HARNACK INEQUALITY FOR FRACTIONAL SUB-LAPLACIANS IN CARNOT GROUPS

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ABSTRACT. In this paper we prove an invariant Harnack inequality on Carnot-Carathéodory balls for fractional powers of sub-Laplacians in Carnot groups. The proof relies on an "abstract" formulation of a technique recently introduced by Caffarelli and Silvestre. In addition, we write explicitly the Poisson kernel for a class of degenerate subelliptic equations in product-type Carnot groups.

CONTENTS

1. INTRODUCTION

In Euclidean spaces, fractional integral operators have been studied in connection with different phenomena that can be described as isotropic diffusion with jumps. We mention, for instance, the thin obstacle problem, phase transition problems, and the study of a general class of conformally covariant operators in conformal geometry: see, for instance, [7], [34] and [10]. Typically, these problems can be reduced, in their simplest form, to the study of the equation

$$(1) \qquad (-\Delta)^{\gamma/2} u = f \quad \text{in } \mathbb{R}^n,$$

where $0 < \gamma < 2$. We remind that the fractional Laplacian in (1) is a nonlocal operator (even more: it is a *antilocal operator*, see [32]). Nevertheless, solutions of (1) share some properties of the solutions of elliptic equations. More precisely:

- $(-\Delta)^{\gamma/2}$ is the infinitesimal generator of a Feller semigroup $\{T_t\}_{t>0}$. This means that, if $0 \leq f \leq 1$, then $0 \leq T_t f \leq 1$ for $t > 0$. By a classical result (see P. Lévy [28], G. A. Hunt [25], Courrège

[11] and Bony-Courrège-Priouret [4] and, for recent accounts, to [1] pp 163-6 or [26]), this is equivalent to say that $(-\Delta)^{\gamma/2}$ belongs to a larger class of pseudodifferential operators satisfying the so-called *positive maximum principle*. We refer to [11] and [4] for an exhaustive discussion; here we restrict ourselves to stress that the positive maximum principle is not the usual maximum principle of potential theory.

- recently, L. Caffarelli & L. Silvestre [6] proved that functions u that are positive *on all of* $\mathbb{R}^n$ and solve the equation $(-\Delta)^{\gamma/2}u = 0$ *in an open set* $\Omega \subset \mathbb{R}^n$ satisfy an invariant local Harnack inequality. Their technique relies on an extension (or 'lifting") procedure, showing ultimately that u can be extended to a function $\tilde{v}$ on $\mathbb{R}^{n+1}$ satisfying a (degenerate) elliptic *differential* equation.

  We remind also that related results have been proved by different methods by N.S. Landkof [27] and K. Bogdan [2].

On the other hand,

- Hunt's theorem in [25] applies to a larger class of pseudodifferential operators in Lie groups;
- sub-Laplacians in Carnot groups (i.e. in connected and simply connected stratified nilpotent Lie groups) exhibit strong analogies with classical Laplace operator in the Euclidean space (for instance Harnack inequality, maximum principle, existence and estimates of the fundamental solution).

It is therefore natural to ask whether Caffarelli & Silvestre's approach can be adapted to prove a Harnack inequality for subelliptic fractional integral equations of the form

$$\mathcal{L}^{\gamma/2}u = 0,$$

where $\mathcal{L}$ is a (positive) sublaplacian in a Carnot group $\mathbb{G}$.

In fact, an "abstract" extension technique akin to that of Caffarelli-Silvestre has been recently developed in a general setting by Stinga & Torrea in [36], under very mild hypotheses on the operator $\mathcal{L}$. In particular, they obtained the Harnack inequality for the (fractional) harmonic oscillator. In addition, using analogous arguments, Stinga & Zhang [37] proved a Harnack inequality for a larger class of fractional integral operators, containing, for instance, Ornstein-Uhlenbeck operators. However, we stress that subelliptic operators in Carnot groups, though, as a matter of fact, fitting in the wide class of "degenerate elliptic operators", do not belong to the class of degenerate operators considered in [37]. Indeed, the degeneration considered in [37] is described by means of A_2-weights that may vanish only on sets of finite Lebesgue measure. On the contrary, subelliptic Laplacians, when considered as degenerate elliptic operators, may in fact degenerate on all the space. In other words, the degeneration induced by weights is a "degeneration of the measure", whereas subelliptic Laplacians could be considered as Laplace-Beltrami operators for a degenerate geometry.

Typically, if we forget the potentials, operators as in [37] have the form

$$\left(-\operatorname{div}\left(|x|^{\alpha}\nabla u\right)\right)^{\gamma/2}, \quad -n < \alpha < n$$

in $\mathbb{R}^n$, whereas, the simplest instance of our operators is provided by the fractional sub-Laplacian of the first Heisenberg group $\mathbb{H}^1$

$$\big(-(\partial_x + 2y\partial_z)^2 u - (\partial_y - 2x\partial_z)^2 u\big)^{\gamma/2}$$

in $\mathbb{R}^3$. Some comments in this sense can be found already in [17].

In this paper we further develop the idea of an abstract approach to the problem. However, the setting of Carnot groups, with a natural notion of group convolution, makes possible to recover, starting from the abstract representation in terms of the spectral resolution, another explicit form of the fractional powers (in terms of convolutions with singular kernels), as well as of the lifting operator (in terms of the convolution with a suitable Poisson kernel).

We like also to mention that, in the special case of Heisenberg groups, an explicit representation of the Poisson kernel is given also in [21] through different methods (group Fourier transform).

To state our main result, we need preliminarily to remind that in any Carnot group we can define a left-invariant distance d_c (the so-called Carnot-Carathéodory distance) that fits the structure of the group. If we denote by $B_c = B_c(x, r)$ ($x \in \mathbb{G}$ and $r > 0$) the metric balls associated with d_c and by $W_{\mathbb{G}}^{s,2}$ the Folland-Stein Sobolev space in $\mathbb{G}$ (see Section 1 for details), then the Harnack inequality for fractional sub-Laplacians takes an invariant intrinsic form. More precisely, we have:

Theorem *Let $-1 < a < 1$ and let $u \in W_{\mathbb{G}}^{1-a,2}(\mathbb{G})$ be given, $u \geq 0$ on all of $\mathbb{G}$. Assume $\mathcal{L}^{(1-a)/2} u = 0$ in an open set $\Omega \subset \mathbb{G}$.*

Then there exist $C, b > 0$ (independent of u) such that the following invariant Harnack inequality holds:

$$\sup_{B_c(x,r)} u \leq C \inf_{B_c(x,r)} u$$

for any metric ball $B_c(x, r)$ such that $B_c(x, br) \subset \Omega$.

Let us sketch briefly the main features of our proof. Basically, still following [6], its core consists in the construction of a $\mathcal{L}$-harmonic "lifting" operator $u = u(x) \to v = v(x, y)$ from $\mathbb{G}$ to $\mathbb{G} \times \mathbb{R}^+$ by means of the spectral resolution of $\mathcal{L}$ in $L^2(\mathbb{G})$ in such a way that u is the trace of the normal derivative of v on $y = 0$. If, in particular, $a = 0$, then this operator is nothing but the semigroup generated by $-\mathcal{L}^{1/2}$.

Subsequently, as in [6], we show that, if $\mathcal{L}^{\frac{1-a}{2}} u = 0$ in an open set Ω then its lifting v can be continued by parity across $y = 0$ to a weak solution $\tilde{v}$ of the equation

$$\tilde{\mathcal{L}}\tilde{v} := -|y|^a \mathcal{L}\tilde{v} + \partial_y(|y|^a \partial_y \tilde{v}) = 0.$$

In addition we show that the lifting operator can be also written as a convolution operator with a positive kernel $P_{\mathbb{G}}$, that is written explicitly. Thus $\tilde{v} \geq 0$ if $u \geq 0$ on all $\mathbb{G}$, and therefore our problem reduces to prove Harnack inequality for a weighted sub-elliptic differential operator. The construction of $P_{\mathbb{G}}$ not only yields the possibility of replacing the assumption $u \in W_{\mathbb{G}}^{1-a,2}(\mathbb{G})$ by some weaker assumptions on the behavior of u at infinity (in the spirit of some remarks in [6]), but provides an explicit form for the

Poisson kernel $P_{\mathbb{G}}(\cdot, y)$ in the half-space $\mathbb{G} \times (0, \infty)$ for $\tilde{\mathcal{L}}$. More precisely, if we denote by $h(t, \cdot)$ the heat kernel associated with $-\mathcal{L}$ as in [14], then

$$P_{\mathbb{G}}(\cdot, y) := C_a \, y^{1-a} \int_0^\infty t^{(a-3)/2} e^{-\frac{y^2}{4t}} h(t, \cdot) \, dt,$$

where

$$C_a = 2^{a-1} \Gamma((1-a)/2)^{-1}.$$

A similar formula appears in [36], but, as long as we know, this representation is new for sublaplacians in Carnot groups.

The paper is organized as follows: in Section 2 we fix our notations for Carnot groups and for Harnack inequality in this setting; in Section 3 we collect some more or less known results on fractional powers of sub-Laplacian in Carnot groups and we prove different representation theorems. Finally, in Section 4 we prove our main results.

2. Preliminary results

A connected and simply connected stratified nilpotent Lie group $(\mathbb{G}, \cdot)$ (in general non-commutative) is said to be a *Carnot group of step* κ if its Lie algebra $\mathfrak{g}$ admits a *step κ stratification*, i.e. there exist linear subspaces $V_1, ..., V_\kappa$ such that

$$(2) \quad \mathfrak{g} = V_1 \oplus ... \oplus V_\kappa, \quad [V_1, V_i] = V_{i+1}, \quad V_\kappa \neq \{0\}, \quad V_i = \{0\} \text{ if } i > \kappa,$$

where $[V_1, V_i]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in V_1$ and $Y \in V_i$. The first layer V_1, the so-called *horizontal layer*, plays a key role in the theory, since it generates $\mathfrak{g}$ by commutation.

In the last few years, Carnot groups have been largely studied in several respects, such as differential geometry, geometric measure theory, subelliptic differential equations, complex variables, optimal control theory, mathematical models in neurosciences, non-holonomic mechanics, robotics.

Beside Euclidean spaces that are the only Abelian Carnot groups, typical examples are provided by Heisenberg groups and upper triangular groups. For a general introduction to Carnot groups from the point of view of the present paper and for further examples, we refer, e.g., to [3], [15] and [35].

