



ALMA MATER STUDIORUM  
UNIVERSITÀ DI BOLOGNA

ARCHIVIO ISTITUZIONALE  
DELLA RICERCA

Alma Mater Studiorum Università di Bologna  
Archivio istituzionale della ricerca

Weihrach complexity in computable analysis

This is the final peer-reviewed author's accepted manuscript (postprint) of the following publication:

*Published Version:*

*Availability:*

This version is available at: <https://hdl.handle.net/11585/618943> since: 2021-11-02

*Published:*

DOI: [http://doi.org/10.1007/978-3-030-59234-9\\_11](http://doi.org/10.1007/978-3-030-59234-9_11)

*Terms of use:*

Some rights reserved. The terms and conditions for the reuse of this version of the manuscript are specified in the publishing policy. For all terms of use and more information see the publisher's website.

This item was downloaded from IRIS Università di Bologna (<https://cris.unibo.it/>).  
When citing, please refer to the published version.

(Article begins on next page)

This is the final peer-reviewed accepted manuscript of:

**V. Brattka, G. Gherardi, A. Pauly (2021) Weihrauch complexity in computable analysis, in *Handbook of Computability and Complexity in Analysis*, Springer, pp. 367-417**

The final published version is available online at: <https://doi.org/10.1017/jsl.2020.76>

Rights / License:

The terms and conditions for the reuse of this version of the manuscript are specified in the publishing policy. For all terms of use and more information see the publisher's website.

*This item was downloaded from IRIS Università di Bologna (<https://cris.unibo.it/>)*

***When citing, please refer to the published version.***

# Weihrauch Complexity in Computable Analysis

Vasco Brattka, Guido Gherardi and Arno Pauly

**Abstract** We provide a self-contained introduction to Weihrauch complexity and its applications to computable analysis. This includes a survey on some classification results and a discussion of its relation to other approaches.

## 11.1 The Algebra of Problems

The Weihrauch lattice offers a framework to classify the uniform computational content of problems and theorems from analysis and other areas of mathematics. This framework can be seen as an attempt to create a calculus of mathematical problems, very much in spirit of Kolmogorov's interpretation of intuitionistic logic [71].

We express mathematical problems with the help of partial multi-valued functions  $f : \subseteq X \rightrightarrows Y$ , which are just relations  $f \subseteq X \times Y$ . It has turned out to be fruitful for our approach to think of these relations as input-output-oriented multi-valued functions  $f : \subseteq X \rightrightarrows Y$ . We consider  $\text{dom}(f) = \{x \in X : f(x) \neq \emptyset\}$  as the set of admissible instances  $x$  of the problem  $f$ , and we consider the corresponding set of function values  $f(x) \subseteq Y$  as the set of possible results. In the case of single-valued  $f$  we identify  $f(x)$  with the corresponding singleton. An example of a mathematical

---

Vasco Brattka  
Faculty of Computer Science, Universität der Bundeswehr München, Germany and  
Department of Mathematics & Applied Mathematics, University of Cape Town, South Africa  
e-mail: Vasco.Brattka@cca-net.de

Guido Gherardi  
Dipartimento di Filosofia e Comunicazione, Università di Bologna, Italy  
e-mail: Guido.Gherardi@unibo.it

Arno Pauly  
Department of Computer Science, Swansea University, United Kingdom  
e-mail: a.m.pauly@swansea.ac.uk

problem that the reader can have in mind as a prototypical case is the zero problem. Obviously, many problems in mathematics can be expressed in terms of solutions of equations of the type  $f(x) = 0$  with a continuous  $f : X \rightarrow \mathbb{R}$ . We formalize this problem.

*Example 11.1.1 (Zero problem).* Let  $X$  be a topological space and let  $\mathcal{C}(X)$  denote the set of continuous  $f : X \rightarrow \mathbb{R}$ . The *zero problem*  $Z_X : \subseteq \mathcal{C}(X) \rightrightarrows X, f \mapsto f^{-1}\{0\}$  is the problem to find a solution  $x \in X$  of an equation of the type  $f(x) = 0$ , given a continuous function  $f : X \rightarrow \mathbb{R}$ . The set  $\text{dom}(Z_X)$  of admissible instances of this problem is the set of all continuous functions  $f$  with a non-empty zero set  $f^{-1}\{0\}$ . The set  $Z_X(f) = f^{-1}\{0\}$  of solutions is the set of all zeros of  $f$ .

Mathematical problems can be combined in various natural ways to obtain new problems. The following definition lists a number of typical algebraic operations that we are going to use. By  $X \sqcup Y := (\{0\} \times X) \cup (\{1\} \times Y)$  we denote the *disjoint union*. By  $X^* := \bigcup_{i=0}^{\infty} (\{i\} \times X^i)$  we denote the *set of words* over  $X$ , where  $X^i := \times_{j=1}^i X$  stands for the  $i$ -fold *Cartesian product* of  $X$  with itself with  $X^0 := \{()\}$ . Here  $()$  stands for the empty tuple or word. By  $\bar{X} := X \cup \{\perp\}$  we denote the *completion* of  $X$ , where  $\perp \notin X$ . We use the set of *natural numbers*  $\mathbb{N} = \{0, 1, 2, \dots\}$ .

**Definition 11.1.2 (Algebraic operations).** Let  $f : \subseteq X \rightrightarrows Y$ ,  $g : \subseteq Z \rightrightarrows W$  and  $h : \subseteq Y \rightrightarrows Z$  be multi-valued functions. We define the following operations (for exactly those inputs given by the specified domains):

1.  $h \circ f : \subseteq X \rightrightarrows Z, (h \circ f)(x) := \{z \in Z : (\exists y \in f(x)) z \in h(y)\}$  and  
 $\text{dom}(h \circ f) := \{x \in \text{dom}(f) : f(x) \subseteq \text{dom}(h)\}$  (composition)
2.  $f \times g : \subseteq X \times Z \rightrightarrows Y \times W, (f \times g)(x, z) := f(x) \times g(z)$  and  
 $\text{dom}(f \times g) := \text{dom}(f) \times \text{dom}(g)$  (product)
3.  $f \sqcup g : \subseteq X \sqcup Z \rightrightarrows Y \sqcup W, (f \sqcup g)(0, x) := \{0\} \times f(x), (f \sqcup g)(1, z) := \{1\} \times g(z)$   
and  $\text{dom}(f \sqcup g) := \text{dom}(f) \sqcup \text{dom}(g)$  (coproduct)
4.  $f \boxplus g : \subseteq X \sqcup Z \rightrightarrows \bar{Y} \times \bar{W}, (f \boxplus g)(0, x) := f(x) \times \bar{W}, (f \boxplus g)(1, z) := \bar{Y} \times g(z)$  and  
 $\text{dom}(f \boxplus g) := \text{dom}(f) \sqcup \text{dom}(g)$  (box sum)
5.  $f \sqcap g : \subseteq X \times Z \rightrightarrows Y \sqcup W, (f \sqcap g)(x, z) := f(x) \sqcup g(z)$  and  
 $\text{dom}(f \sqcap g) := \text{dom}(f) \times \text{dom}(g)$  (meet)
6.  $f + g : \subseteq X \times Z \rightrightarrows \bar{Y} \times \bar{W}, (f + g)(x, z) := (f(x) \times \bar{W}) \cup (\bar{Y} \times g(z))$  and  
 $\text{dom}(f + g) := \text{dom}(f) \times \text{dom}(g)$  (sum)
7.  $f^* : \subseteq X^* \rightrightarrows Y^*, f^*(i, x) := \{i\} \times f^i(x)$  and  
 $\text{dom}(f^*) := \text{dom}(f)^*$  (finite parallelization)
8.  $\hat{f} : \subseteq X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}}, \hat{f}(x_n)_n := \times_{i \in \mathbb{N}} f(x_i)$  and  
 $\text{dom}(\hat{f}) := \text{dom}(f)^{\mathbb{N}}$  (parallelization)

Here  $f^i := \times_{j=1}^i f$  denotes the  $i$ -fold product of  $f$  with itself, where  $f^0 = \text{id}_X$ . It is important to point out that the appropriate definition of the domain of  $h \circ f$  is crucial. If  $x \in \text{dom}(h \circ f)$ , then we require that all possible results  $y \in f(x)$  of  $f$

upon input of  $x$  are supported by  $h$ , i.e.,  $f(x) \subseteq \text{dom}(h)$ . This definition of composition corresponds to our understanding of multi-valued functions as computational problems.<sup>1</sup> We often write for short  $hf$  for the composition  $h \circ f$ .

The reader might notice some relations between the resource-oriented interpretation of linear logic and the way we combine mathematical problems (see Section 11.9.1). Indeed, the following intuitive interpretation of some of our algebraic operations is useful:

1. The composition  $h \circ f$  applies both problems consecutively, first  $f$  and then  $h$ .
2. The product  $f \times g$  provides both problems  $f$  and  $g$  in parallel. For each instance one obtains solutions of both  $f$  and  $g$ .
3. The coproduct  $f \sqcup g$  provides both problems  $f$  and  $g$  as alternatives. For each instance one can select to obtain either a solution of  $f$  or a solution of  $g$ .
4. The meet  $f \sqcap g$  provides either  $f$  or  $g$ . For each instance one either obtains a solution for  $f$  or a solution for  $g$ ; one learns a posteriori which one it is, but one cannot control in advance which one it will be.
5. The sum  $f + g$  provides two potential solutions for given instances of  $f$  and  $g$ , at least one of which has to be correct.
6. The finite parallelization  $f^*$  allows arbitrarily many finite applications of  $f$  in parallel, and with each instance one can select how many applications are to be used in parallel.
7. The parallelization  $\widehat{f}$  allows countably many applications of  $f$  in parallel.

Given the above list of operations we can derive other algebraic operations.

**Definition 11.1.3 (Juxtaposition).** For  $f : \subseteq X \rightrightarrows Y$  and  $g : \subseteq X \rightrightarrows Z$  we denote by  $(f, g) : \subseteq X \rightrightarrows Y \times Z$  the *juxtaposition* of  $f$  and  $g$ , which is defined by  $(f, g) := (f \times g) \circ \Delta_X$ , where  $\Delta_X : X \rightrightarrows X \times X, x \mapsto (x, x)$  denotes the *diagonal* of  $X$ .

Given two problems  $f$  and  $g$  we want to express what it means that  $f$  solves  $g$ .

**Definition 11.1.4 (Solutions).** Let  $f, g : \subseteq X \rightrightarrows Y$  be multi-valued functions. We define  $f \sqsubseteq g : \iff \text{dom}(g) \subseteq \text{dom}(f)$  and  $(\forall x \in \text{dom}(g)) f(x) \subseteq g(x)$ . In this situation we say that  $f$  *solves*  $g$ ,  $f$  is a *strengthening* of  $g$  and  $g$  is a *weakening* of  $f$ .

Intuitively,  $f \sqsubseteq g$  means that all instances of  $g$  are also instances of  $f$ , and on all these common instances  $f$  yields a possible solution of  $g$ . It is clear that the relation  $\sqsubseteq$  yields a preorder, i.e., it is reflexive and transitive.

Many theorems give rise to mathematical problems. In general, a theorem of the logical form

$$(\forall x \in X)(x \in D \implies (\exists y \in Y)P(x, y))$$

translates into the problem

---

<sup>1</sup> The way we define composition turns the multi-valued functions into morphisms of a specific category [96] that is not identical to the usual category of relations.

$$F : \subseteq X \rightrightarrows Y, x \mapsto \{y \in Y : P(x, y)\} \text{ with } \text{dom}(F) := D.$$

That is,  $F$  plays the rôle of a multi-valued Skolem function for the statement of the theorem. The problem  $F$  measures the difficulty of finding a suitable  $y$ , given  $x$ , whereas the condition encapsulated in  $D$  is a purely classical premise that is not meant to bear any constructive content. As an example we mention the intermediate value theorem.

*Example 11.1.5 (Intermediate value theorem).*  $\text{IVT} : \subseteq \mathcal{C}[0, 1] \rightrightarrows [0, 1], f \mapsto f^{-1}\{0\}$ , where  $\text{dom}(\text{IVT})$  contains all  $f \in \mathcal{C}[0, 1]$  with  $f(0) \cdot f(1) < 0$ , is called the *intermediate value theorem*. It is easy to see that  $\mathbb{Z}_{[0,1]} \sqsubseteq \text{IVT}$  holds.

## Bibliographic Remarks

Algebraic operations on multi-valued functions have been used frequently in computable analysis. For instance, composition in the way defined here, product, juxtaposition and parallelization have been used by Brattka [6, 8]. The coproduct operation and finite parallelization were introduced by Pauly in [91]. The meet operation was introduced by Brattka and Gherardi [18], and the box sum was introduced by Dzhafarov [42]. Inspired by the definition of the box sum, the definition of the sum from Brattka, Gherardi and Hölzl [19] appears here for the first time in a modified version that has better properties. The category of multi-valued functions was studied by Pauly [96].

## 11.2 Represented Spaces

In this section we want to provide the data types that we will use for problems  $f : \subseteq X \rightrightarrows Y$ . For a purely topological development of our theory it would be sufficient to consider topological spaces  $X$  and  $Y$ . However, since we want to discuss computability properties too, we need slightly more structure on the spaces  $X$  and  $Y$ , and this structure is provided by representations.

**Definition 11.2.1 (Represented spaces).** A *represented space*  $(X, \delta)$  is a set  $X$  together with a surjective partial function  $\delta : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow X$ .

If  $\delta(p) = x$  then we call  $p$  a *name* for  $x$ , and we reserve the word *representation* for the map  $\delta$  itself. We endow the Baire space  $\mathbb{N}^{\mathbb{N}}$  with its usual product topology of the discrete topology on  $\mathbb{N}$  and we always assume that a represented space  $(X, \delta_X)$  is endowed with the final topology  $\mathcal{O}(X)$  induced by  $\delta_X$  on  $X$ , which is the largest topology on  $X$  that turns  $\delta_X$  into a continuous map. In this situation  $\delta_X$  is automatically a quotient map. Typically, we will deal with *admissible* representations<sup>2</sup>  $\delta_X$  that are not just quotient maps but they are even more closely linked to the topology  $\mathcal{O}(X)$ . In the following we will often just write for short  $X$  for a represented space if the representation is clear from the context or not needed explicitly. We can now formally define problems.

<sup>2</sup> See the chapter “Admissibly Represented Spaces and Qcb-Spaces” by Schröder in this book for more details.

**Definition 11.2.2 (Problems).** We call partial multi-valued functions  $f : \subseteq X \rightrightarrows Y$  on represented spaces  $X, Y$  *problems*, for short.

Properties of problems such as computability and continuity can easily be introduced via realizers.

**Definition 11.2.3 (Realizer).** Given represented spaces  $(X, \delta_X), (Y, \delta_Y)$ , a problem  $f : \subseteq X \rightrightarrows Y$  and a function  $F : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ , we define  $F \vdash f : \iff \delta_Y F \sqsubseteq f \delta_X$ . In this situation we say that  $F$  is a *realizer* of  $f$ .

In other words,  $F$  is a realizer of  $f$  if  $\delta_Y F$  solves  $f \delta_X$ . Obviously, this concept depends on the underlying represented spaces and the notation  $F \vdash f$  is only justified when these are clear from the context.

On the Baire space  $\mathbb{N}^{\mathbb{N}}$  it is clear what a continuous function  $F : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  is. Computability of such functions can be defined via Turing machines in a well-known way. Such properties can now easily be transferred to problems via realizers.

**Definition 11.2.4 (Computability and continuity).** A problem  $f$  is called *computable (continuous)* if it has a computable (continuous) realizer.

We warn the reader that the resulting notion of continuity for single-valued functions is not automatically the topological notion of continuity that is induced by the final topologies of the representations. However, every total single-valued function  $f : X \rightarrow Y$  on represented spaces that is continuous in our sense is also continuous in the usual topological sense with respect to the final topologies, and in all our applications we will use admissible representations for which these two notions even coincide.

Two representations  $\delta_1, \delta_2$  of the same set  $X$  are called *equivalent* if the identity  $\text{id} : (X, \delta_1) \rightarrow (X, \delta_2)$  and its inverse are computable. It is easy to see that equivalent representations yield the same notion of computability and continuity.

By  $\mathcal{C}(X, Y)$  we denote the set of continuous functions  $f : X \rightarrow Y$  in terms of Definition 11.2.4. The category of represented spaces is Cartesian closed, and the same holds for the category of admissibly represented spaces. In particular, we have canonical ways of defining product and function space representations.

In order to define those, we use pairing functions. We define a pairing function  $\langle \cdot, \cdot \rangle : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  by  $\langle p, q \rangle(2n) := p(n)$  and  $\langle p, q \rangle(2n+1) := q(n)$  for  $p, q \in \mathbb{N}^{\mathbb{N}}$  and  $n \in \mathbb{N}$ . We define a pairing function of type  $\langle \cdot, \cdot \rangle : (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  by  $\langle p_0, p_1, p_2, \dots \rangle \langle n, k \rangle := p_n(k)$  for all  $p_i \in \mathbb{N}^{\mathbb{N}}$  and  $n, k \in \mathbb{N}$ , where  $\langle n, k \rangle$  is the standard Cantor pairing defined by  $\langle n, k \rangle := \frac{1}{2}(n+k+1)(n+k) + k$ . Finally, we note that by  $np$  we denote the concatenation of a number  $n \in \mathbb{N}$  with a sequence  $p \in \mathbb{N}^{\mathbb{N}}$ .

We assume that we have some standard representation  $\Phi$  of (a sufficiently large class<sup>3</sup>) of continuous functions, i.e., for any such function  $f : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  there is a  $p \in \mathbb{N}^{\mathbb{N}}$  with  $f = \Phi_p$ . For total functions this representation yields the exponential in the category of (admissibly) represented spaces and satisfies natural versions of the

<sup>3</sup> It suffices to consider all continuous functions  $f : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  with  $G_{\delta}$ -domain since any continuous function can be extended to such a function.

utm- and smn-theorems. For computable  $p$  one obtains the computable functions  $\Phi_p$  with natural domains (see [116] for details). For  $p \in \mathbb{N}^{\mathbb{N}}$  we denote by  $p-1 \in \mathbb{N}^{\mathbb{N}} \cup \mathbb{N}^*$  the sequence or word that is formed by concatenation of  $p(0)-1, p(1)-1, p(2)-1, \dots$  with the understanding that  $-1 = ()$  is the empty word.

**Definition 11.2.5 (Constructions on representation).** Let  $(X, \delta_X)$  and  $(Y, \delta_Y)$  be represented spaces. We define

1.  $\delta_{X \times Y} : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow X \times Y$ ,  $\delta_{X \times Y}(p, q) := (\delta_X(p), \delta_Y(q))$
2.  $\delta_{X \sqcup Y} : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow X \sqcup Y$ ,  $\delta_{X \sqcup Y}(0p) := (0, \delta_X(p))$  and  $\delta_{X \sqcup Y}(1p) := (1, \delta_Y(p))$
3.  $\delta_{X^*} : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow X^*$ ,  $\delta_{X^*}(n\langle p_1, p_2, \dots, p_n \rangle) := (n, (\delta_X(p_1), \delta_X(p_2), \dots, \delta_X(p_n)))$
4.  $\delta_{X^{\mathbb{N}}} : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow X^{\mathbb{N}}$ ,  $\delta_{X^{\mathbb{N}}}(p_0, p_1, p_2, \dots) := (\delta_X(p_n))_{n \in \mathbb{N}}$
5.  $\delta_{\mathcal{C}(X, Y)} : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathcal{C}(X, Y)$  by  $\delta_{\mathcal{C}(X, Y)}(p) = f : \iff \Phi_p \vdash f$
6.  $\delta_{\bar{X}} : \mathbb{N}^{\mathbb{N}} \rightarrow \bar{X}$ ,  $\delta_{\bar{X}}(p) := \delta_X(p-1)$  if  $p-1 \in \text{dom}(\delta_X)$  and  $\delta_{\bar{X}}(p) := \perp$  otherwise.

Many spaces that occur in analysis are actually computable metric spaces. For the definition we assume that the reader knows the notion of a computable (double) sequence of real numbers.

**Definition 11.2.6 (Computable metric spaces and Cauchy representations).**

1. A *computable metric space*  $(X, d, \alpha)$  is a separable metric space  $(X, d)$  with metric  $d : X \times X \rightarrow \mathbb{R}$  and a dense sequence  $\alpha : \mathbb{N} \rightarrow X$  such that  $d \circ (\alpha \times \alpha) : \mathbb{N}^2 \rightarrow \mathbb{R}$  is a computable double sequence of real numbers.
2. We define the *Cauchy representation*  $\delta_X : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow X$  by  $\delta_X(p) := \lim_{n \rightarrow \infty} \alpha p(n)$  and  $\text{dom}(\delta_X) = \{p \in \mathbb{N}^{\mathbb{N}} : (\forall i > j) d(\alpha p(i), \alpha p(j)) < 2^{-j}\}$ .

A standard numbering of the rational numbers  $\mathbb{Q}$  and the Euclidean metric yields the standard Cauchy representation  $\delta_{\mathbb{R}}$  of real numbers. Cauchy representations are examples of admissible representations, and for such representations continuity in the usual topological sense and continuity defined via realizers coincides. In particular,  $\mathcal{C}(\mathbb{R}) := \mathcal{C}(\mathbb{R}, \mathbb{R})$  is the usual set of continuous functions. In the following we often consider  $\mathbb{N}, \mathbb{R}, [0, 1], 2^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}$  and similar spaces as computable metric spaces in the straightforward sense without further mentioning this fact. A *computable Banach space* is just a computable metric space that is additionally a Banach space and such that the linear operations are computable. If the space is additionally a Hilbert space, then it is called a *computable Hilbert space*.

A non-metrizable space that we occasionally need is the Sierpiński space  $\mathbb{S} = \{0, 1\}$ , which is endowed with the topology  $\mathcal{O}(\mathbb{S}) = \{\emptyset, \mathbb{S}, \{1\}\}$ . By  $\hat{n} \in \mathbb{N}^{\mathbb{N}}$  we denote the constant sequence with value  $n \in \mathbb{N}$ .

**Definition 11.2.7 (Sierpiński space).** Let  $\delta_{\mathbb{S}} : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{S}$  be defined by  $\delta_{\mathbb{S}}(\hat{0}) = 0$  and  $\delta_{\mathbb{S}}(p) = 1$  for all  $p \neq \hat{0}$ .

We close this section with a discussion of computability properties of subsets. The most important notion for us is that of a *co-c.e. closed set*. Given a computable metric space  $(X, d, \alpha)$  we denote by  $B(x, r) := \{y \in X : d(x, y) < r\}$  the open ball with center  $x \in X$  and radius  $r \geq 0$ . More specifically, we denote by  $B_{\langle n, (i, k) \rangle} := B(\alpha(n), \frac{i}{k+1})$  a basic open ball.



**Definition 11.2.8 (Co-c.e. closed subsets).** Let  $X$  be a computable metric space. Then  $A \subseteq X$  is called *co-c.e. closed* if  $X \setminus A = \bigcup_{n \in \mathbb{N}} B_{p(n)}$  for some computable  $p \in \mathbb{N}^{\mathbb{N}}$ .

