Non-Abelian Gauge Symmetry and the Higgs Mechanism in F-theory

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Abstract

Singular fiber resolution does not describe the spontaneous breaking of gauge symmetry in F-theory, as the corresponding branch of the moduli space does not exist in the theory. Accordingly, even non-abelian gauge theories have not been fully understood in global F-theory compactifications. We present a systematic discussion of using singularity deformation, which does describe the spontaneous breaking of gauge symmetry in F-theory, to study non-abelian gauge symmetry. Since this branch of the moduli space also exists in the defining M-theory compactification, it provides the only known description of gauge theory states which exists in both pictures; they are string junctions in F-theory. We discuss how global deformations give rise to local deformations, and also give examples where local deformation can be utilized even in models where a global deformation does not exist. Utilizing deformations, we study a number of new examples, including non-perturbative descriptions of $SU(3)$ and $SU(2)$ gauge theories on seven-branes which do not admit a weakly coupled type IIb description. It may be of phenomenological interest that these non-perturbative descriptions do not exist for higher rank $SU(N)$ theories.

1 Introduction

Much has been learned about the landscape of string vacua over the last fifteen years. On one hand, in weakly coupled corners of the landscape there are scenarios for controlled moduli stabilization which are giving rise to increasingly realistic global compactifications, and in many cases provide inspiration for new models of particle physics or early universe cosmology. On the other hand, there has been much formal progress in understanding the physics of compactifications at small volume or strong coupling. We have gained both a clearer view of well-known regions, and also a glimpse of relatively unexplored vistas.

The strongest example of the latter may be in compactifications of F-theory [1], which has been the subject of much study in recent years. In addition to enjoying [2,3] certain model-building advantages for grand unification over their heterotic and type II string counterparts, F-theory compactifications provide a broad view of the landscape; much of their strongly coupled physics can be understood in terms of the geometry of elliptically fibered Calabi-Yau varieties, which can be explicitly constructed and studied as a function of their moduli. Recent works have made significant progress in understanding the physics of F-theory compactifications, including globally consistent models [4–19], $U(1)$ symmetries [20–27, 27–33], instanton corrections [12, 22, 34–41], the physics of codimension two and three singularities [42–50], and chirality inducing G_4-flux [22, 41, 46, 51–63]; there has also been progress in understanding the landscape of six-dimensional F-theory compactifications [24, 64–67].

In this paper we study non-abelian gauge symmetry as it exists in global F-theory compactifications. There, non-abelian sectors (not arising from D3 branes) require a singular compactification geometry X. Lacking both the tools to deal directly with the singular geometry and also a fundamental quantization of M-theory, which defines any F-theory compactification, one must resort to studying the theory on a related smooth manifold $\tilde{X}$ and then inferring the physics as one takes the singular limit $\tilde{X} \rightarrow X$. Since a singular geometry is necessary for a non-abelian gauge sector and $\tilde{X}$ is smooth, the movement in moduli space $X \rightarrow \tilde{X}$ must describe spontaneous symmetry breaking via the Higgs mechanism. The Lie algebraic data of particle states in the broken gauge theory is encoded in the geometry of $\tilde{X}$.

As X is a Calabi-Yau variety, the smoothing processes necessarily correspond to movement in the Kähler or complex structure moduli spaces of X. A common technique is to study F-theory via M-theory on a related singular variety X_M, where $X_M \rightarrow X$ is the limit of vanishing elliptic fiber, and then to blow-up X_M. This branch of the moduli space is often referred to as the M-theory Coulomb branch (which is a bit of a misnomer), but it does not exist in F-theory on X; as such, it does not describe the spontaneous breaking of *any* non-abelian gauge symmetry in F-theory. Instead, the breaking of gauge symmetry in F-theory is accomplished by singularity deformation via movement in complex structure. This description of spontaneous symmetry breaking exists both for M-theory on X_M and F-theory on X.

This paper is organized as follows. First we will review F-theory, as well as the drawbacks of resolution and the advantages of deformation as a technique for its study. In section 2 we present a simple example which demonstrates how the Higgs mechanism operates in global F-theory compactifications. In section 3 we present a general discussion of complex structure deformations useful for studying F-theory, which applies even when the singularities cannot be globally Higgsed, as in the case of non-Higgsable clusters. In section 4 we present new realizations of $\mathfrak{su}(3)$ and $\mathfrak{su}(2)$ gauge algebras on stacks of four and three seven branes; interestingly, these descriptions do not exist in the type IIb limit or for higher rank $\mathfrak{su}(N)$ algebras. Links to codes used to perform the computations in this paper can be found in appendix A.

F-theory and its defining M-theory compactification.

In general, an F-theory compactification to d dimensions on an elliptically fibered Calabi-Yau variety X is *defined* to be the vanishing fiber limit of an M-theory compactification to $d-1$ dimensions on an elliptically[1] fibered Calabi-Yau variety X_M; for us this will be a Weierstrass model. In taking this limit $X_M \to X$, one of the dimensions decompactifies (via a fiberwise T-duality), yielding a d-dimensional theory. This M-theory compactification is often called the "defining M-theory compactification," and a common technique is to study F-theory on X in terms of the defining M-theory compactification on X_M. Via circle compactification, both the d^{th} component of a gauge field and also scalars in F-theory are scalars in the defining M-theory compactification; as such, there are more scalars which can receive expectation values in the M-theory compactification, and thus perhaps more branches of gauge theory moduli space.

The non-abelian gauge structure of these compactifications is determined by the singular geometry of X and X_M. In particular, if X and X_M exhibit singularities along a codimension one locus Z in the base B, both the F-theory and defining M-theory compactifications enjoy a non-abelian gauge sector along Z. If the metric moduli of X or X_M are then continuously varied such that the singular locus becomes smooth, the respective non-abelian gauge theory has been spontaneously broken, and the now smooth geometry encodes features of the broken theory. Since any such smoothing will break the non-abelian gauge theory, it is natural in both M-theory on X_M and its limit of F-theory on X to study all possible smoothings of X_M and X, and also the associated physics.

The M-theory Coulomb branch of singularity resolution.

One such smoothing is singularity resolution via blow-up; this is a classical technique in algebraic geometry which can be used to resolve singularities in X_M. From the point of view of resolution, X_M is on the boundary of a Kähler cone of a family of smooth Calabi-Yau manifolds, and the resolution procedure simply involves moving to the interior of the Kähler cone. For the singularities which are typically[2] resolved in the defining M-theory compactification on X_M, this involves movement in a direction in Kähler moduli space which cause rational curves to appear in the fiber. For a generic such direction, the gauge group G of the theory along Z is broken to $U(1)^{rk(G)}$. This is one Coulomb branch, of perhaps many, of this gauge theory in the defining M-theory compactification; henceforth, in any reference to "the" M-theory Coulomb branch, we will mean the M-theory Coulomb branch obtained by the resolution of singular elliptic fibers. Gauge theoretically, this corresponds to giving expectation values to the scalars in the $d-1$ dimensional theory which are obtained from the d-dimensional theory via reduction of the d^{th} component of the gauge field. Alternatively, expectation values of adjoint scalars in $d-1$ which are also adjoint scalars in d dimensions can give rise to Coulomb branches; see section 3.1. Singularity resolution can be a useful tool because properties of the broken gauge theory are encoded in the geometry of the resolved manifold.

However, there are important drawbacks to using the M-theory Coulomb branch as a tool to study F-theory compactifications. These include:

- *It doesn't exist in F-theory.* The scalars which have expectation values on the M-theory Coulomb branch become components of gauge fields in F-theory, and thus cannot receive expectation values in the F-theory limit. The M-theory Coulomb branch does not lift to F-theory. Therefore, it can

[1] Actually, only a genus-one fibration is required, as explored recently in [68].

[2] We mean the resolution of singular fibers. See [69], which uses resolution of curves in the base to study $d = 6$ $(1,0)$ SCFTs.

describe neither the breaking of gauge symmetry nor massive gauge bosons in F-theory.

- *It is generically obstructed by instantons.* The Coulomb branch of $d = 3$ $\mathcal{N} = 2$ gauge theories is typically lifted by instanton corrections; see [70] for early work and [71, 72] for further analysis. In M-theory compactifications these instanton corrections arise from $M5$-branes wrapped on Cartan divisors of the resolved geometry, as studied, for example, in [73]. Physically, these effects imply that far out on the Coulomb branch there is a scalar potential obstructing the path back to the origin. To our knowledge, most recent works on the Coulomb branch do not take these effects into account.
- *It doesn't encompass the heterotic or type IIb perspective.* Under certain circumstances, an F-theory compactification can be dual to a heterotic compactification [1,74,75] or admit a weakly coupled type IIb description [1,76]. If so, the moduli which spontaneously break the heterotic or type IIb theories become complex structure moduli in F-theory. Thus, the M-theory Coulomb branch, on which Kähler moduli have expectation values, does not encompass the heterotic or type IIb perspectives.

Due to these drawbacks, it is natural to expect that utilizing the M-theory Coulomb branch to study F-theory compactifications will miss important aspects of F-theoretic physics; as we have mentioned, one important omission is that it cannot describe massive gauge bosons in F-theory. Accordingly, it is worthwhile to examine other possible approaches.