Set $m_i = \dim(V_i)$, for $i = 1, \ldots, \kappa$ and $h_i = m_1 + \cdots + m_i$, so that $h_\kappa = n$. For sake of simplicity, we write also $m := m_1$. We denote by Q the *homogeneous dimension* of $\mathbb{G}$, i.e. we set

$$Q := \sum_{i=1}^\kappa i \dim(V_i).$$

If e is the unit element of $(\mathbb{G}, \cdot)$, we remind that the map $X \to X(e)$, that associate with a left-invariant vector field X its value at e, is an isomorphism from $\mathfrak{g}$ to $T\mathbb{G}_e$, in turn identified with $\mathbb{R}^n$. From now on, we shall use systematically these identications. Thus, the horizontal layer defines, by left translation, a fiber bundle $H\mathbb{G}$ over $\mathbb{G}$ (the *horizontal bundle*). Its sections are the *horizontal vector fields*.

We choose now a basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$ adapted to the stratification of $\mathfrak{g}$, i.e. such that

$$e_{h_{j-1}+1}, \ldots, e_{h_j} \text{ is a basis of } V_j \text{ for each } j = 1, \ldots, \kappa.$$

Moreover, let $X = \{X_1, \ldots, X_n\}$ be the family of left invariant vector fields such that $X_i(e) = e_i$, $i = 1, \ldots, n$.

The subbundle of the tangent bundle $T\mathbb{G}$ that is spanned by the vector fields $X_1, \ldots, X_m$ plays a particularly important role in the theory, it is called the *horizontal bundle* $H\mathbb{G}$; the fibers of $H\mathbb{G}$ are

$$H\mathbb{G}_x = \text{span } \{X_1(x), \ldots, X_m(x)\}, \qquad x \in \mathbb{G}.$$

We can endow each fiber of $H\mathbb{G}$ with a scalar product $\langle \cdot, \cdot \rangle x$ and with a norm $|\cdot|_x$ that make the basis $X_1(x), \ldots, X_m(x)$ an orthonormal basis. That is if $v = \sum_{i=1}^{m_1} v_i X_i(x) = (v_1, \ldots, v_m)$ and $w = \sum_{i=1}^{m_1} w_i X_i(x) = (w_1, \ldots, w_m)$ are in $H\mathbb{G}_x$, then $\langle v, w \rangle x := \sum_{j=1}^{m} v_j w_j$ and $|v|_x^2 := \langle v, v \rangle x$.

The sections of $H\mathbb{G}$ are called *horizontal sections*, a vector of $H\mathbb{G}_x$ is an *horizontal vector* while any vector in $T\mathbb{G}_x$ that is not horizontal is a vertical vector. Each horizontal section is identified by its canonical coordinates with respect to this moving frame $X_1(x), \ldots, X_m(x)$. This way, an horizontal section ϕ is identified with a function $\phi = (\phi_1, \ldots, \phi_m) : \mathbb{R}^n \to \mathbb{R}^m$. When dealing with two such sections ϕ and ψ whose argument is not explicitly written, we drop the index x in the scalar product writing $\langle \psi | \phi \rangle$ for $\langle \psi(x), \phi(x) \rangle x$. The same convention is adopted for the norm.

Finally, if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n \equiv \mathbb{G}$ and $x_0 \in \mathbb{G}$ are given, we set

$$\pi_{x_0}(x) = \sum_{j=1}^{m} x_j X_j(x_0).$$

The map $x_0 \to \pi_{x_0}(x)$ is a smooth section of $H\mathbb{G}$.

A Carnot group $\mathbb{G}$ can be always identified, through exponential coordinates, with the Euclidean space $(\mathbb{R}^n, \cdot)$, where n is the dimension of $\mathfrak{g}$, endowed with a suitable group operation. The explicit expression of the group operation $\cdot$ is determined by the Campbell-Hausdorff formula.

For any $x \in \mathbb{G}$, the *(left) translation* $\tau_x : \mathbb{G} \to \mathbb{G}$ is defined as

$$z \mapsto \tau_x z := x \cdot z.$$

For any $\lambda > 0$, the *dilation* $\delta_\lambda : \mathbb{G} \to \mathbb{G}$, is defined as

(3) $$\delta_\lambda(x_1, \ldots, x_n) = (\lambda^{d_1} x_1, \ldots, \lambda^{d_n} x_n),$$

where $d_i \in \mathbb{N}$ is called *homogeneity of the variable x_i* in $\mathbb{G}$ (see [15] Chapter 1) and is defined as

(4) $$d_j = i \quad \text{whenever } h_{i-1} + 1 \leq j \leq h_i,$$

hence $1 = d_1 = \ldots = d_{m_1} < d_{m_1+1} = 2 \leq \ldots \leq d_n = \kappa$.

Through this paper, by $\mathbb{G}$-homogeneity we mean homogeneity with respect to group dilations δ_λ (see again [15] Chapter 1).

The Haar measure of $\mathbb{G} = (\mathbb{R}^n, \cdot)$ is the Lebesgue measure in $\mathbb{R}^n$. If $A \subset \mathbb{G}$ is L-measurable, we write $|A|$ to denote its Lebesgue measure. We notice that Lebesgue measure is invariant under the map $y \to y^{-1}$. Moreover, if $m \geq 0$, we denote by $\mathcal{H}^m$ the m-dimensional Hausdorff measure obtained from the Euclidean distance in $\mathbb{R}^n \simeq \mathbb{G}$.

The following result is contained in [15], Proposition 1.26.

5

Proposition 2.1. *If* $j = 1, \ldots, m$, *the vector fields* X_j *have polynomial coefficients and have the form*

(5)
$$X_j(x) = \partial_j + \sum_{k \geq 1; d_k > 1} p_{j,k}(x) \partial_k,$$

where the $p_{j,k}$ *are* $\mathbb{G}$-*homogeneous polynomials of degree* $d_k - 1$ *for* $d_k > 1$.

Once a basis $X_1, \ldots, X_m$ of the horizontal layer is fixed, we define, for any function $f : \mathbb{G} \to \mathbb{R}$ for which the partial derivatives $X_j f$ exist, the horizontal gradient of f, denoted by $\nabla_{\mathbb{G}} f$, as the horizontal section

$$\nabla_{\mathbb{G}} f := \sum_{i=1}^{m} (X_i f) X_i,$$

whose coordinates are $(X_1 f, \ldots, X_m f)$. Moreover, if $\phi = (\phi_1, \ldots, \phi_m)$ is an horizontal section such that $X_j \phi_j \in L^1_{\text{loc}}(\mathbb{G})$ for $j = 1, \ldots, m$, we define $\text{div}_{\mathbb{G}} \phi$ as the real valued function

$$\text{div}_{\mathbb{G}} (\phi) := -\sum_{j=1}^{m} X_j^* \phi_j = \sum_{j=1}^{m} X_j \phi_j.$$

Following [15], we also adopt the following multi-index notation for higher-order derivatives. If $I = (i_1, \ldots, i_n)$ is a multi–index, we set $X^I = X_1^{i_1} \cdots X_n^{i_n}$. By the Poincaré–Birkhoff–Witt theorem (see, e.g. [5], I.2.7), the differential operators X^I form a basis for the algebra of left invariant differential operators in $\mathbb{G}$. Furthermore, we set $|I| := i_1 + \cdots + i_n$ the order of the differential operator X^I, and $d(I) := d_1 i_1 + \cdots + d_n i_n$ its degree of $\mathbb{G}$-homogeneity with respect to group dilations.

We denote by $X_1, \ldots, X_m$ be a basis of the first layer of $\mathfrak{g}$, we denote by $\mathcal{L}$ the associated positive sub-Laplacian

$$\mathcal{L} := -\sum_{j=1}^{m} X_j^2.$$

It is easy to see that

$$\mathcal{L} u = -\text{div}_{\mathbb{G}} (\nabla_{\mathbb{G}} u).$$

In addition, $\mathcal{L}$ is left-invariant, i.e. for any $x \in \mathbb{G}$, we have

$$\mathcal{L}(u \circ \tau_x) = (\mathcal{L} u) \circ \tau_x.$$

Following e.g. [15], we can define a group convolution in $\mathbb{G}$: if, for instance, $f \in \mathcal{D}(\mathbb{G})$ and $g \in L^1_{\text{loc}}(\mathbb{G})$, we set

(6)
$$f * g(x) := \int f(y) g(y^{-1} x) \, dy \quad \text{for } x \in \mathbb{G}.$$

We remind that, if (say) g is a smooth function and L is a left invariant differential operator, then $L(f * g) = f * Lg$. We remind also that the convolution is again well defined when $f, g \in \mathcal{D}'(\mathbb{G})$, provided at least one of them has compact support (as customary, we denote by $\mathcal{E}'(\mathbb{G})$ the class of compactly supported distributions in $\mathbb{G}$ identified with $\mathbb{R}^n$).

If f is a real function defined in $\mathbb{G}$, we denote by $^{\mathsf{v}} f$ the function defined by $^{\mathsf{v}} f(p) := f(p^{-1})$. If $f = {}^{\mathsf{v}} f$ we say that f is check-invariant.

If $E \subset \mathbb{G}$ is a mesurable set, a notion of $\mathbb{G}$-perimeter measure $|\partial E|_{\mathbb{G}}$ has been introduced in [22]. We refer to [22], [18], [20], [19] for a detailed presentation. For our needs, we restrict ourselves to remind that, if E has locally finite $\mathbb{G}$-perimeter (is a $\mathbb{G}$-Caccioppoli set), then $|\partial E|_{\mathbb{G}}$ is a Radon measure in $\mathbb{G}$, $\mathbb{G}$-homogeneous of degree $Q - 1$. Moreover, the following representation theorem holds (see [9]).

Proposition 2.2. *Let $\Omega \subset \mathbb{G}$ be an open set. If E is a $\mathbb{G}$-Caccioppoli set with Euclidean $\mathbf{C}^1$ boundary, then there is an explicit representation of the $\mathbb{G}$-perimeter in terms of the Euclidean $(n - 1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$*

$$|\partial E|_{\mathbb{G}}(\Omega) = \int_{\partial E \cap \Omega} \left(\sum_{j=1}^{m_1} \langle X_j, n \rangle_{\mathbb{R}^n}^2 \right)^{1/2} d\mathcal{H}^{n-1},$$

where $n = n(x)$ is the Euclidean unit outward normal to ∂E.

We have also the following result that generalizes the classical Divergence Theorem.