For  $X = \mathbb{N}^{\mathbb{N}}$  the co-c.e. closed subsets are also known as  $\Pi_1^0$ -classes. By  $\mathcal{A}(X)$  we denote the set of closed subsets of a topological space  $X$ . The definition of co-c.e. closed subsets of computable metric spaces  $X$  directly leads to a representation  $\psi_-$  of the set  $\mathcal{A}(X)$  defined by  $\psi_-(p) := X \setminus \bigcup_{n=0}^{\infty} B_{p(n)}$ . We denote the represented space  $(\mathcal{A}(X), \psi_-)$  for short by  $\mathcal{A}_-(X)$ . We now formulate two equivalent characterizations of co-c.e. closed sets. For every set  $A \subseteq X$  we denote its *characteristic function* by  $\chi_A : X \rightarrow \mathbb{S}$ , which is defined by  $\chi_A(x) = 1 : \iff x \in A$ .

**Proposition 11.2.9 (Co-c.e. closed sets).** Let  $X$  be a computable metric space and let  $A \subseteq X$ . Then the following are equivalent:

1.  $A$  is co-c.e. closed,
2.  $A = f^{-1}\{0\}$  for some computable  $f : X \rightarrow \mathbb{R}$ ,
3.  $\chi_{X \setminus A} : X \rightarrow \mathbb{S}$  is computable.

These equivalences are uniform, i.e., the maps  $\mathcal{A}_-(X) \rightarrow \mathcal{C}(X, \mathbb{S}), A \mapsto \chi_{X \setminus A}$  and  $\mathcal{C}(X) \rightarrow \mathcal{A}_-(X), f \mapsto f^{-1}\{0\}$  are computable and admit (in the second case multi-valued) computable right inverses.

The third characterization has the advantage that it is the most general of these three, and it works even for arbitrary represented spaces  $X$ . Hence, for such spaces we define  $\psi_-$  by  $\psi_-(p) = A : \iff \delta_{\mathcal{C}(X, \mathbb{S})}(p) = \chi_{X \setminus A}$ . We denote the corresponding represented space  $(\mathcal{A}(X), \psi_-)$  also by  $\mathcal{A}_-(X)$ . Due to Proposition 11.2.9 this notation is consistent with the special definition for computable metric spaces  $X$  above. Besides the notion of a co-c.e. closed subset we also need the notion of a co-c.e. compact subset.

**Definition 11.2.10 (Computable compact subsets).** Let  $X$  be a computable metric space and let  $K \subseteq X$  be compact.

1.  $K$  is called *co-c.e. compact* if  $\{\langle n_1, \dots, n_k \rangle, k \in \mathbb{N} : K \subseteq \bigcup_{i=1}^k B_{n_i}\}$  is c.e.
2.  $K$  is called *computably compact* if  $K$  is co-c.e. compact and there exists a computable sequence that is dense in  $K$ .

Obviously, a computable metric space is computably compact if and only if it is co-c.e. compact. Similarly as in the case of closed sets we can derive a representation  $\kappa_-$  of the set  $\mathcal{K}(X)$  of compact subsets that is based on (1) and a representation  $\kappa$  of compact sets that is based on (2). By  $\mathcal{K}_-(X)$  we denote the represented space  $(\mathcal{K}(X), \kappa_-)$ . Once again there is a more general representation that works for arbitrary represented spaces, but we will not formalize this representation here.

## Bibliographic Remarks

The theory of representations and of computable functions on represented spaces was developed by Kreitz and Weihrauch [72, 73, 117, 113], who also introduced the notion of an admissible representation. Admissible representations in a more general sense have been further studied by Schröder [103], who also recognized the relevance of the Sierpiński space in this context. Computable metric spaces were first introduced by Lacombe [78]. Represented spaces are used as a basic framework for computable analysis [116, 24, 95]. Computability properties of subsets of computable metric spaces were studied since Lacombe [76, 77], and discussions of corresponding representations can be found in [73, 117, 32, 30, 95]. Proposition 11.2.9 for metric spaces is taken from Bratka and Presser [30, Theorem 3.10, Corollary 3.14]. The mentioned representations for compact subsets of metric spaces are also studied in [30]. Representations of subsets for general represented spaces have been studied by Pauly [95].

## 11.3 The Weihrauch Lattice

We now want to define Weihrauch reducibility as a way to compare problems with each other. The goal is that  $f \leq_W g$  expresses the fact that  $f$  can be computed by a single application of  $g$ . We will need two variants of such a reducibility. By  $\text{id} : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  we denote the *identity* of the Baire space. For other sets  $X$  we usually add an index  $X$  and write the *identity* as  $\text{id}_X : X \rightarrow X$ . For  $F, G : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  we define  $\langle F, G \rangle(p) := \langle F(p), G(p) \rangle$ .

**Definition 11.3.1 (Weihrauch reducibility).** Let  $f$  and  $g$  be problems. We define:

1.  $f \leq_W g : \iff (\exists \text{ computable } H, K : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}})(\forall G \vdash g) H(\text{id}, GK) \vdash f$ .
2.  $f \leq_{sW} g : \iff (\exists \text{ computable } H, K : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}})(\forall G \vdash g) HGK \vdash f$ .

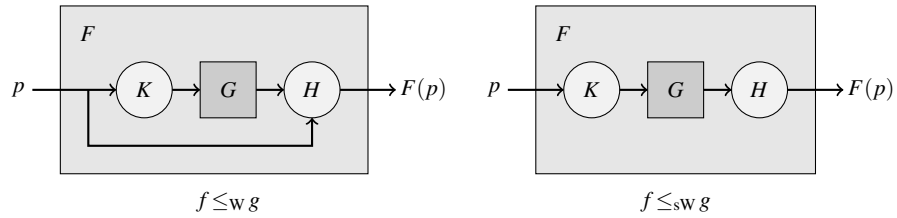
We say that  $f$  is (*strongly*) *Weihrauch reducible* to  $g$ , if  $f \leq_W g$  ( $f \leq_{sW} g$ ) holds.

The diagram in Figure 11.1 illustrates Weihrauch reducibility and its strong counterpart. It is easy to see that  $f \leq_{sW} g$  implies  $f \leq_W g$ . It is also easy to see that  $\leq_W$  and  $\leq_{sW}$  are preorders, i.e., they are reflexive and transitive. We denote the corresponding equivalences by  $\equiv_W$  and  $\equiv_{sW}$ , respectively, and we use the symbols  $<_W$  and  $<_{sW}$  for strict reducibilities, respectively. Similar reducibilities can be defined if the notion of computability is replaced by continuity or other suitable categories. A more categorical characterization of Weihrauch reducibility that mentions neither realizers nor the Baire space is given by the following proposition.

**Proposition 11.3.2.** Let  $f : \subseteq X \rightrightarrows Y$  and  $g : \subseteq Z \rightrightarrows W$  be problems. Then:

1.  $f \leq_W g$  if and only if there are computable  $h : \subseteq V \times W \rightrightarrows Y$  and  $k : \subseteq X \rightrightarrows V \times Z$  for some represented space  $V$  such that  $h \circ (\text{id}_V \times g) \circ k \sqsubseteq f$ .
2.  $f \leq_{sW} g$  if and only if there are computable  $h : \subseteq W \rightrightarrows Y$  and  $k : \subseteq X \rightrightarrows Z$  such that  $h \circ g \circ k \sqsubseteq f$ .

Even though the proof of Proposition 11.3.2 is elementary, there is a subtle point in it. Namely, the proof requires a version of the axiom of choice. In fact, we are freely using the axiom of choice, and mostly we invoke the following version.



**Fig. 11.1** Weihrauch reducibility and strong Weihrauch reducibility.

The *axiom of choice for the Baire space*: every problem  $f$  has a realizer  $F$ .

The fact that Weihrauch reducibility captures the idea of using  $g$  exactly once in the course of the computation is stated in the following theorem that we only formulate in intuitive terms here.

**Theorem 11.3.3 (Generalized Turing oracles).**  $f \leq_W g$  holds if and only if  $f$  can be computed on a (generalized) Turing machine that uses exactly one application of  $g$  in the course of its computation.

We emphasize that  $f \leq_W g$  actually *requires* that the oracle  $g$  is used once in the course of the computation of  $f$ . Hence, using the oracle  $g$  can actually be an obstacle if the domain of  $g$  contains only complicated points.

We note that a characterization of strong Weihrauch reducibility analogous to Theorem 11.3.3 would require discarding all results that were obtained in the course of the computation other than the result of the application of the oracle  $g$ . This would be a rather unnatural way of using oracles, and it indicates why ordinary Weihrauch reducibility is a more appropriate concept from this perspective.

The relation between strong and ordinary Weihrauch reducibility is similar to the relation between one-one and many-one reducibility in classical computability theory, and it can be expressed using the notion of a cylinder.

**Definition 11.3.4 (Cylinder).** A problem  $f$  is called a *cylinder* if  $\text{id} \times f \leq_{sW} f$ .

It is clear that  $f \leq_{sW} \text{id} \times f$  and  $\text{id} \times f \equiv_W f$  hold for all problems  $f$ , whereas  $\text{id} \times f \leq_{sW} f$  is a specific property of  $f$  that allows one to “feed the input through to  $f$ .”

**Proposition 11.3.5 (Cylinder).** A problem  $f$  is a cylinder if and only if for all problems  $g$  the following holds:  $g \leq_W f \iff g \leq_{sW} f$ .

It is important to mention that the definitions of  $\leq_W$  and  $\leq_{sW}$  are invariant under the replacement of represented spaces by equivalent ones [18, Lemma 2.1]. The equivalence classes induced by  $\equiv_W$  and  $\equiv_{sW}$  are called *Weihrauch degrees* and *strong Weihrauch degrees*, respectively. The reducibilities  $\leq_W$  and  $\leq_{sW}$  naturally extend to these degrees.

Most algebraic operations defined in Definition 11.1.2 are monotone with respect to (strong) Weihrauch reducibility. We say that a binary operation  $\square$  on problems is *monotone* with respect to  $\leq_W$  if for all problems  $f_0, f_1, g_0$  and  $g_1$  condition 1. holds, and a unary operation  $\square$  on problems is called a *closure operator* with respect to  $\leq_W$  if for all problems  $f, g$  condition 2. holds:

1.  $(f_0 \leq_W f_1 \text{ and } g_0 \leq_W g_1) \implies f_0 \square g_0 \leq_W f_1 \square g_1$  (monotone)
2.  $f \leq_W f^\square, f^{\square\square} \leq_W f^\square$  and  $(f \leq_W g \implies f^\square \leq_W g^\square)$  (closure operator)

Analogously to monotone, we define *antitone* with a reversed order on one side. Monotonicity and closure operators with respect to  $\leq_{sW}$  are defined analogously.

**Proposition 11.3.6 (Monotonicity and closure operators).** *We obtain:*

1. The binary operations  $\times, \sqcup, \sqcap, \boxplus$  and  $+$  are all monotone with respect to  $\leq_W$  and  $\leq_{sW}$ .
2. The unary operation  $*$  is a closure operator with respect to  $\leq_W$  and monotone with respect to  $\leq_{sW}$ .
3. The unary operation  $\hat{\phantom{x}}$  is a closure operator with respect to  $\leq_W$  and  $\leq_{sW}$ .

In particular, all the mentioned operations extend to operations on degrees.

It is an obvious question whether there is any least and any greatest Weihrauch degree. The first question is easy to answer.

**Definition 11.3.7 (Special Weihrauch degrees).** By  $\mathbf{0}$  we denote the (strong) Weihrauch degree of the nowhere-defined problems, and by  $\mathbf{1}$  we denote the Weihrauch degree of the identity id.

It is easy to see that  $\mathbf{0}$  is exactly the class of all nowhere defined problems, and it is the least (strong) Weihrauch degree. The class  $\mathbf{1}$  characterizes the computable problems in the sense that  $f \leq_W \mathbf{1}$  holds if and only if  $f$  is computable. In many respects  $\mathbf{0}$  and  $\mathbf{1}$  behave algebraically like the numerical constants 0 and 1.

The question of whether there is a greatest Weihrauch degree is less straightforward to answer. If we do not accept the axiom of choice for the Baire space, then the class of problems without realizer forms a natural top element. Since we are accepting the axiom of choice, this natural top element is not available, and we can only add an additional top element to the Weihrauch degrees.<sup>4</sup>

If one is not interested in classifying specific problems with general types  $X, Y$  as they appear in analysis, but if one rather wants to study the structure of Weihrauch degrees as such, then it is sufficient to consider problems of the type  $f : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$  on the Baire space. We make this slightly more precise.

**Lemma 11.3.8 (Realizer version).** *Let  $(X, \delta_X)$  and  $(Y, \delta_Y)$  be represented spaces and let  $f : \subseteq X \rightrightarrows Y$  be a problem. Then the realizer version  $f^r : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$  of  $f$  is defined by  $f^r := \delta_Y^{-1} \circ f \circ \delta_X$ . We have  $f^r \equiv_{sW} f$ .*

<sup>4</sup> See Brattka and Pauly [29] for a more detailed discussion.

This means that every (strong) Weihrauch degree has a representative of type  $f^r : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ . By  $\mathscr{W}$  and  $\mathscr{W}_s$  we denote the set of Weihrauch degrees and strong Weihrauch degrees, respectively, both restricted to problems on the Baire space.<sup>5</sup>

**Theorem 11.3.9 (Weihrauch lattice).** *The Weihrauch degrees  $(\mathscr{W}, \leq_w)$  form a distributive lattice with supremum operation  $\sqcup$ , infimum operation  $\sqcap$  and bottom element  $\mathbf{0}$ .*

Also the strong Weihrauch degrees form a lattice structure, albeit a non-distributive one with a different supremum operation.

**Theorem 11.3.10 (Strong Weihrauch lattice).** *The strong Weihrauch degrees  $(\mathscr{W}_s, \leq_{sw})$  form a non-distributive lattice with supremum operation  $\boxplus$ , infimum operation  $\sqcap$  and bottom element  $\mathbf{0}$ .*

## Bibliographic Remarks

The reducibility that is now called *Weihrauch reducibility* was introduced by Klaus Weihrauch in the late 1980s and appeared in two unpublished technical reports [114, 115]. He supervised six Ph.D. and M.Sc. projects on this topic (by von Stein [110], Mylatz [83, 84], Brattka [5], Hertling [50], Pauly [90]). Most of this material remained unpublished, and the concept featured only in a few early publications by Brattka, Weihrauch and Gherardi [7, 116, 9, 45, 16]. In a more abstract categorical setting a concept related to Weihrauch reducibility has independently been studied by Hirsch [56]. Theorem 11.3.3 is due to Tavana and Weihrauch [111], and we refer the readers to the reference for a precise formulation. The subject took a turn when Gherardi and Marcone [46] defined Weihrauch reducibility in its full generality for problems, and they promoted Weihrauch complexity as a uniform version of reverse mathematics. Brattka and Gherardi [18, 17] continued to study the subject and discovered that the structure is a lower semilattice, whereas Pauly [92] independently provided the coproduct operation and discovered that the structure is a distributive upper semilattice. The notion of a cylinder is taken from Brattka and Gherardi [18]. Several authors continued to investigate the subject from the perspective of reverse mathematics, among them Dorais, Dzhafarov, Hirst, Mileti and Shafer [38, 40, 42] and Hirschfeldt and Jockusch [57, 58]. Hirschfeldt and Jockusch also introduced a generalized version of Weihrauch reducibility that has a built-in closure under composition. Dzhafarov provided the box sum operation  $\boxplus$  and proved that the strong Weihrauch degrees form a non-distributive lattice [42]. Independently, a polynomial-time version of Weihrauch complexity was used by Kawamura and Cook to classify the uniform computational complexity of problems in analysis [65, 66].

## 11.4 Algebraic and Topological Properties

In this section we discuss a number of algebraic and topological notions and their interactions that turned out to be fruitful for the study of the Weihrauch lattice. We mention that while  $f \leq_w \mathbf{1}$  characterizes the computable problems  $f$ , also the relation  $\mathbf{1} \leq_w f$  bears some meaning.

<sup>5</sup> We use the restriction to the Baire space for our formal definition of  $\mathscr{W}$  and  $\mathscr{W}_s$ , since the class of all (strong) Weihrauch degrees of problems with arbitrary type does not form a set.

**Definition 11.4.1 (Pointedness).** We call a problem  $f$  *pointed* if  $\text{id} \leq_{\text{w}} f$  holds. Analogously, we can define *strong pointedness* with the help of  $\leq_{\text{sw}}$  instead of  $\leq_{\text{w}}$ .

It is easy to see that the pointed problems are exactly those with a computable point in their domain. By definition  $f^*$  is always pointed since  $f^0 = \text{id}_{\{\emptyset\}}$ . We introduce some further terminology that can be expressed with the help of the algebraic operations.

**Definition 11.4.2 (Idempotency and parallelizability).** Let  $f$  be a problem.

1. We call  $f$  *idempotent* if  $f \times f \equiv_{\text{w}} f$ .
2. We call  $f$  *parallelizable* if  $\widehat{f} \equiv_{\text{w}} f$ .

Analogously, we define *strong idempotency* and *strong parallelizability* with the help of  $\equiv_{\text{sw}}$  instead of  $\equiv_{\text{w}}$ .

Whether or not a problem is idempotent or parallelizable might be hard to prove in some instances. In the following example the first statement is relatively easy to obtain, whereas the second one is harder to prove (see Theorems 11.7.34 and 11.7.35).

*Example 11.4.3.*  $\widehat{\text{IVT}} \equiv_{\text{sw}} Z_{[0,1]}$  and hence  $Z_{[0,1]}$  is (strongly) parallelizable, but  $\text{IVT}$  is not idempotent.

The following result captures some easy observations. Pointedness is involved here, since  $f^0 = \text{id}_{\{\emptyset\}}$  is pointed for every problem  $f$ .

**Proposition 11.4.4 (Idempotency and parallelizability).** Let  $f$  be a problem. Then:

1.  $f$  (strongly) parallelizable  $\implies f$  (strongly) idempotent.
2.  $f$  pointed and idempotent  $\iff f^* \equiv_{\text{w}} f$ .
3.  $f$  strongly pointed and strongly idempotent  $\iff f^* \equiv_{\text{sw}} f$ .

A less obvious result relates idempotency and parallelizability. In order to formulate this result, we need another definition.

**Definition 11.4.5 (Finite tolerance).** A problem  $f : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$  is called *finitely tolerant* if there is a computable partial function  $T : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  such that for all  $p, q \in \text{dom}(f)$  and  $k \in \mathbb{N}$  with  $(\forall n \geq k)(p(n) = q(n))$  it follows that  $r \in f(q)$  implies  $T\langle r, k \rangle \in f(p)$ . More generally, a problem  $g : \subseteq X \rightrightarrows Y$  can be called *finitely tolerant* if there is some finitely tolerant  $f : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$  with  $f \equiv_{\text{w}} g$ .

Intuitively, finite tolerance means that for two almost identical inputs and a solution for one of these inputs we can compute a solution for the other input. The squashing theorem relates products  $g \times f$  to parallelizations  $\widehat{g}$  of problems.

**Theorem 11.4.6 (Squashing theorem).** For  $f, g : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$  we obtain:

1. If  $\text{dom}(f) = \mathbb{N}^{\mathbb{N}}$  and  $f$  is finitely tolerant, then  $g \times f \leq_{\text{w}} f \implies \widehat{g} \leq_{\text{w}} f$ .

2. If  $\text{dom}(f) = 2^{\mathbb{N}}$  and  $f$  is finitely tolerant, then  $g \times f \leq_{\text{sw}} f \implies \widehat{g} \leq_{\text{sw}} f$ .

We obtain the following immediate corollary.

**Corollary 11.4.7.** *Let  $f : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$  be finitely tolerant. Then we obtain:*

1. For  $\text{dom}(f) = \mathbb{N}^{\mathbb{N}}$ :  $f$  idempotent  $\iff f$  parallelizable.
2. For  $\text{dom}(f) = 2^{\mathbb{N}}$ :  $f$  strongly idempotent  $\iff f$  strongly parallelizable.

Another property that turned out to be quite useful is join-irreducibility. We recall that a problem  $f$  is called *join-irreducible* in the lattice-theoretic sense if  $f \leq_{\text{w}} g \sqcup h$  implies  $f \leq_{\text{w}} g$  or  $f \leq_{\text{w}} h$  for all problems  $g, h$ . We need a countable version of this property. For this purpose we first need to define countable coproducts. For a sequence  $(X_i)_{i \in \mathbb{N}}$  of sets we define the disjoint union by  $\bigsqcup_{i=0}^{\infty} X_i := \bigcup_{i=0}^{\infty} (\{i\} \times X_i)$ . Now we can define the countable coproduct.

**Definition 11.4.8 (Countable coproduct).** Let  $f_i : \subseteq X_i \rightrightarrows Y_i$  be problems for all  $i \in \mathbb{N}$ . Then we define  $\bigsqcup_{i=0}^{\infty} f_i : \subseteq \bigsqcup_{i=0}^{\infty} X_i \rightrightarrows \bigsqcup_{i=0}^{\infty} Y_i$  by  $\bigsqcup_{i=0}^{\infty} f_i(n, x) := \{n\} \times f_n(x)$ .

Now we are prepared to define countable irreducibility.

**Definition 11.4.9 (Countable irreducibility).** A problem  $f$  is called *countably irreducible* if for every sequence  $(g_i)_{i \in \mathbb{N}}$  of problems:  $f \leq_{\text{w}} \bigsqcup_{i=0}^{\infty} g_i \implies (\exists i) f \leq_{\text{w}} g_i$ . Likewise we can define *strong countable irreducibility* with  $\leq_{\text{sw}}$  in place of  $\leq_{\text{w}}$ .

It is clear that every countably irreducible<sup>6</sup> problem is join-irreducible. Another notion that turned out to be fruitful in this context is the notion of a fractal. Roughly speaking, a fractal is a problem that exhibits its full power even if we zoom arbitrarily deep into its domain.

**Definition 11.4.10 (Fractal).** A problem  $f$  is called a *fractal* if there is a problem  $F : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$  such that  $F \equiv_{\text{w}} f$  and  $F|_A \equiv_{\text{w}} F$  holds for every clopen  $A \subseteq \mathbb{N}^{\mathbb{N}}$  with  $A \cap \text{dom}(F) \neq \emptyset$ . Likewise we define a *strong fractal* with  $\equiv_{\text{sw}}$  instead of  $\equiv_{\text{w}}$ . A *total (strong) fractal* is a (strong) fractal where  $F$  can be chosen to be total.

One reason why fractals are useful is captured in the following observation.

**Proposition 11.4.11 (Fractals).** *Every (strong) fractal is (strongly) countably irreducible.*

Some natural problems in the Weihrauch lattice are densely realized in the following sense.

**Definition 11.4.12 (Densely realized).** Let  $(X, \delta_X), (Y, \delta_Y)$  be represented spaces. A problem  $f : \subseteq X \rightrightarrows Y$  is called *densely realized* if  $f^{\text{r}}(p) = \delta_Y^{-1} \circ f \circ \delta_X(p)$  is dense in  $\text{dom}(\delta_Y)$  for all  $p \in \text{dom}(f \circ \delta_X)$ .

