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Two populations mean-field monomer-dimer model

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Abstract

A two populations mean-field monomer-dimer model including both hard-core and attractive interactions between dimers is considered. The pressure density in the thermodynamic limit is proved to satisfy a variational principle. A detailed analysis is made in the limit of one population is much smaller than the other and a ferromagnetic mean-field phase transition is found.

1 Introduction

Monomer-dimer models have been introduced in theoretical physics during the '70s to explain the absorption of diatomic molecules on a two-dimensional layer [21]. Fundamental results were obtained by Heilmann and Lieb, who proved the absence of phase transitions [15] when only the hard-core interaction is taken into account, while the presence of an additional interaction coupling dimers can generate critical behaviours [16]. Monomer-dimers models have been source of a renewed interest in the last years in mathematical physics [1, 2, 11, 13], condensed matter physics [19] and in the applications to computer science [17, 22] and social sciences [7, 10]. The presence of an interaction beyond the hard-core one that couples different dimers is fundamental for the applications where phase transitions are observed [7, 10]. Indeed in [3–5] the authors proved that a mean-field monomer-dimer model exhibits a ferromagnetic phase transition when a sufficiently strong interaction is introduced between pairs of dimers.

In this paper the investigation is extended to the case of a mean-field monomer-dimer model defined over two populations. The methods presented here can be extended to a higher number of populations. This multi-species framework has been already introduced in the context of spin models [8,9,18,20] reveling interesting mathematical features. Multi-species monomer-dimer models are suitable to describe the experimental situation treated in [7, 10], where a mean-field type phase transition has been observed in the percentage of mixed marriages between native people and immigrants. The hard-core interaction between dimers naturally represents the monogamy constraint in marriages, while, as pointed out by the authors of [7], an additional imitative interaction between individuals can be at the origin of the observed critical behaviour.

In this work we consider a mean-field model built on two populations A and B (e.g., the immigrants population and the local one) which takes into account both the imitative and the hard-core interactions. Dimers can be divided into three classes: type A if they link two individuals in A, type B if they link two individuals in B and type AB if they link a mixed couple. When the total size of the system $N = N_A + N_B$ increases, we assume that the relative sizes of the two populations N_A/N, N_B/N take fixed values α, $1-\alpha$. The energy contribution of dimers is tuned by a three dimensional vector $h = (h_A, h_B, h_{AB}) \in \mathbb{R}^3$ where h_A tunes the activity of a dimer of type A and so on. Individuals have also a certain propensity to imitate or counter-imitate the behaviour of the other individuals; this feature is is encoded in an additional contribution to the energy tuned by a 3×3 real matrix J. For example, the entry J_{AB}^{AB} couples dimers of type AB with other dimers of the same type. The main result we obtain is a representation of the pressure density in the thermodynamic limit in terms of a variational problem in $\mathbb{R}^3$ for all the values of the parameters h and J (see Theorem 1 in section 2 for the precise statement). This result is then applied to the case where the only non-zero parameters contributing to the energy are h_{AB} and J_{AB}^{AB}. As a consequence, the only relevant degree of freedom is the density of mixed dimers d_{AB} and the above variational problem leads to a consistency equation of the type

$$f_\alpha(d_{AB}) \,=\, h_{AB} + J_{AB}^{AB}\, d_{AB}\,.$$

Its analytical properties are investigated in detail for small α: the mean-field critical exponent $1/2$ is rigorously found, consistently with the experimental situation analyzed in [7, 10].

The paper is structured as follows. In section 2 we introduce the statistical mechanics model with the basic definitions and we prove the main result: the thermodynamic limit of the pressure density is expressed as a three-dimensional variational problem, where the order parameters are the dimer densities d_A, d_B internal to each population and the mixed dimer density d_{AB}.

In section 3 we focus on three non-zero parameters, α, h_{AB}, J_{AB}^{AB}, and we study in detail the critical behaviour of the system when one population is much larger than the other ($\alpha \to 0$), finding a phase transition with standard mean-field exponents.