F-theory, M-theory, and Singularity Deformation

We will take a different approach in this work. As any smoothing of X_M or X will spontaneously break a non-abelian gauge theory in the respective M-theory and F-theory compactifications, the other natural choice is to break the theory by complex structure deformation. (See [62] for a study of resolution and deformation in the $d = 3$ $\mathcal{N} = 2$ compactifications of M-theory with G_4 fluxes.)

Our method has the advantage that the corresponding branch of the moduli space exists both in an F-theory compactification and its defining M-theory compactification. In the $d = 4$ case, this is simply the gauge theoretic statement that the scalars in four-dimensional chiral multiplets[3] can receive expectation values in *both* the four-dimensional theory *and* its circle compactification to three dimensions, in contrast to the scalars of the M-theory Coulomb branch which can receive expectations values only in three dimensions. Moreover, studying F-theory via complex structure deformation has the advantage that it encompasses and also extends, see section 4, some of the heterotic and type IIb perspectives , since deformations of seven-branes in F-theory via complex structure deformation map to vector bundle and D7-brane moduli spaces in the heterotic and type IIb strings, respectively.

We restrict our attention to utilizing singularity deformation to study the structure of non-abelian gauge theories in F-theory. On this branch of the moduli space in the M-theory description, the massive states of a spontaneously broken non-abelian gauge theory are described by $M2$-branes wrapped on two-cycles in the total space of X_M which extend exactly one dimension in the elliptic fiber. In the F-theory limit $X_M \to X$, these states become the (p, q) string junctions of [78–80]; the Lie algebraic structure of string junctions for certain sets of vanishing cycles was pioneered in the work of Zwiebach and DeWolfe [79].

[3]When we discuss four-dimensional $\mathcal{N} = 1$ theories we refer to the supersymmetric multiplets with scalar fields as chiral multiplets, even though we do not address G-flux and therefore each chiral multiplet comes in a vector pair. This matches the standard language of $\mathcal{N} = 1$ theories in four dimensions even in the case of a non-chiral spectrum; see e.g. the classic paper on non-perturbative effects in supersymmetric QCD [77], which is a non-chiral theory.

The systematic study of such $M2$-brane states and string junctions in terms of the deformation theory of elliptically fibered Calabi-Yau varieties was initiated in our previous work [45]. We continue this program here and will also pursue it in forthcoming works [81, 82]. Thus, for the sake of brevity, we will only refer to the main concepts of [45] as they appear here, instead referring the reader to our previous work for a systematic treatment of those concepts.

2 A Glimpse of the Higgs Mechanism in F-theory

Before proceeding to a systematic discussion of deformations which is sufficient to determine the Lie algebraic data of a spontaneously broken gauge theory in F-theory, let us present a simple global example which will give a useful and intuitive visual picture. As a brief caveat, in this section we will use language and techniques which are less common in the F-theory literature. We will attempt to describe the salient points briefly, but refer the reader who is interested in further details to our previous work [45].

Consider an F-theory compactification to six dimensions on a Calabi-Yau elliptic fibration $\pi : X \to B$ with $B = \mathbb{P}^2$ having homogeneous coordinates (z, t, s). We will consider a Weierstrass model defined by

$$y^2 = x^3 + f\,x + g \tag{2.1}$$

with f and g global sections of $\mathcal{O}(12)$ and $\mathcal{O}(18)$, respectively. For generic f and g the manifold is smooth and the associated F-theory model has no non-abelian gauge symmetry, which requires the presence of a singular codimension one locus Z in B.

Now we would like to engineer a non-abelian gauge theory via tuning the complex structure moduli in f and g. Suppose they take the form $f = z^2\,p_{10}$ and $g = z^3\,p_{15}$ for general polynomials p_i of degree i in the homogeneous coordinates, and furthermore that z divides neither p_{10} nor p_{15}. Then the discriminant $\Delta = 4\,f^3 + 27\,g^2$ for this example is given by

$$\Delta = z^6\,\left(4p_{10}^3 + 27p_{15}^2\right) \equiv z^6\,\Delta_r \tag{2.2}$$

and the theory exhibits an I_0^* singularity along $Z \equiv \{z = 0\}$, a genus 0 curve in $\mathbb{P}^2$. The curve $z = 0$ intersect the remaining discriminant in 30 points, with transversal intersection, as in [45]. This is a consistent global six-dimensional F-theory compactification with[4] $\mathfrak{g}_2$ gauge symmetry[5].

We would like to study the Lie algebraic structure of the gauge theory by performing a complex structure deformation which spontaneously breaks the gauge theory. To do this, we consider the deformation

$$(f, g) \mapsto (f, g + \epsilon\,p_{18}) \tag{2.3}$$

where the degree 18 polynomial p_{18} is chosen such that z does not divide it and $\varepsilon \in \mathbb{C}^*$. The deformed discriminant

$$\Delta = z^6\,\Delta_r + \epsilon\,(54\,p_{15}\,p_{18}\,z^3 + 27\,\epsilon\,p_{18}^2) \equiv z^6\,\Delta_r + \epsilon\,\Delta_\epsilon \tag{2.4}$$

is an I_1 locus. The non-abelian gauge theory has been spontaneously broken. At generic points in

[4]If p_{10} and p_{15} takes special forms the gauge symmetry may be $\mathfrak{so}(7)$ or $\mathfrak{so}(8)$; here it is $\mathfrak{g}_2$.
[5]Henceforth we will use algebra rather than notation, since we are only studying the Lie algebraic structure in the geometry.

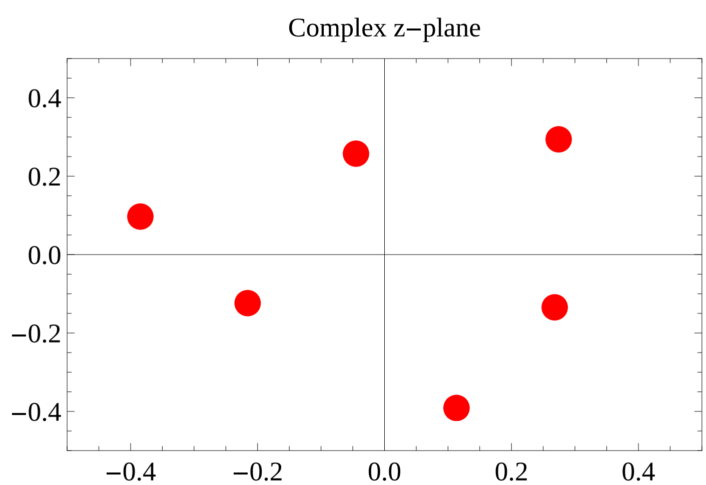
neighborhood of $z=0$, we have $\Delta_r \neq 0$ and $\Delta_\epsilon \neq 0$ and one can study an elliptic fibration over the complex z-plane in this neighborhood; in fact this is a family of elliptic surfaces, where the parameters are the coordinates in the base other than z.

In a generic elliptic surface in this family, prior to deformation, the intersection of the discriminant with the z-plane gives a marked point at $z=0$ with multiplicity 6, which now has blossomed out to give 6 marked points with multiplicity 1 collected around the origin, and whose distance from the origin is set by $\epsilon\,\Delta_\epsilon/\Delta_r$. This is the geometry transverse to the Higgsed seven-brane for a generic point in its worldvolume, and the Lie algebraic data of the Higgsed $\mathfrak{g}_2$ gauge theory must be captured by the geometry. One can study the Lie algebraic structure of two-cycles in a family of elliptic surfaces to recover the data of the broken gauge theory, for example those encoding the gauge theoretic structure of massive W-bosons.

A generic point in a patch containing $z=0$ has $s\neq 0$ (and also $t\neq 0$, but we'll choose to focus on s), and the $\mathbb{C}^*$ action of $\mathbb{P}^2$ can be used to set $s=1$. On this patch the Weierstrass model becomes an elliptic fibration over $\mathbb{C}^2$ with local coordinates (z,t). The deformed discriminant is a very long expression, but it can be computed that z does not divide Δ_ϵ and that its form appears to be generic enough to confirm that this is a minimal Weierstrass model. Furthermore, and a generic t does not lie on the locus $\Delta_\varepsilon = 0$. For any such generic t one can study the Lie algebraic structure of the mentioned elliptic fibration over the complex z-plane.

This geometry, not the blown-up geometry, describes a spontaneously broken $\mathfrak{g}_2$ gauge theory. We will focus on a particular elliptic surface with base the z-plane, and in this elliptic surface the data of an $\mathfrak{so}(8)$ symmetry will be evident; outer monodromy induces an action on this data, which gives rise to gauge algebra $\mathfrak{g}_2$. Let us first see the $\mathfrak{so}(8)$ structure directly.

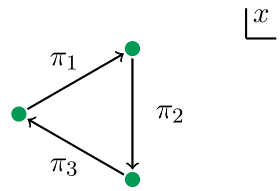
We have a family of elliptic fibrations over the z-plane parametrized by the coordinate t. To perform a concrete analysis we fix the complex structure, taking $p_k = ts^{k-1} + l_k s^k + sz^{k-1} + zt^{k-1} + t^k + z^k$ with $(l_{10}, l_{15}, l_{18}) = (-3, 2, 1)$ and setting $s=1$ via the $\mathbb{C}^*$ action of $\mathbb{P}^2$. The l_k are chosen such that the residual discriminant has a t factor, i.e. $\Delta_r|_{z=0} \sim t$; therefore the I_0^* and I_1 loci have simple normal crossing at $z=t=0$. For any t away from $\Delta_\epsilon = 0$, the Lie algebraic structure should be evident; we choose $t=i$ for simplicity. Varying ϵ from $\epsilon = 0$ to $\epsilon = .1$ the six marked points at the origin blossom out, giving a picture



(2.5)

at $\varepsilon = .1$. The red dots denote the intersection of Δ with the base of this elliptic surface. That is, they are where these seven-branes intersect the z-plane; so we are viewing a cross section of the seven-branes.