Proposition 2.3. *If E is a regular bounded open set with Euclidean $\mathbf{C}^1$ boundary and ϕ is a horizontal vector field, continuously differentiable on $\overline{E}$, then*

$$\int_E \operatorname{div}_{\mathbb{G}} \phi \, dx = \int_{\partial E} \langle \phi, n_{\mathbb{G}} \rangle d|\partial E|_{\mathbb{G}},$$

where $n_{\mathbb{G}}(x)$ is the intrinsic horizontal outward normal to ∂E, given by the (normalized) projection of $n(x)$ on the fiber $H\mathbb{G}_x$ of the horizontal fibre bundle $H\mathbb{G}$.

Remark 2.4. The definition of $n_{\mathbb{G}}$ is well done, since $H\mathbb{G}_x$ is transversal to the tangent space to E at x for $|\partial E|_{\mathbb{G}}$-a.e. $x \in \partial E$ (see [31]).

Definition 2.5. (Carnot-Carathéodory distance) An absolutely continuous curve $\gamma : [0, T] \to \mathbb{G}$ is a *sub-unit curve* with respect to $X_1, \ldots, X_m$ if it is an *horizontal curve*, i.e. if there are real measurable functions $c_1(s), \ldots, c_m(s)$, $s \in [0, T]$ such that

$$\dot{\gamma}(s) = \sum_{j=1}^m c_j(s) X_j(\gamma(s)), \qquad \text{for a.e. } s \in [0, T],$$

and if, in addition,

$$\sum_j c_j^2 \leq 1.$$

If $x, y \in \mathbb{G}$, their Carnot-Carathéodory distance (cc-distance) $d_c(x, y)$ is defined as follows:

$d_c(x, y) = \inf \{ T > 0 : \text{ there is a subunit curve } \gamma \text{ with } \gamma(0) = x, \, \gamma(T) = y \}.$

The set of subunit curves joining x and y is not empty, by Chow-Rashevsky Theorem (see for instance [3], Theorem 9.1.3), since by (2), the rank of the Lie algebra generated by $X_1, \ldots, X_m$ is n; hence d_c is a distance on $\mathbb{G}$ inducing the same topology as the standard Euclidean distance. We shall denote

$B_c(x, r)$ the open balls associated with d_c. The cc-distance is well behaved with respect to left translations and dilations, that is

$$(7) \qquad d_c(z \cdot x, z \cdot y) = d_c(x, y) \quad, \quad d_c(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d_c(x, y)$$

for $x, y, z \in \mathbb{G}$ and $\lambda > 0$.

We have also

$$|B_c(x, r)| = r^Q |B_c(0, 1)| \quad \text{and} \quad |\partial B_c(x, r)|_\mathbb{G}(\mathbb{G}) = r^{Q-1} |\partial B_c(0, 1)|_\mathbb{G}(\mathbb{G}).$$

Denote by Y the vector field $\frac{\partial}{\partial y}$ in $\hat{\mathbb{G}} := \mathbb{G} \times \mathbb{R}$. The Lie group $\hat{\mathbb{G}}$ is a Carnot group; its Lie algebra $\hat{\mathfrak{g}}$ admits the stratification

$$(8) \qquad\qquad\qquad \hat{\mathfrak{g}} = \hat{V}_1 \oplus V_2 \oplus \cdots \oplus V_\kappa,$$

where $\hat{V}_1 = \text{span}\,\{Y, V_1\}$. Since the basis $\{X_1, \dots, X_m\}$ of V_1 has been already fixed once and for all, the associated basis for $\hat{V}_1$ will be $\{X_1, \dots, X_m, Y\}$.

The following statement follows trivially from the definition of Carnot-Carathéodory distance, keeping into account that the coefficients of $X_1, \cdots, X_m$ in $\hat{\mathbb{G}}$ are independent of y.

Lemma 2.6. *Denote by $\hat{B}_c((x, y), r)$ a Carnot-Carathéodory ball in $\hat{\mathbb{G}}$ centered at the point $(x, y) \in \hat{\mathbb{G}}$ and $B_c(x, r)$ the Carnot-Carathéodory ball in $\mathbb{G}$ centered at the point $x \in \mathbb{G}$. Then*

$$\hat{B}_c((x, 0), r) \cap \{y = 0\} = B_c(x, r) \times \{0\}.$$

Moreover, if $(x, y) \in K$, where $K \subset \mathbb{G} \times \mathbb{R}$ is a compact set, and $r \leq r_0$ there exist $\sigma_1, \sigma_2 > 0$ (independent of r and (x, y)) such that

$$\hat{B}_c((x, y), \sigma_1 r) \subset B_c(x, r) \times (y - r, y + r) \subset \hat{B}_c((x, y), \sigma_2 r).$$

Definition 2.7 (see [33], [8])**.** A function $\omega \in L^1_{\text{loc}}(\mathbb{G})$ is said to be a A_2-weight with respect to the cc-metric of $\mathbb{G}$ if

$$\sup_{x \in \mathbb{G},\, r > 0} \fint_{B_c(x, r)} \omega(y)\, dy \cdot \fint_{B_c(x, r)} \omega(y)^{-1}\, dy < \infty.$$

The following remark will be crucial in Section 4.

Remark 2.8. By Lemma 2.6, the function $\omega(x, y) = |y|^a$ is a A_2-weight with respect to the cc-metric of $\mathbb{G} \times \mathbb{R}$ if and only if $-1 < a < 1$.

The following result, is the counterpart in the sub-elliptic framework of a well-known result proved in the the Euclidean setting by Fabes, Kenig & Serapioni (see e.g. [12], Lemma 2.3.5 and Theorem 2.4.6). It can be found in [30]. This idea goes back (at least for the so-called "Grushin type" vector fields) to [16] and [17]. Basically, this is possible thanks to weighted Sobolev-Poincaré inequalities in Carnot groups.

For further results concerning the boundary Harnack principle in Carnot groups we refer to [13].

Theorem 2.9. *Let $\mathbb{G}$ be a Carnot group, and let $\Omega \subset \mathbb{G}$ be an open set. Let now $\omega \in L^1_{\text{loc}}(\mathbb{G})$ be a A_2-weight with respect to the Carnot-Carathéodory metric d_c of $\mathbb{G}$. Then, if $u \in W^{1,2}_\mathbb{G}(\Omega, \omega dx)$ is a weak solution of*

$$(9) \qquad\qquad\qquad \text{div}_\mathbb{G}\,(\omega\,\nabla_\mathbb{G} u) = 0,$$

then u is locally Hölder continuous in Ω. *If, in addition,* $u \geq 0$, *then there exist* $C, b > 0$ *(independent of* u) *such that the following invariant Harnack inequality holds:*

$$\sup_{B_c(x,r)} u \leq C \inf_{B_c(x,r)} u$$

for any metric ball $B_c(x,r)$ *such that* $B_c(x, br) \subset \Omega$.

Suppose now Ω *satisfies the following local condition (S): for any* $x_0 \in \partial\Omega$ *there exist* $r_0 > 0$ *and* $\alpha > 0$ *such that*

$$|B_c(x_0, r) \cap \Omega^c| \geq \alpha |B_c(x_0, r)| \quad \text{for } r < r_0.$$

Then u *is locally Hölder continuous in* $\overline{\Omega}$.

3. Fractional powers of subelliptic Laplacians

Definition 3.1. Let $\alpha \in \mathbb{C}$. We call K_α a kernel of type α (according to Folland [14] and [15]) if it is a distribution which is smooth away from 0 and $\mathbb{G}$-homogeneous of degree $\alpha - Q$.

Remark 3.2. Let K_α be a positive kernel of type α; then there exist $m, M \in \mathbb{R}$, with $0 < m \leq M < \infty$, such that

$$m \, d(y,0)^{\alpha - Q} < K_\alpha(y) < M d(y,0)^{\alpha - Q},$$

for any $y \in \mathbb{G}$.

Proposition 3.3. *Suppose* $Q \geq 3$ *and* $0 < \beta < Q$. *Denote by* $h = h(t,x)$ *the fundamental solution of* $\mathcal{L} + \partial/\partial t$ *(see* [14], *Proposition 3.3). Then the integral*

$$R_\beta(x) = \frac{1}{\Gamma(\beta/2)} \int_0^\infty t^{\frac{\beta}{2} - 1} h(t,x) \, dt$$

converges absolutely for $x \neq 0$. *In addition,* R_β *is a kernel of type* β.
Moreover

  i) R_2 *is the fundamental solution of* $\mathcal{L}$;
  ii) *if* $\alpha \in (0,2)$ *and* $u \in \mathcal{D}(\mathbb{G})$, *then*

$$\mathcal{L}^{\alpha/2} u = \mathcal{L}u * R_{2-\alpha}.$$

  iii) *the kernels* R_α *admit the following convolution rule: if* $\alpha > 0$, $\beta > 0$ *and* $x \neq 0$, *then*

$$R_{\alpha+\beta}(x) = R_\alpha(x) * R_\beta(x).$$

Proof. These results are basically contained in [14]. Let us sketch the proof of ii): by [14], Theorem 3.15, iii), and Proposition 3.18, keeping in mind that $\mathcal{D}(\mathbb{G})$ is contained in the domain of all real powers of $\mathcal{L}$, we obtain

$$\mathcal{L}^{\alpha/2} u = \mathcal{L}^{(\alpha-2)/2} \mathcal{L}u = \mathcal{L}u * R_{2-\alpha}.$$

$\square$

Remark 3.4. If $\beta < 0$, $\beta \notin \{0, -2, -4, \cdots\}$, then again

$$\widetilde{R}_\beta(x) = \frac{\frac{\beta}{2}}{\Gamma(\beta/2)} \int_0^\infty t^{\frac{\beta}{2} - 1} h(t,x) \, dt$$

defines a smooth function in $\mathbb{G} \setminus \{0\}$, since $t \to h(t,x)$ vanishes of infinite order as $t \to 0$ if $x \neq 0$. In addition, $\widetilde{R}_\beta$ is positive and $\mathbb{G}$-homogeneous of

degree $\beta - Q$. However, unlike R_β for $0 < \beta < Q$, $\widetilde{R}_\beta$ is not a kernel of type β, since it does not belong to $L^1_{\mathrm{loc}}(\mathbb{G})$. Integrating by parts, it is easy to see also that, if $0 < \alpha < 2$, then

$$\mathcal{L}R_{2-\alpha} = \widetilde{R}_{-\alpha}$$

for $x \neq 0$.

