance for lumen lifetimes [70], is the log-linear one for the stress process X(t):

$$X(t) = \alpha \exp(\beta t) \tag{20}$$

In which (α,β) are positive RV. Of course, a model such as (20) can be immediately transformed into a linear one by a logarithmic transformation (and defining of course a new threshold, Y′, as Y′ = log(Y)). This example shows that a linear process is more general than it could appear at a first sight.

Of course, there much more complex classes of wear processes, such as the "cumulative wear process", the "Wiener process", and more, which for brevity will be introduced when needed (in particular, the first class in Section 3, the second in Section 5).

Then, a list of the basic models deducible from an adequate SS model, that will be presented in alphabetical order in the following sections, is here presented:

1. *Birnbaum-Saunders model (Section 3)*
2. *Exponential model (Section 4)*
3. *Inverse Gaussian model (Section 5)*
4. *Inverse Weibull model (Section 6)*
5. *Log-Logistic model (Section 7)*
6. *Lognormal model (Section 8)*
7. *"Mixture" DHR or IDHR models (Section 9)*

As to the last point, regarding the so-called "Mixture" DHR or IDHR models, it is the last class of models presented: in Section 9, some DHR or IDHR models generated by mixtures of Exponential or Weibull RV are briefly illustrated. These mixtures represent (as above hinted at) another possible generative mechanism of such models which can assumes a great importance in the applications here considered, since the above discussed uncertainty or randomness of the environment in which the device operates can induce a certain degree oh "heterogeneity" which often produces DHR or IDHR models.

It is remarked that also the Weibull model W(α,β) of Equations (9)–(11) can be inserted among the DHR models, when the shape parameter b is less than 1; such case, however, although not to be forgotten, appears rarely (if ever) justified by an SS generative mechanism. Moreover, in most applications (except perhaps when fitting life data derived from infant mortality) as already stated, the parameter b is greater than 1 (for instance, in Section 10, an application related to lumen life data is presented, in which the above parameter is estimated as β = 7.890). It is also again remarked that the Weibull model covers the Exponential reliability model when β = 1.

Moreover, if not for its HRF behaviour (which is approximately linearly increasing), a brief mention of the Normal (or Gaussian) distribution has to be reminded. Indeed, such model is essential in statistical investigations since the distributions of various basic statistical estimators (such as the "Sample mean") are well approximated, in view of the "Central Limit Theorem" [45], by the Gaussian probability distribution. Here, the Normal distribution will be mainly introduced, since from Section 3, in the framework of stress or wear processes. The PDF of a Normal RV with mean μ (a real number) and SD σ (a positive real number)—whose symbol is N(μ,σ), has the following expression:

$$f(t;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} exp\left[-\frac{1}{2\sigma^2}(t-\mu)^2\right] \tag{21}$$

where $-\infty < t < +\infty$, $-\infty < \mu < +\infty$, σ > 0. The popular key features of the Normal PDF are that it is bell-shaped and symmetrical around the mean μ. The standard Normal PDF, φ(z), and CDF, Φ(z), are often used to recast the Normal PDF and CDF in terms of a Normal RV z with zero mean and unit variance. Φ(z) and φ(z) can be written, respectively as reported here below:

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} exp\left(-\frac{u^2}{2}\right)du; \qquad \varphi(z) = \frac{d\Phi(z)}{dz} \tag{22}$$

It is observed that, if Z is a N(μ,σ), then Z = (X − μ)/σ is a N(0,1) RV, so that: T = μ + σZ. Thus, the PDF and CDF of the N(μ,σ) Gaussian model of a RV T are expressible respectively in terms of φ(z) and Φ(z), as follows:

$$f(t;\mu,\sigma) = \frac{\Phi((t-\mu)/\sigma)}{\sigma};\ F(t;\mu,\sigma) = \Phi((t-\mu)/\sigma) \tag{23}$$

Obviously, the Gaussian distribution is not appropriate for the RV lifetime, as it encompasses the interval t < 0, i.e., negative times. However, it is sometimes adopted provided that μ > 3σ, but its adoption is also strongly limited by its lack of flexibility (only symmetrical, bell shaped PDF can be of course represented by this model; and also in such case a valid alterative, which does not possesses the evident drawback of a Normal model, is a Weibull model W(α,β) with β ≈ 3.6). However, it will be illustrated here not as a lifetime model, but: in the definition of a random succession of stress values leading to a Birnbaum-Saunders model (Section 3); in the framework of a Wiener process leading to an Inverse Gaussian model (Section 5); in the definition of the Lognormal model (Section 8).

3. Birnbaum-Saunders (BS) Model

As hinted at in previous section, in this and in the following Sections 3–9, Section 3 the basic DHR or DHR or IDHR lifetime models are reviewed, as deduced on the basis of stress-strength models. In all of the following expression, mission time t (the argument of the CDF PDF and HRF) is always non-negative.

The BS model [45,105–107] has the expressions of CDF and PDF reported, respectively, here below (with positive parameters α, β):

$$F(t;\alpha,\beta) = \Phi\left\{\frac{1}{\alpha}\left[\left(\frac{t}{\beta}\right)^{\frac{1}{2}} - \left(\frac{\beta}{t}\right)^{\frac{1}{2}}\right]\right\} \tag{24}$$

$$f(t;\alpha,\beta) = \frac{1}{2\sqrt{2\pi}\alpha\beta}\left[\left(\frac{\beta}{t}\right)^{\frac{1}{2}} + \left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right] exp\left[-\frac{1}{2\alpha^2}\left(\frac{t}{\beta} + \frac{\beta}{t} - 2\right)\right] \tag{25}$$

where *Φ(z)* is the standard Normal CDF, and the mean μ and variance σ^2 are equal, respectively, to:

$$\mu = \beta(1 + \alpha^2/2) \tag{26}$$

$$\sigma^2 = \beta^2\ \alpha^2\ [1 + (5/4)\alpha^2] \tag{27}$$

The simplest expression of the HRF is:

$$\mathrm{h(t) = f(t)/(1 - F(t))} \tag{28}$$

In [107] it is shown analytically that the HRF of the BS distribution is always an IDHR model–or an "unimodal HRF" model—whatever the value of the shape parameters, and the "change point" (defined in Section 1, and after which the HRF approaches a positive limit as time diverges) of the HRF can be determined as a solution of a non-linear equation. The authors of [107] have given different numerical tools for the evaluation of the change point.

