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Towards a classification of symplectic automorphisms on manifolds of $K3^{[n]}$ type

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Abstract The present paper is devoted to the classification of symplectic automorphisms of some hyperkähler manifolds. The result presented here is a proof that all finite groups of symplectic automorphisms of manifolds of $K3^{[n]}$ type are contained in Conway's group Co_1.

Keywords Symplectic automorphisms, manifolds of $K3^{[n]}$ type

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Introduction

Automorphisms of symplectic manifolds have attracted a lot of interest, mainly in the two dimensional setting. Automorphisms of $K3$ surfaces preserving the symplectic form have been fully classified, starting from the foundational work of Nikulin [28] and Mukai [27] and of several others eventually has lead to a list of groups and actions on cohomology by Hashimoto [13].
In higher dimensions, far less is known: there are results concerning deformations of automorphisms, such as [5] by Boissière and [23]. There are also previous results in the peculiar case of involutions on four dimensional hyperkähler manifolds by Camere [10], O'Grady [31] and there is also [21]. Symplectic automorphisms have already been addressed in [24] and the recent [9] by Boissière, Camere and Sarti addresses automorphisms that do not preserve the symplectic form.
In the present paper, we restrict to manifolds obtained as smooth deformations of Hilbert schemes of n points on $K3$ surfaces (shorthand: manifolds of $K3^{[n]}$

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type). Our results, obtained with techniques similar to Kondō's approach for $K3$ surfaces in [19], yield a classification of prime order automorphisms preserving the symplectic form and the following two characterisations of finite groups of symplectic automorphisms:

Theorem *Let X be a hyperkähler manifold of $K3^{[n]}$ type and let G be a finite group of symplectic automorphisms of X. Then $G \subset Co_1$ and $S_G(X) = S_G(\Lambda)$ for some conjugacy class of G in Co_1.*

See **Theorem 25** for details. Here Λ is the Leech lattice, i.e. the unique even unimodular negative definite lattice of rank 24 with no elements of square -2 and $Aut(\Lambda)/\pm 1 = Co_1$. This can be seen as an analogue of Mukai's [27] result on $K3$ surfaces, where symplectic automorphisms were realised as subgroups of the Mathieu group $M_{23} \subset Co_1$. The Conway group appears also in some related fields, like derived autoequivalencies of $K3$ surfaces [18, Huybrechts] or sigma models on $K3$ surfaces [12, Gaberdiel, Hohenegger and Volpato]. The following is a somewhat more technical sufficient condition:

Theorem *Let $G \subset Co_0$ be a group of isometries of the Leech lattice with invariant sublattice T of rank at least 4 and the discriminant group of T has less generators than its rank. Then there exist an integer n and a manifold X of $K3^{[n]}$ type such that $G \subset Aut_s(X)$.*

See **Theorem 29**. Based on the behaviour of some isometries not satisfying the above proposition, we conjecture the following.

Conjecture *There is a bijective correspondence between finite groups of symplectic automorphisms of manifolds of $K3^{[n]}$ type (for some n) and subgroups G of Co_1 satisfying the conditions of the above theorem.*

Recently, Huybrechts [18] found a similar result for some special automorphisms. He proves that all subgroups G of Co_1 satisfying a condition on their action on Λ can be obtained as automorphism groups of certain manifolds of $K3^{[n]}$ type, where the automorphisms are induced from autoequivalences of the derived category of a $K3$ surface. In **Lemma 30** we prove that the sufficient condition he imposes on the group action is equivalent to ours. In principle, the results contained in the present paper are sufficient to classify all finite automorphism groups of manifolds of $K3^{[n]}$ type. This task has been recently carried out by Höhn and Mason [15] for four dimensional manifolds, thus providing a higher dimensional analogue of [13].
The structure of the paper is as follows. In **Section 1**, we gather several known results on manifolds of $K3^{[n]}$ type. In **Section 2** we collect some results about lattices, mainly about their discriminant groups. In **Section 3** we prove basic properties of invariant and coinvariant lattices associated to symplectic automorphisms and we use them to prove that finite groups of symplectic automorphisms can be embedded in the Conway group. The same result, but only in dimension four, was proven in [24]. In **Section 4** we address the converse, namely how to obtain automorphisms from subgroups of the Conway group. We provide a sufficient and a necessary condition.

Notations

Let L be a lattice and let $G \subset O(L)$ be a group of isometries. For any lattice L, we denote by $L(n)$ the lattice with the same $\mathbb{Z}$ module structure as L, but with quadratic form multiplied by n. If X is a manifold of $K3^{[n]}$ type , we will call discriminant group of X the group $H^2(X,\mathbb{Z})^\vee/H^2(X,\mathbb{Z})$. We will denote by $Aut_s(X)$ the group of automorphisms of a manifold of $K3^{[n]}$ type that preserve the symplectic form, i.e. symplectic automorphisms. Any isometry between $H^2(X,\mathbb{Z})$ and a lattice L is called a marking.