<sup>6</sup> We note that countable irreducibility is not identical to what is sometimes called  *$\sigma$ -join-irreducibility* since the countable coproduct is not necessarily a countable supremum, as we will see in Theorem 11.5.1.

We note that this notion depends on the representations chosen. It turns out that all problems with discrete output below densely realized problems with totally represented output are computable.

**Proposition 11.4.13 (Densely realized).** *Let  $f : \subseteq X \rightrightarrows Y$  be densely realized, where  $Y$  is a represented space with total representation, and let  $g : \subseteq Z \rightrightarrows \mathbb{N}$  be a problem. If  $g \leq_w f$  holds, then  $g$  is computable.*

## Bibliographic Remarks

The notions of pointedness, idempotency and parallelizability were introduced by Brattka and Gherardi in [18]. Dorais, Dzhafarov, Hirst, Mileti and Shafer [38] defined finitely tolerant problems and proved the squashing theorem (a proof for Theorem 11.4.6 exactly as stated here can be found in [102]). Countable irreducibility was first considered by Brattka, de Brecht and Pauly [14], who also implicitly defined fractals that were later used by Brattka, Gherardi and Marcone [21]. Densely realized problems have been introduced by Brattka, Hendtlass and Kreuzer [22] and Proposition 11.4.13 is due to Brattka and Pauly [29].

## 11.5 Completeness, Composition and Implication

Another obvious question regarding the Weihrauch lattice is whether the lattice is complete or, more generally, which suprema and infima exist. A mostly negative answer is given by the following result.

**Theorem 11.5.1 (Suprema and infima).** *No non-trivial countable suprema exist in the Weihrauch lattice, i.e., a sequence  $(f_n)_{n \in \mathbb{N}}$  of problems has a supremum if and only if this supremum is already a supremum of  $(f_n)_{n \leq k}$  for some  $k \in \mathbb{N}$ . Some non-trivial countable infima exist in the Weihrauch lattice, others do not exist.*

In particular, the Weihrauch lattice is not complete in the lattice-theoretic sense. We can also conclude that  $\bigsqcup_{n=0}^{\infty} f_n$  is typically not the supremum of  $\{f_n : n \in \mathbb{N}\}$  unless it is already a supremum of  $\bigsqcup_{n=0}^k f_n$  for some  $k \in \mathbb{N}$ .

However, it turns out that some important suprema and infima exist in the Weihrauch lattice. We are particularly interested in composition and implication. The composition  $f \circ g$  of problems as it has been defined in Definition 11.1.2 is not an operation on degrees in the same sense in which the other algebraic operations extend to degrees. It requires that the output type of  $g$  fits the input type of  $f$ , and even if the types fit, the operation does not need to be monotone. On the other hand, it is natural to consider a Weihrauch degree  $f * g$  that captures exactly what can be achieved when one first applies  $g$ , possibly followed by some computation, and then one applies  $f$ . That the maximal Weihrauch degree that can be built in this way always exists is the first statement of the following theorem. The second statement captures the minimal degree  $(g \rightarrow f)$  that is needed in advance of  $g$  in order to compute  $f$ . In some sense  $(g \rightarrow f)$  measures how much harder  $f$  is to compute than  $g$ .



**Theorem 11.5.2 (Compositional product and implication).** *Let  $f$  and  $g$  be problems. The following Weihrauch degrees exist:*

1.  $f * g := \max_{\leq_W} \{f_0 \circ g_0 : f_0 \leq_W f, g_0 \leq_W g\}$  *(compositional product)*
2.  $(g \rightarrow f) := \min_{\leq_W} \{h : f \leq_W g * h\}$  *(implication)*

*Maximum and minimum are understood with respect to  $\leq_W$ . Only such  $f_0$  and  $g_0$  are considered that can be composed.*

By definition,  $*$  and  $\rightarrow$  are operations on degrees. It is easy to see that  $*$  is even a monotone operation, whereas  $\rightarrow$  is antitone in the first component and monotone in the second component. In order to prove Theorem 11.5.2 it is useful to define a specific representative of the degree  $f * g$  that we denote by  $f \star g$ . For  $F, G : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  we define  $\langle F \times G \rangle(p, q) := \langle F(p), G(q) \rangle$ .

**Definition 11.5.3 (Compositional product).** Let  $f$  and  $g$  be problems. We define  $f \star g : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$  by  $(f \star g)(p, q) := (\text{id} \times f^r) \circ \Phi_p \circ g^r(q)$  for all  $p, q \in \mathbb{N}^{\mathbb{N}}$ .

This definition captures the intuition that in between  $g$  and  $f$  there is another possible computation  $\Phi_p$ . Of course, this definition does not have the same set-theoretic flavor as the other operations in Definition 11.1.2, and it is not a definition that we typically work with. It is mostly needed in order to prove Theorem 11.5.2, and the working definition of the compositional product  $f * g$  is the one given in Theorem 11.5.2. The following result captures another interesting property of  $f \star g$ .

**Proposition 11.5.4.**  $f * g \equiv_W f \star g$  and  $f \star g$  is always a cylinder. If  $f$  and  $g$  are fractals, then so is  $f \star g$ .

We can also define a strong version of the compositional product. This operation has been studied less and is only known to exist in specific cases. In fact, since  $f \star g$  is always a cylinder, we directly obtain the following corollary of Theorem 11.5.2 and Proposition 11.5.4.

**Corollary 11.5.5.**  $f *_s g := \max_{\leq_{sW}} \{f_0 \circ g_0 : f_0 \leq_{sW} f, g_0 \leq_{sW} g\}$  exists for cylinders  $f, g$ .

The maximum  $f *_s g$  exists also in some cases where  $f, g$  are not cylinders, but we do not claim that it exists in general. The following result summarizes some algebraic properties of compositional products and implications.

**Proposition 11.5.6 (Algebraic properties).**

1.  $*$  is associative but not commutative,  $\rightarrow$  is neither associative nor commutative.
2.  $*_s$  is associative whenever all occurring degrees actually exist.

The operations  $+$ ,  $\sqcap$ ,  $\boxplus$ ,  $\sqcup$ ,  $\times$  and  $*$  are typically ordered as given.

**Proposition 11.5.7 (Order of algebraic operations).** *We obtain:*

1.  $f + g \leq_{sW} f \sqcap g \leq_{sW} f \boxplus g \leq_{sW} f \sqcup g$  and  $f \times g \leq_{sW} f \star g$  for all problems  $f, g$ .

2.  $f \sqcup g \leq_w f \times g$  and  $f \boxplus g \leq_{sw} f \times g$  for all pointed problems  $f, g$ .

The following result expresses in which way compositional product and implication are adjoints of each other.

**Proposition 11.5.8 (Adjointness).**  $f \leq_w g * h \iff (g \rightarrow f) \leq_w h$ .

In the language of lattice theory this result can be expressed such that  $(\mathcal{W}, \geq_w, *)$  is right residuated, and the residual operation is exactly  $\rightarrow$ . It follows from Example 11.7.43 that  $(\mathcal{W}, \geq_w, *)$  is not left residuated and that  $(\mathcal{W}, \geq_w, \times)$  is not residuated. The following result expresses that the Weihrauch lattice is not residuated with respect to the lattice operations  $\sqcup, \sqcap$ .

**Theorem 11.5.9 (Brouwer and Heyting algebras).** *The Weihrauch lattice  $\mathcal{W}$  is neither a Brouwer algebra nor a Heyting algebra.*

Brouwer algebras can be seen as models of intermediate logics that are in between classical logic and intuitionistic logic.

## Bibliographic Remarks

Theorems 11.5.1 and 11.5.9 are due to Higuchi and Pauly [55], who also discussed several variants of the Weihrauch lattice in this regard. The compositional product was introduced by Brattka, Gherardi and Marcone [21]. The implication was introduced by Brattka and Pauly [95], who also proved most other results in this section. They also studied many further algebraic properties of the Weihrauch lattice, including distributivity laws. The results on strong compositional products are taken from Brattka, Henttlass and Kreuzer [22].

## 11.6 Limits and Jumps

A map of particular importance in the Weihrauch lattice is the limit map. Given a Hausdorff space  $X$ , we define the *limit map of the space  $X$*  and the *the limit map* (of the Baire space) by

1.  $\lim_X : \subseteq X^{\mathbb{N}} \rightarrow X, (x_n)_{n \in \mathbb{N}} \mapsto \lim_{n \rightarrow \infty} x_n,$
2.  $\lim : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}, \langle p_0, p_1, p_2, \dots \rangle \mapsto \lim_{n \rightarrow \infty} p_n.$

The domain of  $\lim_X$  consists of all converging sequences in  $X$ . In the special case of the Baire space, we use a tupling with  $p_i \in \mathbb{N}^{\mathbb{N}}$  on the input side for mere reasons of convenience. By  $\lim_{\Delta}$  we denote the restriction of  $\lim$  to eventually constant sequences. It is easy to see that  $\lim_{\mathbb{N}} \equiv_w \lim_{\Delta}$ . It has been noticed that limit maps can be used to characterize limit computable functions and functions computable with finitely many mind changes.

**Proposition 11.6.1 (Limit computability and finite mind change computability).** *For problems  $f$  we obtain:*

1.  $f \leq_W \text{lim} \iff f \text{ limit computable.}$
2.  $f \leq_W \text{lim}_{\mathbb{N}} \iff f \text{ computable with finitely many mind changes.}$

Limit computability and computability with finitely many mind changes can be defined directly with Turing machines that allow two-way output tapes. In the case of limit computable problems the Turing machine can change the content of each output cell finitely many times before it has to stabilize; in the case of problems that are computable with finitely many mind changes, the entire output has to stabilize after finitely many changes. These concepts are well known from learning theory.

One might ask whether  $\text{lim}_X$  for other spaces  $X$  yields different classes of computable problems, but for many spaces  $X$  this is not the case. We recall that a computable metric space  $X$  is called *rich* if there is a *computable embedding*  $\iota : 2^{\mathbb{N}} \hookrightarrow X$ , i.e.,  $\iota$  is injective, and  $\iota$  and its partial inverse  $\iota^{-1}$  are computable.

**Proposition 11.6.2 (Limits).**  $\text{lim}_X \equiv_{s,W} \text{lim}$  for all rich computable metric spaces  $X$ .

Examples of rich computable metric spaces are  $2^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}, \mathbb{R}, \mathbb{R}^{\mathbb{N}}, [0, 1], [0, 1]^{\mathbb{N}}$ , etc. This justifies also the more generic notation  $\text{lim}$  for the limit operation on the Baire space. An interesting property of  $\text{lim}_X$  is its behavior under composition. The following result on  $\text{lim}_{\mathbb{N}}$  can be proved with the help of Theorem 11.7.11, but is also easy to see directly.

**Proposition 11.6.3 (Composition).**  $\text{lim}_{\mathbb{N}} * \text{lim}_{\mathbb{N}} \equiv_W \text{lim}_{\mathbb{N}}$ , i.e., problems that are computable with finitely many mind changes are closed under composition.

The situation for iterations of  $\text{lim}$  is very different. For a problem  $f$  we denote by  $f^{[n]}$  the  $n$ -fold iteration of the compositional product of  $f$  with itself, i.e.,  $f^{[0]} \equiv_W \text{id}$ ,  $f^{[1]} \equiv_W f$ ,  $f^{[2]} \equiv_W f * f$ , etc. By iterations of  $\text{lim}$  one climbs up the Borel hierarchy with every further application of a limit. By  $\Sigma_n^0$  we denote the corresponding Borel class of subsets of  $\mathbb{N}^{\mathbb{N}}$ , i.e.,  $\Sigma_1^0$  is the class of open subsets,  $\Sigma_2^0$  is the class of  $F_{\sigma}$ -subsets and so forth. A function  $F : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  is called  $\Sigma_n^0$ -measurable, if preimages  $F^{-1}(U)$  of open sets  $U$  are  $\Sigma_n^0$ -sets relative to  $\text{dom}(F)$ . Analogously,  $F$  is *effectively*  $\Sigma_n^0$ -measurable if the preimage can be uniformly computed from a description of  $U$ . The  $\Sigma_1^0$ -measurable functions  $F$  are exactly the continuous ones, and the effectively  $\Sigma_1^0$ -measurable functions  $F$  are exactly the computable ones. We can transfer concepts of measurability to problems via realizers.

**Definition 11.6.4 (Effective Borel measurability).** Let  $n \geq 1$ . A problem  $f$  is called (effectively)  $\Sigma_n^0$ -measurable if it has a realizer with the same property.

It can be proved that for computable metric spaces  $X$  and  $Y$  and total functions  $f : X \rightarrow Y$  this yields just the usual (effectively)  $\Sigma_n^0$ -measurable functions as they are known in descriptive set theory [9]. The measurable problems can also easily be characterized in the Weihrauch lattice.

**Theorem 11.6.5 (Effective Borel measurability).**  $f \leq_W \text{lim}^{[n]} \iff f$  is effectively  $\Sigma_{n+1}^0$ -measurable, for all problems  $f$  and  $n \in \mathbb{N}$ .

This theorem can be relativized. We write  $f \leq_W^p g$  if  $f$  is Weihrauch reducible to  $g$  with respect to some oracle  $p \in \mathbb{N}^{\mathbb{N}}$ , which means that the reduction functions  $H, K$  are computable relative to  $p$ . Then  $f \leq_W^p \text{lim}^{[n]}$  holds for some  $p \in \mathbb{N}^{\mathbb{N}}$  if and only if  $f$  is  $\Sigma_{n+1}^0$ -measurable. Theorem 11.6.5 shows that the Weihrauch lattice yields a refinement of the effective Borel hierarchy very much in the same way as many-one reducibility yields a refinement of the Kleene hierarchy. We summarize some of the obvious algebraic properties of  $\text{lim}$ .

**Proposition 11.6.6.**  *$\text{lim}$  is a cylinder, strongly parallelizable, strongly idempotent, finitely tolerant, a strong fractal and (strongly) countably irreducible.*

It is useful to know that there are many problems that are equivalent to  $\text{lim}$ . We mention only a few. By  $\mathbf{J} : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}, p \mapsto p'$  we denote the Turing jump operation, which is injective as a function on the Baire space. By  $\mathbf{EC} : \mathbb{N}^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$  we denote the function that translates enumerations of sets into their characteristic functions, and by  $\mathbf{LPO} : \mathbb{N}^{\mathbb{N}} \rightarrow \{0, 1\}$  we denote the *limited principle of omniscience*<sup>7</sup>:

$$\mathbf{EC}(p)(n) := \begin{cases} 1 & \text{if } (\exists k \in \mathbb{N}) p(k) = n+1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and } \mathbf{LPO}(p) := \begin{cases} 1 & \text{if } (\exists k) p(k) = 0 \\ 0 & \text{otherwise} \end{cases}$$

We also use  $\inf, \sup : \subseteq \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}$ . We obtain the following result that lists some important members of the equivalence class of  $\text{lim}$ .

**Theorem 11.6.7 (Limit).**  $\text{lim} \equiv_{\text{sw}} \inf \equiv_{\text{sw}} \sup \equiv_{\text{sw}} \mathbf{J} \equiv_{\text{sw}} \mathbf{EC} \equiv_{\text{sw}} \widehat{\mathbf{LPO}} \equiv_{\text{sw}} \widehat{\text{lim}}_{\mathbb{N}}$ .

We now use this limit operation to define the jump of a represented space.

**Definition 11.6.8 (Jump of a represented space).** Let  $(X, \delta)$  be a represented space. Then we define its *jump*  $(X', \delta')$  by  $X' := X$  and  $\delta' := \delta \circ \text{lim}$ . Likewise,  $(X^{(n)}, \delta^{(n)})$  denotes the  $n$ -fold jump.

That is, in the new represented space  $X'$  a name of  $x$  with respect to  $\delta'$  is a sequence that converges to a name in the sense of  $\delta$ . Hence, names in  $(X', \delta')$  typically carry less computably accessible information than names in  $(X, \delta)$ . Now the jump of a problem is just the same problem but with the input space replaced by its jump.

**Definition 11.6.9 (Jump of a problem).** Let  $f : \subseteq X \rightrightarrows Y$  be a problem. Then its *jump*  $f' : \subseteq X' \rightrightarrows Y$  is defined to be the same problem with the modified input space  $X'$ . Likewise  $f^{(n)} : \subseteq X^{(n)} \rightrightarrows Y$  denotes the  $n$ -fold jump for  $n \in \mathbb{N}$  with  $f^{(0)} := f$ .

Since the jump  $f'$  has to work with a weaker type of input information, it is typically harder to compute  $f'$  than  $f$ . The study of jumps provides one reason why it is important to keep track of strong Weihrauch reductions. Jumps are monotone with respect to strong Weihrauch reductions, but not with respect to ordinary Weihrauch reductions in general.

<sup>7</sup> LPO is also Weihrauch equivalent to the identity  $\text{id} : \mathbb{S} \rightarrow \{0, 1\}$ .

**Proposition 11.6.10 (Monotonicity).** *For all problems  $f, g$  the following hold:  $f \leq_{sW} f'$  and also  $f \leq_{sW} g \implies f' \leq_{sW} g'$ .*

Trivial examples such as a constant function show that  $f \equiv_{sW} f'$  can happen. For  $f \leq_W g$  all possible reductions between  $f'$  and  $g'$  can occur; the order can even be reversed, i.e.,  $g' <_W f'$  can happen [22, Figure 2]. Surprisingly, there is also a certain inverse of Proposition 11.6.10 that one can prove. We recall that by  $p'$  we denote the Turing jump of  $p$ . Using this concept we can phrase the following theorem.

**Theorem 11.6.11 (Inverting jumps).**  *$f' \leq_W^p g' \implies f \leq_W^{p'} g$  holds for all problems  $f, g$  and  $p \in \mathbb{N}^{\mathbb{N}}$ . An analogous statement holds if Weihrauch reducibility is replaced by strong Weihrauch reducibility in both occurrences.*

The property that  $f \leq_W^p g$  holds for some oracle  $p$  is equivalent to the continuous version of Weihrauch reducibility, where the two reduction functions  $H, K$  just need to be continuous. By Theorem 11.6.11 continuous separations are particularly useful, since they automatically carry over to jumps.

The following result summarizes some algebraic properties of the jump.

**Proposition 11.6.12 (Algebraic properties).** *We obtain  $f' \times g' \equiv_{sW} (f \times g)'$ ,  $\widehat{f'} \equiv_{sW} \widehat{f'}$ ,  $f' \sqcap g' \equiv_{sW} (f \sqcap g)'$ ,  $f' \sqcup g' \leq_{sW} (f \sqcup g)'$  and  $f'^* \leq_{sW} f'^*$  for all problems  $f, g$ .*

One can see that coproducts do not commute with jumps in general, since jumps are join-irreducible.

**Proposition 11.6.13 (Finite tolerance and fractality).**  *$f'$  is finitely tolerant, a strong fractal and (strongly) countably irreducible for every problem  $f$ .*

Sometimes it is useful to have the following characterizations of the jump.

**Proposition 11.6.14 (Cylinder).**  *$f' \equiv_{sW} f *_s \text{lim}$ , and if  $f$  is a cylinder, then  $f'$  is a cylinder and  $f' \equiv_W f' \times \text{lim} \equiv_W f *_s \text{lim}$ .*

In particular,  $f *_s \text{lim}$  always exists. We continue with a discussion of some invariant properties. We call a class  $P$  of problems *invariant* if  $f \leq_W g$  and  $g \in P$  implies  $f \in P$ . Likewise, we define *strong invariance*. We list a number of examples of (strongly) invariant properties that easily follow from results of this section.

**Corollary 11.6.15 (Invariance).** *The following properties of problems are (strongly) invariant: continuity, computability, limit computability, (effective)  $\Sigma_n^0$ -measurability, computability with finitely many mind changes, non-uniform computability (i.e., the class of problems that have some computable output for every computable input in the domain).*

Sometimes it is also useful to use numerical quantities that are preserved by Weihrauch reducibility. Besides the level  $n$  of (effective)  $\Sigma_n^0$ -measurability, we can also use the number of mind changes that are required to compute a problem. Let  $\text{mind}(f)$  denote the minimal number  $n \in \mathbb{N}$  that a Turing machine with two-way output needs in order to compute  $f$  with at most  $n$  mind changes on all inputs (if such a number exists). This property is invariant in the following sense.

**Proposition 11.6.16 (Mind changes).**  $f \leq_W g \implies \text{mind}(f) \leq_W \text{mind}(g)$  for all problems  $f, g$  for which  $\text{mind}(f), \text{mind}(g)$  exist.

Another numerical quantity that is useful as an invariant for strong Weihrauch reducibility is the cardinality of a problem.

**Definition 11.6.17 (Cardinality).** For every problem  $f : \subseteq X \rightrightarrows Y$  we denote by  $\#f$  the maximal cardinality (if it exists) of a set  $M \subseteq \text{dom}(f)$  such that  $\{f(x) : x \in M\}$  contains pairwise disjoint sets.

It is easy to see that the following holds.

**Proposition 11.6.18 (Cardinality).**  $f \leq_{sW} g \implies \#f \leq \#g$  for all problems  $f, g$  with existing cardinality.

Every cylinder  $f$  needs to satisfy  $\#f \geq |\mathbb{N}^{\mathbb{N}}|$ , since  $\#\text{id} = |\mathbb{N}^{\mathbb{N}}|$ , where  $|X|$  denotes the cardinality of the set  $X$ . For instance, it is easy to see that  $\#\lim_{\mathbb{N}} = |\mathbb{N}|$  and  $\#\lim_{\Delta} = |\mathbb{N}^{\mathbb{N}}|$ . We obtain the following.

*Example 11.6.19.*  $\lim_{\mathbb{N}} \equiv_W \lim_{\Delta}$  and  $\lim_{\mathbb{N}} <_{sW} \lim_{\Delta}$ . Moreover,  $\lim_{\Delta}$  is a cylinder, whereas  $\lim_{\mathbb{N}}$  is not.

Next we mention that LPO is in a certain sense the weakest discontinuous problem among all *single-valued* problems.

**Theorem 11.6.20 (Discontinuous single-valued problems).**  $\text{LPO} \leq_W^p f$  for some oracle  $p \iff f$  is discontinuous, for  $f : X \rightarrow Y$  on computable metric spaces  $X, Y$ .

We close this section with the following result that shows that for (certain well-behaved) linear closed operators there is a dichotomy: either they are bounded and computable or  $\lim$  is reducible to them.

**Theorem 11.6.21 (Linear operators).** Let  $T : \subseteq X \rightarrow Y$  be a linear closed operator on computable Banach spaces  $X, Y$ . Let  $(e_n)_{n \in \mathbb{N}}$  be a computable sequence in  $\text{dom}(T)$  whose linear span is dense in  $\text{dom}(T)$  and such that  $(T(e_n))_{n \in \mathbb{N}}$  is computable.