This HRF behaviour is very similar to the one of the Inverse Gaussian model. The two models are indeed rather similar and often undistinguishable in case only a few data are available, as will be recalled also in Section 5, devoted to the derivation of the Inverse Gaussian model.

The derivation of the BS from a SS model is straightforward, since it was introduced on purpose as the result of some of such models, in the framework of the so-called "fatigue-affected lifetimes" analyses. The BS model is indeed sometimes also referred to as "fatigue life" model. The model was obtained from a "discrete" stress process, which takes the accumulation of cracks on a material into account; failure is reached as a certain fixed "critical dimension", y, is exceeded. If one assumes that the component undergoes a so-called "*cumulative wear process*"–i.e., a sequence $\{Z_k;\ k = 1,2, \ldots\ \}$of stresses being RV from a common distribution– and denotes as X_n the total stress after n cycles, then it holds:

$$X_n = \sum\nolimits_{k=0}^{n} Z_k,\ \ n = 1, 2, \ldots, \infty \tag{29}$$

Thus, in the presence of a fixed strength y, the BS model is derived when noticing that the "discrete" failure time N (i.e., the number of cycles after which breakdown takes place) possesses the CDF reported here below—holding for discrete values of $n \in \mathbf{N}$ (= set of natural numbers):

$$\mathrm{P(N \le n) = P(X_n > y)} \tag{30}$$

Indeed the random event ($N > n$) (i.e., "*the device is still working after n cycles*") occur if and only if ($X_n < y$) (i.e., "*the total stress after n cycles is smaller than y*"); then, considering the complementary events, the above relation is obtained.

Let us hypothezise that the stress amplitudes, namely the RV Z_K, are independent Gaussian RV, with $N(\theta,\sigma)$ distribution (of course $\theta > 0$), or that the stress number is so high that the central limit theorem [45] is valid. As a consequence, W_n is (or converges towards, as n diverges) a $N(\theta n,\ \sigma\ n)$ RV, so that:

$$P(N \le n) = 1 - P(W_n \le y) = 1 - \Phi\ [(y - \theta n)/\sigma\ \sqrt{n}] = \Phi\ [(\Phi\ \theta n - y)/\sigma\ \sqrt{n}] \tag{31}$$

(in Equation (31) the symmetry of the standard Normal CDF has been exploited, namely $\Phi\ (-x) = 1 - \Phi\ (x)$) Then, if discrete time n is "transformed" into continuous time t (which can be motivated assuming that the cracks rate ρ is constant in time, so that $n = \rho t$), thereafter the CDF reported here below can be easily derived—where the meaning of the positive constants (α,β) is readily understood:

$$F_T(t;\alpha,\beta) = \Phi\left\{\frac{1}{\alpha}\left[\left(\frac{t}{\beta}\right)^{\frac{1}{2}} - \left(\frac{\beta}{t}\right)^{\frac{1}{2}}\right]\right\} \tag{32}$$

which is indeed a BS CDF.

4. Exponential Model

The widely popular Exponential model is here reported only for the aforementioned reasons, not being a DHR or IDHR model, but the unique possessing a constant HRF, thus being at same time DHR and IHR. Such model, with positive scale parameter λ, possesses the following RF, PDF, HRF:

$$R(t) = \exp[-(\lambda\, t)]; f(t) = \lambda \exp[-(\lambda\, t)]; f(t) = \lambda \quad (33)$$

In various sources, among which [73,82], such model is shown to be deducible by means of a so-called "memoryless" dynamic stress-strength models with "shock type" stresses. Let us assume that:

(1) $X(t)$ is a stochastic process which can be described as a "shock type" stress, the shocks (whose amplitudes are denoted as Z_k) occurring at the time instants T_k;
(2) The device fails only because of the occurrence of a stress, i.e., at the time $t = T_k$ when stress amplitude Z_k is greater than Strength $Y(t) = Y(T_j)$; of course, such failure time is a RV.

It is observed that, in order that the device does not fail in the *whole* interval (0,t), then every stress within the given interval must be smaller than the relevant Strength, i.e., $(Z_j < Y_j)$ must be verified for every index $j = 1, \ldots ,N(t)$, being $Z_k = X(T_k)$, and $Y_k = Y(T_k)$, T_k being the RV "time of k-th stress occurrence".

The RF can be obtained first by conditioning on the event $A_n = [N(t) = n]$:

$$R(t \mid A_n) = P[(X_1 < Y_1) \cap (X_2 < Y_2) \cap \ldots \cap (X_n < Y_n) \mid A_n] \quad (34)$$

The term $R(t \mid A_n)$ is denoted as R_n hereafter.

Then, after calculating functions R_n, the RF $R(t)$ can be attained—in terms of the R_n's and of the distribution of the point process $N(t)$—resorting to the total probability theorem, as follows:

$$R(t) = \sum_{k=0}^{\infty} R_n(t)p(n,t), \;\; t > 0 \quad (35)$$

where $p(n,t) = P[N(t) = n]$. Then, let us assume that stress occur according a Poisson stochastic process, so that its probability distribution is expressed by:

$$p(k,t) = P[N(t) = k] = \frac{(\varphi t)^k}{k!} exp(-\varphi t), \;\; k = 0,1,\ldots,\infty \quad (36)$$

In this case, assuming also that the RV Z_j ($j = 1,..n,..$) and Y_j ($j = 1,..n,..$) are statistically independent of each other and of $N(t)$, then:

$$\begin{aligned} Rn = R(t|\, En) = P[(Z1 < Y1) \cap (Z2 < Y2) \cap \ldots \cap (Zn < Yn)|\, En)] \\ = \textstyle\prod_{k=1}^{n} P(Z_k < Y_k) \end{aligned} \quad (37)$$

and the RF is expressed by means of the constants R_n as:

$$R(t) = \textstyle\sum_{k=0}^{\infty} R_n(t)p(n,t), \;\; t > 0 \quad (38)$$

Then, the following hypotheses are assumed to hold:

- the shock amplitudes, i.s., the RV Z_j, are independent with common CDF $G(z) = P(Z_j < z)$, $\forall j = 1,2,\ldots, n,\ldots$, (independent of time), and PDF $g(z)$;
- the Y_j are independent with common CDF $F(y) = F_y(y) = P(Y_j < y)$, $\forall j = 1,2,\ldots, n,\ldots$, (independent of time), and PDF $f(y)$;