1 Manifolds of $K3^{[n]}$ type

In this section we give a short introduction to manifolds of $K3^{[n]}$ type. They are one of the two known series of hyperkähler manifolds.

Definition 1 Let S be a $K3$ surface and let $S^{[n]}$ denote its Hilbert scheme of points, $n > 1$. Let X be a smooth Kähler deformation of $S^{[n]}$. Then X is called of $K3^{[n]}$ type.

We remark that these manifolds inherit a symplectic form from the $K3$ surface, as shown in [4]. We will mainly need three results on these manifolds, which are the existence of the Beauville-Bogomolov form, the global Torelli theorem and the structure of their Kähler cone.

Theorem 2 *Let X be a manifold of $K3^{[n]}$ type. Then there exists a canonically defined pairing $(\ ,\ )_X$ on $H^2(X,\mathbb{C})$, the Beauville-Bogomolov pairing, and a constant c_X (the Fujiki constant) such that the following holds:*

$$(\alpha,\alpha)_X^n = c_x \int_X \alpha^{2n}. \tag{1}$$

Moreover c_X and $(\ ,\ )_X$ are deformation and birational invariants.

With this form, $H^2(X,\mathbb{Z})$ is isometric to the following

$$L_n := U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1) \oplus (2-2n). \tag{2}$$

Here U is the hyperbolic lattice, $E_8(-1)$ is the unique unimodular even negative definite lattice of rank 8, $(2-2n)$ is $(\mathbb{Z},q)$ with $q(1) = 2-2n$ and $\oplus$ denotes orthogonal direct sum. In analogy to the case of $K3$ surfaces, there exists a non Hausdorff moduli space of marked manifolds.

Definition 3 Let (X,f) be a marked manifold of $K3^{[n]}$ type. Let $\mathcal{M}_n$ be the set $\{(X,f)\}/\sim$ of marked manifolds of $K3^{[n]}$ type where $(X,f) \sim (X',f')$ if and only if there exists an isomorphism $\phi : X \to X'$ such that $\phi^* = f^{-1} \circ f'$.

Definition 4 We define the period domain Ω_n as

$$\Omega_n = \{x \in \mathbb{P}(L_n \otimes \mathbb{C}) \,|\, (x,x)_{L_n} = 0,\ (x+\overline{x}, x+\overline{x})_{L_n} > 0\}. \tag{3}$$

The period map $\mathcal{P} : \mathcal{M}_n \to \Omega_n$ sends a pair (X, f) to the line $f(H^{2,0}(X))$.

Definition 5 Let S be a $K3$ surface. Let $L_M := H^*(S, \mathbb{Z})$ be a lattice, where the pairing is given by intersection on H^2 and duality between H^0 and H^4. We call this lattice the Mukai lattice. It is isometric to $U^4 \oplus E_8(-1)^2$.

Markman [20] proved that the Mukai lattice can be used to classify Hodge isometries obtained by parallel transport:

Theorem 6 *[20, Theorem 9.3] Let X be a manifold of $K3^{[n]}$ type. Then there exists a canonically defined equivalence class of embeddings $\iota_X \colon H^2(X, \mathbb{Z}) \to L_M$. A Hodge isometry $g \colon H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ is a parallel transport operator if and only if $\iota_X = \iota_Y \circ g$ in $O(L_n, L_M)/O(L_M)$*

For $X = Y$, this amounts to saying that g acts as ± 1 on $L_n^\vee / L_n$.

Parallel transport operators can then be used for the following

Theorem 7 (Global Torelli, Huybrechts, Markman and Verbitsky) *Let X and Y be two hyperkähler manifolds of $K3^{[n]}$ type. Suppose $\psi : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ is a parallel transport Hodge isometry. Then there exists a birational map $\phi : X \dashrightarrow Y$.*

Related to this there is also the following useful theorem, due to Huybrechts [20, Theorem 3.2]:

Theorem 8 *Let (X, f) be a marked hyperkähler manifold and (X', g) another marked hyperkähler manifold such that $f^{-1} \circ g$ is a parallel transport Hodge isometry. Then there exists an effective cycle $\Gamma = Z + \sum_j Y_j$ in $X \times X'$ satisfying the following conditions:*

- *Z is the graph of a bimeromorphic map from X to X'.*
- *The codimension of the two projections $\pi_1(Y_j)$ and $\pi_2(Y_j)$ are equal.*
- *The composition $g^{-1} \circ f$ is equal to $\Gamma_* : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$.*
- *The cycles $\pi_i(Y_j)$ are uniruled.*

Finally, the birational geometry of certain manifolds of $K3^{[n]}$ type has been analysed by Bayer and Macrì [2] and was generalised in [25] (see [1] for a different approach) using general properties of the Kähler cone of hyperkähler manifolds.

