Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 54-7 (2021) 363–368

Recursive identification of errors-in-variables models with correlated output noise

Matteo Barbieri * **Roberto Diversi** *

* *Department of Electrical, Electronic, and Information Engineering, University of Bologna, Viale del Risorgimento 2, 40136 Bologna, Italy (e-mail: matteo.barbieri15@unibo.it, roberto.diversi@unibo.it).*

Abstract: The identification of Errors-in-variables (EIV) models refers to systems where the available measurements of their inputs and outputs are corrupted by additive noise. A large variety of solutions are available when dealing with this estimation problem, in particular when the corrupting noises are white processes. However, the number of available solutions decreases when the output noise is assumed as a colored process, which is a case of great practical interest. On the other hand, many applications require estimation algorithms to work on-line, tracking a dynamical system behavior for control, signal processing, or diagnosis. In many cases, they even have to take into account computational constraints. In this paper, we propose an estimation method that is able to both lay out an algorithm to solve the identification problem of EIV systems with arbitrarily correlated output noise and also provide an efficient recursive version that does not make use of variable size matrix inversions.

Keywords: System Identification, Errors-in-variables models, Recursive Identification, Correlated output noise, Compensated Normal Equations, High-Order Yule-Walker Equations.

1. INTRODUCTION

Errors-in-variables (EIV) models, i.e models where both the input and the output are corrupted by additive noise, find application in a broad class of scientific disciplines (Van Huffel, 1997; Van Huffel and Lemmerling, 2002). For this reason, the problem of identifying EIV models has received increasing attention in the last three decades and many identification algorithms are now available (Söderström, 2018).

As can be seen in (Söderström, 2018), most of the proposed identification methods refer to the case of white additive noise on both the input and the output sides. This assumption can be unrealistic in many practical cases since the input noise represents in general a measurement (sensor) noise while the output noise often represents the effect of both measurement noise and process disturbances. Indeed, a white process can be a good representation of a sensor noise but the process disturbance needs to be modelled by a correlated process.

The case of colored output noise is much less treated in the literature. In (Zheng, 2002) a bias-eliminating least squares algorithm is proposed whereas two extended Frisch scheme-based approaches are described in (Söderström, 2008). In (Thil et al., 2008) some methods based on instrumental variable techniques are described by considering finitely correlated output noise while the covariance matching method in (Mossberg, 2008) allows dealing with finitely correlated noise on both the input and output. The paper (Song and Chen, 2008) deals with EIV models corrupted by input and output ARMA noises but the noise-free input signal is required to be an i.i.d. random sequence. The EIV identification problem with colored output noise is solved as a generalized eigenvalue problem in (Diversi and Soverini, 2015). Some papers deal with the identification of ARX, ARARX, and ARMAX models in the presence of input noise (Diversi et al., 2010, 2011, 2013, 2014; Liu and Zhu, 2017). The colored noise case has also been treated in the frequency domain (Pintelon and Schoukens, 2007; Zhang and Pintelon, 2021).

The paper's objective is to develop a recursive identification algorithm for EIV models corrupted by white input noise and arbitrarily correlated output noise, without the use of variable-size matrix inversions. Such algorithms are valuable in a large variety of applications, from automatic control to signal processing, and, in recent years, fault diagnosis and condition monitoring (Isermann, 2006), because of their low demand for computational resources. In particular, this family of recursive algorithms is having an impact in our studies on edge-computing applications for diagnostics and prognostics of automatic machine components (Barbieri et al., 2021).

The simplest way to develop a recursive algorithm is to start from the basic instrumental variable (IV) estimator (Söderström, 2018). When the output noise is arbitrarily correlated, only delayed inputs can be used as instruments. This leads to a set of high-order Yule-Walker (HOYW) equations that can be directly exploited to get an estimate of the system parameters (Söderström and Mahata, 2002; Söderström, 2018). However, the accuracy of the obtained estimates is often poor. Better results can be achieved by exploiting both the so-called compensated normal equations and the HOYW equations, that is, a set of equations where the unknowns are the system parameters and the input-output noise statistics (Söderström, 2008; Diversi and Soverini, 2015). Since our objective is the development of a recursive method, we propose to use only part of the compensated normal equations together with the HOYW equations. In particular, we start from a system of equations involving the system parameters and the variance of the input noise, thus avoiding the autocovariances of the colored output noise. This allows deriving an offline algorithm that can be put in a recursive form by exploiting the so-called overdetermined recursive

10.1016/j.ifacol.2021.08.386

instrumental variable estimator (Friedlander, 1984; Söderström and Stoica, 1989). In this formulation, the algorithm does not require matrix inversions, except for a two-by-two one, and fits within the previously mentioned paper objective.

The paper is organized as follows. The EIV identification problem is stated in Section 2. Section 3 describes a set of noise-compensated equations that can be exploited for identifying EIV models with correlated output noise. These equations lead to the iterative identification algorithm described in Section 4. A recursive version of the identification algorithm is developed in Section 5. The recursive algorithm has been tested employing Monte Carlo simulations and compared with the offline version. The simulation results are shown in Section 6.

2. STATEMENT OF THE PROBLEM

The linear, discrete-time and time-invariant errors-in-variables model under investigation is represented in Figure 1. The noise-free input $u_0(t)$ is linked to the undisturbed output $y_0(t)$ through the difference equation

$$A(z^{-1})\, y_0(t) = B(z^{-1})\, u_0(t) \tag{1}$$

where $A(z^{-1}), B(z^{-1})$ are the following polynomials in the backward shift operator z^{-1}

$$A(z^{-1}) = 1 + a_1\, z^{-1} + \cdots + a_{n_a}\, z^{-n_a} \tag{2}$$

$$B(z^{-1}) = b_0 + b_1\, z^{-1} + \cdots + b_{n_b}\, z^{-n_b}. \tag{3}$$

The available observations $u(t), y(t)$ are corrupted by additive measurement noises $\tilde{u}(t), \tilde{y}(t)$ so that

$$u(t) = u_0(t) + \tilde{u}(t) \tag{4}$$

$$y(t) = y_0(t) + \tilde{y}(t). \tag{5}$$

The following assumptions will be considered as satisfied.