Above each marked point is a singular fiber which has a particular vanishing one-cycle. In [45], we gave a method for reading off these vanishing cycles systematically by choosing the origin to be the base point of the fundamental group and taking straight line paths of approach from the origin to the marked points, reading off the vanishing cycles in the process. This can be done as follows. A Weierstrass equation generically takes the form $y^2 = v_3(x; f, g)$ where v_3 is a cubic polynomial in a coordinate x and f and g are appropriate sections which depend on coordinates in the base. At the base point $z = 0$, the roots of v_3 are the green points



(2.6)

which are the ramification points of the elliptic curve described by the Weierstrass equation, viewed as a two-sheeted cover of the x-plane. The directed paths π_1, π_2, and π_3 describe one-cycles in the elliptic curve above the base point which satisfy $\pi_1 + \pi_2 + \pi_3 = 0$ in homology. Upon taking a straight path of approach from the base point to one of the red singular points in the above figure, v_3 changes and two of the three green points in the x-plane collide upon reaching the red singular point in the z-plane. This determines a vanishing one-cycle. Applying this method in this example, the associated ordered set of vanishing cycles is

$$Z = \{\pi_3, \pi_2, \pi_1, \pi_3, \pi_2, \pi_1\} \tag{2.7}$$

beginning with the right-most point in the upper left quadrant and working clockwise. This is the critical data which serves as input for a Lie algebraic analysis of the deformed geometry.

Let us briefly review the basic concepts that will be needed below. A segment in the z-plane from one marked point to another, which both have the same vanishing cycle, is a two-sphere in the total space[6], since the cycles vanish at the endpoints, giving rise to the north and south pole. In this elliptic surface this two-sphere has self intersection -2, where there is a contribution of -1 from each endpoint. Picking an appropriate $SL(2,\mathbb{Z})$ frame and taking the type IIb limit, this is just a fundamental string stretched between two D7-branes. More generic cycles than this exist, though, which can end on multiple marked points / seven-branes. In the type IIb language, these are the string junctions of [78–80] which end on (p, q) seven-branes, where (p, q) is the one-cycle vanishing over the seven-brane in a chosen basis. Of course, in the M-theory[7] picture of the same setup, these are simply M2-branes wrapping these cycles which have one leg along the fiber and one leg along the base, giving particles in spacetime.

These more generic junctions also have a generalized intersection product, which we wrote down in general in [45]. It is simple to see why the intersection product generalizes beyond that of the string with two endpoints. Segments of a generalized junction along which three prongs join are locally pairs of pants in the total space, and the additional intersections come from the intersections of the wrapped one-cycles in the elliptic fiber at such "junction points." Generalized representations come from two manifolds emanating from the marked points but going off to infinity. Such a junction J carries an "asymptotic charge" $a(J)$, which is just the one-cycle in the elliptic fiber wrapped by the two-manifold as it emanates

[6]Here we mean the total space of X_M; $M2$-branes which wrap those two-cycles become string junctions in the F-theory limit.
[7]Note well that these two cycles in the M-theory picture wrapped by $M2$-branes *are not* the two cycles of the Coulomb branch.

towards infinity; in the type IIb language $a(J)$ is just the asymptotic (p,q)-charge of this state, meaning that it has p units of of F-string charge and q units of D-string charge. $a(J)$ determines what type of defect/seven-brane the two-manifold could end on elsewhere in the geometry; if these loci and that extra defect collide at a codimension two locus, one obtains massless matter associated to the two-manifold. A junction J can be represented as a vector $\in \mathbb{Z}^N$ where N is the number of defects/seven-branes in question, but is never the rank of the gauge group. The entries J_i of $J \in \mathbb{Z}^N$ are the number of junction prongs ending on the seven-brane; the formula for the asymptotic charge is simply $a(J) = \sum_I J_i \pi_i$.

Having set the stage, let us proceed to uncover the root lattice of $\mathfrak{so}(8)$ in this example. States which become massless upon undoing the deformation by sending $\varepsilon \to 0$, that is, the W-bosons of $\mathfrak{so}(8)$, must come from junctions J which shrink to zero size in that limit. As such, there must be no free end emanating to infinity; i.e. $a(J) = 0$. Furthermore, to reproduce the Cartan data as expected from comparison to the Coulomb branch, they must have $(J,J) = -2$ under the intersection product. Concretely, for this ordered set of vanishing cycles the intersection product and its associated I-matrix can be computed from the general formula of [45]; the I-matrix is

$$I \equiv (\cdot,\cdot) = \begin{pmatrix} -1 & 1/2 & -1/2 & 0 & 1/2 & -1/2 \\ 1/2 & -1 & 1/2 & -1/2 & 0 & 1/2 \\ -1/2 & 1/2 & -1 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & -1 & 1/2 & -1/2 \\ 1/2 & 0 & -1/2 & 1/2 & -1 & 1/2 \\ -1/2 & 1/2 & 0 & -1/2 & 1/2 & -1 \end{pmatrix} \tag{2.8}$$

and solving for the set $R = \{J \in \mathbb{Z}^6 \mid a(J) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } (J,J) = -2\}$, one obtains

$$\begin{aligned} R = \{&(-1,-1,-1,0,0,0), (-1,-1,0,0,0,-1), (-1,-1,0,1,1,0), (-1,0,0,0,-1,-1), (-1,0,0,1,0,0), \\ &(-1,0,1,1,0,-1), (0,-1,-1,-1,0,0), (0,-1,-1,0,1,1), (0,-1,0,0,1,0), (0,0,-1,-1,-1,0), \\ &(0,0,-1,0,0,1), (0,0,0,-1,-1,-1), (0,0,0,1,1,1), (0,0,1,0,0,-1), (0,0,1,1,1,0), \\ &(0,1,0,0,-1,0), (0,1,1,0,-1,-1), (0,1,1,1,0,0), (1,0,-1,-1,0,1), (1,0,0,-1,0,0), \\ &(1,0,0,0,1,1), (1,1,0,-1,-1,0), (1,1,0,0,0,1), (1,1,1,0,0,0)\} \end{aligned} \tag{2.9}$$

where a particular entry of a vector indicates the number of prongs on the respective marked point, according to the ordering of Z. Note that the set has $|R| = 24$, matching precisely the number of roots of $\mathfrak{so}(8)$. The subset

$$SR = \{[0,0,0,1,1,1], [0,0,1,0,0,-1], [0,1,0,0,-1,0], [1,0,-1,-1,0,1]\} \tag{2.10}$$

is a set of four simple root junctions, as verified by checking that the I-matrix of these junctions is precisely negative of the Cartan matrix for $\mathfrak{so}(8)$

$$-A_{ij}=(J_i,J_j)=\begin{pmatrix}-2 & 1 & 0 & 0\\ 1 & -2 & 1 & 1\\ 0 & 1 & -2 & 0\\ 0 & 1 & 0 & -2\end{pmatrix}\qquad \text{for}\qquad J_i\in SR. \tag{2.11}$$

One can also check, given the full set of roots above, that there is a highest root determined by these simple roots, and that the associated level diagram generated by Freudenthal's recursion formula[8] is of the correct topology. Accordingly, the levels of the roots are

Level	J	(J,J)	Multiplicity
0	$(1,1,1,0,0,0)$	-2	1
1	$(1,1,0,0,0,1)$	-2	1
2	$(1,1,0,-1,-1,0)$	-2	1
2	$(1,0,0,0,1,1)$	-2	1
2	$(0,1,1,1,0,0)$	-2	1
3	$(1,0,0,-1,0,0)$	-2	1
3	$(0,1,1,0,-1,-1)$	-2	1
3	$(0,0,1,1,1,0)$	-2	1
4	$(1,0,-1,-1,0,1)$	-2	1
4	$(0,0,1,0,0,-1)$	-2	1
4	$(0,1,0,0,-1,0)$	-2	1
4	$(0,0,0,1,1,1)$	-2	1
5	$(0,0,0,0,0,0)$	0	4
6	$(0,0,0,-1,-1,-1)$	-2	1
6	$(0,0,-1,0,0,1)$	-2	1
6	$(0,-1,0,0,1,0)$	-2	1
6	$(-1,0,1,1,0,-1)$	-2	1
7	$(0,0,-1,-1,-1,0)$	-2	1
7	$(0,-1,-1,0,1,1)$	-2	1
7	$(-1,0,0,1,0,0)$	-2	1
8	$(0,-1,-1,-1,0,0)$	-2	1
8	$(-1,0,0,0,-1,-1)$	-2	1
8	$(-1,-1,0,1,1,0)$	-2	1
9	$(-1,-1,0,0,0,-1)$	-2	1
10	$(-1,-1,-1,0,0,0)$	-2	1

This includes, for example, a symmetry on the level diagram indicative of $\mathfrak{so}(8)$ triality; notice that there exist a number of levels with three roots. $M2$-branes on these two-manifolds, which are string junctions in F-theory, give rise to particle states filling out an adjoint of $\mathfrak{so}(8)$. However, we know that this geometry exhibits instead a broken $\mathfrak{g}_2$ gauge symmetry due to outer monodromy; in [45] we showed that codimension two singularities can, in fact, induce an action on junctions upon moving in a family of elliptic surfaces, and that the associated algebraic automorphism is a $\mathbb{Z}_3$ outer automorphism that turns $\mathfrak{so}(8)$ into $\mathfrak{g}_2$. In fact, the local geometry studied in [45] is just the geometry we have considered (with the specific p_k) near the locus $z=t=0$, $s=1$.

