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# Probabilistic Termination by Monadic Affine Sized Typing (Long Version)

Ugo Dal Lago* Charles Grellois†

December 9, 2019

**Abstract**

We introduce a system of monadic affine sized types, which substantially generalise usual sized types, and allows this way to capture probabilistic higher-order programs which terminate almost surely. Going beyond plain, strong normalisation without losing soundness turns out to be a hard task, which cannot be accomplished without a richer, quantitative notion of types, but also without imposing some affinity constraints. The proposed type system is powerful enough to type classic examples of probabilistically terminating programs such as random walks. The way typable programs are proved to be almost surely terminating is based on reducibility, but requires a substantial adaptation of the technique.

## 1 Introduction

Probabilistic models are more and more pervasive in computer science [1, 2, 3]. Moreover, the concept of algorithm, originally assuming determinism, has been relaxed so as to allow probabilistic evolution since the very early days of theoretical computer science [4]. All this has given impetus to research on probabilistic programming languages, which however have been studied at a large scale only in the last twenty years, following advances in randomized computation [5], cryptographic protocol verification [6, 7], and machine learning [8]. Probabilistic programs can be seen as ordinary programs in which specific instructions are provided to make the program evolve probabilistically rather than deterministically. The typical example are instructions for sampling from a given distribution toolset, or for performing probabilistic choice.

One of the most crucial properties a program should satisfy is *termination*: the execution process should be guaranteed to end. In (non)deterministic computation, this is easy to formalize, since any possible computation path is only considered qualitatively, and termination is a boolean predicate on programs: any non-deterministic program either terminates – in must or may sense – or it does not. In probabilistic programs, on the other hand, any terminating computation path is attributed a probability, and thus termination becomes a *quantitative* property. It is therefore natural to consider a program terminating when its terminating paths form a set of measure one or, equivalently, when it terminates with maximal probability. This is dubbed "almost sure termination" (AST for short) in the literature [9], and many techniques for automatically and semi-automatically checking programs for AST have been introduced in the last years [10, 11, 12, 13]. All of them, however, focus on imperative programs; while probabilistic functional programming languages are nowadays among the most successful ones in the realm of probabilistic programming [8]. It is not clear at all whether the existing techniques for imperative languages could be easily applied to functional ones, especially when higher-order functions are involved.

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In this paper, we introduce a system of monadic affine sized types for a simple probabilistic $\lambda$-calculus with recursion and show that it guarantees the AST property for all typable programs. The type system, described in Section 4, can be seen as a non-trivial variation on Hughes et al.'s sized types [14], whose main novelties are the following:

- Types are generalised so as to be *monadic*, this way encapsulating the kind of information we need to type non-trivial examples. This information, in particular, is taken advantage of when typing recursive programs.
- Typing rules are *affine*: higher-order variables cannot be freely duplicated. This is quite similar to what happens when characterising polynomial time functions by restricting higher-order languages akin to the $\lambda$-calculus [15]. Without affinity, the type system is bound to be unsound for AST.

The necessity of both these variations is discussed in Section 2 below. The main result of this paper is that typability in monadic affine sized types entails AST, a property which is proved using an adaptation of the Girard-Tait reducibility technique [16]. This adaptation is technically involved, as it needs substantial modifications allowing to deal with possibly infinite and probabilistic computations. In particular, every reducibility set must be parametrized by a quantitative parameter $p$ guaranteeing that terms belonging to this set terminate with probability at least $p$. The idea of parametrizing such sets already appears in work by the first author and Hofmann [17], in which a notion of realizability parametrized by resource monoids is considered. These realizability models are however studied in relation to linear logic and to the complexity of normalisation, and do not fit as such to our setting, even if they provided some crucial inspiration. In our approach, the fact that recursively-defined terms are AST comes from a continuity argument on this parameter: we can prove, by unfolding such terms, that they terminate with probability $p$ for every $p < 1$, and continuity then allows to take the limit and deduce that they are AST. This soundness result is technically speaking the main contribution of this paper, and is described in Section 6.

## 1.1 Related Works

Sized types have been originally introduced by Hughes, Pareto, and Sabry [14] in the context of reactive programming. A series of papers by Barthe and colleagues [18, 19, 20] presents sized types in a way similar to the one we will adopt here, although still for a deterministic functional language. Contrary to the other works on sized types, their type system is proved to admit a decidable type inference, see the unpublished tutorial [19]. Abel developed independently of Barthe and colleagues a similar type system featuring size informations [21]. These three lines of work allow polymorphism, arbitrary inductive data constructors, and ordinal sizes, so that data such as infinite trees can be manipulated. These three features will be absent of our system, in order to focus the challenge on the treatment of probabilistic recursive programs. Another interesting approach is the one of Xi's Dependent ML [22], in which a system of lightweight dependent types allows a more liberal treatment of the notion of size, over which arithmetic or conditional operations may in particular be applied. Termination is ensured by checking during typing that a given metrics decreases during recursive calls. This system is well-adapted for practical termination checking and can be extended with mutual recursion, inductive types and polymorphism, but does not feature ordinal sizes. See [21] for a detailed comparison of the previously cited systems. Some works along these lines are able to deal with coinductive data, as well as inductive ones [14, 18, 21]. They are related to Amadio and Coupet-Grimal's work on guarded types ensuring productivity of infinite structures such as streams [23]. None of these works deal with probabilistic computation, and in particular with almost sure termination.

There has been a lot of interest, recently, about probabilistic termination as a verification problem in the context of imperative programming [10, 11, 12, 13]. All of them deal, invariably, with some form of while-style language without higher-order functions. A possible approach is to reduce AST for probabilistic programs to termination of non-deterministic programs [10]. Another one is to extend the concept of ranking function to the probabilistic case. Bournez and Garnier obtained in this way the notion of Lyapunov ranking function [24], but such functions capture a notion more restrictive than AST: *positive* almost sure termination, meaning that the

program is AST and terminates in expected finite time. To capture AST, the notion of ranking supermartingale [25] has been used. Note that the use of ranking supermartingales allows to deal with programs which are both probabilistic and non-deterministic [11, 13] and even to reason about programs with real-valued variables [12].

Some recent work by Cappai, the first author, and Parisen Toldin [26, 27] introduce type systems ensuring that all typable programs can be evaluated in probabilistic polynomial time. This is too restrictive for our purposes. On the one hand, we aim at termination, and restricting to polynomial time algorithms would be an overkill. On the other hand, the above-mentioned type systems guarantee that the length of *all* probabilistic branches are uniformly bounded (by the same polynomial). In our setting, this would restrict the focus to terms in which infinite computations are forbidden, while we simply want the set of such computations to have probability 0. In fact, the results we present in this paper can be seen as a first step towards a type system characterizing average polynomial time, in the style of implicit computational complexity [**?**].

## 2 Why is Monadic Affine Typing Necessary?

In this section, we justify the design choices that guided us in the development of our type system. As we will see, the nature of AST requires a significant and non-trivial extension of the system of sized types originally introduced to ensure termination in the deterministic case [14].

**Sized Types for Deterministic Programs.** The simply-typed $\lambda$-calculus endowed with a typed recursion operator $\mathsf{letrec}$ and appropriate constructs for the natural numbers, sometimes called $\mathsf{PCF}$, is already Turing-complete, so that there is no hope to prove it strongly normalizing. Sized types [14] refine the simple type system by enriching base types with annotations, so as to ensure the termination of any recursive definition. Let us explain the idea of sizes in the simple, yet informative case in which the base type is $\mathsf{Nat}$. Sizes are defined by the grammar

$$\mathfrak{s} \quad ::= \quad \mathfrak{i} \;\big|\; \infty \;\big|\; \widehat{\mathfrak{s}}$$

where $\mathfrak{i}$ is a size variable and $\widehat{\mathfrak{s}}$ is the successor of the size $\mathfrak{s}$ — with $\widehat{\infty} = \infty$. These sizes permit to consider decorations $\mathsf{Nat}^{\mathfrak{s}}$ of the base type $\mathsf{Nat}$, whose elements are natural numbers of size at most $\mathfrak{s}$. The type system ensures that the only constant value of type $\mathsf{Nat}^{\widehat{\mathfrak{i}}}$ is $\mathsf{0}$, that the only constant values of type $\mathsf{Nat}^{\widehat{\widehat{\mathfrak{i}}}}$ are $\mathsf{0}$ or $\underline{1} = \mathsf{S}\ \mathsf{0}$, and so on. The type $\mathsf{Nat}^{\infty}$ is the one of all natural numbers, and is therefore often denoted as $\mathsf{Nat}$.

