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Classification of Degenerate Verma Modules for $E(5, 10)$

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Abstract: Given a Lie superalgebra $\mathfrak{g}$ with a subalgebra $\mathfrak{g}_{\geq 0}$, and a finite-dimensional irreducible $\mathfrak{g}_{\geq 0}$-module F, the induced $\mathfrak{g}$-module $M(F) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_{\geq 0})} F$ is called a finite Verma module. In the present paper we classify the non-irreducible finite Verma modules over the largest exceptional linearly compact Lie superalgebra $\mathfrak{g} = E(5, 10)$ with the subalgebra $\mathfrak{g}_{\geq 0}$ of minimal codimension. This is done via classification of all singular vectors in the modules $M(F)$. Besides known singular vectors of degree 1,2,3,4 and 5, we discover two new singular vectors, of degrees 7 and 11. We show that the corresponding morphisms of finite Verma modules of degree 1,4,7, and 11 can be arranged in an infinite number of bilateral infinite complexes, which may be viewed as "exceptional" de Rham complexes for $E(5, 10)$.

1. Introduction

Recall that a linearly compact Lie (super)algebra $\mathfrak{g}$ is defined by the property that, viewed as a vector space, $\mathfrak{g}$ is linearly compact. According to E. Cartan's classification, the list of infinite-dimensional simple linearly compact Lie algebras consists of four Lie–Cartan series: W_n, S_n, H_n, and K_n.

The infinite-dimensional simple linearly compact Lie superalgebras were classified in [6] and explicitely described in [5]; all their maximal open subalgebras were classified in [4]. The complete list consists of ten "classical" series (which include the Lie–Cartan series), and five exceptional examples, denoted by $E(1, 6)$, $E(3, 6)$, $E(3, 8)$, $E(4, 4)$, and $E(5, 10)$. With the exception of $E(4, 4)$, these Lie superalgebras carry a $\mathbb{Z}$–gradation, compatible with the parity:

$$\mathfrak{g} = \bigoplus_{j \geq -d} \mathfrak{g}_j,$$

where $d = 2$ for $E(1, 6)$, $E(3, 6)$ and $E(5, 10)$, and $d = 3$ for $E(3, 8)$. Then $\mathfrak{g}_{\geq 0} := \oplus_{j \geq 0} \mathfrak{g}_j$ is a maximal open subalgebra of $\mathfrak{g}$ of minimal codimension. In the case of

$\mathfrak{g} = E(3,6)$ and $E(3,8)$ the subalgebra $\mathfrak{g}_0$ is isomorphic to $\mathfrak{sl}_3 \oplus \mathfrak{sl}_2 \oplus \mathbb{C}$, and for $\mathfrak{g} = E(5,10)$, $\mathfrak{g}_0$ is isomorphic to $\mathfrak{sl}_5$, which hints to connections to particle physics [8].

Let F be an irreducible finite-dimensional $\mathfrak{g}_0$-module, extend it to $\mathfrak{g}_{\geq 0}$ by letting all $\mathfrak{g}_j$ with $j > 0$ act by 0, and consider the *finite Verma* $\mathfrak{g}$*-module*

$$M(F) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_{\geq 0})} F,$$

where $M(F)$ is viewed as a vector space with discrete topology. These modules are especially interesting since their topological duals are linearly compact.

The first problem of representation theory of linearly compact Lie superalgebras is to classify their degenerate (i.e., non-irreducible) finite Verma modules and morphisms between them. This is equivalent to classification of *singular vectors* in these modules, i.e., those which are annihilated by $\mathfrak{g}_j$ with $j \geq 1$. This problem was solved for Lie algebras W_n, S_n and H_n by Rudakov [11,12]. In particular, he showed that the degenerate finite W_n-modules form the de Rham complex in a formal neighborhood of 0 in $\mathbb{C}^n$ (rather its topological dual).

In a series of papers [7,8,10] this problem was solved for the exceptional linearly compact Lie superalgebra $E(3,6)$. It turned out that all the morphisms between the degenerate finite Verma modules over $E(3,6)$ can be arranged in an infinite number of complexes, and cohomology of these complexes was computed in [8] as well. The most difficult technical part of this work is [10], where all singular vectors have been classified.

In the subsequent paper [9] a solution to this problem was announced for $E(3,8)$, and a conjecture on classification of degenerate finite Verma modules for $E(5,10)$ was posed, motivated by the singular vectors of degree 1 constructed there (the degree on $M(F) = \mathcal{U}(\mathfrak{g}_{<0}) \otimes F$ is induced by the degree on $\mathfrak{g}_{<0} = \oplus_{j<0}\mathfrak{g}_j$). In a more recent paper [13] it was proved that these are all singular vectors of degree 1, and also some singular vectors of degree 2,3,4 and 5 have been constructed. In the subsequent paper [2] it was shown that the singular vectors of degree less than or equal to 3 constructed by Rudakov are all singular vectors of degree less than or equal to 3. Actually, the morphisms of degrees 2, 3 and 5 corresponding to singular vectors constructed in [13] are composition of morphisms of degree 1 and 4, and the morphisms of degree 1 and 4 can be arranged in an infinite number of infinite complexes [13]. However, in Fig. 2 of [13] there are two notable gaps in the complexes.

The key discovery of the present paper is the existence of morphisms of degree 7 and 11, which fill these gaps (see Fig. 4). Moreover, we show that there are no further singular vectors (Theorem 10.1), thereby proving the conjecture from [9] on classification of degenerate finite Verma modules over $E(5,10)$.

The proof of Theorem 10.1 goes as follows. First, using a result from [12] on S_n-modules for $n = 5$, which is the even part of $E(5,10)$, we show that there are no singular vectors of degree greater than 14. Next we find that for degrees between 11 and 14 there is only one singular vector, it has degree 11 and defines a morphism from $M(\mathbb{C}^5)$ to $M(\mathbb{C}^{5*})$, where $\mathbb{C}^5$ is the standard $\mathfrak{sl}_5$-module and $\mathbb{C}^{5*}$ its dual. After that, using the techniques of [2], we show that in degrees between 6 and 10 the only singular vector has degree 7 and it defines a morphism from $M(S^2\mathbb{C}^5)$ to $M(S^2\mathbb{C}^{5*})$. These are precisely the two morphisms, missing in Fig. 2 of [13]. Finally, we show that in degrees less than or equal to 6 there are no other singular vectors as compared to [13]. The calculations involve solution of large systems of linear equations, which are performed with the aid of computer. Note also that the construction of morphisms is facilitated by the duality, constructed in [3], such that the morphism $M(F) \to M(F_1)$ induces the morphism $M(F_1)^* \to M(F)^*$ and for $E(5, 10)$ has the property that $M(F)^* = M(F^*)$.

We have learned recently that Daniele Brilli obtained in [1] the upper bound 12 on the degrees of singular vectors for finite Verma modules over $E(5, 10)$, using the techniques of representation theory of Lie pseudoalgebras.

2. Preliminaries

We let $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ be the set of non-negative integers and for $n \in \mathbb{N}$ we set $[n] = \{i \in \mathbb{N} \mid 1 \le i \le n\}$.

We consider the simple linearly compact Lie superalgebra of exceptional type $\mathfrak{g} = E(5, 10)$ whose even and odd parts are as follows: $\mathfrak{g}_{\bar{0}}$ consists of zero-divergence vector fields in five (even) indeterminates $x_1, \dots, x_5$, i.e.,

$$\mathfrak{g}_{\bar{0}} = S_5 = \{X = \sum_{i=1}^{5} f_i \partial_i \mid f_i \in \mathbb{C}[[x_1, \dots, x_5]], \operatorname{div}(X) = 0\},$$

where $\partial_i = \partial_{x_i}$, and $\mathfrak{g}_{\bar{1}} = \Omega^2_{cl}$ consists of closed two-forms in the five indeterminates $x_1, \dots, x_5$. The bracket between a vector field and a two-form is given by the Lie derivative and for $f, g \in \mathbb{C}[[x_1, \dots, x_5]]$ we have

$$[f dx_i \wedge dx_j, g dx_k \wedge dx_l] = \varepsilon_{ijkl} f g \partial_{t_{ijkl}}$$

where, for $i, j, k, l \in [5]$, ε_{ijkl} and t_{ijkl} are defined as follows: if $|\{i, j, k, l\}| = 4$ we let $t_{ijkl} \in [5]$ be such that $|\{i, j, k, l, t_{ijkl}\}| = 5$ and ε_{ijkl} be the sign of the permutation (i, j, k, l, t_{ijkl}). If $|\{i, j, k, l\}| < 4$ then $\varepsilon_{ijkl} = 0$.

From now on we shall denote $dx_i \wedge dx_j$ simply by d_{ij}.

The Lie superalgebra $\mathfrak{g}$ has a consistent, irreducible, transitive $\mathbb{Z}$-grading of depth 2 where, for $k \in \mathbb{N}$,

$$\mathfrak{g}_{2k-2} = \langle f \partial_i \mid i = 1, \dots, 5,\ f \in \mathbb{C}[[x_1, \dots, x_5]]_k \rangle \cap S_5$$
$$\mathfrak{g}_{2k-1} = \langle f d_{ij} \mid i, j = 1, \dots, 5,\ f \in \mathbb{C}[[x_1, \dots, x_5]]_k \rangle \cap \Omega^2_{cl}$$

where by $\mathbb{C}[[x_1, \dots, x_5]]_k$ we denote the homogeneous component of $\mathbb{C}[[x_1, \dots, x_5]]$ of degree k.

Note that $\mathfrak{g}_0 \cong \mathfrak{sl}_5$, $\mathfrak{g}_{-2} \cong (\mathbb{C}^5)^*$, $\mathfrak{g}_{-1} \cong \bigwedge^2 \mathbb{C}^5$ as $\mathfrak{g}_0$-modules (where $\mathbb{C}^5$ denotes the standard $\mathfrak{sl}_5$-module). We set $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$, $\mathfrak{g}_+ = \oplus_{j>0} \mathfrak{g}_j$ and $\mathfrak{g}_{\ge 0} = \mathfrak{g}_0 \oplus \mathfrak{g}_+$.

We denote by U (resp. U_-) the universal enveloping algebra of $\mathfrak{g}$ (resp. $\mathfrak{g}_-$). Note that U_- is a $\mathfrak{g}_0$-module with respect to the adjoint action: for $x \in \mathfrak{g}_0$ and $u \in U_-$,

$$x.u = [x, u] = xu - ux.$$

We also point out that the $\mathbb{Z}$-grading of $\mathfrak{g}$ induces a $\mathbb{Z}$-grading on the enveloping algebra U_-. It is customary, though, to invert the sign of the degrees hence getting a grading over $\mathbb{N}$. Note that the homogeneous component $(U_-)_d$ of degree d of U_- under this grading is a $\mathfrak{g}_0$-submodule.

We fix the Borel subalgebra $\langle x_i \partial_j, h_{ij} = x_i \partial_i - x_j \partial_j \mid i < j \rangle$ of $\mathfrak{g}_0$ and we consider the usual base of the corresponding root system given by $\{\alpha_{12}, \dots, \alpha_{45}\}$. We let Λ be the weight lattice of $\mathfrak{sl}_5$ and we express all weights of $\mathfrak{sl}_5$ using their coordinates with respect to the fundamental weights $\varphi_{12}, \varphi_{23}, \varphi_{34}, \varphi_{45}$, i.e., for $\lambda \in \Lambda$ we write $\lambda = (\lambda_{12}, \dots, \lambda_{45})$ for some $\lambda_{i\,i+1} \in \mathbb{Z}$ to mean $\lambda = \lambda_{12}\varphi_{12} + \dots + \lambda_{45}\varphi_{45}$.

If $\lambda \in \Lambda$ is a weight, we use the following convention: for all $1 \leq i < j \leq 5$ we let

$$\lambda_{ij} = \sum_{k=i}^{j-1} \lambda_{k\,k+1}.$$

If V is a $\mathfrak{sl}_5$-module and $v \in V$ is a weight vector we denote by $\lambda(v)$ the weight of v and by $\lambda_{ij}(v) = (\lambda(v))_{ij}$.

If $\lambda = (a, b, c, d) \in \Lambda$ is a dominant weight, i.e. $a, b, c, d \geq 0$, let us denote by $F(\lambda) = F(a, b, c, d)$ the irreducible $\mathfrak{sl}_5$-module of highest weight λ. In this paper we always think of $F(a, b, c, d)$ as the irreducible submodule of

$$\mathrm{Sym}^a(\mathbb{C}^5) \otimes \mathrm{Sym}^b(\textstyle\bigwedge^2(\mathbb{C}^5)) \otimes \mathrm{Sym}^c(\textstyle\bigwedge^2(\mathbb{C}^5)^*) \otimes \mathrm{Sym}^d((\mathbb{C}^5)^*)$$

generated by the highest weight vector $x_1^a x_{12}^b x_{45}^{*\,c} x_5^{*d}$, where $\{x_1, \dots, x_5\}$ denotes the standard basis of $\mathbb{C}^5$, $x_{ij} = x_i \wedge x_j$, and x_i^* and x_{ij}^* are the corresponding dual basis elements. Besides, for a weight $\lambda = (a, b, c, d)$ we let $\lambda^* = (d, c, b, a)$, so that $F(\lambda)^* \cong F(\lambda^*)$.

Notice that, as a $\mathfrak{g}_0$-module, $\mathfrak{g}_1 \cong F(1, 1, 0, 0)$ and that $x_5 d_{45}$ is a lowest weight vector in $\mathfrak{g}_1$. Moreover, for $j \geq 1$, we have $\mathfrak{g}_j = \mathfrak{g}_1^j$.

3. Generalized Verma Modules and Morphisms

We recall the definition and some properties of (generalized) Verma modules over $E(5, 10)$, most of which hold in the generality of arbitrary $\mathbb{Z}$–graded Lie superalgebras (for some detailed proofs see [2]).

Given a $\mathfrak{g}_0$-module V, we extend it to a $\mathfrak{g}_{\geq 0}$-module by letting $\mathfrak{g}_+$ act trivially, and define

$$M(V) = U \otimes_{U(\mathfrak{g}_{\geq 0})} V.$$

Note that $M(V)$ has a $\mathfrak{g}$-module structure by multiplication on the left, and is called the (generalized) Verma module associated to V. We also observe that $M(V) \cong U_- \otimes_{\mathbb{C}} V$ as $\mathfrak{g}_0$-modules.

If the $\mathfrak{g}_0$-module V is finite-dimensional and irreducible, then we call $M(V)$ a finite Verma module (it is finitely-generated as a U_--module). We denote by $M(\lambda)$ the finite Verma module $M(F(\lambda))$. A finite Verma module is said to be non-degenerate if it is irreducible and degenerate otherwise.

Definition 3.1. We say that an element $w \in M(V)$ is *homogeneous* of degree d if $w \in (U_-)_d \otimes V$.

Definition 3.2. A vector $w \in M(V)$ is called a *singular* vector if it satisfies the following conditions:

(i) $x_i \partial_{i+1} w = 0$ for every $i = 1, \dots, 4$;
(ii) $zw = 0$ for every $z \in \mathfrak{g}_1$;
(iii) w does not lie in V.

We observe that the homogeneous components of positive degree of a singular vector are singular vectors. The same holds for its weight components. From now on we will thus assume that a singular vector is a homogeneous weight vector unless otherwise specified. Notice that if condition (i) is satisfied then condition (ii) holds if $x_5 d_{45} w = 0$ since $x_5 d_{45}$ is a lowest weight vector in $\mathfrak{g}_1$.

We recall that a finite Verma module $M(V)$ is degenerate if and only if it contains a singular vector [2, Proposition 3.3].

Degenerate Verma modules can be described in terms of morphisms. A morphism $\varphi : M(V) \to M(W)$ can always be associated to an element $\Phi \in U_- \otimes \mathrm{Hom}(V, W)$ as follows: for $u \in U_-$ and $v \in V$ we let

$$\varphi(u \otimes v) = u\Phi(v)$$

where, if $\Phi = \sum_i u_i \otimes \theta_i$ with $u_i \in U_-, \theta_i \in \mathrm{Hom}(V, W)$, we let $\Phi(v) = \sum_i u_i \otimes \theta_i(v)$. We will say that φ (or Φ) is a morphism of degree d if $u_i \in (U_-)_d$ for every i.

The following proposition characterizes morphisms between Verma modules.

Proposition 3.3. [9,13] *Let $\varphi : M(V) \to M(W)$ be the linear map associated with the element $\Phi \in U_- \otimes \mathrm{Hom}(V, W)$. Then φ is a morphism of $\mathfrak{g}$-modules if and only if the following conditions hold:*

(a) $\mathfrak{g}_0.\Phi = 0$;
(b) $X\varphi(v) = 0$ *for every* $X \in \mathfrak{g}_1$ *and for every* $v \in V$.

We observe that, if $M(V)$ is a finite Verma module and condition (a) holds, it is enough to verify condition (b) for an element X generating $\mathfrak{g}_1$ as a $\mathfrak{g}_0$-module and for v a highest weight vector in V.

We recall that a finite Verma module $M(\mu)$ contains a singular vector if and only if there exist a finite Verma module $M(\lambda)$ and a morphism $\varphi : M(\lambda) \to M(\mu)$ of positive degree [2, Proposition 3.5].

We recall the following duality on finite Verma modules which is established in [3] in a much wider generality.

Theorem 3.4. *Let $\varphi : M(\lambda) \to M(\mu)$ be a morphism of $\mathfrak{g}$-modules of degree d. Then there exists a dual morphism $\varphi^* : M(\mu^*) \to M(\lambda^*)$ of the same degree d. Equivalently, if $M(\lambda)$ contains a singular vector of degree d and weight μ, then $M(\mu^*)$ contains a singular vector of degree d and weight λ^*.*

Remark 3.5. Let $\varphi : M(V) \to M(W)$ be a linear map of degree d associated to an element $\Phi \in U_- \otimes \mathrm{Hom}(V, W)$ that satisfies condition (a) of Proposition 3.3. Then there exists a $\mathfrak{g}_0$-morphism $\psi : (U_-)_d^* \to \mathrm{Hom}(V, W)$ such that $\Phi = \sum_i u_i \otimes \psi(u_i^*)$ where $\{u_i, i \in I\}$ is any basis of $(U_-)_d$ and $\{u_i^*, i \in I\}$ is the corresponding dual basis.

Definition 3.6. Let $M(\mu)$ be a finite Verma module and let $\pi : M(\mu) \to U_- \otimes F(\mu)_\mu$ be the natural projection, $F(\mu)_\mu$ being the weight space of $F(\mu)$ of weight μ. Given a singular vector $w \in M(\mu)$, we call $\pi(w)$ the *leading* term of w.

It is shown in [2] that the leading term of a singular vector is non-zero, and therefore a singular vector is uniquely determined by its leading term.

The action of $E(5, 10)$ on a module M restricts to an action of its even part on M. It is therefore natural to take into account the structure of M as an S_5-module also. In order to do this we consider the grading on S_5 given by $\deg x_i = 2$ and $\deg(\partial_i) = -2$ to be consistent with the embedding of S_5 in $E(5, 10)$. The definition of a Verma module for

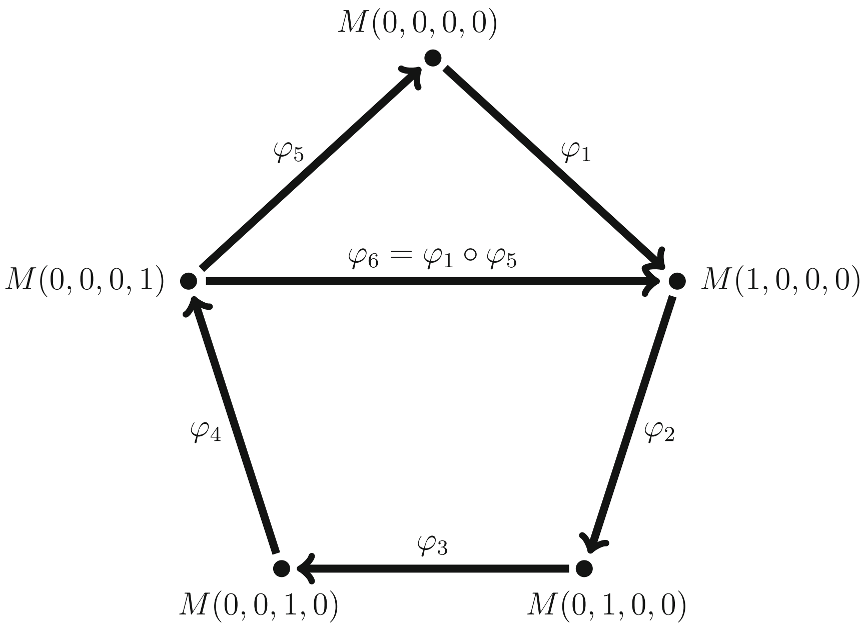

Fig. 1. All non-zero morphisms between finite Verma modules for S_5. External morphisms have degree 2, and the internal one has degree 4. The morphisms $\varphi_1, \ldots, \varphi_6$ correspond to the singular vectors in R1, …, R6 in Theorem 3.7

S_5 is analogous to the one for $E(5, 10)$. Rudakov classified all singular vectors for the infinite-dimensional Lie algebra S_n in [12] and we recall here his results in the special case of S_5.

Theorem 3.7 [12]. *The following is a complete list (up to multiplication by a scalar) of singular vectors w in finite Verma modules $M(\lambda)$ for S_5.*

R1. $\lambda = (1, 0, 0, 0)$, $w = \partial_1 \otimes x_1 + \partial_2 \otimes x_2 + \partial_3 \otimes x_3 + \partial_4 \otimes x_4 + \partial_5 \otimes x_5$;
R2. $\lambda = (0, 1, 0, 0)$, $w = \partial_2 \otimes x_{12} + \partial_3 \otimes x_{13} + \partial_4 \otimes x_{14} + \partial_5 \otimes x_{15}$;
R3. $\lambda = (0, 0, 1, 0)$, $w = \partial_3 \otimes x^*_{45} + \partial_4 \otimes x^*_{53} + \partial_5 \otimes x^*_{34}$;
R4. $\lambda = (0, 0, 0, 1)$, $w = \partial_4 \otimes x^*_5 - \partial_5 \otimes x^*_4$;
R5. $\lambda = (0, 0, 0, 0)$, $w = \partial_5 \otimes 1$;
R6. $\lambda = (1, 0, 0, 0)$, $w = \partial_5(\partial_1 \otimes x_1 + \partial_2 \otimes x_2 + \partial_3 \otimes x_3 + \partial_4 \otimes x_4 + \partial_5 \otimes x_5)$;

Theorem 3.7 provides the diagram of all non-zero morphisms between finite Verma modules for S_5 shown in Fig. 1.

4. A First Bound

Let $\Omega = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}$ and, if $p = \{i, j\} \in \Omega$ with $i < j$, then we let $d_p = d_{ij} = dx_i \wedge dx_j$. In order to avoid cumbersome notation, when no confusion may arise we will denote in this section the subset $\{i, j\}$ simply as ij.