- **A1.** The dynamic system (1) is asymptotically stable.
- **A2.** All system modes are observable and controllable, i.e. $A(z^{-1})$ and $B(z^{-1})$ have no common factor.
- **A3.** The polynomial degrees n_a and n_b are *a priori* known.
- **A4.** The noise–free input $u_0(t)$ is a zero–mean ergodic process or a quasi–stationary bounded deterministic signal (Ljung, 1999). Moreover, $u_0(t)$ is assumed as persistently exciting of sufficiently high order.
- **A5.** The input noise $\tilde{u}(t)$ is a zero–mean ergodic white process with unknown variance $\tilde{\sigma}_u^2$.
- **A6.** The output noise $\tilde{y}(t)$ is an arbitrarily autocorrelated zero–mean ergodic process with unknown autocovariances.
- **A7.** $\tilde{u}(t)$ and $\tilde{y}(t)$ are mutually uncorrelated and uncorrelated with the noise–free signals $u_0(t), y_0(t)$.

The problem to be solved is the following.

Problem 1. *Let $u(1), y(1), u(2), y(2), \ldots, u(N), y(N)$ be a set of noisy input–output observations. Determine a recursive estimate of the system parameters $a_1, \ldots, a_{n_a}, b_0, \ldots, b_{n_b}$.*

Remark 1. *Conditions guaranteeing the identifiability of the EIV model* (1)-(5) *are discussed in (Anderson and Deistler, 1984; Aguëro and Goodwin, 2008; Söderström, 2018).*

2.1 Notations

In the sequel, the autocovariance of a scalar random process $x(t)$ will be denoted as $r_x(\tau) = E\,[x(\tau)\, x(t-\tau)], \tau = 0, \pm 1, \pm 2 \ldots$. The covariance matrix of a random vector $x(t)$, the cross–covariance matrix between two random vectors $x(t)$ and $y(t)$ and the cross-covariance vector between a random vector $x(t)$ and a scalar random process $z(t)$ are

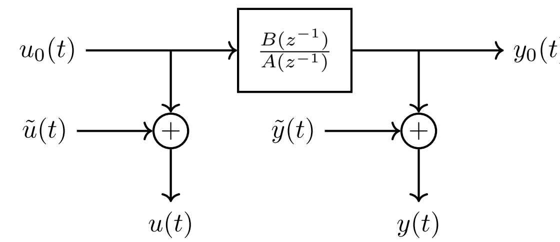

Fig. 1. EIV model

denoted as $R_{xx} = E\,[x(t)\, x^T(t)]$, $R_{xy} = E\,[x(t)\, y^T(t)]$ and $r_{xz} = E\,[x(t)\, z(t)]$. The sample estimates of the above matrices and vector that can be obtained from N measurements will be denoted as $\hat{R}_{xx} = \frac{1}{N}\sum_{t=1}^{N} x(t)\, x^T(t)$, $\hat{R}_{xy} = \frac{1}{N}\sum_{t=1}^{N} x(t)\, y^T(t)$ and $\hat{r}_{xz} = \frac{1}{N}\sum_{t=1}^{N} x(t)\, z(t)$.

3. A SET OF NOISE-COMPENSATED EQUATIONS

The EIV model (1)-(5) can be rewritten in the form

$$y_0(t) = \varphi_0^T(t)\, \theta_0 \tag{6}$$

$$\varphi(t) = \varphi_0(t) + \tilde{\varphi}(t). \tag{7}$$

where

$$\begin{aligned} \varphi(t) &= [\,-y(t-1) \,\cdots\, -y(t-n_a)\; u(t) \,\cdots\, u(t-n_b)\,]^T \\ &= [\,\varphi_y^T(t)\;\, \varphi_u^T(t)\,]^T \end{aligned} \tag{8}$$

$$\begin{aligned} \varphi_0(t) &= [\,-y_0(t-1) \,\cdots\, -y_0(t-n_a)\; u_0(t) \,\cdots\, u_0(t-n_b)\,]^T \\ &= [\,\varphi_{y0}^T(t)\;\, \varphi_{u0}^T(t)\,]^T \end{aligned} \tag{9}$$

$$\begin{aligned} \tilde{\varphi}(t) &= [\,-\tilde{y}(t-1) \,\cdots\, -\tilde{y}(t-n_a)\; \tilde{u}(t) \,\cdots\, \tilde{u}(t-n_b)\,]^T \\ &= [\,\varphi_{\tilde{y}}^T(t)\;\, \varphi_{\tilde{u}}^T(t)\,]^T \end{aligned} \tag{10}$$

$$\theta_0 = \left[\, a_1 \,\cdots\, a_{n_a}\;\, b_0 \,\cdots\, b_{n_b} \,\right]^T = \left[\, \theta_a^T\;\, \theta_b^T \,\right]^T. \tag{11}$$

By inserting (5) and (7) into (6) we get

$$y(t) = \varphi^T(t)\, \theta_0 + \tilde{y}(t) - \tilde{\varphi}^T(t)\, \theta_0. \tag{12}$$

Multiplying both sides of (12) by $\varphi(t)$ and taking expectations leads to

$$r_{\varphi y} = R_{\varphi\varphi}\, \theta_0 + r_{\varphi\tilde{y}} - R_{\varphi\tilde{\varphi}}\, \theta_0. \tag{13}$$