Finally, in the Appendix we give an alternative proof for the existence of thermodynamic limit of the pressure density in the case $J = 0$, $h_A + h_B \geq 2h_{AB}$. This proof, which easily applies also to the standard single population case, uses a convexity inequality and is based on the Gaussian representation for the partition function [6].

2 Model and main result

Consider a system composed by N sites divided into two populations of sizes N_A and N_B respectively, $N_A + N_B = N$. We assume that the ratios $\alpha = N_A/N$ and $1 - \alpha = N_B/N$ are fixed when the total size N of the system varies. A *monomer-dimer configuration* can be identified with a set Δ of edges that satisfies a hard-core condition:

$$e = \{i,j\} \in \Delta \ , \ e' = \{i',j'\} \in \Delta \quad \Rightarrow \quad e \cap e' = \emptyset \tag{1}$$

Given the configuration Δ (see Figure 1), the edges in Δ are called dimers and they can be partitioned into three families: denote by D_A the number of dimers having both endpoints in A, by D_B the number of dimers having both endpoints in B and by D_{AB} the number of dimers having one endpoint in A and the other one in B. Monomers, namely sites free of dimers, can be partitioned into two families: denote by M_A, M_B the number of monomers in A, B respectively. Observe that

$$2\,D_A + D_{AB} + M_A = N_A \quad , \quad 2\,D_B + D_{AB} + M_B = N_B \ . \tag{2}$$

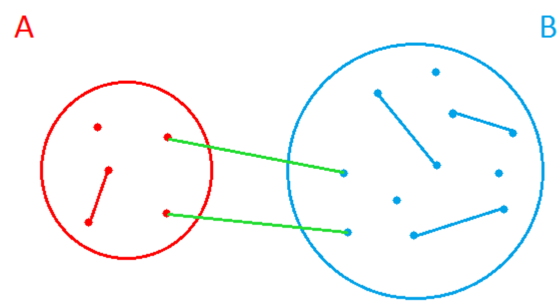

Figure 1: A monomer-dimer configuration on two populations of sizes $N_A = 5$, $N_B = 11$. In this example there are $D_A = 1$ dimers internal to population A, $D_B = 3$ dimers internal to population B and $D_{AB} = 2$ mixed dimers.

We denote by $\mathscr{D}_N$ the set of all possible monomer-dimer configurations on N sites. For a given configuration $\Delta \in \mathscr{D}_N$, D denotes the vector of the cardinalities of the three families of dimers

$$D := \begin{pmatrix} D_A \\ D_B \\ D_{AB} \end{pmatrix} , \tag{3}$$

while

$$|D| := D_A + D_B + D_{AB} \tag{4}$$

represents the total number of dimers. The Hamiltonian function is defined as

$$H_N(D) = -h \cdot D - \frac{1}{2N} JD \cdot D \tag{5}$$

where $\cdot$ denotes the standard scalar product in $\mathbb{R}^3$, the dimer vector field h tunes the activity of dimers while the coupling matrix J tunes the interaction between sites according to the types of dimers they host:

$$h = \begin{pmatrix} h_A \\ h_B \\ h_{AB} \end{pmatrix} \quad J = \begin{pmatrix} J_A^A & J_A^B & J_A^{AB} \\ J_B^A & J_B^B & J_B^{AB} \\ J_{AB}^A & J_{AB}^B & J_{AB}^{AB} \end{pmatrix} . \tag{6}$$

The partition function of the model is

$$Z_N \equiv Z_N(h, J, \alpha) = \sum_{\Delta \in \mathscr{D}_N} N^{-|D|} \, e^{-H_N(D)} \,. \tag{7}$$

Since the number of dimers $|D|$ is at most N, the Hamiltonian is of order N. On the other hand the term $N^{-|D|}$ guarantees that the entropy, namely the logarithm of $\Phi_N(D)$ (defined in equation (21)), is of order N. As we will see in Theorem 1 and in the Appendix, these two facts ensure a well defined thermodynamic limit of the model. Without loss of generality we assume the inverse temperature $\beta = 1$, since this parameter can be absorbed in h and J. Given $f : \mathscr{D}_N \to \mathbb{R}$ we call expected value of f with respect to the Gibbs measure the quantity

$$\langle f \rangle_N := \frac{1}{Z_N} \sum_{\Delta \in \mathscr{D}_N} N^{-|D|} e^{-H_N(D)} \, f(\Delta) \,. \tag{8}$$