Let V be a finite dimensional $\mathfrak{g}_0$-module. For all $k \geq 0$ we let

$$M_k(V) = \mathbb{C}[\partial] \sum_{j \leq k} \sum_{p_1, \ldots, p_j \in \Omega} \mathbb{C} d_{p_1} \cdots d_{p_j} \otimes V.$$

Note that $M_k(V)$ is not an $E(5, 10)$-submodule of $M(V)$. Nevertheless the following result holds.

Proposition 4.1. *For all $k = 0, 1, \ldots, 10$ the subspace $M_k(V)$ is an S_5-module.*

Proof. It is enough to show that for all $X \in S_5$, $1 \leq j \leq k$, $p_1, \ldots, p_j \in \Omega$, and $v \in V$

$$Xd_{p_1} \cdots d_{p_j} \otimes v \in M_j(V), \tag{1}$$

since $M_j(V) \subseteq M_k(V)$. We also show that

$$[X, d_{p_1}]d_{p_2} \cdots d_{p_j} \otimes v \in M_j(V) \tag{2}$$

and we prove that (1) and (2) hold simultaneously by a double induction on j and $\deg X$.

If $j = 1$ then (2) is trivial and (1) follows from (2).

If $\deg X = -2$ then (1) and (2) are both trivial, so we assume that $j \geq 2$ and $\deg X \geq 0$.

We have

$$Xd_{p_1} \ldots d_{p_j} \otimes v = [X, d_{p_1}]d_{p_2} \cdots d_{p_j} \otimes v + d_{p_1} X d_{p_2} \cdots d_{p_j} \otimes v.$$

The latter summand clearly lies in $M_j(V)$ by induction on j and so (1) will follow from (2). We have

$$[X, d_{p_1}]d_{p_2} \cdots d_{p_j} \otimes v = -d_{p_2}[X, d_{p_1}]d_{p_3} \cdots d_{p_j} \otimes v + [[X, d_{p_1}], d_{p_2}]d_{p_3} \cdots d_{p_j} \otimes v.$$

The former summand lies in $M_j(V)$ by induction on j and the latter by induction on $\deg X$: the result follows. □

By Proposition 4.1 we have a filtration

$$\{0\} = M_{-1}(V) \subseteq \mathbb{C}[\partial] \otimes V = M_0(V) \subseteq M_1(V) \subseteq \cdots \subseteq M_{10}(V) = M(V)$$

of S_5-modules and we let

$$N_k(V) = M_k(V)/M_{k-1}(V)$$

for all $k = 0, \ldots, 10$.

Proposition 4.2. *For all $k = 0, \ldots, 10$ and for any total order $\prec$ on Ω we have*

$$N_k(V) \cong \mathbb{C}[\partial] \otimes \bigoplus_{p_1 \prec \cdots \prec p_k} \mathbb{C} d_{p_1} \cdots d_{p_k} \otimes V$$

as $\mathbb{C}$-vector spaces.

Proof. For all $p_1, \ldots, p_k \in \Omega$ and every permutation σ of the indices $\{1, \ldots, k\}$ we have

$$d_{p_1} d_{p_2} \cdots d_{p_k} - \varepsilon(\sigma) d_{p_{\sigma(1)}} \cdots d_{p_{\sigma(k)}} \in M_{k-1}(V) \tag{3}$$

and so $N_k(V)$ is generated as $\mathbb{C}[\partial]$-module by the elements $d_{p_1} \cdots d_{p_k} \otimes v$ for all $p_1 \prec \cdots \prec p_k$ and all $v \in V$. The result follows by Poincaré–Birkhoff–Witt theorem for $U(\mathfrak{g}_-)$. □

Next we observe that the subspace

$$F_k(V) = \bigoplus_{p_1 \prec \cdots \prec p_k} \mathbb{C} d_{p_1} \cdots d_{p_k} \otimes V$$

of $N_k(V)$ also has a special structure:

Proposition 4.3. *The subspace $F_k(V)$ of $N_k(V)$ is an $\mathfrak{sl}_5$-module annihilated by $(S_5)_{>0}$. The S_5-module $N_k(V)$ is the finite Verma module for S_5 induced by $F_k(V)$, i.e.*

$$N_k(V) = M(F_k(V)).$$

Proof. The subspace $F_k(V)$ of $N_k(V)$ is an $\mathfrak{sl}_5$-module since $\mathfrak{g}_{-1}$ is a $\mathfrak{g}_0$-module, and by the definition of $N_k(V)$. The fact that $F_k(V)$ is annihilated by $(S_5)_{>0}$ follows easily by degree reasons. The second part follows from Proposition 4.2 and the first part. □

This result together with Theorem 3.7 allows us to determine a first bound on the degree of singular vectors for $E(5, 10)$.

Corollary 4.4. *Let $M(V)$ be a finite Verma module for $E(5, 10)$ and $w \in M(V)$ be a singular vector. Then w has degree at most 14.*

Proof. Let k be minimal such that $w \in M_k(V)$. Then w is a fortiori either a highest weight vector in $F_k(V)$ or a singular vector in the S_5-Verma module $N_k(V)$, and as such it has degree at most 4. It follows that w has degree at most $k + 4$, where $k \leq 10$. □

5. Singular Vectors of Degree Greater than 10

The description of singular vectors for S_5 allows us to give a much more precise description of possible singular vectors for $E(5, 10)$ of degree greater than 10. We fix a total order $\prec$ on the set $\Omega = \{\{1, 2\}, \{1, 3\}, \ldots, \{4, 5\}\}$. If $I = \{p_1, \ldots, p_j\} \subseteq \Omega$ with $p_1 \prec \cdots \prec p_j$ we let $d_I^{\prec} = d_{p_1} \cdots d_{p_j}$. We let

$$L_h^{\prec}(V) = \mathbb{C}[\partial] \bigoplus_{I : |I| = h} \mathbb{C} d_I^{\prec} \otimes V.$$

By construction we have

$$M_h(V) = L_h^{\prec}(V) \oplus M_{h-1}(V)$$

for all $h = 0, \ldots, 10$ and in particular

$$M(V) = \bigoplus_{h=0}^{10} L_h^{\prec}(V).$$

Every non-zero vector w in $M(V)$ can be expressed uniquely in the following form:

$$w = w_h^{\prec} + w_{h-1}^{\prec} + \cdots + w_0^{\prec},$$

for some $h = 0, \ldots, 10$, with $w_h^{\prec} \neq 0$ and $w_j^{\prec} \in L_j^{\prec}(V)$ for all $j \leq h$. We say in this case that w has *height* h and we call $w_h^{\prec}$ the *highest term* of w. Note that the height of an element does not depend on the order $\prec$, while its highest term does.

If $M = (m_1, \ldots, m_5) \in \mathbb{N}^5$ we let $\partial^M = \partial_1^{m_1} \cdots \partial_5^{m_5}$ and $|M| = m_1 + \cdots + m_5$. Moreover we let $e_1 = (1, 0, 0, 0, 0)$, $e_2 = (0, 1, 0, 0, 0), \ldots, e_5 = (0, 0, 0, 0, 1)$.

If w is homogeneous of degree d, then the term $w_j^{\prec}$ has the following form

$$w_j^{\prec} = \sum_{I \subseteq \Omega \colon |I|=j} \; \sum_{M \in \mathbb{N}^5 \colon |M| = \frac{d-j}{2}} \partial^M d_I^{\prec} \otimes v_{M,I}, \tag{4}$$

where $v_{M,I} \in V$.

Note that if w is homogeneous of height h, then $w_j^{\prec} = 0$ for all $j \not\equiv h \mod 2$. Observe that, by construction, if w has height h, then

$$w \equiv w_h^{\prec} \mod M_{h-1}(V),$$

and, in particular, w and $w_h^{\prec}$ lie in the same class in $N_h(V)$. Theorem 3.7 provides us the following description of possible singular vectors for $E(5, 10)$.

Corollary 5.1. *Let $w \in M(V)$ be a singular vector of degree d and height h, and let $w_h^{\prec}$ be its highest term. Let $w_j^{\prec}$ be as in (4). Then one of the following applies:*

(i) $d = h$;

(ii) $d = h + 2$ *and there exists* $i \in [5]$ *such that*

$$\sum_{I \colon |I|=h} d_I^{\prec} v_{e_i,I} \neq 0$$

is a highest weight vector for $\mathfrak{sl}_5$ *in* $N_h(V)$ *and*

$$w_h^{\prec} = \sum_{j=i}^{5} \partial_j \sum_{I \colon |I|=h} d_I^{\prec} \otimes v_{e_j,I}$$

with

$$\lambda(w) = \begin{cases} (0,0,0,0) & \text{if } i = 1; \\ (1,0,0,0) & \text{if } i = 2; \\ (0,1,0,0) & \text{if } i = 3; \\ (0,0,1,0) & \text{if } i = 4; \\ (0,0,0,1) & \text{if } i = 5; \end{cases} \quad \text{and} \quad \lambda\Big(\sum_{I \colon |I|=h} d_I^{\prec} v_{e_i,I}\Big) = \begin{cases} (1,0,0,0) & \text{if } i = 1; \\ (0,1,0,0) & \text{if } i = 2; \\ (0,0,1,0) & \text{if } i = 3; \\ (0,0,0,1) & \text{if } i = 4; \\ (0,0,0,0) & \text{if } i = 5; \end{cases}$$

(iii) $d = h + 4$,

$$\sum_{I \colon |I|=h} d_I v_{e_1+e_5,I} \neq 0$$

is a highest weight vector for $\mathfrak{sl}_5$ *in* $N_h(V)$ *and*

$$w_h^{\prec} = \partial_5 \sum_{j=1}^{5} \partial_j \sum_{I \colon |I|=h} d_I^{\prec} \otimes v_{e_j+e_5,I}$$

with $\lambda(w) = (0, 0, 0, 1)$ *and* $\lambda\big(\sum_{I \colon |I|=h} d_I v_{e_1+e_5,I}\big) = (1, 0, 0, 0)$.

Proof. This is a straightforward consequence of Theorem 3.7. We know that $N_h(V)$ is a Verma module for S_5 and $w_h^{\prec}$ is annihilated by $(S_5)_{>0}$ and by $x_i\partial_j$ for all $i < j$. In particular, if $d \neq h$, we have that the class of $w_h^{\prec}$ in $N_h(V)$ is a genuine singular vector for S_5: the classification of singular vectors in Theorem 3.7 then completes the proof. Note that if $d = h$, then the class of $w_h^{\prec}$ in $N_h(V)$ is actually a highest weight vector in $F_h(V)$, i.e. the $\mathfrak{sl}_5$-module we are inducing from. □

In this section we classify all possible singular vectors with degree strictly bigger than height, i.e. we treat the cases $d = h + 2$ and $d = h + 4$ in Corollary 5.1, and, in particular, we find all singular vectors of degree greater than 10. We fix the lexicographic order on $\Omega = \{\{1, 2\}, \{1, 3\}, \dots, \{4, 5\}\}$, i.e. we set

$$\{1, 2\} \prec \{1, 3\} \prec \{1, 4\} \prec \{1, 5\} \prec \{2, 3\} \prec \{2, 4\} \prec \{2, 5\} \prec \{3, 4\} \prec \{3, 5\} \prec \{4, 5\}$$

and we simply write $L_h(V)$ instead of $L_h^{\prec}(V)$, w_h instead of $w_h^{\prec}$ and d_I instead of $d_I^{\prec}$.

Remark 5.2. The following inclusions are immediate from the definition of the action of $\mathfrak{g}_0$ and $\mathfrak{g}_1$ on $M(V)$:

$$\mathfrak{g}_0.L_h(V) \subseteq L_h(V) \oplus L_{h-2}(V) \tag{5}$$
$$\mathfrak{g}_1.L_h(V) \subseteq L_{h+1}(V) \oplus L_{h-1}(V) \oplus L_{h-3}(V). \tag{6}$$

Due to (5), for $X \in \mathfrak{g}_0$ and $w \in L_h(V)$, we adopt the following notation:

$$Xw = X^0 w + X^{-2} w \tag{7}$$

with $X^0 w \in L_h(V)$ and $X^{-2} w \in L_{h-2}(V)$. Similarly, due to (6), for $X \in \mathfrak{g}_1$ and $w \in L_h(V)$ we write:

$$Xw = X^1 w + X^{-1} w + X^{-3} w \tag{8}$$

with $X^1 w \in L_{h+1}(V)$, $X^{-1} w \in L_{h-1}(V)$ and $X^{-3} w \in L_{h-3}(V)$. The following simple observation will be crucial in the sequel.

Remark 5.3. Let $w \in M(V)$ be a singular vector of height h. Then for all $X \in \mathfrak{g}_1$ we have

$$X^1 w_h = 0 \tag{9}$$
$$X^{-1} w_h + X^1 w_{h-2} = 0. \tag{10}$$

Moreover, for all $i = 1, 2, 3, 4$ and $E_i = x_i \partial_{i+1} \in \mathfrak{g}_0$ we have

$$E_i^0 w_h = 0 \tag{11}$$
$$E_i^{-2} w_h + E_i^0 w_{h-2} = 0. \tag{12}$$

It will be convenient to rephrase (9) in the following equivalent way: for all $X \in \mathfrak{g}_1$ we have

$$X w_h \equiv 0 \mod M_h(V). \tag{13}$$

Proposition 5.4. *Let w be a singular vector in $M(F)$ with height h and degree d with $d = h + 4$. Then $d = 14$ and $F = F(1, 0, 0, 0)$ is the standard representation of $\mathfrak{g}_0$.*

Proof. By Corollary 5.1 we have

$$w_h = \partial_5 \sum_{i,I} \partial_i d_I \otimes v_{i,I}.$$

By applying (13) with $X = x_k d_{kj}$ and all $k \neq j$, we deduce that if $v_{i,I} \neq 0$ then I must contain all pairs containing i and all pairs containing 5, and, in particular, w has height at least 7 since $v_{1,I} \neq 0$ for some I. If we apply (13) with $X = x_1 d_{23} + x_2 d_{13}$, we obtain

$$-\partial_5 d_{23} \sum_I d_I v_{1,I} - \partial_5 d_{13} \sum_I d_I v_{2,I} \equiv 0 \mod M_h(V).$$

We deduce that, if $v_{1,I} \neq 0$ and I does not contain 23, then it necessarily contains 24, since all terms in the second summand do, and one can similarly show that I must contain 34 using $X = x_1 d_{23} - x_3 d_{12}$. Permuting the roles of 2, 3, 4, this argument shows that I must contain at least two of the three pairs 23, 24, 34 and hence w has height at least 9. A singular vector of height 9 and degree 13 produces a morphism $\varphi : M(0, 0, 0, 1) \to M(\lambda)$ for some λ, by Corollary 5.1. The dual morphism $\varphi^* : M(\lambda^*) \to M(1, 0, 0, 0)$ is also a morphism of degree 13 and so we necessarily have $\lambda = (1, 0, 0, 0)$. Therefore, if $v_{1,I} \neq 0$ then the weight of d_I must be $(0, 0, 0, 0)$, but one can easily check that there are no I with $|I| = 9$ such that $\lambda(d_I) = (0, 0, 0, 0)$ (see [2, §6] for an easy way to compute the weight of the d_I's).

If w has height 10 and degree 14, then by Corollary 5.1, and an argument analogous to the previous one shows that $w \in M(1, 0, 0, 0)$. □

Now we can rule out the only left case with $d = h + 4$.

Proposition 5.5. *Let w be a singular vector of degree d and height h. Then $d < h + 4$.*

Proof. By Propositions 5.1 and 5.4 we can assume that $d = 14$, $h = 10$ and

$$w_{10} = \partial_5(\alpha_1 \partial_1 d_\Omega \otimes x_1 + \cdots + \alpha_5 \partial_5 d_\Omega \otimes x_5),$$

for some $\alpha_1, \ldots, \alpha_5 \in \mathbb{C}$ with $\alpha_1 \neq 0$, and that w_8 has the following form:

$$w_8 = \sum_{I:|I|=8} \sum_{M:|M|=3} \sum_{k=1}^{5} \alpha_{M,I,k} \partial^M d_I \otimes x_k,$$

for some $\alpha_{M,I,k} \in \mathbb{C}$. If we expand

$$x_5 d_{45}(w_{10} + w_8) = \sum \beta_{M,I,k} \partial^M d_I \otimes x_k,$$

by (10) we obtain the relation

$$\beta_{(1,0,0,1,0),\Omega\setminus\{23\},4} = -\alpha_4 - \alpha_{(1,0,0,1,1),\Omega\setminus\{23,45\},4} = 0.$$

Similarly, if we expand

$$x_4 d_{45}(w_{10} + w_8) = \sum \gamma_{M,I,k} \partial^M d_I \otimes x_k,$$

by (10) we obtain te relation

$$\gamma_{(1,0,0,0,1),\Omega\setminus\{23\},4} = \alpha_1 - \alpha_4 - \alpha_{(1,0,0,1,1),\Omega\setminus\{23,45\},4} = 0,$$

and hence $\alpha_1 = 0$, a contradiction. □

Next target is to deal with the case $d = h + 2$ in Corollary 5.1.

Proposition 5.6. *Let w be a singular vector in $M(V)$ of degree d and height h, with $d = h + 2$. Then $h \geq 8$. Moreover, if $h = 8$ and*

$$w_8 = \sum_{j=i}^{5} \partial_j \sum_{I:|I|=8} d_I \otimes v_{j,I}$$

as in Corollary 5.1, then $v_{j,I} \neq 0$ only if $\lambda(\partial_j d_I) = (0, 0, 0, 0)$.

Proof. By (13) we have for all $k \neq l$

$$x_k d_{kl}\, w_h \equiv -d_{kl} \sum_I d_I \otimes v_{k,I} \equiv 0 \mod M_h(V).$$

This implies that, if $v_{k,I} \neq 0$, then $\{k,l\} \in I$ for all $l \neq k$. In particular, we immediately deduce that $h \geq 4$ and

$$w_h \equiv \sum_j \partial_j d_{1j} \cdots \hat{d_{jj}} \cdots d_{5j} \sum_{I_j} d_{I_j} \otimes v_{j,I_j} \mod M_{h-1}(V),$$

where I_1 runs through all subsets of $\{\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}\}$ of cardinality $h - 4$, and similarly for $I_2, \ldots, I_5$, where the vectors v_{j,I_j} have been reindexed. Now let k, l, m be distinct integers in [5] and use again (13) with the element $X = x_k d_{lm} + x_l d_{km}$. We obtain:

$$d_{lm} d_{1k} \cdots \hat{d_{kk}} \cdots d_{5k} \sum_{I_k} d_{I_k} \otimes v_{k,I_k} + d_{km} d_{1l} \cdots \hat{d_{ll}} \cdots d_{5l} \sum_{I_l} d_{I_l} \otimes v_{l,I_l} \equiv 0 \mod M_h(V)$$

Again, by (3), this implies that, if $v_{k,I_k} \neq 0$, then I_k contains $\{l,m\}$ (in which case the corresponding summand is zero), or it contains both pairs $\{l,r\}$ and $\{l,s\}$, where $\{k,l,m,r,s\} = [5]$. It follows that I_k must contain at least two pairs containing l (since if it does not contain one such pair it must contain the other two). This implies that I_k contains at least four pairs, and this completes the proof that $h \geq 8$.

If $h = 8$, by the previous argument, the two missing pairs in I_k must contain the four elements distinct from k exactly once, and so the weight of $\partial_k d_{1k} \cdots \hat{d}_{kk} \cdots d_{5k} d_{I_k}$ is $(0, 0, 0, 0)$. □

We can now tackle the case of singular vectors of height 8 and degree 10.

Proposition 5.7. *There are no singular vectors $M(V)$ of height 8 and degree 10.*

Proof. Assume by contradiction that w is a singular vector of height 8 and degree 10. For distinct $i, j, k, l \in [5]$, with $i < j$ and $k < l$ we let $d^{\vee}_{jk,lm} = d_{\Omega \setminus \{jk,lm\}}$. For example $d^{\vee}_{14,25} = d_{12} d_{13} d_{15} d_{23} d_{24} d_{34} d_{35} d_{45}$. By Proposition 5.6 we have that w_8 can be expressed in the following way

$$w_8 = \sum_{i,j,k,l,m} \partial_i d^{\vee}_{jk,lm} \otimes v_{i,jk,lm}$$

where the sum is over all distinct $i, j, k, l, m \in [5]$ such that $j < k, l$, and $l < m$ (so we have exactly 15 summands). We also adopt the convention $v_{i,lm,jk} = v_{i,jk,lm}$

Fig. 2. The lexicographic order

for notational convenience. By construction, we immediately have $(x_i\, d_{ik})^1 w_8 = 0$ for all $i \neq k$. We will therefore consider elements in $\mathfrak{g}_1$ of the form $x_i d_{jk} + x_j d_{ik}$ and $x_i d_{jk} - x_k d_{ij}$ (for all $i < j < k$), which will allow us to deduce that $w_8 = 0$. To perform this computation efficiently we need the following notation.

Let

$$\eta_{ij} = \begin{cases} 1 & \text{if } i + j = 5 \\ 0 & \text{otherwise.} \end{cases}$$

The reason for introducing this function is the following: let $d(ij, kl)$ be the distance between the pairs ij and kl in the lexicographic order (i.e. in the graph represented in Fig. 2; then one can easily check that for all $i < j < k$ we have

$$(-1)^{d(ik, jk)} = (-1)^{\eta_{ij}+1} \tag{14}$$

and

$$(-1)^{d(ij, jk)} = (-1)^{\eta_{ij}+k+j+1}. \tag{15}$$

Let i, j, k, l, m be distinct such that $i < j < k$ and $l < m$. We have

$$(x_i d_{jk} + x_j d_{ik})^1 w_8 \equiv -d_{jk} d^{\vee}_{jk,lm} \otimes v_{i,jk,lm} - d_{ik} d^{\vee}_{ik,lm} \otimes v_{j,ik,lm} \mod M_8(V)$$

By (3) and (14) we have

$$v_{i,jk,lm} = \begin{cases} (-1)^{\eta_{ij}+1} v_{j,ik,lm} & \text{if } i < l < j \\ (-1)^{\eta_{ij}} v_{j,ik,lm} & \text{otherwise.} \end{cases} \tag{16}$$

Similarly, applying $x_i d_{jk} - x_k d_{ij}$ we obtain

$$(x_i d_{jk} - x_k d_{ij})^1 w_8 \equiv -d_{jk} d^{\vee}_{jk,lm} \otimes v_{i,jk,lm} + d_{ij} d^{\vee}_{ij,lm} \otimes v_{k,ij,lm} \mod M_8(V)$$

and by (3) and (15) we have

$$v_{i,jk,lm} = \begin{cases} (-1)^{\eta_{ij}+k+j} v_{k,ij,lm} & \text{if } i < l < j \\ (-1)^{\eta_{ij}+k+j+1} v_{k,ij,lm} & \text{otherwise.} \end{cases} \tag{17}$$

By repeated application of Eq. (16) we obtain

- $v_{1,23,45} = v_{2,13,45} = v_{4,15,23}$
- $v_{1,24,35} = v_{2,14,35} = -v_{3,15,24}$
- $v_{1,25,34} = v_{2,15,34} = -v_{3,14,25}$
- $v_{2,13,45} = v_{4,13,25}$
- $v_{2,14,35} = -v_{3,14,25}$
- $v_{2,15,34} = -v_{3,15,24}$
- $v_{3,12,45} = v_{4,12,35}$

and by repeated application of Eq. (17) we obtain

- $v_{1,23,45} = v_{3,12,45} = v_{5,14,23}$
- $v_{1,24,35} = -v_{4,12,35} = v_{5,13,24}$
- $v_{1,25,34} = v_{5,12,34} = -v_{4,13,25}$
- $v_{2,13,45} = v_{5,13,24}$
- $v_{2,14,35} = v_{5,14,23}$
- $v_{2,15,34} = -v_{4,15,23}$
- $v_{3,12,45} = v_{5,12,34}$.