Then:

$$R_n = w^n \quad (39)$$

where w is the following constant, representing the probability that the event (Z < Y) occurs denoting by Z a generic one of the Z_j RV (and the same for Y and Y_j)—evaluated as using the same approach when the SS models were introduced (Equation (14)):

$$w = P(Z < Y) = \int_0^{\infty} g(z)(1 - F(z))dz \quad (40)$$

Under the above hypotheses, the wear process can be defined as a "memory-less" one, since wear at age t does not depend on previously occurred shocks. It is so reasonable that such model can be applied to electronics devices. Finally, as shown in [82], using the series expansion of the exponential function appearing in the assumed Poisson law $p(n,t)$, the simple result is obtained:

$$R(t) = \exp[-\varphi\, t(1 - w)] = \exp(-\varphi\theta t) \quad (41)$$

having defined θ as the elementary failure probability: $\theta = 1 - w = P(Z > Y)$, i.e., the probability that the generic strength Y_j is smaller than the generic stress Z_j. In case that the strength is a constant, y, then $\theta = P(Z > y) = 1 - G(z)$. In any case, the above RF of (41) is clearly an Exponential one, i.e., it may be expressed as:

$$R(t) = \exp(-\lambda t) \quad (42)$$

with parameter $\lambda = \varphi\theta$; i.e., the hazard rate is expressed by the product of the mean stress occurrence (φ) and the elementary failure probability (θ).

Also the fact that such model has a constant hazard rate might seem once more against naïve expectation, as one could guess the applied stresses involve an increasing hazard rate (since the number of stresses increases in time). This confirms that a RM can be hardly guessed via pure intuition.

5. Inverse Gaussian (IG) Model

The PDF of the IG model–by a proper choice of the parameters μ and λ (both having the dimensions of time)—is expressed by [106–111]:

$$f(t;\mu,\lambda) = \sqrt{\frac{\lambda}{2\pi t^3}} exp\left[-\frac{\lambda}{2\mu^2 t}(t-\mu)^2\right],\ \mu,\ \lambda > 0 \quad (43)$$

i.e., the IG PDF as in Section 2. Using the above recalled Gaussian CDF $\Phi(x)$, the RF and HRF are given by the following expressions:

$$R(t;\mu,\lambda) = \Phi\left[\sqrt{\left(\frac{\lambda}{t}\right)}\left(1-\frac{t}{\mu}\right)\right] - exp\left[\frac{2\lambda}{\mu}\right]\Phi\left[-\sqrt{\frac{\lambda}{t}}\left(1+\frac{t}{\mu}\right)\right] \quad (44)$$

$$h(t) = f(t)/(R(t)) \quad (45)$$

It has been shown that the HRF first increases, reaching its maximum at a time t^* which is not analytically expressible, then approaches the positive limit $\lambda/(2\mu^2)$ as t diverges. Thus, the IG model exhibits a slight difference compared to the LN and LL reliability models, since the HRF of the the LN and LL reliability models goes to zero as $t \rightarrow \infty$. On the other hand, the IG model exhibits a similarity with the BS model, since this latter appears close to the IG model in its attainment, too.

Mean and variance are:

$$E[T] = \mu;\ \sigma^2 = \mu/\lambda \quad (46)$$

The IG model can be derived from a "stress-strength" (SS) model, too. Such SS stems from a Wiener Stress process combined with a deterministic Strength. The IG distribution [108] has been introduced as the first passage time of a Wiener process [111], i.e., a Gaussian stochastic process with independent increments Let us hypothesize that the stochastic process X(t) describing wear is a Wiener process with "drift". Then, X(t) satisfies the differential equation:

$$-dX/dt = \mu + \Theta(t) \quad (47)$$

where μ is a positive constant and Θ (t) is a Normal stochastic process characterized by the following mean and covariance functions (in terms of the Dirac function $\delta(t)$):

$$E[\Theta (t)] = 0, Cov[\Theta (t) \Theta (t - u)] = v\delta(u) \ (v > 0) \tag{48}$$

being $v > 0$ the so-called"diffusion constant". Let us assume that the wear process is due to a stress associated with the above Wiener process and a deterministic strength y; then, the corresponding lifetime RV coincides with the 1st instant, T, at which the stochastic process $W(t)$ overcomes the barrier y, i.e.,:

$$T = \inf \{t: t > 0, X(t) = y\} \tag{49}$$

If, with no loss of generality, the constant y equal to 1 (a model with a different value of y can be rearranged into a model with y = 1 by a simple change of units), then the above IG model is obtained. As pointed out at Section 3, the IG is quite similar to the BS model, also because they are both IDHR (see above); this may be addressed to the fact that they can be obtained in a similar way from Normal wear processes, too. However, as also reported in [106], the BS model was attained in a more general form—dropping the assumption of Normal crack amplitude and of s-independence—demonstrating also that the BS PDF can be derived via a mixture of two proper IG PDFs.

6. Inverse Weibull (IW) Model

The name of the IW model [45,73,112] comes from the fact that it can be seen as the distribution to the reciprocal of a Weibull RV. The Reliability function of the IW model is:

$$R(t;\alpha,\beta) = 1 - exp\left[-(\alpha t)^{-\beta}\right] \tag{50}$$

The PDF and the HRF of a IW RV, with positive parameters α and β are respectively:

$$f(t;\alpha,\beta) = \alpha\beta(\alpha t)^{-(\beta+1)} exp\left[-(\alpha t)^{-\beta}\right] \tag{51}$$

$$h(t;\alpha,\beta) = \frac{\alpha\beta(\alpha t)^{-(\beta+1)} exp\left[-(\alpha t)^{-\beta}\right]}{1 - exp\left[-(\alpha t)^{-\beta}\right]} \tag{52}$$

Also such function is of the "IDHR" family; in [112] Erto has deduced two reasonable Stress-Strength models which can originate the IW model, and has proven that the HR peak value of a IW model is derived at a mission time value falling within an interval limited by the following bounds: $t_m = \left[\frac{\beta}{\beta+1}\right]^{1/\beta}/\alpha$ (the mode of the IW distribution) and $t_n = \beta^{1/\beta}/\alpha$; the HRF becomes infinitesimal as t diverges. By setting $\theta = 1/\alpha$, the following expressions of the mean (existing if $\beta > 1$ only) and the variance (existing if $\beta > 2$ only)are obtained, respectively:

$$E[X] = \theta\, \Gamma(1 - 1/\beta) = \mu;\ Var[X] = \theta^2\, \Gamma(1 - 2/\beta) - \mu^2 \tag{53}$$

where Γ(z) is the "Euler-Gamma" Function evaluate at z.