1. If  $T$  is bounded, then  $T$  is computable.
2. If  $T$  is unbounded, then  $\lim \leq_W T$ .

## Bibliographic Remarks

The relation between Weihrauch reducibility and the Borel hierarchy was established by Brattka, including Proposition 11.6.2 and Theorem 11.6.5 [9, 5] (see also [31]). Limit computable functions and functions that are computable by finite mind changes were originally introduced to computable analysis by Ziegler [119, 118]. The relation of these classes to the Weihrauch lattice was studied by Brattka and Gherardi [18, 17], including Corollary 11.6.15 and Proposition 11.6.16 and by Brattka, de Brecht and Pauly [14], including Proposition 11.6.3. Theorem 11.6.7 collects results from the last-mentioned groups of authors. The concept of a jump of a problem was introduced by Brattka,

Gherardi and Marcone [21], who also proved most related results that are included here, except Theorem 11.6.11, which is due to Brattka, Hölzl and Kuyper [25], and the proof is essentially based on a jump control theorem by Brattka, Hendtlass and Kreuzer [22]. The number of mind changes has a topological counterpart, namely the level of a problem as studied by Hertling [50, 49]; it essentially measures the level in the Hausdorff difference hierarchy. Proposition 11.6.18 is due to Brattka, Gherardi and Hölzl [19, Proposition 3.6]. Theorem 11.6.20 is due to Weihrauch [115, Theorem 3.7]. Theorem 11.6.21 is due to Brattka [7, Theorem 4.3] and can be seen as a uniform version of the first main theorem of Pour-El and Richards [101].

## 11.7 Choice

The choice problem  $C_X$  of a given space  $X$  is the problem of finding a point in a given closed  $A \subseteq X$ . By choosing appropriate spaces  $X$  one obtains several important Weihrauch degrees.

**Definition 11.7.1 (Choice).** The problem  $C_X : \subseteq \mathcal{A}_-(X) \rightrightarrows X, A \mapsto A$  with  $\text{dom}(C_X) := \{A : A \neq \emptyset\}$  is called the *choice problem* of the represented space  $X$ .

Here the description  $A \mapsto A$  of the map is to be read such that on the input side  $A \in \mathcal{A}_-(X)$  is a point of the input space, whereas on the output side it is a subset  $A \subseteq X$  of possible results. Typically,  $X$  will be a computable metric space, and the reader can think of closed sets being represented by enumerations of balls whose union exhausts the complement, as described in the first item of Proposition 11.2.9.

Likewise, one can use the second characterization of Proposition 11.2.9 to define a representation of closed sets via preimages of continuous functions  $f : X \rightarrow \mathbb{R}$ . The uniformity statement in Proposition 11.2.9 yields the conclusion that the choice problem is nothing but the zero problem that we introduced in Example 11.1.1.

**Corollary 11.7.2.**  $C_X \equiv_{\text{sW}} Z_X$  for every computable metric space  $X$ .

Certain relations between spaces transfer to the corresponding choice problems. We mention two such properties in the following result.

**Proposition 11.7.3 (Subsets and surjections).** Let  $X, Y$  be represented spaces.

1. If  $A \subseteq X$  is co-c.e. closed, then  $C_A \leq_{\text{sW}} C_X$ .
2. If there is a computable surjection  $s : X \rightarrow Y$ , then  $C_Y \leq_{\text{sW}} C_X$ .

The choice problem has been studied in many variants that are typically restrictions to closed subsets with certain extra properties. We list a number of examples.

**Definition 11.7.4 (Variants of choice).**

1.  $UC_X$  is  $C_X$  restricted to singletons (*unique choice*)
2.  $CC_X$  is  $C_X$  restricted to connected sets (*connected choice*)
3.  $PWCC_X$  is  $C_X$  restricted to pathwise connected sets (*pathw. connected choice*)
4.  $XC_X$  is  $C_X$  restricted to convex sets (*convex choice*)

5.  $\text{PC}_X$  is  $\text{C}_X$  restricted to sets with positive measure (positive choice)
6.  $\text{AoUC}_X$  is  $\text{C}_X$  restricted to sets of the form  $\{x\}$  or  $X$  (all-or-unique choice)
7.  $\text{ACC}_X$  is  $\text{C}_X$  restricted to sets of the form  $X \setminus \{x\}$  or  $X$  (all-or-co-unique choice)
8.  $\text{CFC}_X$  is  $\text{C}_X$  restricted to co-finite sets (co-finite choice)

In some of these examples some additional structure is required on  $X$ . For instance, for convex choice one would assume that  $X$  is a vector space, and for positive choice one would expect that  $X$  is endowed with a fixed Borel measure. In the case of  $\mathbb{N}^{\mathbb{N}}$  and  $\mathbb{R}$  we assume that the product measure of the geometric probability measure on  $\mathbb{N}$  and the Lebesgue measure are used, respectively. In the case of  $\text{UC}_X$  and  $\text{AoUC}_X$  we assume that  $X$  is a  $T_1$ -space, and in the case of  $\text{ACC}_X$  and  $\text{CFC}_X$  we assume that  $X$  is endowed with a discrete topology. The choice problem  $\text{C}_X$  is a fractal for many spaces  $X$  and often a total fractal for compact  $X$ .

**Proposition 11.7.5 (Fractality).**

1.  $\text{C}_{\mathbb{N}}, \text{C}_{\mathbb{R}}, \text{PC}_{\mathbb{R}}$  and  $\text{C}_{\mathbb{N}^{\mathbb{N}}}$  are fractals,
2.  $\text{C}_{2^{\mathbb{N}}}, \text{PC}_{2^{\mathbb{N}}}, \text{CC}_{[0,1]}$  and  $\text{XC}_{[0,1]^{n+1}}$  are total fractals for all  $n \in \mathbb{N}$ .

In particular, all the mentioned problems are countably irreducible and hence join-irreducible.

While it is easy to see that  $\text{C}_X$  is a cylinder for many spaces  $X$ , it follows from the fact that there are only countably many pairwise different sets of positive measure that  $\#\text{PC}_{2^{\mathbb{N}}} = \#\text{PC}_{\mathbb{R}} = \#\text{PC}_{\mathbb{N}^{\mathbb{N}}} = \#\text{C}_{\mathbb{N}} = |\mathbb{N}|$ , and hence all the mentioned problems are not cylinders. It requires more sophisticated arguments to show that  $\text{CC}_{[0,1]}$  is not a cylinder despite the fact that  $\#\text{CC}_{[0,1]} = |\mathbb{N}^{\mathbb{N}}|$ .

**Proposition 11.7.6 (Cylinders).**

1.  $\text{C}_{2^{\mathbb{N}}}, \text{C}_{\mathbb{R}}, \text{C}_{\mathbb{N}^{\mathbb{N}}}$  are cylinders,
2.  $\text{C}_{\mathbb{N}}, \text{PC}_{2^{\mathbb{N}}}, \text{PC}_{\mathbb{R}}, \text{PC}_{\mathbb{N}^{\mathbb{N}}}$  and  $\text{CC}_{[0,1]}$  are not cylinders.

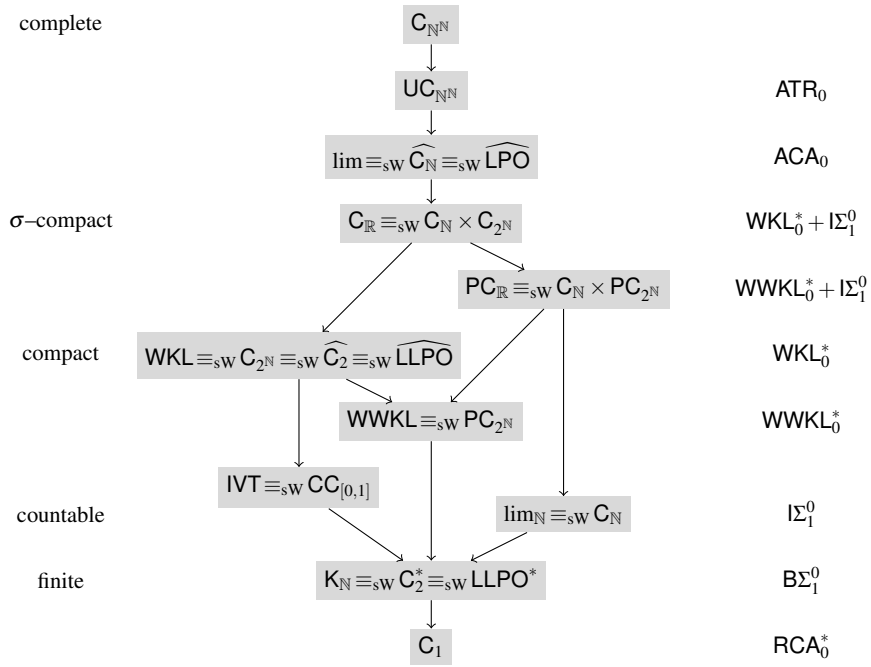
We note that there is also a choice problem  $\text{K}_X : \subseteq \mathcal{K}(X) \Rightarrow X, K \mapsto K$  that is called *compact choice*. Unlike the other choice problems we do not just restrict  $\text{C}_X$  to compact sets here, but we also increase the input information, i.e., the input set  $K$  is actually described as a compact set.

With the help of the choice problem  $\text{C}_X$  for different spaces  $X$  we obtain several important Weihrauch degrees. In the following result we indicate how the most important choice problems appear naturally as upper bounds for certain topological properties of the underlying space. We call  $X$  *computably countable* if there is a computable surjection  $s : \mathbb{N} \rightarrow X$ , and we say that a computable metric space is *computably  $\sigma$ -compact* if there is a computable sequence  $(K_i)_{i \in \mathbb{N}}$  of compact sets  $K_i \subseteq X$  such that  $X = \bigcup_{i \in \mathbb{N}} K_i$ .

**Proposition 11.7.7 (Spaces).** *Let  $X$  be a computable metric space.*

1.  $\text{C}_X \leq_{\text{sW}} \text{C}_{\mathbb{N}^{\mathbb{N}}}$  if  $X$  is complete,
2.  $\text{C}_X \leq_{\text{sW}} \text{C}_{\mathbb{R}}$  if  $X$  is computably  $\sigma$ -compact,





**Fig. 11.2** Basic choice problems together with corresponding reverse mathematics systems (see Subsection 11.9.3) and topological properties (every arrow indicates a strong Weihrauch reduction; no additional ordinary Weihrauch reductions hold besides those that follow from transitivity; the arrows in the diagram point in the direction of computations and implicit logical implications and hence in the inverse direction of the corresponding reductions).

3.  $C_X \leq_{sW} C_{2^{\mathbb{N}}}$  if  $X$  is computably compact,
4.  $C_X \leq_{sW} C_{\mathbb{N}}$  if  $X$  is computably countable,
5.  $C_X \leq_{sW} K_{\mathbb{N}}$  if  $X$  is finite.

We will discuss these cones in individual subsections below. The diagram in Figure 11.2 displays several basic choice problems in the Weihrauch lattice. The corresponding systems from reverse mathematics are discussed later in Subsection 11.9.3.

The reader who is mostly interested in classifications of theorems in analysis can continue reading in Section 11.8 from here on. In the remainder of this section we continue discussing systematically choice principles and their properties.

### 11.7.1 Composition and Non-determinism

In this section we discuss a different perspective on choice that can be seen as a type conversion and that is related to non-determinism. Firstly, we define non-deterministic computability with some advice space  $R$ .

**Definition 11.7.8 (Non-deterministic computability).** Let  $(X, \delta_X), (Y, \delta_Y)$  be represented spaces and  $R \subseteq \mathbb{N}^{\mathbb{N}}$ . Then  $f : \subseteq X \rightrightarrows Y$  is called *non-deterministically computable with advice space  $R$*  if there exist computable functions  $F : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  and  $S : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{S}$  such that  $\langle \text{dom}(f\delta_X) \times R \rangle \subseteq \text{dom}(S)$  and for each  $p \in \text{dom}(f\delta_X)$ :

1.  $R_p := \{r \in R : S\langle p, r \rangle = 0\} \neq \emptyset$ ,
2.  $r \in R_p \implies \delta_Y F\langle p, r \rangle \in f\delta_X(p)$ .

If  $R = 2^{\mathbb{N}}$ , then we say for short that  $f$  is *non-deterministically computable*. If in this case we strengthen the first condition to  $\mu(R_p) > 0$  with the uniform measure  $\mu$ , then we say that  $f$  is *Las Vegas computable*.

Intuitively, the machine can access an arbitrary oracle  $r \in R$  besides the input  $p$ . Here  $R_p$  is the set of successful oracles for input  $p$ . On input  $p$  together with such successful oracles  $r \in R_p$  the computable realizer  $F$  produces a correct output. On the other hand, the computable  $S$  eventually rejects unsuccessful oracles  $r$ , i.e.,  $S\langle p, r \rangle = 1$  for such  $r$ . The importance of non-determinism in our context is based on the following observation.

**Theorem 11.7.9 (Non-determinism).**  $f \leq_{\text{w}} \mathbf{C}_R \iff f$  is non-deterministically computable with advice space  $R$ , for every  $R \subseteq \mathbb{N}^{\mathbb{N}}$ .

In the case of  $R = \mathbb{N}$  it is not too hard to see that we obtain exactly the functions that are computable with finitely many mind changes. We summarize some important classes of functions that can be characterized by an appropriate version of choice.

**Corollary 11.7.10 (Notions of computability).** Let  $f$  be a problem. Then:

1.  $f \leq_{\text{w}} \mathbf{C}_{\mathbb{N}} \iff f$  is computable with finitely many mind changes,
2.  $f \leq_{\text{w}} \mathbf{C}_{2^{\mathbb{N}}} \iff f$  is non-deterministically computable,
3.  $f \leq_{\text{w}} \mathbf{PC}_{2^{\mathbb{N}}} \iff f$  is Las Vegas computable.

In particular, all the given properties of  $f$  are invariant.

One benefit of characterizing the choice problem with the help of non-deterministic computations is that it is very easy to consider compositions of non-deterministic computations, and hence one obtains a simple proof of the following result that is much harder to prove directly.

**Theorem 11.7.11 (Independent choice).** We obtain  $\mathbf{C}_R * \mathbf{C}_S \leq_{\text{w}} \mathbf{C}_{R \times S}$  and  $\mathbf{PC}_R * \mathbf{PC}_S \leq_{\text{w}} \mathbf{PC}_{R \times S}$  for all  $R, S \subseteq \mathbb{N}^{\mathbb{N}}$ .

In the case of positive choice one needs an invocation of Fubini's theorem besides the composition of the two non-deterministic computations. We obtain the following important corollary that, in particular, shows that the notions of computability listed in Corollary 11.7.10 are very natural.

**Corollary 11.7.12 (Composition).**  $\mathbf{C}_{\mathbb{N}}, \mathbf{C}_{2^{\mathbb{N}}}, \mathbf{C}_{\mathbb{R}}, \mathbf{C}_{\mathbb{N}^{\mathbb{N}}}, \mathbf{UC}_{\mathbb{N}^{\mathbb{N}}}, \mathbf{PC}_{2^{\mathbb{N}}}, \mathbf{PC}_{\mathbb{R}}$  and  $\mathbf{PC}_{\mathbb{N}^{\mathbb{N}}}$  are closed under compositional product and hence, in particular, idempotent.

## 11.7.2 Choice on Natural Numbers

An important equivalence class is the class of choice on natural numbers. We summarize some of its characterizations. In particular, we use the complementary minimum function  $\min^c : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}, p \mapsto \min\{n \in \mathbb{N} : (\forall k) p(k) \neq n\}$  and the maximum function  $\max : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}, p \mapsto \max\{p(n) : n \in \mathbb{N}\}$ .

**Theorem 11.7.13 (Choice on  $\mathbb{N}$ ).**  $\mathbf{UC}_{\mathbb{N}} \equiv_{sW} \mathbf{C}_{\mathbb{N}} \equiv_{sW} \mathbf{C}_{\mathbb{Q}} \equiv_{sW} \mathbf{lim}_{\mathbb{N}} \equiv_{sW} \min^c \equiv_{sW} \max$ .

Here  $\mathbb{Q}$  can be endowed with the discrete or the Euclidean topology. The ordinary Weihrauch degree of  $\mathbf{C}_{\mathbb{N}}$  has some further members that occasionally appear.

**Theorem 11.7.14.**  $\mathbf{CFC}_{\mathbb{N}} \equiv_W \mathbf{C}_{\mathbb{N}} \equiv_W \mathbf{lim}_{\Delta}$  and  $\mathbf{CFC}_{\mathbb{N}} <_{sW} \mathbf{C}_{\mathbb{N}} <_{sW} \mathbf{lim}_{\Delta}$ .

Here the strictness results follow from Proposition 11.6.18 since  $\#\mathbf{CFC}_{\mathbb{N}} = 1$ ,  $\#\mathbf{C}_{\mathbb{N}} = |\mathbb{N}|$  and  $\#\mathbf{lim}_{\Delta} = |\mathbb{N}^{\mathbb{N}}|$ .

An important result related to choice on natural numbers shows that it can not contribute anything to the computation of total fractals if it is applied first (possibly followed by another problem).

**Theorem 11.7.15 (Choice elimination).**  $f \leq_W g * \mathbf{C}_{\mathbb{N}} \implies f \leq_W g$ , for every total fractal  $f$  and every problem  $g$ .

The proof of this theorem is based on the Baire category theorem. In light of Proposition 11.7.5 we obtain the following corollary.

**Corollary 11.7.16 (Separations).**  $\mathbf{CC}_{[0,1]} \not\leq_W \mathbf{C}_{\mathbb{N}}$  and  $\mathbf{PC}_{2^{\mathbb{N}}} \not\leq_W \mathbf{C}_{\mathbb{N}}$ .

We use the identification  $n = \{0, 1, \dots, n-1\}$  for all  $n \in \mathbb{N}$ , and we also consider the finite choice problems  $\mathbf{C}_n$ . It is clear that  $\mathbf{C}_0 \equiv_W \mathbf{0}$  and  $\mathbf{C}_1 \equiv_W \mathbf{1}$ . The particular case of  $\mathbf{C}_2$  is related to LLPO, which is the counterpart of the *lesser limited principle of omniscience* as it is known from constructive analysis.

$$\mathbf{LLPO} : \subseteq 2^{\mathbb{N}} \rightrightarrows \{0, 1\}, \mathbf{LLPO}(p) \ni \begin{cases} 0 & \iff (\forall n) p(2n) = 0 \\ 1 & \iff (\forall n) p(2n+1) = 0 \end{cases}$$

with  $\text{dom}(\mathbf{LLPO}) := \{p \in 2^{\mathbb{N}} : p(k) \neq 0 \text{ for at most one } k\}$ .

**Proposition 11.7.17 (Principles of omniscience).**  $\mathbf{C}_2 \equiv_{sW} \mathbf{LLPO} <_W \mathbf{LPO} <_W \mathbf{C}_{\mathbb{N}}$ .

Likewise, one can define problems  $\text{MLPO}_n$  that are equivalent to  $\mathbf{C}_n$  and problems  $\text{LPO}_n$  that are equivalent to  $\text{ACC}_n$ . These yield an increasing and a decreasing chain of problems, respectively.

**Proposition 11.7.18 (Finite choice).** *For every  $n > 2$  and every  $p \in \mathbb{N}^{\mathbb{N}}$  we obtain  $\text{ACC}_{\mathbb{N}} <_{\mathbb{W}}^p \text{ACC}_{n+1} <_{\mathbb{W}}^p \text{ACC}_n <_{\mathbb{W}}^p \text{ACC}_2 = \mathbf{C}_2 <_{\mathbb{W}}^p \mathbf{C}_n <_{\mathbb{W}}^p \mathbf{C}_{n+1} <_{\mathbb{W}}^p \mathbf{C}_{\mathbb{N}}$ .*

The mere fact that  $\mathbf{C}_{n+1} \not\leq_{\mathbb{W}} \mathbf{C}_n$  holds follows since  $\text{mind}(\mathbf{C}_n) = n - 1$  for all  $n \geq 1$ . While the power of choice increases with the finite cardinality, we can compensate cardinality by sufficiently many parallel copies of  $\mathbf{C}_2$ , as the following result shows.

**Theorem 11.7.19 (Cardinality versus products).**  $\mathbf{C}_{n+1} \leq_{\text{sW}} \mathbf{C}_2^n$  for all  $n \in \mathbb{N}$ .

It is easy to see that also  $\mathbf{C}_2^n \leq_{\text{sW}} \mathbf{C}_{2^n}$  holds. This implies the second equivalence in the following result.

**Proposition 11.7.20 (Compact choice).**  $\mathbf{K}_{\mathbb{N}} \equiv_{\text{sW}} \mathbf{C}_2^* \equiv_{\text{sW}} \mathbf{C}_n^*$  for all  $n \geq 2$ .

We note that  $\mathbf{C}_2^* <_{\mathbb{W}} \mathbf{C}_{\mathbb{N}}$ , since  $\mathbf{C}_{\mathbb{N}}$  is a fractal by Proposition 11.7.5 and hence countably irreducible.

**Corollary 11.7.21 (Compact versus closed choice).**  $\mathbf{K}_{\mathbb{N}} <_{\mathbb{W}} \mathbf{C}_{\mathbb{N}}$ .

We use the minimum function  $\min : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}, p \mapsto \min\{p(n) : n \in \mathbb{N}\}$  in order to express the last result of this subsection.

**Proposition 11.7.22 (Minimum).**  $\text{LPO}^* \equiv_{\text{sW}} \min$ .

### 11.7.3 Choice on the Cantor Space

Choice on the Cantor space  $2^{\mathbb{N}}$  is closely related to weak Kőnig's lemma, which states that every infinite binary tree  $T \subseteq 2^*$  has an infinite path  $p \in 2^{\mathbb{N}}$  (formally, a binary tree is a subset of  $2^*$  closed downward with respect to the partial order induced by the prefix relation). By  $\text{Tr}$  we denote the set of binary trees  $T \subseteq 2^*$  (represented by their characteristic functions  $\chi_T : 2^* \rightarrow 2$ ), and by  $[T]$  we denote the set of infinite paths  $p \in 2^{\mathbb{N}}$  of  $T$ . Now we formalize weak Kőnig's lemma as the problem  $\text{WKL} : \subseteq \text{Tr} \rightrightarrows 2^{\mathbb{N}}, T \mapsto [T]$ , where  $\text{dom}(\text{WKL})$  consists of all infinite binary trees.

It is well known that the map  $[\cdot] : \text{Tr} \rightarrow \mathcal{A}_-(2^{\mathbb{N}}), T \mapsto [T]$  is computable, surjective and has a computable multi-valued right-inverse. This yields the first equivalence in the following theorem.

**Theorem 11.7.23 (Choice on the Cantor space).** *For every rich computably compact computable metric space  $X$ :  $\text{WKL} \equiv_{\text{sW}} \mathbf{C}_{2^{\mathbb{N}}} \equiv_{\text{sW}} \mathbf{C}_X \equiv_{\text{sW}} \widehat{\mathbf{C}}_2 \equiv_{\text{sW}} \widehat{\text{LLPO}}$ .*

In particular,  $\mathbf{C}_{2^{\mathbb{N}}}$  is parallelizable, and the problem of finding a path in a binary tree can be reduced to countably many binary choices, i.e.,  $\text{WKL} \leq_{\text{W}} \widehat{\mathbf{C}}_2$ . In the following corollary we list the choice problem for some important examples of rich computably compact computable metric spaces.

**Corollary 11.7.24.**  $\mathbf{C}_{2^{\mathbb{N}}} \equiv_{\text{sW}} \mathbf{C}_{[0,1]^{\mathbb{N}}} \equiv_{\text{sW}} \mathbf{C}_{[0,1]^n}$  for all  $n \geq 1$ .