Let us introduce the definitions needed to state our main result. Denote by Ω_α the set of $d = (d_A, d_B, d_{AB})^T \in (\mathbb{R}_+)^3$ such that

$$2d_A + d_{AB} \le \alpha \,, \;\; 2d_B + d_{AB} \le 1 - \alpha \,. \tag{9}$$

The above constraints on the vector d reflect the hard-core relations (2). Set

$$\gamma(x) := \exp(x \log x - x) \,, \quad x \ge 0 \tag{10}$$

and define the following functions

$$\begin{aligned} s(d; \alpha) := \; & \log \gamma(\alpha) + \log \gamma(1 - \alpha) - \log \gamma(\alpha - 2d_A - d_{AB}) + \\ & - \log \gamma(1 - \alpha - 2d_B - d_{AB}) - \log \gamma(d_A) - \log \gamma(d_B) + \\ & - \log \gamma(d_{AB}) - d_A \log 2 - d_B \log 2 \end{aligned} \tag{11}$$

$$\epsilon(d; h, J) := -h \cdot d - \frac{1}{2} \, Jd \cdot d \tag{12}$$

$$\psi(d; h, J, \alpha) := s(d; \alpha) - \epsilon(d; h, J) \,. \tag{13}$$

The functions ψ, s, ϵ represent respectively the variational pressure, entropy and energy densities.

Theorem 1. *For all $\alpha \in (0, 1)$, $h \in \mathbb{R}^3$ and $J \in \mathbb{R}^{3\times 3}$, there exists*

$$\lim_{N \to \infty} \frac{1}{N} \log Z_N(h, J, \alpha) = \max_{d \in \Omega_\alpha} \psi(d; h, J, \alpha) =: p(h, J, \alpha) \tag{14}$$

The function $\psi(d; h, J, \alpha)$ attains its maximum in at least one point $d^* = d^*(h, J, \alpha) \in \Omega_\alpha$ which solves the following fixed point system:

$$\begin{cases} d_A = \frac{w_A}{2} \, m_A^2 \\ d_B = \frac{w_B}{2} \, m_B^2 \\ d_{AB} = w_{AB} \, m_A \, m_B \end{cases} \tag{15}$$

where we denote

$$m_A = \alpha - 2d_A - d_{AB} \,, \quad m_B = 1 - \alpha - 2d_B - d_{AB} \,, \tag{16}$$

$$w_A = e^{h_A + J_A d} \;, \quad w_B = e^{h_B + J_B d} \;, \quad w_{AB} = e^{h_{AB} + J_{AB} d} \;. \tag{17}$$

At $J = 0$ the system (15) has a unique solution $d^* = g(h, \alpha) \in \Omega_\alpha$ which is an analytic function of the parameters h, α. Clearly at any J the system (15) rewrites as

$$d \;=\; g(h + Jd\,,\, \alpha)\;. \tag{18}$$

Provided that d^* is differentiable, $\nabla_h\, p = d^*$ and there exists

$$\lim_{N\to\infty} \frac{1}{N}\,\langle\, D\,\rangle_N \;=\; d^*\;. \tag{19}$$

Proof. The number of configurations $\Delta \in \mathscr{D}_N$ with given cardinalities D_A, D_B, D_{AB} can be computed by a standard combinatorial argument. Therefore the partition function rewrites as

$$Z_N \;=\; \sum_{D_A=0}^{N_A/2}\;\sum_{D_B=0}^{N_B/2}\;\sum_{D_{AB}=0}^{(N_A-2D_A)\wedge(N_B-2D_B)} \phi_N(D)\, e^{-H_N(D)} \tag{20}$$

with

$$\phi_N(D) := \frac{N_A!\,N_B!\,N{-}|D|}{(N_A - 2D_A - D_{AB})!\,(N_B - 2D_B - D_{AB})!\,D_A!\,D_B!\,D_{AB}!\,2^{D_A}\,2^{D_B}} \tag{21}$$