6.1. Derivation of the IW Model from a SS Model: Case of Stress Being a Weibull RV

Let us assume that the RV stress X is time-independent and that its distribution is a Weibull distributed RV. Here and in Section 10, for mathematical convenience, the following alternative expression for describing the Weibull distribution is adopted, denoted as W′ (θ, β), with CDF:

$$F(x) = 1 - exp[-(x/\theta)^{\beta}] \tag{54}$$

which is of course equivalent to the W(α,β) model in (9), i.e., to the CDF: $F(x) = 1 - exp(-\alpha x^b)$ by denoting: $\alpha = 1/\theta^\beta$.

Then, let the stress, X be a W'(υ,ϖ) RV, thereby having CDF:

$$F_X(x) = P(X \leq x) = 1 - exp[-(x/\upsilon)^{\varpi}]\ (\upsilon,\varpi) > 0) \quad (55)$$

Let us assume that the strength y(t) is a deterministic decreasing function of time, described by a process with the popular "inverse power" model [22–28,73,104] as:

$$y = y(t) = \kappa\ /t^\eta,\ (\eta,\kappa > 0) \quad (56)$$

Then, the RF at time t is expressible in terms of the above Weibull CDF of stress X:

$$R(t) = P[y(t) > X] = F_X[y(t)] = 1 - \exp\{-\ [\kappa\ /(\upsilon\ t^\eta)]^{\varpi}\} = 1 - \exp\{-[1/\alpha t]^\beta\} \quad (57)$$

Then, the following RF is obtained, denoting: $\beta = \eta\varpi$ and $\alpha = (\upsilon/\kappa)^{1/\eta}$:

$$R(t) = 1 - \exp\{-[1/\alpha t]^\beta\} \quad (58)$$

Thus, clearly an IW model has been deduced, where both parameter α and parameter β are positive.

6.2. Derivation of the IW Model from A SS Model: Case of Stress Being a Weibull Stochastic Process

Let us assume that the strength y is a deterministic constant, and that the stress X = X(t) is a random function of time (stochastic process), following a Weibull W'(θ,β) model, with CDF—denoted as $F_X(x,t)$ for evidencing time-dependency—characterized by time-dependent scale parameter $\theta = \theta$ (t) (and a constant shape parameter $\beta > 0$):

$$F_X(x,t) = P[X(t) \leq x] = 1 - \exp\{-[x/\theta(t)]^\beta\}\ (\beta > 0) \quad (59)$$

Let us further assume that $q(t)$ is an increasing power function of time:

$$\theta(t) = kt^m\ (k,m > 0) \quad (60)$$

The hypothesis under (60) appears to be reasonable, implying that the mean value of stress is an increasing function of time as well. Indeed, under the Weibull W'(θ,β) model of (54) one obtains the following expression for the mean value of X(t), in terms of the Gamma Function Γ(x)):

$$E[X(t)] = \theta(t)\ \Gamma(1 + 1/\beta) = kt^m\ \Gamma(1 + 1/\beta) \quad (61)$$

which is increasing in time as a power function, since β is a constant. Thus, the reliability function at time t is:

$$R(t) = P[X(t) \leq y] = 1 - \exp\{-[y/\theta\ (t)]^\beta\} = 1 - \exp\{-[y/kt^m]^\beta\} = 1 - \exp\{-[1/\eta t]^\gamma\} \quad (62)$$

where positive constants, h and g, are evidently depending on (b,k,m,y). Once more, T is an IW RV. It is interesting to remark that—by means of an analogous methodology—the same author, with Palumbo, in [113] derived a similar, less known, DHR model, called "*hyperbolic reliability model*" (also discussed in [73]).

6.3. Derivation of the IW Model from A Linear Stress Process

As reported above, one of the most adopted analytical expressions for stress X(t) is the one of a linear process of (17), here reported again:

$$X(t) = \beta t \quad (63)$$

in which β is a (positive) RV. Assuming that such RV follows a Weibull model (as has been shown to be a valid assumption for this parameter, in a similar stress analysis framework for lumen data, in [70]). Let also assume that the device's strength is a constant y (which is the case for the above lumen data in [70]: see also Section 10); then, it is easy to deduce that the failure time $T = y/\beta$ of (18), being proportional to the reciprocal of a Weibull RV, follows an IW model.

7. Log-Logistic (LL) Model

The LL model is derived here according to two different approaches.

According to the first approach, reported in [73], and tackled here first, the LL model is derived within a "Weibull Stress-Strength model in the following way.

Let us assume that X and Y are two Weibull RV with the same value of shape parameter β, and with scale parameters θ for the Strength X, and α for the Stress Y. As a consequence the CDF of X and Y can be expressed as follows:

$$G(x) = 1 - \exp[-(x/\theta)^{\beta}];\ F(y) = 1 - \exp[-(y/\alpha)^{\beta}] \tag{64}$$

As for time dependence, it is reasonable to consider the following "Inverse power" characterization of the Strength scale parameter α with time t, in which k and m are positive constants:

$$\alpha = \alpha(t) = k/t^{m} \tag{65}$$

As a matter of fact, the expectation of the Weibull RV Y is $\mu = \alpha\Gamma(1+1/\beta)$, namely $\propto \alpha$. Hence, due to relationship (65), Y is a decreasing power function of time t. Hence, after trivial manipulations, the RF is recast in the form of the following Log-logistic (LL) model [45,114–117]:

$$R(t) = 1/[1 + (\lambda t)^{b}] \tag{66}$$

where $b = m\beta$; $\lambda = (\theta/k)^{1/m}$.

The LL model belongs to the IDHR or (less frequently) to the DHR family of reliability models, depending on the value of the shape parameter b. Indeed, its hazard rate function $h(t)$ has the following expression:

$$h(t) = b\lambda^{b}t^{b-1}\ /[1 + (\lambda t)^{b}] \tag{67}$$

which is always decreasing with time if $b \leq 1$; first increasing, then decreasing with time if $b > 1$.

Denoting by $c = 1/b$, the mean value (existing if $b > 1$ only) and the standard deviation SD (existing—like the variance—if $b > 2$ only) can be written, respectively, as follows:

$$E(X) = \frac{c\pi}{b\sin(c\pi)};\ D[X] = E[X]CV[X] \tag{68}$$

where CV is the Coefficient of Variation, that in this case has the following expression:

$$CV[X] = \sqrt{\left(\frac{b}{\pi}\right)\tan\left(\frac{b}{\pi}\right) - 1} \tag{69}$$

There is also a quite different possible origin of LL distribution, i.e., a mixture of a Weibull RV, with a Gamma model as a mixing distribution" (see Section 9).