[8]Given the highest weight of a Lie algebra representation and the simple roots, Freudenthal's recursion formula can be utilized to generate all of the weights in a representation, along with their multiplicities. See [45] for further discussion.

Finally, though we have a broken $\mathfrak{g}_2$ gauge theory, let us study a few non-adjoint representations of $\mathfrak{so}(8)$ at the level of junctions for the sake of illustration. These arise from two-manifolds J in this elliptic surface which carry some asymptotic charge $a(J)$. For simple representations, all J making up the representations will have the same self-intersection (J, J); this is not true, of course, for higher dimensional representations, whose weights may not all have the same length. Studying three different sets of junctions, each with[9] $(J, J) = -1$ but with $a(J) = \pi_1$, $a(J) = \pi_2$ and $a(J) = \pi_3$, gives three different representations. From these sets the level diagrams of the junctions can be determined. They are

$a(J)$	Level 0	Level 1	Level 2	Level 3	Level 4	Level 5	Level 6
π_1	$(0,1,1,0,-1,0)$	$(0,0,1,0,0,0)$	$(0,0,0,0,0,1)$	$(0,0,0,-1,-1,0)$, $(-1,0,1,1,0,0)$	$(-1,0,1,0,-1,-1)$	$(-1,0,0,0,-1,0)$	$(-1,-1,0,0,0,0)$
π_2	$(0,0,1,1,0,0)$	$(0,0,1,0,-1,-1)$	$(0,0,0,0,-1,0)$	$(0,-1,0,0,0,0)$, $(1,0,1,1,-1,-1)$	$(-1,-1,1,1,0,-1)$	$(-1,-1,0,1,0,0)$	$(-1,-1,0,0,-1,-1)$
π_3	$(1,0,0,0,0,0)$	$(0,0,1,1,0,-1)$	$(0,0,0,1,0,0)$	$(0,0,0,0,-1,-1)$,$(0,-1,0,1,1,0)$	$(0,-1,0,0,0,-1)$	$(0,-1,-1,0,0,0)$	$(-1,-1,0,1,0,-1)$

where the highest weight junctions are those at level 0. These junctions are vectors in $\mathbb{Z}^6$, but since they represent some representation of the D_4 algebra, which is of rank 4, there must be a linear map from $\mathbb{Z}^6$ to $\mathbb{Z}^4$ which maps junctions to weights in the Dynkin basis. We wrote down such a general map in [45], which in this example is computed to be

$$F = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 \end{pmatrix}. \tag{2.12}$$

Applying this map to the highest weight junctions above, these eight dimensional representations have Dynkin labels given by $(0, 0, 1, 0)$, $(1, 0, 0, 0)$, and $(0, 0, 0, 1)$. Recall that $\mathfrak{so}(8)$ has three eight dimensional representations, which are permuted by triality. Their highest weights have precisely these Dynkin labels; these sets of junctions fill out those representations.

[9]Note that in the code referenced in appendix A we choose a basis of one-cycles such that $\pi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\pi_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in order to perform concrete computations; from this one computes $\pi_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ from the fact that $\pi_1 + \pi_2 + \pi_3$ is trivial in homology.

3 Gauge Theories and Singularity Deformation

We have just seen how singularity deformation describes the spontaneous breaking of gauge symmetry in F-theory and how structures of the broken gauge theory can be identified in the deformed geometry.

In this section we would like to make some brief, but we think important, comments about physics, the relationship between the geometric and gauge theory moduli spaces, and also the only known description for describing both massless and massive gauge states in F-theory. We will then develop a general framework for performing deformations which allow one to see these structures in the geometry.

3.1 The F-theoretic and M-theoretic Physics of Deformation

For the sake of clarity we would like to again state the points about the M-theoretic and F-theoretic physics of singularity deformation; some may exist elsewhere in the literature. Though we will utilize $d = 3, 4$ language exclusively, many of these comments apply to other dimensions as well.

Gauge Theoretic and Geometric Moduli Spaces

A $d = 4$ compactification of F-theory on X and its defining M-theory compactification on an elliptically fibered Calabi-Yau variety X_M give rise to $d = 4$ $\mathcal{N} = 1$ and $d = 3$ $\mathcal{N} = 2$ gauge theories, respectively, and movement in these moduli spaces occurs by movement in corresponding geometric moduli spaces.

Since the resolution of singular fibers in X_M is not consistent with the limit of vanishing fiber $X_M \to X$ which defines F-theory, this branch of the moduli space does not exist[10] in F-theory, and thus

1) The resolution of singular fibers does not describe the spontaneous breaking of any gauge symmetry in F-theory.

Since it can break the $d = 3$ $\mathcal{N} = 2$ gauge theories but not the $d = 4$ $\mathcal{N} = 1$ gauge theories, this is sufficient to show that those Kähler moduli which perform this breaking must correspond to the scalars obtained from the dimensional reduction of the 4^{th} component of the gauge field, as is well known.

Alternatively, both X_M and X can undergo deformation of complex structure. This means that the associated moduli can break both the $d = 3$ $\mathcal{N} = 2$ gauge theories of M-theory on X_M and also the $d = 4$ $\mathcal{N} = 1$ gauge theories of F-theory on X. As such, these

2) Complex structure moduli determine the expectation values of scalar fields in $d = 4$ chiral multiplets and their dimensional reductions to three dimensions,

though it remains to work out a general map. Thus, it is complex structure deformations that determine the spontaneous breaking of gauge theories arising in four-dimensional F-theory compactifications.

[10] A more physical way to state its non-existence is as follows. In F-theory compactifications gravitons can propagate in $\mathbb{R}^{3,1}$ and also the six-dimensional base B of X; if the rational curves of resolution were to appear gravitons could propagate in them, also, for a total of twelve dimensions of graviton propagation. Since no such twelve-dimensional theory exists, it must be impossible to do this in F-theory. In M-theory on $\mathbb{R}^{2,1} \times X_M$ the smooth elliptic fibers and also the rational curves of resolution are available for graviton propagation, for a total of 11 dimensions, which is of course perfectly consistent.

Gauge States in F-theory are String Junctions

An immediately corollary of the fact that resolution of singular fibers does not break any gauge symmetry in F-theory is that

$$\textit{3) Resolution of singular fibers cannot describe massive gauge states in F-theory,} \tag{3.1}$$

for example massive W-bosons. This is not ideal, as a preferred description of the theory should be able to describe both massive and massless gauge states!

Alternatively, as complex structure deformation does describe the spontaneous breaking of gauge symmetry in F-theory, it does describe those massive W-bosons (and their superpartners). In the deformed M-theory picture, these are $M2$-brane states wrapped on two-cycles with one leg on the fiber and one leg on the base; in the F-theory limit these objects lose a dimension and become string junctions. Thus, from these arguments we see unambiguously that

4) Gauge states in F-theory arising from singular fibers are string junctions, possibly in a massless limit. (3.2)

It is reasonable to expect that significant progress in F-theory can be made by utilizing this description, since it exists both in a defining M-theory compactification and its F-theory limit.

In addition, there are known examples [43] of six-dimensional F-theory compactifications where anomaly cancellation requires a multiplicity factor for matter fields which is not captured by a simple resolution, since this factor is related to the versal deformation space near the singularities. It would be interesting to see if our techniques correctly capture this multiplicity. Furthermore, the Higgs couplings in the low energy effective action of a four-dimensional $\mathcal{N} = 1$ theory are critical for a proper understanding of the physics; in F-theory the expectation values of Higgs fields, which affect the Higgs couplings, typically depend on complex structure moduli, and never on the Kähler moduli of resolution.

Higgs and Coulomb Branches of Singularity Deformation

To what subgroups can singularity deformation break G in F-theory? This depends as usual on the representation which performs the Higgsing and also which of its components receive expectation values, but the possibility exists for both Higgs branches and Coulomb branches in $d = 4$ gauge theories. In particular, if the field which obtains an expectation value is the scalar component of a four-dimensional adjoint chiral multiplet, the theory is broken to $U(1)^{rk(G)}$; this is a four-dimensional Coulomb branch which can also exist in three dimensions, where one might call it the "M-theory Coulomb branch of singularity deformation," in contrast to its resolution counterpart. Thus,

5) There can be an M-theory Coulomb branch which lifts to F-theory;
It is the one of deformation, not resolution.

On this branch non-abelian gauge states are described by string junctions. Since the Cartan $U(1)$'s of G have not been broken, this gauge theoretic description geometrically requires the existence of deformations which do not change $h^{1,1}(X)$ or $h^{1,1}(X_M)$; these are known to exist in certain examples. For example, in the threefold case see [83]; there these deformations completely break the non-abelian part of the gauge theory, but a $U(1)^{rk(G)}$ subgroup remains, along with a finite collection of conifolds.