Definition 3.5. We set (we remind that $R_\beta > 0$ for $0 < \beta < Q$)

$$\rho(x) = R_{2-\alpha}^{1/(2-\alpha-Q)}.$$

It is easy to see that ρ is an $\mathbb{G}$-homogeneous norm in $\mathbb{G}$, smooth outside of the origin. In addition, $d(x,y) := \rho(y^{-1}x)$ is a quasi-distance in $\mathbb{G}$. In turn, d is equivalent to the Carnot-Carathéodory distance on G, as well as to any other $\mathbb{G}$-homogeneous left invariant distance on $\mathbb{G}$.

A standard argument based on group dilations yields the following estimates.

Proposition 3.6. *Denote by* $B_\rho = B_\rho(x, r)$ *the metric balls given by* ρ. *We have:*

(10) $$md_c(x, y) \leq d(x, y) \leq Md_c(x, y) \quad \text{for all } x, y \in \mathbb{G};$$

(11) $$mr^Q \leq |B_\rho(x, r)| \leq Mr^Q;$$

(12) $$mr^{Q-1} \leq \mathcal{H}_{\mathbb{G}}^{Q-1}(\partial B_\rho(x, r)) \leq Mr^{Q-1}.$$

Definition 3.7. We denote by $x \to {}^{\mathrm{w}}x$ the "semicheck" map

$$(x_1, \ldots, x_n) \to ((-1)^{d_1} x_1, (-1)^{d_2} x_2, \ldots, (-1)^{d_n} x_n).$$

From now on, we adopt the following notation:${}^{\mathrm{w}}f(x, t) := f({}^{\mathrm{w}}x, t)$ for any function f defined in $\mathbb{G} \times \mathbb{R}$.

Theorem 3.8. *We have:*

    i) *if* $j = 1, \ldots, m$, *then* $X_j{}^{\mathrm{w}} = -{}^{\mathrm{w}}X_j$. *In particular,* $\mathcal{L}^{\mathrm{w}} = {}^{\mathrm{w}}\mathcal{L}$;
    ii) *if* h *is the fundamental solution of* $\partial_t + \mathcal{L}$, *then* ${}^{\mathrm{w}}h = h$;
    iii) *if* $\alpha > 0$, $R_\alpha = {}^{\mathrm{w}}R_\alpha$ *and* $\tilde{R}_{-\alpha} = {}^{\mathrm{w}}\tilde{R}_{-\alpha}$. *In particular,* ${}^{\mathrm{w}}\rho = \rho$;
    iv) $d_c({}^{\mathrm{w}}x, {}^{\mathrm{w}}y) = d_c(x, y)$ *for all* $x, y \in \mathbb{G}$.
    v) *if* $E \subset \mathbb{G}$ *is a* $\mathbb{G}$-*Cacciopoli set, then the perimeter measure* $|\partial E|_{\mathbb{G}}$ *is semicheck-invariant.*

Proof. The core of the proof relies in the following identity. If p_{jk} are the polynomials defined in Proposition 2.1, then

(13) $$p_{j,k}({}^{\mathrm{w}}x) = (-1)^{d_k-1} p_{j,k}(x).$$

To prove (13), we remind that $p_{j,k}$ is a $\mathbb{G}$-homogeneous polynomials of degree $d_k - 1$. Let now $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index, and let x^α be an arbitrary $\mathbb{G}$-homogeneous monomial of degree $d_k - 1$, i.e. assume

(14) $$d_1\alpha_1 + \cdots + d_n\alpha_n = d_k - 1.$$

We have but to show that (13) holds for x^α.

If $\ell = 1, \ldots, n$, we set $I_\ell := \{i \,;\, d_i = \ell\}$. Gathering in (14) the terms with $d_i = \ell$, identity (14) becomes

$$(15) \qquad \sum_\ell d_\ell \big(\sum_{i \in I_\ell} \alpha_i\big) = d_k - 1.$$

Then

$$(^{\mathrm{w}}x)^\alpha = (-1)^{\sum_\ell d_\ell(\sum_{i \in I_\ell} \alpha_i)} x^\alpha,$$

and the assertion follows by (15).

Let us prove now i). If u is a (say) smooth function, by (13), we have

$$X_j(^{\mathrm{w}}u(x)) = X_j(u(^{\mathrm{w}}x))$$
$$= -(\partial_j u)(^{\mathrm{w}}x) + \sum_{d_k > 1} p_{j,k}(x)(-1)^{d_k}(\partial_k u)(^{\mathrm{w}}x)$$
$$= -(\partial_j u)(^{\mathrm{w}}x) - \sum_{d_k > 1} p_{j,k}(^{\mathrm{w}}x)(\partial_k u)(^{\mathrm{w}}x)$$
$$= -(X_j u)(^{\mathrm{w}}x) = -^{\mathrm{w}}(X_j u)(x).$$

In order to prove ii), let us show preliminarily that $h(t, {}^{\mathrm{w}}x)$ is still a fundamental solution of $\partial_t + \mathcal{L}$. Indeed, if $u \in \mathcal{D}(\mathbb{R} \times \mathbb{G})$, we have

$$\langle (\partial_t + \mathcal{L})h(t, {}^{\mathrm{w}}x) | u(t, x) \rangle = \langle h(t, {}^{\mathrm{w}}x) | (-\partial_t + \mathcal{L})u(t, x) \rangle$$
$$= \langle h | ^{\mathrm{w}}(-\partial_t + \mathcal{L})u \rangle = \langle h | (-\partial_t + \mathcal{L})^{\mathrm{w}}u \rangle$$
$$= \langle (\partial_t + \mathcal{L})h | ^{\mathrm{w}}u \rangle = {}^{\mathrm{w}}u(0,0) = u(0,0).$$

Therefore, the function

$$h_0 := h - h^{\mathrm{w}}$$

vanishes at $t = 0$ and solves $(\partial_t + \mathcal{L})h_0 = 0$, being in particular smooth in $\mathbb{R} \times \mathbb{G}$, by the hypoellipticity of $\partial_t + \mathcal{L}$ ([24]). By [14], Corollary 3.5, $h_0(t, x) \to 0$ as $x \to \infty$ uniformly for t in a bounded interval. Thus we can apply the standard "parabolic" maximum principle to conclude that $h_0 \equiv 0$, and then ii) follows.

The proof of iii) is straightforward. To prove iv), it is enough to show that, if $x, y \in \mathbb{G}$ and γ is a horizontal curve joining x and y with sub-Riemannian length $\ell(\gamma)$, then $^{\mathrm{w}}\gamma$ is still horizontal, $\ell(^{\mathrm{w}}\gamma) = \ell(\gamma)$, and, obviously, joins x and y.

By assumption, we can write

$$\gamma'(t) = \sum_{j=1}^m a_j(t) X_j(\gamma(t)), \quad t \in [0,1],$$

i.e, if for any $p \in \mathbb{G}$ we write p_ℓ for the ℓ-th component of p in exponential coordinates, for $\ell = 1, 2, \ldots, n$, then

$$(16) \qquad \gamma'_\ell = \sum_{j=1}^m a_j(X_j(\gamma))_\ell = \sum_{j=1}^m a_j(e_j + \sum_{d_k > 1} p_{j,k}(\gamma)e_k)_\ell,$$

with

$$\int_0^1 \big(\sum_j a_j^2(t)\big)^{1/2} dt = \ell(\gamma).$$

Notice that (16) reads as follows:

$$(17) \qquad \gamma'_\ell = \begin{cases} a_\ell & \text{if } 1 \le \ell \le m \\ \displaystyle\sum_j a_j p_{j,\ell}(\gamma) & \text{if } \ell > m. \end{cases}$$

Our assertion will follow by showing that

$$(18) \qquad (^{\mathrm{w}}\gamma)'(t) = -\sum_{j=1}^{m} a_j(t) X_j(^{\mathrm{w}}\gamma(t)), \quad t \in [0,1],$$

Indeed, by (13),

$$X_j(^{\mathrm{w}}\gamma) = e_j + \sum_{d_k > 1} p_{j,k}(^{\mathrm{w}}\gamma) e_k$$

$$= (-1)^{d_1 - 1} e_j + \sum_{d_k > 1} (-1)^{d_k - 1} p_{j,k}(\gamma) e_k,$$

so that, keeping in mind (17),

$$(\sum_{j=1}^{m} a_j X_j(^{\mathrm{w}}\gamma))_\ell = \begin{cases} -(-1)^{d_1} a_\ell = -(^{\mathrm{w}}\gamma)_\ell & \text{if } 1 \le \ell \le m \\ -(-1)^{d_\ell} \displaystyle\sum_j a_j p_{j,\ell}(\gamma) = -(^{\mathrm{w}}\gamma)_\ell & \text{if } \ell > m. \end{cases}$$

This proves (18) and achieves the proof of the theorem, since v) is a straight-forward consequence of i). □

Corollary 3.9. *If $\alpha > 0$ and $j = 1, \ldots, m$, then*

$$^{\mathrm{w}}(X_j R_\alpha) = -X_j R_\alpha \quad and \quad ^{\mathrm{w}}(X_j \tilde{R}_{-\alpha}) = -X_j \tilde{R}_{-\alpha}.$$

We follow the guidelines of [14], Section 3. We have:

Theorem 3.10. *The operator $\mathcal{L}$ is a positive self-adjoint operator with domain $W_{\mathbb{G}}^{2,2}(\mathbb{G})$. Denote now by $\{E(\lambda)\}$ the spectral resolution of $\mathcal{L}$ in $L^2(\mathbb{G})$. If $\alpha > 0$ then*

$$\mathcal{L}^{\alpha/2} = \int_0^{+\infty} \lambda^{\alpha/2} dE(\lambda)$$

with domain

$$W_{\mathbb{G}}^{\alpha,2}(\mathbb{G}) := \{u \in L^2(\mathbb{G}) \; : \; \int_0^{+\infty} \lambda^\alpha d\langle E(\lambda)u, u \rangle < \infty\},$$

endowed with the graph norm.

Finally, if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n \equiv \mathbb{G}$ and $x_0 \in \mathbb{G}$ are given, we set

$$\pi_{x_0}(x) = \sum_{j=1}^{m_1} x_j X_j(x_0).$$

The map $x_0 \to \pi_{x_0}(x)$ is a smooth section of $H\mathbb{G}$.