The crucial rule of the sized type system, which we present here following Barthe et al. [18], allows one to type recursive definitions as follows:

$$\frac{\Gamma, f \,:\, \mathsf{Nat}^{\mathfrak{i}} \to \sigma \vdash M \,:\, \mathsf{Nat}^{\widehat{\mathfrak{i}}} \to \sigma[\widehat{\mathfrak{i}}/\mathfrak{i}] \qquad \mathfrak{i} \text{ pos } \sigma}{\Gamma \vdash \mathsf{letrec}\ f \;=\; M \,:\, \mathsf{Nat}^{\mathfrak{s}} \to \sigma[\mathfrak{s}/\mathfrak{i}]} \tag{1}$$

This typing rule ensures that, to recursively define the function $f = M$, the term $M$ taking an input of size $\widehat{\mathfrak{i}}$ calls $f$ on inputs of *strictly lesser* size $\mathfrak{i}$. This is for instance the case when typing the program

$$M_{DBL} \;=\; \mathsf{letrec}\ f \;=\; \lambda x.\mathsf{case}\ x\ \mathsf{of}\ \big\{\, \mathsf{S} \to \lambda y.\mathsf{S}\ \mathsf{S}\ (f\ y) \;\big|\; \mathsf{0} \to \mathsf{0} \,\big\}$$

computing recursively the double of an input integer, as the hypothesis of the fixpoint rule in a typing derivation of $M_{DBL}$ is

$$f \,:\, \mathsf{Nat}^{\mathfrak{i}} \to \mathsf{Nat} \vdash \lambda x.\mathsf{case}\ x\ \mathsf{of}\ \big\{\, \mathsf{S} \to \lambda y.\mathsf{S}\ \mathsf{S}\ (f\ y) \;\big|\; \mathsf{0} \to \mathsf{0} \,\big\} \,:\, \mathsf{Nat}^{\widehat{\mathfrak{i}}} \to \mathsf{Nat}$$

The fact that $f$ is called on an input $y$ of strictly lesser size $\mathfrak{i}$ is ensured by the rule typing the $\mathsf{case}$ construction:

$$\frac{\Gamma \vdash x \,:\, \mathsf{Nat}^{\widehat{\mathfrak{i}}} \qquad \Gamma \vdash \lambda y.\mathsf{S}\ \mathsf{S}\ (f\ y) \,:\, \mathsf{Nat}^{\mathfrak{i}} \to \mathsf{Nat} \qquad \Gamma \vdash \mathsf{0} \,:\, \mathsf{Nat}}{\Gamma \vdash \mathsf{case}\ x\ \mathsf{of}\ \big\{\, \mathsf{S} \to \lambda y.\mathsf{S}\ \mathsf{S}\ (f\ y) \;\big|\; \mathsf{0} \to \mathsf{0} \,\big\} \,:\, \mathsf{Nat}}$$

where $\Gamma = f : \mathsf{Nat}^{\mathsf{i}} \to \mathsf{Nat},\ x : \mathsf{Nat}^{\widehat{\mathsf{i}}}$. The soundness of sized types for strong normalization allows to conclude that $M_{DBL}$ is indeed SN.

**A Naïve Generalization to Probabilistic Terms.** The aim of this paper is to obtain a probabilistic, *quantitative* counterpart to this soundness result for sized types. Note that unlike the result for sized types, which was focusing on *all* reduction strategies of terms, we only consider a *call-by-value* calculus[1]. Terms can now contain a probabilistic choice operator $\oplus_p$, such that $M \oplus_p N$ reduces to the term $M$ with probability $p \in \mathbb{R}_{[0,1]}$, and to $N$ with probability $1-p$. The language and its operational semantics will be defined more extensively in Section 3. Suppose for the moment that we type the choice operator in a naïve way:

$$\text{Choice} \quad \frac{\Gamma \vdash M : \sigma \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash M \oplus_p N : \sigma}$$

On the one hand, the original system of sized types features subtyping, which allows some flexibility to "unify" the types of $M$ and $N$ to $\sigma$. On the other hand, it is easy to realise that *all* probabilistic branches would have to be terminating, without any hope of capturing interesting AST programs: nothing has been done to capture the *quantitative* nature of probabilistic termination. An instance of a term which is not strongly normalizing but which is almost-surely terminating — meaning that it normalizes with probability 1 — is

$$M_{BIAS} = \left(\mathsf{letrec}\ f = \lambda x.\mathsf{case}\ x\ \mathsf{of}\ \left\{ \mathsf{S} \to \lambda y.f(y) \oplus_{\frac{2}{3}} (f(\mathsf{S}\,\mathsf{S}\,y)) \ \middle|\ 0 \to 0 \right\}\right)\ \underline{n} \tag{2}$$

simulating a *biased random walk* which, on $x = m+1$, goes to $m$ with probability $\frac{2}{3}$ and to $m+2$ with probability $\frac{1}{3}$. The naïve generalization of the sized type system only allows us to type the body of the recursive definition as follows:

$$f : \mathsf{Nat}^{\widehat{\widehat{\mathsf{i}}}} \to \mathsf{Nat}^{\infty} \vdash \lambda y.f(y) \oplus_{\frac{2}{3}} (f(\mathsf{S}\,\mathsf{S}\,y))) : \mathsf{Nat}^{\widehat{\mathsf{i}}} \to \mathsf{Nat}^{\infty} \tag{3}$$

and thus does not allow us to deduce any relevant information on the *quantitative* termination of this term: nothing tells us that the recursive call $f(\mathsf{S}\,\mathsf{S}\,y)$ is performed with a relatively low probability.

**A Monadic Type System.** Along the evaluation of $M_{BIAS}$, there is *indeed* a quantity which decreases during each recursive call to the function $f$: the *average* size of the input on which the call is performed. Indeed, on an input of size $\widehat{\mathsf{i}}$, $f$ calls itself on an input of smaller size $\mathsf{i}$ with probability $\frac{2}{3}$, and on an input of greater size $\widehat{\widehat{\mathsf{i}}}$ with probability only $\frac{1}{3}$. To capture such a relevant *quantitative* information on the recursive calls of $f$, and with the aim to capture almost sure termination, we introduce a *monadic* type system, in which *distributions of types* can be used to type in a finer way the functions to be used recursively. Contexts $\Gamma \mid \Theta$ will be generated by a context $\Gamma$ attributing sized types to any number of variables, while $\Theta$ will attribute a *distribution* of sized types to at most *one* variable — typically the one we want to use to recursively define a function. In such a context, terms will be typed by a distribution type, formed by combining the Dirac distributions of types introduced in the Axiom rules using the following rule for probabilistic choice:

$$\text{Choice} \quad \frac{\Gamma \mid \Theta \vdash M : \mu \qquad \Gamma \mid \Psi \vdash N : \nu \qquad \langle \mu \rangle = \langle \nu \rangle}{\Gamma \mid \Theta \oplus_p \Psi \vdash M \oplus_p N : \mu \oplus_p \nu}$$

The guard condition $\langle \mu \rangle = \langle \nu \rangle$ ensures that $\mu$ and $\nu$ are distributions of types decorating of the *same* simple type. Without this condition, there is no hope to aim for a decidable type inference algorithm.

[1] Please notice that choosing a reduction strategy is crucial in a probabilistic setting, otherwise one risks getting nasty forms of non-confluence [28].