Similarly as for choice on natural numbers there is a choice elimination result for  $\mathbf{C}_{2^{\mathbb{N}}}$ . This result can be proved using compactness properties.

**Theorem 11.7.25 (Choice elimination).**  $f \leq_{\text{W}} \mathbf{C}_{2^{\mathbb{N}}} * g \implies f \leq_{\text{W}} g$  for every single-valued problem  $f : X \rightarrow Y$  with a computable metric space  $Y$  and every  $g$ .

This result can also be generalized to admissibly represented spaces  $Y$ . We obtain the following important special case.

**Corollary 11.7.26 (Single-valuedness).**  $f \leq_{\text{W}} \mathbf{C}_{2^{\mathbb{N}}} \implies f$  computable, for all single-valued problems  $f : X \rightarrow Y$  with a computable metric space  $Y$ .

In particular this applies to  $f = \mathbf{UC}_{2^{\mathbb{N}}}$ . Since  $\text{lim}_{\mathbb{N}}$  is a single-valued problem in the equivalence class of  $\mathbf{C}_{\mathbb{N}}$ , we also get the following conclusion.

**Corollary 11.7.27 (Separation).**  $\mathbf{C}_{\mathbb{N}} \not\leq_{\text{W}} \mathbf{C}_{2^{\mathbb{N}}}$ .

The so-called weak weak König's lemma WWKL is WKL restricted to trees  $T$  such that  $\mu([T]) > 0$ . It is easy to see that it is equivalent to  $\text{PC}_{2^{\mathbb{N}}}$ .

**Theorem 11.7.28 (Positive choice on the Cantor space).**

$\text{WWKL} \equiv_{\text{sW}} \text{PC}_{2^{\mathbb{N}}} \equiv_{\text{sW}} \text{PC}_{[0,1]}$ .

One can use a result of Jockusch and Soare [64, Theorem 5.3] that essentially shows that WKL cannot be computed with an advice set of positive measure, in order to separate  $\text{PC}_{2^{\mathbb{N}}}$  and  $\mathbf{C}_{2^{\mathbb{N}}}$ .

**Proposition 11.7.29 (Positive choice versus choice).**  $\text{PC}_{2^{\mathbb{N}}} <_{\text{W}} \mathbf{C}_{2^{\mathbb{N}}}$ .

Since  $\mathbf{C}_2 \leq_{\text{W}} \text{PC}_{2^{\mathbb{N}}} <_{\text{W}} \mathbf{C}_{2^{\mathbb{N}}} \equiv_{\text{W}} \widehat{\mathbf{C}}_2$ , it is clear that  $\text{PC}_{2^{\mathbb{N}}}$  is not parallelizable.

**Corollary 11.7.30 (Parallelizability).**  $\text{PC}_{2^{\mathbb{N}}}$  is not parallelizable and we obtain  $\widehat{\text{PC}}_{2^{\mathbb{N}}} \equiv_{\text{sW}} \mathbf{C}_{2^{\mathbb{N}}}$ .

Also quantitative versions of WWKL have been considered, and by  $\varepsilon$ -WWKL for  $\varepsilon \in (0, 1)$  we denote WWKL restricted to trees with  $\mu([T]) > \varepsilon$ .

**Theorem 11.7.31 (Quantitative WWKL).**  $\varepsilon\text{-WWKL} \leq_{\text{W}} \delta\text{-WWKL} \iff \varepsilon \geq \delta$ .

We continue with the discussion of further special versions of choice related to  $\mathbf{C}_{2^{\mathbb{N}}}$ . It is not obvious at all that connected choice  $\mathbf{CC}_{[0,1]^n}$  is in the same equivalence class as  $\mathbf{C}_{2^{\mathbb{N}}}$  for dimension  $n \geq 2$ .

**Theorem 11.7.32 (Connected choice).**  $\text{CC}_{[0,1]^n} \equiv_{\text{sW}} \text{PWCC}_{[0,1]^{n+1}} \equiv_{\text{sW}} \mathbf{C}_{2^{\mathbb{N}}}$  for  $n \geq 2$ .

The map  $A \mapsto (A \times [0, 1] \times \{0\}) \cup (A \times A \times [0, 1]) \cup ([0, 1] \times A \times \{1\})$  shows that one can map each closed subset  $A \subseteq [0, 1]$  to a pathwise connected closed subset  $B \subseteq [0, 1]^3$ , and given a point in the latter set one can reconstruct a point in the former set. This proves the previous statement on  $\text{PWCC}_{[0,1]^n}$  and  $\text{CC}_{[0,1]^n}$  for  $n \geq 3$ . Only the two-dimensional case needs a more sophisticated argument, and in the case of  $\text{PWCC}_{[0,1]^2}$  the Weihrauch degree is not known.

**Problem 11.7.33 (Pathwise connected choice).** Does  $\text{PWCC}_{[0,1]^2} \equiv_{\text{W}} \mathbf{C}_{2^{\mathbb{N}}}$  hold?

The one-dimensional case of connected choice yields the degree of the intermediate value theorem.

**Theorem 11.7.34 (Intermediate value theorem).**  $\text{CC}_{[0,1]} \equiv_{\text{sW}} \text{IVT}$ .

We mention a fact that was already stated in Example 11.4.3.

**Theorem 11.7.35 (Idempotency).**  $\text{CC}_{[0,1]}$  is not idempotent.

While connected choice is very stable with respect to the dimension of the space, this is not so for convex choice as the following result shows.

**Theorem 11.7.36 (Convex choice).**  $\text{XC}_{[0,1]^n} <_{\text{W}} \text{XC}_{[0,1]^{n+1}}$  for all  $n \in \mathbb{N}$ .

Convex choice is not closed under composition, as the following result shows.

**Theorem 11.7.37 (Composition of convex choice).**  $\text{XC}_{[0,1]^n}$  is not closed under compositional product  $*$  and  $\text{XC}_{[0,1]} * \text{XC}_{[0,1]} \not\leq_{\text{W}} \text{XC}_{[0,1]^n}$  for all  $n \geq 1$ .

We mention that compact choice  $\text{K}_X$  does not lead to anything new on rich computable metric spaces.

**Theorem 11.7.38 (Compact choice).**  $\text{K}_X \equiv_{\text{sW}} \mathbf{C}_{2^{\mathbb{N}}}$  for all rich computable metric spaces  $X$ .

In particular, this implies  $\text{K}_{2^{\mathbb{N}}} \equiv_{\text{sW}} \text{K}_{\mathbb{R}} \equiv_{\text{sW}} \text{K}_{\mathbb{N}^{\mathbb{N}}} \equiv_{\text{sW}} \mathbf{C}_{2^{\mathbb{N}}}$ . We close this section by mentioning that  $\text{CC}_{[0,1]}$  and  $\text{PC}_{2^{\mathbb{N}}}$  are both upper bounds of  $\text{K}_{\mathbb{N}}$ .

**Proposition 11.7.39 (Upper bound of compact choice).**  $\text{K}_{\mathbb{N}} \leq_{\text{W}} \text{CC}_{[0,1]} \sqcap \text{PC}_{2^{\mathbb{N}}}$ .

### 11.7.4 Choice on Euclidean Space

In this section we discuss  $\text{C}_{\mathbb{R}}$  and related problems. The basic observation is that  $\text{C}_{\mathbb{R}}$  can be described with the help of  $\text{C}_{2^{\mathbb{N}}}$  and  $\text{C}_{\mathbb{N}}$  in several different ways.

**Theorem 11.7.40 (Choice on Euclidean space).**  $C_{\mathbb{R}} \equiv_{sW} C_{\mathbb{R}^n} \equiv_{sW} C_{2^{\mathbb{N}} \times \mathbb{N}} \equiv_{sW} C_{2^{\mathbb{N}}} \times C_{\mathbb{N}} \equiv_{sW} C_{2^{\mathbb{N}}} \star C_{\mathbb{N}} \equiv_{sW} C_{\mathbb{N}} \star C_{2^{\mathbb{N}}}$  for all  $n \geq 1$ .

The results regarding  $\star$  follow with the help of Theorem 11.7.11. We have deliberately used the symbol  $\star$  and not  $*$ , since the degrees with  $\star$  are cylinders and hence we obtain strong equivalences. Theorem 11.7.40 shows that Theorem 11.7.25 is applicable to  $C_{\mathbb{R}}$ , and we obtain the following conclusion.

**Corollary 11.7.41 (Single-valuedness).**  $f \leq_W C_{\mathbb{R}} \implies f \leq_W C_{\mathbb{N}}$ , for all single-valued problems  $f : X \rightarrow Y$  with a computable metric space  $Y$ .

This result applies in particular to  $UC_{\mathbb{R}}$  and implies  $UC_{\mathbb{R}} \equiv_W C_{\mathbb{N}}$ . Now we discuss an important upper bound on  $C_{\mathbb{R}}$ . The low basis theorem of Jockusch and Soare states that every computable infinite binary tree has a low path. We recall that  $p \in \mathbb{N}^{\mathbb{N}}$  is called *low* if  $p' \leq_T \theta'$  holds, i.e., if the halting problem relative to  $p$  is not more difficult than the ordinary halting problem. Lowness is represented by the problem  $L := J^{-1} \circ \text{lim}$  since  $p$  is low if and only if there is a computable  $q$  such that  $p = L(q)$ . It is clear that  $L <_W \text{lim}$  (since  $J^{-1}$  is computable and not every limit computable  $p$  is low). The following result can be seen as a uniform version of the low basis theorem.

**Theorem 11.7.42 (Uniform low basis theorem).**  $C_{\mathbb{R}} <_{sW} L$ .

The strictness follows for instance from Corollary 11.7.41 since  $L$  is single-valued. We mention an interesting algebraic example of how infima and suprema of the degrees of  $C_{\mathbb{N}}$  and  $C_{2^{\mathbb{N}}}$  interact.

*Example 11.7.43.* We obtain

1.  $(C_{2^{\mathbb{N}}} \sqcap C_{\mathbb{N}}) * (C_{2^{\mathbb{N}}} \sqcup C_{\mathbb{N}}) \equiv_W (C_{2^{\mathbb{N}}} \sqcap C_{\mathbb{N}}) \times (C_{2^{\mathbb{N}}} \sqcup C_{\mathbb{N}}) \equiv_W C_{2^{\mathbb{N}}} \sqcup C_{\mathbb{N}}$ .
2.  $(C_{2^{\mathbb{N}}} \sqcup C_{\mathbb{N}}) * (C_{2^{\mathbb{N}}} \sqcap C_{\mathbb{N}}) \equiv_W C_{2^{\mathbb{N}}} * C_{\mathbb{N}} \equiv_W C_{2^{\mathbb{N}}} \times C_{\mathbb{N}}$ .

We note that  $C_{2^{\mathbb{N}}} \sqcup C_{\mathbb{N}} <_W C_{2^{\mathbb{N}}} \times C_{\mathbb{N}} \equiv_W C_{\mathbb{R}}$  since the right-hand degree is join-irreducible and since  $C_{2^{\mathbb{N}}}$  and  $C_{\mathbb{N}}$  are incomparable. We formulate a counterpart of Theorem 11.7.40 for  $PC_{\mathbb{R}}$ .

**Theorem 11.7.44 (Positive choice on Euclidean space).**  $PC_{\mathbb{R}} \equiv_{sW} PC_{2^{\mathbb{N}} \times \mathbb{N}} \equiv_{sW} PC_{2^{\mathbb{N}}} \times C_{\mathbb{N}} \equiv_W PC_{2^{\mathbb{N}}} * C_{\mathbb{N}} \equiv_W C_{\mathbb{N}} * PC_{2^{\mathbb{N}}}$  for all  $n \geq 1$ .

We mention that in this case we cannot simply replace  $*$  by  $\star$  and  $\equiv_W$  by  $\equiv_{sW}$ , since  $PC_{\mathbb{R}}$  is not a cylinder. We note that  $C_{\mathbb{N}} <_W C_{\mathbb{R}} <_W \text{lim} \equiv_W \widehat{C_{\mathbb{N}}}$  implies the following.

**Corollary 11.7.45 (Parallelizability).**  $C_{\mathbb{R}}$  and  $PC_{\mathbb{R}}$  are not parallelizable, and we obtain  $\widehat{PC_{\mathbb{R}}} \equiv_{sW} \widehat{C_{\mathbb{R}}} \equiv_{sW} \text{lim}$ .

### 11.7.5 Choice on the Baire Space

Choice on the Baire space is the upper bound of all choice problems of complete computable metric spaces. In fact, we obtain the following.

**Theorem 11.7.46 (Non- $\sigma$ -compact spaces).**  $C_{\mathbb{N}^{\mathbb{N}}} \equiv_W^p C_X$  for some oracle  $p \in \mathbb{N}^{\mathbb{N}}$  if  $X$  is a separable complete metric space that is not  $\sigma$ -compact.

We list a number of choice problems that fall into the equivalence class of  $C_{\mathbb{N}^{\mathbb{N}}}$ . We assume that these spaces are represented as computable metric spaces in the standard way.

**Theorem 11.7.47 (Baire space).**  $C_{\mathbb{N}^{\mathbb{N}}} \equiv_{sW} C_{\mathbb{R}^{\mathbb{N}}} \equiv_{sW} C_{\mathbb{R} \setminus \mathbb{Q}} \equiv_{sW} C_{\ell_p} \equiv_{sW} C_{\mathcal{C}[0,1]}$  for all computable  $p \geq 1$ .

Also the single-valued problems below  $C_{\mathbb{N}^{\mathbb{N}}}$  have a very natural characterization.

**Theorem 11.7.48 (Single-valuedness).**  $f \leq_W C_{\mathbb{N}^{\mathbb{N}}} \iff f$  is effectively Borel measurable, for  $f : X \rightarrow Y$  on complete computable metric spaces.

Similarly to Theorem 11.6.5 this result can be relativized. We mention that it is easy to see that  $C_{\mathbb{N}^{\mathbb{N}}}$  is parallelizable.

**Proposition 11.7.49 (Parallelizability).**  $C_{\mathbb{N}^{\mathbb{N}}}$  is strongly parallelizable.

We briefly mention  $UC_{\mathbb{N}^{\mathbb{N}}}$ , the unique version of choice on the Baire space. It is easy to see that  $\lim_{<_W} UC_{\mathbb{N}^{\mathbb{N}}}$  holds. It follows from a basis theorem of Kreisel that  $UC_{\mathbb{N}^{\mathbb{N}}}$  is strictly weaker than  $C_{\mathbb{N}^{\mathbb{N}}}$ .

**Proposition 11.7.50 (Unique choice).**  $\lim^{(n)} <_W UC_{\mathbb{N}^{\mathbb{N}}} <_W C_{\mathbb{N}^{\mathbb{N}}}$  for all  $n \in \mathbb{N}$ .

We close with the following characterization of positive choice on the Baire space.

**Theorem 11.7.51 (Positive choice).**  $PC_{\mathbb{N}^{\mathbb{N}}} \equiv_{sW} PC_{\mathbb{R}}$ .

### 11.7.6 Jumps of Choice

In order to characterize the jump of choice we need the *cluster point problem*  $CL_X : \subseteq X^{\mathbb{N}} \rightrightarrows X, (x_n)_{n \in \mathbb{N}} \mapsto \{x \in X : x \text{ is a cluster point of } (x_n)_{n \in \mathbb{N}}\}$ . This problem fully characterizes the jump of  $C_X$  on computable metric spaces  $X$ . If we restrict  $CL_X$  to such sequences  $(x_n)_{n \in \mathbb{N}}$  whose range  $\{x_n : n \in \mathbb{N}\}$  has a compact closure, then we denote it by  $BWT_X : \subseteq X^{\mathbb{N}} \rightrightarrows X$  since it can be seen as a problem that realizes the Bolzano-Weierstraß theorem.

**Theorem 11.7.52 (Jump of choice).**  $C'_X \equiv_{sW} CL_X$  and  $K'_X \equiv_{sW} BWT_X$  for all computable metric spaces  $X$ .



$\text{BWT}_2 = \text{CL}_2$  is also known as the *infinite pigeonhole problem*. Many properties of problems can be transferred to jumps. However, this is often not so for properties that involve compositional products. We recall that by  $f^{[n]}$  we denote the  $n$ -fold compositional product of  $f$  with itself and by  $f^{(n)}$  the  $n$ -fold jump.

**Theorem 11.7.53 (Composition).** *We obtain:*

1.  $\mathbf{C}'_{\mathbb{N}} * \mathbf{C}'_{\mathbb{N}} \equiv_{\mathbf{W}} \mathbf{C}'_{\mathbb{N}}$ .
2.  $\mathbf{C}'_{2^{\mathbb{N}}} * \mathbf{C}'_{2^{\mathbb{N}}} \equiv_{\mathbf{W}} \mathbf{C}''_{2^{\mathbb{N}}}$  and more generally  $\mathbf{C}'_{2^{\mathbb{N}}}^{[n]} \equiv_{\mathbf{W}} \mathbf{C}_{2^{\mathbb{N}}}^{(n)}$  for all  $n \geq 1$ .
3.  $\text{PC}'_{2^{\mathbb{N}}} * \text{PC}'_{2^{\mathbb{N}}} \equiv_{\mathbf{W}} \text{PC}'_{\mathbb{R}} * \text{PC}'_{\mathbb{R}} \equiv_{\mathbf{W}} \text{PC}'_{\mathbb{R}}$ .

It is perhaps surprising that compositions behave very differently in the probabilistic case and in the non-probabilistic case. The difference between  $\mathbf{C}_{2^{\mathbb{N}}}$  and  $\text{PC}_{2^{\mathbb{N}}}$  is also underlined by the third statement in the following result that strengthens the negative statement of Proposition 11.7.29.

**Theorem 11.7.54 (Separations).** *We obtain for all  $n \in \mathbb{N}$ :*

1.  $\mathbf{C}_2^{(n+1)} \not\leq_{\mathbf{W}} \lim^{(n)}$ .
2.  $\text{LPO}^{(n)} \not\leq_{\mathbf{W}} \mathbf{C}_{2^{\mathbb{N}}}^{(n)}$ .
3.  $\mathbf{C}_{2^{\mathbb{N}}} \not\leq_{\mathbf{W}} \text{PC}_{2^{\mathbb{N}}}^{(n)}$ .

It follows from the first statement that  $\mathbf{C}_2 \leq_{\mathbf{W}} \text{PC}_{2^{\mathbb{N}}} \leq_{\mathbf{W}} \mathbf{C}_{2^{\mathbb{N}}}$  all climb up the Borel hierarchy one step with every jump. This statement even holds relative to any oracle, and hence  $\mathbf{C}_2^{(n)}$  is not  $\Sigma_{n+1}^0$ -measurable. We obtain the following alternating hierarchies.

**Theorem 11.7.55 (Alternating hierarchies).** *For all  $n \in \mathbb{N}$  we obtain*

1.  $\mathbf{C}_2^{(n)} <_{\mathbf{W}} \text{LPO}^{(n)} <_{\mathbf{W}} \mathbf{C}_2^{(n+1)}$ ,
2.  $\mathbf{K}_{\mathbb{N}}^{(n)} <_{\mathbf{W}} \mathbf{C}_{\mathbb{N}}^{(n)} <_{\mathbf{W}} \mathbf{K}_{\mathbb{N}}^{(n+1)}$ ,
3.  $\mathbf{C}_{2^{\mathbb{N}}}^{(n)} <_{\mathbf{W}} \lim^{(n)} <_{\mathbf{W}} \mathbf{C}_{2^{\mathbb{N}}}^{(n+1)}$ .

*Analogous statements hold with  $\leq_{\text{sW}}$  in place of  $\leq_{\mathbf{W}}$ .*

Choice on the Baire space is an example of a choice problem that is stable under jump.

**Theorem 11.7.56 (Baire space).**  $\mathbf{C}'_{\mathbb{N}^{\mathbb{N}}} \equiv_{\text{sW}} \mathbf{C}_{\mathbb{N}^{\mathbb{N}}}$  and  $\text{UC}'_{\mathbb{N}^{\mathbb{N}}} \equiv_{\text{sW}} \text{UC}_{\mathbb{N}^{\mathbb{N}}}$ .

The following example shows that a straightforward jump inversion theorem does not hold in the Weihrauch lattice.

*Example 11.7.57.*  $\lim <_{\mathbf{W}} \mathbf{C}'_{2^{\mathbb{N}}} \sqcup \mathbf{C}'_{\mathbb{N}}$ , but there is no problem  $f$  with  $f' \equiv_{\mathbf{W}} \mathbf{C}'_{2^{\mathbb{N}}} \sqcup \mathbf{C}'_{\mathbb{N}}$ .

The latter holds since  $f'$  is join-irreducible and  $\mathbf{C}'_{2^{\mathbb{N}}}$  and  $\mathbf{C}'_{\mathbb{N}}$  are incomparable. While Theorem 11.7.54 shows that  $\mathbf{C}_{2^{\mathbb{N}}}$  has no jump of positive choice as upper bound, this is different for  $\text{CC}_{[0,1]}$  as the following result shows.

**Proposition 11.7.58 (Upper bounds).**  $\text{CC}_{[0,1]} \sqcup \text{C}_{\mathbb{N}} \leq_{\text{w}} \text{PC}'_{2^{\mathbb{N}}} \sqcap \text{C}'_{\mathbb{N}}$ .

This result is contrasted by  $\text{CC}_{[0,1]} \not\leq_{\text{w}} \text{PC}_{2^{\mathbb{N}}} \sqcup \text{C}_{\mathbb{N}}$ , which holds because  $\text{CC}_{[0,1]}$  is not reducible to any of the problems on the right-hand side and since it is join-irreducible by Propositions 11.7.5 and 11.4.11.

For  $\frac{1}{2}$ -WWKL we can improve the statement that follows from Corollary 11.7.26 by the following result, which can be proved with a majority vote argument.

**Theorem 11.7.59 (Single-valuedness).**  $f \leq_{\text{w}} \frac{1}{2}$ -WWKL<sup>(n)</sup>  $\implies f$  computable, for all single-valued problems  $f : X \rightarrow Y$  with a computable metric space  $Y$  and  $n \in \mathbb{N}$ .

We note that  $\frac{1}{2}$ -WWKL cannot be replaced by WWKL in this result, since  $\lim_{\mathbb{N}} \leq_{\text{w}} \text{WWKL}'$  holds as a consequence of Proposition 11.7.58.

### 11.7.7 All-or-Unique Choice

We briefly discuss all-or-unique choice in this section. The problem  $\text{AoUC}_{[0,1]}$  is located between LLPO and LPO and related to *robust division*, which is defined as the problem  $\text{RDIV} : [0, 1] \times [0, 1] \rightrightarrows [0, 1]$  with  $\text{RDIV}(x, y) := \{\frac{x}{\max(x, y)}\}$  if  $y \neq 0$  and  $\text{RDIV}(x, y) := [0, 1]$  otherwise. We now obtain the following characterization.

**Proposition 11.7.60 (All-or-unique choice).**  $\text{C}_2 <_{\text{w}} \text{AoUC}_{[0,1]} \equiv_{\text{sw}} \text{RDIV} <_{\text{w}} \text{LPO}$ .