Let X be a manifold of $K3^{[n]}$ type and let $H^2(X, \mathbb{Z}) \to L_M$ be the primitive embedding described in [20, Theorem 9.3]. Let v be a generator of the orthogonal complement of H^2 under this embedding. For every divisor D we denote by T_D the primitive rank 2 lattice containing v and D inside L_M.

Definition 9 Let X, D and T_D be as above. The divisor D is a wall divisor if there exists $r_D \in T_D$ such that one of the following is satisfied:

- $r_D^2 = -2$ and $0 \leq (v, r_D) \leq v^2/2$.
- $0 \leq r_D^2 v^2 \leq (v, r_D)^2 < (v^2/2)^2$.

Theorem 10 *[25, Theorem 1.3 and Proposition 1.5] Let X be a manifold of $K3^{[n]}$ type and let $\mathcal{W}$ be the set of wall divisors on X. Then the Kähler cone of X is one of the connected components of the following set*

$$\{x \in H^2(X, \mathbb{R}),\, x^2 > 0,\, (x, w) \neq 0 \,\forall\, w \in \mathcal{W}\}. \tag{4}$$

2 Lattice theory

In this section we sketch the lattice theory used in the proofs of the main theorems. The interested reader can consult [29] for prerequisites on discriminant groups and forms. By lattice we mean a free $\mathbb{Z}$ module equipped with a non degenerate bilinear form. We call it even if the associated quadratic form takes only even values. Given an element $v \in L$, we denote by $div(v)$ the positive generator of the ideal (v, L) and we call it the divisibility of v.

2.1 Discriminant groups

For a lattice L its discriminant group is $A_L := L^\vee/L$. Let $l(A_L)$ denote the length of this group. If the lattice L is even, A_L has a bilinear form with values in $\mathbb{Q}/\mathbb{Z}$ induced from the bilinear form on L. Many properties of the associated quadratic form on $\mathbb{Q}/2\mathbb{Z}$, called discriminant form, were found by Nikulin in [29]. If L is a lattice, we denote its discriminant form by q_{A_L}. If (l_+, l_-) is the signature of L, the integer $l_+ - l_-$ is called signature of q_{A_L} and, modulo 8, it is well defined.
Here we will make often use of the following facts concerning primitive embeddings.

Lemma 11 *[29, Proposition 1.15.1] Let S and N be even lattices of signature (s_+, s_-) resp. (n_+, n_-). Primitive embeddings of S into N are determined by the sets $(H_S, H_N, \gamma, K, \gamma_K)$, where K is an even lattice with signature $(n_+ - s_+, n_- - s_-)$ and discriminant form $-\delta$ where $\delta \cong (q_{A_S} \oplus -q_{A_N})_{|\Gamma_\gamma^\perp/\Gamma_\gamma}$ and $\gamma_K : q_K \to (-\delta)$ is an isometry.*
Moreover two such sets $(H_S, H_N, \gamma, K, \gamma_K)$ and $(H'_S, H'_N, \gamma', K', \gamma'_K)$ determine isomorphic sublattices if and only if

- *$H_S = \lambda H'_S$, $\lambda \in O(q_S)$,*
- *$\exists \epsilon \in O(q_{A_N})$ and $\psi \in Isom(K, K')$ such that $\gamma' = \epsilon \circ \gamma$ and $\overline{\epsilon} \circ \gamma_K = \gamma'_K \circ \overline{\psi}$, where $\overline{\epsilon}$ and $\overline{\psi}$ are the isometries induced among discriminant groups.*

Here Γ_γ is the graph of γ. For most purposes, the following simplified version will suffice:

Lemma 12 *Let S be an even lattice of signature (s_+, s_-). The existence of a primitive embedding of S into some unimodular lattice L of signature (l_+, l_-) is equivalent to the existence of a lattice M of signature (m_+, m_-) and discriminant form q_{A_M} such that the following are satisfied:*

- *$s_+ + m_+ = l_+$ and $s_- + m_- = l_-$.*
- *$A_M \cong A_S$ and $q_{A_M} = -q_{A_S}$.*

We will also use a result on the existence of lattices, the following is a simplified version of [29, Theorem 1.10.1]

Lemma 13 *Suppose* (A_T, q_{A_T}) *and* (t_+, t_-) *are invariants satisfying the following:*

- $sign(q_{A_T}) \equiv t_+ - t_- \ mod\, 8$.
- $t_+ \geq 0$, $t_- \geq 0$ *and* $t_+ + t_- \geq l(A_T)$.
- *There exists an even lattice* T' *of rank* $t_+ + t_-$ *and discriminant form* (A_T, q_{A_T}).