All these equations together imply that all $v_{i,jk,lm}$ vanish. □

We now consider the case of a singular vector w of height 9 and degree 11. In this case, as in the proof of Proposition 5.6, we can immediately deduce that w_9 must have the following form

$$w_9 = \sum_{i,j,k} \partial_i d^\vee_{jk} \otimes v_{i,jk}, \tag{18}$$

where the sum is over all distinct i, j, k with $j < k$ (a total of 30 summands) and $d^\vee_{jk} = d_{\Omega \setminus \{j,k\}}$. As in the case of height 8, we can now proceed by applying all elements in $\mathfrak{g}_1$ of the form $x_i d_{jk} + x_j d_{ik}$ and $x_i d_{jk} - x_k d_{ij}$.

Lemma 5.8. *If w is a singular vector of height 9 and degree 11 with highest term w_9 as in* (18)*, then*

- $v_{1,23} = v_{2,13} = v_{3,12}$;
- $v_{1,24} = v_{2,14} = -v_{4,12}$;
- $v_{1,25} = v_{2,15} = v_{5,12}$;
- $v_{1,34} = v_{3,14} = v_{4,13}$;
- $v_{1,35} = v_{3,15} = -v_{5,13}$;
- $v_{1,45} = -v_{4,15} = -v_{5,14}$;
- $v_{2,34} = -v_{3,24} = -v_{4,23}$;
- $v_{2,35} = -v_{3,25} = v_{5,23}$;
- $v_{2,45} = v_{4,25} = v_{5,24}$;
- $v_{3,45} = v_{4,35} = v_{5,34}$.

Proof. All equalities are obtained using (3) and (13) applying elements $x_i d_{jk} + x_j d_{ik}$ and $x_i d_{jk} - x_k d_{ij}$. For example we have

$$(x_2 d_{35} + x_3 d_{25}) w_9 \equiv -d_{35} d^\vee_{35} \otimes v_{2,35} - d_{25} d^\vee_{25} \otimes v_{3,25} \equiv d_\Omega(-v_{2,35} - v_{3,25}) \mod M_8(V),$$

hence $v_{2,35} = -v_{3,25}$. All other equalities can be obtained similarly. □

Thanks to Lemma 5.8 the highest term of the singular vector assumes the following form:

$$\begin{aligned} w_9 = {} & \partial_1 (d^\vee_{23} \otimes u_1 - d^\vee_{24} \otimes u_2 + d^\vee_{25} \otimes u_3 - d^\vee_{34} \otimes u_4 + d^\vee_{35} \otimes u_5 - d^\vee_{45} \otimes u_6) \\ & + \partial_2 (d^\vee_{13} \otimes u_1 - d^\vee_{14} \otimes u_2 + d^\vee_{15} \otimes u_3 - d^\vee_{34} \otimes u_7 + d^\vee_{35} \otimes u_8 - d^\vee_{45} \otimes u_9) \\ & + \partial_3 (d^\vee_{12} \otimes u_1 - d^\vee_{14} \otimes u_4 + d^\vee_{15} \otimes u_5 + d^\vee_{24} \otimes u_7 - d^\vee_{25} \otimes u_8 - d^\vee_{45} \otimes u_{10}) \\ & + \partial_4 (d^\vee_{12} \otimes u_2 - d^\vee_{13} \otimes u_4 + d^\vee_{15} \otimes u_6 + d^\vee_{23} \otimes u_7 - d^\vee_{25} \otimes u_9 - d^\vee_{35} \otimes u_{10}) \\ & + \partial_5 (d^\vee_{12} \otimes u_3 - d^\vee_{13} \otimes u_5 + d^\vee_{14} \otimes u_6 + d^\vee_{23} \otimes u_8 - d^\vee_{24} \otimes u_9 - d^\vee_{34} \otimes u_{10}). \end{aligned} \tag{19}$$

for suitable elements $u_1, \ldots, u_{10} \in V$.

Lemma 5.9. *Let $E_i = x_i\partial_{i+1} \in \mathfrak{g}_0$ and w be a singular vector of height 9 and degree 11 with w_9 as in (19) above. Then*

- *E_1 annihilates $u_1, u_2, u_3, u_4, u_5, u_6, u_{10}$, $E_1.u_7 = u_4$, $E_1.u_8 = u_5$, $E_1.u_9 = u_6$.*
- *E_2 annihilates $u_1, u_2, u_3, u_6, u_7, u_8, u_9$, $E_1.u_4 = u_2$, $E_1.u_5 = u_3$, $E_1u_{10} = u_9$.*
- *E_3 annihilates $u_1, u_3, u_4, u_5, u_7, u_8, u_{10}$, $E_1.u_2 = u_1$, $E_1.u_6 = u_5$, $E_1u_9 = u_8$.*
- *E_4 annihilates $u_1, u_2, u_4, u_6, u_7, u_9, u_{10}$, $E_1.u_3 = u_2$, $E_1.u_5 = u_4$, $E_1u_8 = u_7$.*

Proof. Recall the definition of E_1^0 from (7). By (11) we have

$$\begin{aligned}
0 &= E_1^0 w_9 \\
&= \partial_2(d_{34}^\vee \otimes u_4 - d_{35}^\vee \otimes u_5 + d_{45}^\vee \otimes u_6) + \partial_3(-d_{24}^\vee \otimes u_4 + d_{25}^\vee \otimes u_5) \\
&\quad + \partial_4(-d_{23}^\vee \otimes u_4 + \partial_4 d_{25}^\vee \otimes u_6) + \partial_5(-d_{23}^\vee \otimes u_5 + d_{24}^\vee \otimes u_6) \\
&\quad + \partial_1(d_{23}^\vee \otimes E_1.u_1 - d_{24}^\vee \otimes E_1.u_2 + d_{25}^\vee \otimes E_1.u_3 - d_{34}^\vee \otimes E_1.u_4 \\
&\quad + d_{35}^\vee \otimes E_1.u_5 - d_{45}^\vee \otimes E_1.u_6) \\
&\quad + \partial_2(d_{13}^\vee \otimes E_1.u_1 - d_{14}^\vee \otimes E_1.u_2 + d_{15}^\vee \otimes E_1.u_3 - d_{34}^\vee \otimes E_1.u_7 \\
&\quad + d_{35}^\vee \otimes E_1.u_8 - d_{45}^\vee \otimes E_1.u_9) \\
&\quad + \partial_3(d_{12}^\vee \otimes E_1.u_1 - d_{14}^\vee \otimes E_1.u_4 + d_{15}^\vee \otimes E_1.u_5 + d_{24}^\vee \otimes E_1.u_7 \\
&\quad - d_{25}^\vee \otimes E_1.u_8 - d_{45}^\vee \otimes E_1.u_{10}) \\
&\quad + \partial_4(d_{12}^\vee \otimes E_1.u_2 - d_{13}^\vee \otimes E_1.u_4 + d_{15}^\vee \otimes E_1.u_6 + d_{23}^\vee \otimes E_1.u_7 \\
&\quad - d_{25}^\vee \otimes e_1.u_9 - d_{35}^\vee \otimes E_1.u_{10}) \\
&\quad + \partial_5(d_{12}^\vee \otimes E_1.u_3 - d_{13}^\vee \otimes E_1.u_5 + d_{14}^\vee \otimes E_1.u_6 + d_{23}^\vee \otimes E_1.u_8 \\
&\quad - d_{24}^\vee \otimes E_1.u_9 - d_{34}^\vee \otimes E_1.u_{10}).
\end{aligned}$$

The result for E_1 follows. The other statements are obtained similarly. □

Lemma 5.9 is depicted in Fig. 3, where an arrow from u_i to u_k labelled E_j means $E_j.u_i = u_k$ and the absence of an arrow labelled E_j coming out from u_i means $E_j.u_i = 0$.

Proposition 5.10. *Let $w \in M(V)$ be a singular vector of height 9 and degree 11 and let w_9 be as in (19). Then $u_1 = \cdots = u_6 = 0$, u_7 is a highest weight vector, $V = F(0, 0, 0, 1)$ and $\lambda(w) = (1, 0, 0, 0)$.*

Proof. We first show that one of the following applies:

(1) u_1 is a highest weight vector, $V = F(0, 0, 1, 0)$ and $\lambda(w) = (0, 0, 0, 0)$;
(2) $u_1 = \cdots = u_6 = 0$, u_7 is a highest weight vector, $V = F(0, 0, 0, 1)$ and $\lambda(w) = (1, 0, 0, 0)$.
(3) $u_1 = u_2 = \cdots = u_9 = 0$, u_{10} is a highest weight vector, $V = F(0, 0, 0, 0)$ and $\lambda(w) = (0, 1, 0, 0)$.

If $u_1 \neq 0$ then it is a highest weight vector in V by Lemma 5.9, and by Corollary 5.1 we necessarily have $\lambda(w) = (0, 0, 0, 0)$ and hence $\lambda(u_1) = (0, 0, 1, 0)$. If $u_1 = 0$ and $u_2 \neq 0$, then u_2 is a highest weight vector by Lemma 5.9, and by Corollary 5.1 we necessarily have $\lambda(w) = (0, 0, 0, 0)$ and hence $\lambda(u_2) = (0, 1, -1, 1)$, which is impossible since it is not a dominant weight. Similarly, if $u_1, u_2 = 0$ and $u_3 \neq 0$ then u_3 would be a highest weight vector of weight $(0, 1, 0, -1)$.

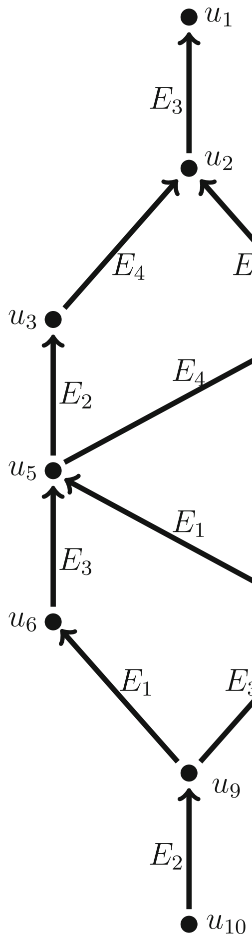

Fig. 3. The action of the E_i's on the elements u_j's

If $u_1 = u_2 = u_3 = 0$ and $u_4 \neq 0$ then u_4 would be a highest weight vector of weight $(1, -1, 0, 1)$.

If $u_1 = u_2 = u_3 = u_4 = 0$ then then u_5 would be a highest weight vector of weight $(1, -1, 1, -1)$.

If $u_1 = \cdots = u_5 = 0$ then u_6 would be a highest weight vector of weight $(1, 0, -1, 0)$.

If $u_1 = \cdots = u_6 = 0$ and $u_7 \neq 0$ then by Corollary 5.1 we have $\lambda(w) = (1, 0, 0, 0)$ and so $\lambda(u_7) = (0, 0, 0, 1)$.

If $u_1 = \cdots = u_7 = 0$ and $u_8 \neq 0$, then $\lambda(w) = (1, 0, 0, 0)$ by Corollary 5.1 and hence $\lambda(u_8) = (0, 0, 1, -1)$.

If $u_1 = \cdots = u_8 = 0$ and $u_9 \neq 0$, then $\lambda(w) = (1, 0, 0, 0)$ by Corollary 5.1 and hence $\lambda(u_9) = (0, 1, -1, 0)$.

Finally, if $u_1 = \cdots = u_9 = 0$ then $u_{10} \neq 0$ is a highest weight vector, $\lambda(w) = (0, 1, 0, 0)$ by Corollary 5.1 and so $\lambda(u_{10}) = (0, 0, 0, 0)$.

Now we show that cases (1) and (3) can not occur. Observe that by Theorem 3.4 it is enough to show that case (3) does not occur. In this case we have:

$$w_9 = (\partial_3 d^{\vee}_{45} + \partial_4 d^{\vee}_{35} + \partial_5 d^{\vee}_{34}) \otimes u,$$

where u is a generator of the trivial $\mathfrak{g}_0$-module. By construction w_9 satisfies (9) for all $X \in \mathfrak{g}_1$ and (11) for all i. We will therefore take into account also (10) and (12) showing that there exists no w_7 which satisfies these equations. We start computing

$$(x_5 d_{45})^{-1} w_9 \equiv (-\partial_1 d^{\vee}_{23,34} - \partial_2 d^{\vee}_{13,34} + \partial_3 d^{\vee}_{13,24} + \partial_4 d^{\vee}_{13,23}) \otimes u \mod M_7(V).$$

We have

$$w_7 = \sum_{i \leq j} \partial_i \partial_j \sum_{p_1 \prec p_2 \prec p_3} d^{\vee}_{p_1, p_2, p_3} \otimes v_{i,j,\{p_1,p_2,p_3\}}, \tag{20}$$

and

$$(x_5 d_{45})^1 w_7 \equiv \sum_i (1 + \delta_{i,5}) \partial_i \sum_{p_1 \prec p_2 \prec \{4,5\}} d^{\vee}_{p_1, p_2} d_{45} \otimes v_{i,5,\{p_1,p_2,45\}} \mod M_7(V).$$

We deduce in particular that $v_{4,5,\{13,23,45\}} = -u \neq 0$ by (10).

Next observe that $x_4 \partial_5 w_9 = 0$ and

$$(x_4 \partial_5)_0 \partial_4 \partial_5 d^{\vee}_{13,23,45} \otimes u = -\partial_5^2 d^{\vee}_{13,23,45} \otimes u \mod M_6(V),$$

and no other term of w_7 in (20) can "produce" a summand $\partial_5^2 d^{\vee}_{13,23,45} \otimes u$ by applying $x_4 \partial_5$. This would imply $v_{4,5,\{13,23,45\}} = 0$ by (12), a contradiction. □

Proposition 5.10 leads us to the following surprising discovery.

Theorem 5.11. *The following vector is a (unique up to multiplication by a scalar) singular vector in* $M(0, 0, 0, 1)$ *of degree 11, height 9, and weight* $(1, 0, 0, 0)$*:*

$$\begin{aligned} w[11] = d_{12} d_{13} d_{14} d_{15} \Big(& - \partial_2 d_{23} d_{24} d_{25} d_{35} d_{45} \otimes x_5^* - \partial_2 d_{23} d_{24} d_{25} d_{34} d_{45} \otimes x_4^* \\ & - \partial_2 d_{23} d_{24} d_{25} d_{34} d_{35} \otimes x_3^* + \partial_3 d_{23} d_{25} d_{34} d_{35} d_{45} \otimes x_5^* + \partial_3 d_{23} d_{24} d_{34} d_{35} d_{45} \otimes x_4^* \\ & + \partial_3 d_{23} d_{24} d_{25} d_{34} d_{35} \otimes x_2^* + \partial_4 d_{24} d_{25} d_{34} d_{35} d_{45} \otimes x_5^* - \partial_4 d_{23} d_{24} d_{34} d_{35} d_{45} \otimes x_3^* \\ & + \partial_4 d_{23} d_{24} d_{25} d_{34} d_{45} \otimes x_2^* - \partial_5 d_{24} d_{25} d_{34} d_{35} d_{45} \otimes x_4^* - \partial_5 d_{23} d_{25} d_{34} d_{35} d_{45} \otimes x_3^* \\ & + \partial_5 d_{23} d_{24} d_{25} d_{35} d_{45} \otimes x_2^* - \partial_1 \partial_2 d_{23} d_{24} d_{25} \otimes x_2^* + \partial_2^2 d_{23} d_{24} d_{25} \otimes x_1^* + \partial_1 \partial_3 d_{23} d_{25} d_{34} \otimes x_2^* \\ & - \partial_2 \partial_3 d_{23} d_{25} d_{34} \otimes x_1^* + \partial_1 \partial_4 d_{24} d_{25} d_{34} \otimes x_2^* - \partial_2 \partial_4 d_{24} d_{25} d_{34} \otimes x_1^* - \partial_1 \partial_3 d_{23} d_{24} d_{35} \otimes x_2^* \\ & + \partial_2 \partial_3 d_{23} d_{24} d_{35} \otimes x_1^* + \partial_1 \partial_5 d_{24} d_{25} d_{35} \otimes x_2^* - \partial_2 \partial_5 d_{24} d_{25} d_{35} \otimes x_1^* - \partial_1 \partial_3 d_{23} d_{34} d_{35} \otimes x_3^* \\ & + \partial_3^2 d_{23} d_{34} d_{35} \otimes x_1^* - \partial_1 \partial_4 d_{24} d_{34} d_{35} \otimes x_3^* + \partial_3 \partial_4 d_{24} d_{34} d_{35} \otimes x_1^* - \partial_1 \partial_5 d_{25} d_{34} d_{35} \otimes x_3^* \\ & + \partial_3 \partial_5 d_{25} d_{34} d_{35} \otimes x_1^* - \partial_1 \partial_4 d_{23} d_{24} d_{45} \otimes x_2^* + \partial_2 \partial_4 d_{23} d_{24} d_{45} \otimes x_1^* - \partial_1 \partial_5 d_{23} d_{25} d_{45} \otimes x_2^* \\ & + \partial_2 \partial_5 d_{23} d_{25} d_{45} \otimes x_1^* - \partial_1 \partial_3 d_{23} d_{34} d_{45} \otimes x_4^* + \partial_3 \partial_4 d_{23} d_{34} d_{45} \otimes x_1^* - \partial_1 \partial_4 d_{24} d_{34} d_{45} \otimes x_4^* \\ & + \partial_4^2 d_{24} d_{34} d_{45} \otimes x_1^* - \partial_1 \partial_5 d_{25} d_{34} d_{45} \otimes x_4^* + \partial_4 \partial_5 d_{25} d_{34} d_{45} \otimes x_1^* - \partial_1 \partial_3 d_{23} d_{35} d_{45} \otimes x_5^* \\ & + \partial_3 \partial_5 d_{23} d_{35} d_{45} \otimes x_1^* - \partial_1 \partial_4 d_{24} d_{35} d_{45} \otimes x_5^* + \partial_4 \partial_5 d_{24} d_{35} d_{45} \otimes x_1^* - \partial_1 \partial_5 d_{25} d_{35} d_{45} \otimes x_5^* \\ & + \partial_5^2 d_{25} d_{35} d_{45} \otimes x_1^* - \partial_1^2 \partial_3 d_{23} \otimes x_2^* + \partial_1 \partial_2 \partial_3 d_{23} \otimes x_1^* - \partial_1^2 \partial_4 d_{24} \otimes x_2^* \\ & + \partial_1 \partial_2 \partial_4 d_{24} \otimes x_1^* - \partial_1^2 \partial_5 d_{25} \otimes x_2^* + \partial_1 \partial_2 \partial_5 d_{25} \otimes x_1^* \Big). \end{aligned}$$

Proof. We prefer to omit the long but elementary computations that show that this is indeed a singular vector. Its uniqueness follows from the fact that the term w_9 is determined up to a scalar by Lemma 5.9 and Proposition 5.10. So if w' is another singular vector with $w_9 = w'_9$ then $w - w'$ would be a singular vector of degree 11 with height at most 7 and this would contradict Proposition 5.5. □

The last possible case with $d > h$ is ruled out by the following result.

Proposition 5.12. *There are no singular vectors of height 10 and degree 12.*

Proof. By hypothesis we are in one of the five cases in (ii) of Corollary 5.1. Suppose there exists a singular vector w of height 10 and degree 12 in $M(1, 0, 0, 0)$ with weight $(0, 0, 0, 0)$, i.e., by Theorem 3.7,

$$w = \sum_{i=1}^{5} \partial_i d_\Omega \otimes x_i.$$

Then there exists a morphism φ of $E(5, 10)$-modules,

$$\varphi : M(0, 0, 0, 0) \to M(1, 0, 0, 0).$$

By duality (see Theorem 3.4), there exists a morphism

$$\varphi^* : M(0, 0, 0, 1) \to M(0, 0, 0, 0),$$

i.e., a singular vector $\bar{w}$ of degree 12, of weight $(0, 0, 0, 1)$ in $M(0, 0, 0, 0)$. Then, by Theorem 3.7 and Proposition 5.5, we have: $\bar{w}_{10} = \partial_5 d_\Omega \otimes 1$ with 1 the highest weight vector in $F(0, 0, 0, 0)$. Let

$$\bar{w}_8 = \sum_{i,j,k,l} \alpha_{ij} \partial_i \partial_j d^\vee_{k5,\ell 5} \otimes 1 + \sum_{i,j,k,l,t} \partial_i \partial_5 (\beta_{i,jk} d^\vee_{jk,\ell t} \otimes 1 + \beta_{i,j\ell} d^\vee_{jl,kt} \otimes 1 + \beta_{i,jt} d^\vee_{jt,k\ell} \otimes 1)$$

for some $\alpha_{ij}, \beta_{i,rs} \in \mathbb{C}$, where the first sum is over all $\{i, j, k, l\} = [4]$ with $i < j$ and $k < l$, and the second sum is over all $\{i, j, k, l, t\} = [5]$ with $j < k < l < t$. We apply condition (10) with $h = 10$, using the following elements X in $\mathfrak{g}_1$:

i) $X = x_5 d_{45}$ hence getting $\beta_{2,45} = -1 = \beta_{3,45}$;
ii) $X = x_5 d_{13} + x_3 d_{15}$ hence getting $\alpha_{23} = 1 = -\beta_{2,45}$;
iii) $X = x_5 d_{12} - x_1 d_{25}$ hence getting $\alpha_{13} = -1 = \beta_{3,45}$;
iv) $X = x_1 d_{25} + x_2 d_{15}$ hence getting $\alpha_{13} = \alpha_{23}$.

These conditions lead to a contradiction, we therefore conclude that there is no singular vector of degree 12 and height 10 as in R1 and R5, of Theorem 3.7.