**The Fixpoint Rule.** Using these monadic types, instead of the insufficiently informative typing (3), we can derive the sequent

$$f \,:\, \left\{ \left(\mathsf{Nat}^{\mathsf{i}} \to \mathsf{Nat}^{\infty}\right)^{\frac{2}{3}},\ \left(\mathsf{Nat}^{\widehat{\widehat{\mathsf{i}}}} \to \mathsf{Nat}^{\infty}\right)^{\frac{1}{3}} \right\} \vdash \lambda y.f(y) \oplus_{\frac{2}{3}} (f(\mathsf{S}\,\mathsf{S}\,y))) \;:\; \mathsf{Nat}^{\widehat{\mathsf{i}}} \to \mathsf{Nat}^{\infty} \tag{4}$$

in which the type of $f$ contains finer information on the sizes of arguments over which it is called recursively, and with which probability. This information enables us to perform a first switch from a qualitative to a quantitative notion of termination: we will adapt the hypothesis

$$\Gamma, f \,:\, \mathsf{Nat}^{\mathsf{i}} \to \sigma \vdash M \,:\, \mathsf{Nat}^{\widehat{\mathsf{i}}} \to \sigma[\widehat{\mathsf{i}}/\mathsf{i}] \tag{5}$$

of the original fix rule (1) of sized types, expressing that $f$ is called on an argument of size one less than the one on which $M$ is called, to a condition meaning that there is probability 1 to call $f$ on arguments of a lesser size *after enough iterations of recursive calls*. We therefore define a random walk associated to the distribution type $\mu$ of $f$, the *sized walk* associated to $\mu$, and which is as follows for the typing (4):

- the random walk starts on 1, corresponding to the size $\widehat{\mathsf{i}}$,
- on an integer $n+1$, the random walk jumps to $n$ with probability $\frac{2}{3}$ and to $n+2$ with probability $\frac{1}{3}$,
- 0 is stationary: on it, the random walk loops.

This random walk – as all sized walks will be – is an instance of *one-counter Markov decision problem* [29], so that it is decidable in polynomial time whether the walk reaches 0 with probability 1. We will therefore replace the hypothesis (5) of the $\mathsf{letrec}$ rule by the quantitative counterpart we just sketched, obtaining

$$\mathsf{letrec}\quad \frac{\begin{array}{c}\left\{\, (\mathsf{Nat}^{\mathfrak{s}_j} \to \nu[\mathfrak{s}_j/\mathsf{i}])^{p_j} \;\middle|\; j \in \mathcal{J} \,\right\} \text{induces an AST sized walk} \\ \Gamma \,|\, f \,:\, \left\{\, (\mathsf{Nat}^{\mathfrak{s}_j} \to \nu[\mathfrak{s}_j/\mathsf{i}])^{p_j} \;\middle|\; j \in \mathcal{J} \,\right\} \vdash V \,:\, \mathsf{Nat}^{\widehat{\mathsf{i}}} \to \nu[\widehat{\mathsf{i}}/\mathsf{i}]\end{array}}{\Gamma,\, \Delta \,|\, \Theta \vdash \mathsf{letrec}\ f \;=\; V \,:\, \mathsf{Nat}^{\mathfrak{r}} \to \nu[\mathfrak{r}/\mathsf{i}]}$$

where we omit two additional technical conditions to be found in Section 4 and which justify the weakening on contexts incorporated to this rule. The resulting type system allows to type a varieties of examples, among which the following program computing the geometric distribution over the natural numbers:

$$M_{EXP} \;=\; \left(\mathsf{letrec}\ f \;=\; \lambda x.x \oplus_{\frac{1}{2}} \mathsf{S}\ \ (f\ x)\right)\ \mathsf{0} \tag{6}$$

and for which the decreasing quantity is the size of the set of probabilistic branches of the term making recursive calls to $f$. Another example is the unbiased random walk

$$M_{UNB} \;=\; \left(\mathsf{letrec}\ f \;=\; \lambda x.\mathsf{case}\ x\ \mathsf{of}\ \left\{ \mathsf{S} \to \lambda y.f(y) \oplus_{\frac{1}{2}} (f(\mathsf{S}\,\mathsf{S}\,y))) \;\middle|\; \mathsf{0} \to \mathsf{0} \right\}\right)\ \underline{n} \tag{7}$$

for which there is no clear notion of decreasing measure during recursive calls, but which yet terminates almost surely, as witnessed by the sized walk associated to an appropriate derivation in the sized type system. We therefore claim that the use of this external guard condition on associated sized walks, allowing us to give a general condition of termination, is satisfying as it both captures an interesting class of examples, and is computable in polynomial time.

In Section 6, we prove that this shift from a qualitative to a quantitative hypothesis in the type system results in a shift from the soundness for strong normalization of the original sized type system to a soundness for its quantitative counterpart: *almost-sure termination*.

**Why Affinity?** To ensure the soundness of the letrec rule, we need one more structural restriction on the type system. For the sized walk argument to be adequate, we must ensure that the recursive calls of $f$ are indeed precisely modelled by the sized walk, and this is not the case when considering for instance the following term:

$$M_{NAFF} \;=\; \Big(\mathsf{letrec}\ f \;=\; \lambda x.\mathsf{case}\ x\ \mathsf{of}\ \Big\{\mathsf{S} \to \lambda y.f(y) \oplus_{\frac{2}{3}} (f(\mathsf{S}\,\mathsf{S}\,y)\,;\, f(\mathsf{S}\,\mathsf{S}\,y)) \;\Big|\; 0 \to 0\Big\}\Big)\ \underline{n} \tag{8}$$

where the sequential composition ; is defined in this call-by-value calculus as

$$M\,;\,N \;=\; (\lambda x.\lambda y.0)\ M\ N$$

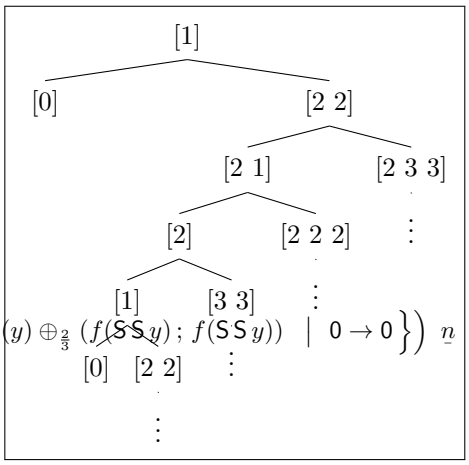


Figure 1: A Tree of Recursive Calls.

Note that $M_{NAFF}$ calls recursively $f$ *twice* in the right branch of its probabilistic choice, and is not therefore modelled appropriately by the sized walk associated to its type. In fact, we would need a generalized notion of random walk to model the recursive calls of this process; it would be a random walk on *stacks* of integers. In the case where $n = 1$, the recursive calls to $f$ can indeed be represented by a tree of stacks as depicted in Figure 1, where leftmost edges have probability $\frac{2}{3}$ and rightmost ones $\frac{1}{3}$. The root indicates that the first call on $f$ was on the integer 1. From it, there is either a call of $f$ on 0 which terminates, or *two* calls on 2 which are put into a stack of calls, and so on. We could prove that, without the *affine* restriction we are about to formulate, the term $M_{NAFF}$ is typable with monadic sized types and the fixpoint rule we just designed. However, this term is not almost-surely terminating. Notice, indeed, that the *sum* of the integers appearing in a stack labelling a node of the tree in Figure 1 decreases by 1 when the left edge of probability $\frac{2}{3}$ is taken, and increases by *at least* 3 when the right edge of probability $\frac{1}{3}$ is taken. It follows that the expected increase of the sum of the elements of the stack during one step is at least $-1 \times \frac{2}{3} + 3 \times \frac{1}{3} = \frac{1}{3} > 0$. This implies that the probability that $f$ is called on an input of size 0 after enough iterations is strictly less than 1, so that the term $M_{NAFF}$ cannot be almost surely terminating.

Such general random processes have stacks as states and are rather complex to analyse. To the best of the authors' knowledge, they do not seem to have been considered in the literature. We also believe that the complexity of determining whether 0 can be reached almost surely in such a process, if decidable, would be very high. This leads us to the design of an *affine* type system, in which the management of contexts ensures that a given probabilistic branch of a term may only use at most once a given higher-order symbol. We do not however formulate restrictions on variables of simple type Nat, as affinity is only used on the letrec rule and thus on higher-order symbols. Remark that this is in the spirit of certain systems from implicit computational complexity [15, 30].