Then there exists an even lattice T *of signature* (t_+, t_-) *and discriminant form* (A_T, q_{A_T}).

Proof We will apply the stronger result contained in [29, Theorem 1.10.1], where the existence of a lattice is equivalent to four conditions. The first two in the above cited theorem are true by hypothesis. The last two are implied by the existence of T': indeed they are conditions involving only the discriminant form and, as T' exists, they are satisfied. Therefore, there exists a lattice T as above.

The uniqueness of a lattice is usually difficult to prove, but here are two special cases:

Lemma 14 *[29, Corollary 1.13.3] Let* S *be an even indefinite lattice with signature* (t_+, t_-) *and discriminant form* q_{A_S}. *Then all even lattices with the same signature and discriminant form are isometric to it if* $t_+ + t_- \geq 2 + l(A_S)$.

Let L be a lattice such that $A_L = (\mathbb{Z}_{/(2)})^r$. It is called a 2-modular lattice and its isometry class is easily determined using its discriminant form. Let us denote by Δ the discrete invariant, which is 0 if the discriminant form takes only values 0 and 1, and it is 1 otherwise.

Theorem 15 *[30, Theorem 4.3.1 and 4.3.2] A 2-modular indefinite lattice is uniquely determined by its rank, its signature, the length of its discriminant group and* Δ.

2.2 Isometries, Invariant and Co-invariant Lattices

In this subsection we analyse two kind of lattices linked to an isometry, namely the co-invariant and invariant lattices.

Definition 16 Let L be a lattice and let $G \subset O(L)$. Then we define $T_G(L) = L^G$ as the invariant lattice of G and $S_G(L) = T_G(L)^\perp$ as the co-invariant lattice.

We will be mainly interested in the following lattices and groups:

Definition 17 Let L be an even lattice and let $G \subset O(L)$. Then L is a Leech type lattice with respect to G if the following are satisfied:

- L is negative definite.
- L contains no vectors of square -2.

– G acts trivially on A_L.
– $S_G(L) = L$.

Moreover we call (L, G) a Leech pair and G a Leech type group.

Leech type lattices of rank at most 11 have been classified by Nikulin in [29, Section 1.14] and the only such lattice is $E_8(-2)$. The easiest examples of such lattices can be constructed as follows. Let N be a unimodular negative definite even lattice with no elements of square -2 and let $G \subset O(N)$. Then $(S_G(N), G)$ is a Leech pair.

Lemma 18 *Let L be a lattice, and let $G \subset O(L)$. Then the following hold:*

– *$T_G(L)$ contains $\sum_{g\in G} gv$ for all $v \in L$.*
– *$S_G(L)$ contains $v - gv$ for all $v \in L$ and all $g \in G$.*
– *If L is definite then $T_G(L)$ and $S_G(L)$ are nondegenerate.*
– *$L/(T_G(L) \oplus S_G(L))$ is of $|G|$-torsion.*

Proof It is obvious that $\sum_{g\in G} gv$ is G-invariant for all $v \in L$, moreover $T_G(L) \otimes \mathbb{Q}$ is generated by such elements. For $w \in T_G(L)$ we have $(w, v) = (gw, gv) = (w, gv)$ for all $v \in L$ and all $g \in G$. Therefore $v - gv$ is orthogonal to all G-invariant vectors, hence it lies in $S_G(L)$ and $S_G(L) \otimes \mathbb{Q}$ is generated by such elements. Obviously whenever L is definite all of its sublattices are nondegenerate. Let $v \in L$, we can write $|G|v = \sum_{g\in G} g(v) + \sum_{g\in G}(v - g(v))$, where the first term lies in $T_G(L)$ and the second in $S_G(L)$.

Remark 19 Let L be a lattice and let $L \subset L'$ be a primitive embedding into a unimodular lattice. Let $g \in O(L)$ be an isometry which acts trivially on A_L. Then there exists an isometry $\overline{g} \in O(L')$ such that $\overline{g}_{|L} = g$ and $\overline{g}_{|L^\perp} = Id$.

3 From symplectic automorphisms to the Conway group

Definition 20 Let X be a manifold of $K3^{[n]}$ type and let $G \subset Aut(X)$. We let $T_G(X)$ inside $H^2(X, \mathbb{Z})$ be the sublattice fixed by the induced action of G on $H^2(X, \mathbb{Z})$. Moreover we define the co-invariant lattice $S_G(X) \subset H^2(X, \mathbb{Z})$ as $T_G(X)^\perp$.