In some respects  $\text{AoUC}_{[0,1]}$  is closer to  $\text{C}_2$  than to LPO, at least with respect to the following upper bounds.

**Theorem 11.7.61 (Upper bound).**  $\text{AoUC}_{[0,1]} \leq_{\text{w}} \text{CC}_{[0,1]} \sqcap \text{PC}_{2^{\mathbb{N}}}$ .

In the diagram in Figure 11.2  $\text{AoUC}_{[0,1]}$  would be in a similar position to  $\text{K}_{\mathbb{N}}$ , however it is incomparable to  $\text{K}_{\mathbb{N}}$  since  $\text{AoUC}_{[0,1]}$  is countably irreducible and  $\text{K}_{\mathbb{N}} \not\leq_{\text{w}} \text{LPO}$ . We continue with a number of separation results that involve  $\text{AoUC}_{[0,1]}$ .

**Theorem 11.7.62 (Separation).**

1.  $\text{XC}_{[0,1]} * \text{AoUC}_{[0,1]} \not\leq_{\text{w}} \text{XC}_{[0,1]}^n$  for all  $n \in \mathbb{N}$ .
2.  $\text{C}_2 * \text{AoUC}_{[0,1]} \not\leq_{\text{w}} \text{AoUC}_{[0,1]}^*$ .
3.  $\text{C}_2 \times \text{AoUC}_{[0,1]} \not\leq_{\text{w}} \text{CC}_{[0,1]}$ .

These separation results have a number of interesting consequences. The first statement implies Theorem 11.7.37, and the third statement implies Theorem 11.7.35. Since  $\text{C}_2 \leq_{\text{w}} \text{AoUC}_{[0,1]}$  we can conclude the following from the second statement.

**Corollary 11.7.63 (Composition).**  $\text{AoUC}_{[0,1]}^*$  is not closed under compositional product.

Surprisingly, a composition of  $\text{AoUC}_{[0,1]}^*$  with itself yields a new problem that is closed under compositional product.

**Theorem 11.7.64 (Double composition).** We obtain:

$$\text{AoUC}_{[0,1]}^* * \text{AoUC}_{[0,1]}^* \equiv_{\text{w}} \text{AoUC}_{[0,1]}^* * \text{AoUC}_{[0,1]}^* * \text{AoUC}_{[0,1]}^*.$$

## Bibliographic Remarks

The study of choice problems in the Weihrauch lattice has been started by Gherardi and Marcone [46] and Brattka and Gherardi [18, 17]. Indirectly, Weihrauch [115] already studied choice problems in the form of versions of MLPO and LPO; Propositions 11.7.17 and 11.7.18 are due to him. Theorem 11.7.19 is due to Pauly [91]. Non-deterministically computable functions have been introduced to computable analysis by Ziegler [119, 118], and the relation to choice problems has been established by Brattka, de Brecht and Pauly [14], who also started to study choice problems more systematically. Many results in this section are due to them. The study of positive choice was initiated by Brattka and Pauly [28] and more systematically continued by Brattka, Gherardi and Hölzl [19]. This subject was independently studied by Dorais, Dzhafarov, Hirst, Mileti and Shafer [38], who also introduced the quantitative version of WWKL and proved Theorem 11.7.31. Connected choice was mostly studied by Brattka, Le Roux, Miller and Pauly [27] and convex choice by Le Roux and Pauly [79], who also proved Theorem 11.7.15. The study of jumps of choice is due to Brattka, Gherardi and Marcone [21], and the statement on  $K_X$  in Theorem 11.7.52 is due to Brattka, Cettolo, Gherardi, Marcone and Schröder [15]. The statement on positive choice in Theorem 11.7.53 is due to Bienvenu and Kuyper [3]. Pauly started the study of all-or-unique choice [91, 93], and some of the separation results in this regard are due to Kihara and Pauly [69]. Theorem 11.7.37 is due to Kihara [68].

## 11.8 Classifications

In this section we present results on the classification of theorems. Most of these theorems originate from analysis. We interpret theorems as problems as explained after Definition 11.1.4. For many theorems one can derive upper bounds using the following observation.

**Theorem 11.8.1 (Upper bounds).** *Let  $X, Y$  be represented spaces and  $A \subseteq X \times Y$  co-c.e. closed. If  $(\forall x \in X)(\exists y \in Y)(x, y) \in A$  holds, then the corresponding problem  $F : X \rightrightarrows Y, x \mapsto \{y \in Y : (x, y) \in A\}$  satisfies  $F \leq_W C_Y$ .*

In combination with Proposition 11.7.7 one can thus derive upper bounds on theorems by exploiting topological properties of  $Y$ . We essentially group our classifications according to related choice problems.

The equivalence class of choice on natural numbers contains many theorems that are typically proved with the help of the Baire category theorem.

**Theorem 11.8.2 (Choice on the natural numbers).** *The following are all Weihrauch equivalent to each other:*

1. Choice on natural numbers  $C_{\mathbb{N}}$ .
2. The Baire category theorem  $BCT_1$ .
3. Banach's inverse mapping theorem  $BIM_{\ell_2, \ell_2}$ .
4. The open mapping theorem for  $\ell_2$ .
5. The closed graph theorem for  $\ell_2$ .
6. The uniform boundedness theorem on non-singleton computable Banach spaces.
7. The Lebesgue covering lemma for  $[0, 1]$ .

8. *The partial identity from continuous functions to analytic functions.*

In most cases these theorems are interpreted as problems in a straightforward way. We only provide some examples and refer the reader to the references for exact definitions. For instance, Banach's inverse mapping theorem on computable Banach spaces  $X, Y$  is formalized as  $\mathbf{BIM}_{X,Y} : \subseteq \mathcal{C}(X, Y) \rightarrow \mathcal{C}(Y, X), T \mapsto T^{-1}$ , restricted to bijective, linear, bounded  $T$ . We always obtain  $\mathbf{BIM}_{X,Y} \leq_w \mathbf{C}_{\mathbb{N}}$ , and in the case of  $X = Y = \ell_2$  the theorem actually attains the maximal complexity. For finite-dimensional  $X, Y$  it is, however, computable. Similar remarks apply in the cases of the open mapping theorem and the closed graph theorem. The Baire category theorem and the uniform boundedness theorem are even equivalent to  $\mathbf{C}_{\mathbb{N}}$  for all complete computable metric spaces and non-singleton computable Banach spaces, respectively.

In the case of some theorems it can happen that one logical formulation of the theorem and the contrapositive formulation carry different computational content. In such a situation it might not always be clear which form is more natural, and perhaps both forms have applications. Such an example is the Baire category theorem, which we can formalize at least in two ways. By  $A^\circ$  we denote the *interior* of the set  $A$ .

1.  $\mathbf{BCT}_0 : \subseteq \mathcal{A}_-(X)^{\mathbb{N}} \rightrightarrows X, (A_n)_{n \in \mathbb{N}} \mapsto \{x \in X : x \notin \bigcup_{n=0}^{\infty} A_n\}$ ,  
with  $\text{dom}(\mathbf{BCT}_0) := \{(A_n)_{n \in \mathbb{N}} : A_n^\circ = \emptyset\}$ .
2.  $\mathbf{BCT}_1 : \subseteq \mathcal{A}_-(X)^{\mathbb{N}} \rightrightarrows \mathbb{N}, (A_n)_{n \in \mathbb{N}} \mapsto \{n \in \mathbb{N} : A_n^\circ \neq \emptyset\}$ ,  
with  $\text{dom}(\mathbf{BCT}_1) := \{(A_n)_{n \in \mathbb{N}} : X = \bigcup_{n=0}^{\infty} A_n\}$ .

While  $\mathbf{BCT}_1$  is in the equivalence class of  $\mathbf{C}_{\mathbb{N}}$ , it is easy to see that  $\mathbf{BCT}_0$  is computable. Nevertheless the jump of  $\mathbf{BCT}_0$  has interesting applications that we mention below. Similarly to the Baire category theorem, also the Heine-Borel covering theorem can be formalized in at least two ways. Here  $\mathcal{O}(X)$  denotes the set of open subsets of  $X$  seen as the complements of the elements of  $\mathcal{A}_-(X)$ , i.e., every open set is represented by an enumeration of basic open balls whose union coincides with the set.

1.  $\mathbf{HBC}_0 : \subseteq \mathcal{O}([0, 1])^{\mathbb{N}} \rightrightarrows \mathbb{N}, (U_n)_{n \in \mathbb{N}} \mapsto \{k \in \mathbb{N} : [0, 1] \subseteq \bigcup_{n=0}^k U_n\}$ ,  
with  $\text{dom}(\mathbf{HBC}_0) := \{(U_n)_{n \in \mathbb{N}} : [0, 1] \subseteq \bigcup_{n=0}^{\infty} U_n\}$ .
2.  $\mathbf{HBC}_1 : \subseteq \mathcal{O}([0, 1])^{\mathbb{N}} \rightrightarrows [0, 1], (U_n)_{n \in \mathbb{N}} \mapsto \{x \in [0, 1] : x \notin \bigcup_{n=0}^{\infty} U_n\}$ ,  
with  $\text{dom}(\mathbf{HBC}_1) := \{(U_n)_{n \in \mathbb{N}} : (\forall k) [0, 1] \not\subseteq \bigcup_{n=0}^k U_n\}$ .

Once again, it is easy to see that  $\mathbf{HBC}_0$  is computable, and  $\mathbf{HBC}_1$  is in the equivalence class of choice on the Cantor space.

**Theorem 11.8.3 (Choice on the Cantor space).** *The following are all strongly Weihrauch equivalent to each other:*

1. *Choice on the Cantor space*  $\mathbf{C}_{2^{\mathbb{N}}}$ .
2. *Weak König's lemma*  $\mathbf{WKL}$ .
3. *The Hahn-Banach theorem*.
4. *The Heine-Borel covering theorem*  $\mathbf{HBC}_1$ .
5. *The theorem of the maximum*  $\mathbf{MAX}$ .

6. The Brouwer fixed-point theorem  $\text{BFT}_n$  for dimension  $n \geq 2$ .
7. The Brouwer fixed-point theorem  $\text{BFT}_\infty$  for the Hilbert cube  $[0, 1]^\mathbb{N}$ .
8. Finding connectedness components of sets  $A \subseteq [0, 1]^n$  for  $n \geq 1$ .
9. The parallelization  $\widehat{\text{IVT}}$  of the intermediate value theorem.
10. Determinacy of Gale-Stewart games in  $2^\mathbb{N}$  with closed winning sets.

In the case of the Hahn-Banach theorem the underlying separable Banach space is part of the input information. No space of maximal complexity is known in this case. For certain spaces (such as computable Hilbert spaces) the Hahn-Banach theorem is computable. For two further theorems mentioned above we provide formalizations as problems.

1.  $\text{MAX} : \mathcal{C}[0, 1] \rightrightarrows \mathbb{R}, f \mapsto \{x \in [0, 1] : f(x) = \max f([0, 1])\}$ .
2.  $\text{BFT}_n : \mathcal{C}([0, 1]^n, [0, 1]^n) \rightrightarrows [0, 1]^n, f \mapsto \{x \in [0, 1]^n : f(x) = x\}$ .

The Brouwer fixed-point theorem of dimension  $n = 1$  is equivalent to the intermediate value theorem, i.e.,  $\text{BFT}_1 \equiv_{\text{sw}} \text{IVT} \equiv_{\text{sw}} \text{CC}_{[0,1]}$ . We note that classifications such as the one in Theorem 11.8.3 lead to simple proofs of classically known non-uniform results in computable analysis. We mention some examples.

**Corollary 11.8.4 (Non-uniform results).**

1. There exists an infinite binary tree without computable paths (Kleene [70]).
2. There is a computable function  $f : [0, 1] \rightarrow \mathbb{R}$  that attains its maximum only at non-computable points  $x \in [0, 1]$  (Lacombe [77, Theorems VI and VII] and Specker [109]).
3. There is a computable function  $f : [0, 1]^2 \rightarrow [0, 1]^2$  that has no computable fixed-point  $x \in [0, 1]^2$  (Orevkov [88] and Baigger [2]).
4. There is a computable sequence  $(f_n)_{n \in \mathbb{N}}$  of functions  $f_n : [0, 1] \rightarrow \mathbb{R}$  with  $f_n(0) \cdot f_n(1) < 0$  for all  $n \in \mathbb{N}$  such that there is no computable sequence  $(x_n)_{n \in \mathbb{N}}$  with  $f_n(x_n) = 0$  (Pour-El and Richards [101, Example 8a]).

Once one has one of these negative results, all the others follow immediately by Theorem 11.8.3. On the other hand, also positive non-uniform results can be derived from Theorem 11.8.3. For instance, every computable function  $f : [0, 1]^2 \rightarrow [0, 1]^2$  has a low fixed-point. Such non-uniform results hold analogously for other classifications presented here, but we are not going to discuss them in detail.

A theorem that is often proved with the help of the Brouwer fixed-point theorem is the Nash equilibria existence theorem. Its computational content is significantly weaker than that of the Brouwer fixed-point theorem.

**Theorem 11.8.5 (All-or-unique choice).** *The following are strongly Weihrauch equivalent to each other:*

1. The finite parallelization  $\text{AoUC}_{[0,1]}^*$  of all-or-unique choice.
2. The Nash equilibria existence theorem  $\text{NASH}$ .

Here  $\text{NASH}_{n,m} : \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n} \Rightarrow \mathbb{R}^n \times \mathbb{R}^m$  is the map that maps a bi-matrix game  $(A, B)$  to a pair of strategies that form a Nash equilibrium of  $(A, B)$ , and  $\text{NASH} := \bigsqcup_{n,m \in \mathbb{N}} \text{NASH}_{n,m}$ .

The equivalence class of choice on Euclidean space contains a theorem that we mention in the following result without further definitions.

**Theorem 11.8.6 (Choice on Euclidean space).** *The following are Weihrauch equivalent to each other:*

1. Choice on Euclidean space  $\text{C}_{\mathbb{R}}$ .
2. Frostman's lemma on the existence of measures.

The Vitali covering theorem is a theorem that has even been studied in three different logical versions. We consider  $\text{Int} := (\mathbb{Q}^2)^{\mathbb{N}}$  as the set of sequences  $\mathcal{I} = (I_n)_{n \in \mathbb{N}}$  of rational intervals  $I_n = (a, b)$ . We say that  $\mathcal{I}$  is a *Vitali cover* of a set  $A \subseteq \mathbb{R}$  if for every  $x \in A$  and  $\varepsilon > 0$  there is some  $n \in \mathbb{N}$  with  $x \in I_n$  and  $\text{diam}(I_n) < \varepsilon$ . We write  $\mathcal{J} \sqsubseteq \mathcal{I}$  if  $\mathcal{J}$  is a subsequence of  $\mathcal{I}$  of pairwise disjoint intervals. We consider the following three formalizations of the Vitali covering theorem:

1.  $\text{VCT}_0 : \subseteq \text{Int} \Rightarrow \text{Int}, \mathcal{I} \mapsto \{\mathcal{J} : \mathcal{J} \sqsubseteq \mathcal{I} \text{ with } \mu([0, 1] \setminus \bigcup \mathcal{J}) = 0\}$  and  $\text{dom}(\text{VCT}_0)$  contains all  $\mathcal{I} \in \text{Int}$  that are Vitali covers of  $[0, 1]$ .
2.  $\text{VCT}_1 : \subseteq \text{Int} \Rightarrow [0, 1], \mathcal{I} \mapsto [0, 1] \setminus \bigcup \mathcal{I}$  and  $\text{dom}(\text{VCT}_1)$  contains all  $\mathcal{I} \in \text{Int}$  that are Vitali covers of  $[0, 1]$  and without a  $\mathcal{J} \sqsubseteq \mathcal{I}$  with  $\mu([0, 1] \setminus \bigcup \mathcal{J}) = 0$ .
3.  $\text{VCT}_2 : \subseteq \text{Int} \Rightarrow [0, 1], \mathcal{I} \mapsto \{x \in [0, 1] : (\exists \varepsilon > 0)(\forall n)(x \notin I_n \text{ or } \text{diam}(I_n) \geq \varepsilon)\}$  and  $\text{dom}(\text{VCT}_2)$  contains all  $\mathcal{I}$  without a  $\mathcal{J} \sqsubseteq \mathcal{I}$  with  $\mu([0, 1] \setminus \bigcup \mathcal{J}) = 0$ .

It turns out that  $\text{VCT}_0$  is computable and  $\text{VCT}_1$  and  $\text{VCT}_2$  are equivalent to different versions of positive choice.

**Theorem 11.8.7 (Positive choice).** *The following are all strongly Weihrauch equivalent to each other:*

1. Positive choice on the Cantor space  $\text{PC}_{2^{\mathbb{N}}}$ .
2. Weak weak König's Lemma  $\text{WWKL}$ .
3. The Vitali covering theorem  $\text{VCT}_1$ .

*The following are strongly Weihrauch equivalent to each other:*

1. Positive choice on Euclidean space  $\text{PC}_{\mathbb{R}}$ .
2. The Vitali covering theorem  $\text{VCT}_2$ .

Convex choice is equivalent to the Browder-Göhde-Kirk fixed-point theorem, which is formalized as  $\text{BGK}_K : \subseteq \mathcal{C}(K, K) \Rightarrow K, f \mapsto \{x \in K : f(x) = x\}$ , where  $K \subseteq H$  is compact and convex,  $H$  is a computable Hilbert space and  $\text{dom}(\text{BGK}_K)$  consists of all non-expansive continuous maps  $f : K \rightarrow K$ . More general versions of the theorem have been studied, but for simplicity we state only this basic result.

**Theorem 11.8.8 (Convex choice).** *Let  $H$  be a computable Hilbert space and  $K \subseteq H$  convex and computably compact. The following are Weihrauch equivalent to each other:*

1. Convex choice  $\text{XC}_K$ .
2. The Browder-Göhde-Kirk fixed-point theorem  $\text{BGK}_K$  on  $K$ .

Another important equivalence class is that of the limit map.

**Theorem 11.8.9 (The limit).** *The following are Weihrauch equivalent to each other:*

1. The limit map  $\text{lim}$  on the Baire space (or every other rich computable metric space).
2. The monotone convergence theorem  $\text{MCT} : \subseteq \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}, (x_n)_{n \in \mathbb{N}} \mapsto \sup_{n \in \mathbb{N}} x_n$ .
3. The operator of differentiation  $d : \subseteq \mathcal{C}[0, 1] \rightarrow \mathcal{C}[0, 1], f \mapsto f'$ .
4. The Fréchet-Riesz representation theorem for  $\ell_2$ .
5. The Radon-Nikodym theorem.
6. The parallelization  $\widehat{\text{BIM}}$  of Banach's inverse mapping theorem.
7. Finding a basis of a countable vector space.
8. Finding a connected component of a countable graph.
9. The partial identity from infinitely differentiable functions to Schwartz functions.

Of course, the Banach inverse mapping theorem can be replaced by any other problem from Theorem 11.8.2. We mention that the reduction  $\text{lim} \leq_w d$  follows easily with Theorem 11.6.21. Several theorems also fall into the equivalence class of the jump of choice on the Cantor space. Here  $\text{KL}$  is defined as  $\text{WKL}$  but for finitely branching trees  $T \subseteq \mathbb{N}^*$ .

**Theorem 11.8.10 (Jump of choice on the Cantor space).** *The following are all strongly Weihrauch equivalent to each other:*

1. The jump  $\text{C}'_{2^{\mathbb{N}}}$  of choice on the Cantor space.
2. König's lemma  $\text{KL}$ .
3. The Bolzano-Weierstraß theorem  $\text{BWT}_{\mathbb{R}}$  on Euclidean space.
4. The Arzelá-Ascoli theorem for functions  $f : [0, 1] \rightarrow [0, 1]$ .
5. Determinacy of Gale-Stewart games in  $2^{\mathbb{N}}$  with winning sets that are differences of open sets.

A natural problem that is known to be equivalent to higher jumps of choice on the Cantor space is the parallelization of Ramsey's theorem. We summarize some results on this theorem.  $\text{RT}_n^k : k^{[\mathbb{N}]^n} \rightrightarrows 2^{\mathbb{N}}$  denotes the problem that maps every coloring  $c : [\mathbb{N}]^n \rightarrow k$  (of the  $n$ -element subsets of  $\mathbb{N}$  with  $k$  colors) to an infinite set  $H \subseteq \mathbb{N}$  that is homogeneous for  $c$ .

**Theorem 11.8.11 (Ramsey's theorem).**  $\text{C}_2^{(n)} <_w \text{RT}_k^n <_w \text{RT}_{k+1}^n <_w \text{C}_{2^{\mathbb{N}}}^{(n)}$  for all  $n, k \geq 2$ . The reductions also hold in the case  $n = 1$ , but the first one is not strict in this case.

This result can be proved with the help of the squashing theorem (Theorem 11.4.6). Since  $\text{C}_{2^{\mathbb{N}}}^{(n)}$  is the parallelization of  $\text{C}_2^{(n)}$  we obtain the following corollary.

**Corollary 11.8.12 (Ramsey's theorem).**  $C_{2^{\mathbb{N}}}^{(n)} \equiv_W \widehat{RT}_k^n$  for all  $n \geq 1$  and  $k \geq 2$ .

Higher levels of the Weihrauch lattices are not yet all too well explored. This is currently a topic of further research, and we mention one result along these lines.

**Theorem 11.8.13 (Choice on the Baire space).** *The following are Weihrauch equivalent to each other:*

1. Choice on the Baire space  $C_{\mathbb{N}^{\mathbb{N}}}$ .
2. The perfect subtree theorem.

At the end of this section we demonstrate how some problems from computability theory can be classified in the Weihrauch lattice. We consider in particular the following (for  $X \subseteq \mathbb{N}$  with at least two elements and a standard numbering  $\varphi^p$  of the computable functions on natural numbers relative to  $p$ ):

1.  $DNC_X : \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}, p \mapsto \{q \in X^{\mathbb{N}} : (\forall n) \varphi_n^p(n) \neq q(n)\}$ .
2. 1-GEN :  $2^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}}, p \mapsto \{q : q \text{ is 1-generic relative to } p\}$ .
3. MLR :  $2^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}}, p \mapsto \{q : q \text{ is Martin-Löf random relative to } p\}$ .
4. PA :  $2^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}}, p \mapsto \{q : q \text{ is of PA degree relative to } p\}$ .
5. COH :  $(2^{\mathbb{N}})^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}}, (R_i)_{i \in \mathbb{N}} \mapsto \{A : A \text{ is cohesive for } (R_i)_{i \in \mathbb{N}}\}$ .

The first observation is that  $DNC_n$  is just the parallelization of  $ACC_n$ .

**Theorem 11.8.14 (Diagonal non-computability).**  $DNC_n \equiv_{sW} \widehat{ACC}_n$  for all  $n \geq 2$  and  $n = \mathbb{N}$ .

The jump  $BCT'_0$  of the computable version of the Baire category theorem  $BCT_0$  is closely related to 1-genericity and the problem  $\Pi_1^0 G$  of  $\Pi_1^0$ -genericity, which we do not define here.

**Theorem 11.8.15 (Genericity).**  $1\text{-GEN} <_W BCT'_0 \equiv_{sW} \Pi_1^0 G <_W L$ .

We note that all the problems from computability theory mentioned here, except  $DNC_n$ , are densely realized. Hence we can apply Proposition 11.4.13.