From (7) and Assumptions A5-A7 it follows that

$$r_{\varphi y} = R_{\varphi\varphi}\, \theta_0 + r_{\tilde{\varphi}\tilde{y}} - R_{\tilde{\varphi}\tilde{\varphi}}\, \theta_0, \tag{14}$$

where

$$r_{\tilde{\varphi}\tilde{y}} = \begin{bmatrix} r_{\varphi_{\tilde{y}}\tilde{y}} \\ 0_{(n_b+1)\times 1} \end{bmatrix} \quad R_{\tilde{\varphi}\tilde{\varphi}} = \begin{bmatrix} R_{\varphi_{\tilde{y}}\varphi_{\tilde{y}}} & 0_{n_a\times(n_b+1)} \\ 0_{(n_b+1)\times n_a} & \tilde{\sigma}_u^2\, I_{n_b+1} \end{bmatrix}, \tag{15}$$

and

$$r_{\varphi_{\tilde{y}}\tilde{y}} = [\,-r_{\tilde{y}}(1)\; -r_{\tilde{y}}(2) \,\cdots\, -r_{\tilde{y}}(n_a)\,]^T \tag{16}$$

$$R_{\varphi_{\tilde{y}}\varphi_{\tilde{y}}} = \begin{bmatrix} r_{\tilde{y}}(0) & r_{\tilde{y}}(1) \cdots r_{\tilde{y}}(n_a-1) \\ r_{\tilde{y}}(1) & \ddots & \\ \vdots & & \ddots \\ r_{\tilde{y}}(n_a-1) & & r_{\tilde{y}}(0) \end{bmatrix}. \tag{17}$$

Since

$$r_{\varphi y} = \begin{bmatrix} r_{\varphi_y y} \\ r_{\varphi_u y} \end{bmatrix} \quad R_{\varphi\varphi} = \begin{bmatrix} R_{\varphi_y\varphi_y} & R_{\varphi_y\varphi_u} \\ R_{\varphi_y\varphi_u}^T & R_{\varphi_u\varphi_u} \end{bmatrix}, \tag{18}$$

by taking into account (15), relation (14) can be splitted into two parts as follows

$$r_{\varphi_y y} = \left[R_{\varphi_y\varphi_y} - R_{\varphi_{\tilde{y}}\varphi_{\tilde{y}}} \;\; R_{\varphi_y\varphi_u}\right]\theta_0 + r_{\varphi_{\tilde{y}}\tilde{y}} \tag{19}$$

$$r_{\varphi_u y} = \left[R^T_{\varphi_y\varphi_u} \;\; R_{\varphi_u\varphi_u} - \tilde{\sigma}^2_u I_{n_b+1}\right]\theta_0. \tag{20}$$

The quantities appearing in (18) can be estimated directly from the available observations so that (19) and (20) can be seen as sets of equations where the unknowns are the model parameters θ_0, the output noise autocovariances $r_{\tilde{y}}(0), r_{\tilde{y}}(1), \ldots, r_{\tilde{y}}(n_a)$ and the input noise variance $\tilde{\sigma}^2_u$. In particular, the set (19) involves n_a equations and $2n_a + n_b + 2$ unknowns while the set (20) involves $n_b + 1$ equations and $n_a + n_b + 2$ unknowns. The set (19)-(20) is often called 'compensated normal equations' since its use for estimating θ_0 requires an estimate of the input-output noise statistics. However, as we have more unknowns than equations, a set of additional equations is required to solve the EIV identification problem.

Let us define the $q \times 1$ vector of delayed input observations

$$\varphi^q_u(t) = [u(t-n_b-1)\; u(t-n_b-2)\; \ldots\; u(t-n_b-q)]^T. \tag{21}$$

Multiplying both sides of (12) by $\varphi^q_u(t)$ and taking the expectations we get

$$r_{\varphi^q_u y} = R_{\varphi^q_u\varphi}\,\theta_0 + r_{\varphi^q_u\tilde{y}} - R_{\varphi^q_u\tilde{\varphi}}\,\theta_0. \tag{22}$$

Because of (4) and Assumptions A5-A7 we have $r_{\varphi^q_u\tilde{y}} = 0$ and $R_{\varphi^q_u\tilde{\varphi}} = 0$ so that

$$r_{\varphi^q_u y} = R_{\varphi^q_u\varphi}\,\theta_0. \tag{23}$$

Relation (23) represents a set of q high-order Yule-Walker (HOYW) equations that can be used in conjunction with the compensated normal equations (19), (20) in order to get a system of $n_a + n_b + 1 + q$ equations with $2n_a + nb + 2$ unknowns. Therefore, by properly choose q, the system (19), (20), (23) becomes a set of noise-compensated equations that can be exploited to solve Problem 1.

Remark 2. *When the output noise $\tilde{y}(t)$ is arbitrarily autocorrelated the choice of delayed inputs* (21) *is the only one leading to a set of HOYW equations like* (23). *In fact, the use of delayed outputs increases both the number of equations and the number of unknowns (autocovariances of $\tilde{y}(t)$) (Söderström, 2008).*

Remark 3. *As $r_{\varphi^q_u y}$ and $R_{\varphi^q_u\varphi}$ can be estimated from the available observations, Eq.* (23) *could be used directly to solve Problem 1 i.e. to to get an estimate of θ_0 provided that $q \geq n_a + n_b + 1$. This approach can also be seen as a basic instrumental variable (IV) method using delayed noisy inputs as instruments (Söderström and Mahata, 2002; Söderström, 2018). However, it has been shown that the use of both the normal and the HOYW equations leads to a better estimation accuracy (Söderström, 2008; Thil et al., 2008; Diversi and Soverini, 2015).*

4. AN OFFLINE IDENTIFICATION ALGORITHM

Bearing in mind our final objective, that is a recursive identification algorithm, we will exploit (20) and (23) but not (19), thus avoiding the use of the output noise autocovariances $r_{\tilde{y}}(0), \ldots, r_{\tilde{y}}(n_a)$. This choice will lead to an offline identification algorithm that allows a recursive form, as shown in the following. Moreover, the estimation of the output noise autocovariances is often bad so that their use does not necessarily improve the estimate of θ_0 (Söderström, 2008).