Another restriction imposed by this reduction of almost-sure termination checking for higher-order programs to almost-sure termination checking for one-counter Markov decision processes is the fact that we do not allow a general form of nested recursion. This restriction is encoded in the system by allowing *at most* one variable to have a distribution of types in the context. It follows that programs making use of mutual recursion can not be typed in this system.

# 3 A Simple Probabilistic Functional Programming Language

We consider the language $\lambda_{\oplus}$, which is an extension of the $\lambda$-calculus with recursion, constructors for the natural numbers, and a choice operator. In this section, we introduce this language and its operational semantics, and use them to define the crucial notion of *almost-sure termination*.

**Terms and Values.** Given a set of variables $\mathcal{X}$, terms and values of the language $\lambda_\oplus$ are defined by mutual induction as follows:

$$\begin{array}{llll} \text{Terms:} & M, N, \ldots & ::= & V \mid V\ W \mid \mathsf{let}\ x = M\ \mathsf{in}\ N \mid M \oplus_p N \\ & & & \mid \mathsf{case}\ V\ \mathsf{of}\ \{\mathsf{S} \to W \mid \mathsf{0} \to Z\} \\ \\ \text{Values:} & V, W, Z, \ldots & ::= & x \mid \mathsf{0} \mid \mathsf{S}\ V \mid \lambda x.M \mid \mathsf{letrec}\ f = V \end{array}$$

where $x,\ f \in \mathcal{X}$, $p \in ]0,1[$. When $p = \frac{1}{2}$, we often write $\oplus$ as a shorthand for $\oplus_{\frac{1}{2}}$. The set of terms is denoted $\Lambda_\oplus$ and the set of values is denoted $\Lambda_\oplus^V$. Terms of the calculus are assumed to be in A-normal form [31]. This allows to formulate crucial definitions in a simpler way, concentrating in the Let construct the study of the probabilistic behaviour of terms. We claim that all traditional constructions can be encoded in this formalism. For instance, the usual application $M\ N$ of two terms can be harmlessly recovered via the encoding $\mathsf{let}\ x = M\ \mathsf{in}\ (\mathsf{let}\ y = N\ \mathsf{in}\ x\ y)$. In the sequel, we write $c\ \overrightarrow{V}$ when a value may be either $\mathsf{0}$ or of the shape $\mathsf{S}\ V$.

**Term Distributions.** The introduction of a probabilistic choice operator in the syntax leads to a *probabilistic* reduction relation. It is therefore meaningful to consider the (operational) semantics of a term as a *distribution* of values modelling the outcome of *all* the finite probabilistic reduction paths of the term. For instance, the term $M_{EXP}$ defined in (6) evaluates to the term distribution assigning probability $\frac{1}{2^{n+1}}$ to the value $\underline{n}$. Let us define this notion more formally:

**Definition 1 (Distribution)** *A* distribution *on $X$ is a function $\mathscr{D} : X \to [0,1]$ satisfying the constraint $\sum \mathscr{D} = \sum_{x \in X} \mathscr{D}(x) \leq 1$, where $\sum \mathscr{D}$ is called the* sum *of the distribution $\mathscr{D}$. We say that $\mathscr{D}$ is* proper *precisely when $\sum \mathscr{D} = 1$. We denote by $\mathcal{P}$ the set of all distributions, would they be proper or not. We define the* support *$\mathsf{S}(\mathscr{D})$ of a distribution $\mathscr{D}$ as: $\mathsf{S}(\mathscr{D}) = \{x \in X \mid \mathscr{D}(x) > 0\}$. When $\mathsf{S}(\mathscr{D})$ consists only of closed terms, we say that $\mathscr{D}$ is a* closed *distribution. When it is finite, we say that $\mathscr{D}$ is a* finite *distribution. We call* Dirac *a proper distribution $\mathscr{D}$ such that $\mathsf{S}(\mathscr{D})$ is a singleton. We denote by $0$ the null distribution, mapping every term to the probability $0$.*

When $X = \Lambda_\oplus$, we say that $\mathscr{D}$ is a *term distribution.* In the sequel, we will use a more practical notion of *representation* of distributions, which enumerates the terms with their probabilities as a family of assignments. For technical reasons, notably related to the subject reduction property, we will also need *pseudo-representations*, which are essentially multiset-like decompositions of the representation of a distribution.

**Definition 2 (Representations and Pseudo-Representations)** *Let $\mathscr{D} \in \mathcal{P}$ be of support $\{x_i \mid i \in \mathcal{I}\}$, where $x_i = x_j$ implies $i = j$ for every $i, j \in \mathcal{I}$. The* representation *of $\mathscr{D}$ is the set $\mathscr{D} = \left\{ x_i^{\mathscr{D}(x_i)} \mid i \in \mathcal{I} \right\}$ where $x_i^{\mathscr{D}(x_i)}$ is just an intuitive way to write the pair $(x_i, \mathscr{D}(x_i))$. A* pseudo-representation *of $\mathscr{D}$ is any multiset $\left[ y_j^{p_j} \mid j \in \mathcal{J} \right]$ such that*

$$\forall j \in \mathcal{J},\quad y_j \in \mathsf{S}(\mathscr{D}) \qquad\qquad \forall i \in \mathcal{I},\quad \mathscr{D}(x_i) = \sum_{y_j = x_i} p_j.$$

*By abuse of notation, we will simply write $\mathscr{D} = \left[ y_j^{p_j} \mid j \in \mathcal{J} \right]$ to mean that $\mathscr{D}$ admits $\left[ y_j^{p_j} \mid j \in \mathcal{J} \right]$ as pseudo-representation. Any distribution has a canonical pseudo-representation obtained by simply replacing the set-theoretic notation with the multiset-theoretic one.*

**Definition 3 ($\omega$-CPO of distributions)** *We define the pointwise order on distributions over $X$ as*

$$\mathscr{D} \preccurlyeq \mathscr{E} \qquad \text{if and only if} \qquad \forall x \in X,\quad \mathscr{D}(x) \leq \mathscr{E}(x).$$

*This turns $(\mathcal{P}, \preccurlyeq)$ into a partial order. This partial order is an $\omega$-CPO, but not a lattice as the join of two distributions does not necessarily exist. The bottom element of this $\omega$-CPO is the null distribution $0$.*

**Definition 4 (Operations on distributions)** *Given a distribution $\mathscr{D}$ and a real number $\alpha \leq 1$, we define the distribution $\alpha \cdot \mathscr{D}$ as $x \mapsto \alpha \cdot \mathscr{D}(x)$. We similarly define the sum $\mathscr{D} + \mathscr{E}$ of two distributions over a same set $X$ as the function $x \mapsto \mathscr{D}(x) + \mathscr{E}(x)$. Note that this is a total operation on functions $X \to \mathbb{R}$, but a partial operation on distributions: it is defined if and only if $\sum \mathscr{D} + \sum \mathscr{E} \leq 1$. When $\mathscr{D} \preccurlyeq \mathscr{E}$, we define the partial operation of difference of distributions $\mathscr{E} - \mathscr{D}$ as the function $V \mapsto \mathscr{E}(V) - \mathscr{D}(V)$. We naturally extend these operations to representations and pseudo-representations of distributions.*

**Definition 5 (Value Decomposition of a Term Distribution)** *Let $\mathscr{D}$ be a term distribution. We write its* value decomposition *as $\mathscr{D} \stackrel{VD}{=} \mathscr{D}_{|V} + \mathscr{D}_{|T}$, where $\mathscr{D}_{|V}$ is the maximal subdistribution of $\mathscr{D}$ whose support consists of values, and $\mathscr{D}_{|T} = \mathscr{D} - \mathscr{D}_{|V}$ is the subdistribution of "non-values" contained in $\mathscr{D}$.*

**Operational Semantics.** The semantics of a term will be the value distribution to which it reduces via the probabilistic reduction relation, iterated up to the limit. As a first step, we define the call-by-value reduction relation $\to_v \subseteq \mathcal{P} \times \mathbb{R}^{\Lambda_\oplus}$ on Figure 2. The relation $\to_v$ is in fact a relation on distributions:

**Lemma 1** *Let $\mathscr{D}$ be a distribution such that $\mathscr{D} \to_v \mathscr{E}$. Then $\mathscr{E}$ is a distribution.*

Note that we write Dirac distributions simply as terms on the left side of $\to_v$, to improve readability. As usual, we denote by $\to_v^n$ the $n$-th iterate of the relation $\to_v$, with $\to_v^0$ being the identity relation. We then define the relation $\Rrightarrow_v^n$ as follows. Let $\mathscr{D} \to_v^n \mathscr{E} \stackrel{VD}{=} \mathscr{E}_{|V} + \mathscr{E}_{|T}$. Then $\mathscr{D} \Rrightarrow_v^n \mathscr{E}_{|V}$. Note that, for every $n \in \mathbb{N}$ and $\mathscr{D} \in \mathcal{P}$, there is a unique distribution $\mathscr{E}$ such that $\mathscr{D} \to_v^n \mathscr{E}$. Moreover, $\mathscr{E}_{|V}$ is the only distribution such that $\mathscr{D} \Rrightarrow_v^n \mathscr{E}_{|V}$.