8. Lognormal Model

The Lognormal (LN) model [45,118–122] has spread over the last years, even in lifetime data studies. The LN PDF with parameters (ξ,δ) and argument t can be written as follows:

$$f(t; \xi,\delta) = (1/\delta\sqrt{2\pi})\varphi[(\log(t) - \xi)/\delta] \tag{70}$$

where: $\xi = E[\log(T)]$, $\delta = DS[\log(T)]$ ($\delta > 0$)

Indicating with $\Phi(z)$ and $\varphi(z)$, respectively, the standard Normal PDF and CDF, the LN RF and HRF can be written, respectively, as reported here below:

$$R(t;\, \xi,\delta) = 1 - \Phi[(\log(t) - \xi)/\delta] \tag{71}$$

$$h(t;\, \xi,\delta) = f(t;\, \xi,\delta)/R(t;\, \xi,\delta) \tag{72}$$

The time variation of HRF, since it is rather cumbersome to deduce analytically, has been the object of discussion (and, sometimes, mistakes) in literature. As a matter of fact, some articles like [119] treated it specifically.

The LN model is of the IDHR type: first, the HRF rises from zero, later on it drops towards zero (a graphical example will be illustrated in Section 10). Differently from the LL model (which is indeed very similar to the LN one), no analytical evaluation of the "change point" can be done (as also occurs for the IW and the BS model).

The expressions of the mean and the SD are respectively the following:

$$\mu = exp\left(\xi + \frac{\delta^2}{2}\right) \tag{73}$$

$$\sigma = \mu\left\{exp\left(\delta^2\right) - 1\right\}^{\frac{1}{2}} \tag{74}$$

The literature reports several approaches to derive the LN model theoretically grounds from wear processes, whereas in practice several applicative articles proved its effectiveness in matching experimental data from remarkably various fields: for instance, focusing on duration data, from the lifetime model of microelectronic components to the first wedding time of people.

It should be pointed out that, among the widely used reliability models, the LN features the highest values of "skewness coefficient" [45] for a fixed value of coefficient of variation (CV). Moreover, its flexibility is enhanced by two properties that are rarely mentioned, namely:

(i) if the shape parameter β of the LN(α,β) model is low enough (more precisely, in practice if $\beta < 0.3$) the LN PDF tends to symmetry and may approximate well also a Normal model with the same mean (*as can be shown in an analytical way, resorting to the series expansion of $y = exp(x)$ for $x \to 0$*);
(ii) the CV, $\nu = \sigma/\mu$, varies over a broad interval: in more detail, $\nu = 1$ when $\beta = 0.8325$, as for the Exponential model, to which the LN model is so close in this case that the two models cannot almost be distinguished one from another; this motivates the applications of the LN model for microelectronic components [118]. Also Section 10 will show an application of the LN model for electronics components.

Hereafter, let us illustrate three noteworthy models which lead to the LN model for lifetime

8.1. Stress Process with Linear Function Yielding the Lognormal Model

As already discussed in Sections 2 and 6, a linear Stress process may be expressed as:

$$X(t) = \beta t \tag{75}$$

being β a RV, with $\beta > 0$ almost surely.

An illustrative instance of a linear function Stress process—leading to a LN survival distribution—is the Strength model of relationship (75) with β following the LN distribution; let us also assume that Y follows the LN distribution and β and Y are statistically independent. As a consequence, obviously the lifetime RV T matches the following relationships:

$$\beta T = Y \to T = Y/\beta \tag{76}$$

Hence, as the ratio of two LN RV is also a LN RV (like the difference of two Normal RVs is also a Normal random variable [45]), T is a LN RV, whose parameters are easily derived.

If the RV Y is deterministic (i.e.,: Y = y, constant), then once more T is a LN RV.

8.2. Stress Process with Log-Linear Function Yielding the Lognormal Model

The Log-linear model for a stress process X(t)—being widely applied, too—was presented above (see relationship (20)) in the following form (recalled here for the sake of convenience):

$$\mathrm{X(t)} = \alpha \exp(\beta \mathrm{t}) \tag{77}$$

In which (α, β) are positive RV. It is easily demonstrated that, If the RV Y is deterministic then lifetime T is a LN RV if: α is LN RV, and β is a Gaussian RV, also in case (α,β) are not independent RV.

The same happens if, in addition, Y is a LN RV, for the above recalled property that the ratio of two LN RV is also a LN RV.

8.3. Stress Process with Power Function Yielding to the Lognormal Model

Analogous outcomes are attained when stress X(t) is a "power function" of time, like the following:

$$\mathrm{X(t)} = \beta \cdot t^c \tag{78}$$

Let β again be a Lognormal-distributed RV, and c a real constant; in this case, with an LN strength Y, the lifetime is again LN. This is easily demonstrated, recalling that if T is an LN RV, then T^c is also an LN RV, regardless of the value of the real exponent c.

9. A Brief Account of Mixture Models Leading to DHR or IDHR Models

Here, we describe a peculiar family of models deduced from the combination of two models (or more), i.e., "*Mixture models*", based on a random hazard rate that appears to be particularly suited to electronic devices.

In fact, a big deal of models are obtained by letting some parameters of the lifetime PDF vary randomly among components, taking heterogeneity of material or production process or random variability of environment where components work into account. As reported in some basilar studies [43,123–132], a theory which can explain satisfactorily such variation is the one based upon a "*random HRF*" approach, such as the one proposed in [130], in which a so-called "Proportional Hazard Model" [22] is characterized by a stochastic parameter, say Ω. Such RV Ω has the role of expressing the role of the above introduced random variability, so that the effective HRF can be expressed as a function of time (as always) and also of the given RV W, so that a "conditional" HRF is H(t | Ω) defined as follows (the capital letter *H* is used here instead of *h*, as a reminder that such HRF is, in this framework, a random function):

$$\mathrm{H}(\mathrm{t} \mid \Omega) = \Omega\, \mathrm{B(t)} \tag{79}$$

In (79), H(t | Ω) denotes the above said conditional HRF, i.e., the HRF conditioned to the positive random factor W, and B(t) is the "baseline" HRF (a non-random function, possessing all the properties of a typical HRF). Here, a motivation is given to justify the above multiplicative relation. It is well known that, based upon the above discussed well known models from reliability standards for electronics devices [94,95], the HRF—generally considered as a constant, *B*—is evaluated as the result of a multiplicative model:

$$B = h_b h_1 h_2 \dots h_p \tag{80}$$

where:

$-h_b$ is the "base" HRF (it is a positive constant for an Exponential model, in all other cases is a time function as in [79]); *p* (*p* > 1) is an integer value depending on the component;

$-h_1$, h_2,h_p are the so-called "environment factors". In practice, the environment factors are positive constants which account for all the factors (environment, production, quality, applied voltage and others) affecting the final value of the HRF, so they are to be considered, in the most general case, as RV in order to account for the environment randomness for electronics devices already discussed in Section 1. If the product h_1 $h_{2..}$, h_p is denoted by Ω, the above model (79) is obtained.