The set of equations (20), (23) can be written in the compact form

$$\begin{bmatrix} r_{\varphi_u y} \\ r_{\varphi^q_u y} \end{bmatrix} = \begin{bmatrix} R^T_{\varphi_y\varphi_u} \;\; R_{\varphi_u\varphi_u} \\ R_{\varphi^q_u\varphi} \end{bmatrix}\theta_0 + \tilde{\sigma}^2_u\, J\,\theta_0 \tag{24}$$

where

$$J = \begin{bmatrix} 0_{(n_b+1)\times n_a} \;\; I_{n_b+1} \\ 0_{q\times(n_a+n_b+1)} \end{bmatrix}. \tag{25}$$

The number of equations is $nb + 1 + q$ while the number of unknowns is $n_a + n_b + 2$, the parameters θ_0 and the input noise variance $\tilde{\sigma}^2_u$. Then, the integer q in (21) must be chosen such that $q \geq n_a + 1$. Let us rewrite (24) as follows

$$\rho = R\,\theta_0 + \tilde{\sigma}^2_u\, J\,\theta_0, \tag{26}$$

with obvious meaning of the terms ρ and R. If an estimate $\hat{\tilde{\sigma}}^2_u$ of $\tilde{\sigma}^2_u$ were available, an estimate $\hat{\theta}$ of θ_0 could be computed

$$\hat{\theta} = \left(R - \tilde{\sigma}^2_u\, J\right)^+ \rho \tag{27}$$

where R^+ denotes the pseudoinverse of R. Conversely, if an estimate $\hat{\theta}$ were known, the input noise variance could be estimated by exploiting the first $n_b + 1$ relations of (24):

$$\hat{\tilde{\sigma}}^2_u = \frac{\hat{\theta}^T_b\left(R^T_{\varphi_y\varphi_u}\hat{\theta}_a + R_{\varphi_u\varphi_u}\hat{\theta}_b - r_{\varphi_u y}\right)}{\hat{\theta}^T_b\hat{\theta}_b}, \tag{28}$$

where $\hat{\theta} = \left[\,\hat{\theta}^T_a \;\; \hat{\theta}^T_b\,\right]^T$ (see (11)). The system (24) can thus be seen as a set of noise-compensated equations because the effect of the additive input noise is compensated in (27) by using the estimate (28). Relations (27) and (28) lead to the following iterative identification algorithm.

Algorithm 1.

(1) *Starting from the available observations $u(1), \ldots, u(N)$ and $y(1), \ldots, y(N)$, compute the sample estimates $\hat{r}_{\varphi_u y}$, $\hat{R}_{\varphi_y\varphi_u}$, $\hat{R}_{\varphi_u\varphi_u}$, $\hat{r}_{\varphi^q_u y}$ and $\hat{R}_{\varphi^q_u\varphi}$. Then, define*

$$\hat{\rho} = \begin{bmatrix} \hat{r}_{\varphi_u y} \\ \hat{r}_{\varphi^q_u y} \end{bmatrix} \qquad \hat{R} = \begin{bmatrix} \hat{R}^T_{\varphi_y\varphi_u} \;\; \hat{R}_{\varphi_u\varphi_u} \\ \hat{R}_{\varphi^q_u\varphi} \end{bmatrix}$$

(2) *Compute an initial estimate $\hat{\theta}^0$ of θ_0 and set $\hat{\theta}^k = \hat{\theta}^0 = \left[\,\hat{\theta}_a{}^{kT} \;\; \hat{\theta}_b{}^{kT}\,\right]^T$.*

(3) *Compute the following estimate of the input noise variance*

$$\hat{\tilde{\sigma}}^{2^k}_u = \frac{\hat{\theta}_b{}^{kT}\left(\hat{R}^T_{\varphi_y\varphi_u}\hat{\theta}^k_a + \hat{R}_{\varphi_u\varphi_u}\hat{\theta}^k_b - \hat{r}_{\varphi_u y}\right)}{\hat{\theta}_b{}^{kT}\hat{\theta}^k_b}, \tag{29}$$

(4) *Update the parameter estimate as follows*

$$\hat{\theta}^{k+1} = \left(\hat{R} - \hat{\tilde{\sigma}}^{2k}_u\, J\right)^+ \hat{\rho} \tag{30}$$

(5) *Set $\hat{\theta}^k = \hat{\theta}^{k+1}$ and go to step (3).*

(6) *Repeat steps 3–5 until*

$$\frac{\|\hat{\theta}^{k+1} - \hat{\theta}^k\|}{\|\hat{\theta}^{k+1}\|} < \varepsilon, \tag{31}$$

where ε is an assigned convergence threshold.

Remark 4. *A possible initial estimate $\hat{\theta}^0$ can be obtained from*

$$\hat{r}_{\varphi^q_u y} = \hat{R}_{\varphi^q_u\varphi}\,\hat{\theta}^0 \tag{32}$$

by choosing $q \geq n_a + n_b + 1$, see Remark 3.