12

Theorem 3.11. *If u belongs to the Schwartz class $S(\mathbb{G})$ (see [15], Section I.D), and $0 < \alpha < 2$, then $\mathcal{L}^{\alpha/2} u \in L^2(\mathbb{G})$, and*

$$\mathcal{L}^{\alpha/2} u(x) = \int_{\mathbb{G}} \left(u(xy) - u(x) - \omega(y) \langle \nabla_{\mathbb{G}} u(x), \pi_x(y) \rangle \right) \widetilde{R}_{-\alpha}(y) \, dy$$

$$= \text{P.V.} \int_{\mathbb{G}} (u(y) - u(x)) \widetilde{R}_{-\alpha}(y^{-1} x) \, dy,$$

where ω is the characteristic function of the unit ball $B_\rho(0, 1)$.

Proof. First of all, we notice that the map

$$y \to (u(xy) - u(x) - \omega(y) \langle \nabla_{\mathbb{G}} u(x), \pi_x(y) \rangle) \widetilde{R}_{-\alpha}(y)$$

belongs to $L^1(\mathbb{G})$. Indeed,

$$(u(xy) - u(x) - \omega(y) \langle \nabla_{\mathbb{G}} u(x), \pi_x(y) \rangle) \widetilde{R}_{-\alpha}(y) = O(\rho(y)^{-Q-\alpha})$$

as $y \to \infty$, and

$$(u(xy) - u(x) - \omega(y) \langle \nabla_{\mathbb{G}} u(x), \pi_x(y) \rangle) \widetilde{R}_{-\alpha}(y) = O(\rho(y)^{-Q+2-\alpha})$$

as $y \to 0$, since ([15], (1.37))

$$u(xy) - u(x) - \langle \nabla_{\mathbb{G}} u(x), \pi_x(y) \rangle = O(\rho(y)^2)$$

If $\varepsilon > 0$, keeping in mind that both ρ and $\widetilde{R}_{-\alpha}$ are check-invariant, we can write

$$\int_{\rho(y^{-1}x) > \varepsilon} (u(y) - u(x)) \widetilde{R}_{-\alpha}(y^{-1} x) \, dy = \int_{\rho(y) > \varepsilon} (u(xy) - u(x)) \widetilde{R}_{-\alpha}(y) \, dy.$$

Notice both integral are absolutely convergent, since $y \to (u(xy) - u(x)) \widetilde{R}_{-\alpha}(y)$ is a smooth function away from the origin and $(u(xy) - u(x)) \widetilde{R}_{-\alpha}(y) = O(\rho(y)^{-Q-\alpha})$ as $y \to \infty$. On the other hand, the map

$$y \to \omega(y) \langle \nabla_{\mathbb{G}} u(x), \pi_x(y) \rangle \widetilde{R}_{-\alpha}(y)$$

(that belongs to $L^1(\{\rho(y) > \varepsilon\})$) has zero integral, since $\omega(y) \widetilde{R}_{-\alpha}(y)$ is check-invariant, whereas $\langle \nabla_{\mathbb{G}} u(x), y^{-1} \rangle = \langle \nabla_{\mathbb{G}} u(x), \pi_x(y) \rangle$. Therefore, we can write

$$\int_{\rho(y^{-1}x) > \varepsilon} (u(y) - u(x)) \widetilde{R}_{-\alpha}(y^{-1} x) \, dy$$

$$= \int_{\rho(y) > \varepsilon} \left(u(xy) - u(x) - \omega(y) \langle \nabla_{\mathbb{G}} u(x), \pi_x(y) \rangle \right) \widetilde{R}_{-\alpha}(y) \, dy,$$

so that

$$\int_{\mathbb{G}} \left(u(xy) - u(x) - \omega(y) \langle \nabla_{\mathbb{G}} u(x), \pi_x(y) \rangle \right) \widetilde{R}_{-\alpha}(y) \, dy = \text{P.V.} \int_{\mathbb{G}} (u(y) - u(x)) \widetilde{R}_{-\alpha}(y^{-1} x) \, dy.$$

We want to show now that

(19)
$$\int_{\rho(y^{-1}x) > \varepsilon} (u(y) - u(x)) \widetilde{R}_{-\alpha}(y^{-1} x) \, dy$$

$$= \int_{\rho(y^{-1}x) > \varepsilon} \mathcal{L}u(y) R_{2-\alpha}(x^{-1} y) \, dy + o(1)$$

as $\varepsilon \to 0$. Notice both integrals absolutely converge at infinity.

Take now $R > \varepsilon$. By Green identity (see e.g. [3], formula (5.43b)), we have

$$\int_{\varepsilon < \rho(y^{-1}x) < R} (u(y) - u(x))\widetilde{R}_{-\alpha}(y^{-1}x)\, dy$$

$$= \int_{\varepsilon < \rho(y^{-1}x) < R} (u(y) - u(x))\mathcal{L}R_{2-\alpha}(y^{-1}x)\, dy$$

$$= \int_{\varepsilon < \rho(y^{-1}x) < R} \mathcal{L}u(y)R_{2-\alpha}(x^{-1}y)\, dy$$

$$+ \int_{\varepsilon = \rho(y^{-1}x)} R_{2-\alpha}(x^{-1}y)\sum_j X_j(u(y) - u(x))\langle X_j, \nu\rangle\, d\mathcal{H}^{n-1}(y)$$

$$- \int_{\varepsilon = \rho(y^{-1}x)} (u(y) - u(x))\sum_j X_j R_{2-\alpha}(x^{-1}y)\langle X_j, \nu\rangle\, d\mathcal{H}^{n-1}(y)$$

$$+ \int_{R = \rho(y^{-1}x)} R_{2-\alpha}(x^{-1}y)\sum_j X_j(u(y) - u(x))\langle X_j, \nu\rangle\, d\mathcal{H}^{n-1}(y)$$

$$- \int_{R = \rho(y^{-1}x)} (u(y) - u(x))\sum_j X_j R_{2-\alpha}(x^{-1}y)\langle X_j, \nu\rangle\, d\mathcal{H}^{n-1}(y)$$

$$= \int_{\varepsilon < \rho(y^{-1}x) < R} \mathcal{L}u(y)R_{2-\alpha}(x^{-1}y)\, dy$$

$$+ I^1(\varepsilon) + I^2(\varepsilon) + J^1(R) + J^2(R),$$

where ν in the outward unit normal to $\{\varepsilon < \rho(y^{-1}x) < R\}$. Obviously, J^1 vanishes as $R \to \infty$. Again, by Remark 3.2, if R is large, we have

$$|J^2(R)| \le C|u(x)|R^{1-\alpha-Q}\int_{R=\rho(y^{-1}x)} \sum_j |\langle X_j, \nu\rangle|\, d\mathcal{H}^{n-1}(y)$$

$$\le C|u(x)|R^{1-\alpha-Q}\int_{R=\rho(y^{-1}x)} d\mathcal{H}_{\mathbb{G}}^{Q-1}(y) \quad \text{(by Proposition 2.2)}$$

$$= O(R^{-\alpha}),$$

by (12). Thus we can take above the limit as $R \to \infty$ and we get

$$\int_{\varepsilon < \rho(y^{-1}x)} (u(y) - u(x))\widetilde{R}_{-\alpha}(y^{-1}x)\, dy$$

$$= \int_{\varepsilon < \rho(y^{-1}x)} \mathcal{L}u(y)R_{2-\alpha}(x^{-1}y)\, dy + I^1(\varepsilon) + I^2(\varepsilon)$$

(notice again both integrals are absolutely convergent).

Thus, (19) will follow by showing that $I^1(\varepsilon) + I^2(\varepsilon) = o(1)$ as $\varepsilon \to 0$.

Consider $I^1(\varepsilon)$. First of all, we notice that

$$(20) \qquad \int_{\varepsilon = \rho(y^{-1}x)} R_{2-\alpha}(x^{-1}y)\langle X_j, \nu\rangle\, d\mathcal{H}^{n-1}(y)$$

$$= \int_{\varepsilon = \rho(y)} R_{2-\alpha}(y)\langle X_j, \nu\rangle\, d\mathcal{H}^{n-1}(y) = 0$$

14

for $j = 1, \ldots, m$. Indeed we can write

$$\int_{\varepsilon=\rho} R_{2-\alpha} \langle X_j, \nu \rangle \, d\mathcal{H}^{n-1}$$
$$= \int_{\varepsilon=\rho} R_{2-\alpha}(X_j R_{2-\alpha}) |\nabla R_{2-\alpha}|^{-1} \, d\mathcal{H}^{n-1},$$

and (20) follows, since ρ, $R_{2-\alpha}$, $|\nabla R_{2-\alpha}|$, and $\mathcal{H}^{n-1}$ are even under the change of variables $y \to {}^wy$, whereas $X_j R_{2-\alpha}$ is odd. Thus, by Proposition 2.2, we can write

$$I^1(\varepsilon) = \int_{\varepsilon=\rho(y)} R_{2-\alpha}(y) \sum_j (X_j u)(xy) \langle X_j, \nu \rangle \, d\mathcal{H}^{n-1}(y)$$
$$= \int_{\varepsilon=\rho(y)} R_{2-\alpha}(y) \sum_j [X_j u(xy) - X_j u(x)] \langle X_j, \nu \rangle \, d\mathcal{H}^{n-1}(y)$$
$$\leq C \max |X^2 u| \, \varepsilon^{3-\alpha-Q} \int_{\varepsilon=\rho} d\mathcal{H}_{\mathbb{G}}^{Q-1}(y)$$
$$= O(\varepsilon^{2-\alpha}) = o(1) \quad \text{as } \varepsilon \to 0.$$

Finally, $I^2(\varepsilon)$ can be estimated by similar arguments. We write

$$I^2(\varepsilon) = -\int_{\varepsilon=\rho} (u(xy) - u(x)) \sum_j (X_j R_{2-\alpha}) \langle X_j, \nu \rangle \, d\mathcal{H}^{n-1}(y),$$

and we notice that, if $1 \leq \ell \leq m$

$$\int_{\varepsilon=\rho} y_\ell \sum_j (X_j R_{2-\alpha}) \langle X_j, \nu \rangle \, d\mathcal{H}^{n-1}(y) = 0.$$

Indeed, keeping again in mind Proposition 2.2, we have

$$\int_{\varepsilon=\rho} y_\ell \sum_j (X_j R_{2-\alpha}) \langle X_j, \nu \rangle \, d\mathcal{H}^{n-1}(y)$$
$$= \int_{\varepsilon=\rho} y_\ell \sum_j |X_j R_{2-\alpha}|^2 |\nabla R_{2-\alpha}|^{-1} \, d\mathcal{H}^{n-1}(y)$$
$$= \int_{\varepsilon=\rho} y_\ell \, |\nabla_{\mathbb{G}} R_{2-\alpha}| (\sum_j \langle X_j, \nu \rangle^2)^{1/2} \, d\mathcal{H}^{n-1}(y)$$
$$= \int_{\varepsilon=\rho} y_\ell \, |\nabla_{\mathbb{G}} R_{2-\alpha}| \, d\mathcal{H}_{\mathbb{G}}^{Q-1}(y) = 0$$

since both $|\nabla_{\mathbb{G}} R_{2-\alpha}|$ and $\mathcal{H}_{\mathbb{G}}^{Q-1}$ are even with respect to the change of variable $y \to {}^ww$, whereas y_ℓ is odd.