We wish to remark that the map

$$Aut(X) \xrightarrow{\nu} O(H^2(X, \mathbb{Z})) \tag{5}$$

is injective for manifolds of $K3^{[n]}$ type. This was proven by Beauville for Hilbert schemes themselves [3, Lemma 3] and Hassett and Tschinkel proved that this kernel is a deformation invariant [14, Theorem 2.1]. We have the following exact sequence for any finite group G of Hodge isometries on $H^2(X, \mathbb{Z})$:

$$1 \to G_0 \to G \xrightarrow{\pi} \Gamma_m \to 1, \tag{6}$$

where $\Gamma_m \subset U(1)$ is a cyclic group of order m. In fact the action of G on $H^{2,0}$ is the action of a finite group on $\mathbb{C}$.

Lemma 21 *Let X be a manifold of $K3^{[n]}$ type and let $G \subset Aut(X)$ be a finite group. Then the following hold:*

1. *$g \in G$ acts trivially on $T(X)$ if and only if $g \in G_0$.*
2. *The representation of Γ_m on $T(X)\otimes\mathbb{Q}$ splits as the direct sum of irreducible representations of the cyclic group Γ_m having maximal rank (i.e. of rank $\phi(m)$).*

Proof The proof goes exactly as [21, Lemma 3.4]. See [3, Proposition 6] or [22, Lemma 7.1.4] for further reference.

Definition 22 Let $G \subset O(L_n)$ be a finite group and let X be a manifold of $K3^{[n]}$ type such that $Pic(X) \cong S_G(L_n)$. Suppose moreover that G consists of parallel transport operators on X. A numerical wall divisor in $S_G(L_n)$ is the image under the above isometry of a wall divisor of X.

Lemma 23 *Let $f \in O(L_n)$ be a finite order isometry such that f acts as $-Id$ on A_{L_n} and $S_f(L_n)$ is negative definite. Then $S_f(L_n)$ contains a numerical wall divisor.*

Proof For any $v \in L_n$ of square $2-2n$ and $div(v) = 2n-2$, the reflection R_v is an isometry which acts as $-Id$ on A_{L_n} and therefore g defined by $f = R_v \circ g$ acts trivially on it. Furthermore, we can find a manifold X of $K3^{[n]}$ type such that g and f are also (symplectic) Hodge isometries for the Hodge structure that X induces on L_n (it is enough that $S_f(L_n) \oplus \mathbb{Z}v \subset Pic(X)$). This implies that $S_g(L_n)$ is also negative definite, as $T_g(L_n)$ contains an invariant positive class of $H^{1,1}(X)$ and $H^{2,0}(X) \oplus H^{0,2}(X)$.
Let $f(v) = -v+(2n-2)w$. This implies $v-g(v) = (2-2n)w$, i.e. $w \in S_g(L_n)$. Moreover $t := (v - f(v))/2 = v - (n-1)w \in S_f(L_n)$ is a non zero element as v is primitive. Since f is an isometry, $(v,w) = (n-1)w^2 \leq 0$ because $w \in S_g(L_n)$. Therefore $2-2n \leq t^2 < 0$ and $div(t) \geq n-1$. Let us now choose a manifold Y of $K3^{[n]}$ type with $Pic(Y) \cong S_f(L_n)$ such that f is a parallel transport operator. Let s be a generator of $(H^2)^\perp$ in L_M. The reflection by v is a parallel transport operator, hence either $\frac{s+t}{div(t)}$ or $\frac{s-t}{div(t)}$ lie in L_M. In any case, the primitive lattice containing s and t satisfies **Definition 9**, therefore t is a numerical wall divisor.

We remark that the trivial action on the discriminant group is specific to the symplectic case, as it relies on the fact that $S_f(L_n)$ is negative definite. If f comes from a nonsymplectic automorphism, then $S_f(L_n)$ is not negative definite and there are examples where f acts nontrivially on the discriminant group of the manifold, such as [26, Example 3.7].

Lemma 24 *Let X be a hyperkähler manifold and let $G \subset Aut_s(X)$ be a finite group. Then the following assertions are true:*

1. *$S_G(X)$ is nondegenerate and negative definite.*
2. *$T(X) \subset T_G(X)$ and $S_G(X) \subset S(X)$.*
3. *$S_G(X)$ contains no wall divisors*

4. *The action of G on $A_{S_G(X)}$ is trivial.*

Proof The proof of the first three items goes exactly as in [21, Lemma 3.5] with wall divisors taking the role of -2 elements. For the reader's convenience we sketch it here. The invariant lattice $T_G(X)$ contains $T(X)$ because G is symplectic and, after tensoring with $\mathbb{R}$, it contains an invariant Kähler class because G is finite. Therefore its orthogonal $S_G(X)$ is negative definite. Since $T_G(X)\otimes\mathbb{R}$ contains a Kähler class, its orthogonal can not contain wall divisors. For the final statement, suppose that an element g of G acts nontrivially on A_{L_n}. Since it is a parallel transport operator, it must act as $-Id$ by [20, Lemma 9.2] and **Lemma 23** implies that $S_g(X)$ would contain a wall divisor.