Remark 5. *While iterating steps 3-5 it is advisable to perform a check on the estimation of $\tilde{\sigma}^{2k}_u$ and keep it within the range $(0, \hat{\sigma}^2_u)$, where $\hat{\sigma}^2_u$ is the top left element of $\hat{R}_{\varphi_u\varphi_u}$, i.e. an estimate of the noisy input variance σ^2_u. This condition is easily obtained since the value of $\tilde{\sigma}^2_u$ is surely positive and from* (4) *and Assumptions A7 we have $\sigma^2_u = \sigma^2_{u_0} + \tilde{\sigma}^2_u$.*

Remark 6. *Note that Eq.* (26) *is bilinear in the unknowns θ_0 and $\tilde{\sigma}^2_u$ and steps (3) and (4) of Algorithm 1 represents the solution of two separate least squares problems. Therefore, Algorithm 1 leads to the minimization of the loss function $\|\rho - (R - \tilde{\sigma}^2_u\, J)\theta_0\|^2_2$ with respect to θ_0 and $\tilde{\sigma}^2_u$.*

5. RECURSIVE EIV IDENTIFICATION

The recursive formulation of Algorithm 1 starts from equation (26), which is reformulated as follows:

$$\theta = R^{+}\,\rho - \tilde{\sigma}_u^2\,R^{+}\,J\,\theta. \tag{33}$$

Equation (33) can be seen as the iteration of the computation of $\hat{\theta}$ at time t given its value at time $t-1$:

$$\hat{\theta}(t) = \hat{R}^{+}(t)\,\hat{\rho}(t) - \hat{\tilde{\sigma}}_u^2(t)\,\hat{R}^{+}(t)\,J\,\hat{\theta}(t-1). \tag{34}$$

For the sake of compactness the time index is indicated as superscript from now on (e.g. $\hat{\theta}(t) = \hat{\theta}^t, y(t) = y^t$). The main goal now is to get suitable recursive update of the quantities at time t involved in (34) starting from their values at time $t-1$, with particular attention to the pseudo-inverse $\hat{R}^{t^+}$. (Friedlander, 1984) provides the tools to update the pseudo-inverse without an actual inverse computation, only using the inversion of a two-by-two matrix, a well-known formulation. It was introduced to provide a recursive version of the standard overdetermined recursive instrumental variable (ORIV) and in the following we will adapt the calculation to our estimation problem. Firstly, we redefine (34) by splitting the pseudo-inverse into its components

$$\hat{\theta}^t = \hat{P}^t\hat{R}^{t^T}\hat{\rho}^t - \hat{\tilde{\sigma}}_u^{2^t}\,\hat{P}^t\hat{R}^{t^T}\,J\,\hat{\theta}^{t-1}, \tag{35}$$

with $\hat{P}^{-1} = \hat{R}^T\hat{R}$. The original ORIV method does not take into account sample covariance matrices and vectors like $\hat{R}$ and $\hat{\rho}$, that are required by Algorithm 1 in order to solve the identification problem. Then, the update of all of the terms involved becomes:

$$\begin{aligned}\hat{\rho}^t &= \frac{1}{t-n_b-q}\sum_{s=n_b+q+1}^{t}\bar{\varphi}_u^s y^s \\ &= \frac{1}{t-n_b-q}\left[\sum_{s=n_b+q+1}^{t-1}\bar{\varphi}_u^s y^s + \bar{\varphi}_u^s y^s\right] \\ &= \frac{\tau-1}{\tau}\hat{\rho}^t + \frac{1}{\tau}\bar{\varphi}_u^t y^t, \quad .\end{aligned} \tag{36}$$

with $\tau = t - n_b - q$, and the $(n_b+q)\times 1$ vector containing all the involved samples of $u(t)$, $\bar{\varphi}_u^t = [\varphi_u^{t^T}\ \varphi_u^{q^{t^T}}]^T$. The computations for $\hat{R}^t$ are analogous and produce the following recursive update:

$$\hat{R}^t = \frac{\tau-1}{\tau}\hat{R}^{t-1} + \frac{1}{\tau}\bar{\varphi}_u^t\varphi^{t^T} \tag{37}$$

On the other hand, $\hat{P}^t$ is computed with a similar concept in mind, but in this case we start from the decomposition of its inverse:

$$\begin{aligned}\hat{P}^{t^{-1}} &= \\ &= \left[\frac{\tau-1}{\tau}\hat{R}^{t-1^T} + \frac{1}{\tau}\varphi^t\bar{\varphi}_u^{t^T}\right]\left[\frac{\tau-1}{\tau}\hat{R}^{t-1} + \frac{1}{\tau}\bar{\varphi}_u^t\varphi^{t^T}\right] \\ &= \frac{(\tau-1)^2}{\tau^2}\hat{P}^{t-1^{-1}} + \frac{\tau-1}{\tau^2}\varphi^t\eta^{t^T} + \frac{\tau-1}{\tau^2}\eta^t\varphi^{t^T} \\ &\quad + \frac{1}{\tau^2}\varphi^t\bar{\varphi}_u^{t^T}\bar{\varphi}_u^t\varphi^{t^T} \\ &= \frac{(\tau-1)^2}{\tau^2}\hat{P}^{t^{-1}} + \\ &\quad + \frac{1}{\tau^2}(\eta^t \quad \varphi^t)\begin{pmatrix}0 & \tau-1\\ \tau-1 & \bar{\varphi}_u^{t^T}\bar{\varphi}_u^t\end{pmatrix}\begin{pmatrix}\eta^{t^T}\\ \varphi^{t^T}\end{pmatrix} \\ &= \frac{(\tau-1)^2}{\tau^2}\hat{P}^{t^{-1}} + \frac{1}{\tau^2}\phi^t\Lambda^{t^{-1}}\phi^{t^T}\end{aligned} \tag{38}$$

where $\eta^t = R^{t-1^T}\bar{\varphi}_u^t$, $\phi^t = (\eta^t \quad \varphi^t)$ and $\Lambda^{t^{-1}}$ is:

$$\Lambda^{t^{-1}} = \begin{pmatrix}0 & \tau-1\\ \tau-1 & \bar{\varphi}_u^{t^T}\bar{\varphi}_u^t\end{pmatrix}. \tag{39}$$