A straightforward model is obtained if, following [130], a Weibull model is assumed for the hazard rate, with shape parameter b:

$$H(t \mid \Omega) = \Omega\, bt^{b-1}, (b > 0) \tag{81}$$

For the characterization of the random parameter W, distributions definite only or positive values of their argument must be considered. Let the PDF of the RV Ω be denoted by p(ω). Applying the total probability theorem in the continuous case, it is straightforward to obtain the (unconditional) reliability function, which is a "mixture" [17,45] of the RF according to the PDF p(ω):

$$R(t) = \int_0^\infty exp\left(-\omega t^b\right) p(\omega) d\omega \tag{82}$$

The result of the integral (82) possesses a closed analytical form only in some cases, for instance when Z has a Gamma or an Inverse Gaussian PDF. In particular, let us assume that p(w) is a Gamma PDF:

$$p(\omega; r, s) = k \frac{\omega^{r-1} \ \exp\left(-\frac{\omega}{s}\right)}{s^r} \tag{83}$$

where k = Γ(r), the Gamma special function evaluated at s. Then, it is easy to show that the non-conditional RF R(t) and HRF h(t) may be expressed by:

$$R(t) = \frac{1}{\left(1 + st^b\right)^r};\ h(t) = \frac{rsbt^{b-1}}{\left(1 + st^b\right)} \tag{84}$$

Such RF and HRF are part of the *Burr* family of distributions [45], which gives as a particular case the LL model of Section 7. So, a Weibull model can give origin to a LL model in two unexpectedly different ways, i.e., a model (illustrated in Section 7) based upon a Stress-Strength approach, or a mixture model illustrated here. In the framework of the present study, the importance of such model is that it belongs to the DHR or the IDHR model as already analysed. There is an apparent contradiction, or inconsistency (which often happens with mixtures) that an unconditional HRF is decreasing for large t, even if it is originated by a Weibull model with an increasing HRF: such contradiction can be clearly understood if it is interpreted from a Bayesian point of view, i.e., adopting a subjective approach to probability [43,73,124].

10. On the Consequences of Mistaken Model Identification in Terms of the Hazard Rate Function. A Numerical Example from a Real Dataset

As hinted at above, in this final section (devoted to real data analysis, the data—and the relevant statistical lifetime distribution parameters—being reported in [70]), the consequences, in terms of the hazard rate function behaviour of mistaken model identification are discussed. This is performed by means of a simple numerical and graphical example relevant to real electronics devices' lifetime data. In this case, the Weibull model is compared first to a LN model, as reported in [70], regarding a comparison of statistical models for the lumen lifetime distribution of high power white Light emitting diode (LED). As highlighted in [70], LED is one of the highly reliable products developed in the past few decades, and significant advances have been reached for the manufacturing technologies of LED. In [70], indeed, it is discusses how traditional reliability assessment techniques have severe limitation on this highly-reliable electronic devices with data few lifetime data;

on the other hand, analytical degradation processes models are known for them, which are of great help in evaluating device reliability.

In this application, the authors report that both the LN and Weibull model appear adequate for the statistical fitting of a set of lumen lifetime data (they also consider the Normal model, which will be hinted at afterwards in the following). In particular, the lifetime data are "pseudo failure times", obtained from the estimated process of a wear model discussed above for lumen data, in eqn. (20). The wear model is extrapolated, for each unit under observation, to the critical failure threshold, which is defined in such application as the one corresponding to a 30% light decrease.

The RF expression of the LN and Weibull models are again reported here. Recalling that $\Phi(z)$ denotes the standard Normal CDF for any real number z, the LN RF is expressed as follows (see also (71)):

$$R(t;\, \xi,\delta)= 1 - \Phi[(\log(t) - \xi)/\delta] \tag{85}$$

Measuring the lumen lifetime in hours/10^4, the parameters of the above expression are so estimated from the real data, as reported in [70]: $\xi = 2.3054$; $\delta = 0.149415$.

The RF expression of the Weibull model here adopted is the W′(θ,β) one of (54):

$$R(t) = \exp[-(t/\theta)^{\beta}] \tag{86}$$

The following values of the parameters (θ,β) of the above expression also provide a RF which fits very well the same data (again expressed in hours/10^4): $\theta = 10.5464$, $\beta = 7.890$.

The fact that both models are adequate to represent the dataset is confirmed, as reported in [70], by adequate statistical tests, such as the "Akaike Information Criterion" [24], but is also evident graphically. In Figure 1, the RF of the two models (blue curve for the LN, black curve for Weibull model) are reported. In practice, they are almost indistinguishable.

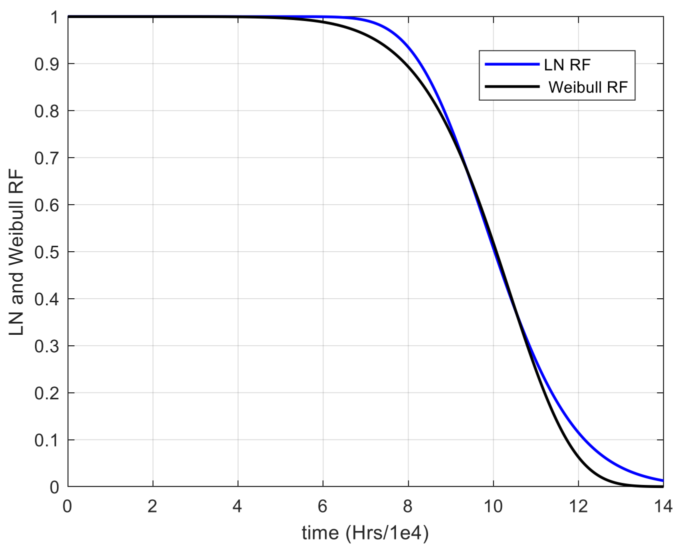

Figure 1. The RF curves of the LN and Weibull models for lumen data after [70].