As we are interested in the limit $N_A, N_B \to \infty$ (while keeping fixed the ratio), in order to simplify the computations, we approximate the factorial by the continuous function γ defined in (10). We denote by $\tilde\phi_N$ the function obtained from ϕ_N by substituting any factorial $n!$ with $\gamma(n)$, then we denote by $\tilde Z_N$ the partition function obtained from Z_N by substituting ϕ_N with $\tilde\phi_N$. The Stirling approximation and elementary computations give the following properties of γ:

i. $1 \vee \sqrt{2\pi n} \;\leq\; n!/\gamma(n) \;\leq\; 1 \vee e^{\frac{1}{12}}\sqrt{2\pi n} \quad \forall n \in \mathbb{N}$

ii. $\frac{d}{dx}\log\gamma(x) = \log x\,, \quad \log\gamma(x)$ is convex

iii. $\frac{1}{N}\log\gamma(Nx) = \log\gamma(x) + x\,\log N$

By i. it follows that

$$\frac{1}{N}\log Z_N \;=\; \frac{1}{N}\log\tilde Z_N \;+\; \mathcal{O}\left(\frac{\log N}{N}\right)\;, \tag{22}$$

by a standard argument

$$\frac{1}{N}\log\tilde Z_N \;=\; \max_{D\in N\Omega_\alpha} \frac{1}{N}\Big(\log\tilde\phi_N(D) - H_N(D)\Big) \;+\; \mathcal{O}\left(\frac{\log N}{N}\right) \tag{23}$$

and using iii. a direct computation shows that for every $N \in \mathbb{N}$

$$\frac{1}{N}\Big(\log\tilde\phi_N(Nd) - H_N(Nd)\Big) \;=\; \psi(d; h, J, \alpha)\,, \quad d \in \Omega_\alpha\;. \tag{24}$$

Therefore there exists

$$\lim_{N\to\infty}\frac{1}{N}\log Z_N = \max_{d\in\Omega_\alpha}\psi(d; h, J, \alpha)\;.$$

Using ii. one can easily compute

$$\nabla_d\, s \;=\; \left(\; \log \frac{m_A^2}{2d_A}\;,\; \log \frac{m_B^2}{2d_B}\;,\; \log \frac{m_A m_B}{d_{AB}}\;\right) \tag{25}$$

$$-\nabla_d\, \epsilon \;=\; (\, h_A + J_A \cdot d\;,\; h_B + J_B \cdot d\;,\; h_{AB} + J_{AB} \cdot d\,) \tag{26}$$

therefore

$$\nabla_d\, \psi(d; h, J, \alpha) \;=\; 0 \;\Leftrightarrow\; d \text{ is a solution of } (15)\;.$$

The first derivatives of $p(h, J, \alpha) \;=\; \psi(d^*(h, J, \alpha); h, J, \alpha)$ can be easily computed since $\nabla_d\, \psi(d^*; h, J, \alpha) = 0$.

3 The limit $\alpha \to 0$

In this section we choose a particular framework that simplifies the mathematical treatment of the problem and allows a detailed analysis of the thermodynamic properties of the system. The most peculiar parameters of the model are h_{AB} and J_{AB}^{AB}, describing respectively the AB-dimer field and the interaction between pairs of AB-dimers, indeed they have no correspondence in a bipopulated Ising model [18]. Moreover we focus on the case where one population is much smaller than the other ($\alpha \to 0$). Thus in this section we set $h_A = h_B = 0$, $J_A^A = J_B^B = J_A^B = J_B^A = J_A^{AB} = J_{AB}^A = J_B^{AB} = J_{AB}^B = 0$ and we consider only the remaining coefficients h_{AB} and J_{AB}^{AB}. From now on, with a slight abuse of notation, we will denote