Now assume that w is a singular vector as in R2, i.e., by Corollary 5.1, $w = \sum_{i=2}^{5} \partial_i d_\Omega \otimes x_{1i}$, i.e., that there exists a morphism φ, of degree 12, of $E(5, 10)$-modules:

$$\varphi : M(1, 0, 0, 0) \to M(0, 1, 0, 0).$$

By duality this means that there exists a morphism

$$\varphi^* : M(0, 0, 1, 0) \to M(0, 0, 0, 1)$$

of degree 12, i.e., a singular vector $\bar{w}$ of degree 12 and weight $(0, 0, 1, 0)$ in $M(0, 0, 0, 1)$. By Theorem 3.7 and Proposition 5.7, $\bar{w}$ is necessarily as in R4, with height 10, i.e.,

$$\bar{w}_{10} = \partial_4 d_\Omega \otimes x_5^* - \partial_5 d_\Omega \otimes x_4^*.$$

We have:

$$(x_5 d_{45})^{-1}(\bar{w}_{10}) = \partial_4(d_{12}^\vee \otimes x_3^* + d_{13}^\vee \otimes x_2^* + d_{23}^\vee \otimes x_1^*) + (\partial_3 d_{12}^\vee + \partial_2 d_{13}^\vee + \partial_1 d_{23}^\vee) \otimes x_4^*.$$

By condition (10) with $h = 10$ and $X = x_5 d_{45}$ it follows that in the expression of $\bar{w}_8$ the summand $\partial_4 \partial_5 d_{23,45}^\vee \otimes x_1^*$ must appear with coefficient equal to 1. Now we have:

$$E_4(\partial_4 \partial_5 d_{23,45}^\vee \otimes x_1^*) = (E_4)^0(\partial_4 \partial_5 d_{23,45}^\vee \otimes x_1^*) = -\partial_5^2 d_{23,45}^\vee \otimes x_1^*.$$

This contradicts condition (12) for $h = 10$. Indeed, one can see that no term in $E_4^{-2}\bar{w}_{10} + E_4^0 \bar{w}_8$ can cancel the summand $\partial_5^2 d_{23,45}^\vee \otimes x_1^*$.

Finally, let us assume that there exists a singular vector of degree 12 and height 10, as in R3, i.e.,

$$w_{10} = \partial_3 d_\Omega \otimes x_{45}^* + \partial_4 d_\Omega \otimes x_{53}^* + \partial_5 d_\Omega \otimes x_{34}^*.$$

Then we have:

$$\begin{aligned}(x_5 d_{45})^{-1}(w_{10}) =& \partial_3(d_{12}^\vee \otimes x_{34}^* + d_{13}^\vee \otimes x_{24}^* + d_{23}^\vee \otimes x_{14}^*) \\ &+ \partial_4(-d_{13}^\vee \otimes x_{23}^* - d_{23}^\vee \otimes x_{13}^*) - (\partial_3 d_{12}^\vee + \partial_2 d_{13}^\vee + \partial_1 d_{23}^\vee) \otimes x_{34}^*.\end{aligned}$$

Therefore, similarly as above, by condition (10) with $h = 10$ and $X = x_5 d_{45}$, in the expression of w_8 the summand $\partial_4 \partial_5 d_{23,45}^\vee x_{13}^*$ must appear with coefficient 1. Then we have:

$$E_4(\partial_4 \partial_5 d_{23,45}^\vee \otimes x_{13}^*) = (E_4)^0(\partial_4 \partial_5 d_{23,45}^\vee \otimes x_{13}^*) = -\partial_5^2 d_{23,45}^\vee \otimes x_{13}^*.$$

This contradicts condition (12) for $h = 10$. Indeed, one can see that no term in $E_4^{-2} w_{10} + E_4^0 w_8$ can cancel the summand $\partial_5^2 d_{23,45}^\vee \otimes x_{13}^*$. □

6. Properties of ω_I

In order to study morphisms between finite Verma modules and to better understand their structure as $\mathfrak{g}_0$-modules, a particular basis of U_- has been introduced in [2]. The main goal of this section is to show that this basis is also extremely useful when considering the action of the whole $\mathfrak{g}$ on a Verma module. We recall some technical notation needed to give an explicit definition of such a basis. We refer the reader to [2, §5] for further details.

We recall that $(U_-)_d$ denotes the homogeneous component of U_- of degree d. We let

$$\mathcal{I}_d = \{I = (I_1, \ldots, I_d) : I_l = (i_l, j_l) \text{ with } 1 \le i_l, j_l \le 5 \text{ for every } l = 1, \ldots, d\}.$$

If $I = (I_1, \ldots, I_d) \in \mathcal{I}_d$ we let $d_I = d_{I_1} \cdots d_{I_d} \in (U_-)_d$, with $d_{I_l} = d_{i_l j_l}$. Note that this notation is slightly different from the one adopted in Sects. 4 and 5.

We let $\mathcal{S}_d$ be the set of subsets of $[d]$ of cardinality 2, so that $|\mathcal{S}_d| = \binom{d}{2}$.

Note that elements in $\mathcal{I}_d$ are ordered tuples of ordered pairs, while elements in $\mathcal{S}_d$ are unordered tuples of unordered pairs.

If $\{k, l\} \in \mathcal{S}_d$ and $I \in \mathcal{I}_d$ we let $t_{I_k,I_l} = t_{i_k,j_k,i_l,j_l}$ and $\varepsilon_{I_k,I_l} = \varepsilon_{i_k,j_k,i_l,j_l}$.

We also let

$$D_{\{k,l\}}(I) = \frac{1}{2}(-1)^{l+k}\varepsilon_{I_k,I_l}\partial_{t_{I_k,I_l}} \in (U_-)_2.$$

Definition 6.1. A subset S of $\mathcal{S}_d$ is *self-intersection free* if its elements are pairwise disjoint.

For example $S = \{\{1, 3\}, \{2, 5\}, \{4, 7\}\}$ is self-intersection free while $\{\{1, 3\}, \{2, 5\}, \{3, 7\}\}$ is not. We denote by SIF_d the set of self-intersection free subsets of $\mathcal{S}_d$.

Definition 6.2. Let $\{k, l\}, \{h, m\} \in \mathcal{S}_d$ be disjoint pairs. We say that $\{k, l\}$ and $\{h, m\}$ *cross* if exactly one element in $\{k, l\}$ is between h and m. If $S \in \mathrm{SIF}_d$ we let the crossing number $c(S)$ of S be the number of pairs of elements in S that cross.

Definition 6.3. Let $S = \{S_1, \dots, S_r\} \in \mathrm{SIF}_d$. We let

$$D_S(I) = \prod_{j=1}^{r} D_{S_j}(I) \in (U_-)_{2r}$$

if $r \geq 2$ and $D_\emptyset(I) = 1$ (note that the order of multiplication is irrelevant as the elements $D_{S_j}(I)$ commute among themselves).

Definition 6.4. For $I = (I_1, \dots, I_d) \in \mathcal{I}_d$ and $S = \{S_1, \dots, S_r\} \in \mathrm{SIF}_d$ we let $C_S(I) \in \mathcal{I}_{d-2r}$ be obtained from I by removing all I_j such that $j \in S_k$ for some $k \in [r]$.

Definition 6.5. For all $I \in \mathcal{I}_d$ we let

$$\omega_I = \sum_{S\in \mathrm{SIF}_d} (-1)^{c(S)} D_S(I)\, d_{C_S(I)} \in (U_-)_d.$$

If $I \in \mathcal{I}_d$ we let $x_I = x_{I_1} \wedge \cdots \wedge x_{I_d} \in \bigwedge^d(\bigwedge^2(\mathbb{C}^5))$. The main properties of the elements ω_I have been obtained in [2, Proposition 5.6 and Theorem 5.8] and can be summarized in the following result.

Proposition 6.6. *Let $d = 0, \dots, 10$. Then the map $\varphi : \bigwedge^d(\bigwedge^2(\mathbb{C}^5)) \to U_-$, given by*

$$\varphi(x_{I_1} \wedge \cdots \wedge x_{I_d}) = \omega_{I_1,\dots,I_d}$$

for all $(I_1, \dots, I_d) \in \mathcal{I}_d$, is a (well-defined) injective morphism of $\mathfrak{g}_0$-modules.

We will also need the following very useful notation. Let $I \in \mathcal{I}_d$ and $J \in \mathcal{I}_c$, with $c \leq d$. If there exists $K \in \mathcal{I}_{d-c}$ such that $x_I = x_J \wedge x_K \neq 0$ we let $\omega_{I\setminus J} = \omega_K$, and we let $\omega_{I\setminus J} = 0$ if such K does not exist. Note that this notation is well-defined also thanks to Proposition 6.6.

For example, in order to compute $\omega_{(12,24,35,54)\setminus(24,45)}$ we observe that $x_{12,24,35,54} = x_{24,45} \wedge x_{12,35}$, therefore $\omega_{(12,24,35,54)\setminus(24,45)} = \omega_{12,35}$.

Instead of the explicit definition of the elements ω_I given in Definition 6.5, we will need some (equivalent) recursive properties that they satisfy.

Lemma 6.7. *Let $I = (I_1, \ldots, I_d)$. Then for all $k > 1$ we have*

$$\sum_{S\in SIF_d:\,\{1,k\}\in S} (-1)^{c(S)} D_S(I)\, d_{C_S(I)} = -\frac{1}{2}\varepsilon_{I_1,I_k}\partial_{t_{I_1,I_k}}\omega_{(I_2,\ldots,I_d)\setminus I_k}.$$

Furthermore

$$\omega_I = d_{I_1}\omega_{(I_2,\ldots,I_d)} - \frac{1}{2}\sum_{k=2}^{d}\varepsilon_{I_1,I_k}\partial_{t_{I_1,I_k}}\omega_{(I_2,\ldots,I_d)\setminus I_k} \tag{21}$$

Proof. We prove the first statement for all $I \in \mathcal{I}_d$ by induction on k. If $k = 2$ we have $D_{\{1,2\}}(I) = -\frac{1}{2}\varepsilon_{I_1,I_2}\partial_{t_{I_1,I_2}}$ and so, letting $J = (J_1, \ldots, J_{d-2}) = (I_3, \ldots, I_d)$ we have

$$\begin{aligned}\sum_{S\in \mathrm{SIF}_d:\,\{1,2\}\in S} (-1)^{c(S)} D_S(I)\, d_{C_S(I)} &= -\frac{1}{2}\varepsilon_{I_1,I_2}\partial_{t_{I_1,I_2}} \sum_{S\in \mathrm{SIF}_{d-2}} (-1)^{c(S)} D_S(J) d_{C_S(J)}\\ &= -\frac{1}{2}\varepsilon_{I_1,I_2}\partial_{t_{I_1,I_2}}\omega_{I_3,\ldots,I_d}\\ &= -\frac{1}{2}\varepsilon_{I_1,I_2}\partial_{t_{I_1,I_2}}\omega_{(I_2,\ldots,I_d)\setminus I_2}.\end{aligned}$$

If $k > 2$ we let $J = (I_1, \ldots, I_{k-2}, I_k, I_{k-1}, I_{k+1}, \ldots, I_d)$ be obtained from I by swapping I_k and I_{k-1}. We also observe that swapping k with $k-1$ provides a bijection $S \mapsto S'$ between elements in SIF_d containing $\{1, k\}$ and elements in SIF_d containing $\{1, k-1\}$; we also observe that by this bijection we have $d_{C_S(I)} = d_{C_{S'}(J)}$ and $(-1)^{c(S)} D_S(I) = -(-1)^{c(S')} D_{S'}(J)$: indeed if there exists l such that $\{k,l\} \in S$ then $(-1)^{c(S)} = -(-1)^{c(S')}$ and $D_S(I) = D_{S'}(J)$, and if such element l does not exist then $(-1)^{c(S)} = (-1)^{c(S')}$ and $D_S(I) = -D_{S'}(J)$. Therefore, using the induction hypothesis, we have

$$\begin{aligned}\sum_{S\in \mathrm{SIF}_d:\,\{1,k\}\in S} (-1)^{c(S)} D_S(I)\, d_{C_S(I)} &= -\sum_{S'\in \mathrm{SIF}_d:\,\{1,k-1\}\in S'} (-1)^{c(S')} D_{S'}(J)\, d_{C_S(J)}\\ &= \frac{1}{2}\varepsilon_{J_1,J_{k-1}}\partial_{t_{J_1,J_{k-1}}}\omega_{(J_2,\ldots,J_d)\setminus J_{k-1}}\\ &= -\frac{1}{2}\varepsilon_{I_1,I_k}\partial_{t_{I_1,I_k}}\omega_{(I_2,\ldots,I_d)\setminus I_k}.\end{aligned}$$

Equation (21) now follows from the first part observing that the first summand in the right-hand side of (21) is just

$$\sum_{S\in \mathrm{SIF}_d:\,\{1,k\}\notin S\,\forall k} (-1)^{c(S)} D_S(I)\, d_{C_S(I)}.$$

□

The following result is probably the easiest way to handle and compute the elements ω_I in a recursive way.

Proposition 6.8. *Let $I = (I_1, \ldots, I_d)$. Then*

$$\omega_I = \frac{1}{d}\sum_{j=1}^{d} d_{I_j}\omega_{I\setminus I_j}.$$

Proof. By (21) and Proposition 6.6 we have

$$\begin{aligned}\omega_I &= \frac{1}{d}\sum_{j=1}^{d}(-1)^{j+1}\omega_{I_j,I_1,\ldots,\hat{I}_j,\ldots,I_d}\\ &= \frac{1}{d}\sum_{j=1}^{d}(-1)^{j+1}\Big(d_{I_j}\omega_{I_1,\ldots,\hat{I}_j,\ldots,I_d} - \frac{1}{2}\sum_{k\neq j}\varepsilon_{I_j,I_k}\partial_{t_{I_j,I_k}}\omega_{(I_1,\ldots,\hat{I}_j,\ldots,I_d)\setminus I_k}\Big)\\ &= \frac{1}{d}\sum_{j=1}^{d}d_{I_j}\omega_{I\setminus I_j} - \frac{1}{2}\sum_{k\neq j}\varepsilon_{I_j,I_k}\partial_{t_{I_j,I_k}}\omega_{I\setminus(I_j,I_k)}\\ &= \frac{1}{d}\sum_{j=1}^{d}d_{I_j}\omega_{I\setminus I_j},\end{aligned}$$

since, clearly, $\omega_{I\setminus(I_j,I_k)} = -\omega_{I\setminus(I_k,I_j)}$ for all $k\neq j$. □

The following is an immediate consequence which is not needed in the sequel but sheds more light on the symmetric nature of the elements ω_I's.

Corollary 6.9. *We have*

$$\omega_{I_1,\ldots,I_d} = \frac{1}{d!}\sum_{\sigma\in S_d}\varepsilon_\sigma d_{I_{\sigma(1)}}\cdots d_{I_{\sigma(d)}}.$$

Proof. We proceed by induction on d, the result being trivial for $d=1$.

We have

$$\begin{aligned}\frac{1}{d!}\sum_{\sigma\in S_d}\varepsilon_\sigma d_{I_{\sigma(1)}}\cdots d_{I_{\sigma(d)}} &= \frac{1}{d!}\sum_{j=1}^{d}\sum_{\sigma\in S([n]\setminus j)}(-1)^{j-1}\varepsilon_\sigma d_{I_j}d_{I_{\sigma(1)}}\cdots\hat{d}_{I_{\sigma(j)}}\cdots d_{I_{\sigma(d)}}\\ &= \frac{1}{d}\sum_{j=1}^{d}(-1)^{j-1}d_{I_j}\sum_{\sigma\in S([n]\setminus j)}\varepsilon_\sigma d_{I_{\sigma(1)}}\cdots\hat{d}_{I_{\sigma(j)}}\cdots d_{I_{\sigma(d)}}\\ &= \frac{1}{d}\sum_{j=1}^{d}(-1)^{j-1}d_{I_j}\omega_{I_1,\ldots,\hat{I}_j,\ldots,I_d}\\ &= \frac{1}{d}\sum_{j=1}^{d}d_{I_j}\omega_{I\setminus I_j}.\end{aligned}$$

□

We now reformulate (21) in a way which is more suitable for our next arguments.

Lemma 6.10. *Let* $I\in\mathcal{I}_d$ *and let* $\{i,j,r,s,t\}=\{1,2,3,4,5\}$ *be such that* $\varepsilon_{ij,rs} = \varepsilon_{ij,st} = \varepsilon_{ij,tr} = 1$. *Then*

$$d_{ij}\omega_I = \omega_{ij,I} + \frac{1}{2}\partial_r\omega_{I\setminus st} + \frac{1}{2}\partial_s\omega_{I\setminus tr} + \frac{1}{2}\partial_t\omega_{I\setminus rs}.$$

Next target is to study the commutator between an element in $\mathfrak{g}_1$ of the form $x_p d_{pq}$ and a generic element ω_I. In order to simplify the reading of the arguments we prefer to show the proof explicitly in the special case $x_5 d_{45}$.

Lemma 6.11. *We have*

$$\sum_{k\neq j}\big[[x_5d_{45}, d_{I_k}], d_{I_j}\big]\omega_{I\setminus(I_k,I_j)} = \sum_j\big[[x_5d_{45}, d_{I_j}], \omega_{I\setminus I_j}\big] - 3\partial_4\omega_{I\setminus(12,23,31)} + \frac{3}{2}\sum_{(\alpha,\beta,\gamma)\in S_3}\partial_\alpha\omega_{I\setminus(\alpha\beta,\beta\gamma,\gamma4)}.$$

Proof. We first notice that in the left-hand side we have nonzero contributions only for those k such that $I_k = 12, 23, 31$ (up to order). We compute the contribution of $I_k = 12$, the others will be similar. We have

$$\sum_j[x_5\partial_3, d_{I_j}]\omega_{I\setminus(12,I_j)} = d_{25}\omega_{I\setminus(12,23)} + d_{51}\omega_{I\setminus(12,31)} + d_{54}\omega_{I\setminus(12,34)}$$

and by Lemma 6.10 and Proposition 6.6 we have

$$\begin{aligned}\sum_j[x_5\partial_3, d_{I_j}]\omega_{I\setminus(12,I_j)} &= \omega_{25,I\setminus(12,23)} + \omega_{51,I\setminus(12,31)} + \omega_{54,I\setminus(12,34)}\\ &+ \frac{1}{2}\big(\partial_1\omega_{I\setminus(12,23,34)} + \partial_3\omega_{I\setminus(12,23,41)} + \partial_4\omega_{I\setminus(12,23,13)} + \partial_2\omega_{I\setminus(12,31,34)}\\ &\quad + \partial_3\omega_{I\setminus(12,31,42)} + \partial_4\omega_{I\setminus(12,31,23)}\\ &+ \partial_1\omega_{I\setminus(12,34,32)} + \partial_3\omega_{I\setminus(12,34,21)} + \partial_2\omega_{I\setminus(12,34,13)}\big)\\ &= [x_5\partial_3, \omega_{I\setminus 12}] - \partial_4\omega_{I\setminus(12,23,31)} + \partial_1\omega_{I\setminus(12,23,34)} + \partial_2\omega_{I\setminus(21,13,34)}\\ &\quad + \frac{1}{2}\partial_3\omega_{I\setminus(32,21,14)} + \frac{1}{2}\partial_3\omega_{I\setminus(31,12,24)}\end{aligned}$$

The contributions of $I_k = 23, 31$ are similarly computed and the result follows. □

Theorem 6.12. *For all $I \in \mathcal{I}_d$ we have*

$$[x_5d_{45}, \omega_I] = \sum_{j=1}^{d}\Big(\frac{1}{2}\big[[x_5d_{45}, d_{I_j}], \omega_{I\setminus I_j}\big] + \omega_{I\setminus I_j}[x_5d_{45}, d_{I_j}]\Big) + \frac{1}{2}\partial_4\omega_{I\setminus(12,23,31)} - \frac{1}{4}\sum_{(\alpha,\beta,\gamma)\in S_3}\partial_\alpha\omega_{I\setminus(\alpha\beta,\beta\gamma,\gamma4)}.$$

Proof. We proceed by induction on d, the case $d = 1$ being easy. Note that by induction hypothesis we can assume that

$$[x_5d_{45}, \omega_{I\setminus I_j}] = \sum_{k\neq j}\Big(\frac{1}{2}\big[[x_5d_{45}, d_{I_k}], \omega_{I\setminus(I_j,I_k)}\big] + \omega_{I\setminus(I_j,I_k)}[x_5d_{45}, d_{I_k}]\Big) + \frac{1}{2}\partial_4\omega_{I\setminus(I_j,12,23,31)} - \frac{1}{4}\sum_{(\alpha,\beta,\gamma)\in S_3}\partial_\alpha\omega_{I\setminus(I_j,\alpha\beta,\beta\gamma,\gamma4)}.$$

Using Proposition 6.8 and the induction hypothesis we have:

$$\begin{aligned}
[x_5d_{45},\omega_I] &= \frac{1}{d}\sum_{j=1}^{d}\big([x_5d_{45},d_{I_j}]\omega_{I\setminus I_j} - d_{I_j}[x_5d_{45},\omega_{I\setminus I_j}]\big)\\
&= \frac{1}{d}\sum_{j=1}^{d}\Big(\big[[x_5d_{45},d_{I_j}],\omega_{I\setminus I_j}\big] + \omega_{I\setminus I_j}[x_5d_{45},d_{I_j}]\\
&\qquad - d_{I_j}\sum_{k\neq j}\big(\frac{1}{2}\big[[x_5d_{45},d_{I_k}],\omega_{I\setminus(I_j,I_k)}\big] + \omega_{I\setminus(I_j,I_k)}[x_5d_{45},d_{I_k}]\big)\\
&\qquad + \frac{1}{2}\partial_4\omega_{I\setminus(I_j,12,23,31)} - \frac{1}{4}\sum_{(\alpha,\beta,\gamma)\in S_3}\partial_\alpha\omega_{I\setminus(I_j,\alpha\beta,\beta\gamma,\gamma4)}\Big)\\
&= \frac{1}{d}\sum_{j=1}^{d}\big(\omega_{I\setminus I_j}[x_5d_{45},d_{I_j}] - d_{I_j}\sum_{k\neq j}\omega_{I\setminus(I_j,I_k)}[x_5d_{45},d_{I_k}]\big)\\
&\quad + \frac{1}{d}\sum_{j=1}^{d}\big(\big[[x_5d_{45},d_{I_j}],\omega_{I\setminus I_j}\big] - d_{I_j}\sum_{k\neq j}\frac{1}{2}\big[[x_5d_{45},d_{I_k}],\omega_{I\setminus(I_j,I_k)}\big]\big)\\
&\quad + \frac{1}{d}\sum_{j=1}^{d}d_{I_j}\big(-\frac{1}{2}\partial_4\omega_{I\setminus(I_j,12,23,31)} + \frac{1}{4}\sum_{(\alpha,\beta,\gamma)\in S_3}\partial_\alpha\omega_{I\setminus(I_j,\alpha\beta,\beta\gamma,\gamma4)}\big)
\end{aligned}$$

We split this formula into three parts (according to the last three lines above): the first part is

$$\begin{aligned}
&\frac{1}{d}\sum_{j=1}^{d}\big(\omega_{I\setminus I_j}[x_5d_{45},d_{I_j}] - d_{I_j}\sum_{k\neq j}\omega_{I\setminus(I_j,I_k)}[x_5d_{45},d_{I_k}]\big)\\
&\quad = \frac{1}{d}\big(\sum_{j=1}^{d}\omega_{I\setminus I_j}[x_5d_{45},d_{I_j}] + \sum_{k=1}^{d}\sum_{j\neq k}d_{I_j}\omega_{I\setminus(I_k,I_j)}[x_5d_{45},d_{I_k}]\big)\\
&\quad = \frac{1}{d}\big(\sum_{j=1}^{d}\omega_{I\setminus I_j}[x_5d_{45},d_{I_j}] + \sum_{k=1}^{d}(d-1)\omega_{I\setminus I_k}[x_5d_{45},d_{I_k}]\big)\\
&\quad = \sum_{j=1}^{d}\omega_{I\setminus I_j}[x_5d_{45},d_{I_j}].
\end{aligned}$$