Moreover, the curves of the two corresponding PDFs are reported in Figure 2, which also highlights the very strong similarity of the two models (the time axis has been showed on a larger time interval with respect to Figure 1 for the presence of "large tails" in the two PDF curves).

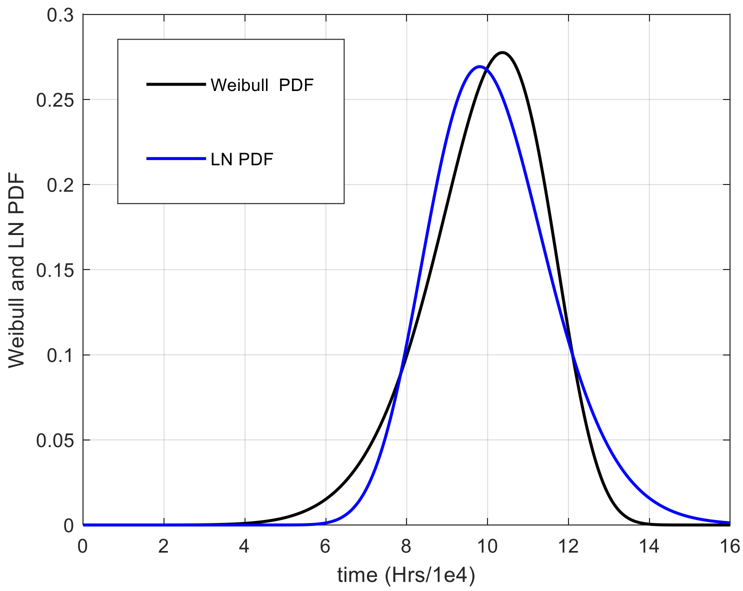

Figure 2. The PDF curves of the LN and Weibull models for lumen data after [70].

Such similarity is largely confirmed by the values the p-quantiles (It is recalled that the p-quantile of a given a RV is a value x* such that p100% of the observed value of the RV fall below x*. In the case of a continuous RV as in the present case, if F(x) is the CDF of the RV under study, x* it is the unique solution of F(x*) = p) of the two different reliability models, some of them (for p = 0.05, 0.50, 0.95) are shown in Table 1. As well known, the 0.50-quantile is also known as the Median of the distribution. The two medians, as expected, are equal in practice. It is significant that also the p-quantiles for large values of p (such as $p = 0.95$) are very close.

Table 1. Values of some p-quantiles of the LN and the Weibull reliability models for lumen data after [70] (for $p = 0.05, 0.50, 0.95$).

p	LN	Weibull
0.05	7.8428	7.2379
0.50	10.0278	10.0677
0.95	12.8215	12.1199

On the other hand, the time variations of the two HRF functions are shown in Figure 3, illustrating the remarkable differences between the two models under this respect. Such difference becomes larger and larger as mission times become larger than the median value ($t \approx 10 \times 10^4$ h). In order to better appreciate the LN HRF behavior, since its values are much smaller than those of the Weibull one, the curve of the LN HRF alone is also reported in Figure 4. As apparent, for large times the LN HRF tends to become nearly constant (and very slowly decreasing for very large times), while the Weibull one sharply increases.

It is perhaps interesting to remark that in some cases [73] the difference between the HRF for large mission times may be put in relation to the presence of larger tails, and thus larger extreme quantiles (i.e., the p-quantiles with p approaching 1), for the model with a decreasing or a much smaller HRF (as in this case happens to the LN model with respect to the Weibull one). This does not occur in the present application, as already shown, for instance, by the similar values the 0.95 quantiles of the two different reliability models (as it could be anticipated, the 0.95-quantile of the LN model is larger than the 0.95-quantile of the Weibull one, but only marginally, 12.8215 for the LN model vs. 12.1199 for Weibull one).

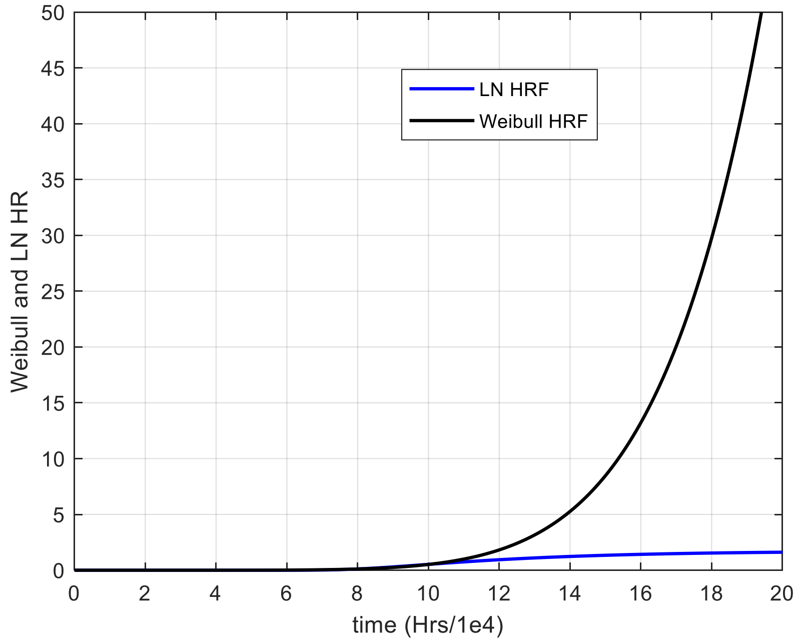

Figure 3. The two HRF corresponding to the two RF of Figure 1 and PDF of Figure 2.