Therefore, by applying the inverse operator and the well-known matrix inversion lemma, we obtain

$$\begin{aligned}\hat{P}^t &= \frac{\tau^2}{(\tau-1)^2}\hat{P}^{t-1} - \frac{1}{(\tau-1)^2}\hat{P}^{t-1}\phi^t \\ &\times\left[\Lambda^t + \frac{1}{(\tau-1)^2}\phi^{t^T}\hat{P}^{t-1}\phi^t\right]^{-1}\frac{\tau^2}{(\tau-1)^2}\phi^{t^T}\hat{P}^{t-1}\end{aligned} \tag{40}$$

where given

$$\Lambda^t = \frac{1}{(\tau-1)^2}\begin{pmatrix}-\bar{\varphi}_u^{t^T}\bar{\varphi}_u^t & \tau-1\\ \tau-1 & 0\end{pmatrix} = \frac{1}{(\tau-1)^2}\bar{\Lambda}^t \tag{41}$$

we have the update of $\hat{P}^t$:

$$\begin{aligned}\hat{P}^t &= \frac{\tau^2}{(\tau-1)^2}(\hat{P}^{t-1} - \hat{P}^{t-1}\phi^t \\ &\times\left[\phi^{t^T}\hat{P}^{t-1}\phi^t + \bar{\Lambda}^t\right]^{-1}\phi^{t^T}\hat{P}^{t-1}).\end{aligned} \tag{42}$$

Finally, with all the updates computed, the whole recursive algorithm can be summarised as follows

Algorithm 2.

(1) $\eta^t = \hat{R}^{t-1^T}\bar{\varphi}_u^t$

(2) $\phi^t = (\eta^t \quad \varphi^t)$

(3) $\bar{\Lambda}^t = \begin{pmatrix}-\bar{\varphi}_u^{t^T}\bar{\varphi}_u^t & \tau-1\\ \tau-1 & 0\end{pmatrix}$

(4) $v^t = \begin{pmatrix}\bar{\varphi}_u^{t^T}\hat{\rho}^{t-1}\\ y^t\end{pmatrix}$

(5) $K^t = \hat{P}^{t-1}\phi^t\left[\phi^{t^T}\hat{P}^{t-1}\phi^t + \bar{\Lambda}^t\right]^{-1}$

(6) $\hat{\rho}^t = \frac{\tau-1}{\tau}\hat{\rho}^{t-1} + \frac{1}{\tau}\bar{\varphi}_u^t y^t$

(7) $\hat{R}^t = \frac{\tau-1}{\tau}\hat{R}^{t-1} + \frac{1}{\tau}\bar{\varphi}_u^t\varphi^{t^T}$

(8) $\hat{P}^t = \frac{\tau^2}{(\tau-1)^2}\left(\hat{P}^{t-1} - K^t\phi^{t^T}\hat{P}^{t-1}\right)$

(9) $\hat{\tilde{\sigma}}_u^{2t} = \dfrac{\hat{\theta}_b^{t^T}\left(\hat{R}^t_{\phi_y\phi_u}\hat{\theta}_a^t + \hat{R}^t_{\phi_u\phi_u}\hat{\theta}_b^t - \hat{r}^t_{\varphi_u y}\right)}{\hat{\theta}_b^{t^T}\hat{\theta}_b^t}$

(10) $\hat{\theta}^t = \hat{P}^t\hat{R}^{t^T}\hat{\rho}^t - \hat{\tilde{\sigma}}_u^{2t}\hat{P}^t\hat{R}^{t^T}J\,\hat{\theta}^{t-1}$

The initial step may be defined in the following way

$$\begin{aligned}\hat{\theta}^0 &= \alpha\mathbf{1}, & \hat{P}^0 &= \psi I_{n_a+n_b},\\ \hat{\rho}^0 &= 0, & \hat{R}^0 &= 0,\end{aligned} \tag{43}$$

with α and ψ any small and large positive number, respectively, and $\mathbf{1}$ a vector of all ones with the same size of θ.

Remark 7. *The algorithm requires the inverse of a 2×2 matrix at point (5). This can be easily tackled during the algorithm implementation by defining*

$$\Gamma = \left[\phi^{\mathrm{T}}(t)\hat{P}(t-1)\phi(t) + \bar{\Lambda}(t)\right] = \begin{bmatrix}\Gamma_{11} & \Gamma_{12}\\ \Gamma_{21} & \Gamma_{22}\end{bmatrix},$$

and then, by the well known formula obtain the inverse:

$$\Gamma^{-1} = \frac{1}{\Gamma_{11}\Gamma_{22} - \Gamma_{12}\Gamma_{21}}\begin{bmatrix}\Gamma_{22} & -\Gamma_{12}\\ -\Gamma_{21} & \Gamma_{11}\end{bmatrix}.$$

Remark 8. *While iterating the algorithm it is advisable to perform a check on the estimation $\hat{\tilde{\sigma}}_u^{2t}$ and keep it within the range $(0, \hat{\sigma}_u^2)$, with the latter being one of the diagonal elements of $\hat{R}_{\phi_u\phi_u}$ (see Remark 5).*

The algorithm in this formulation is not able to track parameters changes online when τ becomes bigger and bigger. Typically, a forgetting factor is inserted in the various updates to avoid this issue, however in our case it would invalidate the estimation of $\tilde{\sigma}_u^2$. The simplest way to implement a behaviour similar to the insertion of a forgetting factor in this recursive form is to fix τ to a predefined time window.