**Lemma 2** *Let $n, m \in \mathbb{N}$ with $n < m$. Let $\mathscr{D}_n$ (resp $\mathscr{D}_m$) be the distribution such that $M \to_v^n \mathscr{D}_n$ (resp $M \to_v^m \mathscr{D}_m$). Then $\mathscr{D}_n \preccurlyeq \mathscr{D}_m$.*

**Lemma 3** *Let $n, m \in \mathbb{N}$ with $n < m$. Let $\mathscr{D}_n$ (resp $\mathscr{D}_m$) be the distribution such that $M \Rrightarrow_v^n \mathscr{D}_n$ (resp $M \Rrightarrow_v^m \mathscr{D}_m$). Then $\mathscr{D}_n \preccurlyeq \mathscr{D}_m$.*

**Definition 6 (Semantics of a Term, of a Distribution)** *The semantics of a distribution $\mathscr{D}$ is the distribution $[\![ \mathscr{D} ]\!] = \sup_{n \in \mathbb{N}} \left( \left\{ \mathscr{D}_n \mid \mathscr{D} \Rrightarrow_v^n \mathscr{D}_n \right\} \right)$. This supremum exists thanks to Lemma 3, combined with the fact that $(\mathcal{P}, \preccurlyeq)$ is an $\omega$-CPO. We define the semantics of a term $M$ as $[\![ M ]\!] = [\![ \{ M^1 \} ]\!]$.*

**Corollary 1** *Let $n \in \mathbb{N}$ and $\mathscr{D}_n$ be such that $M \Rrightarrow_v^n \mathscr{D}_n$. Then $\mathscr{D}_n \preccurlyeq [\![ M ]\!]$.*

We now have all the ingredients required to define the central concept of this paper, the one of almost-surely terminating term:

**Definition 7 (Almost-Sure Termination)** *We say that a term $M$ is* almost-surely terminating *precisely when $\sum [\![ M ]\!] = 1$.*

Before we terminate this section, let us formulate the following lemma on the operational semantics of the $\mathsf{let}$ construction, which will be used in the proof of typing soundness for monadic affine sized types:

**Lemma 4** *Suppose that $M \Rrightarrow_v^n \left[ V^{p_i} \mid i \in \mathcal{I} \right]$ and that, for every $i \in \mathcal{I}$, $N[V_i/x] \Rrightarrow_v^m \mathscr{E}_i$. Then $\mathsf{let}\ x = M\ \mathsf{in}\ N \Rrightarrow_v^{n+m+1} \sum_{i \in \mathcal{I}} p_i \cdot \mathscr{E}_i$.*

**Proof.** Easy from the definition of $\Rrightarrow_v$ and of $\to_v$ in the case of $\mathsf{let}$. □

$$\frac{}{\mathsf{let}\ x\ =\ V\ \mathsf{in}\ M\ \ \to_v\ \ \left\{ (M[V/x])^1 \right\}} \qquad \frac{}{(\lambda x.M)\ V\ \ \to_v\ \ \left\{ (M[V/x])^1 \right\}}$$

$$\frac{}{M\ \oplus_p\ N\ \ \to_v\ \ \left\{ M^p,\, N^{1-p} \right\}}$$

$$\frac{M\ \ \to_v\ \ \left\{ L_i^{p_i}\ \middle|\ i \in \mathcal{I} \right\}}{\mathsf{let}\ x\ =\ M\ \mathsf{in}\ N\ \ \to_v\ \ \left\{ (\mathsf{let}\ x\ =\ L_i\ \mathsf{in}\ N)^{p_i}\ \middle|\ i \in \mathcal{I} \right\}}$$

$$\frac{}{\mathsf{case}\ \mathsf{S}\ V\ \mathsf{of}\ \{\, \mathsf{S} \to W\ \mid\ \mathsf{0} \to Z \,\}\ \ \to_v\ \ \left\{ (W\ V)^1 \right\}}$$

$$\frac{}{\mathsf{case}\ \mathsf{0}\ \mathsf{of}\ \{\, \mathsf{S} \to W\ \mid\ \mathsf{0} \to Z \,\}\ \ \to_v\ \ \left\{ (Z)^1 \right\}}$$

$$\frac{}{(\mathsf{letrec}\ f\ =\ V)\ \left(c\ \overrightarrow{W}\right)\ \ \to_v\ \ \left\{ \left(V[(\mathsf{letrec}\ f\ =\ V)/f]\ \left(c\ \overrightarrow{W}\right)\right)^1 \right\}}$$

$$\frac{\mathscr{D}\ \stackrel{VD}{=}\ \left\{ M_j^{p_j}\ \middle|\ j \in \mathcal{J} \right\} + \mathscr{D}_{|V} \qquad \forall j \in \mathcal{J},\ \ M_j\ \ \to_v\ \ \mathscr{E}_j}{\mathscr{D}\ \ \to_v\ \ \left(\sum_{j \in \mathcal{J}} p_j \cdot \mathscr{E}_j\right) + \mathscr{D}_{|V}}$$

Figure 2: Call-by-value reduction relation $\to_v$ on distributions.

# 4 Monadic Affine Sized Typing

Following the discussion from Section 2, we introduce in this section a non-trivial lifting of sized types to our probabilistic setting. As a first step, we design an *affine* simple type system for $\lambda_\oplus$. This means that no higher-order variable may be used more than once in the same probabilistic branch. However, variables of base type $\mathsf{Nat}$ may be used freely. In spite of this restriction, the resulting system allows to type terms corresponding to any probabilistic Turing machine. In Section 4.2, we introduce a more sophisticated type system, which will be *monadic* and affine, and which will be sound for almost-sure termination as we prove in Section 6.

## 4.1 Affine Simple Types for $\lambda_\oplus$

The terms of the language $\lambda_\oplus$ can be typed using a variant of the simple types of the $\lambda$-calculus, extended to type $\mathsf{letrec}$ and $\oplus_p$, but also restricted to an *affine* management of contexts. Recall that the constraint of affinity ensures that a given higher-order symbol is used at most *once* in a probabilistic branch. We define simple types over the base type $\mathsf{Nat}$ in the usual way: $\kappa, \kappa', \ldots ::= \mathsf{Nat} \mid \kappa \to \kappa'$ where, by convention, the arrow associates to the right. Contexts $\Gamma, \Delta, \ldots$ are sequences of simply-typed variables $x :: \kappa$. We write sequents as $\Gamma \vdash M :: \kappa$ to distinguish these sequents from the ones using distribution types appearing later in this section. Before giving the rules of the type system, we need to define two policies for contracting contexts: an affine and a general one.