$$h := h_{AB}\;, \quad J := J_{AB}^{AB} > 0$$

and the mixed dimer density

$$d := d_{AB} = \frac{D_{AB}}{N} \in [0, \alpha]$$

In this framework the degrees of freedom of the variational problem (14) reduces from three to one, since d_A, d_B are explicit functions of $d_{AB} \equiv d$ as can be easily observed by looking to the consistency equation (15). Precisely, by setting $x_\alpha(d) := m_A = \sqrt{2d_A}$, $y_\alpha(d) := m_B = \sqrt{2d_B}$ one can easily see that $x_\alpha(d)$, $y_\alpha(d)$ are the positive solutions of the following quadratic equations respectively

$$x^2 + x - (\alpha - d) = 0 \quad , \quad y^2 + y - (1 - \alpha - d) = 0 \tag{27}$$

namely

$$x_\alpha(d) = \frac{-1 + \sqrt{1 + 4(\alpha - d)}}{2} \quad , \quad y_\alpha(d) = \frac{-1 + \sqrt{1 + 4(1 - \alpha - d)}}{2}\;. \tag{28}$$

Then one can easily prove from Theorem 1 that

$$p(h, J, \alpha) \;=\; \max_{d \in (0, \alpha)} \psi_1(d; h, J, \alpha) \tag{29}$$

where ψ_1 coincides with the function ψ defined by equation (13) evaluated at

$$\begin{pmatrix} d_A \\ d_B \\ d_{AB} \end{pmatrix} = \begin{pmatrix} x_\alpha(d)^2/2 \\ y_\alpha(d)^2/2 \\ d \end{pmatrix} . \tag{30}$$

Any solution $d^* = d^*(h, J, \alpha)$ of the one-dimensional variational problem (29) satisfies the fixed point equation

$$d = \exp(h + Jd)\, x_\alpha(d)\, y_\alpha(d) \tag{31}$$

It is convenient to set $f_\alpha(d) := \log d - \log x_\alpha(d) - \log y_\alpha(d)$ and rewrite equation (31) as $f_\alpha(d) = h + Jd$. Fix $\alpha \in (0, 1)$. f_α is the inverse function of a sigmoid function[1]. Therefore the point (d_c, h_c, J_c) such that $f''_\alpha(d_c) = 0$, $f'_\alpha(d_c) = J_c$, $f_\alpha(d_c) = h_c + J_c\, d_c$ is the critical point of the system, where the density d^* branches from one to two values (see Figure 2).

For small values of α, the following estimates for the critical point can be obtained by expanding $f_\alpha(d)$ as $\alpha \to 0$:

$$d_c(\alpha) = \frac{\alpha}{2} + \mathcal{O}(\alpha^3) \tag{32}$$

$$J_c(\alpha) = \frac{4}{\alpha} + \mathcal{O}(\alpha) \tag{33}$$

$$h_c(\alpha) = -2 - \log \frac{\sqrt{5} - 1}{2} + \mathcal{O}(\alpha) \tag{34}$$

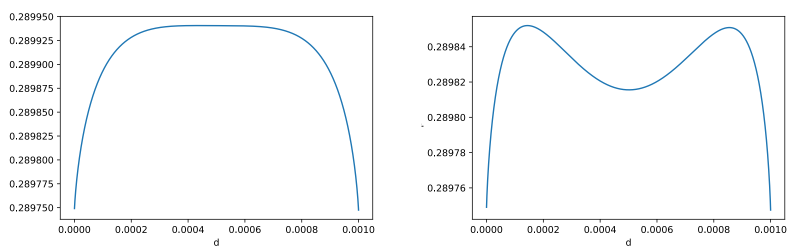

Figure 2: Plots of the variational pressure ψ_1 versus d, for $\alpha = 10^{-3}$ and different values of the parameters: critical parameters $J = J_c$, $h = h_c$ on the left-hand side; parameters $J = J_c + 10^3$, $h = h_c - d_c\,(J - J_c)$ on the right-hand side. The number of global maximum points of ψ_1, that identify the phases of the system (see eq. (29)), passes from one to two when we move the parameters (J, h) away from the critical point along a suitable curve.