The third part is

$$\begin{aligned}
&\frac{1}{d}\sum_{j=1}^{d}d_{I_j}\big(-\frac{1}{2}\partial_4\omega_{I\setminus(I_j,12,23,31)} + \frac{1}{4}\sum_{(\alpha,\beta,\gamma)\in S_3}\partial_\alpha\omega_{I\setminus(I_j,\alpha\beta,\beta\gamma,\gamma4)}\big)\\
&\quad = \frac{1}{2d}\partial_4\sum_{j=1}^{d}d_{I_j}\omega_{I\setminus(12,23,31,I_j)} - \frac{1}{4d}\sum_{(\alpha,\beta,\gamma)\in S_3}\partial_\alpha\sum_{j=1}^{d}d_{I_j}\omega_{I\setminus(\alpha\beta,\beta\gamma,\gamma4,I_j)}
\end{aligned}$$

$$= \frac{d-3}{2d} \partial_4 \omega_{I \setminus (12,23,31)} - \frac{d-3}{4d} \sum_{(\alpha,\beta,\gamma) \in S_3} \partial_\alpha \omega_{I \setminus (\alpha\beta,\beta\gamma,\gamma 4)}.$$

In order to compute the second part we notice, using Lemma 6.11, that the following holds:

$$\begin{aligned}
-d_{I_j} \sum_{k,j} \big[[x_5 d_{45}, d_{I_k}], \omega_{I \setminus (I_j, I_k)}\big] &= d_{I_j} \sum_{k,j} \big[[x_5 d_{45}, d_{I_k}], \omega_{I \setminus (I_k, I_j)}\big] \\
&= \sum_{j,k} \Big(\big[[x_5 d_{45}, d_{I_k}], d_{I_j} \omega_{I \setminus (I_k, I_j)}\big] - \big[[x_5 d_{45}, d_{I_k}], d_{I_j}\big] \omega_{I \setminus (I_k, I_j)} \Big) \\
&= \sum_k \big[[x_5 d_{45}, d_{I_k}], \sum_j d_{I_j} \omega_{I \setminus (I_k, I_j)}\big] - \sum_{j,k} \big[[x_5 d_{45}, d_{I_k}], d_{I_j}\big] \omega_{I \setminus (I_k, I_j)} \\
&= (d-1) \sum_k \big[[x_5 d_{45}, d_{I_k}], \omega_{I \setminus I_k}\big] - \sum_j \big[[x_5 d_{45}, d_{I_j}], \omega_{I \setminus I_j}\big] + 3 \partial_4 \omega_{I \setminus (12,23,31)} \\
&\quad - \frac{3}{2} \sum_{(\alpha,\beta,\gamma) \in S_3} \partial_\alpha \omega_{I \setminus (\alpha\beta,\beta\gamma,\gamma 4)} \\
&= (d-2) \sum_k \big[[x_5 d_{45}, d_{I_k}], \omega_{I \setminus I_k}\big] + 3 \partial_4 \omega_{I \setminus (12,23,31)} - \frac{3}{2} \sum_{(\alpha,\beta,\gamma) \in S_3} \partial_\alpha \omega_{I \setminus (\alpha\beta,\beta\gamma,\gamma 4)}.
\end{aligned}$$

Therefore the whole second part is

$$\begin{aligned}
&\frac{1}{d} \sum_{j=1}^{d} \Big(\big[[x_5 d_{45}, d_{I_j}], \omega_{I \setminus I_j}\big] - d_{I_j} \sum_{k \neq j} \frac{1}{2} \big[[x_5 d_{45}, d_{I_k}], \omega_{I \setminus (I_j, I_k)}\big] \Big) \\
&= \frac{1}{d} \Big(\sum_j \big[[x_5 d_{45}, d_{I_j}], \omega_{I \setminus I_j}\big] + \frac{1}{2}(d-2) \sum_k \big[[x_5 d_{45}, d_{I_k}], \omega_{I \setminus I_k}\big] + \frac{3}{2} \partial_4 \omega_{I \setminus (12,23,31)} \\
&\quad - \frac{3}{4} \sum_{(\alpha,\beta,\gamma) \in S_3} \partial_\alpha \omega_{I \setminus (\alpha\beta,\beta\gamma,\gamma 4)} \Big) \\
&= \frac{1}{2} \sum_j \big[[x_5 d_{45}, d_{I_j}], \omega_{I \setminus I_j} + \frac{3}{2d} \partial_4 \omega_{I \setminus (12,23,31)} - \frac{3}{4d} \sum_{(\alpha,\beta,\gamma) \in S_3} \partial_\alpha \omega_{I \setminus (\alpha\beta,\beta\gamma,\gamma 4)}.
\end{aligned}$$

The sum of the three parts gives the result. □

One can analogously prove the following result.

Theorem 6.13. *Let* $\{p, q, a, b, c\} = \{1, 2, 3, 4, 5\}$ *and* $I \in \mathcal{I}_d$. *Then*

$$\begin{aligned}
[x_p d_{pq}, \omega_I] = \sum_{j=1}^{d} \Big(\frac{1}{2} \big[[x_p d_{pq}, d_{I_j}], \omega_{I \setminus I_j}\big] + \omega_{I \setminus I_j} [x_p d_{pq}, d_{I_j}] \Big) \\
- \frac{1}{2} \partial_q \omega_{I \setminus (ab,bc,ca)} + \frac{1}{4} \sum_{(\alpha,\beta,\gamma) \in S(a,b,c)} \partial_\alpha \omega_{I \setminus (\alpha\beta,\beta\gamma,\gamma q)},
\end{aligned}$$

where $S(a, b, c)$ *denotes the set of permutations of* $\{a, b, c\}$.

7. The Fundamental Equations

We are now going to use Theorem 6.13 to study possible morphisms of finite Verma modules $\varphi : M(V) \to M(W)$. Let $\sim$ be the equivalence relation on $\mathcal{I}_d$ such that $I \sim I'$ if and only if $\omega_I = \pm\omega_{I'}$, i.e. if I' can be obtained from I by permuting the pairs in I and the elements in each pair. By Proposition 3.3, Remark 3.5 and Proposition 6.6 we know that if a morphism has degree d then it can be expressed in the following way

$$\varphi(v) = \sum_{l \le d/2} \sum_{I \in \mathcal{I}_{d-2l}/\sim} \sum_{1 \le r_1 \le \cdots \le r_l \le 5} \partial_{r_1} \cdots \partial_{r_l} \omega_I \otimes \theta_I^{r_1,\dots,r_l}(v) \tag{22}$$

where the $\theta_I^{r_1,\dots,r_l} : V \to W$ are such that the map

$$\mathrm{Sym}^l(\mathbb{C}^5) \otimes \textstyle\bigwedge^{d-2l}(\bigwedge^2((\mathbb{C}^5)^*)) \to \mathrm{Hom}(V, W)$$

given by

$$x_{r_1} \cdots x_{r_l} \otimes x_{I_1}^* \wedge \cdots \wedge x_{I_{d-2l}}^* \mapsto \theta_{I_1,\dots,I_{d-2l}}^{r_1,\dots,r_l} \tag{23}$$

is a (well-defined) morphism of $\mathfrak{g}_0$-modules. This fact permits us to easily compute the action of $\mathfrak{g}_0$ on the morphisms $\theta_I^{r_1,\dots,r_l}$'s. For example we have

$$x_1\partial_2.\theta^{2,3}_{12,13,14,23} = \theta^{1,3}_{12,13,14,23} - \theta^{2,3}_{12,23,14,23} - \theta^{2,3}_{12,13,24,23} = \theta^{1,3}_{12,13,14,23} + \theta^{2,3}_{12,13,23,24}.$$

A technical lemma is in order.

Lemma 7.1. *For all distinct $\alpha, \beta, \gamma, p \in [5]$ we have:*

$$\sum_{J \in \mathcal{I}_d/\sim} [x_p\partial_\gamma, \omega_J] \otimes \theta_{\alpha\beta,J} = -\sum_{J \in \mathcal{I}_d/\sim} \omega_J \otimes (x_p\partial_\gamma.\theta_{\alpha\beta,J}).$$

and

$$\sum_{K \in \mathcal{I}_d/\sim} (-\partial_\gamma\omega_K \otimes \theta^p_{\alpha\beta,K} + \sum_{t=1}^5 \partial_t[x_p\partial_\gamma, \omega_K] \otimes \theta^t_{\alpha\beta,K}) = -\sum_{t=1}^5 \sum_{K \in \mathcal{I}_d/\sim} \partial_t\omega_K \otimes (x_p\partial_\gamma.\theta^t_{\alpha\beta,K}).$$

Proof. The first equation follows from the following observation. By Proposition 6.6 and (23) we have that if $[x_p\partial_\gamma, \omega_J] = \sum_{J'} a_{J,J'}\omega_{J'}$ then $x_p\partial_\gamma.\theta_{J'} = -\sum_J a_{J,J'}\theta_J$ and hence also

$$x_p\partial_\gamma.\theta_{\alpha\beta,J'} = -\sum_J a_{J,J'}\theta_{\alpha\beta,J},$$

since α and β are distinct from p. We can conclude that

$$\sum_J [x_p\partial_\gamma, \omega_J] \otimes \theta_{\alpha\beta,J} = \sum_{J'} \omega_{J'} \otimes \sum_J a_{J,J'}\theta_{\alpha\beta,J} = -\sum_{J'} \omega_{J'} \otimes (x_p\partial_\gamma.\theta_{\alpha\beta,J'}).$$

In order to prove the second equation we proceed in a similar way. If $[x_p\partial_\gamma, \omega_K] = \sum_{K'} a_{K,K'}\omega_{K'}$ we have $x_p\partial_\gamma.\theta_{K'} = -\sum_K a_{K,K'}\theta_K$ and also $x_p\partial_\gamma.\theta_{\alpha\beta,K'} =$

$-\sum_K a_{K,K'}\theta_{\alpha\beta,K}$. Therefore, if $t \neq \gamma$ we have $x_p\partial_\gamma.\theta^t_{\alpha\beta,K'} = -\sum_K a_{K,K'}\theta^t_{\alpha\beta,K}$ and for $t = \gamma$ we have

$$(x_p\partial_\gamma.\theta^\gamma_{\alpha\beta,K'}) = \theta^p_{\alpha\beta,K'} - \sum_K a_{K,K'}\theta^\gamma_{\alpha\beta,K}.$$

So we can compute

$$\begin{aligned}
&\sum_K(-\partial_\gamma\omega_K \otimes \theta^p_{\alpha\beta,K} + \sum_t \partial_t[x_p\partial_\gamma, \omega_K] \otimes \theta^t_{\alpha\beta,K})\\
&\quad= \sum_K(-\partial_\gamma\omega_K \otimes \theta^p_{\alpha\beta,K} + \sum_t \partial_t \sum_{K'} a_{K,K'}\omega'_K \otimes \theta^t_{\alpha\beta,K})\\
&\quad= \sum_{K'} \partial_\gamma\omega_{K'} \otimes (\theta^p_{\alpha\beta,K'} + \sum_K a_{K,K'}\theta^\gamma_{\alpha\beta,K}) + \sum_{t\neq\gamma}\sum_{K'} \partial_t\omega_{K'} \otimes \sum_K a_{K,K'}\theta^t_{\alpha\beta,K}\\
&\quad= -\sum_t\sum_{K'} \partial_t\omega'_K \otimes (x_p\partial_\gamma.\theta^\gamma_{\alpha\beta,K'}).
\end{aligned}$$

□

We let $C(a, b, c) = \{(a, b, c), (b, c, a), (c, a, b)\}$ the set of cyclic permutations of (a, b, c). From now on when we write $\sum_{\alpha\beta\gamma}$ we always mean the sum over $(\alpha, \beta, \gamma) \in C(a, b, c)$.

Lemma 7.2. *Let $\varphi : M(V) \to M(W)$ be a morphism of finite Verma modules as in* (22) *and (p, q, a, b, c) be any permutation of* [5]. *Then*

$$\begin{aligned}
x_p d_{pq} \sum_{I\in\mathcal{I}_d/\sim} \omega_I \otimes \theta_I(v) = &\sum_{J\in\mathcal{I}_{d-1}/\sim} \omega_J \otimes \frac{1}{2}\varepsilon_{pqabc}\sum_{\alpha\beta\gamma}\\
&\big(-(x_p\partial_\gamma.\theta_{\alpha\beta,J})(v) + 2x_p\partial_\gamma.(\theta_{\alpha\beta,J}(v))\big)\\
&+ \sum_{K\in\mathcal{I}_{d-3}/\sim} \partial_q\omega_K \otimes -\frac{1}{2}\theta_{ab,bc,ca,K}(v)\\
&+ \sum_{\alpha\beta\gamma} \partial_\alpha\omega_K \otimes \frac{1}{4}\big(\theta_{\alpha\beta,\beta\gamma,\gamma q,K}(v) + \theta_{\alpha\gamma,\gamma\beta,\beta q,K}(v)\big).
\end{aligned}$$

Proof. Theorem 6.13 can be reformulated in the following more convenient way

$$\begin{aligned}
[x_p d_{pq}, \omega_I] = &\frac{1}{2}\varepsilon_{pqabc} \sum_{(\alpha,\beta,\gamma)\in C(a,b,c)} \big([x_p\partial_\gamma, \omega_{I\setminus\alpha\beta}] + 2\omega_{I\setminus\alpha\beta}\, x_p\partial_\gamma\big)\\
&- \frac{1}{2}\partial_q\omega_{I\setminus(ab,bc,ca)} + \frac{1}{4}\sum_{(\alpha,\beta,\gamma)\in C(a,b,c)} \partial_\alpha(\omega_{I\setminus(\alpha\beta,\beta\gamma,\gamma q)} + \omega_{I\setminus(\alpha\gamma,\gamma\beta,\beta q)}).
\end{aligned} \tag{24}$$

We can therefore compute

$$\begin{aligned}
x_p d_{pq} \sum_{I\in\mathcal{I}_d/\sim} \omega_I \otimes \theta_I(v) = &\frac{1}{2}\varepsilon_{pqabc} \sum_{I\in\mathcal{I}_d/\sim}\ \sum_{(\alpha,\beta,\gamma)\in C(a,b,c)} \big([x_p\partial_\gamma, \omega_{I\setminus\alpha\beta}] \otimes \theta_I(v) + 2\omega_{I\setminus\alpha\beta} \otimes x_p\partial_\gamma.(\theta_I(v))\big)\\
&+ \sum_{I\in\mathcal{I}_d/\sim} \Big(-\frac{1}{2}\partial_q\omega_{I\setminus(ab,bc,ca)} \otimes \theta_I(v) + \frac{1}{4}\sum_{(\alpha,\beta,\gamma)\in C(a,b,c)} \partial_\alpha(\omega_{I\setminus(\alpha\beta,\beta\gamma,\gamma q)} + \omega_{I\setminus(\alpha\gamma,\gamma\beta,\beta q)}) \otimes \theta_I(v)\Big)
\end{aligned}$$

$$= \frac{1}{2}\varepsilon_{pqabc} \sum_{J\in\mathcal{I}_{d-1}/\sim} \sum_{(\alpha,\beta,\gamma)\in C(a,b,c)} \Big([x_p\partial_\gamma, \omega_J]\otimes\theta_{\alpha\beta,J}(v) + 2\omega_J\otimes x_p\partial_\gamma.(\theta_{\alpha\beta,J}(v))\Big)$$
$$+ \sum_{K\in\mathcal{I}_{d-3}} \Big(-\frac{1}{2}\partial_q\omega_K\otimes\theta_{ab,bc,ca,K}(v) + \frac{1}{4}\sum_{(\alpha,\beta,\gamma)\in C(a,b,c)} \partial_\alpha\omega_K\otimes(\theta_{\alpha\beta,\beta\gamma,\gamma q,K}(v)+\theta_{\alpha\gamma,\gamma\beta,\beta q,K}(v))\Big)$$
$$= \frac{1}{2}\varepsilon_{pqabc} \sum_{J\in\mathcal{I}_{d-1}/\sim} \sum_{(\alpha,\beta,\gamma)\in C(a,b,c)} \Big(-\omega_J\otimes(x_p\partial_\gamma.\theta_{\alpha\beta,J})(v) + 2\omega_J\otimes x_p\partial_\gamma.(\theta_{\alpha\beta,J}(v))\Big)$$
$$+ \sum_{K\in\mathcal{I}_{d-3}} \Big(-\frac{1}{2}\partial_q\omega_K\otimes\theta_{ab,bc,ca,K}(v) + \frac{1}{4}\sum_{(\alpha,\beta,\gamma)\in C(a,b,c)} \partial_\alpha\omega_K\otimes(\theta_{\alpha\beta,\beta\gamma,\gamma q,K}(v)+\theta_{\alpha\gamma,\gamma\beta,\beta q,K}(v))\Big),$$

where we have used Lemma 7.1. □

Lemma 7.3. *Let $\varphi : M(V) \to M(W)$ be a morphism of finite Verma modules as in* (22). *Then*

$$x_p d_{pq}\sum_{t=1}^{5}\sum_{I\in\mathcal{I}_{d-2}/\sim}\partial_t\omega_I\otimes\theta_I^t(v) = \sum_{J\in\mathcal{I}_{d-1}/\sim}\omega_J\otimes -\theta^p_{J\setminus pq}(v)$$
$$+\sum_{t=1}^{5}\sum_{K\in\mathcal{I}_{d-3}/\sim}\partial_t\omega_K\otimes\frac{1}{2}\varepsilon_{pqabc}\sum_{\alpha\beta\gamma}\big(-(x_p\partial_\gamma.\theta^t_{\alpha\beta,K})(v)+2x_p\partial_\gamma.(\theta^t_{\alpha\beta,K}(v))\big)$$
$$+\sum_t\sum_{L\in\mathcal{I}_{d-5}/\sim}\partial_t\partial_q\omega_L\otimes -\frac{1}{2}\theta^t_{ab,bc,ca,L}(v)$$
$$+\sum_{\alpha\beta\gamma}\partial_t\partial_\alpha\omega_L\otimes\frac{1}{4}\big(\theta^t_{\alpha\beta,\beta\gamma,\gamma q,L}(v)+\theta^t_{\alpha\gamma,\gamma\beta,\beta q,L}(v)\big)$$

Proof. By Lemma 6.10, Lemma 7.1 and (24) we have

$$x_p d_{pq}\sum_{t=1}^{5}\sum_{I\in\mathcal{I}_{d-2}/\sim}\partial_t\omega_I\otimes\theta_I^t(v) =$$
$$= \sum_{I\in\mathcal{I}_{d-2}/\sim}(-\omega_{pq,I}-\frac{1}{2}\varepsilon_{pqabc}\sum_{\alpha\beta\gamma}\partial_\gamma\omega_{I\setminus\alpha\beta})\otimes\theta_I^p(v)$$
$$+\sum_{t=1}^{5}\sum_{I\in\mathcal{I}_{d-2}/\sim}\frac{1}{2}\varepsilon_{pqabc}\partial_t\sum_{\alpha\beta\gamma}\big([x_p\partial_\gamma,\omega_{I\setminus\alpha\beta}]\otimes\theta_I^t(v)+2\omega_{I\setminus\alpha\beta}\otimes x_p\partial_\gamma.(\theta_I^t(v))\big)$$
$$+\sum_t\sum_{I\in\mathcal{I}_{d-2}/\sim}\Big(-\frac{1}{2}\partial_t\partial_q\omega_{I\setminus(ab,bc,ca)}\otimes\theta_I^t(v)+\frac{1}{4}\sum_{\alpha\beta\gamma}\partial_t\partial_\alpha\big(\omega_{I\setminus(\alpha\beta,\beta\gamma,\gamma q)}+\omega_{I\setminus(\alpha\gamma,\gamma\beta,\beta q)}\big)\otimes\theta_I^t(v)\Big)$$
$$=\sum_{J\in\mathcal{I}_{d-1}/\sim}\omega_J\otimes-\theta^p_{J\setminus pq}(v)$$
$$+\frac{1}{2}\varepsilon_{pqabc}\sum_{\alpha\beta\gamma}\sum_{K\in\mathcal{I}_{d-3}/\sim}\Big(-\partial_\gamma\omega_K\otimes\theta^p_{\alpha\beta,K}+\sum_{t=1}^{5}\partial_t[x_p\partial_\gamma,\omega_K]\otimes\theta^t_{\alpha\beta,K}$$
$$+\sum_{t=1}^{5}2\partial_t\omega_K\otimes x_p\partial_\gamma.(\theta^t_{\alpha\beta,K}(v))\Big)$$
$$+\sum_t\sum_{L\in\mathcal{I}_{d-5}/\sim}\partial_t\partial_q\omega_L\otimes-\frac{1}{2}\theta^t_{ab,bc,ca,L}(v)+\sum_{\alpha\beta\gamma}\partial_t\partial_\alpha\omega_L\otimes\frac{1}{4}\big(\theta^t_{\alpha\beta,\beta\gamma,\gamma q,L}(v)+\theta^t_{\alpha\gamma,\gamma\beta,\beta q,L}(v)\big)$$
$$=\sum_{J\in\mathcal{I}_{d-1}/\sim}\omega_J\otimes-\theta^p_{J\setminus pq}(v)$$

$$+\frac{1}{2}\varepsilon_{pqabc}\sum_{t=1}^{5}\sum_{K\in\mathcal{I}_{d-3}/\sim}\partial_t\omega_K\otimes\sum_{\alpha\beta\gamma}-(x_p\partial_\gamma.\theta^t_{\alpha\beta,K})(v)$$
$$+2x_p\partial_\gamma.(\theta^t_{\alpha\beta,K}(v))$$
$$+\sum_t\sum_{L\in\mathcal{I}_{d-5}/\sim}\partial_t\partial_q\omega_L\otimes-\frac{1}{2}\theta^t_{ab,bc,ca,L}(v)$$
$$+\sum_{\alpha\beta\gamma}\partial_t\partial_\alpha\omega_L\otimes\frac{1}{4}\big(\theta^t_{\alpha\beta,\beta\gamma,\gamma q,L}(v)+\theta^t_{\alpha\gamma,\gamma\beta,\beta q,L}(v)\big).$$

□

The following result is fundamental in our study.