6. SIMULATION RESULTS

The proposed identification algorithm in both offline and recursive form is tested on input-output sequences generated by the following second-order EIV models, one considered in (Söderström and Mahata, 2002; Thil et al., 2008), and the other one proposed by us. They are model 1

$$y_0(t) = \frac{z^{-1} + 0.5\,z^{-2}}{1 - 1.5\,z^{-1} + 0.7\,z^{-2}}\,u_0(t)$$

and model 2

$$y_0(t) = \frac{1.5z^{-1} - 0.9\,z^{-2}}{1 - 0.2\,z^{-1} - 0.48\,z^{-2}}\,u_0(t)$$

where the noiseless input is an autoregressive process given by

$$u_0(t) = \frac{1}{1 - 0.2\,z^{-1} + 0.5\,z^{-2}}\,e(t)$$

and $e(t)$ is a white process with variance $\sigma_e^2 = 1$. Then, the input noise $\tilde{u}(t)$ is a white process whereas the output noise $\tilde{y}(t)$ is a first order ARMA process given by

$$\tilde{y}(t) = \frac{1 + 0.3\,z^{-1}}{1 - 0.6\,z^{-1}}\,w(t),$$

where $w(t)$ is a white process. The variances $\tilde{\sigma}_u^2$ and $\tilde{\sigma}_w^2$ are chosen in order to test the algorithms with SNR of 10dB and 5dB. The corresponding input noise variance values and the algorithm estimation performances are presented in Table 1 showing the parameters mean and standard deviation obtained after a Monte Carlo simulation of 1000 runs with $N = 3000$ samples each. The number of HOYW equations is set to $q = 10$. Algorithm 1 is initialised as stated in Remark 4, while Algorithm 2 is initialised with $\alpha = 0.01$ and $\psi = 10$. The table shows very promising results concerning the models parameter estimation, producing reliable estimations at both SNRs with both algorithms. Besides, the estimation of the noise input variance is weaker, but still reliable and within the standard deviation range. However, the result highlight is that Algorithm 2 has almost the same estimation performance as Algorithm 1, suggesting an acceptable trade-off between computation efficiency and estimation accuracy when deploying the recursive version. Figure 2 presents the evolution of the recursive algorithm (Algorithm 2) estimation during a run of the Monte Carlo test with model 1. It shows the convergence of the parameter estimation with just $N = 3000$ samples at 5 dB SNR. Also, it shows how limiting $\hat{\tilde{\sigma}}_u^2$ into its physical boundaries affects the quantity evolution during the iterations. Finally, Figure 3 presents the tracking behaviour of the recursive estimator when the time window is fixed to $\tau = 3000$ samples. It shows that the step change from model 1 to model 2 does not impair the estimation and after a transient the algorithm is back on track.

7. CONCLUSION

In this paper, we presented a novel estimation algorithm for the identification of EIV models under the presence of white input noise and colored output noise in both its batch and recursive form. The method makes use of a combination of part of the noise compensated normal equations and HOYW equations involving delayed samples of the noisy input to estimate the underlying model parameters. This structure permits the derivation of an "inverse-free" recursive form of the algorithm, evolved from the overdetermined recursive instrumental variable method proposed in (Friedlander, 1984). Finally, the presented simulation results show promising performances in model estimation for both the batch and recursive form, even in scenarios with low SNR, validating our proposition. In this regard, the recursive version of the algorithm results useful for practitioners in applications where online, real-time tracking is required or computational constraints are involved: signal processing, control, and system diagnostics are the first that comes to mind.

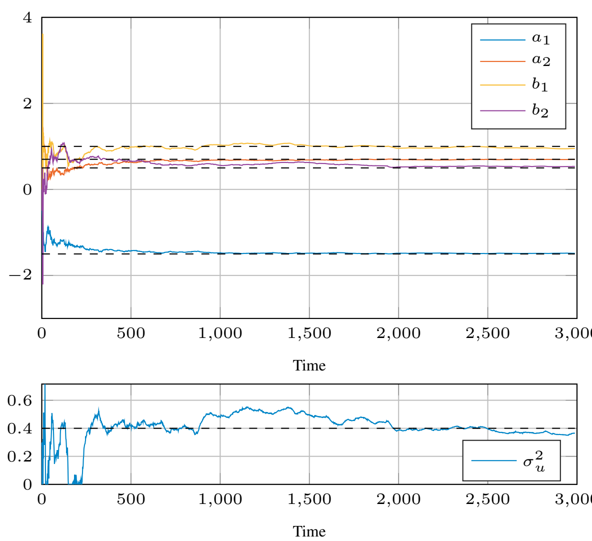

Fig. 2. Estimation of Model 1 under SNR of 5dB with Algorithm 2, $\hat{\theta}$ and $\hat{\tilde{\sigma}}_u^2$ evolution over time.

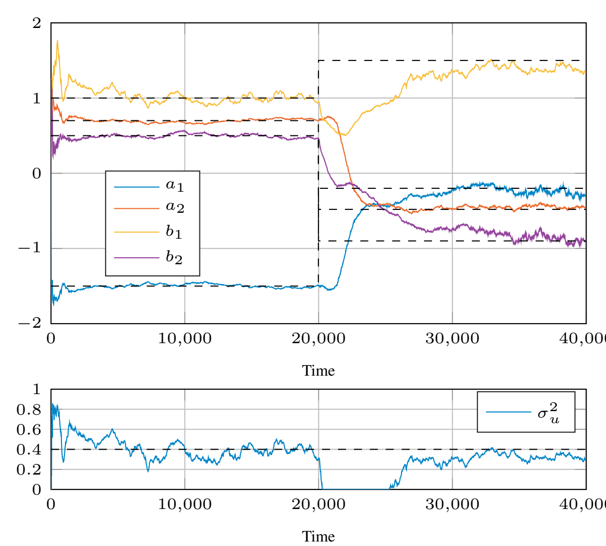

Fig. 3. Estimation of the parameter vector when changing from model 1 to 2 under SNR of 5dB with Algorithm 2 with $\tau = 3000$, $\hat{\theta}$ and $\hat{\tilde{\sigma}}_u^2$ evolution over time.