**Context Contraction.** The contraction $\Gamma \cup \Delta$ of two contexts is a non-affine operation, and is partially defined as follows:

- $x :: \kappa \in \Gamma \setminus \Delta \implies x :: \kappa \in \Gamma \cup \Delta$,

$$\text{Var}\quad \frac{}{\Gamma, x :: \kappa \vdash x :: \kappa} \qquad \frac{\Gamma \vdash V :: \mathsf{Nat}}{\Gamma \vdash \mathsf{S}\, V :: \mathsf{Nat}} \qquad \frac{}{\Gamma \vdash \mathsf{0} :: \mathsf{Nat}}$$

$$\lambda\quad \frac{\Gamma, x :: \kappa \vdash M :: \kappa'}{\Gamma \vdash \lambda x.M :: \kappa \to \kappa'} \qquad \frac{\Gamma \vdash V :: \kappa \to \kappa' \qquad \Delta \vdash W :: \kappa}{\Gamma \uplus \Delta \vdash V\, W :: \kappa'}\quad \text{App}$$

$$\text{Choice}\quad \frac{\Gamma \vdash M :: \kappa \qquad \Delta \vdash N :: \kappa}{\Gamma \cup \Delta \vdash M \oplus_p N :: \kappa}$$

$$\text{Let}\quad \frac{\Gamma \vdash M :: \kappa \qquad \Delta, x :: \kappa \vdash N :: \kappa'}{\Gamma \uplus \Delta \vdash \mathsf{let}\ x = M\ \mathsf{in}\ N :: \kappa'}$$

$$\text{Case}\quad \frac{\Gamma \vdash V :: \mathsf{Nat} \qquad \Delta \vdash W :: \mathsf{Nat} \to \kappa \qquad \Delta \vdash Z :: \kappa}{\Gamma \uplus \Delta \vdash \mathsf{case}\ V\ \mathsf{of}\ \{\mathsf{S} \to W \mid \mathsf{0} \to Z\} :: \kappa}$$

$$\mathsf{letrec}\quad \frac{\Gamma, f :: \mathsf{Nat} \to \kappa \vdash V :: \mathsf{Nat} \to \kappa \qquad \forall x \in \Gamma,\ x :: \mathsf{Nat}}{\Gamma \vdash \mathsf{letrec}\ f = V :: \mathsf{Nat} \to \kappa}$$

Figure 3: Affine simple types for $\lambda_\oplus$.

- $x :: \kappa \in \Delta \setminus \Gamma \implies x :: \kappa \in \Gamma \cup \Delta$ ,
- if $x :: \kappa \in \Gamma$ and $x :: \kappa' \in \Delta$,
  - if $\kappa = \kappa'$, $x :: \kappa \in \Gamma \cup \Delta$,
  - else the operation is undefined.

This operation will be used to contract contexts in the rule typing the choice operation $\oplus_p$: indeed, we allow a same higher-order variable $f$ to occur both in $M$ and in $N$ when forming $M \oplus_p N$, as both terms correspond to different probabilistic branches.

**Affine contraction of contexts.** The affine contraction $\Gamma \uplus \Delta$ will be used in all rules but the one for $\oplus_p$. It is partially defined as follows:

- $x :: \kappa \in \Gamma \setminus \Delta \implies x :: \kappa \in \Gamma \uplus \Delta$,
- $x :: \kappa \in \Delta \setminus \Gamma \implies x :: \kappa \in \Gamma \uplus \Delta$ ,
- if $x :: \kappa \in \Gamma$ and $x :: \kappa' \in \Delta$,
  - if $\kappa = \kappa' = \mathsf{Nat}$, $x :: \kappa \in \Gamma \uplus \Delta$,
  - in any other case, the operation is undefined.

As we explained earlier, only variables of base type $\mathsf{Nat}$ may be contracted.

**The Affine Type System.** The affine simple type system is then defined in Figure 3. All the rules are quite standard. Higher-order variables can occur at most once in any probabilistic branch because all binary typing rules – except probabilistic choice – treat contexts affinely. We set $\Lambda_\oplus^V(\Gamma, \kappa) = \{V \in \Lambda_\oplus^V \mid \Gamma \vdash V :: \kappa\}$ and $\Lambda_\oplus(\Gamma, \kappa) = \{M \in \Lambda_\oplus \mid \Gamma \vdash M :: \kappa\}$. We simply write $\Lambda_\oplus^V(\kappa) = \Lambda_\oplus^V(\emptyset, \kappa)$ and $\Lambda_\oplus(\kappa) = \Lambda_\oplus(\emptyset, \kappa)$ when the terms or values are closed. These closed, typable terms enjoy subject reduction and the progress property.

$$\frac{}{\mathfrak{i}\ \mathsf{pos}\ \mathsf{Nat}^{\mathfrak{s}}} \qquad \frac{\mathfrak{i}\ \mathsf{neg}\ \sigma \qquad \mathfrak{i}\ \mathsf{pos}\ \mu}{\mathfrak{i}\ \mathsf{pos}\ \sigma \to \mu} \qquad \frac{\forall i \in \mathcal{I},\ \ \mathfrak{i}\ \mathsf{pos}\ \sigma_i}{\mathfrak{i}\ \mathsf{pos}\ \left\{ \sigma_i^{p_i} \ \middle|\ i \in \mathcal{I} \right\}}$$

$$\frac{\mathfrak{i} \notin \mathfrak{s}}{\mathfrak{i}\ \mathsf{neg}\ \mathsf{Nat}^{\mathfrak{s}}} \qquad \frac{\mathfrak{i}\ \mathsf{pos}\ \sigma \qquad \mathfrak{i}\ \mathsf{neg}\ \mu}{\mathfrak{i}\ \mathsf{neg}\ \sigma \to \mu} \qquad \frac{\forall i \in \mathcal{I},\ \ \mathfrak{i}\ \mathsf{neg}\ \sigma_i}{\mathfrak{i}\ \mathsf{neg}\ \left\{ \sigma_i^{p_i} \ \middle|\ i \in \mathcal{I} \right\}}$$

Figure 4: Positive and negative occurrences of a size variable in a size type.

## 4.2 Monadic Affine Sized Types

This section is devoted to giving the basic definitions and results about monadic affine sized types (MASTs, for short), which can be seen as decorations of the affine simple types with some *size information*.

**Sized Types.** We consider a set $\mathcal{S}$ of *size variables*, denoted $\mathfrak{i}, \mathfrak{j}, \ldots$ and define *sizes* (called *stages* in [18]) as:

$$\mathfrak{s}, \mathfrak{r} \quad ::= \quad \mathfrak{i} \ \middle|\ \infty \ \middle|\ \widehat{\mathfrak{s}}$$

where $\widehat{\cdot}$ denotes the *successor* operation. We denote the iterations of $\widehat{\cdot}$ as follows: $\widehat{\widehat{\mathfrak{s}}}$ is denoted $\widehat{\mathfrak{s}}^2$, $\widehat{\widehat{\widehat{\mathfrak{s}}}}$ is denoted $\widehat{\mathfrak{s}}^3$,and so on. By definition, at most one variable $\mathfrak{i} \in \mathcal{S}$ appears in a given size $\mathfrak{s}$. We call it its *spine variable*, denoted as $\mathrm{spine}(\mathfrak{s})$. We write $\mathrm{spine}(\mathfrak{s}) = \emptyset$ when there is no variable in $\mathfrak{s}$. An order $\preccurlyeq$ on sizes can be defined as follows:

$$\frac{}{\mathfrak{s} \preccurlyeq \mathfrak{s}} \qquad \frac{\mathfrak{s} \preccurlyeq \mathfrak{r} \qquad \mathfrak{r} \preccurlyeq \mathfrak{t}}{\mathfrak{s} \preccurlyeq \mathfrak{t}} \qquad \frac{}{\mathfrak{s} \preccurlyeq \widehat{\mathfrak{s}}} \qquad \frac{}{\mathfrak{s} \preccurlyeq \infty}$$

Notice that these rules imply notably that $\widehat{\infty}$ is equivalent to $\infty$, i.e., $\widehat{\infty} \preccurlyeq \infty$ and $\infty \preccurlyeq \widehat{\infty}$. We consider sizes modulo this equivalence. We can now define sized types and distribution types by mutual induction, calling distributions of (sized) types the distributions over the set of sized types:

**Definition 8 (Sized Types, Distribution Types)** *Sized types and distribution types are defined by mutual induction, contextually with the function $\langle\cdot\rangle$ which maps any sized or distribution type to its* underlying *affine type.*

$$\begin{array}{lrcl}
\textit{Sized types:} & \sigma, \tau & ::= & \sigma \to \mu \ \middle|\ \mathsf{Nat}^{\mathfrak{s}} \\
\textit{Distribution types:} & \mu, \nu & ::= & \left\{ \sigma_i^{p_i} \ \middle|\ i \in \mathcal{I} \right\}, \\
\textit{Underlying map:} & \langle \sigma \to \mu \rangle & = & \langle \sigma \rangle \to \langle \mu \rangle \\
 & \langle \mathsf{Nat}^{\mathfrak{s}} \rangle & = & \mathsf{Nat} \\
 & \langle \left\{ \sigma_i^{p_i} \ \middle|\ i \in \mathcal{I} \right\} \rangle & = & \langle \sigma_j \rangle
\end{array}$$

*For distribution types we require additionally that $\sum_{i \in \mathcal{I}} p_i \leq 1$, that $\mathcal{I}$ is a finite non-empty set, and that $\langle \sigma_i \rangle = \langle \sigma_j \rangle$ for every $i, j \in \mathcal{I}$. In the last equation, $j$ is any element of $\mathcal{I}$.*

The definition of sized types is *monadic* in that a higher-order sized type is of the shape $\sigma \to \mu$ where $\sigma$ is again a sized type, and $\mu$ is a *distribution* of sized types.