Corollary 7.4. *If $\varphi : M(V) \to M(W)$ is a morphism of finite Verma modules as in (22), then for all p, q, a, b, c such that $\{p, q, a, b, c\} = [5]$ we have*

$$-\theta^p_{J\setminus pq}(v)+\frac{1}{2}\varepsilon_{pqabc}\sum_{\alpha\beta\gamma}\big(-(x_p\partial_\gamma.\theta_{\alpha\beta,J})(v)+2x_p\partial_\gamma.(\theta_{\alpha\beta,J}(v))\big)=0 \quad (25)$$

for all $J \in \mathcal{I}_{d-1}$ and

$$\frac{1}{4}\big(\theta_{ab,bc,cq,K}(v)+\theta_{ac,cb,bq,K}(v)\big)-\theta^{a,p}_{K\setminus pq}(v)$$
$$+\frac{1}{2}\varepsilon_{pqabc}\sum_{\alpha\beta\gamma}\big(-(x_p\partial_\gamma.\theta^a_{\alpha\beta,K})(v)+2x_p\partial_\gamma.(\theta^a_{\alpha\beta,K}(v))\big)=0, \quad (26)$$

$$-2\theta^{p,p}_{K\setminus pq}(v)+\frac{1}{2}\varepsilon_{pqabc}\sum_{\alpha\beta\gamma}\big(-(x_p\partial_\gamma.\theta^p_{\alpha\beta,K})(v)+2x_p\partial_\gamma.(\theta^p_{\alpha\beta,K}(v))\big)=0, \quad (27)$$

$$-2\theta^{p,q}_{K\setminus pq}(v)-\theta_{ab,bc,ca,K}(v)$$
$$+\varepsilon_{pqabc}\big(-(x_p\partial_\gamma.\theta^q_{\alpha\beta,K})(v)+2x_p\partial_\gamma.(\theta^q_{\alpha\beta,K}(v))\big)=0, \quad (28)$$

for all $K \in \mathcal{I}_{d-3}$.

Proof. Recall that $\varphi(v)=\sum_{l\le d/2}\sum_{I\in\mathcal{I}_{d-2l}/\sim}\sum_{1\le r_1\le\cdots\le r_l\le 5}\partial_{r_1}\cdots\partial_{r_l}\omega_I\otimes\theta_I^{r_1,\dots,r_l}(v)$. By Proposition 3.3 we have $x_pd_{pq}\varphi(v) = 0$ and if we expand

$$x_pd_{pq}\varphi(v)=\sum_{l\le(d-1)/2}\sum_{1\le r_1\le\cdots\le r_l\le 5}\sum_{I\in\mathcal{I}_{d-1-2l}/\sim}\partial_{r_1}\cdots\partial_{r_l}\omega_I\otimes v_{(r_1,\dots,r_l),I}$$

we have that all vectors $v_{(r_1,\dots,r_l),I}$ must be 0. By Lemmas 7.2 and 7.3 we have for all $J \in \mathcal{I}_{d-1}$

$$v_{(),J}=-\theta^p_{J\setminus pq}(v)+\frac{1}{2}\varepsilon_{pqabc}\sum_{\alpha\beta\gamma}\big(-(x_p\partial_\gamma.\theta_{\alpha\beta,J})(v)+2x_p\partial_\gamma.(\theta_{\alpha\beta,J}(v))\big)$$

hence (25) follows. Moreover, for all $K \in \mathcal{I}_{d-3}$, by Lemmas 7.2 and 7.3 we have

$$v_{(a),K}=\frac{1}{4}\big(\theta_{ab,bc,cq,K}(v)+\theta_{ac,cb,bq,K}(v)\big)-\theta^{a,p}_{K\setminus pq}(v)$$
$$+\frac{1}{2}\varepsilon_{pqabc}\sum_{\alpha\beta\gamma}\big(-(x_p\partial_\gamma.\theta^a_{\alpha\beta,K})(v)+2x_p\partial_\gamma.(\theta^a_{\alpha\beta,K}(v))\big).$$

Note that in this case we have an additional term $-\theta^{a,p}_{K\setminus pq}(v)$ which is produced by

$$x_p d_{pq}\, \partial_a \partial_p \omega_{K\setminus pq} \otimes \theta_{K\setminus pq}(v).$$

Equation (26) follows. Equations (27) and (28) are obtained similarly by considering $v_{(p),K}$ and $v_{(q),K}$. □

Corollary 7.4 can be slightly simplified if v is a highest weight vector in V. For all n, m we let

$$\chi_{n>m} = \begin{cases} 1 & \text{if } n > m; \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 7.5. *Let $\varphi : M(V) \to M(W)$ be a morphism of finite Verma modules as in (22) and let $s \in V$ be a highest weight vector. Then for all $\{p, q, a, b, c\}$ we have:*

$$-2\theta^{p}_{J\setminus pq}(s) + \varepsilon_{pqabc} \sum_{\alpha\beta\gamma} \big((-1)^{\chi_{p>\gamma}} (x_p \partial_\gamma .\theta_{\alpha\beta,J})(s) \\ +2\chi_{p>\gamma} x_p \partial_\gamma .(\theta_{\alpha\beta,J}(s))\big) = 0 \tag{29}$$

for all $J \in \mathcal{I}_{d-1}$ and

$$-4\theta^{a,p}_{K\setminus pq}(s) + \big(\theta_{ab,bc,cq,K}(s) + \theta_{ac,cb,bq,K}(s)\big) + 2\varepsilon_{pqabc} \sum_{\alpha\beta\gamma} \big((-1)^{\chi_{p>\gamma}} (x_p \partial_\gamma .\theta^{a}_{\alpha\beta,K})(s) \\ +2\chi_{p>\gamma} x_p \partial_\gamma .(\theta^{a}_{\alpha\beta,K}(s))\big) = 0, \tag{30}$$

$$-4\theta^{p,p}_{K\setminus pq}(s) + \varepsilon_{pqabc} \sum_{\alpha\beta\gamma} \big((-1)^{\chi_{p>\gamma}} (x_p \partial_\gamma .\theta^{p}_{\alpha\beta,K})(s) + 2\chi_{p>\gamma} x_p \partial_\gamma .(\theta^{p}_{\alpha\beta,K}(s))\big) = 0, \tag{31}$$

$$-2\theta^{p,q}_{K\setminus pq} - \theta_{ab,bc,ca,K}(s) \\ +\varepsilon_{pqabc}\big((-1)^{\chi_{p>\gamma}} (x_p \partial_\gamma .\theta^{q}_{\alpha\beta,K})(s) + 2\chi_{p>\gamma} x_p \partial_\gamma .(\theta^{q}_{\alpha\beta,K}(s))\big) = 0 \tag{32}$$

for all $K \in \mathcal{I}_{d-3}$.

Proof. We prove Equation (29), the others are similar. By (25), if $p > \gamma$ we clearly have

$$(-1)^{\chi_{p>\gamma}} (x_p \partial_\gamma .\theta_{\alpha\beta,J})(s) + 2\chi_{p>\gamma} x_p \partial_\gamma .(\theta_{\alpha\beta,J}(s)) = -(x_p \partial_\gamma .\theta_{\alpha\beta,J})(s) + 2x_p \partial_\gamma .(\theta_{\alpha\beta,J}(s)).$$

If $p < \gamma$ we have

$$-(x_p \partial_\gamma .\theta_{\alpha\beta,J})(s) + 2x_p \partial_\gamma .(\theta_{\alpha\beta,J}(s)) = (x_p \partial_\gamma .\theta_{\alpha\beta,J})(s)$$

since s is a highest weight vector, and the result follows. □

Now observe that all (non trivial) summands in any equation appearing in Corollary 7.5 have the same weight, and we call it the weight of the equation. Next result shows that if $\varphi : M(\lambda) \to M(\mu)$ is a morphism between finite Verma modules, then every equation of weight μ in Corollary 7.5 can be further simplified.

Corollary 7.6. *Let $\varphi : M(\lambda) \to M(\mu)$ be a morphism of finite Verma modules. If $J \in \mathcal{I}_{d-1}$ and a, b, c, p, q are such that Equation (29) has weight μ, then*

$$-2\theta^{p}_{J\setminus pq}(s) + \varepsilon_{pqabc}\sum_{\alpha\beta\gamma}(-1)^{\chi_{p>\gamma}}(x_p\partial_\gamma.\theta_{\alpha\beta,J})(s) = 0. \quad (33)$$

If K and a, b, c, p, q are such that Equation (30) has weight μ, then

$$\begin{aligned} &-4\theta^{a,p}_{K\setminus pq}(s) + \theta_{ab,bc,cq,K}(s) + \theta_{ac,cb,bq,K}(s) \\ &+2\varepsilon_{pqabc}\sum_{\alpha\beta\gamma}(-1)^{\chi_{p>\gamma}}(x_p\partial_\gamma.\theta^{a}_{\alpha\beta,K})(s) = 0. \end{aligned} \quad (34)$$

If K and a, b, c, p, q are such that Equation (31) has weight μ then

$$-4\theta^{p,p}_{K\setminus pq}(s) + \varepsilon_{pqabc}\sum_{\alpha\beta\gamma}(-1)^{\chi_{p>\gamma}}(x_p\partial_\gamma.\theta^{p}_{\alpha\beta,K})(s) = 0. \quad (35)$$

If K and a, b, c, p, q are such that Equation (32) has weight μ then

$$-2\theta^{p,q}_{K\setminus pq}(s) - \theta_{ab,bc,ca,K}(s) + \varepsilon_{pqabc}(-1)^{\chi_{p>\gamma}}(x_p\partial_\gamma.\theta^{q}_{\alpha\beta,K})(s) = 0. \quad (36)$$

Note that all equations appearing in Corollary 7.6 do not depend on the weights λ and μ: this observation will be the keypoint of our final classification.

8. Singular Vectors of Degree Between 5 and 10

If $w \in M(\mu)$ is a singular vector of degree d at most 10 we know that it also has height d by the results in Sect. 4. In particular we can express it as

$$w = \sum_{l\le d/2}\ \sum_{I\in\mathcal{I}_{d-2l}/\sim}\ \sum_{1\le r_1\le\cdots\le r_l\le 5} \partial_{r_1}\cdots\partial_{r_l}\omega_I \otimes \theta^{r_1,\ldots,r_l}_{I}(s), \quad (37)$$

where s is a highest weight vector in $F(\lambda)$.

Lemma 8.1. *If $w \in M(\mu)$ is a singular vector of degree and height d as in (37), then there exists $I_0 \in \mathcal{I}_d$ such that $\theta_{I_0}(s) \neq 0$ is a highest weight vector in $F(\mu)$.*

Proof. Since w has height d, we know that there exists $I \in \mathcal{I}_d$ such that $\theta_I(s) \neq 0$. Among all such I's choose I_0 such that $\theta_{I_0}(s)$ has maximal weight. Applying $x_i\partial_{i+1}$ to w we obtain a term $\omega_{I_0} \otimes x_i\partial_{i+1}.\theta_{I_0}(s)$, which cannot be simplified by any other term. Therefore $x_i\partial_{i+1}.\theta_{I_0}(s) = 0$. □

If we fix any possible I_0 as in Lemma 8.1, we can consider all equations in Corollary 7.6 with weight μ and we observe that these equations do not depend on μ. For example, if we choose $I_0 = (12, 24, 34, 45)$, we can consider Equation (33) with $(a, b, c, p, q) = (1, 2, 4, 5, 3)$ and $J = (25, 34, 45)$, getting

$$2\theta_{14,24,25,34}(s) + 2\theta_{12,24,34,45}(s) = 0,$$

and also with $(a, b, c, p, q) = (2, 4, 5, 3, 1)$ and $J = (14, 23, 34)$, getting

$$2\theta_{14,24,25,34}(s) = 0,$$

and deducing $\theta_{I_0}(s) = 0$.

Theorem 8.2. *Let*

$$w = \sum_{l \le d/2} \sum_{I \in \mathcal{I}_{d-2l}/\sim} \sum_{1 \le r_1 \le \cdots \le r_l \le 5} \partial_{r_1} \cdots \partial_{r_l} \omega_I \otimes \theta_I^{r_1,\dots,r_l}(s)$$

be a singular vector of degree d, with $5 \le d \le 10$ *and let* $I_0 \in \mathcal{I}_d$ *be such that* $\theta_{I_0}(s) \neq 0$ *is a highest weight vector. Then* $d = 7$ *and* $I_0 \sim (12, 13, 14, 15, 25, 35, 45)$ *or* $d = 5$ *and* $I_0 \sim (12, 13, 14, 15, 45)$ *or* $I_0 \sim (12, 15, 25, 35, 45)$.

Proof. The proof is based on Corollary 7.6. The set of Eqs. (33)–(36) of weight $\lambda(\theta_{I_0}(s))$ provides a system of homogeneous linear equations among all $\theta_I(s)$, $\theta_J^r(s)$ and $\theta_K^{r_1,r_2}(s)$ (with $I \in \mathcal{I}_d$, $J \in \mathcal{I}_{d-2}$ and $K \in \mathcal{I}_{d-4}$) such that θ_I, θ_J^r and $\theta_K^{r_1,r_2}$ have the same weight of θ_{I_0}, and which do not depend on (the weight of) s. This system can be solved with the help of a computer in all possible cases and one can check that it implies $\theta_{I_0}(s) = 0$ in all cases, but in the three exceptions stated above.

We add a few words to explain what happens in the most complicated case, i.e., $d = 10$ and $I_0 = (12, 13, 14, 15, 23, 24, 25, 34, 35, 45)$. In this case 86 variables are involved: $\theta_{I_0}(s)$, 15 vectors of the form $\theta_J^r(s)$, and 70 vectors of the form $\theta_K^{r_1,r_2}(s)$ with $r_1 \neq r_2$. Equations (33) and (35) do not provide any condition. In Equation (34) we can choose (a, b, c, p, q) to be any permutation in S_5 and $K = (ac, ap, aq, bp, bq, cp, pq)$, getting 120 linear equations among our 86 variables, and in Equation (36) we can choose (a, b, c, p, q) to be any permutation in S_5 (with $a < b < c$ to avoid repeated equations) and $K = (ap, aq, bp, bq, cp, cq, pq)$ getting 20 more equations. This system of 140 equations implies that all 86 variables involved vanish. □

Now we study the exceptions given by Theorem 8.2. The case of degree 7 leads to another new singular vector.

Theorem 8.3. *The following vector* $w[7] \in M(0, 0, 0, 2)$ *of weight* $(2, 0, 0, 0)$ *is the unique (up to a scalar factor) singular vector of degree 7 in a finite Verma module:*

$$\begin{aligned} w[7] = d_{12}d_{13}d_{14}d_{15}\Big(&d_{23}d_{24}d_{25} \otimes (x_2^*)^2 - d_{23}d_{25}d_{34} \otimes x_2^*x_3^* - d_{24}d_{25}d_{34} \otimes x_2^*x_4^* + d_{23}d_{24}d_{35} \otimes x_2^*x_3^* \\ &- d_{24}d_{25}d_{35} \otimes x_2^*x_5^* + d_{23}d_{34}d_{35} \otimes (x_3^*)^2 + d_{24}d_{34}d_{35} \otimes x_3^*x_4^* + d_{25}d_{34}d_{35} \otimes x_3^*x_5^* + d_{23}d_{24}d_{45} \otimes x_2^*x_4^* \\ &+ d_{23}d_{25}d_{45} \otimes x_2^*x_5^* + d_{23}d_{34}d_{45} \otimes x_3^*x_4^* + d_{24}d_{34}d_{45} \otimes (x_4^*)^2 + d_{25}d_{34}d_{45} \otimes x_4^*x_5^* + d_{23}d_{35}d_{45} \otimes x_3^*x_5^* \\ &+ d_{24}d_{35}d_{45} \otimes x_4^*x_5^* + d_{25}d_{35}d_{45} \otimes (x_5^*)^2 + \partial_1 d_{23} \otimes x_2^*x_3^* + \partial_1 d_{24} \otimes x_2^*x_4^* + \partial_1 d_{25} \otimes x_2^*x_5^* - \partial_2 d_{23} \otimes x_1^*x_3^* \\ &- \partial_2 d_{24} \otimes x_1^*x_4^* - \partial_2 d_{25} \otimes x_1^*x_5^* + \partial_3 d_{23} \otimes x_1^*x_2^* - \partial_3 d_{34} \otimes x_1^*x_4^* - \partial_3 d_{35} \otimes x_1^*x_5^* \\ &+ \partial_4 d_{24} \otimes x_1^*x_2^* + \partial_4 d_{34} \otimes x_1^*x_3^* - \partial_4 d_{45} \otimes x_1^*x_5^* + \partial_5 d_{25} \otimes x_1^*x_2^* + \partial_5 d_{35} \otimes x_1^*x_3^* + \partial_5 d_{45} \otimes x_1^*x_4^*\Big). \end{aligned}$$

Proof. Let $I_0 = (12, 13, 14, 15, 25, 35, 45)$. In this case Equations (33)–(36) of weight $\lambda(\theta_{I_0}(s))$ provide the following homogeneous linear relations:

$$\begin{aligned} \theta_{13,14,15,25,35,45}(s) = -2\theta^2_{14,15,25,35}(s) = 2\theta^2_{12,13,15,25,45}(s) = \\ -2\theta^3_{13,14,15,25,35}(s) = 2\theta^3_{12,13,15,35,45}(s) \\ = -2\theta^4_{13,14,15,25,45}(s) = 2\theta^4_{12,14,15,35,45}(s) = -4\theta^{2,2}_{12,15,25}(s) = -4\theta^{2,3}_{13,15,25}(s) \\ = -4\theta^{2,3}_{12,15,35}(s) = -4\theta^{3,3}_{13,15,35}(s) = -4\theta^{2,4}_{14,15,25}(s) = -4\theta^{2,4}_{12,15,45}(s) \\ = -4\theta^{3,4}_{14,15,35}(s) = -4\theta^{3,4}_{13,15,45}(s) = -4\theta^{4,4}_{14,15,45}(s). \end{aligned} \tag{38}$$

We use Equation (29) to determine the weight $\mu = (\mu_{12}, \mu_{23}, \mu_{34}, \mu_{45})$. Taking $(a, b, c, p, q) = (1, 2, 3, 4, 5)$ and $J = (13, 14, 15, 25, 35, 45)$ in (29) we obtain $\mu_{34} = 0$, using (38).

Taking $(a, b, c, p, q) = (4, 5, 1, 3, 2)$ and $J = (12, 13, 14, 15, 25, 35,)$ in (29) we obtain $\mu_{13} = 0$.

Taking $(a, b, c, p, q) = (1, 2, 4, 5, 3)$ and $J = (13, 14, 15, 25, 35, 45)$ in (29) we have

$$\begin{aligned}
0 &= -2\theta^5_{(13,14,15,25,35,45)\setminus(53)} - (x_5\partial_4.\theta_{12,13,14,15,25,35,45})(s) + 2x_5\partial_4(\theta_{12,13,14,15,25,35,45}(s)) \\
&\quad - (x_5\partial_1.\theta_{24,13,14,15,25,35,45})(s) \\
&= 2\theta^5_{13,14,15,25,45}(s) + \theta_{12,13,14,15,25,35,45}(s) + \theta_{12,13,14,15,25,34,45}(s) \\
&\quad + 2x_5\partial_4.(\theta_{12,13,14,15,25,35,45}(s)) \\
&\quad + \theta_{24,13,14,15,21,35,45}(s) \\
&= 2\theta^5_{13,14,15,25,45}(s) + 2\theta_{12,13,14,15,25,35,45}(s) + \theta_{12,13,14,15,25,34,45}(s) \\
&\quad + 2x_5\partial_4.(\theta_{12,13,14,15,25,35,45}(s)).
\end{aligned}$$

Finally we can apply $x_4\partial_5$ to this equation and use (38) to conclude

$$\begin{aligned}
0 &= 2\theta^4_{13,14,15,25,45}(s) - 2\theta_{12,13,14,15,25,35,45}(s) - \theta_{12,13,14,15,25,35,45}(s) \\
&\quad + 2\mu_{45}\theta_{12,13,14,15,25,35,45}(s) \\
&= 2(\mu_{45} - 2)\theta_{12,13,14,15,25,35,45}(s).
\end{aligned}$$

This shows that the only possible singular vector of degree 7 sits in $M(0, 0, 0, 2)$ and has weight $(0, 0, 0, 2) + \lambda(\omega_{12,13,14,15,25,35,45}) = (2, 0, 0, 0)$. The uniqueness of such singular vector follows from Lemma 8.1, since there are no other $I \in \mathcal{I}_7$ such that $\lambda(I) = \lambda(I_0)$. The fact that the displayed vector is indeed a singular vector can be checked with a long and technical calculation.

Note that (38) is consistent with the vector $w[7]$ since one can check that

$$\begin{aligned}
&4d_{12}d_{13}d_{14}d_{15}d_{25}d_{35}d_{45} = \\
&\ 4\omega_{13,14,15,25,35,45} - 2\partial_2\omega_{12,14,15,25,35} + 2\partial_2\omega_{12,13,15,25,45} \\
&\quad - 2\partial_3\omega_{13,14,15,25,35} + 2\partial_3\omega_{12,13,15,35,45} \\
&\quad - 2\partial_4\omega_{13,14,15,25,45} + 2\partial_4\omega_{12,14,15,35,45} - \partial_2^2\omega_{12,15,25} - \partial_2\partial_3\omega_{13,15,25} \\
&\quad - \partial_2\partial_3\omega_{12,15,35} - \partial_3^2\omega_{13,15,35} \\
&\quad - \partial_2\partial_4\omega_{14,15,25} - \partial_2\partial_4\omega_{12,15,45} - \partial_3\partial_4\omega_{14,15,35} - \partial_3\partial_4\omega_{13,15,45} - \partial_4^2\omega_{14,15,45}.
\end{aligned}$$

□

The two possible cases in degree 5 given by Theorem 8.2 are dual to each other. They lead to singular vectors which were already known to Rudakov in [13].

Theorem 8.4. *Let w be a singular vector of degree 5 in $M(\mu)$ of weight λ. Then one of the following occurs.*

(1) $\mu = (0, 0, 1, 0)$, $\lambda = (3, 0, 0, 0)$ *and*

$$w = w[5_{CD}] = d_{12}d_{13}d_{14}d_{15}(d_{45} \otimes x^*_{45} + d_{35} \otimes x^*_{35} + d_{25} \otimes x^*_{25} + d_{24} \otimes x^*_{24} + d_{23} \otimes x^*_{23});$$

(2) $\mu = (0, 0, 0, 3)$, $\lambda = (0, 1, 0, 0)$ *and* $w = w[5_{EA}] = d_{12}w[4_E]$, *where* $w[4_E]$ *is explicitly described in Sect. 11.*

Proof. By Theorem 8.2 we can assume that $\theta_{I_0}(s)$ is a highest weight vector with $I_0 = (12, 13, 14, 15, 45)$ or $I_0 = (12, 15, 25, 35, 45)$ and by Theorem 3.4 it is enough to show that the case $I_0 = (12, 13, 14, 15, 45)$ leads to conditions (1).

In this case Eqs. (33)–(36) easily provide the following relations:

$$\theta_{12,13,14,15,45}(s) = -2\theta^2_{12,14,15}(s) = -2\theta^3_{13,14,15}(s). \tag{39}$$

We use Equation (29) three times to show that necessarily $\mu = (0, 0, 1, 0)$. We first use Equation (29) with $(a, b, c, p, q) = (4, 5, 1, 3, 2)$ and $J = (12, 13, 14, 15)$. All terms but one vanish and we obtain

$$x_3\partial_1.(\theta_{45,12,13,14,15}(s)) = 0,$$

and so $\mu_{1,2} = \mu_{2,3} = 0$. Using Equation (9) with $(a, b, c, p, q) = (1, 2, 3, 4, 5)$ and $J = (13, 14, 15, 45)$, we obtain

$$\begin{aligned} 0 = &-2\theta^4_{(13,14,15,45)\setminus(45)} - (x_4\partial_3.\theta_{12,13,14,15,45})(s) \\ &+ 2x_4\partial_3.(\theta_{12,13,14,15,45}(s)) - (x_4\partial_2.\theta_{31,13,14,15,45})(s) \\ &- 2x_4\partial_2.(\theta_{31,13,14,15,45}(s)) - (x_4\partial_1.\theta_{23,13,14,15,45})(s) - 2x_4\partial_1.(\theta_{23,13,14,15,45}(s)) \\ = &\ 2\theta^4_{13,14,15}(s) + \theta_{12,13,14,15,35}(s) + 2x_4\partial_3.(\theta_{12,13,14,15,45}(s)), \end{aligned}$$

where we have used the fact that $\theta_{23,13,14,15,45}(s) = 0$ since it has weight greater than $\theta_{12,13,14,15,45}(s)$. Applying $x_3\partial_4$ to the previous equation, we obtain

$$-2\theta^3_{13,14,15}(s) - \theta_{12,13,14,15,45}(s) + 2\mu_{34}\theta_{12,13,14,15,45}(s) = 0.$$

By (39) we can conclude that $\mu_{34} = 1$.