Alternatively, depending on the representation theory there can also be Higgs branches in F-theory compactifications and their defining M-theory compactifications where the gauge symmetry is completely Higgsed to nothing. Whether a particular deformation gives a Higgs branch or a Coulomb branch depends on the details of the geometry.

3.2 Deformation Types for Weierstrass Models

Having emphasized important aspects of the F-theoretic and M-theoretic physics of deformation, let us discuss certain deformation types of Weierstrass models which will be useful. There has been much study in the mathematics literature of resolution and deformations of the surface A-D-E (Kleinian) singularities [84–87] and also of complete intersections isolated points [88]. The deformations of relevance in this context are the deformations of the elliptic fibration $X \to B$ around the singular loci.

Consider a Weierstrass model for a Calabi-Yau elliptic fibration $\pi : X \to B$. The defining equation for such a variety is given by

$$y^2 = x^3 + f\,x + g \tag{3.3}$$

with sections $f \in \Gamma(K_B^{-4})$ and $g \in \Gamma(K_B^{-6})$. The singular fibers occur over points in the base in the discriminant locus $\Delta \equiv 4f^3 + 27g^2 = 0$. For the singularities relevant for F-theory compactifications with non-abelian gauge symmetry, the discriminant takes the form

$$\Delta = z^N\,\Delta_r \tag{3.4}$$

where Δ_r is easily computed in examples. The singular fiber above the locus $z = 0$ is determined by Kodaira's classification of singular fibers in codimension one. In many cases this singular fiber corresponds to an ADE singularity[11]; the only counterexample is a type II fiber, which we will explicitly show does not carry a gauge algebra using deformation. We call Δ_r the *residual discriminant*, which itself may be a reducible subvariety in B. Its form determines the precise structure of singular fibers in codimension two and three, as well as the associated physics. For example, though the representation theory of matter fields is determined by codimension one data, it is the structure of Δ_r that determines which matter fields become massless in codimension two. See [45] for discussion on this point.

For a Weierstrass model[12], it is simple to see an organizing principle for the deformations: one can deform both f and g as

$$f \mapsto f + \varepsilon_f, \qquad g \mapsto g + \varepsilon_g \tag{3.5}$$

where ε_f and ε_g are sections of the same bundles as f and g, of course. Such a deformation deforms the discriminant locus as

$$\Delta \mapsto \Delta + 4\epsilon_f\left(3f^2 + 3f\epsilon_f + \epsilon_f^2\right) + 27\epsilon_g\left(\epsilon_g + 2g\right) \tag{3.6}$$

We call such a deformation a *fg-deformation* and will call the deformation of the discriminant the *deformed discriminant*. In some cases these general deformations will not be needed to uncover gauge structure and

[11]An interesting fact is that N is never the rank of the gauge group; as such root junctions are represented as vectors in $\mathbb{Z}^N$ which nevertheless only span a $rk(G)$-dimensional subspace.

[12]Similar deformations could also be done for the sections a_n of a Tate model.

we will consider an *f-deformation* or a *g-deformation*, defined by $\varepsilon_g = 0$ and $\varepsilon_f = 0$, respectively; that is, these deform either f or g, but not both. The deformed discriminant is easily derived from the more general form.

Having discussed the general ways to deform a Weierstrass equation — by deforming f, g, or both — we now categorize certain types of deformations to set a common language for discussing examples. Taking a singular variety with $\Delta = z^N \Delta_r$, a general deformation gives $\Delta \mapsto z^N \Delta_r + \Delta_\varepsilon$ for some Δ_ε.

Suppose that z does not divide Δ_ε. Then in small neighborhoods of $z = 0$ away from $\Delta_\varepsilon \neq 0$ and $\Delta_r \neq 0$ the deformed discriminant is

$$z^N \Delta_r + \Delta_\varepsilon = \Delta_r \left(z^N + \Delta_\varepsilon/\Delta_r\right) \tag{3.7}$$

and we see that $\Delta \sim (z^N + \tilde{\varepsilon})$ for $\tilde{\varepsilon} \equiv \Delta_\varepsilon/\Delta_r$. This gives a family of elliptic surfaces with base the complex z-plane, where the family is determined by base coordinates via $\tilde{\varepsilon}$; for a generic member of the family, $\tilde{\varepsilon} \in \mathbb{C}^*$. We call such a deformation a *completely Higgsing deformation* since for a generic member of the family the singularity at $z = 0$ has been completely smoothed out; the associated non-abelian part of the gauge theory has been Higgsed[13]. More specifically, the N coincident components of z^N have been deformed into N distinct components at the N^{th} roots of $\tilde{\varepsilon}$. The singular fiber above each component has a vanishing one-cycle, and by identifying an ordered set of such vanishing cycles, the new finite volume two-cycles whose structure reproduces the structure of the Higgsed gauge algebra can be systematically constructed [45]. We call f-deformations and g-deformations which are completely Higgsing *completely Higgsing f-deformations* and *completely Higgsing g-deformations*, respectively. Finally, we note that a completely Higgsing deformation may not actually smooth all singularities: we have named it to denote the complete Higgsing of a particular codimension one gauge theory along $z = 0$; there may, of course, be other gauge theories on other components of the discriminant.

In some cases it will be useful to seek deformations which keep certain structures intact. We will not use such deformations in this work, but one was utilized in the $\mathfrak{g}_2$ example of [45] and will be utilized in our upcoming works [81, 82]. We call an fg-deformation with

$$f \mapsto f + \varepsilon_{f,r}\, \Delta_r \qquad \text{and} \qquad g \mapsto g + \varepsilon_{g,r}\, \Delta_r \tag{3.8}$$

a *Δ_r-deformation.* In such a case the deformed discriminant is

$$\Delta = \Delta_r \left[z^N + 4\epsilon_{f,r}\left(3f\Delta_r\epsilon_{f,r} + \Delta_r^2\epsilon_{f,r}^2 + 3f^2\right) + 27\epsilon_{g,r}\left(\Delta_r\epsilon_{g,r} + 2g\right)\right] \tag{3.9}$$

and we see the additional property enjoyed by a Δ_r-deformation: after deformation, Δ_r is still a component of the discriminant. Of course, in a compact model such a deformation requires the existence of appropriate sections $\varepsilon_{f,r} \in \Gamma(\mathcal{O}(N\, D_z) \otimes K_B^8)$ and $\varepsilon_{g,r} \in \Gamma(\mathcal{O}(N\, D_z) \otimes K_B^6)$, where D_z is the divisor class of the $z = 0$ locus. If Δ_r has a particular component Δ_i whose codimension two intersection with $z = 0$ is of interest, we may also consider a *Δ_i-deformation*, defined by $\varepsilon_f = \varepsilon_{f,i}\, \Delta_i$ and $\varepsilon_g = \varepsilon_{g,i}\, \Delta_i$, the existence of which requires the existence of appropriate sections $\varepsilon_{f,i}$ and $\varepsilon_{g,i}$. In such a case Δ_i remains a component of the discriminant after deformation. In some cases it may be that a Δ_i-deformation exists, even if a Δ_r-

[13] Though in certain examples it may be possible for abelian subgroups to survive.

deformation does not. Note that both of these cases can be restricted to f-deformations and g-deformations by setting $\varepsilon_{g,i}$ and $\varepsilon_{f,i}$ to zero, respectively.

We will find that Δ_r- and Δ_i-deformations are useful for studying outer automorphisms on codimension one algebras induced by monodromy around codimension two singularities. As in our previous work, we will call such monodromy *O-monodromy*, which we emphasize is different than the monodromy induced by the Picard-Lefschetz action on vanishing cycles. The general method we will employ to study O-monodromy is to deform a codimension one singularity and analyze its Lie algebraic structure, and then to encircle another codimension one locus which intersects $z = 0$ at the codimension two point of interest prior to deformation. Herein lies the advantage of Δ_r- and Δ_i-deformations: after deformation, Δ_r or Δ_i are still components of the discriminant, and one of these is the codimension one locus which will be encircled to study O-monodromy. If we had performed a generic deformation which was not a Δ_r or Δ_i deformation, it would not be clear what to encircle.

3.3 Local vs. Global Deformations and non-Higgsable Clusters

While in many examples we use the above technology to deform local Weierstrass models, our techniques can be used also in compact geometries, in particular for global compactifications of M-theory and F-theory. In fact, we have already presented such a compact example in section 2.

In the following subsection we discuss two examples demonstrating interesting features of global and local deformations. In the first, the local deformation is actually the restriction of a global deformation. In the second example we show how to use our local techniques, together with a global geometry analysis, to deduce global information about gauge algebras and matter representations, even in cases where no global deformation is possible. These are "non-Higgsable clusters" defined and classified in [67].