Fixing α close to zero and moving the parameters (h, J) towards their critical values, along the half line $h - h_c(\alpha) = -d_c(\alpha)\,\big(J - J_c(\alpha)\big)$,

[1] It is easy to check that $f_\alpha(d) \to -\infty$ as $d \searrow 0$, $f_\alpha(d) \to \infty$ as $d \nearrow \alpha$, $f'_\alpha > 0$, f''_α vanishes exactly once.

$J \geq J_c$, the mixed dimer density $d^*(h, J, \alpha)$ exhibits the following critical behaviour:

$$d^*(h, J, \alpha) - d_c(\alpha) \;=\; C(\alpha)\,\sqrt{J - J_c(\alpha)} \;+\; \mathcal{O}\Big((J - J_c(\alpha))^{3/2}\Big) \qquad (35)$$

with $C(\alpha) = \sqrt{\frac{3}{16}\alpha^3 + \mathcal{O}(\alpha^6)}$. This fact can be proven using the Taylor expansion of $f_\alpha(d)$ around $d = d_c(\alpha)$ up to the third order.

Remark 1. The expansion (35) describes the mean-field critical behaviour with respect to the coupling J for fixed α. However one can also fix J and move α around the critical point. For example let's take $J = \alpha\,(1-\alpha)\,J'$ with $J' \gg 1$. In this case we obtain

$$d - d_c \;=\; C(J')\,\sqrt{\alpha - \alpha_c} \;+\; \mathcal{O}\big((\alpha - \alpha_c)^{3/2}\big) \qquad (36)$$

as $\alpha \to \alpha_c$, $h = h_c - d_c\,(\alpha - \alpha_c)$ and

$$\alpha_c = \frac{2}{\sqrt{J'}} \;+\; \mathcal{O}\left(\frac{1}{J'}\right) \qquad (37)$$

$$h_c = -2 - \log\frac{\sqrt{5}-1}{2} \;+\; \mathcal{O}\left(\frac{1}{\sqrt{J'}}\right) \qquad (38)$$

$$d_c = \frac{1}{\sqrt{J'}} \;+\; \mathcal{O}\left(\frac{1}{J'^{\,3/2}}\right). \qquad (39)$$

The critical behaviour (36) clearly has no counterpart in the single population case. This behaviour has been observed in the experimental situation [7], where the authors find that relation (36), with suitable parameters, fits well the data.

Remark 2. Equation (36) is a consequence of the fact that at the critical point the lowest order non vanishing derivative of the variational pressure ψ_1 in (29) is the fourth one. This fact suggests that the fluctuations of the order parameter at the critical point follows the standard mean field theory [3, 12]. From the above considerations we expect the fluctuations to scale as $N^{3/4}$ and to converge to a quartic exponential distribution.

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Appendix

Here we give a directed proof of the existence of the thermodynamic limit for the pressure density in the particular case

$$J = 0\;, \quad W = \begin{pmatrix} w_A & w_{AB} \\ w_{AB} & w_B \end{pmatrix} = \begin{pmatrix} e^{h_A} & e^{h_{AB}} \\ e^{h_{AB}} & e^{h_B} \end{pmatrix} > 0\;. \qquad (40)$$

where $W > 0$ means that the matrix W is positive definite. This proof is independent from Theorem 1 and the strategy follows a basic idea introduced in [14] in the context of Spin Glass Theory. In this case the partition function (7) admits a representation in terms of Gaussian moments:

$$Z_N = \sum_{\Delta\in\mathscr{D}_N} \left(\frac{w_A}{N}\right)^{D_A} \left(\frac{w_B}{N}\right)^{D_B} \left(\frac{w_{AB}}{N}\right)^{D_{AB}} = \mathbb{E}\left[(1+\xi_A)^{N_A}(1+\xi_B)^{N_B}\right] , \tag{41}$$

where $\xi = (\xi_A, \xi_B)$ is a centred Gaussian vector of covariance matrix $\frac{1}{N}W$ (the hypothesis of positive definiteness is crucial) and $\mathbb{E}$ denotes the expectation operator. The representation (41) is based on the Isserlis-Wick formula, see [6] (Proposition 2.2) for the proof.