**Proposition 11.8.16.** *1-GEN, MLR, PA, COH and  $BCT'_0$  are densely realized and hence  $ACC_{\mathbb{N}}$  and  $C_2$  are not Weihrauch reducible to any of them.*

This means that these problems are very different from all the theorems from analysis mentioned above that are all above  $C_2$  in the Weihrauch lattice. Hence, it is interesting that some of these densely realized problems can be characterized as implications (i.e., as “quotients”) of problems above  $C_2$ .

**Theorem 11.8.17 (Randomness, Peano arithmetic, cohesiveness).** *We obtain:*

1.  $MLR \equiv_W (C_{\mathbb{N}} \rightarrow WWKL) \equiv_W (C_{\mathbb{N}} \rightarrow PC_{2^{\mathbb{N}}})$ .
2.  $PA \equiv_W (C'_{\mathbb{N}} \rightarrow WKL) \equiv_W (C'_{\mathbb{N}} \rightarrow C_{2^{\mathbb{N}}})$ .
3.  $COH \equiv_W (\text{lim} \rightarrow KL) \equiv_W (\text{lim} \rightarrow C'_{2^{\mathbb{N}}})$ .

We close this section by mentioning that one can apply results from computability theory such as the theorem of van Lambalgen to conclude that some of the above mentioned problems are closed under composition.

**Proposition 11.8.18.** *MLR and 1-GEN are closed under compositional product  $*$ .*



## Bibliographic Remarks

The intermediate value theorem and theorems from functional analysis were studied by Brattka and Gherardi [17, 12, 10]. The Lebesgue covering lemma for  $[0, 1]$  has been classified by Brattka, Gherardi and Hölzl [19]. The equivalence of the Hahn-Banach theorem and weak König's lemma was proved by Gherardi and Marcone [46, 11]. The Brouwer fixed-point theorem and the problem of finding connectedness components was studied by Brattka, Le Roux, Miller and Pauly [27]. The theorem of the maximum has been studied by Brattka [13] and the Fréchet-Riesz representation theorem by Brattka and Yoshikawa [13, 33]. The classification of the Nash equilibria existence theorem is due to Pauly [91, 93]. Frostman's lemma was studied by Pauly and Fouché [99] and Vitali's covering theorem by Brattka, Gherardi, Hölzl and Pauly [20]. The Browder-Göhde-Kirk fixed-point theorem was classified by Neumann [85]. The operator of differentiation was studied by von Stein [110], and the degrees of many operations on sets that are not mentioned here were classified by Brattka and Gherardi [16]. The identities to analytic and Schwartz functions have been classified by Pauly and Steinberg [100]. The analysis of the Radon-Nikodym theorem is due to Hoyrup, Rojas and Weihrauch [60]. The problems of finding a basis of a countable vector space and of finding a connected component of a countable graph were studied by Gura, Hirst and Mummert [47, 59]. The Bolzano-Weierstraß theorem was studied by Brattka, Gherardi and Marcone [21]. König's lemma was studied by Brattka and Rakotoniaina [31] and Gale-Stewart games by Le Roux and Pauly [80]. Ramsey's theorem was studied in the Weihrauch lattice by Dorais, Dzhabarov, Hirst, Mileti and Shafer [38, 40, 41], by Brattka and Rakotoniaina [31, 102], by Patey [89] and by Hirschfeldt and Jockusch [58]. The uniform content of problems for partial and linear orders that are closely related to Ramsey's theorem for pairs was studied by Astor, Dzhabarov, Solomon and Suggs [1]. The classification of the perfect subtree theorem was initiated by Marcone [26], and the result mentioned here is unpublished. The results on problems from computability theory including a systematic study of the Baire category theorem are due to Brattka, Hendtlass and Kreuzer [22, 23] and Brattka and Pauly [29]. The problem of diagonally non-computable functions was also studied by Higuchi and Kihara [54].

## 11.9 Relations to Other Theories

In this section we discuss very briefly the relation between Weihrauch complexity and other theories and we provide some further references.

### 11.9.1 Linear Logic

There is an apparent similarity between some algebraic operations on problems and the resource-oriented interpretation of some logical operations in (intuitionistic) linear logic that was noticed early on. Table 11.1 provides a dictionary on these relations.

However, it seems that other algebraic operations on problems, such as the compositional product  $*$ , do not have any obvious counterpart in the standard approach to linear logic. The compositional product could be seen as a non-commutative conjunction. There does not seem to be any straightforward interpretation of the Weihrauch lattice as a model for (intuitionistic) linear logic.

logical operation in linear logic	algebraic operation on problems
$\otimes$ multiplicative conjunction	$\times$ product
$\&$ additive conjunction	$\sqcup$ coproduct
$\oplus$ additive disjunction	$\sqcap$ infimum
$\wp$ multiplicative disjunction	$+$ sum
$!$ bang	$\hat{\phantom{x}}$ parallelization, $*$ finite parallelization

**Table 11.1** Linear logic versus the algebra of problems.

Several researchers have independently noticed that Gödel’s Dialectica interpretation has some formal similarity to Weihrauch reducibility. This observation has not yet been formally exploited.

### 11.9.2 Medvedev Lattice, Many-One and Turing Semilattices

The Medvedev lattice has also been considered as a calculus of problems. Here problems are understood to be subsets  $A, B \subseteq \mathbb{N}^{\mathbb{N}}$  of the Baire space and  $A$  is called *Medvedev reducible* to  $B$ , in symbols  $A \leq_s B$ , if there exists a partial computable function  $F : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  with  $B \subseteq \text{dom}(F)$  such that  $F(B) \subseteq A$ . The supremum operation of this lattice is defined by  $A \oplus B := \langle A, B \rangle$  and the infimum operation by  $A \otimes B := 0A \cup 1B$ .

The relation between the Weihrauch lattice and the Medvedev lattice can be expressed from both perspectives:

1. The Medvedev lattice is a special case of the Weihrauch lattice for problems  $f : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$  that are constant.
2. The Weihrauch lattice is a generalization of the Medvedev lattice for “relativized” problems  $A_p \subseteq \mathbb{N}^{\mathbb{N}}$  that depend on a parameter  $p \in \mathbb{N}^{\mathbb{N}}$ .

This point of view translates into a formal embedding of the Medvedev preorder into the Weihrauch preorder (the first result mentioned in Theorem 11.9.1). In fact, we can embed the Medvedev lattice also with order reversed into the Weihrauch lattice, and we list both embeddings here.

**Theorem 11.9.1 (Embedding the Medvedev lattice).** *Let  $A, B \subseteq \mathbb{N}^{\mathbb{N}}$ .*

1.  $c_A : \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}, p \mapsto A$  satisfies  $A \leq_s B \iff c_A \leq_W c_B$  with  $c_{A \oplus B} \equiv_W c_A \times c_B$  and  $c_{A \otimes B} \equiv_W c_A \sqcap c_B$ .
2.  $d_A : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}, p \mapsto \hat{0}$  with  $\text{dom}(d_A) := A$  satisfies  $A \leq_s B \iff d_B \leq_W d_A$  with  $d_{A \oplus B} \equiv_W d_A \sqcap d_B$  and  $d_{A \otimes B} \equiv_W d_A \sqcup d_B$ .

*In both cases all Weihrauch reductions and equivalences can be replaced by strong ones, in which case  $\sqcup$  has to be replaced by  $\boxplus$  in 2.*

The second, reverse embedding is even a lattice embedding since it preserves suprema and infima in the reverse order. The first embedding is also a lattice embedding if considered as an embedding into the parallelized Weihrauch degrees. These

embeddings were studied by Brattka and Gherardi [18], Higuchi and Pauly [55] and Dzhafarov [42]. Since the Turing degrees and the enumeration degrees can be embedded into the Medvedev lattice, it follows that they can also be embedded into the Weihrauch lattice via the abovementioned embeddings.

We note that the Medvedev lattice has been used by Downey, Greenberg, Jockusch, Milans, Lewis and others [39, 63] in order to study problems from computability theory, such as MLR, 1-GEN, PA and  $\text{DNC}_n$  in their unrelativized form (for computable inputs). The advantage of the Weihrauch lattice is that these problems can be studied in this lattice together with problems such as WKL and WWKL that depend on parameters (i.e., the input tree) in an essential way. Finally, we mention that the Muchnik lattice, which is the non-uniform counterpart of the Medvedev lattice, has also been used to classify problems, see Simpson [107].

Also the many-one semilattice can be embedded into the Weihrauch lattice, albeit in a slightly less natural way than the Turing semilattice. The construction starts with a non-canonical choice of two Turing incomparable points. As usual we denote *many-one reducibility* between sets  $A, B \subseteq \mathbb{N}$  by  $\leq_m$  and we recall that  $A \oplus B := \{2n : n \in A\} \cup \{2n+1 : n \in B\}$  is the supremum with respect to many-one reducibility.

**Proposition 11.9.2 (Embedding of the many-one semilattice).** *Let  $p, q \in \mathbb{N}^{\mathbb{N}}$  be Turing incomparable and, for  $A \subseteq \mathbb{N}$ , define  $m_A : \mathbb{N} \rightarrow \{p, q\}$  by  $m_A(n) = p : \iff n \in A$ . Then we obtain  $A \leq_m B \iff m_A \leq_W m_B$  and  $m_{A \oplus B} \equiv_W m_A \sqcup m_B$  for all  $A, B \subseteq \mathbb{N}$ , i.e.,  $A \mapsto m_A$  is a join-semilattice embedding.*

### 11.9.3 Reverse Mathematics

Reverse mathematics is a proof-theoretic approach that aims to classify theorems according to the axioms that are needed to prove these theorems in second-order arithmetic [106]. Many theorems from various areas of mathematics have been classified in this approach. Most axiom systems that are used in reverse mathematics have counterparts in the Weihrauch lattice (see also Figure 11.2):

- $\text{B}\Sigma_n^0$  ( $\Sigma_n^0$ -boundedness):  $\text{B}\Sigma_2^0$  is equivalent to the *regularity principle*  $\text{R}\Sigma_1^0$  over a very weak system [48], and it corresponds to  $\text{K}'_{\mathbb{N}}$  by Theorem 11.7.52. Hence  $\text{B}\Sigma_n^0$  can be seen as the counterpart of  $\text{K}'_{\mathbb{N}}^{(n-1)}$ .
- $\text{I}\Sigma_n^0$  ( $\Sigma_n^0$ -induction) is equivalent to the *least number principle*  $\text{L}\Pi_n^0$  over a very weak system [48].  $\text{L}\Pi_1^0$  directly translates into  $\text{min}^c$  as a problem and hence  $\text{I}\Sigma_n^0$  corresponds to  $\text{C}'_{\mathbb{N}}^{(n-1)}$  by Theorem 11.8.2.
- $\text{RCA}_0^*$  (recursive comprehension) stands for the usual system  $\text{RCA}_0$  but with  $\text{I}\Sigma_0^0$  instead of  $\text{I}\Sigma_1^0$ . It corresponds to  $\text{C}_1$  (the computable problems).
- $\text{WKL}_0^*$  and  $\text{WWKL}_0^*$ , by which we mean  $\text{RCA}_0^*$  plus weak König's lemma and weak weak König's lemma, respectively, correspond directly to the problems WKL and WWKL.

- $ACA_0$  (arithmetic comprehension) corresponds to the problems  $\text{lim}$  (and its finite compositions  $\text{lim}^{[n]}$  with  $n \in \mathbb{N}$ ). Sometimes, a uniform version  $ACA'_0$  of  $ACA_0$  is used [57], which corresponds to  $\bigsqcup_{n \in \mathbb{N}} \text{lim}^{[n]}$ .
- $ATR_0$  (arithmetical transfinite recursion) corresponds to  $UC_{\mathbb{N}\mathbb{N}}$  and  $C_{\mathbb{N}\mathbb{N}}$  (see Theorem 11.8.13). This topic is still very much research in progress.

Counterparts in the Weihrauch lattice of higher systems such as  $\Pi_1^1\text{-CA}_0$  ( $\Pi_1^1$ -comprehension) have not yet been systematically studied. By Theorem 11.7.55 we have  $K_{\mathbb{N}}^{(n)} <_W C_{\mathbb{N}}^{(n)} <_W K_{\mathbb{N}}^{(n+1)}$  in analogy to  $B\Sigma_n^0 \leftarrow I\Sigma_n^0 \leftarrow B\Sigma_{n+1}^0$ .

Reverse mathematics is based on a proof-theoretic approach, whereas classifications in the Weihrauch lattice are based on a computational approach. Besides this we note the following distinguishing features:

1. *Resource sensitivity*: classifications in reverse mathematics do not distinguish between a single application, a finite number of consecutive applications or a finite number of parallel applications of a theorem, since classical logic is used (as opposed to linear logic).
2. *Uniformity*: classifications in reverse mathematics only capture the non-uniform content of problems, i.e., the way output parameters depend on input parameters in the worst case. Again this is due to the usage of classical logic (as opposed to intuitionistic logic).

For instance, a number of theorems that are non-uniformly computable in the sense that there is a computable output for every computable input are provable over  $RCA_0$  in reverse mathematics, even though they are not computable in a uniform way. This includes the intermediate value theorem  $IVT$ , the Baire category theorem  $BCT_1$  and others. Due to the lack of uniformity reverse mathematics also cannot distinguish between theorems and their contrapositive forms. For instance the version  $HBC_0$  of the Heine-Borel covering theorem is computable, while  $HBC_1 \equiv_W WKL$ . In reverse mathematics, the Heine-Borel theorem is equivalent to  $WKL_0$  over  $RCA_0$  irrespectively of whether we consider the analogue of  $HBC_0$  or  $HBC_1$ . In other words: classifications in reverse mathematics automatically capture the most complicated contrapositive form.

It is remarkable that despite these explicable differences most classifications in the Weihrauch lattice can be seen as uniform and resource sensitive refinements of classifications in reverse mathematics. This seems to confirm a “computations as proofs” paradigm (as opposed to the well-known “proofs as computations” paradigm in intuitionistic logic).

#### ***11.9.4 Constructive Reverse Mathematics***

Constructive reverse mathematics, as proposed by Ishihara [61], classifies problems in the Bishop approach to constructive analysis, which is based on intuitionistic logic. Due to the usage of intuitionistic logic this approach is fully uniform, but it is

even less resource sensitive compared to classical reverse mathematics. This is due to the fact that typically the *axiom of countable choice* can be used freely, which amounts to a free usage of parallelization in the Weihrauch lattice. In this sense, the Weihrauch complexity approach is closer to a hypothetical version of constructive reverse mathematics with intuitionistic linear logic. The classifications in constructive reverse mathematics are captured by their equivalence to certain constructively unacceptable principles:

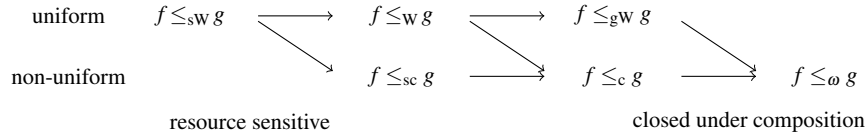
1. LLPO (the *lesser limited principle of omniscience*) is the theorem that corresponds to our problem LLPO. In the presence of countable choice it corresponds to WKL by Theorem 11.7.23.
2. LPO (the *limited principle of omniscience*) is the theorem that corresponds to our problem LPO. In the presence of countable choice and due to the availability of composition it corresponds to  $\text{lim}^{(n)}$  with  $n \in \mathbb{N}$  by Theorem 11.6.7 (and hence to  $\text{ACA}_0$  in classical reverse mathematics).
3. MP (*Markov's principle*),  $\text{BD-}\mathbb{N}$  (the *boundedness problem*) and some other principles that are rejected in constructive analysis correspond to computable (and hence continuous) problems in the Weihrauch lattice.

In conclusion, this means that the Weihrauch complexity approach is finer than constructive reverse mathematics in terms of resource sensitivity, but coarser when it comes to distinctions that are based on computable principles such as MP and  $\text{BD-}\mathbb{N}$ . In order to translate these heuristic observations into formal theorems, one needs to fix an axiomatic framework for constructive analysis. Some results in this direction have been obtained by Kuyper [75], Fujiwara [43] and Uftring [112].

### 11.9.5 Other Reducibilities

Hirschfeldt and Jockusch [57, 58, 108] have introduced a number of further reducibilities that are related to Weihrauch reducibility. For one, there are non-uniform versions of Weihrauch reducibility and strong Weihrauch reducibility, which are called *computable reducibility* and *strong computable reducibility*, in symbols  $\leq_c$  and  $\leq_{sc}$ , as well as a reducibility  $\leq_\omega$  that is based on Turing ideals. On the other hand, they introduced a concept of *generalized Weihrauch reducibility* that has a built-in closure under composition. This operation can be formalized as a closure operator  $f \mapsto f^\circ$  in the Weihrauch lattice. Likewise, Brattka and Gherardi [18] and Higuchi and Pauly [55] studied variants of Weihrauch reducibility with a built-in parallelization. These and further reducibilities allow one to interpolate between Weihrauch complexity and reverse mathematics in the sense that one can choose a reduction that captures a particular degree of uniformity and resource sensitivity (see Figure 11.3).

Yet another related reducibility that originates from descriptive set theory is *Wadge reducibility*, which is defined via preimages. Given  $A, B \subseteq \mathbb{N}^{\mathbb{N}}$ , we say that  $A$  is *Wadge reducible* to  $B$ , in symbols  $A \leq_w B$ , if there exists a continuous function



**Fig. 11.3** Implications between notions of reducibility.

$f : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  such that  $A = f^{-1}(B)$ . Hence, Wadge reducibility is the (topological) analogue of many-one reducibility on the Baire space. Weihrauch reducibility can be seen as a (computable) analogue of this reduction for multi-valued functions.

In early work by Weihrauch [114, 115] and by Hertling [50] mostly the continuous version of Weihrauch reducibility was considered. In particular, Hertling [50, 51] completely characterized continuous (strong) Weihrauch and Wadge degrees of certain functions with discrete image in terms of preorders on labeled forests. Kudinov, Selivanov and Zhukov [74] and Hertling and Selivanov [52] have studied the decidability and complexity of some initial segments of these preorders. Such preorders and versions of Weihrauch reducibility have also been used in descriptive set theory, e.g., by Carroy [37].

### 11.9.6 Descriptive Set Theory

Descriptive set theory studies the complexity of subsets of and functions between separable complete metric spaces. Wadge reducibility has been established as a critical tool here, and reasonable classes of subsets are typically closed downwards under Wadge reducibility. For functions, Weihrauch reducibility can play the analogous role. As demanded by Moschovakis [81], this treatment covers both the effective and the non-effective case simultaneously, with the former implying the latter via relativization.

Many typical classes of functions even have complete problems under Weihrauch reducibility. Theorem 11.6.5 provides an example; another one is related to  $(\Sigma_{n+2}^0, \Sigma_{n+2}^0)$ -measurability, which is defined such that preimages of  $\Sigma_{n+2}^0$ -sets are  $\Sigma_{n+2}^0$ -sets (see Pauly and de Brecht [97] and Kihara [67]).

**Theorem 11.9.3 (Effective  $(\Sigma_{n+2}^0, \Sigma_{n+2}^0)$ -measurability).**  $f \leq_w \mathbf{C}_{\mathbb{N}}^{(n)} \iff f$  is effectively  $(\Sigma_{n+2}^0, \Sigma_{n+2}^0)$ -measurable, for all  $f : X \rightarrow Y$  on complete computable metric spaces  $X, Y$  with  $n = 0$  and for  $f : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  with  $n \in \mathbb{N}$ .

There is a subtle but crucial issue with relativization here: relativizing the theorem covers the case where the preimage map of  $f$  from  $\Sigma_{n+2}^0$ -sets to  $\Sigma_{n+2}^0$ -sets is continuous, rather than merely being well defined. For  $n \in \{0, 1\}$  the theorem of Jayne and Rogers [62] and a theorem by Semmes [105] show that these cases are equivalent. For  $n > 1$ , the question of whether the cases are equivalent is open and

equivalent to the generalized conjecture of Jayne and Rogers. This is discussed in some more detail in [94].

Weihrauch complete problems for function classes correspond to game characterizations in descriptive set theory. A general account of the latter is provided by Motto Ros [82], and the link to Weihrauch reducibility is made by Nobrega and Pauly [87].

More generally, the theory of Weihrauch degrees is closely linked to a programme to extend descriptive set theory from Polish spaces to larger classes of spaces, such as represented spaces. Such an endeavor was called for and started by Selivanov [104]. De Brecht introduced the quasi-Polish spaces [34] and demonstrated that many results from descriptive set theory remain valid in this setting. A further extension is possible using the formalism of jump operators (de Brecht [35]) or computable endofunctors (Pauly and de Brecht [98]), both of which are closely related to each other and to Weihrauch degrees.

### 11.9.7 Other Models of Computability

We have already seen in Sections 11.6 and 11.7 that other models of computability can be characterized in the Weihrauch lattice. This includes the classes of problems that are computable with finitely many mind changes, limit computable, non-deterministically computable and Las Vegas computable.

There are completely different algebraic models of computability such as the Blum-Shub-Smale machines [4] (BSS machines). Due to their algebraic nature, the class of functions computable by these machines lack certain completeness properties and cannot be characterized exactly in the Weihrauch lattice. However, some tight upper bounds have been found by Neumann and Pauly [86].

**Theorem 11.9.4 (Algebraic computation).** *If  $f : \subseteq \mathbb{R}^* \rightarrow \mathbb{R}^*$  is computable on a BSS machine, then  $f \leq_W \mathbf{C}_{\mathbb{N}}$  and there is a function  $f : \subseteq \mathbb{R}^* \rightarrow \mathbb{R}^*$  that is computable on a BSS machine and satisfies  $f \equiv_W \mathbf{C}_{\mathbb{N}}$ .*

Hertling and Weihrauch [53, 49] have studied how the number of tests that are performed is related to degeneracies in computations. Yet a further class of machines can be obtained if one allows infinite computation time of higher order. Weihrauch computability was generalized to this context by Carl [36] and Galeotti and Nobrega [44].

**Acknowledgements** Vasco Brattka has been supported by the National Research Foundation of South Africa (Grant Number 115269). Guido Gherardi has been supported by MIUR-PRIN project *Analysis of Program Analyses* (ASPRA, ID 201784YSZ5.004).