A Simple Example of a Global to Local Map

Consider an F-theory compactification to four dimensions on an elliptically fibered Calabi-Yau fourfold with $B = \mathbb{P}^3$. We consider a Weierstrass model; since $K_B = \mathcal{O}_B(-4)$ we have $f \in \Gamma(\mathcal{O}_B(16))$ and $g \in \Gamma(\mathcal{O}_B(24))$ in homogeneous coordinates (z, x_1, x_2, x_3) on $\mathbb{P}^3$. Suppose that f and g are chosen such that they have order of vanishing zero around $z = 0$, but the discriminant has order of vanishing 5. In such a case there is an I_5 fiber above a generic point in the locus $z = 0$, and the discriminant takes the form

$$\Delta = z^5\, \Delta_r. \tag{3.10}$$

This is appropriate for a global F-theory compactification with an $\mathfrak{su}(5)$ factor in the gauge algebra.

We would like to perform a complex structure deformation of the compact manifold which is a completely Higgsing g-deformation. Specifically, we consider the deformation given by

$$(f, g) \mapsto (f, g + \varepsilon\, [x_1^{24} + x_2^{24} + x_3^{24}]) \qquad\qquad \varepsilon \in \mathbb{C}^*, \tag{3.11}$$

which is an allowed one-parameter deformation of the global geometry. The deformed discriminant is

$$\Delta = z^5\, \Delta_r + 54\, \varepsilon\, g \left(x_1^{24} + x_2^{24} + x_3^{24}\right) + 27\, \epsilon^2\, \left(x_1^{24} + x_2^{24} + x_3^{24}\right)^2 \tag{3.12}$$

The structure of this equation matches the general form (3.6), Δ is an I_1 locus, and the non-abelian part of the gauge theory along $z = 0$ is completely Higgsed.

How does this deformation map to a deformation of a local Weierstrass model, as will be studied throughout this work? Since $z = x_1 = x_2 = x_3 = 0$ is not in $\mathbb{P}^3$ any patch containing $z = 0$ must have at least one $x_i \neq 0$. Suppose it is $x_1 \neq 0$. On that patch, the $\mathbb{C}^*$ action of $\mathbb{P}^3$ can be used to set that $x_1 = 1$, and at every point on that patch the deformation appears as

$$(f, g) \mapsto (f, g + \tilde{\varepsilon}) \tag{3.13}$$

with $\tilde{\varepsilon} = \varepsilon\,[1 + x_2^{24} + x_3^{24}]$; this gives a family of elliptic surfaces over the z-plane determined by x_2, x_3. For generic x_2, x_3, $(1 + x_2^{24} + x_3^{24}) \neq 0$; this means that a generic member of the family of elliptic surface is deformed by this global deformation. For a particular member of the family, $\tilde{\varepsilon}$ is simply a number and we have obtained a local description of the elliptic fibration over the z-plane at this point.

In many examples we will simply deform g by adding a number $\varepsilon \in \mathbb{C}^*$; here we see have seen how deformations of such a local model are realized in a global context.

From local deformation to global information, even when a global deformation does not exist: the case of non-Higgsable clusters

In [67] the authors consider Calabi-Yau varieties $X \to B$ which are resolutions of maximally Higgsed (minimal) Weierstrass models. They analyze all connected configurations of curves C of negative self-intersection in B which carry non-Higgsable gauge algebra: these curves are necessarily smooth, rational and of negative self intersection, say $-m$. Since the Weierstrass model is assumed to be minimal, $m \leq 12$. For example, if $B = \mathbb{F}_m$ is a Hirzebruch surface with $m \geq 3$, then the "infinity section" C_∞ with ${C_\infty}^2 = -m$ must carry a gauge algebra, and are necessarily non-Higgsable clusters.

Under general conditions, assumed in [67], the vanishing orders of f and g near the curve C are also determined by the geometry and can be calculated explicitly. Table 2 in [67] describes the (connected) non-Higgsable clusters consisting of a single curve:

$-m$	Gauge Algebra	Matter	(f, g, Δ)
-3	$\mathfrak{su}(3)$	0	$(2, 2, 4)$
-4	$\mathfrak{so}(8)$	0	$(2, 3, 6)$
-5	$\mathfrak{f}_4$	0	$(3, 4, 8)$
-6	$\mathfrak{e}_6$	0	$(3, 4, 8)$
-7	$\mathfrak{e}_7$	$\frac{1}{2}\mathbf{56}$	$(3, 5, 9)$
-8	$\mathfrak{e}_7$	0	$(3, 5, 9)$
-12	$\mathfrak{e}_8$	0	$(4, 5, 10)$

We show that our method can be applied also in this situation, namely even when a global deformation (Higgsing) does not exist. Let $\mathcal{V}$ in B be a sufficiently small neighborhood (in the complex topology) such that $C \subset \mathcal{V}$. Let $\mathcal{D} \subset \mathcal{V}$ a divisor which intersects C in a smooth point and let $\mathcal{U} = \mathcal{V} \setminus \mathcal{D}$. Then the elliptic fibration restricted to $\mathcal{U}$ is an affine Weierstrass model $W_0 \to \mathcal{U}$: $y^2 = x^3 + f(z)\,x + g(z)$, where z is the local parameter of C, and the vanishing orders of f and g along C are described in the fourth column.

We can then deform this affine Weierstrass model as described in the previous paper [45]; see section 4.1 for the case $m = 3$. Note that when $m \neq 5, 7$ the curve C does not intersect the other components of the discriminant: we then obtain, in particular, the gauge algebras in the second column of the above Table. If $m = 5$, the same geometric analysis described in [67] tells us the points are branched points of an outer monodromy and that the contribution to the matter along C is zero. In the forthcoming paper [82] we derive the explicit matter contribution if $m = 7$, as well as the non-localized matter for $m = 5$.

4 Codimension One Gauge Structure: Less Studied Fibers

The codimension one fibers utilized in gauge theories or F-theory are typically of type I_n, I_n^*, IV^*, III^*, II^*, corresponding to gauge algebras $\mathfrak{su}(N)$, $\mathfrak{so}(2(N+4))$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, and $\mathfrak{e}_8$, respectively. For I_n, I_n^*, IV^*, III^*, and II^* fibers the relevant data has already been obtained in the literature; for explicit deformations of Calabi-Yau elliptic fibrations, see [45], and for other examples and ordered sets of vanishing cycles, see some of the original literature on string junctions [78, 79].

In this section we will deform Kodaira's other codimension one singular fibers[14], of type IV, III, and II and recover the corresponding gauge algebras $\mathfrak{su}(3)$, $\mathfrak{su}(2)$ and $\emptyset$, respectively. Interestingly, these $\mathfrak{su}(3)$ and $\mathfrak{su}(2)$ gauge algebras arise from four and three sets of seven-branes, respectively, as opposed to the three and two sets in type IIb; this is an intrinsically F-theoretic realization of these algebras.

4.1 SU(3) Gauge Symmetry from a Type IV Fiber

In this example we study the gauge symmetry realized by a type IV fiber in codimension one. We find this example intriguing: by deforming the geometry we will see the appearance of an $\mathfrak{su}(3)$ gauge algebra realized by string junctions ending on *four* seven-branes, as opposed to the standard three seven-branes in the case of an I_3 fiber, which becomes three coincident D7-branes in the weakly coupled type IIb picture. In the case of threefolds or higher dimension, a type IV fiber can either realize $\mathfrak{su}(3)$ or $\mathfrak{sp}(1) \cong \mathfrak{su}(2)$ due to outer monodromy.

We start with the case of a surface, or equivalently with the analysis of a singular fiber of type IV around a general point of the discriminant locus. The general Weierstrass form, the singular fiber can be realized by

$$f = z^2 \qquad g = z^2. \tag{4.1}$$

and the discriminant is $\Delta = z^4\left(4z^2+27\right)$. Since we performed a completely Higgsing g-deformation in section 2, let us utilize a completely Higgsing fg-deformation for the sake of illustration. The fg-deformation $(f,g) \mapsto (f+\varepsilon, g+\varepsilon)$ has an associated deformed discriminant

$$\Delta = \left(z^2+\epsilon\right)^2\left(4z^2+4\epsilon+27\right). \tag{4.2}$$

We see that this deformation is not completely Higgsing. Instead, we study the deformation

$$(f,g) \mapsto (f+2\varepsilon, g+\varepsilon), \tag{4.3}$$

with associated deformed discriminant

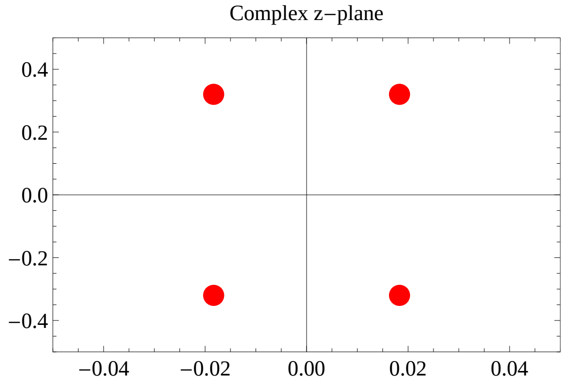
$$\Delta = 4z^6+27z^4+6\,\varepsilon\, z^2\left(4z^2+9\right)+3\,\varepsilon^2\,\left(16z^2+9\right)+32\,\varepsilon^3 \tag{4.4}$$

which appears to have broken the degeneracy. In all, the single discriminant component with multiplicity four has been deformed into four discriminant components with multiplicity one.