Now consider the set $Q = \{\xi \in \mathbb{R}^2 \,:\, 1+\xi_A > 0,\, 1+\xi_B > 0\}$ and define a modified partition function

$$Z_N^* = \mathbb{E}\left[(1+\xi_A)^{N_A}(1+\xi_B)^{N_B}\, \mathbb{1}_Q(\xi)\right] . \tag{42}$$

Z_N^* can be rewritten as an integral over $\xi \in Q$, with integrand function proportional to $\exp(N\,f(\xi))$ and

$$f(\xi) = -\frac{1}{2}\langle W^{-1}\xi, \xi\rangle + \alpha \log|1+\xi_A| + (1-\alpha)\log|1+\xi_B| \,.$$

Since f approaches its global maximum on $\mathbb{R}^2$ only for $\xi_A \geq 0$, $\xi_B \geq 0$, standard Laplace type estimates implies that

$$\frac{Z_N}{Z_N^*} \to 1 \quad \text{as } N \to \infty \,. \tag{43}$$

Hence we can restrict our attention to the sequence $\log Z_N^*$, $N \in \mathbb{N}$. We claim that

Proposition 1. *For every $N_1, N_2, N \in \mathbb{N}$ such that $N = N_1 + N_2$, it holds that*

$$Z_{N_1}^*\, Z_{N_2}^* \,\leq\, Z_N^* \,. \tag{44}$$

Then the sequence $\log Z_N^*$ is super-additive and the “monotonic” convergence of the pressure density will follow immediately by Fekete’s lemma and equation (43):

Corollary 1. *Under the hypothesis* (40)*, there exists*

$$\lim_{N\to\infty} \frac{1}{N}\log Z_N \,=\, \sup_N \frac{1}{N}\log Z_N^* \tag{45}$$

Only the proposition 1 remains to be proven.

Proof of the proposition 1. The strategy for the proof follows the basic ideas introduced in [14] for mean field spin models. For a fixed N consider two integers N_1, N_2, such that $N = N_1 + N_2$ and set

$$\gamma = N_1/N\,,\ 1-\gamma = N_2/N\,,$$

We decompose each of the two parts of the system N_1, N_2 in two populations A, B according to the fixed ratio α, namely according to the relation

$$N_i = \alpha N_i + (1-\alpha)N_i =: N_{iA} + N_{iB}\,, \quad i = 1,2$$

Now we introduce two *independent* centred Gaussian vectors:

$$\xi_i = (\xi_{iA}\,,\,\xi_{iB}) \;\text{ with covariance matrix } \frac{1}{N_i}\,W\,, \quad i=1,2$$

and we prove the following lemmas.

Lemma 1.

$$\gamma\,\xi_1 + (1-\gamma)\,\xi_2 \overset{d}{=} \xi$$

Proof. Since ξ_1, ξ_2 are independent centred Gaussian vectors, $\xi' := \gamma\,\xi_1 + (1-\gamma)\,\xi_2$ is a centred Gaussian vector. Its covariance matrix is:

$$\gamma^2\,\frac{W}{N_1} + (1-\gamma)^2\,\frac{W}{N_2} \;=\; \gamma\,\frac{W}{N} + (1-\gamma)\,\frac{W}{N} \;=\; \frac{W}{N}\ ,$$

the same of ξ. □

Lemma 2.

$$(1+x)^{\gamma}\,(1+y)^{1-\gamma} \le 1 + \gamma x + (1-\gamma)y \quad \forall\, x > -1,\ y > -1, \gamma \in (0,1)$$

Proof. Consider the function $f(x,y) = (1+x)^{\gamma}\,(1+y)^{1-\gamma}$ and its Taylor polynomial of first order at $(0,0)$, $P(x,y) = 1 + \gamma x + (1-\gamma)y$. The Hessian matrix of f is negative defined for $x > -1$, $y > -1$ (it has zero determinant and negative trace), hence $f(x,y) \le P(x,y)$. □

Finally the proof of proposition 1 follows easily using the independence of ξ_1, ξ_2, lemma 2 and lemma 1 . □