Therefore, keeping in mind Taylor inequality in $\mathbb{G}$ (see, e.g. [15] Theorem 1.37), as well as Remark 3.2 and, again, Proposition 2.2, we can write

$$|I^2(\varepsilon)|$$
$$= \left| \int_{\varepsilon=\rho} (u(xy) - u(x) - \sum_\ell X_\ell u(x) y_\ell) \sum_j (X_j R_{2-\alpha}) \langle X_j, \nu \rangle \, d\mathcal{H}^{n-1}(y) \right|$$
$$= O(\varepsilon^{3-\alpha-Q}) \mathcal{H}_{\mathbb{G}}^{Q-1}(\{\varepsilon = \rho\}) = O(\varepsilon^{2-\alpha}) = o(1).$$

15

This achieves the proof of (19). Taking the limit as $\varepsilon \to 0$ in (19), and keeping in mind that $R_{2-\alpha}\mathcal{L}u \in L^1(\mathbb{G})$, we get eventually

$$\text{P.V} \int_{\mathbb{G}} (u(y) - u(x)) \widetilde{R}_{-\alpha}(y^{-1}x)\, dy = \int_{\mathbb{G}} \mathcal{L}u(y) R_{2-\alpha}(x^{-1}y)\, dy$$
$$= \mathcal{L}^{\alpha/2}u,$$

by Proposition 3.3. This achieves the proof of the theorem. □

4. MAIN RESULTS

Proposition 4.1 (see also Caffarelli & Silvestre [6])**.** *If* $-\infty < \alpha < 1$, *the boundary value problem*

$$(21) \qquad \begin{cases} -t^\alpha \phi'' + \phi = 0 \\ \phi(0) = 1 \\ \lim_{t \to +\infty} \phi(t) = 0 \end{cases}$$

has a solution $\phi \in \mathbf{C}^{2-\alpha}([0,\infty))$ *of the form*

$$\phi(t) = c_\alpha\, t^{1/2} K_{1/2k}(k^{-1}t^k),$$

where $c_\alpha := 2^{1-1/2k}\Gamma(1/2k)^{-1}k^{-1/2k} > 0$ *is a positive constant,* $k = \frac{2-\alpha}{2}$, *and* $K_{1/2k}$ *is the modified Bessel function of second kind (see [38]). We know that*

  i) $0 < \phi < 1$. *Moreover* $\phi'(t)$ *has a finite limit as* $t \to 0$ *and, recursively,*

  $$t^{\alpha+h-2}\phi^{(h)}(t) \text{ has a finite limit as } t \to 0$$

  for $h = 2, 3, \dots$;
  ii) $\phi' \in L^2((0,\infty))$;
  iii) $\phi(t) = c\sqrt{\frac{\pi k}{2}}\, t^{\alpha/2}\, e^{-t^k/k}\big(1 + O(\frac{1}{t})\big)$ *as* $t \to \infty$;
  iv) $\phi^{(h)}(t) = c_h\, t^{\alpha(1-h)/2}\, e^{-t^k/k}\big(1 + o(1)\big)$ *as* $t \to \infty$ *for* $h = 1, 2, \dots$.

Proof. By iteration, we can reduce ourselves to prove the assertion for $h = 1$. Since ϕ is convex, $\phi'(t) \to 0$ as $t \to \infty$ and we can write

$$\phi'(t) = \int_t^\infty s^{-\alpha}\phi(s)\, ds = c\sqrt{\frac{\pi k}{2}} \int_t^\infty s^{-\alpha/2}\, e^{-s^k/k}\big(1 + o(1)\big)\, ds.$$

Then the estimate follows by the de l'Hôpital's rule. □

Remark 4.2. The exact value of $\phi'(0)$ can be explicitly computed keeping in mind that

$$\phi'(0) = c_\alpha \int_0^\infty s^{-\alpha+1/2} K_{1/2k}\big(\frac{1}{k}s^k\big)\, ds = \frac{c_\alpha}{k} \int_0^\infty t^{(a+1)/2} K_{1/2k}\big(\frac{1}{k}t\big)\, dt,$$

and that the last integral in turn can be explicitly evaluated by [23], 6.561 (16).

Put $\theta := (1-a)^{a-1}$. If $u \in W^{1-a,2}(\mathbb{G})$, for $y > 0$ we set

$$(22) \qquad v(\cdot, y) := \phi(\theta y^{1-a} \mathcal{L}^{(1-a)/2})u := \int_0^\infty \phi(\theta y^{1-a}\lambda^{(1-a)/2})\, dE(\lambda)u,$$

Notice v is well defined since ϕ is continuous and bounded in $[0,\infty)$.

Choose now
$$\alpha = -\frac{2a}{1-a}.$$

Proposition 4.3. *Set $\Sigma_+ = \mathbb{G}_x \times (0,1)_y$ and $\Sigma_+^\varepsilon = \mathbb{G}_x \times (\varepsilon,1)_y$. If*
$$s \geq 1 - \frac{a+1}{2} \quad and \quad u \in W^{s,2}(\mathbb{G}),$$
then $v \in W_{\hat{\mathbb{G}}}^{1,2}(\Sigma_+; y^a dx\, dy)$ and

(23)
$$\|v\|_{W_{\hat{G}}^{1,2}(\Sigma_+; y^a dx\, dy)} \leq C\, \|u\|_{W^{s,2}(\mathbb{G})}.$$

Moreover, if
$$s \geq 2 - \frac{a+1}{2} \quad and \quad u \in W^{s,2}(\mathbb{G}),$$
then $v \in W_{\hat{G}}^{2,2}(\Sigma_+^\varepsilon; y^a dx\, dy)$ for any $\varepsilon > 0$.

Proof. The function v belongs to $L^2(\Sigma_+; y^a dx\, dy)$. Indeed

$$\|v\|_{L^2(\Sigma_+; y^a dx\, dy)}^2 = \int_0^1 dy\, y^a \|v(y,\cdot)\|_{L^2(\mathbb{R}^n)}^2$$

$$= \int_0^1 dy\, y^a \int_0^\infty \phi^2(\theta y^{1-a} \lambda^{(1-a)/2})\, d\|E(\lambda)u\|^2 \leq C\|u\|_{L^2(\mathbb{G})}^2,$$

since ϕ is bounded.

On the other hand, if $\varepsilon \geq 0$,

$$\|v\|_{W_{\hat{G}}^{k,2}(\Sigma_+; y^a dx\, dy)}^2 = \sum_{0 \leq h \leq k} \sum_{|\beta| \leq k-h} \|\partial_y^k X^\beta v\|_{L^2(\Sigma_+; y^a dx\, dy)}^2$$

$$= \sum_{0 \leq h \leq k} \int_\varepsilon^1 dy\, y^a \int_\mathbb{G} dx \sum_{|\beta| \leq k-h} \|X^\beta \partial_y^h v\|^2$$

$$= \sum_{0 \leq h \leq k} \int_\varepsilon^1 dy\, y^a \|\partial_y^h v\|_{W^{k-h,2}(\mathbb{G})}^2$$

$$\approx \sum_{0 \leq h \leq k} \int_\varepsilon^1 dy\, y^a \int_\mathbb{G} dx \|\mathcal{L}^{(k-h)/2} \partial_y^h v\|_{W^{k-h,2}(\mathbb{G})}^2$$

$$= \sum_{0 \leq h \leq k} \int_\varepsilon^1 dy\, y^a \int_0^\infty \lambda^{k-h} |\partial_y^h \phi(\theta y^{1-a} \lambda^{(1-a)/2})|^2\, d\|E(\lambda)u\|^2.$$

Recalling that
$$\sum_{j=1}^h m_j = m$$
and
$$\sum_{j=1}^h j m_j = h$$
the last term can be estimated by a sum of terms of the form

$$\int_\varepsilon^1 dy\, y^a \int_0^\infty \lambda^{k-h+(1-a)m} y^{2m(1-a)-2h} |\phi^{(h)}(\theta y^{1-a} \lambda^{(1-a)/2})|^2\, d\|E(\lambda)u\|^2,$$

with $m \leq h$. If we put $y\sqrt{\lambda} = \tau$, the last term is estimated by

$$\int_0^\infty d\|E(\lambda)u\|^2 \lambda^{-\frac{a}{2}+k-h+(1-a)m-(1-a)m+h-\frac{1}{2}}$$

$$\cdot \int_{\varepsilon\sqrt{\lambda}}^\infty \tau^{2m(1-a)-2h+2a} |\phi^{(h)}(\theta\tau^{1-a})|^2 d(\tau^{1-a})$$

(24)

$$= \int_0^\infty d\|E(\lambda)u\|^2 \lambda^{k-\frac{a+1}{2}}$$

$$\cdot \int_{(\varepsilon\sqrt{\lambda})^{1-a}}^\infty s^{2m-2(h-a)/(1-a)} |\phi^{(h)}(\theta s)|^2 ds$$

Consider now the case $k = 1$ (and therefore $h = m = 1$, since the case $h = 0$ yields the L^2-estimate we have already proved). Then we can take $\varepsilon = 0$ and the last term becomes

$$\int_0^\infty d\|E(\lambda)u\|^2 \lambda^{1-\frac{a+1}{2}} \cdot \int_0^\infty |\phi'(\theta s)|^2 ds$$

$$\leq \int_0^\infty d\|E(\lambda)u\|^2 (1 + \lambda^s) \cdot \int_0^\infty |\phi'(\theta s)|^2 ds \leq C \|u\|_{W^{s,2}(\mathbb{G})}^2,$$

by ii) above.