Taking $\varepsilon = .1$ and going to a neighborhood of $z=0$ in order to see the four deformed components (and

[14] Links to computational codes and illustrative films for these examples are provided in appendix A.

not the other two z-roots of Δ) the intersection of the deformed discriminant with the z-plane is given by



and we see that, indeed, the gauge theory previously at $z = 0$ has been completely Higgsed. Picking the base point to be to at the origin and following straight line paths of approach to the singular fibers, beginning with the upper left and working clockwise, determines an ordered set of vanishing cycles

$$Z_{IV} = \{\pi_1, \pi_3, \pi_1, \pi_3\}. \tag{4.5}$$

¿From the generic formula of [45], the I-matrix for this ordered set of vanishing cycles is

$$I = (\cdot,\cdot) = \begin{pmatrix} -1 & 1/2 & 0 & 1/2 \\ 1/2 & -1 & -1/2 & 0 \\ 0 & -1/2 & -1 & 1/2 \\ 1/2 & 0 & 1/2 & -1 \end{pmatrix}. \tag{4.6}$$

Per usual, the roots are $\{J \in \mathbb{Z}^4 \mid (J,J) = -2 \text{ and } a(J) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$. They are given by

$$(-1,-1,1,1), (-1,0,1,0), (0,-1,0,1), (0,1,0,-1), (1,0,-1,0), (1,1,-1,-1), (0,0,0,0), (0,0,0,0) \tag{4.7}$$

to which the Cartan generators $(0,0,0,0), (0,0,0,0)$ have been added. For one possible set of simple roots, the root diagram is given by

Level	J	(J,J)	Multiplicity
0	$(1,1,-1,-1)$	-2	1
1	$(1,0,-1,0)$	-2	1
1	$(0,1,0,-1)$	-2	1
2	$(0,0,0,0)$	0	2
3	$(0,-1,0,1)$	-2	1
3	$(-1,0,1,0)$	-2	1
4	$(-1,-1,1,1)$	-2	1

and we see the structure of an $\mathfrak{su}(3)$ algebra, in full agreement with expectations from resolution. The linear map $F : \mathbb{Z}^4 \to \mathbb{Z}^2$ from weight junctions to Dynkin labels is given by $F = \begin{pmatrix} 0 & 1 & 1 & -1 \\ 1 & -1 & -1 & 0 \end{pmatrix}$. From the level diagram for the roots, we see that the simple roots in this Weyl chamber are $\{(1,0,-1,0), (0,1,0,-1)\}$.

We would now like to uncover junctions in the fundamental and antifundamental representations. Consider junctions $J \in \mathbb{Z}^4$ with $a(J) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and $(J,J) = -1$, and also recall that we have chosen a basis of one-cycles such that $\pi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\pi_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. There are three such junctions. The level diagram of these junctions is given by

Level	J	(J,J)	Multiplicity
0	$(0,1,-1,-1)$	-1	1
1	$(0,0,-1,0)$	-1	1
2	$(-1,0,0,0)$	-1	1

and by acting with F on the transpose of the highest weight junction $(0,1,-1,-1)$ we get (after taking the transpose) $(1,0)$, the Dynkin labels of the fundamental. The level diagram of junctions $J \in \mathbb{Z}^4$ with $a(J) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $(J,J) = -1$ is

Level	J	(J,J)	Multiplicity
0	$(1,0,0,0)$	-1	1
1	$(0,0,1,0)$	-1	1
2	$(0,-1,1,1)$	-1	1

and by applying F to the highest weight junction we see that these junctions comprise the antifundamental.

In finding the fundamental and the antifundamental, we have identified highest weight junctions with corresponding Dynkin labels $(1\,0)$ and $(0\,1)$; from these we can build *any* representation of $\mathfrak{su}(3)$. Let us do this in a few examples for the sake of illustration. One ten dimensional representation has a highest weight with Dynkin labels $(3,0)$; this is three times the Dynkin labels of the highest weight of the fundamental, signifying that the 10 is the totally symmetric part of the third tensor power of the fundamental. Taking the highest weight junction $(0,1,-1,-1)$, multiplying by three to get the highest weight junction of the 10, and applying Freudenthal's formula, we obtain the full 10, as

Level	J	(J,J)	Multiplicity
0	$(0,3,-3,-3)$	-9	1
1	$(0,2,-3,-2)$	-5	1
2	$(0,1,-3,-1)$	-5	1
2	$(-1,2,-2,-2)$	-5	1
3	$(0,0,-3,0)$	-9	1
3	$(-1,1,-2,-1)$	-3	1
4	$(-1,0,-2,0)$	-5	1
4	$(-2,1,-1,-1)$	-5	1
5	$(-2,0,-1,0)$	-5	1
6	$(-3,0,0,0)$	-9	1

Alternatively, consider the 27 dimensional representation whose highest weight has Dynkin labels $(2,2)$. Adding two copies of the highest weight junctions of both the fundamental and antifundamental gives the highest should give the highest weight junction of the 27; indeed, applying Freudenthal's formula we see that it does. These junctions are given by

Level	J	(J,J)	Multiplicity
0	$(2,2,-2,-2)$	-8	1
1	$(2,1,-2,-1)$	-6	1
1	$(1,2,-1,-2)$	-6	1
2	$(2,0,-2,0)$	-8	1
2	$(1,1,-1,-1)$	-2	2
2	$(0,2,0,-2)$	-8	1
3	$(1,0,-1,0)$	-2	2
3	$(0,1,0,-1)$	-2	2
4	$(1,-1,-1,1)$	-6	1
4	$(0,0,0,0)$	0	3
4	$(-1,1,1,-1)$	-6	1
5	$(0,-1,0,1)$	-2	2
5	$(-1,0,1,0)$	-2	2
6	$(0,-2,0,2)$	-8	1
6	$(-1,-1,1,1)$	-2	2
6	$(-2,0,2,0)$	-8	1
7	$(-1,-2,1,2)$	-6	1
7	$(-2,-1,2,1)$	-6	1
8	$(-2,-2,2,2)$	-8	1

and we could also build any other representation of $\mathfrak{su}(3)$ in a similar manner.

An explicit global to local map. For the sake of illustration, let us briefly show an example global model into which this local deformation embeds; this is only one of many, of course. Consider an elliptic threefold with base $B = \mathbb{P}^2$; f and g are sections of $\Gamma(\mathcal{O}(12))$ and $\Gamma(\mathcal{O}(18))$, respectively and thus are degree 12 and degree 18 polynomials in the homogeneous coordinates (z,t,x_1) on $\mathbb{P}^2$. Write $f = z^2\,p_{10}$ and $g = z^2\,p_{16}$, with p_i homogenous polynomials of degree i. Deform the manifold by $(f,g) \mapsto (f + 2\varepsilon\,p_{12}, g + \varepsilon\,p_{18})$. Around a general point of $z = 0$ the local geometry is equivalent to that defined by the initial data (4.1) and the deformation (4.3). The gauge algebra associated to the IV singular locus will be then either $\mathfrak{su}(3)$ or generically $\mathfrak{sp}(1) \cong \mathfrak{su}(2)$ if there exists outer monodromy, as in [45].

$\mathfrak{su}(3)$ ***gauge algebra from deformation in a non-Higgsable Cluster.*** As discussed in section 3.3, an elliptic fibration over $\mathbb{F}_3$ necessarily gives rise to a non-Higgsable cluster with a type IV fiber along the infinity section. It is a simple exercise to write down such a compact elliptic fibration: in a neighborhood of the locus $z = 0$ of the infinity section C_∞, the elliptic fibration $W_0 \to \mathcal{U}$ is precisely the local elliptic surface defined by (4.1) (see also the vanishing data in the last column of the Table in section 3.3). Since this *very same* elliptic surface can be obtained via restriction from geometries which do (see above) or do not admit a global deformation, the local deformation (4.3) uncovers the same topological structures in both cases!

It makes perfect sense physically that the local deformation of the non-Higgsable cluster uncovers the correct Lie algebraic structure, even though a global deformation of the geometry does not exist. Physically, this is just the statement that the geometry still encodes the massless W-bosons, even though there is no flat direction in the moduli space which would spontaneously break the gauge symmetry.

4.2 SU(2) Gauge Symmetry from a Type III Fiber

In this section determine the gauge algebra associated to a deformed type III fiber. From the resolution picture, we know that the algebra should be $\mathfrak{su}(2)$. However, we will see that this algebra is realized from three seven-branes, rather than two.