Table 1. True and estimated values of the model coefficients and of the input noise variance. Monte Carlo simulation of 1000 runs performed with $N = 3000$ and $q = 10$.

	a_1	a_2	b_1	b_2	$\sigma^2_{\tilde{u}}$
		Model 1			10 dB-5 dB
True	-1.5	0.7	1	0.5	0.13-0.4
Algorithm 1 10dB	-1.497 ± 0.020	0.698 ± 0.016	0.995 ± 0.072	0.498 ± 0.027	0.117 ± 0.081
Algorithm 1 5dB	-1.490 ± 0.036	0.692 ± 0.029	0.979 ± 0.123	0.499 ± 0.044	0.358 ± 0.153
Algorithm 2 10dB	-1.497 ± 0.020	0.698 ± 0.016	0.995 ± 0.071	0.498 ± 0.026	0.117 ± 0.081
Algorithm 2 5dB	-1.490 ± 0.037	0.692 ± 0.031	0.977 ± 0.124	0.499 ± 0.043	0.356 ± 0.156
		Model 2			10 dB-5 dB
True	-0.2	-0.48	1.5	-0.9	0.15-0.4
Algorithm 1 10dB	-0.200 ± 0.048	-0.471 ± 0.031	1.482 ± 0.077	-0.888 ± 0.062	0.136 ± 0.048
Algorithm 1 5dB	-0.201 ± 0.090	-0.450 ± 0.059	1.444 ± 0.121	-0.860 ± 0.117	0.357 ± 0.093
Algorithm 2 10dB	-0.199 ± 0.046	-0.470 ± 0.020	1.481 ± 0.078	-0.885 ± 0.057	0.135 ± 0.017
Algorithm 2 5dB	-0.191 ± 0.100	-0.443 ± 0.063	1.437 ± 0.143	-0.844 ± 0.126	0.354 ± 0.049

REFERENCES

Aguëro, J.C. and Goodwin, G.C. (2008). Identifiability of errors in variables dynamic systems. *Automatica*, 44(2), 371–382.

Anderson, B. and Deistler, M. (1984). Identifiability of dynamic errors in variables models. *Journal of Time Series Analysis*, 5, 1–13.

Barbieri, M., Nguyen, K.T., Diversi, R., Medjaher, K., and Tilli, A. (2021). RUL prediction for automatic machines: a mixed edge-cloud solution based on model-of-signals and particle filtering techniques. *Journal of Intelligent Manufacturing*, 32, 1421–1440.

Diversi, R., Bartolini, A., Tilli, A., Beneventi, F., and Benini, L. (2013). SCC thermal model identification via advanced bias-compensated least-squares. In *Proc. Des., Autom. Test Eur. Conf. Exhib (DATE 2013)*, 230–235.

Diversi, R., Guidorzi, R., and Soverini, U. (2010). Identification of ARX and ARARX models in the presence of input and output noises. *European Journal of Control*, 16(3), 242–255.

Diversi, R., Guidorzi, R., and Soverini, U. (2011). Identification of ARMAX models with noisy input and output. *IFAC Proceedings Volumes*, 44(1), 13121–13126.

Diversi, R. and Soverini, U. (2015). Identification of errors-in-variables models with colored output noise. In *2015 European Control Conference*, 1778–1783.

Diversi, R., Tilli, A., Bartolini, A., Beneventi, F., and Benini, L. (2014). Bias-compensated least squares identification of distributed thermal models for many-core systems-on-chip. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 27(10), 915–924.

Friedlander, B. (1984). The overdetermined recursive instrumental variable method. *IEEE Transactions on Automatic Control*, 29(4), 353–356.

Isermann, R. (2006). *Fault-diagnosis systems: an introduction from fault detection to fault tolerance*. Springer Science & Business Media.

Liu, X. and Zhu, Y. (2017). Identification of errors-in-variables systems: an asymptotic approach. *International Journal of Adaptive Control and Signal Processing*, 31(8), 1126–1138.

Ljung, L. (1999). *System identification – Theory for the user*. Prentice-hall.

Mossberg, M. (2008). Errors-in-variables identification through covariance matching: Analysis of a colored measurement noise case. In *2008 American Control Conference*, 1310–1315.

Pintelon, R. and Schoukens, J. (2007). Frequency domain maximum likelihood estimation of linear dynamic errors-in-variables models. *Automatica*, 43(4), 621–630.

Söderström, T. (2008). Extending the frisch scheme for errors-in-variables identification to correlated noise. *International Journal of Adaptive Control and Signal Processing*, 22(1), 55–73.

Söderström, T. (2018). *Errors-in-Variables Methods in System Identification*. Springer.

Söderström, T. and Mahata, K. (2002). On instrumental variable and total least squares approaches for identification of noisy systems. *International Journal of Control*, 75(6), 381–389.

Söderström, T. and Stoica, P. (1989). *System Identification*. Prentice Hall.

Song, Q.J. and Chen, H.F. (2008). Identification of errors-in-variables systems with ARMA observation noises. *Systems & Control Letters*, 57(5), 420–424.

Thil, S., Gilson, M., and Garnier, H. (2008). On instrumental variable-based methods for errors-in-variables model identification. *IFAC Proceedings Volumes*, 41(2), 426–431.

Van Huffel, S. (ed.) (1997). *Recent Advances in Total Least Squares Techniques and Errors–in–Variables Modelling*. SIAM.

Van Huffel, S. and Lemmerling, P. (eds.) (2002). *Total Least Squares Techniques and Errors–in–Variables Modelling: Analysis, Algorithms and Applications*. Kluwer Academic Publishers.

Zhang, E. and Pintelon, R. (2021). Identification of dynamic errors-in-variables systems with quasi-stationary input and colored noise. *Automatica*, 123(1).

Zheng, W.X. (2002). A bias correction method for identification of linear dynamic errors–in–variables models. *IEEE Transactions on Automatic Control*, 47(7), 1142–1147.