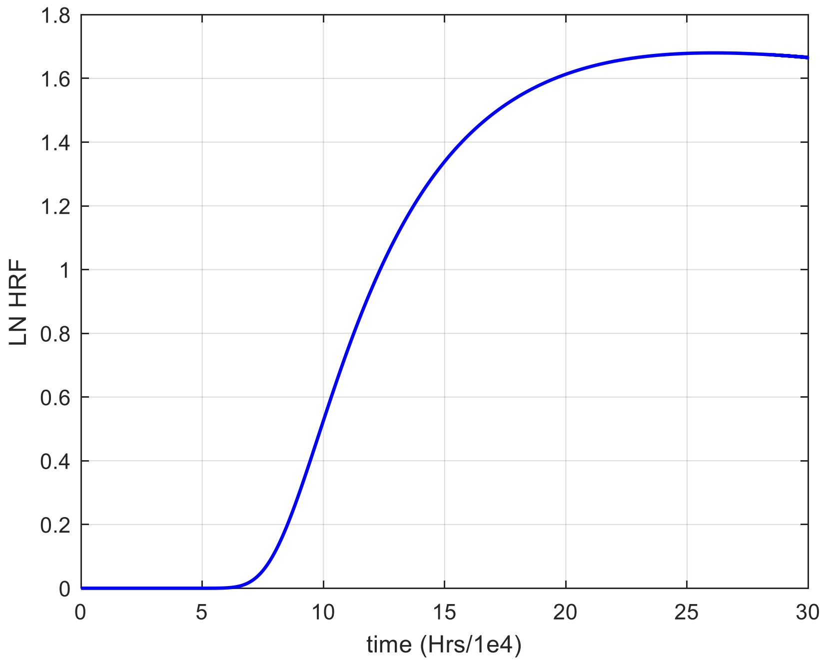

Figure 4. The HRF curves of the LN model for lumen data.

In conclusion, it can be assessed that—in this and similar cases—if a Weibull model is (wrongly) assumed or estimated instead of a ”true” LN model, the HRF for large mission time maybe grossly overestimated, and such mistake may cause avoidable and costly maintenance actions: indeed, as intuitive in view of the physical meaning of the HRF, maintenance strategies based upon the device’s age are effective only in case of devices possessing an increasing HRF [85–93].

11. Conclusions

Aim of the paper is to present a review in reliability studies and applications, which may help in understanding why, even if often contrary to intuition and seldom used in the field of electronics, a decreasing hazard rate function (at least for relatively large mission times) may be the best suited to describe the true model behind a given failure mechanism. This is accomplished by analyzing a review of reasonable physical motivations behind the identification of a reliability model. These motivations lead to the identification of probabilistic life models, originated by the knowledge or estimation of failure mechanisms, and in particular to the so-called "stress-strength" models in order to characterize device "aging".

This approach may help not only in selecting the "right" reliability model, but has other "practical" application, since the HRF is essential for setting the most adequate maintenance actions on the device.

The author believe that such kind of studies should be further developed, for instance extending the hypotheses assumed for the wear processes (e.g., allowing for non-homogeneous Poisson processes in derivations such as those provided in Section 4), in order to derive possibly even more adequate models for high reliability devices, which offer very few data for a model selection.

It seems also necessary, developing works as in [133], explore more theoretical properties which could establish a closer relation between the device HRF variation in time and the properties of wear acting on the device, as well as developing more complex classes of degradation processes such as the Wiener Process with Random Effects, the Gamma, and the Inverse Gaussian processes [74,133–135]. Finally, for lack of space, some other interesting approaches were not illustrated here, but at least two alternative methodologies should be mentioned such as the so-called Multi-State System reliability analysis [136–138], and Machine Learning Approach [139–142].

In the paper we have focused our analysis on Decreasing Hazard Rate models, since there are many cases—which have also been justified on a theoretical basis and illustrated via applicative examples in the former Section—in which a decreasing hazard rate function may be the best suited to describe the true model behind a given failure mechanism (at least for relatively large mission times). Of course, the problem of system degradation is investigated in the literature—also in the field of electronic components—by resorting to approaches different from the one followed here, depending on the various applications. Among these different approaches, one of the possible models for investigating the system degradation is the Multi-State System analysis approach [136–138]. In strong summary, Multi-State System reliability analysis is a kind of reliability analysis which considers multiple possible states of the system, whereby both the system and its components are allowed to assume more than two levels of performance. Through multi-state reliability models provide, generally speaking, more realistic and more precise representations of engineering systems, they are much more complex and present major difficulties in system definition and performance evaluation. A suggestion for a further future broadening of the present investigation is thus considering and comparing Multi-State System based approach in reliability analysis with the Decreasing Hazard Rate models.

The Machine Learning approach is based upon a quite different philosophy, which is only minimally related to the probabilistic approach adopted here, since it does not specify a RM; nonetheless, it leads to an "Indirect Reliability Prediction" [142] which is somewhat similar to the "Indirect Reliability Assessment" discussed in the present paper. The Machine Learning Algorithms appear to be adequate, for instance, to power device reliability prediction in extreme conditions, as discussed in [142], and surely appear worth of being further developed for such kind of applications.

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Glossary

BS	Birnbaum–Saunders (distribution)
CDF	Cumulative distribution function
CLT	Central limit theorem
CV	Coefficient of variation
DHR	Decreasing hazard rate
D[Y]	Standard deviation of the RV Y
E[Y]	Expectation of the RV Y
$F(\,)$	Generic CDF
$f(\,)$	Generic PDF
$f(x)$, $F(x)$	PDF and CDF of Stress
$g(y)$, $G(y)$	PDF and CDF of Strength
HRF	Hazard Rate Function
IDHR	First increasing, then decreasing hazard rate
IG	Inverse Gaussian (distribution)
IHR	Increasing hazard rate
IW	Inverse Weibull (distribution)
LL	Log-logistic (distribution)
LN	Lognormal (distribution)
MTTF	Mean Time to Failure
$N(\alpha,\beta)$	Normal (Gaussian) random variable with mean α and standard deviation β
PDF	Probability density function
RF	Reliability function
RM	Reliability model
$R(t)$	Reliability function at mission time t
RRF	Residual reliability function
$R(t \mid s)$	Residual reliability function at mission time t, for a device aged s time units
RV	Random variable
SD, σ	Standard deviation
SS	Stress-Strength
Var, σ^2	Variance
$W(t)$	Wear process at time t acting on a device
$W(\alpha,\beta)$	Weibull model with CDF: $F(x) = 1 - exp(-\alpha x^{\beta})$
$W'(\theta,\beta)$	Weibull alternative form, with CDF: $F(x) = 1 - exp(-(x/\theta)^{\beta})$
$\Gamma(\,)$	Euler's Gamma function
μ	Mean value (Expectation)
$\Phi(z)$	Standard normal CDF
$\varphi(z)$	Standard normal PDF
Remark:	*the symbol "log" always denotes natural logarithm.*

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