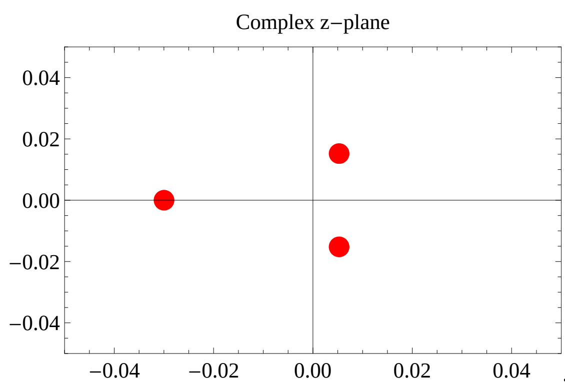
The Weierstrass model we study which exhibits this singular fiber is

$$f = z \qquad g = z^2. \tag{4.8}$$

and the discriminant is $\Delta = z^3\,(27z+4)$. The deformations $(f,g) \mapsto (f+\varepsilon, g+\varepsilon)$ completely break the degeneracy. The deformed discriminant is

$$\Delta = 4z^3 + 27z^4 + 66\,\varepsilon\, z^2 + 3\,\varepsilon^2\,(4z+9) + 4\,\varepsilon^3. \tag{4.9}$$

Taking $\varepsilon = .001$ and a neighborhood of $z=0$ the discriminant appears as



and we see that the theory is completely Higgsed, since the three degenerate defects at $z=0$ are now non-degenerate. Utilizing a similar technique as in the previous section, the ordered set of vanishing cycles is determined to be

$$Z_{III} = \{\pi_2, \pi_1, \pi_3\}, \tag{4.10}$$

beginning with the leftmost defect and working clockwise around the origin. The I-matrix is given by:

$$I = (\cdot,\cdot) = \begin{pmatrix} -1 & 1/2 & -1/2 \\ 1/2 & -1 & 1/2 \\ -1/2 & 1/2 & -1 \end{pmatrix} \tag{4.11}$$

and the roots are $J \in \mathbb{Z}^3$ such that $(J,J) = -2$ and $a(J) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, as usual. The root diagram is given by

Level	J	(J,J)	Multiplicity
0	$(1,1,1)$	-2	1
1	$(0,0,0)$	0	1
2	$(-1,-1,-1)$	-2	1

which matches an $\mathfrak{su}(2)$ algebra, as expected. Note that unlike the W_+ and W_- bosons of an $\mathfrak{su}(2)$ algebra from an I_2 fiber, those arising from a type III fiber are three pronged string junctions!

For an $\mathfrak{su}(2)$ algebra there is no way to distinguish a fundamental from an antifundamental. However, there are two simple sets of junctions which can be distinguished, but nevertheless map to (1) under the map F, which here is given by $F = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}$. These are the sets

$$\begin{gathered} \left\{ J \mid (J,J) = -1 \text{ and } a(J) = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right) \right\} = \{(0,1,1), (-1,0,0)\} \\ \left\{ J \mid (J,J) = -1 \text{ and } a(J) = \left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix} \right) \right\} = \{(1,0,0), (0,-1,-1)\} \end{gathered} \tag{4.12}$$

We see that there is geometric data which differentiates between two different realizations of doublets, despite being the same Lie algebra representation; this phenomenon and associated physics implications were discussed[15] in [89].

Of course, from this data higher dimensional representations can be built up as in section 4.1. For example, taking the junction $(0,1,1)$ and multiplying by four should give junctions in the fourth symmetric tensor power of the 2 of $\mathfrak{su}(2)$; that is, the 5. Indeed, Freudenthal's formula applied to $(0,4,4)$ gives

Level	J	(J,J)	Multiplicity
0	$(0,4,4)$	-16	1
1	$(-1,3,3)$	-10	1
2	$(-2,2,2)$	-8	1
3	$(-3,1,1)$	-10	1
4	$(-4,0,0)$	-16	1

which are the junctions in the 5 of $\mathfrak{su}(2)$.

4.3 No Gauge Symmetry from a Type II Fiber

Finally, we would like to briefly discuss type II fibers. These are interesting because though they are in Kodaira's classification of singular fibers in codimension one, they are known to carry no gauge algebra, despite having $\Delta \sim z^2$. We will see that deformation quite easily recovers this fact.

The local Weierstrass model we study which realized this singular fiber is defined by

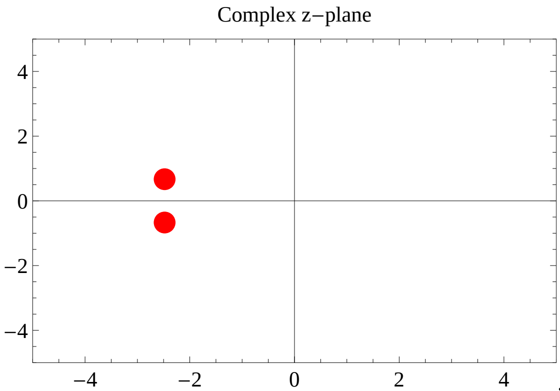
$$f = z \qquad g = z \tag{4.13}$$

with $\Delta = z^2\,(4z + 27)$. Consider the completely Higgsing fg-deformation $(f,g) \mapsto (f + 2\,\varepsilon, g + \varepsilon)$. The deformed discriminant is

$$\Delta = 4z^3 + 27z^2 + \epsilon\,(24z^2 + 54z) + \epsilon^2\,(48z + 27) + 32\,\epsilon^3 \tag{4.14}$$

Taking $\varepsilon = 2$, the the discriminant in a neighborhood of $z = 0$ is given by

[15] There it was related to additional constraints on $\mathfrak{su}(2)$ gauge theories (in another context) which are necessary and sufficient for anomaly cancellation in nucleated D-brane theories.



and taking straight line paths of approach from the origin, the vanishing cycles are determined to be

$$Z_{II} = \{\pi_3, \pi_1\}, \tag{4.15}$$

where π_3 (π_1) is associated to the discriminant component in the upper (lower) left quadrant. The I-matrix is given by

$$I = (\cdot,\cdot) = \begin{pmatrix} -1 & -1/2 \\ -1/2 & -1 \end{pmatrix} \tag{4.16}$$

With these vanishing cycles and this I-matrix, it is easy to see that there are no junctions J with $(J, J) = -2$ and $a(J) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; i.e. there are no two-manifolds which would be roots, and therefore there is no gauge algebra. From the non-trivial structure of vanishing cycles, however, codimension two collisons of a type II fiber with another fiber could yield an interesting structure of matter representations.

5 Conclusions

In this paper we have studied non-abelian gauge symmetry in F-theory compactifications on an elliptically fibered Calabi-Yau variety X. We have emphasized that the technique we utilize — singularity deformation by movement in the complex structure moduli space of X — is an ideal technique for studying the structure of non-abelian gauge theories in F-theory, because this branch of the moduli space exists in both F-theory and its defining M-theory compactification, and furthermore encompasses and extends the descriptions of gauge theories in compactifications with a heterotic dual or a weakly coupled type IIb limit.

In the second part of the introduction we reviewed the definition of F-theory in terms of M-theory, the drawbacks of the M-theory Coulomb branch of singularity resolution as a technique to study F-theory, and the advantages of utilizing complex structure deformation. These, along with the discussions of section 3.1 lead to the following important points, some of which are known:

- The resolution of singular fibers does not describe the spontaneous breaking of any gauge symmetry in F-theory.
- There can be an M-theory Coulomb branch which lifts to F-theory. However, it is the one of deformation, not resolution; the latter does not lift.
- Complex structure moduli determine the expectation values of scalar fields in F-theory. In $d = 4$ these are the scalar fields in chiral multiplets.
- Singularity resolution cannot describe massive gauge states in F-theory.
- The gauge states in F-theory arising from singular fibers – which are the ones typically studied in the literature – *are* string junctions, and in the zero size limit if they are massless.

These physical points motivate the approach of the rest of the work. In section 2 we determined the Lie algebraic structure of a spontaneously broken global model in a simple example. In section 3 we discussed the physics of deformations and presented a systematic discussion of complex structure deformations which are useful for studying gauge theoretic data. We also show how to use a local deformation, together with a global geometry analysis, to deduce global information about gauge groups and matter representations, even in cases where no global deformation is possible; i.e. the case of non-Higgsable clusters. In section 4 we presented three new examples of using deformation theory to read off gauge theoretic data; these are for the less studied type IV, III, and II fibers, which realize $\mathfrak{su}(3)$, $\mathfrak{su}(2)$ and $\emptyset$ gauge theories, respectively. Interestingly, the $\mathfrak{su}(3)$ and $\mathfrak{su}(2)$ theories are realized by states ending on four and three seven-branes, respectively, in contrast with the $D7$-brane case. Some number of roots of these algebras arise from junctions with more than two prongs, providing additional direct evidence that they do not have a weakly coupled type IIb description.

It is natural to expect that certain aspects of the physics of four-dimensional F-theory compactifications will be elucidated by utilizing the branch of the moduli space that exists in the theory. Some are immediately clear; e.g. G_4-fluxes in $d = 3$ $\mathcal{N} = 2$ M-theory compactifications which lift to Lorentz-invariant configurations in F-theory have precisely one leg along the fiber [90]. What physical objects are these G_4-fluxes "along"? They should be along the $M2$-branes of the deformation picture, which wrap precisely one dimension of the elliptic fiber. This simple physical point is obscured in the resolution picture, where $M2$-branes states wrap two-cycles which are entirely in the resolved fiber, and it is not clear what the necessary "one leg along the fiber" flux condition means physically.

We believe that many other interesting aspects of F-theory compactifications will be better understood in the deformation picture. We leave such investigations to future work.

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A Freely Available Computer Packages

In completing this work, and also our previous work [45], we have written computer codes to perform many computations. These codes are publcly available at

`http://www.jhhalverson.com/deformations`

and we hope that the interested reader finds them useful. There you can find codes for all of the examples in this paper. We have tried to make them clear via including these examples and also comments, but encourage questions on the codes if these are not sufficient.

We have utilized two types of codes. The first are Mathematica notebooks which make it simple to read off vanishing cycles in deformed geometries. This data can then be fed into our package `py-junctions` which can perform a number of Lie algebraic computations. Though it is written in Python, it is best executed through a SAGE terminal, since it utilizes packages which are automatically included in SAGE.

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