The definition of the fixpoint will refer to the notion of *positivity* of a size variable in a sized or distribution type. We define positive and negative occurrences of a size variable in such a type in Figure 4.

**Contexts and Operations on Them.** Contexts are sequences of variables together with a sized type, and at most one distinguished variable with a distribution type:

**Definition 9 (Contexts)** *Contexts are of the shape $\Gamma \,|\, \Theta$, with*

$$\begin{array}{lrcll}
\text{Sized contexts:} & \Gamma, \Delta, \ldots & ::= & \emptyset \ \middle|\ x : \sigma, \Gamma & (x \notin \mathit{dom}(\Gamma)) \\
\text{Distribution contexts:} & \Theta, \Psi, \ldots & ::= & \emptyset \ \middle|\ x : \mu &
\end{array}$$

*As usual, we define the* domain $dom(\Gamma)$ *of a sized context* $\Gamma$ *by induction:* $dom(\emptyset) = \emptyset$ *and* $dom(x : \sigma, \Gamma) = \{x\} \uplus dom(\Gamma)$. *We proceed similarly for the domain* $dom(\Theta)$ *of a distribution context* $\Theta$. *When a sized context* $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$ $(n \geq 1)$ *is such that there is a simple type* $\kappa$ *with* $\forall i \in \{1, \ldots, n\}, \ \langle \sigma_i \rangle = \kappa$, *we say that* $\Gamma$ *is* uniform *of simple type* $\kappa$. *We write this as* $\langle \Gamma \rangle = \kappa$.

*We write* $\Gamma, \Delta$ *for the* disjoint union *of these sized contexts: it is defined whenever* $dom(\Gamma) \cap dom(\Delta) = \emptyset$. *We proceed similarly for* $\Theta, \Psi$, *but note that due to the restriction on the cardinality of such contexts, there is the additional requirement that* $\Theta = \emptyset$ *or* $\Psi = \emptyset$.

*We finally define* contexts *as pairs* $\Gamma \,|\, \Theta$ *of a sized context and of a distribution context, with the constraint that* $dom(\Gamma) \cap dom(\Theta) = \emptyset$.

**Definition 10 (Probabilistic Sum)** *Let* $\mu$ *and* $\nu$ *be two distribution types. We define their probabilistic sum* $\mu \oplus_p \nu$ *as the distribution type* $p \cdot \mu + (1-p) \cdot \nu$. *We extend this operation to a* partial *operation on distribution contexts:*

- *For two distribution types* $\mu$ *and* $\nu$ *such that* $\langle \mu \rangle = \langle \nu \rangle$, *we define* $(x : \mu) \oplus_p (x : \nu) \ = \ x : \mu \oplus_p \nu$,
- $(x : \mu) \oplus_p \emptyset \ = \ x : p \cdot \mu$,
- $\emptyset \oplus_p (x : \mu) \ = \ x : (1-p) \cdot \mu$,
- *In any other case, the operation is undefined.*

**Definition 11 (Weighted Sum of Distribution Contexts)** *Let* $(\Theta_i)_{i \in \mathcal{I}}$ *be a non-empty family of distribution contexts and* $(p_i)_{i \in \mathcal{I}}$ *be a family of reals of* $[0, 1]$. *We define the weighted sum* $\sum_{i \in \mathcal{I}} p_i \cdot \Theta_i$ *as the distribution context* $x : \sum_{i \in \mathcal{I}} p_i \cdot \mu_i$ *when the following conditions are met:*

1. $\exists x, \ \ \forall i \in \mathcal{I}, \ \Theta_i = x : \mu_i$,
2. $\forall (i, j) \in \mathcal{I}^2, \ \ \langle \Theta_i \rangle = \langle \Theta_j \rangle$,
3. *and* $\sum_{i \in \mathcal{I}} p_i \leq 1$,

*In any other case, the operation is undefined.*

**Definition 12 (Substitution of a Size Variable)** *We define the substitution* $\mathfrak{s}[\mathfrak{r}/\mathfrak{i}]$ *of a size variable in a size as follows:*

$$\mathfrak{i}[\mathfrak{r}/\mathfrak{i}] \ = \ \mathfrak{r} \qquad \mathfrak{j}[\mathfrak{r}/\mathfrak{i}] \ = \ \mathfrak{j} \qquad \infty[\mathfrak{r}/\mathfrak{i}] \ = \ \infty \qquad \widehat{\mathfrak{s}}[\mathfrak{r}/\mathfrak{i}] \ = \ \widehat{\mathfrak{s}[\mathfrak{r}/\mathfrak{i}]}$$

*where* $\mathfrak{i} \neq \mathfrak{j}$. *We then define the substitution* $\sigma[\mathfrak{s}/\mathfrak{i}]$ *(resp.* $\mu[\mathfrak{s}/\mathfrak{i}]$*) of a size variable* $\mathfrak{i}$ *by a size* $\mathfrak{s}$ *in a sized or distribution type as:*

$$(\sigma \to \mu)\,[\mathfrak{s}/\mathfrak{i}] \ = \ \sigma[\mathfrak{s}/\mathfrak{i}] \to \mu[\mathfrak{s}/\mathfrak{i}] \qquad\qquad (\mathsf{Nat}^{\mathfrak{s}})\,[\mathfrak{r}/\mathfrak{i}] \ = \ \mathsf{Nat}^{\mathfrak{s}[\mathfrak{r}/\mathfrak{i}]}$$

$$\left(\left\{ \sigma_i^{p_i} \ \middle| \ i \in \mathcal{I} \right\}\right)[\mathfrak{s}/\mathfrak{i}] \ = \ \left\{ (\sigma_i[\mathfrak{s}/\mathfrak{i}])^{p_i} \ \middle| \ i \in \mathcal{I} \right\}$$

*We define the substitution of a size variable in a sized or distribution context in the obvious way:*

$$\emptyset[\mathfrak{s}/\mathfrak{i}] \ = \ \emptyset \qquad\qquad (x : \sigma, \Gamma)\,[\mathfrak{s}/\mathfrak{i}] \ = \ x : \sigma[\mathfrak{s}/\mathfrak{i}], \Gamma[\mathfrak{s}/\mathfrak{i}]$$

$$(x : \mu)\,[\mathfrak{s}/\mathfrak{i}] \ = \ x : \mu[\mathfrak{s}/\mathfrak{i}]$$

**Lemma 5**

1. $(\mu \oplus_p \nu)\,[\mathfrak{s}/\mathfrak{i}] \ = \ \mu[\mathfrak{s}/\mathfrak{i}] \oplus_p \nu[\mathfrak{s}/\mathfrak{i}]$
2. *For distribution contexts,* $(\Theta \oplus_p \Psi)\,[\mathfrak{s}/\mathfrak{i}] \ = \ \Theta[\mathfrak{s}/\mathfrak{i}] \oplus_p \Psi[\mathfrak{s}/\mathfrak{i}]$
3. *For distribution contexts,* $\left(\sum_{i \in \mathcal{I}} p_i \cdot \Gamma_i\right)[\mathfrak{s}/\mathfrak{i}] \ = \ \sum_{i \in \mathcal{I}} p_i \cdot \Gamma_i[\mathfrak{s}/\mathfrak{i}]$