References

[1] D. Alberici, "A cluster expansion approach to the Heilmann-Lieb liquid crystal model", *Journal of Statistical Physics* 162(3), 761-791 (2016)

[2] D. Alberici, P. Contucci, "Solution of the monomer-dimer model on locally tree-like graphs. Rigorous results", *Communications in Mathematical Physics* 331, 975-1003 (2014)

[3] D. Alberici, P. Contucci, M. Fedele, E. Mingione, "Limit theorems for monomer-dimer mean-field models with attractive potential", *Communications in Mathematical Physic* 346(3), 781-799 (2016)

[4] D. Alberici, P. Contucci, E. Mingione, "A mean-field monomer-dimer model with attractive interaction. Exact solution and rigorous results", *Journal of Mathematical Physics* 55, 1-27 (2014)

[5] D. Alberici, P. Contucci, E. Mingione, "The exact solution of a mean-field monomer-dimer model with attractive potential", *Europhysics Letters* 106, 1-5 (2014)

[6] D. Alberici, P. Contucci, E. Mingione, "A mean-field monomer-dimer model with randomness. Exact solution and rigorous results", *Journal of Statistical Physics* 160, 1721-1732 (2015)

[7] A. Barra, P. Contucci, R. Sandell, C. Vernia, "An analysis of a large dataset on immigrant integration in Spain. The statistical mechanics perspective on social action", *Scientific Reports* 4, 4174 (2014)

[8] A. Barra, P. Contucci, E. Mingione, D. Tantari, "Multi-species mean-field spin-glasses. Rigorous results", *Annales Henri Poincaré* 16(3), 691-708 (2015)

[9] A. Barra, G. Genovese, F. Guerra, "Equilibrium statistical mechanics of bipartite spin systems",*Journal of Physics A: Mathematical and Theoretical* 44, 24 (2011)

[10] P. Contucci, C. Vernia, "Alienation in italian cities. Social network fragmentation from collective data", preprint: https://arxiv.org/pdf/1410.0501.pdf

[11] M. Disertori, A. Giuliani, "The nematic phase of a system of long hard rods", *Communications in Mathematical Physics* **323**(1), 143-175 (2013)

[12] R.S. Ellis, C.M. Newman, "Limit theorems for sums of dependent random variables occurring in statistical mechanics", *Probability Theory and Related Fields* 44, 117-139 (1978)

[13] A. Giuliani, I. Jauslin, E.H. Lieb, "A Pfaffian formula for monomer-dimer partition functions", *Journal of Statistical Physics* 163(2), 211-238 (2016)

[14] F. Guerra, F.L. Toninelli, "The thermodynamic limit in mean field spin glass models", *Communications in Mathematical Physics* 230(1), 71-79 (2002)

[15] O.J. Heilmann, E.H. Lieb, "Theory of monomer-dimer systems", *Communications in Mathematical Physics* 25, 190-232 (1972)

[16] O.J. Heilmann, E.H. Lieb, "Lattice models for liquid crystals", *Journal of Statistical Physics* 20, 680-693 (1979)

[17] R. Karp, M. Sipser, "Maximum matchings in sparse random graphs", *Proceedings of the 22nd Annual Symposium on Foundations of Computer Science, IEEE Computer Society Press*, 364-375 (1981)

[18] M. Fedele, P. Contucci, "Scaling Limits for Multi-species Statistical Mechanics Mean-Field Models", *Journal of Statistical Physics* 144, 1186 (2011)

[19] R. Moessner, K. S. Raman, "Quantum dimer models", *Introduction to frustrated magnetism, Springer Series in Solid-State Sciences* 164, 437-479 (2011)

[20] D. Panchenko, "The free energy in a multi-species Sherrington-Kirkpatrick model", *Annals of Probability* 43(6), (2015)

[21] J.K. Roberts, "Some properties of mobile and immobile adsorbed films", *Proceedings of the Cambridge Philosophical Society* 34, 399-411 (1938)

[22] L. Zdeborová, M. Mézard, "The number of matchings in random graphs", *Journal of Statistical Mechanics* 5, P05003 (2006)