Notice that elements D of square -2 are wall divisors for every n, as the lattice generated by v and D in L_M satisfies the conditions of **Definition 9**, see also [20, Proposition 9.12].

We are now ready to prove the main result of this section:

Theorem 25 *Let X be a manifold of $K3^{[n]}$ type and let $G \subset Aut_s(X)$ be a finite group. Then there exists an injection $G \subset Co_0$ not meeting $-Id$ and such that $S_G(X) = S_G(\Lambda)$ for some conjugacy class of G in Co_0.*

Proof The first part of the Theorem is a generalisation of [24, Theorem 1.1], however we follow a different proof which simplifies the classification and yields also the second part of the Theorem. We follow [18, Theorem 0.1]. The first step is just giving a primitive embedding of $S_G(X)$ into $U \oplus \Lambda$ and extending the action of G trivially on $S_G(X)^{\perp}$.

Now, whatever the value of n, elements of square -2 are always numerical wall divisors, hence they do not lie in $S_G(X)$. This is sufficient to find a special embedding $S_G(X) \subset U \oplus \Lambda$ such that the action of G is nontrivial only inside Λ and $S_G(X) \cong S_G(\Lambda)$. Furthermore, we have $G \subset Aut(\Lambda)/\{\pm 1\} = Co_1$ because $-Id$ has a coinvariant lattice of rank 24.

4 From the Conway group to symplectic automorphisms

Theorem 26 *Let G be a finite subgroup of $O(L_n)$. Then G is induced by a symplectic subgroup of $Aut(X)$ for some X of $K3^{[n]}$ type if and only if the following hold:*

1. *$S_G(L_n)$ is non degenerate and negative definite.*
2. *$S_G(L_n)$ contains no numerical wall divisor.*
3. *G acts trivially on A_{L_n}.*

Proof The "only if"-direction is **Lemma 24**. For the "if"-direction, assume G satisfies 1), 2) and 3). By the surjectivity of the period map, there exists a marked (X, f) of $K3^{[n]}$ type such that f induces isometries $T(X) \cong T_G(L_n)$ and $Pic(X) \cong S_G(L_n)$.

Since $S_G(L_n)$ contains no numerical wall divisors, it follows that $Pic(X)$ contains no wall divisors and therefore $\mathcal{K}_X = \mathcal{BK}_X = \mathcal{C}_X$. In particular, we have a

G invariant Kähler class which is also orthogonal to $Pic(X)$, hence X contains no effective (rational) curves nor divisors.
For $g \in G$, we consider the marked varieties (X, f) and $(X, g \circ f)$. Since g acts trivially on the discriminant group of X, these two marked manifolds lie in the same connected component of the moduli space, see [20, Corollary 9.5 and Theorem 9.8]. They also have the same period, hence by **Theorem 8** we have $f^{-1} \circ g \circ f = \Gamma_*$. Here $\Gamma = Z + \sum_j Y_j$ in $X \times X$, where Z is the graph of a bimeromorphic map from X to itself and Y_j's are cycles with $codim(\pi_i(Y_j)) \geq 1$.
As X contains no effective divisors, all Y_j's contained in Γ have $codim(\pi_i(Y_j)) > 1$, thus implying $g = \Gamma_* = Z_*$ on $H^2(X, \mathbb{Z})$. Now the bimeromorphic map Z is biregular since $\mathcal{K}_X = \mathcal{C}_X$.

Remark 27 A wall divisor D on a manifold X of $K3^{[n]}$ type has either square -2 or $div(D) > 1$ and it divides $2(n-1)$. So, if G is a cyclic group of order coprime with $2(n-1)$, the only numerical wall divisors that $S_G(L_n)$ could contain are those of square -2.

Proposition 28 *Let (S, G) be a pair consisting of a Leech type lattice and its Leech automorphism group as in* **Definition 17**. *Let moreover $S \subset N$ be an embedding into a Niemeier lattice.*
Suppose there exists a primitive embedding $S \to L_n$ and suppose that the order of G is coprime to $2(n-1)$.
Then G extends to a group of automorphisms on some X of $K3^{[n]}$ type.