Consider now the case $k = 2$. In this case, we take $\varepsilon > 0$ and we split the last integral in (24) as

$$\int_0^1 d\|E(\lambda)u\|^2 \cdots + \int_1^\infty d\|E(\lambda)u\|^2 \cdots := I_1 + I_2.$$

Obviously,

$$I_2 \leq \int_1^\infty d\|E(\lambda)u\|^2 \int_{\varepsilon^{1-a}}^\infty s^{2m-2(h-a)/(1-a)} |\phi^{(h)}(\theta s)|^2 ds < \infty,$$

since $\phi^{(h)}(s)$ vanishes exponentially as $s \to \infty$. Analogously,

(25)

$$I_1 \leq \int_0^1 d\|E(\lambda)u\|^2 \lambda^{2-\frac{a+1}{2}} \int_{(\varepsilon\sqrt{\lambda})^{1-a}}^1 \cdots ds$$

$$+ \int_0^1 d\|E(\lambda)u\|^2 \lambda^{2-\frac{a+1}{2}} \int_1^\infty \cdots ds$$

Clearly, the second term in (25) is finite, again since since $\phi^{(h)}(s)$ vanishes exponentially as $s \to \infty$. Thus, we are reduced to estimate

$$\int_0^1 d\|E(\lambda)u\|^2 \lambda^{2-\frac{a+1}{2}}$$

$$\cdot \int_{(\varepsilon\sqrt{\lambda})^{1-a}}^1 s^{2m-2(h-a)/(1-a)-2\alpha-2h+4} |s^{\alpha+h-2}\phi^{(h)}(\theta s)|^2 ds$$

$$\leq C \int_0^1 d\|E(\lambda)u\|^2 \lambda^{2-\frac{a+1}{2}}$$

$$\cdot \int_{(\varepsilon\sqrt{\lambda})^{1-a}}^1 s^{2m-2(h-a)/(1-a)-2\alpha-2h+4} ds,$$

18

by Proposition 4.1, i). If we keep in mind that

$$\int_0^1 d\|E(\lambda)u\|^2 \lambda^{2-\frac{a+1}{2}} < \infty$$

since $u \in W_{\mathbb{G}}^{s,2}(\mathbb{G})$, with $s \geq 2 - \frac{a+1}{2}$, to achieve the proof of the proposition we have but to show that

$$2 - \frac{a+1}{2} + (1-a)\big(2m - 2(h-a)/(1-a) - 2\alpha - 2h + 5\big)$$
$$= (m-h)(1-a) - h + 4 \geq (m-h)(1-a) + 2 > 0.$$

On the other hand, if $h = 1$, then necessarily $m = 1$, so that $(m-h)(1-a) + 2 = 2$, whereas, if $h = 2$, then either $m = 1$ or $m = 2$. In the first case $(m-h)(1-a) + 2 = a+1 > 0$. Finally, if $m = 2$, then $(m-h)(1-a) + 2 = 2$, achieving the proof of the proposition. $\qquad\square$

Theorem 4.4 (generalized subordination identity). *If $u \in L^2(\mathbb{G})$ and $y > 0$, we set*

$$v(\cdot, y) := \phi(\theta y^{1-a}\, \mathcal{L}^{(1-a)/2})u := \int_0^\infty \phi(\theta y^{1-a}\, \lambda^{(1-a)/2})\, dE(\lambda)u,$$

where $\theta := (1-a)^{a-1}$ (we remind that ϕ is bounded, and therefore $v \in L^2(\mathbb{G})$ for $y > 0$).

We denote by $h(t, \cdot)$ the heat kernel associated with $-\mathcal{L}$ as in [14], and by $P_{\mathbb{G}}(\cdot, y)$ the "Poisson kernel"

$$(26) \qquad P_{\mathbb{G}}(\cdot, y) := C_a\, y^{1-a} \int_0^\infty t^{(a-3)/2} e^{-\frac{y^2}{4t}} h(t, \cdot)\, dt,$$

where

$$C_a = \frac{2^{a-1}}{\Gamma((1-a)/2)}.$$

Then

$$P_{\mathbb{G}}(\cdot, y) \geq 0$$

by [25], [14], Theorem 3.1, and

$$(27) \qquad v(\cdot, y) = u * P_{\mathbb{G}}(\cdot, y).$$

Proof. By identity (8), p. 182 of [38], if $\nu > 0$ and $z > 0$, we can write

$$K_\nu(z) = \frac{1}{2} \int_0^\infty \xi^{-\nu-1} e^{-\frac{1}{2}z(\xi+\frac{1}{\xi})}\, d\xi.$$

Then (keeping also in mind the definition of θ)

$$\phi(\theta z) := \frac{1}{2} c_\alpha \theta^{1/2} z^{1/2} \int_0^\infty \xi^{-\frac{1}{2k}-1} e^{-\frac{\theta^k}{2k} z^k(\xi+\frac{1}{\xi})}\, d\xi$$
$$= \frac{1}{2} c_\alpha \theta^{1/2} z^{1/2} \int_0^\infty \xi^{-\frac{1}{2k}-1} e^{-\frac{\theta^k}{2k} z^k \xi}\, e^{-\frac{\theta^k}{2k}\frac{z^k}{\xi}}\, d\xi$$
$$= 2^{(a-3)/2} c_\alpha z^{1/2} \theta^{1/2} \int_0^\infty \tau^{(a-3)/2} e^{-\tau z^k}\, e^{-\frac{z^k}{4\tau}}\, d\tau \quad \text{(putting } \tfrac{\theta^k}{2k}\xi = \tau\text{)}.$$

19

Hence

$$\phi(\theta\lambda^{\frac{1-a}{2}}y^{1-a}) = 2^{(a-3)/2}c_\alpha\lambda^{\frac{1-a}{4}}y^{(1-a)/2}\theta^{1/2}\int_0^\infty \tau^{(a-3)/2}e^{-\tau\sqrt{\lambda}y}\,e^{-\frac{\sqrt{\lambda}y}{4\tau}}\,d\tau$$

$$= 2^{(a-3)/2}c_\alpha\theta^{1/2}y^{1-a}\int_0^\infty t^{(a-3)/2}e^{-\lambda t}\,e^{-\frac{y^2}{4t}}\,dt,$$

putting $y\tau = \sqrt{\lambda}t$. In other words, $\lambda \to \phi(\theta\lambda^{\frac{1-a}{2}}y^{1-a})$ is, up to a multiplicative constant, the Laplace transform of $t \to t^{(a-3)/2}\,e^{-\frac{y^2}{4t}}$.

For sake of brevity we set $C_a := 2^{(a-3)/2}c_\alpha\theta^{1/2}$ (we remind that α depends on a). Thus we can write now

$$v(\cdot,y) = C_a y^{1-a}\int_0^\infty \Big(\int_0^\infty t^{(a-3)/2}e^{-\lambda t}\,e^{-\frac{y^2}{4t}}\,dt\Big)dE(\lambda)u$$

$$= C_a y^{1-a}\int_0^\infty t^{(a-3)/2}e^{-\frac{y^2}{4t}}\Big(\int_0^\infty e^{-\lambda t}dE(\lambda)u\Big)\,dt$$

$$= C_a y^{1-a}\int_0^\infty t^{(a-3)/2}e^{-\frac{y^2}{4t}}u * h(t,\cdot)\,dt = u * P_{\mathbb{G}}(\cdot,y).$$

$\square$

Remark 4.5. Formulas (27) and (26) make possible to give a different and more explicit representation of the lifting v of u. On the other hand, the estimates of $h(t,\cdot)$ proved in [14] and [15] yield analogous estimates for $P_{\mathbb{G}}$. Indeed, if I is a multi-index, then, if $\rho := \rho(x)$,

$$|X^I P_{\mathbb{G}}(x,y)| \le C\int_0^\infty t^{(a-3)/2}|X^I h(t,x)|\,dt$$

$$= C\rho^{a-1}\int_0^\infty \tau^{(a-3)/2}|X^I h(\tau\rho^2,x)|\,d\tau$$

By [15], identity (1.73), we write now

$$X^I h(\tau\rho^2,x) = (\sqrt{\tau}\rho)^{-Q-d(I)}|X^I h(1,\frac{x}{\sqrt{\tau}\rho})|,$$

and we notice that, since $h(1,\cdot) \in \mathcal{S}(\mathbb{G})$ (by [15], Proposition 1.74), if $N > 0$, then

$$|X^I h(1,\frac{x}{\sqrt{\tau}\rho})| \le C(1 + \frac{1}{\sqrt{\tau}})^{-N} \le C\frac{\tau^{N/2}}{1 + \tau^{N/2}}.$$

Thus, eventually,

$$|X^I P_{\mathbb{G}}(x,y)| \le C\rho^{a-1-Q-d(I)}\int_0^\infty \tau^{(a-3-Q-d(I))/2}\frac{\tau^{N/2}}{1 + \tau^{N/2}}\cdot d\tau$$

$$\le C\rho^{a-1-Q-d(I)}$$

for large ρ. Then the lifting convolution $u * P_{\mathbb{G}}$ is well defined as long as $u(x)$ does not grow too fast as $x \to \infty$. We refer to [6] for similar growth conditions in the Euclidean setting.