Similarly, by Equation (9) with $(a, b, c, p, q) = (1, 2, 3, 5, 4)$ and $J = (13, 14, 15, 45)$, we obtain

$$0 = -2\theta^5_{(13,14,15,45)\setminus(54)} + (x_5\partial_3.\theta_{12,13,14,15,45})(s) - 2x_5\partial_3.(\theta_{12,13,14,15,45}(s))$$

and applying $x_3\partial_5$ and then using (39) we obtain

$$0 = -2\theta^3_{13,14,15}(s) + \theta_{12,13,14,15,45}(s) - 2\mu_{35}\theta_{12,13,14,15,45}(s) = 2(1 - \mu_{35})\theta_{12,13,14,15,45}(s).$$

So $\mu_{35} = 1$ and $\mu_{45} = \mu_{35} - \mu_{34} = 0$. The weight of w is $\lambda = \mu + \lambda(\omega_{12,13,14,15,45}) = (0, 0, 1, 0) + (3, 0, -1, 0) = (3, 0, 0, 0)$. The uniqueness follows by Lemma 8.1 and the verification that $d_{12}d_{13}d_{14}d_{15}(d_{45} \otimes x^*_{45} + d_{35} \otimes x^*_{35} + d_{25} \otimes x^*_{25} + d_{24} \otimes x^*_{24} + d_{23} \otimes x^*_{23})$ is actually such a singular vector is left to the reader. This shows, by duality, that there exists a (unique) singular vector in $M(0, 0, 0, 3)$ of weight $(0, 1, 0, 0)$, and one can check that it is given by $d_{12}w[4_E]$. □

9. Degree 4

The last case to be considered concerns singular vectors of degree and height 4.

Proposition 9.1. *Let w be a singular vector of degree 4 as in (22) and let $I_0 \in \mathcal{I}_4$ be such that $\theta_{I_0}(s) \neq 0$ is a highest weight vector in $M(\mu)$. Then $I_0 \sim (12, 13, 14, 15)$ or $I_0 \sim (15, 25, 35, 45)$.*

Proof. We make use of the duality in Theorem 3.4 to consider nearly a half of the cases. Indeed let w be a singular vector in $M(\mu)$ of weight λ such that $\theta_{I_0}(s)$ is a highest weight vector (of weight μ). Also consider the dual singular vector w^* in $M(\lambda^*)$ of weight μ^* and assume that $w^* = \varphi^*(s^*)$, where s^* is a highest weight vector in $M(\lambda^*)$, can be expressed as in (22) with θ^*'s instead of θ's, and with $\theta^*_{J_0}(s')$ of weight λ'. Then $\lambda(\theta_{I_0}) = \mu - \lambda$ and $\lambda(\theta^*_{J_0}) = \lambda^* - \mu^* = -\lambda(\theta_{I_0})^*$. In particular if there are no singular vectors such that $\theta_{I_0}(s) \neq 0$ for all I_0 such that $\lambda(\theta_{I_0}) = \lambda$ then there are no singular vectors such that $\theta_{I_0}(s) \neq 0$ for all I_0 such that $\lambda(\theta_{I_0}) = -\lambda^*$.

As in Theorem 8.2, we can use Eqs. (33)–(36), but in this case we have some more exceptions which must be considered apart.

More precisely we have that $\theta_{I_0}(s) = 0$ in all but the following cases (and their duals):

(1) (12, 13, 14, 15);
(2) (12, 14, 15, 23), (12, 13, 15, 24), (12, 13, 14, 25);
(3) (13, 23, 34, 35);
(4) (14, 23, 34, 35), (13, 24, 34, 35), (13, 23, 34, 45);
(5) (14, 24, 34, 35), (13, 24, 34, 45), (14, 23, 34, 45);
(6) (14, 24, 34, 45);
(7) (15, 24, 34, 45), (14, 34, 25, 45), (14, 24, 35, 45);
(8) (15, 25, 34, 45), (14, 25, 35, 45), (15, 24, 35, 45).

These exceptions have been grouped according to their weight. We now analyze all these cases.

(1) This is the case which is not excluded by the statement.
(2) Equations (33)–(36) provide $\theta_{12,14,15,23} = \theta_{12,13,14,25} = -\theta_{12,13,15,24}$. In this case Equation (29) with $a = 2$, $b = 3$, $c = 1$, $p = 5$, $q = 4$ and $J = (12, 14, 15)$ gives:

$$x_5\partial_2.(\theta_{31,12,14,15}(s)) + x_5\partial_1.(\theta_{23,12,14,15}(s)) = 0.$$

If we apply the vector field $x_1\partial_5$, we get the following equation:

$$x_1\partial_2.(\theta_{31,12,14,15}(s)) + \mu_{15}(\theta_{23,12,14,15}(s)) = 0,$$

where we used that if $\theta_{23,12,14,15}(s)$ has the highest weight, then $\theta_{35,12,14,15}(s) = \theta_{31,52,14,15}(s) = \theta_{31,12,54,15}(s) = 0$. Hence we get

$$-\theta_{32,12,14,15}(s) - \theta_{31,12,24,15}(s) - \theta_{31,12,14,15}(s) + \mu_{15}(\theta_{23,12,14,15}(s)) = 0,$$

i.e., $(-\mu_{15} - 3)(\theta_{12,14,15,23}(s)) = 0$, a contradiction.
(3) In this case Equation (29) with $a = 1$, $b = 3$, $c = 2$, $p = 5$, $q = 4$ and $J = (23, 34, 35)$ gives:

$$-x_5\partial_2.(\theta_{13,23,34,35}(s)) + x_5\partial_3.(\theta_{12,23,34,35}(s)) = 0.$$

If we apply the vector field $x_2\partial_5$, we get the following equation:

$$-\mu_{25}(\theta_{13,23,34,35}(s)) + x_2\partial_3.(\theta_{12,23,34,35}(s)) = 0,$$

where we used that if $\theta_{13,23,34,35}(s)$ has the highest weight then $\theta_{15,23,34,35}(s) = 0$. Hence we get

$$(-\mu_{25} - 1)(\theta_{13,23,34,35}(s)) = 0,$$

a contradiction.

(4) In this case Eqs. (33)–(36) provide $\theta_{14,23,34,35} = \theta_{13,24,34,35} = \theta_{13,23,34,45}$. Equation (29) with $a = 1$, $b = 2$, $c = 3$, $p = 5$, $q = 4$ and $J = (24, 34, 35)$ gives:

$$-(x_5\partial_1.\theta_{23,24,34,35})(s) + (x_5\partial_2.\theta_{31,24,34,35})(s) + (x_5\partial_3.\theta_{12,24,34,35})(s) \\ +2x_5\partial_2.(\theta_{31,24,34,35}(s)) + 2x_5\partial_3.(\theta_{12,24,34,35}(s)) = 0,$$

where we used that, if $\theta_{13,24,34,35}(s)$ has the highest weight, then $\theta_{23,24,34,35}(s) = 0$. This is equivalent to the following:

$$2\theta_{13,23,24,34}(s) - 2x_5\partial_2.(\theta_{13,24,34,35}(s)) + 2x_5\partial_3.(\theta_{12,24,34,35}(s)) = 0.$$

If we apply the vector field $x_2\partial_5$ and use the equality $\theta_{13,24,34,35}(s) = \theta_{13,23,34,45}(s)$, we get the following equation:

$$-\mu_{25}(\theta_{13,24,34,35}(s)) + x_2\partial_3.(\theta_{12,24,34,35}(s)) = 0,$$

where we used that, if $\theta_{13,24,34,35}(s)$ has the highest weight, then $\theta_{15,24,34,35}(s) = 0 = \theta_{12,34,35,45}(s)$. Hence we get

$$(-\mu_{25} - 1)(\theta_{13,24,34,35}(s)) = 0,$$

a contradiction.

(5) In this case Eqs. (33)–(36) provide $\theta_{14,24,34,35} = \theta_{14,23,34,45} = \theta_{13,24,34,45}$. Equation (25) with $a = 1$, $b = 4$, $c = 3$, $p = 5$, $q = 2$ and $J = (24, 34, 35)$ gives:

$$-x_5\partial_4.(\theta_{13,24,34,35}(s)) + x_5\partial_3.(\theta_{14,24,34,35}(s)) = 0.$$

If we apply the vector field $x_3\partial_5$, we get the following equation:

$$-x_3\partial_4.(\theta_{13,24,34,35}(s)) + \mu_{35}(\theta_{14,24,34,35}(s)) = 0,$$

where we used that if $\theta_{14,24,34,35}(s)$ has the highest weight, then $\theta_{15,24,34,35}(s) = 0 = \theta_{13,24,35,45}(s)$. Hence, using the hypothesis $\theta_{14,24,34,35}(s) = \theta_{13,24,34,45}(s)$, we get

$$(2 + \mu_{35})(\theta_{14,24,34,35}(s)) = 0,$$

a contradiction.

(6) In this case Equation (29) with $a = 1$, $b = 4$, $c = 3$, $p = 5$, $q = 2$ and $J = (24, 34, 45)$ gives:

$$-x_5\partial_4.(\theta_{13,24,34,45}(s)) + x_5\partial_3.(\theta_{14,24,34,45}(s)) = 0.$$

If we apply the vector field $x_3\partial_5$, we get the following equation:

$$-x_3\partial_4.(\theta_{13,24,34,45}(s)) + \mu_{35}(\theta_{14,24,34,45}(s)) = 0,$$

where we used that if $\theta_{14,24,34,45}(s)$ has the highest weight, then $\theta_{15,24,34,45}(s) = 0$. Hence we get

$$(1 + \mu_{35})(\theta_{14,24,34,45}(s)) = 0,$$

a contradiction.

(7) In this case Eqs. (33)–(36) provide $\theta_{15,24,34,45} = \theta_{14,25,34,45} = \theta_{14,24,35,45}$. Equation (29) with $a = 1$, $b = 2$, $c = 4$, $p = 5$, $q = 3$ and $J = (15, 34, 45)$ gives:

$$\begin{aligned}&-(x_5\partial_1.\theta_{24,15,34,45})(s) - (x_5\partial_2.\theta_{41,15,34,45})(s) - (x_5\partial_4.\theta_{12,15,34,45})(s)\\ &+ 2x_5\partial_1.(\theta_{24,15,34,45}(s)) + 2x_5\partial_2.(\theta_{41,15,34,45}(s))\\ &+ 2x_5\partial_4.(\theta_{12,15,34,45}(s)) = 0,\end{aligned}$$

which is equivalent to the following equation:

$$\begin{aligned}-2\theta_{14,15,24,34}(s) + 2\theta_{12,14,34,45}(s) - 2x_5\partial_1.(\theta_{15,24,34,45}(s)) - 2x_5\partial_2.(\theta_{14,15,34,45}(s))\\ + 2x_5\partial_4.(\theta_{12,15,34,45}(s)) = 0.\end{aligned}$$

If we apply the vector field $x_1\partial_5$ we get the following equation:

$$\begin{aligned}-2\theta_{54,15,24,34}(s) - 2\theta_{52,14,34,45}(s) - 2\mu_{15}(\theta_{15,24,34,45}(s)) - 2x_1\partial_2.(\theta_{14,15,34,45}(s))\\ + 2x_1\partial_4.(\theta_{12,15,34,45}(s)) = 0,\end{aligned}$$

where we used that if $\theta_{15,24,34,45}(s)$ has highest weight then $\theta_{15,25,34,45}(s) = 0$. Hence we get

$$(-6 - 2\mu_{15})(\theta_{15,24,34,45}(s)) = 0,$$

a contradiction.

(8) We have by Eqs. (33)–(36) $\theta_{15,25,34,45} = \theta_{15,24,35,45} = \theta_{14,25,35,45}$ and $\theta^1_{15,45} = \theta^2_{25,45} = \theta^3_{35,45} = 0$. In this case Equation (29) with $a = 1$, $b = 2$, $c = 4$, $p = 5$, $q = 3$ and $J = (25, 35, 45)$ gives:

$$\begin{aligned}-2\theta^5_{J\setminus\{53\}}(s) - (x_5\partial_1.\theta_{24,25,35,45})(s) - (x_5\partial_2.\theta_{41,25,35,45})(s) - (x_5\partial_4.\theta_{12,25,35,45})(s)\\ -2x_5\partial_2.(\theta_{14,25,35,45}(s)) + 2x_5\partial_4.(\theta_{12,25,35,45}(s)) = 0\end{aligned}$$

where we used that if $\theta_{14,25,35,45}(s)$ has highest weight then $\theta_{24,25,35,45}(s) = 0$. Hence we have:

$$\begin{aligned}-2\theta^5_{J\setminus\{53\}}(s) + 2\theta_{12,24,35,45}(s) + \theta_{24,25,31,45}(s) + \theta_{41,25,32,45}(s) + 2\theta_{14,24,25,35}(s)\\ +\theta_{12,25,34,45}(s) - 2x_5\partial_2.(\theta_{14,25,35,45}(s)) + 2x_5\partial_4.(\theta_{12,25,35,45}(s)) = 0.\end{aligned}$$

If we apply the vector field $x_2\partial_5$ we obtain the following equation:

$$\begin{aligned}-2\theta_{15,24,35,45}(s) - \theta_{41,25,35,45}(s) - 2\theta_{14,54,25,35}(s)\\ -\theta_{15,25,34,45}(s) - 2\mu_{25}(\theta_{14,25,35,45}(s)) - 2\theta_{14,25,35,45}(s) = 0\end{aligned}$$

where we used $\theta^2_{25,45}(s) = 0$ and that if $\theta_{14,25,35,45}(s)$ has highest weight then $\theta_{15,25,35,45}(s) = 0$. Now, using the hypotheses $\theta_{15,25,34,45} = \theta_{15,24,35,45} = \theta_{14,25,35,45}$, we get:

$$-2(\mu_{25} + 1)\theta_{14,25,35,45},$$

a contradiction.

□

The following result completes the study of singular vectors of degree 4

Theorem 9.2. *Let w be a singular vector in $M(\mu)$ of weight λ and degree 4. Then one of the following occurs:*

(1) $\mu = (n, 0, 0, 0)$, $\lambda = (n+3, 0, 0, 0)$ *and* $w = d_{12}d_{13}d_{14}d_{15} \otimes x_1^n$ *for some* $n \in \mathbb{N}$.
(2) $\mu = (0, 0, 0, n+3)$, $\lambda = (0, 0, 0, n)$ *and* $w = w[4_E]$ *(see Sect. 11) for some* $n \in \mathbb{N}$.

Proof. By Proposition 9.1 we know that we can assume that w is as in (22) with $\theta_{12,13,14,15}(s)$ a highest weight vector. By (29) with $(a, b, c, p, q) = (1, 2, 3, 5, 4)$ and $J = (12, 14, 15)$ we immediately get $x_5\partial_2.(\theta_{12,13,14,15}(s)) = 0$ (recalling that $\theta_{12,23,14,15}(s) = 0$ for weight reasons) and therefore $\mu = (n, 0, 0, 0)$ for some n and $\lambda = \lambda(\omega_{12,13,14,15}) + (n, 0, 0, 0) = (n+3, 0, 0, 0)$. The uniqueness follows by Lemma 8.1. It is a trivial check that the vector $d_{12}d_{13}d_{14}d_{15} \otimes x_1^n$ is such a singular vector. By duality we have that the other possible case in Proposition 9.1 leads to a unique singular vector in $M(0, 0, 0, n+3)$ of weight $(0, 0, 0, n)$. The fact that this vector is actually $w[4_E]$ displayed in Sect. 10 is a huge verification that can be made with a computer. □

10. Conclusions

As a result of discussions in the previous sections we are now in a position to state a complete classification of singular vectors in finite Verma modules for $E(5, 10)$.

Theorem 10.1. *The following is a complete classification of singular vectors and corresponding morphisms for finite Verma modules.*

In degree 1 we have

- $w[1_A] = d_{12} \otimes x_1^m x_{12}^n \in M(m, n, 0, 0)$ *for all* $m, n \geq 0$, *giving a morphism*

$$\varphi[1_A] : M(m, n+1, 0, 0) \to M(m, n, 0, 0);$$

- $w[1_B] = d_{15} \otimes x_1^m (x_5^*)^{n+1} + d_{14} \otimes x_1^m x_4^* (x_5^*)^n + d_{13} \otimes x_1^m x_3^* (x_5^*)^n + d_{12} \otimes x_1^m x_2^* (x_5^*)^n$ *for all* $m, n \geq 0$, *giving a morphism*

$$\varphi[1_B] : M(m+1, 0, 0, n) \to M(m, 0, 0, n+1);$$

- $w[1_C] = \sum_{i<j} d_{ij} \otimes x_{ij}^* (x_{45}^*)^m (x_5 *)^n$ *for all* $m, n \geq 0$, *giving a morphism*

$$\varphi[1_C] : M(0, 0, m, n) \to M(0, 0, m+1, n).$$

In degree 2 we have

- $w[2_{BA}] = \sum_{j>1} d_{12}d_{1j} \otimes x_1^m x_j^* \in M(m, 0, 0, 1)$ *for all* $m \geq 0$, *giving a morphism*

$$\varphi[2_{BA}] = \varphi[1_B] \circ \varphi[1_A] : M(m+1, 1, 0, 0) \to M(m, 0, 0, 1);$$

- $w[2_{CB}] = \sum_{j>1} \sum_{h<k} d_{1j}d_{hk} \otimes x_{hk}^* x_j^* (x_5^*)^n \in M(0, 0, 1, n+1)$ *for all* $n \geq 0$, *giving a morphism*

$$\varphi[2_{CB}] = \varphi[1_C] \circ \varphi[1_B] : M(1, 0, 0, n) \to M(0, 0, 1, n+1);$$

- $w[2_{CA}] = \sum_{i<j} d_{12}d_{ij} \otimes x_{ij}^* \in M(0, 0, 1, 0)$, *giving a morphism*

$$\varphi[2_{CA}] = \varphi[1_C] \circ \varphi[1_A] : M(0, 1, 0, 0) \to M(0, 0, 1, 0).$$

In degree 3 we have

- $w[3_{CBA}] = \sum_{j>1} \sum_{k<l} d_{12}d_{1j}d_{kl} \otimes x_j^* x_{kl}^* \in M(0, 0, 1, 1)$, *giving a morphism*

$$\varphi[3_{CBA}]) = \varphi[1_C] \circ \varphi[1_B] \circ \varphi[1_A] : M(1, 1, 0, 0) \to M(0, 0, 1, 1).$$

In degree 4 we have

- $w[4_D] = d_{12}d_{13}d_{14}d_{15} \otimes x_1^m \in M(m, 0, 0, 0)$ *for all* $m \geq 0$, *giving a morphism*

$$\varphi[4_D] : M(m + 3, 0, 0, 0) \to M(m, 0, 0, 0);$$

- $w[4_E] \in M(0, 0, 0, n + 3)$ *displayed in Section 11, giving a morphism*

$$\varphi[4_E] : M(0, 0, 0, n) \to M(0, 0, 0, n + 3).$$

In degree 5 we have

- $w[5_{CD}] = d_{12}d_{13}d_{14}d_{15} \sum_{2<i<j} d_{ij} \otimes x_{ij}^*$, *giving a morphism*

$$\varphi[5_{CD}] = \varphi[1_C] \circ \varphi : [4_D] : M(3, 0, 0, 0) \to M(0, 0, 1, 0);$$

- $w[5_{EA}] = d_{12} w[4_E] \in M(0, 0, 0, 3)$, *giving a morphism*

$$\varphi[5_{EA}] = \varphi[4_E] \circ \varphi[1_A] : M(0, 1, 0, 0) \to M(0, 0, 0, 3).$$

In degree 7 we have

- $w[7] \in M(0, 0, 0, 2)$ *given in Theorem 8.3, giving a morphism*

$$\varphi[7] : M(2, 0, 0, 0) \to M(0, 0, 0, 2).$$

In degree 11 we have

- $w[11] \in M(0, 0, 0, 1)$ *given in Theorem 5.11, giving a morphism*

$$\varphi[11] : M(1, 0, 0, 0) \to M(0, 0, 0, 1).$$

Proof. In [9] singular vectors of degree 1 were constructed, and in [13] it was shown that there are no other ones. In [13] singular vectors of degree 2,3,4 and 5 were constructed, and it was shown in [2] that there are no other ones of degree 2 and 3. In the present paper we show that there are no other singular vectors of degree 4 and 5. Namely, singular vectors of degree 4 are classified in Section 9, singular vectors of degree that is equal to height, which is greater than 5, are classified in Section 8, and singular vectors of degree greater than height are classified in Section 5. □

This theorem gives an affirmative answer to the conjecture posed in [9].

Corollary 10.2. *All degenerate finite Verma modules over* $E(5, 10)$ *are of the form* $M(m, n, 0, 0)$, $M(0, 0, m, n)$ *or* $M(m, 0, 0, n)$, *where* $m, n \in \mathbb{N}$.

Figure 4 represents all morphisms between finite degenerate Verma modules, which are not compositions of other morphisms.