Proof This is an immediate consequence of **Theorem 26**: first of all we consider a chain of embeddings $S \subset L_n \subset L_M$. As G acts trivially on A_S, we can extend G to a group of isometries of L_M acting trivially on $S^{\perp_{L_M}}$, therefore also on $S^{\perp_{L_n}}$. Thus we have $S_G(L_n) \cong S$. Moreover, since S is a Leech type lattice contained in a negative definite lattice N, the other conditions of **Theorem 26** are satisfied.

Theorem 29 *Let $G \subset Co_0$ be a group of isometries such that $rk(S_G(\Lambda)) \leq 20$ and $rk(T_G(\Lambda)) > l(A_{T_G(\Lambda)})$. Then there exist an integer n and a manifold X of $K3^{[n]}$ type such that $G \subset Aut_s(X)$ and $S_G(X) \cong S_G(\Lambda)$.*

Proof First of all, by the existence of the lattice $T_G(\Lambda)$, we have a lattice T of signature $(4, rk(T_G(\Lambda)) - 4)$ and $q_{A_T} = q_{A_{T_G(\Lambda)}}$ by **Lemma 13**. Then, by **Lemma 12**, there is a primitive embedding of $S_G(\Lambda)$ into the Mukai lattice L_M whose orthogonal is T. We have $rk(T) > l(A_T)$. This implies that there exists an element $v \in T$ such that $(v + s)/r \in L_M$ implies $r = 1$ for all $s \in S_G(\Lambda)$. After adding some positive element of T, we have $v^2 \geq 2$ (and it can actually take infinitely many positive values). Let now $n = (v^2 + 2)/2$, the lattice $v^\perp$ is isometric to L_n and all elements of $S_G(\Lambda)$ have divisibility 1 in L_n. Since $S_G(\Lambda)$ contains no elements of square -2 and no elements of divisibility at least 2, it contains also no numerical wall divisors. Therefore **Theorem 26** applies and we obtain our claim.

The condition $rk(T_G(\Lambda)) > l(T_G(\Lambda)$ is actually equivalent to $rk(S_G(\Lambda)) + l(A_{S_G(\Lambda)}) \leq 23$. This equivalence is trivial if $rk(S_G(\Lambda)) \geq 12$ and it still holds otherwise because the only smaller Leech type lattice is $E_8(-2)$.

With the above proposition, we can construct automorphism groups from several subgroups of the Conway group. In [18, Theorem 0.2], Huybrechts also finds a sufficient condition to obtain a group of symplectic automorphisms on a manifold of $K3^{[n]}$ type from a subgroup of the Conway group, the following shows that his condition is equivalent to ours:

Lemma 30 *Let M be a negative definite lattice of rank at most 20. Then the following are equivalent:*

- *M embeds primitively in Λ and its orthogonal T satisfies $rk(T) > l(A_T)$.*
- *M embeds primitively in L_M and its orthogonal R satisfies $rk(R) > l(A_R)$.*
- *M embeds primitively into a lattice P of signature $(1, 20)$ with $l(A_P) \leq 2$.*
- *M embeds primitively in L_M and its orthogonal R contains a positive definite lattice Γ of rank 3 with $l(\Gamma) \leq 2$.*

Proof The last two conditions are trivially equivalent, after noticing that both Γ and P embed into L_M and taking $\Gamma = P^{\perp}$.
The first two conditions are also equivalent by **Lemma 13** and **Lemma 12**. Let now $\Gamma \subset R$ with $l(\Gamma) \leq 2$. This means that there exists an element $t \in \Gamma$ such that $t/p \notin \Gamma^{\vee}$ for any prime p and also $t/p \notin R^{\vee}$, i.e. $l(A_R) < rk(R)$. Equivalently if $l(A_R) < rk(R)$ there exists a t as above and we define Γ as any lattice containing t, v and w, where these three are linearly independent.

One is naturally interested in the case of a group $G \subset Aut(\Lambda)$ such that $rk(T_G(\Lambda)) = l(A_{T_G(\Lambda)})$. The following proposition actually shows that, if ever we could obtain G as an automorphism of manifolds of $K3^{[n]}$ type, this would happen only for finitely many n.