Moreover, if u is sufficiently smooth,

$$
\begin{aligned}
y^a \frac{v(x,y) - v(x,0)}{y} &= y^a \frac{u * P_{\mathbb{G}}(\cdot, y) - u(x)}{y} \\
&= \left(C_a \int_0^\infty t^{(a-3)/2} e^{-\frac{y^2}{4t}} u * h(t, \cdot) \, dt \right. \\
&\qquad \left. - C_a u(x) \int_{\mathbb{G}} \int_0^\infty t^{(a-3)/2} e^{-\frac{y^2}{4t}} h(t, \xi^{-1} x) \, dt d\xi \right) \\
&= C_a \int_{\mathbb{G}} \int_0^\infty t^{(a-3)/2} e^{-\frac{y^2}{4t}} h(t, \xi^{-1} x) \, dt (u(\xi) - u(x)) d\xi
\end{aligned}
$$

(28)

On the other hand

$$
(29) \qquad \lim_{y \to 0^+} C_a \int_0^\infty t^{(a-3)/2} e^{-\frac{y^2}{4t}} h(t, \xi^{-1} x) \, dt = \tilde{C}_a \tilde{R}_{a-1}.
$$

Thus

$$
\lim_{y \to 0^+} y^a \frac{v(x,y) - v(x,0)}{y} = C_a \int_{\mathbb{G}} (u(\xi) - u(x)) \tilde{R}_{a-1}(\xi) d\xi = \tilde{C}_a \mathcal{L}^{\frac{1-a}{2}} u(x).
$$

Theorem 4.6. *Let $u \in W_{\mathbb{G}}^{1-a,2}(\mathbb{G})$ be given, $u \geq 0$, and assume $\mathcal{L}^{(1-a)/2} u = 0$ in an open set Ω. With the notations of Theorem 4.4, we denote by $\hat{v}$ the function on $\hat{\mathbb{G}}$ obtained by continuing v by parity across $y = 0$. Then*

    i) $\hat{v} \geq 0$;
    ii) $\hat{v} \in W_{\hat{\mathbb{G}},\text{loc}}^{1,2}(\hat{\Omega}; y^a dx \, dy)$, *where* $\hat{\Omega} := \Omega \times (-1,1)$;
    iii) $\hat{v}$ *is a weak solution of the equation*

$$
\text{div}_{\hat{\mathbb{G}}} \left(|y|^a \nabla_{\hat{\mathbb{G}}} v \right) = 0 \quad \text{in } \hat{\Omega}.
$$

Proof. Statement i) follows from previous Theorem 4.4.

The proofs of ii) and iii) are divided in several steps.

Step 1. From now on, we write $\Sigma_- := \mathbb{G} \times (-1,0)$ and $\Sigma_-^\varepsilon := \mathbb{G} \times (-1, -\varepsilon)$. If $\eta > 0$, we set

$$
u_\eta := (1 + \eta \mathcal{L})^{-1} u := \int_0^\infty (1 + \eta \lambda)^{-1} dE(\lambda) u.
$$

Then $u_\eta \in W_{\mathbb{G}}^{3-a,2}(\mathbb{G})$ so that, with the notation of (22), by Proposition 4.3, $\hat{v}_\eta \in W_{\hat{\mathbb{G}},\text{loc}}^{2,2}(\Sigma_\pm; y^a dx \, dy)$. Moreover, just performing computations, we see that

$$
\text{div}_{\hat{\mathbb{G}}} \left(|y|^a \nabla_{\hat{\mathbb{G}}} \hat{v}_\eta \right) = 0
$$

in $\Sigma_\pm$. Moreover, if $\psi \in \mathcal{D}(\Sigma_\pm)$, then

$$
\int_{\Sigma_\pm^\varepsilon} \langle \nabla_{\hat{\mathbb{G}}} \hat{v}_\eta, \nabla_{\hat{\mathbb{G}}} \psi \rangle |y|^a dx \, dy = 0.
$$

Step 2. The function $\hat{v}$ belongs to both $W_{\hat{\mathbb{G}}}^{1,2}(\Sigma_\pm; y^a dx \, dy)$ (by Proposition 4.3) and in addition

$$
\int_{\Sigma_\pm^\varepsilon} \langle \nabla_{\mathbb{G}} \hat{v}, \nabla_{\hat{\mathbb{G}}} \psi \rangle |y|^a dx \, dy = 0
$$

for any $\psi \in \mathcal{D}(\hat{\Sigma}_\pm)$. Indeed, by (23), we have but to notice that $u_\eta \to u$ in $W_{\mathbb{G}}^{1-a,2}(\mathbb{G})$. Indeed

$$\|u_\eta - u\|_{W_{\mathbb{G}}^{1-a,2}(\mathbb{G})}^2 = \int_0^\infty \lambda^{1-a} |(1+\eta\lambda)^{-1} - 1|^2 \, d\|E(\lambda)u\|^2 \to 0$$

as $\eta \to 0$, by dominated convergence theorem. Since the function $y \to |y|^a$ is smooth away from $\{y = 0\}$, then $\hat{v}_\eta$ is smooth in $\Sigma_\pm$, by classical Hörmander's theorem ([24]).

We notice that this argument going through regularization, equation in non-divergence form, integration by parts and variational equation is required by our abstract arguments that hides the divergence structure of the equation.

Step 3. Because of the properties of A_2-weights, $\hat{v} \in W_{\hat{\mathbb{G}},\mathrm{loc}}^{1,1}(\Sigma_\pm) \cap L_{\mathrm{loc}}^1(\Sigma)$. Moreover, with an obvious meaning of symbols,

$$(30) \qquad\qquad X_j \hat{v} = \widehat{X_j v} \quad \text{in } \Sigma, \text{ for } j = 1, \ldots, m$$

and

$$(31) \qquad\qquad \partial_y \hat{v} = \pm \widehat{\partial_y v} \quad \text{in } \Sigma_\pm.$$

Clearly, this yields $\hat{v} \in W_{\hat{\mathbb{G}},\mathrm{loc}}^{1,2}(\Sigma; y^a dx\, dy)$ and therefore ii) holds. Now, (30) is obvious. As for (31), if $\psi \in \mathcal{D}(\hat{\Omega})$, by divergence theorem

$$\int_\Sigma \hat{v}(\partial_y \psi) \, dxdy = \lim_{\varepsilon \to 0} \int_{\Sigma_+^\varepsilon} \hat{v}(\partial_y \psi) \, dxdy + \lim_{\varepsilon \to 0} \int_{\Sigma_-^\varepsilon} \hat{v}(\partial_y \psi) \, dxdy$$

$$(32) \qquad = \lim_{\varepsilon \to 0} \int_\Omega v(\cdot, \varepsilon)\psi(\cdot, \varepsilon) \, dx - \lim_{\varepsilon \to 0} \int_\Omega v(\cdot, -\varepsilon)\psi(\cdot, \varepsilon) \, dx$$

$$\qquad - \lim_{\varepsilon \to 0} \int_{\Sigma_+^\varepsilon} (\partial_y v)\psi \, dxdy + \lim_{\varepsilon \to 0} \int_{\Sigma_-^\varepsilon} (\widehat{\partial_y v})\psi \, dxdy.$$

Since $\hat{v}$ is locally Hölder continuous up to $y = 0$

$$\lim_{\varepsilon \to 0} \int_\Omega v(\cdot, \varepsilon)\psi(\cdot, \varepsilon) \, dx - \lim_{\varepsilon \to 0} \int_\Omega v(\cdot, -\varepsilon)\psi(\cdot, \varepsilon) \, dx = 0$$

and the assertion follows.

Step 4. By divergence theorem, if $\varepsilon \in (0, 1)$ and $\psi \in \mathcal{D}(\hat{\Omega})$, then

$$(33) \qquad \int_{\Sigma_\pm^\varepsilon} \langle \nabla_{\hat{\mathbb{G}}} \hat{v}, \nabla_{\hat{\mathbb{G}}} \psi \rangle |y|^a dx\, dy = \int_\Omega \varepsilon^a \partial_y \hat{v}(x, \pm\varepsilon)\psi(x, \pm\varepsilon) \, dx$$

Take now the limit as $\varepsilon \to 0$. Clearly

$$\int_{\Sigma_\pm^\varepsilon} \langle \nabla_{\hat{\mathbb{G}}} \hat{v}, \nabla_{\hat{\mathbb{G}}} \psi \rangle |y|^a dx\, dy \to \int_{\hat{\Omega}} \langle \nabla_{\hat{\mathbb{G}}} \hat{v}, \nabla_{\hat{\mathbb{G}}} \psi \rangle |y|^a dx\, dy$$

as $\varepsilon \to 0$. If we show that

$$(34) \qquad \varepsilon^a \partial_y \hat{v}(x, \pm\varepsilon) \to (1-a)^a \phi'(0) \mathcal{L}^{\frac{1-a}{2}} u \quad \text{in } L^2(\mathbb{G}),$$

then assertion iii) follows since $\mathcal{L}^{\frac{1-a}{2}} u$ vanishes on $\mathrm{supp}\, \psi$.

To prove (34), we write

$$\|\varepsilon^a \partial_y \hat{v}(x, \pm\varepsilon) - \phi'(0)\mathcal{L}^{\frac{1-a}{2}}u\|^2_{L^2(\mathbb{G})}$$

$$= (1-a)^{2a} \int_0^\infty |\phi'(\theta\lambda^{\frac{1-a}{2}}\varepsilon^{1-a}) - \phi'(0)|^2 \lambda^{1-a} d\|E(\lambda)u\|^2,$$

and the assertion follows since ϕ' is bounded.

This achieves the proof of the theorem. $\qquad\square$

Remark 4.7. By Theorem 2.9, $\hat{v}$ is locally Hölder continuous, and hence its trace $\hat{v}(\cdot, 0)$ on $\{y = 0\}$ is well defined and it is straightforward to see that $v(\cdot, 0) = u$.

On the other hand, by a classical interpolation theorem ([29], Theorem 10.1), if $\hat{v}$ belongs to $W^{1,2}_{\mathbb{G},\mathrm{loc}}(\hat{\Omega}; y^a dx\, dy)$, then its trace u belongs to $W^{1-a,2}_{\mathbb{G},\mathrm{loc}}(\Omega)$. This shows that our assumption $u \in W^{1-a,2}_{\mathbb{G}}(\mathbb{G})$ is optimal as long as we are concerned with local regularity.

Theorem 4.8. *Let* $-1 < a < 1$ *and let* $u \in W^{1-a,2}_{\mathbb{G}}(\mathbb{G})$ *be given,* $u \geq 0$ *on all of* $\mathbb{G}$*. Assume* $\mathcal{L}^{(1-a)/2}u = 0$ *in an open set* $\Omega \subset \mathbb{G}$*.*

Then there exist $C, b > 0$ *(independent of* u*) such that the following invariant Harnack inequality holds:*

$$\sup_{B_c(x,r)} u \leq C \inf_{B_c(x,r)} u$$

for any metric ball $B_c(x, r)$ *such that* $B_c(x, br) \subset \Omega$*.*

Proof. Let C, b be as in Theorem 2.9. By Theorems 4.6, [12] and by Lemma 2.6, we have:

$$\sup_{B_c(x,r)} u = \sup_{\hat{B}_c((x,0),r)\cap\{y=0\}} \hat{v} \leq \sup_{\hat{B}_c((x,0),r)} \hat{v} \leq C \inf_{\hat{B}_c((x,0),r)} \hat{v}$$

$$\leq \inf_{\hat{B}_c((x,0),r)\cap\{y=0\}} \hat{v} = \inf_{B_c(x,r)} u.$$

$$\square$$

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