## References

1. Astor, E.P., Dzhafarov, D.D., Solomon, R., Suggs, J.: The uniform content of partial and linear orders. *Annals of Pure and Applied Logic* **168**(6), 1153 – 1171 (2017)
2. Baigger, G.: Die Nichtkonstruktivität des Brouwerschen Fixpunktsatzes. *Archive for Mathematical Logic* **25**, 183–188 (1985)
3. Bienvenu, L., Kuyper, R.: Parallel and serial jumps of Weak König’s Lemma. In: A. Day, M. Fellows, N. Greenberg, B. Khossainov, A. Melnikov, F. Rosamond (eds.) *Computability and Complexity: Essays Dedicated to Rodney G. Downey on the Occasion of His 60th Birthday, Lecture Notes in Computer Science*, vol. 10010, pp. 201–217. Springer, Cham (2017)
4. Blum, L., Cucker, F., Shub, M., Smale, S.: *Complexity and Real Computation*. Springer, New York (1998)
5. Brattka, V.: *Grade der Nichtstetigkeit in der Analysis*. Diplomarbeit. Fachbereich Informatik, FernUniversität Hagen (1993)
6. Brattka, V.: *Recursive and computable operations over topological structures*. Ph.D. thesis. Department of Computer Science, University of Hagen, Hagen, Germany (1998)
7. Brattka, V.: Computable invariance. *Theoretical Computer Science* **210**, 3–20 (1999)
8. Brattka, V.: Computability over topological structures. In: S.B. Cooper, S.S. Goncharov (eds.) *Computability and Models*, pp. 93–136. Kluwer Academic Publishers, New York (2003)
9. Brattka, V.: Effective Borel measurability and reducibility of functions. *Mathematical Logic Quarterly* **51**(1), 19–44 (2005)
10. Brattka, V.: Computable versions of the uniform boundedness theorem. In: Z. Chatzidakis, P. Koepke, W. Pohlers (eds.) *Logic Colloquium 2002, Lecture Notes in Logic*, vol. 27, pp. 130–151. Association for Symbolic Logic, Urbana (2006)
11. Brattka, V.: Borel complexity and computability of the Hahn-Banach Theorem. *Archive for Mathematical Logic* **46**(7–8), 547–564 (2008)
12. Brattka, V.: A computable version of Banach’s inverse mapping theorem. *Annals of Pure and Applied Logic* **157**, 85–96 (2009)
13. Brattka, V.: Computability and analysis, a historical approach. In: A. Beckmann, L. Bienvenu, N. Jonoska (eds.) *Pursuit of the Universal, Lecture Notes in Computer Science*, vol. 9709, pp. 45–57. Springer, Switzerland (2016), 12th Conference on Computability in Europe, CiE 2016, Paris, France, June 27 - July 1, 2016
14. Brattka, V., de Brecht, M., Pauly, A.: Closed choice and a uniform low basis theorem. *Annals of Pure and Applied Logic* **163**, 986–1008 (2012)
15. Brattka, V., Cettolo, A., Gherardi, G., Marcone, A., Schröder, M.: Addendum to: “The Bolzano-Weierstrass theorem is the jump of weak König’s lemma”. *Annals of Pure and Applied Logic* **168**(8), 1605–1608 (2017)
16. Brattka, V., Gherardi, G.: Borel complexity of topological operations on computable metric spaces. *Journal of Logic and Computation* **19**(1), 45–76 (2009)
17. Brattka, V., Gherardi, G.: Effective choice and boundedness principles in computable analysis. *The Bulletin of Symbolic Logic* **17**(1), 73–117 (2011)
18. Brattka, V., Gherardi, G.: Weihrauch degrees, omniscience principles and weak computability. *The Journal of Symbolic Logic* **76**(1), 143–176 (2011)
19. Brattka, V., Gherardi, G., Hölzl, R.: Probabilistic computability and choice. *Information and Computation* **242**, 249–286 (2015)
20. Brattka, V., Gherardi, G., Hölzl, R., Pauly, A.: The Vitali covering theorem in the Weihrauch lattice. In: A. Day, M. Fellows, N. Greenberg, B. Khossainov, A. Melnikov, F. Rosamond (eds.) *Computability and Complexity: Essays Dedicated to Rodney G. Downey on the Occasion of His 60th Birthday, Lecture Notes in Computer Science*, vol. 10010, pp. 188–200. Springer, Cham (2017)
21. Brattka, V., Gherardi, G., Marcone, A.: The Bolzano-Weierstrass theorem is the jump of weak König’s lemma. *Annals of Pure and Applied Logic* **163**, 623–655 (2012)



22. Brattka, V., Hendtlass, M., Kreuzer, A.P.: On the uniform computational content of computability theory. *Theory of Computing Systems* **61**(4), 1376–1426 (2017)
23. Brattka, V., Hendtlass, M., Kreuzer, A.P.: On the uniform computational content of the Baire category theorem. *Notre Dame Journal of Formal Logic* **59**(4), 605–636 (2018)
24. Brattka, V., Hertling, P., Weihrauch, K.: A tutorial on computable analysis. In: S.B. Cooper, B. Löwe, A. Sorbi (eds.) *New Computational Paradigms: Changing Conceptions of What Is Computable*, pp. 425–491. Springer, New York (2008)
25. Brattka, V., Hölzl, R., Kuyper, R.: Monte Carlo computability. In: H. Vollmer, B. Vallée (eds.) *34th Symposium on Theoretical Aspects of Computer Science (STACS 2017), Leibniz International Proceedings in Informatics (LIPIcs)*, vol. 66, pp. 17:1–17:14. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany (2017)
26. Brattka, V., Kawamura, A., Marcone, A., Pauly, A. (eds.): Measuring the Complexity of Computational Content: Weihrauch Reducibility and Reverse Analysis (Dagstuhl Seminar 15392), *Dagstuhl Reports*, vol. 5. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, Dagstuhl, Germany (2016)
27. Brattka, V., Le Roux, S., Miller, J.S., Pauly, A.: Connected choice and the Brouwer fixed point theorem. *Journal of Mathematical Logic* **19**(1), 1–46 (2019)
28. Brattka, V., Pauly, A.: Computation with advice. In: X. Zheng, N. Zhong (eds.) *CCA 2010, Proceedings of the Seventh International Conference on Computability and Complexity in Analysis, Electronic Proceedings in Theoretical Computer Science*, vol. 24, pp. 41–55 (2010)
29. Brattka, V., Pauly, A.: On the algebraic structure of Weihrauch degrees. *Logical Methods in Computer Science* **14**(4:4), 1–36 (2018)
30. Brattka, V., Presser, G.: Computability on subsets of metric spaces. *Theoretical Computer Science* **305**, 43–76 (2003)
31. Brattka, V., Rakotoniaina, T.: On the uniform computational content of Ramsey’s theorem. *Journal of Symbolic Logic* **82**(4), 1278–1316 (2017)
32. Brattka, V., Weihrauch, K.: Computability on subsets of Euclidean space I: Closed and compact subsets. *Theoretical Computer Science* **219**, 65–93 (1999)
33. Brattka, V., Yoshikawa, A.: Towards computability of elliptic boundary value problems in variational formulation. *Journal of Complexity* **22**(6), 858–880 (2006)
34. de Brecht, M.: Quasi-Polish spaces. *Ann. Pure Appl. Logic* **164**(3), 356–381 (2013)
35. de Brecht, M.: Levels of discontinuity, limit-computability, and jump operators. In: V. Brattka, H. Diener, D. Spreen (eds.) *Logic, Computation, Hierarchies, Ontos Mathematical Logic*, pp. 93–122. Walter de Gruyter, Boston (2014)
36. Carl, M.: Generalized effective reducibility. In: A. Beckmann, L. Bienvenu, N. Jonoska (eds.) *Pursuit of the Universal, Lecture Notes in Computer Science*, vol. 9709, pp. 225–233. Springer, Switzerland (2016), 12th Conference on Computability in Europe, CiE 2016, Paris, France, June 27 - July 1, 2016
37. Carroy, R.: A quasi-order on continuous functions. *Journal of Symbolic Logic* **78**(2), 663–648 (2013)
38. Dorais, F.G., Dzhafarov, D.D., Hirst, J.L., Mileti, J.R., Shafer, P.: On uniform relationships between combinatorial problems. *Transactions of the American Mathematical Society* **368**(2), 1321–1359 (2016)
39. Downey, R.G., Greenberg, N., Jockusch Jr., C.G., Milans, K.G.: Binary subtrees with few labeled paths. *Combinatorica* **31**(3), 285–303 (2011)
40. Dzhafarov, D.D.: Cohesive avoidance and strong reductions. *Proceedings of the American Mathematical Society* **143**(2), 869–876 (2015)
41. Dzhafarov, D.D.: Strong reductions between combinatorial principles. *Journal of Symbolic Logic* **81**(4), 1405–1431 (2016)
42. Dzhafarov, D.D.: Joins in the strong Weihrauch degrees. *Mathematical Research Letters* **26**(3), 749–767 (2019)
43. Fujiwara, M.: Weihrauch and constructive reducibility between existence statements. *Computability* **10**(to appear) (2021)

44. Galeotti, L., Nobrega, H.: Towards computable analysis on the generalized real line. In: J. Kari, F. Manea, I. Petre (eds.) *Unveiling Dynamics and Complexity, Lecture Notes in Computer Science*, vol. 10307, pp. 246–257. Springer, Cham (2017), 13th Conference on Computability in Europe, CiE 2017, Turku, Finland, June 12–16, 2017
45. Gherardi, G.: Effective Borel degrees of some topological functions. *Mathematical Logic Quarterly* **52**(6), 625–642 (2006)
46. Gherardi, G., Marcone, A.: How incomputable is the separable Hahn-Banach theorem? *Notre Dame Journal of Formal Logic* **50**(4), 393–425 (2009)
47. Gura, K., Hirst, J.L., Mummert, C.: On the existence of a connected component of a graph. *Computability* **4**(2), 103–117 (2015)
48. Hájek, P., Pudlák, P.: *Metamathematics of First-Order Arithmetic. Perspectives in Mathematical Logic*. Springer-Verlag, Berlin (1993)
49. Hertling, P.: Topological complexity with continuous operations. *Journal of Complexity* **12**, 315–338 (1996)
50. Hertling, P.: *Unstetigkeitsgrade von Funktionen in der effektiven Analysis*. Dissertation. Fachbereich Informatik, FernUniversität Hagen (1996)
51. Hertling, P.: Forests describing Wadge degrees and topological Weihrauch degrees of certain classes of functions and relations. *Computability* **9**(3–4), 249–307 (2020)
52. Hertling, P., Selivanov, V.: Complexity issues for preorders on finite labeled forests. In: V. Brattka, H. Diener, D. Spreen (eds.) *Logic, Computation, Hierarchies, Ontos Mathematical Logic*, pp. 165–190. Walter de Gruyter, Boston (2014)
53. Hertling, P., Weihrauch, K.: Levels of degeneracy and exact lower complexity bounds for geometric algorithms. In: *Proceedings of the Sixth Canadian Conference on Computational Geometry*, pp. 237–242. University of Saskatchewan (1994), Saskatoon, Saskatchewan, August 2–6, 1994
54. Higuchi, K., Kihara, T.: Inside the Muchnik degrees II: The degree structures induced by the arithmetical hierarchy of countably continuous functions. *Annals of Pure and Applied Logic* **165**(6), 1201–1241 (2014)
55. Higuchi, K., Pauly, A.: The degree structure of Weihrauch reducibility. *Log. Methods Comput. Sci.* **9**(2), 2:02, 17 (2013)
56. Hirsch, M.D.: *Applications of topology to lower bound estimates in computer science*. Ph.D. thesis. University of California, Berkeley (1990)
57. Hirschfeldt, D.R.: *Slicing the Truth: On the Computable and Reverse Mathematics of Combinatorial Principles, Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore*, vol. 28. World Scientific, Singapore (2015)
58. Hirschfeldt, D.R., Jockusch, C.G.: On notions of computability-theoretic reduction between  $\Pi_2^1$  principles. *Journal of Mathematical Logic* **16**(1), 1650,002, 59 (2016)
59. Hirst, J.L., Mummert, C.: Reverse mathematics of matroids. In: A. Day, M. Fellows, N. Greenberg, B. Khossainov, A. Melnikov, F. Rosamond (eds.) *Computability and Complexity: Essays Dedicated to Rodney G. Downey on the Occasion of His 60th Birthday, Lecture Notes in Computer Science*, vol. 10010, pp. 143–159. Springer, Cham (2017)
60. Hoyrup, M., Rojas, C., Weihrauch, K.: Computability of the Radon-Nikodym derivative. *Computability* **1**(1), 3–13 (2012)
61. Ishihara, H.: Reverse mathematics in Bishop’s constructive mathematics. *Philosophia Scientiae, Cahier special* **6**, 43–59 (2006)
62. Jayne, J.E., Rogers, C.A.: First level Borel functions and isomorphisms. *Journal de Mathématiques Pures et Appliquées. Neuvième Série* **61**(2), 177–205 (1982)
63. Jockusch Jr., C.G., Lewis, A.E.M.: Diagonally non-computable functions and bi-immunity. *Journal of Symbolic Logic* **78**(3), 977–988 (2013)
64. Jockusch Jr., C.G., Soare, R.I.:  $\Pi_1^0$  classes and degrees of theories. *Transactions of the American Mathematical Society* **173**, 33–56 (1972)
65. Kawamura, A.: Lipschitz continuous ordinary differential equations are polynomial-space complete. *Computational Complexity* **19**(2), 305–332 (2010)

66. Kawamura, A., Cook, S.: Complexity theory for operators in analysis. In: Proceedings of the 42nd ACM Symposium on Theory of Computing, *STOC '10*, pp. 495–502. ACM, New York (2010)
67. Kihara, T.: Decomposing Borel functions using the Shore-Slaman join theorem. *Fundamenta Mathematicae* **230**, 1–13 (2015)
68. Kihara, T.: All-or-unique choice and convex choice (2016), unpublished notes
69. Kihara, T., Pauly, A.: Dividing by zero - how bad is it, really? In: P. Faliszewski, A. Muscholl, R. Niedermeier (eds.) 41st International Symposium on Mathematical Foundations of Computer Science (MFCS 2016), *Leibniz International Proceedings in Informatics (LIPIcs)*, vol. 58, pp. 58:1–58:14. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany (2016)
70. Kleene, S.C.: Recursive functions and intuitionistic mathematics. In: Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1, pp. 679–685. Amer. Math. Soc., Providence, R. I. (1952)
71. Kolmogoroff, A.: Zur Deutung der intuitionistischen Logik. *Mathematische Zeitschrift* **35**(1), 58–65 (1932)
72. Kreitz, C., Weihrauch, K.: Theory of representations. *Theoretical Computer Science* **38**, 35–53 (1985)
73. Kreitz, C., Weihrauch, K.: Compactness in constructive analysis revisited. *Annals of Pure and Applied Logic* **36**, 29–38 (1987)
74. Kudinov, O.V., Selivanov, V.L., Zhukov, A.V.: Undecidability in Weihrauch degrees. In: F. Ferreira, B. Löwe, E. Mayordomo, L. Mendes Gomes (eds.) Programs, Proofs, Processes, *Lecture Notes in Computer Science*, vol. 6158, pp. 256–265. Springer, Berlin (2010), 6th Conference on Computability in Europe, CiE 2010, Ponta Delgada, Azores, Portugal, June/July 2010
75. Kuyper, R.: On Weihrauch reducibility and intuitionistic reverse mathematics. *Journal of Symbolic Logic* **82**(4), 1438–1458 (2017)
76. Lacombe, D.: Classes récursivement fermés et fonctions majorantes. *Comptes Rendus Académie des Sciences Paris* **240**, 716–718 (1955)
77. Lacombe, D.: Les ensembles récursivement ouverts ou fermés, et leurs applications à l'Analyse récursive. *Comptes Rendus Académie des Sciences Paris* **245**, 1040–1043 (1957)
78. Lacombe, D.: Quelques procédés de définition en topologie récursive. In: A. Heyting (ed.) *Constructivity in Mathematics*, pp. 129–158. North-Holland, Amsterdam (1959), colloquium at Amsterdam, 1957
79. Le Roux, S., Pauly, A.: Finite choice, convex choice and finding roots. *Logical Methods in Computer Science* **11**(4), 4:6, 31 (2015)
80. Le Roux, S., Pauly, A.: Weihrauch degrees of finding equilibria in sequential games (extended abstract). In: A. Beckmann, V. Mitrană, M. Soskova (eds.) *Evolving Computability, Lecture Notes in Computer Science*, vol. 9136, pp. 246–257. Springer, Cham (2015), 11th Conference on Computability in Europe, CiE 2015, Bucharest, Romania, June 29–July 3, 2015
81. Moschovakis, Y.N.: Classical descriptive set theory as a refinement of effective descriptive set theory. *Annals of Pure and Applied Logic* **162**(3), 243–255 (2010)
82. Motto Ros, L.: Reduction games and reducibilities for sets of reals (abstract). In: U. Berger, V. Brattka, V. Selivanov, D. Spreen, H. Tsuiki (eds.) *Computing with Infinite Data: Topological and Logical Foundations (Dagstuhl Seminar 11411)*, p. 28 (2012)
83. Mylatz, U.: Vergleich unstetiger Funktionen in der Analysis. Diplomarbeit. Fachbereich Informatik, FernUniversität Hagen (1992)
84. Mylatz, U.: Vergleich unstetiger Funktionen: “Principle of Omniscience” und Vollständigkeit in der  $C$ -hierarchie. Ph.D. thesis. Faculty for Mathematics and Computer Science, University Hagen, Hagen, Germany (2006)
85. Neumann, E.: Computational problems in metric fixed point theory and their Weihrauch degrees. *Logical Methods in Computer Science* **11**, 4:20,44 (2015)
86. Neumann, E., Pauly, A.: A topological view on algebraic computation models. *Journal of Complexity* **44**(Supplement C), 1–22 (2018)

87. Nobrega, H., Pauly, A.: Game characterizations and lower cones in the Weihrauch degrees. *Logical Methods in Computer Science* **15**(3), Paper No. 11, 29 (2019)
88. Orevkov, V.P.: A constructive mapping of the square onto itself displacing every constructive point (Russian). *Doklady Akademii Nauk* **152**, 55–58 (1963)
89. Patey, L.: The weakness of being cohesive, thin or free in reverse mathematics. *Israel Journal of Mathematics* **216**, 905–955 (2016)
90. Pauly, A.: Methoden zum Vergleich der Unstetigkeit von Funktionen. M.Sc. thesis. FernUniversität Hagen (2007)
91. Pauly, A.: How incomputable is finding Nash equilibria? *Journal of Universal Computer Science* **16**(18), 2686–2710 (2010)
92. Pauly, A.: On the (semi)lattices induced by continuous reducibilities. *Mathematical Logic Quarterly* **56**(5), 488–502 (2010)
93. Pauly, A.: Computable metamathematics and its application to game theory. Ph.D. thesis. University of Cambridge, Computer Laboratory, Clare College, Cambridge (2011)
94. Pauly, A.: The descriptive theory of represented spaces. arXiv 1408.5329 (2014)
95. Pauly, A.: On the topological aspects of the theory of represented spaces. *Computability* **5**(2), 159–180 (2016)
96. Pauly, A.: Many-one reductions and the category of multivalued functions. *Mathematical Structures in Computer Science* **27**(3), 376–404 (2017)
97. Pauly, A., de Brecht, M.: Non-deterministic computation and the Jayne-Rogers theorem. In: B. Löwe, G. Winskel (eds.) *Proceedings 8th International Workshop on Developments in Computational Models, DCM 2012, Cambridge, United Kingdom, 17 June 2012.*, *Electronic Proceedings in Theoretical Computer Science*, vol. 143, pp. 87–96 (2014)
98. Pauly, A., de Brecht, M.: Descriptive set theory in the category of represented spaces. In: *30th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pp. 438–449 (2015)
99. Pauly, A., Fouché, W.L.: How constructive is constructing measures? *Journal of Logic & Analysis* **9**(c3), 1–44 (2017)
100. Pauly, A., Steinberg, F.: Comparing representations for function spaces in computable analysis. *Theory of Computing Systems* **62**, 557–582 (2018)
101. Pour-El, M.B., Richards, J.I.: Computability in Analysis and Physics. *Perspectives in Mathematical Logic*. Springer, Berlin (1989)
102. Rakotoniaina, T.: On the computational strength of Ramsey’s theorem. Ph.D. thesis. Department of Mathematics and Applied Mathematics, University of Cape Town, Rondebosch, South Africa (2015)
103. Schröder, M.: Extended admissibility. *Theoretical Computer Science* **284**(2), 519–538 (2002)
104. Selivanov, V.L.: Difference hierarchy in  $\varphi$ -spaces. *Algebra and Logic* **43**(4), 238–248 (2004)
105. Semmes, B.T.: A game for the Borel functions. Ph.D. thesis. Institute for Logic, Language and Computation, University of Amsterdam (2009)
106. Simpson, S.G.: Subsystems of Second Order Arithmetic, 2nd edn. *Perspectives in Logic*. Cambridge University Press (2009)
107. Simpson, S.G.: Degrees of unsolvability: a tutorial. In: *Evolving Computability, Lecture Notes in Computer Science*, vol. 9136, pp. 83–94. Springer, Cham (2015)
108. Solomon, R.: Computable reductions and reverse mathematics. In: A. Beckmann, L. Bienvenu, N. Jonoska (eds.) *Pursuit of the Universal, Lecture Notes in Computer Science*, vol. 9709, pp. 182–191. Springer, Switzerland (2016), 12th Conference on Computability in Europe, CiE 2016, Paris, France, June 27 - July 1, 2016
109. Specker, E.: Der Satz vom Maximum in der rekursiven Analysis. In: A. Heyting (ed.) *Constructivity in Mathematics, Studies in Logic and the Foundations of Mathematics*, pp. 254–265. North-Holland, Amsterdam (1959), proc. Colloq., Amsterdam, Aug. 26–31, 1957
110. von Stein, T.: Vergleich nicht konstruktiv lösbarer Probleme in der Analysis. Diplomarbeit. Fachbereich Informatik, FernUniversität Hagen (1989)
111. Tavana, N.R., Weihrauch, K.: Turing machines on represented sets, a model of computation for analysis. *Logical Methods in Computer Science* **7**(2), 2:19, 21 (2011)

112. Uftring, P.: The characterization of Weihrauch reducibility in systems containing  $E\text{-PA}^0 + \text{QF-AC}^{0,0}$ . *The Journal of Symbolic Logic* (to appear) (2021)
113. Weihrauch, K.: Computability, *EATCS Monographs on Theoretical Computer Science*, vol. 9. Springer, Berlin (1987)
114. Weihrauch, K.: The degrees of discontinuity of some translators between representations of the real numbers. Technical Report TR-92-050, International Computer Science Institute, Berkeley (1992)
115. Weihrauch, K.: The TTE-interpretation of three hierarchies of omniscience principles. *Informatik Berichte 130*, FernUniversität Hagen, Hagen (1992)
116. Weihrauch, K.: *Computable Analysis*. Springer, Berlin (2000)
117. Weihrauch, K., Kreitz, C.: Representations of the real numbers and of the open subsets of the set of real numbers. *Annals of Pure and Applied Logic* **35**, 247–260 (1987)
118. Ziegler, M.: Real hypercomputation and continuity. *Theory of Computing Systems* **41**(1), 177–206 (2007)
119. Ziegler, M.: Revising type-2 computation and degrees of discontinuity. In: D. Cenzer, R. Dillhage, T. Grubba, K. Weihrauch (eds.) *Proceedings of the Third International Conference on Computability and Complexity in Analysis, Electronic Notes in Theoretical Computer Science*, vol. 167, pp. 255–274. Elsevier, Amsterdam (2007), CCA 2006, Gainesville, Florida, USA, November 1–5, 2006

There is an electronic version of a bibliography on Weihrauch complexity<sup>8</sup>.

---

<sup>8</sup> See <http://cca-net.de/publications/weibib.php>