Proposition 31 *Let $G \subset O(L_n)$ be a group such that $M := S_G(L_n)$ is negative definite and G acts trivially on A_{L_n}. Suppose moreover that $l(A_M) + rk(M) = 24$. Let $\{t\}_i$ be a set of primitive elements of M such that $[t_i/div(t_i)]$ are the nontrivial part of A_M and $|t_i^2| \leq div(t_i)^2(n+3)/2$. Then M contains a wall divisor for any manifold X with $M \subset Pic(X)$.*

Proof Let $L_n \subset L_M$ and let v be a generator of $L_n^{\perp}$. Let $T := M^{\perp}$ in L_M. The condition $l(A_M) + rk(M) = 24$ is equivalent to $rk(T) = l(A_T)$. Therefore, there exists an element $t \in M$ such that $(v + t)/a \in L_M$, where $a = GCD(div_M(t), 2n-2) \neq 1$. Moreover, $t = t_i + div(t)w$ for some i and some $w \in M$. Let now X be a manifold of $K3^{[n]}$ type with $M \subset Pic(X)$ and the embedding $L_n \subset L_M$ coincides with the natural one on X. By our assumption on t_i, the vector $(v + t_i)/a$ is a wall divisor as in **Definition 9**, hence we are done.

In particular, if we analyse what happens for three interesting lattices with $l(A_T) + rk(T) = 24$, we see that they never occur as coinvariant lattices for symplectic automorphisms. This is some evidence for the following:

Conjecture 32 *There is a bijective correspondence between finite groups of symplectic automorphisms of manifolds of $K3^{[n]}$ type (for some n) and subgroups G of Co_1 satisfying the conditions of* **Theorem 29**.

The three lattices we will analyse are coinvariant lattices for isometries of order 2 or 3 and they are the Barnes-Wall lattice $BW_{16}(-1)$, defined in [11, Section 4.10], the lattice $S_{3.exo}$ defined below and the lattice $D_{12}^{+}(-2)$, where D_{12}^{+} is an (odd) unimodular overlattice of D_{12}.

Lemma 33 *Let $v \in L_M$ be a primitive element of square $2n-2$ and let $BW_{16}(-1)$ be primitively embedded inside $L_n := v^{\perp}$. Then there exists an element t inside BW_{16} such that $\frac{v+t}{2} \in L_M$ and t is a numerical wall divisor.*

Proof We want to apply **Proposition 31**, and we use the fact that the discriminant group of BW_{16} can be generated by elements t_i of square at most 12, as computed in [11, Section 6.5]. Moreover, by **Theorem 15**, the orthogonal of BW_{16} inside L_M is isometric to $U(2)^4$ hence n must be odd and the inequality of **Proposition 31** is satisfied.

Let us take the lattice $N = E_8(-1)^3$ and let f be an order three permutation of the copies of $E_8(-1)$ in it. We set $S_{3.exo} := S_f(N)$.

Lemma 34 *Let $v \in L_M$ be a primitive element of square $2n-2$, where $n \equiv 1 \mod 3$. Let $S_{3.exo}$ be primitively embedded inside $L_n := v^{\perp}$. Then there exists an element t inside $S_{3.exo}$ such that $\frac{v+t}{3} \in L_M$ and t is a numerical wall divisor.*

Proof The proof goes as in **Lemma 33**. Here the set $\{t_i\}$ is given by elements of the form $2e - f - g$, where $S_{3.exo} \subset E_8(-1)^3$, e is a root of one copy of E_8 and f and g are the corresponding roots in the other two copies. These have squares $-36, -24$ or -12 which satisfy the bound in **Proposition 31** if $n \geq 5$. Observing that t^2 and $2n-2$ must be congruent modulo 9, we are done also for $n \leq 4$.

Lemma 35 *Let $v \in L_M$ be a primitive element of square $2n-2$ and let $D_{12}^{+}(-2)$ be primitively embedded inside $L_n := v^{\perp}$. Then there exists an element t inside D_{12}^{+} such that $\frac{v+t}{2} \in L_M$ and $\langle v, \frac{v+t}{2} \rangle$ is primitive and isometric to $\begin{pmatrix} 2n-2 & n-1 \\ n-1 & -2 \end{pmatrix}$, i.e. t is a numerical wall divisor.*

Proof In this case, **Proposition 31** is not sufficient, therefore we have to apply **Lemma 11** to the lattice $D_{12}^{+}(-2)$ several times and we use the fact that there exists an isometry between isometric elements of $A_{D_{12}^{+}(-2)}$. First of all, up to isometry there is only one primitive embedding of $D_{12}^{+}(-2)$ inside L_M, whose orthogonal is $T := U(2)^4 \oplus (-2)^4$ by the classification of 2-modular lattices in **Theorem 15**. Let $v \in T$ be as above, the lattice spanned by v and $D_{12}^{+}(-2)$ is primitive inside L_M and is isometric to $U \oplus (-2)^{11}$ again by **Theorem 15**. Once more, also the primitive embedding of $D_{12}^{+}(-2)$ inside $U \oplus (-2)^{11}$ is unique up to isometry. One such embedding is given by taking an element t of square $-2n-6$ and adding $\frac{v+t}{2}$ to $v \oplus D_{12}^{+}(-2)$, which is our claim.

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