Since a singular vector of weight μ in a finite Verma module with highest weight λ corresponds to a non-zero morphism $M(\mu) \to M(\lambda)$, we can construct infinite sequences of morphisms as in Fig. 4. All of these sequences are complexes (i.e. all compositions of consecutive morphisms vanish), except for the one through the origin; the latter becomes a complex if we replace the sequence

$$M(0, 1, 0, 0) \to M(0, 0, 0, 0) \to M(1, 0, 0, 0)$$

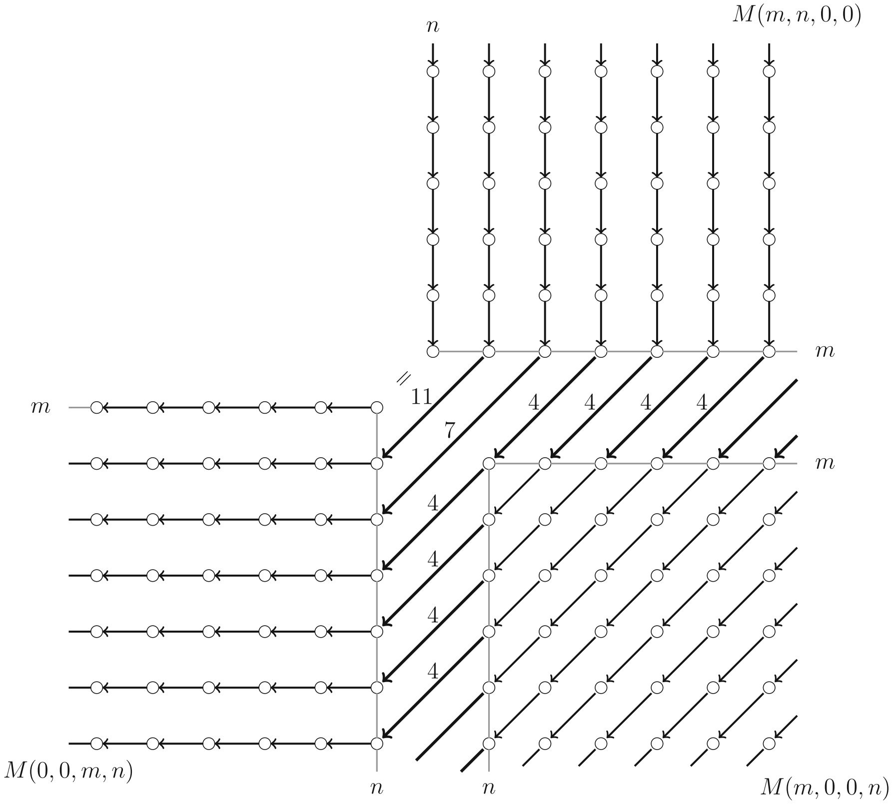

Fig. 4. All nonzero morphisms between finite Verma modules for $E(5, 10)$ which are not compositions of other morphisms. Morphisms of degree > 1 are labelled by their degree

of morphisms of degree 1 by their composition of degree 2. This claim, when morphisms of degree 7 and 11 are not involved, follows from [13]; if they are involved, it follows since there are no morphisms of degree 8 and 12.

Figure 5 represents all morphisms between finite degenerate Verma modules of degree 2 and 3; the corresponding bilateral infinite sequences shown in this picture are complexes, since any possible composition of two of these morphisms does not appear in Theorem 10.1.

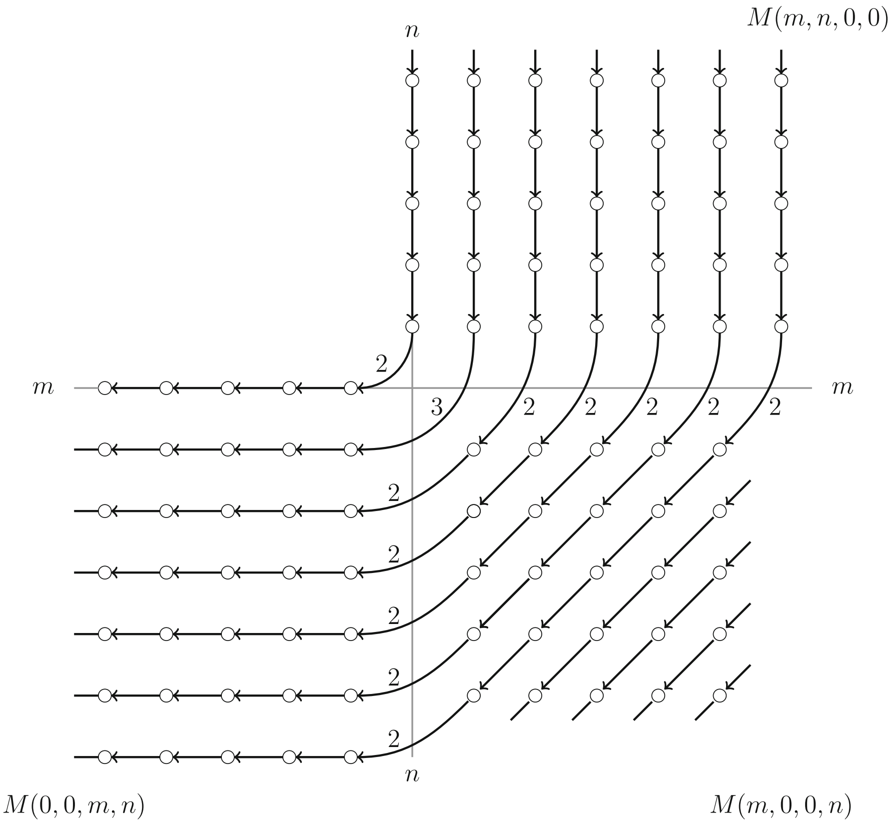

Fig. 5. All nonzero morphisms between finite Verma modules for $E(5, 10)$ of degree 2 and 3 and their infinite bilateral complexes

11. The singular vector in $M(0, 0, 0, n + 3)$ of degree 4 and weight $(0, 0, 0, n)$

The following is the singular vector (which has been found and checked with the aid of a computer) in $M(0, 0, 0, n + 3)$ of degree 4 and weight $(0, 0, 0, n)$. Here we denote x_i^* by f_i and we omit all tensor product symbols.

$$\begin{aligned}
w[4_E] = {} & d_{12}d_{13}d_{14}d_{15} f_1^3 f_5^n + d_{12}d_{14}d_{15}d_{23} f_1^2 f_2 f_5^n + d_{13}d_{14}d_{15}d_{23} f_1^2 f_3 f_5^n - d_{12}d_{13}d_{15}d_{24} f_1^2 f_2 f_5^n \\
& + d_{13}d_{14}d_{15}d_{24} f_1^2 f_4 f_5^n + d_{12}d_{15}d_{23}d_{24} f_1 f_2^2 f_5^n + d_{13}d_{15}d_{23}d_{24} f_1 f_2 f_3 f_5^n + d_{14}d_{15}d_{23}d_{24} f_1 f_2 f_4 f_5^n \\
& + d_{12}d_{13}d_{14}d_{25} f_1^2 f_2 f_5^n + d_{13}d_{14}d_{15}d_{25} f_1^2 f_5^{n+1} - d_{12}d_{14}d_{23}d_{25} f_1 f_2^2 f_5^n - d_{13}d_{14}d_{23}d_{25} f_1 f_2 f_3 f_5^n \\
& + d_{14}d_{15}d_{23}d_{25} f_1 f_2 f_5^{n+1} + d_{12}d_{13}d_{24}d_{25} f_1 f_2^2 f_5^n - d_{13}d_{14}d_{24}d_{25} f_1 f_2 f_4 f_5^n - d_{13}d_{15}d_{24}d_{25} f_1 f_2 f_5^{n+1} \\
& + d_{12}d_{23}d_{24}d_{25} f_2^3 f_5^n + d_{13}d_{23}d_{24}d_{25} f_2^2 f_3 f_5^n + d_{14}d_{23}d_{24}d_{25} f_2^2 f_4 f_5^n + d_{15}d_{23}d_{24}d_{25} f_2^2 f_5^{n+1} \\
& - d_{12}d_{13}d_{15}d_{34} f_1^2 f_3 f_5^n - d_{12}d_{14}d_{15}d_{34} f_1^2 f_4 f_5^n + d_{12}d_{15}d_{23}d_{34} f_1 f_2 f_3 f_5^n + d_{13}d_{15}d_{23}d_{34} f_1 f_3^2 f_5^n \\
& + d_{14}d_{15}d_{23}d_{34} f_1 f_3 f_4 f_5^n + d_{12}d_{15}d_{24}d_{34} f_1 f_2 f_4 f_5^n + d_{13}d_{15}d_{24}d_{34} f_1 f_3 f_4 f_5^n + d_{14}d_{15}d_{24}d_{34} f_1 f_4^2 f_5^n \\
& - d_{12}d_{13}d_{25}d_{34} f_1 f_2 f_3 f_5^n - d_{12}d_{14}d_{25}d_{34} f_1 f_2 f_4 f_5^n + d_{13}d_{15}d_{25}d_{34} f_1 f_3 f_5^{n+1} + d_{14}d_{15}d_{25}d_{34} f_1 f_4 f_5^{n+1} \\
& - d_{12}d_{23}d_{25}d_{34} f_2^2 f_3 f_5^n - d_{13}d_{23}d_{25}d_{34} f_2 f_3^2 f_5^n - d_{14}d_{23}d_{25}d_{34} f_2 f_3 f_4 f_5^n - d_{15}d_{23}d_{25}d_{34} f_2 f_3 f_5^{n+1} \\
& - d_{12}d_{24}d_{25}d_{34} f_2^2 f_4 f_5^n - d_{13}d_{24}d_{25}d_{34} f_2 f_3 f_4 f_5^n - d_{14}d_{24}d_{25}d_{34} f_2 f_4^2 f_5^n - d_{15}d_{24}d_{25}d_{34} f_2 f_4 f_5^{n+1} \\
& + d_{12}d_{13}d_{14}d_{35} f_1^2 f_3 f_5^n - d_{12}d_{14}d_{15}d_{35} f_1^2 f_5^{n+1} - d_{12}d_{14}d_{23}d_{35} f_1 f_2 f_3 f_5^n - d_{13}d_{14}d_{23}d_{35} f_1 f_3^2 f_5^n
\end{aligned}$$

$$+ d_{14}d_{15}d_{23}d_{35} f_1 f_3 f_5^{n+1} + d_{12}d_{13}d_{24}d_{35} f_1 f_2 f_3 f_5^{n} - d_{13}d_{14}d_{24}d_{35} f_1 f_3 f_4 f_5^{n} + d_{12}d_{15}d_{24}d_{35} f_1 f_2 f_5^{n+1}$$
$$+ d_{14}d_{15}d_{24}d_{35} f_1 f_4 f_5^{n+1} + d_{12}d_{23}d_{24}d_{35} f_2^2 f_3 f_5^{n} + d_{13}d_{23}d_{24}d_{35} f_2 f_3^2 f_5^{n} + d_{14}d_{23}d_{24}d_{35} f_2 f_3 f_4 f_5^{n}$$
$$+ d_{15}d_{23}d_{24}d_{35} f_2 f_3 f_5^{n+1} - d_{12}d_{14}d_{25}d_{35} f_1 f_2 f_5^{n+1} - d_{13}d_{14}d_{25}d_{35} f_1 f_3 f_5^{n+1} + d_{14}d_{15}d_{25}d_{35} f_1 f_5^{n+2}$$
$$- d_{12}d_{24}d_{25}d_{35} f_2^2 f_5^{n+1} - d_{13}d_{24}d_{25}d_{35} f_2 f_3 f_5^{n+1} - d_{14}d_{24}d_{25}d_{35} f_2 f_4 f_5^{n+1} - d_{15}d_{24}d_{25}d_{35} f_2 f_5^{n+2}$$
$$+ d_{12}d_{13}d_{34}d_{35} f_1 f_3^2 f_5^{n} + d_{12}d_{14}d_{34}d_{35} f_1 f_3 f_4 f_5^{n} + d_{12}d_{15}d_{34}d_{35} f_1 f_3 f_5^{n+1} + d_{12}d_{23}d_{34}d_{35} f_2 f_3^2 f_5^{n}$$
$$+ d_{13}d_{23}d_{34}d_{35} f_3^3 f_5^{n} + d_{14}d_{23}d_{34}d_{35} f_3^2 f_4 f_5^{n} + d_{15}d_{23}d_{34}d_{35} f_3^2 f_5^{n+1} + d_{12}d_{24}d_{34}d_{35} f_2 f_3 f_4 f_5^{n}$$
$$+ d_{13}d_{24}d_{34}d_{35} f_3^2 f_4 f_5^{n} + d_{14}d_{24}d_{34}d_{35} f_3 f_4^2 f_5^{n} + d_{15}d_{24}d_{34}d_{35} f_3 f_4 f_5^{n+1} + d_{12}d_{25}d_{34}d_{35} f_2 f_3 f_5^{n+1}$$
$$+ d_{13}d_{25}d_{34}d_{35} f_3^2 f_5^{n+1} + d_{14}d_{25}d_{34}d_{35} f_3 f_4 f_5^{n+1} + d_{15}d_{25}d_{34}d_{35} f_3 f_5^{n+2} + d_{12}d_{13}d_{14}d_{45} f_1^2 f_4 f_5^{n}$$
$$+ d_{12}d_{13}d_{15}d_{45} f_1^2 f_5^{n+1} - d_{12}d_{14}d_{23}d_{45} f_1 f_2 f_4 f_5^{n} - d_{13}d_{14}d_{23}d_{45} f_1 f_3 f_4 f_5^{n} - d_{12}d_{15}d_{23}d_{45} f_1 f_2 f_5^{n+1}$$
$$- d_{13}d_{15}d_{23}d_{45} f_1 f_3 f_5^{n+1} + d_{12}d_{13}d_{24}d_{45} f_1 f_2 f_4 f_5^{n} - d_{13}d_{14}d_{24}d_{45} f_1 f_4^2 f_5^{n} - d_{13}d_{15}d_{24}d_{45} f_1 f_4 f_5^{n+1}$$
$$+ d_{12}d_{23}d_{24}d_{45} f_2^2 f_4 f_5^{n} + d_{13}d_{23}d_{24}d_{45} f_2 f_3 f_4 f_5^{n} + d_{14}d_{23}d_{24}d_{45} f_2 f_4^2 f_5^{n} + d_{15}d_{23}d_{24}d_{45} f_2 f_4 f_5^{n+1}$$
$$+ d_{12}d_{13}d_{25}d_{45} f_1 f_2 f_5^{n+1} - d_{13}d_{14}d_{25}d_{45} f_1 f_4 f_5^{n+1} - d_{13}d_{15}d_{25}d_{45} f_1 f_5^{n+2} + d_{12}d_{23}d_{25}d_{45} f_2^2 f_5^{n+1}$$
$$+ d_{13}d_{23}d_{25}d_{45} f_2 f_3 f_5^{n+1} + d_{14}d_{23}d_{25}d_{45} f_2 f_4 f_5^{n+1} + d_{15}d_{23}d_{25}d_{45} f_2 f_5^{n+2} + d_{12}d_{13}d_{34}d_{45} f_1 f_3 f_4 f_5^{n}$$
$$+ d_{12}d_{14}d_{34}d_{45} f_1 f_4^2 f_5^{n} + d_{12}d_{15}d_{34}d_{45} f_1 f_4 f_5^{n+1} + d_{12}d_{23}d_{34}d_{45} f_2 f_3 f_4 f_5^{n} + d_{13}d_{23}d_{34}d_{45} f_3^2 f_4 f_5^{n}$$
$$+ d_{14}d_{23}d_{34}d_{45} f_3 f_4^2 f_5^{n} + d_{15}d_{23}d_{34}d_{45} f_3 f_4 f_5^{n+1} + d_{12}d_{24}d_{34}d_{45} f_2 f_4^2 f_5^{n} + d_{13}d_{24}d_{34}d_{45} f_3 f_4^2 f_5^{n}$$
$$+ d_{14}d_{24}d_{34}d_{45} f_4^3 f_5^{n} + d_{15}d_{24}d_{34}d_{45} f_4^2 f_5^{n+1} + d_{12}d_{25}d_{34}d_{45} f_2 f_4 f_5^{n+1} + d_{13}d_{25}d_{34}d_{45} f_3 f_4 f_5^{n+1}$$
$$+ d_{14}d_{25}d_{34}d_{45} f_4^2 f_5^{n+1} + d_{15}d_{25}d_{34}d_{45} f_4 f_5^{n+2} + d_{12}d_{13}d_{35}d_{45} f_1 f_3 f_5^{n+1} + d_{12}d_{14}d_{35}d_{45} f_1 f_4 f_5^{n+1}$$
$$+ d_{12}d_{15}d_{35}d_{45} f_1 f_5^{n+2} + d_{12}d_{23}d_{35}d_{45} f_2 f_3 f_5^{n+1} + d_{13}d_{23}d_{35}d_{45} f_3^2 f_5^{n+1} + d_{14}d_{23}d_{35}d_{45} f_3 f_4 f_5^{n+1}$$
$$+ d_{15}d_{23}d_{35}d_{45} f_3 f_5^{n+2} + d_{12}d_{24}d_{35}d_{45} f_2 f_4 f_5^{n+1} + d_{13}d_{24}d_{35}d_{45} f_3 f_4 f_5^{n+1} + d_{14}d_{24}d_{35}d_{45} f_4^2 f_5^{n+1}$$
$$+ d_{15}d_{24}d_{35}d_{45} f_4 f_5^{n+2} + d_{12}d_{25}d_{35}d_{45} f_2 f_5^{n+2} + d_{13}d_{25}d_{35}d_{45} f_3 f_5^{n+2} + d_{14}d_{25}d_{35}d_{45} f_4 f_5^{n+2}$$
$$+ d_{15}d_{25}d_{35}d_{45} f_5^{n+3} + \partial_1 d_{12}d_{13} f_1 f_2 f_3 f_5^{n} + \partial_1 d_{12}d_{14} f_1 f_2 f_4 f_5^{n} + \partial_1 d_{12}d_{15} f_1 f_2 f_5^{n+1}$$
$$+ \partial_1 d_{12}d_{23} f_2^2 f_3 f_5^{n} + \partial_1 d_{13}d_{23} f_2 f_3^2 f_5^{n} + \partial_1 d_{14}d_{23} f_2 f_3 f_4 f_5^{n} + \partial_1 d_{15}d_{23} f_2 f_3 f_5^{n+1}$$
$$+ \partial_1 d_{12}d_{24} f_2^2 f_4 f_5^{n} + \partial_1 d_{13}d_{24} f_2 f_3 f_4 f_5^{n} + \partial_1 d_{14}d_{24} f_2 f_4^2 f_5^{n} + \partial_1 d_{15}d_{24} f_2 f_4 f_5^{n+1}$$
$$+ \partial_1 d_{12}d_{25} f_2^2 f_5^{n+1} + \partial_1 d_{13}d_{25} f_2 f_3 f_5^{n+1} + \partial_1 d_{14}d_{25} f_2 f_4 f_5^{n+1} + \partial_1 d_{15}d_{25} f_2 f_5^{n+2}$$
$$- \partial_2 d_{12}d_{13} f_1^2 f_3 f_5^{n} - \partial_2 d_{12}d_{14} f_1^2 f_4 f_5^{n} - \partial_2 d_{12}d_{15} f_1^2 f_5^{n+1} - \partial_2 d_{12}d_{23} f_1 f_2 f_3 f_5^{n}$$
$$- \partial_2 d_{13}d_{23} f_1 f_3^2 f_5^{n} - \partial_2 d_{14}d_{23} f_1 f_3 f_4 f_5^{n} - \partial_2 d_{15}d_{23} f_1 f_3 f_5^{n+1} - \partial_2 d_{12}d_{24} f_1 f_2 f_4 f_5^{n}$$
$$- \partial_2 d_{13}d_{24} f_1 f_3 f_4 f_5^{n} - \partial_2 d_{14}d_{24} f_1 f_4^2 f_5^{n} - \partial_2 d_{15}d_{24} f_1 f_4 f_5^{n+1} - \partial_2 d_{12}d_{25} f_1 f_2 f_5^{n+1}$$
$$- \partial_2 d_{13}d_{25} f_1 f_3 f_5^{n+1} - \partial_2 d_{14}d_{25} f_1 f_4 f_5^{n+1} - \partial_2 d_{15}d_{25} f_1 f_5^{n+2} + \partial_3 d_{12}d_{13} f_1^2 f_2 f_5^{n}$$
$$- \partial_3 d_{13}d_{14} f_1^2 f_4 f_5^{n} - \partial_3 d_{13}d_{15} f_1^2 f_5^{n+1} + \partial_3 d_{12}d_{23} f_1 f_2^2 f_5^{n} + \partial_3 d_{13}d_{23} f_1 f_2 f_3 f_5^{n}$$
$$+ \partial_3 d_{14}d_{23} f_1 f_2 f_4 f_5^{n} + \partial_3 d_{15}d_{23} f_1 f_2 f_5^{n+1} - \partial_3 d_{12}d_{34} f_1 f_2 f_4 f_5^{n} - \partial_3 d_{13}d_{34} f_1 f_3 f_4 f_5^{n}$$
$$- \partial_3 d_{14}d_{34} f_1 f_4^2 f_5^{n} - \partial_3 d_{15}d_{34} f_1 f_4 f_5^{n+1} - \partial_3 d_{12}d_{35} f_1 f_2 f_5^{n+1} - \partial_3 d_{13}d_{35} f_1 f_3 f_5^{n+1}$$
$$- \partial_3 d_{14}d_{35} f_1 f_4 f_5^{n+1} - \partial_3 d_{15}d_{35} f_1 f_5^{n+2} + \partial_4 d_{12}d_{14} f_1^2 f_2 f_5^{n} + \partial_4 d_{13}d_{14} f_1^2 f_3 f_5^{n}$$
$$- \partial_4 d_{14}d_{15} f_1^2 f_5^{n+1} + \partial_4 d_{12}d_{24} f_1 f_2^2 f_5^{n} + \partial_4 d_{13}d_{24} f_1 f_2 f_3 f_5^{n} + \partial_4 d_{14}d_{24} f_1 f_2 f_4 f_5^{n}$$
$$+ \partial_4 d_{15}d_{24} f_1 f_2 f_5^{n+1} + \partial_4 d_{12}d_{34} f_1 f_2 f_3 f_5^{n} + \partial_4 d_{13}d_{34} f_1 f_3^2 f_5^{n} + \partial_4 d_{14}d_{34} f_1 f_3 f_4 f_5^{n}$$
$$+ \partial_4 d_{15}d_{34} f_1 f_3 f_5^{n+1} - \partial_4 d_{12}d_{45} f_1 f_2 f_5^{n+1} - \partial_4 d_{13}d_{45} f_1 f_3 f_5^{n+1} - \partial_4 d_{14}d_{45} f_1 f_4 f_5^{n+1}$$
$$- \partial_4 d_{15}d_{45} f_1 f_5^{n+2} + \partial_5 d_{12}d_{15} f_1^2 f_2 f_5^{n} + \partial_5 d_{13}d_{15} f_1^2 f_3 f_5^{n} + \partial_5 d_{14}d_{15} f_1^2 f_4 f_5^{n}$$
$$+ \partial_5 d_{12}d_{25} f_1 f_2^2 f_5^{n} + \partial_5 d_{13}d_{25} f_1 f_2 f_3 f_5^{n} + \partial_5 d_{14}d_{25} f_1 f_2 f_4 f_5^{n} + \partial_5 d_{15}d_{25} f_1 f_2 f_5^{n+1}$$
$$+ \partial_5 d_{12}d_{35} f_1 f_2 f_3 f_5^{n} + \partial_5 d_{13}d_{35} f_1 f_3^2 f_5^{n} + \partial_5 d_{14}d_{35} f_1 f_3 f_4 f_5^{n} + \partial_5 d_{15}d_{35} f_1 f_3 f_5^{n+1}$$
$$+ \partial_5 d_{12}d_{45} f_1 f_2 f_4 f_5^{n} + \partial_5 d_{13}d_{45} f_1 f_3 f_4 f_5^{n} + \partial_5 d_{14}d_{45} f_1 f_4^2 f_5^{n} + \partial_5 d_{15}d_{45} f_1 f_4 f_5^{n+1}.$$

We recall that the construction of this vector is also sketched by Rudakov in [13, §4].

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