**Proof.** 1. Let $\mu = \left\{ \sigma_i^{p'_i} \;\middle|\; i \in \mathcal{I} \right\}$ and $\nu = \left\{ \tau_j^{p''_j} \;\middle|\; j \in \mathcal{J} \right\}$. Then

$$\begin{array}{rl} & \mu[\mathfrak{s}/\mathfrak{i}] \oplus_p \nu[\mathfrak{s}/\mathfrak{i}] \\ = & \left\{ \sigma_i^{p'_i} \;\middle|\; i \in \mathcal{I} \right\}[\mathfrak{s}/\mathfrak{i}] \oplus_p \left\{ \tau_j^{p''_j} \;\middle|\; j \in \mathcal{J} \right\}[\mathfrak{s}/\mathfrak{i}] \\ = & \left\{ (\sigma_i[\mathfrak{s}/\mathfrak{i}])^{p'_i} \;\middle|\; i \in \mathcal{I} \right\} \oplus_p \left\{ (\tau_j[\mathfrak{s}/\mathfrak{i}])^{p''_j} \;\middle|\; j \in \mathcal{J} \right\} \\ = & \left[ (\sigma_i[\mathfrak{s}/\mathfrak{i}])^{pp'_i} \;\middle|\; i \in \mathcal{I} \right] + \left[ (\tau_j[\mathfrak{s}/\mathfrak{i}])^{(1-p)p''_j} \;\middle|\; j \in \mathcal{J} \right] \\ = & \left( \left[ (\sigma_i)^{pp'_i} \;\middle|\; i \in \mathcal{I} \right] + \left[ (\tau_j)^{(1-p)p''_j} \;\middle|\; j \in \mathcal{J} \right] \right)[\mathfrak{s}/\mathfrak{i}] \\ = & (\mu \oplus_p \nu)[\mathfrak{s}/\mathfrak{i}] \end{array}$$

2. Suppose that $\Theta = x : \mu$ and that $\Psi = x : \nu$. Then $\Theta \oplus_p \Psi = x : \mu \oplus_p \nu$. It follows from (1) that $\Theta[\mathfrak{s}/\mathfrak{i}] \oplus_p \Psi[\mathfrak{s}/\mathfrak{i}] = x : \mu[\mathfrak{s}/\mathfrak{i}] \oplus_p \nu[\mathfrak{s}/\mathfrak{i}] = x : (\mu \oplus_p \nu)[\mathfrak{s}/\mathfrak{i}] = (\Theta \oplus_p \Psi)[\mathfrak{s}/\mathfrak{i}]$

3. The proof is similar to the previous cases.

□

A subtyping relation allows to lift the order $\preccurlyeq$ on sizes to monadic sized types:

**Definition 13 (Subtyping)** *We define the subtyping relation $\sqsubseteq$ on sized types and distribution types as follows:*

$$\frac{}{\sigma \sqsubseteq \sigma} \qquad \frac{\mathfrak{s} \preccurlyeq \mathfrak{r}}{\mathsf{Nat}^{\mathfrak{s}} \sqsubseteq \mathsf{Nat}^{\mathfrak{r}}} \qquad \frac{\tau \sqsubseteq \sigma \qquad \mu \sqsubseteq \nu}{\sigma \to \mu \;\sqsubseteq\; \tau \to \nu}$$

$$\frac{\exists f : \mathcal{I} \to \mathcal{J},\ \left(\forall i \in \mathcal{I},\ \sigma_i \sqsubseteq \tau_{f(i)}\right) \text{ and } \left(\forall j \in \mathcal{J},\ \sum_{i \in f^{-1}(j)} p_i \leq p'_j\right)}{\left\{ \sigma_i^{p_i} \;\middle|\; i \in \mathcal{I} \right\} \sqsubseteq \left\{ \tau_j^{p'_j} \;\middle|\; j \in \mathcal{J} \right\}}$$

**Sized Walks and Distribution Types.** As we explained in Section 2, the rule typing $\mathsf{letrec}$ in the monadic, affine type system relies on an external decision procedure, computable in polynomial time. This procedure ensures that the *sized walk* — a particular instance of *one-counter Markov decision process* (OC-MDP, see [29]), but which does not make use of non-determinism — associated to the type of the recursive function of interest indeed ensures almost sure termination. Let us now define the sized walk associated to a distribution type $\mu$. We then make precise the connection with OC-MDPs, from which the computability in polynomial time of the almost-sure termination of the random walk follows.

**Definition 14 (Sized Walk)** *Let $\mathcal{I} \subseteq_{fin} \mathbb{N}$ be a finite set of integers. Let $\{p_i\}_{i \in \mathcal{I}}$ be such that $\sum_{i \in \mathcal{I}} p_i \leq 1$. These parameters define a Markov chain whose set of states is $\mathbb{N}$ and whose transition relation is defined as follows:*
- *the state $0 \in \mathbb{N}$ is stationary (i.e. one goes from $0$ to $0$ with probability $1$),*
- *from the state $s+1 \in \mathbb{N}$ one moves:*
  - *to the state $s+i$ with probability $p_i$, for every $i \in \mathcal{I}$;*
  - *to $0$ with probability $1 - \left(\sum_{i \in \mathcal{I}} p_i\right)$.*

*We call this Markov chain the* sized walk *on $\mathbb{N}$ associated to $\left(\mathcal{I}, (p_i)_{i \in \mathcal{I}}\right)$. A sized walk is* almost surely terminating *when it reaches $0$ with probability 1 from any initial state.*

Notably, checking whether a sized walk is terminating is relatively easy:

**Proposition 1 (Decidability of AST for Sized Walks)** *It is decidable in polynomial time whether a sized walk is AST.*

**Proof.** See Section 4.3. □

**Definition 15 (From Types to Sized Walks)** *Consider a distribution type* $\mu = \left\{ (\mathsf{Nat}^{\mathfrak{s}_j} \to \nu_j)^{p_j} \;\middle|\; j \in \mathcal{J} \right\}$ *such that* $\forall j \in \mathcal{J},\ \mathrm{spine}(\mathfrak{s}_j) = \mathsf{i}$. *Then* $\mu$ *induces a sized walk, defined as follows. First, by definition,* $\mathfrak{s}_j$ *must be of the shape* $\widehat{\mathsf{i}}^{k_j}$ *with* $k_j \geq 0$ *for every* $j \in \mathcal{J}$. *We set* $\mathcal{I} = \left\{ k_j \;\middle|\; j \in \mathcal{J} \right\}$ *and* $q_{k_j} = p_j$ *for every* $j \in \mathcal{J}$. *The sized walk induced by the distribution type* $\mu$ *is then the sized walk associated to* $(\mathcal{I}, (q_i)_{i \in \mathcal{I}})$.

**Example 1** *Let* $\mu = \left\{ \left(\mathsf{Nat}^{\mathsf{i}} \to \mathsf{Nat}^{\infty}\right)^{\frac{1}{2}}, \left(\mathsf{Nat}^{\widehat{\mathsf{i}}^2} \to \mathsf{Nat}^{\infty}\right)^{\frac{1}{3}} \right\}$. *Then the induced sized walk is the one associated to* $\left(\{0,2\}, \left(p_0 = \frac{1}{2}, p_2 = \frac{1}{3}\right)\right)$. *In other words, it is the random walk on* $\mathbb{N}$ *which is stationary on* $0$, *and which on non-null integers* $i+1$ *moves to* $i$ *with probability* $\frac{1}{2}$, *to* $i+2$ *with probability* $\frac{1}{3}$, *and jumps to* $0$ *with probability* $\frac{1}{6}$. *Note that the type* $\mu$, *and therefore the associated sized walk, models a recursive function which calls itself on a size lesser by one unit with probability* $\frac{1}{2}$, *on a size greater by one unit with probability* $\frac{1}{3}$, *and which does not call itself with probability* $\frac{1}{6}$.

**Typing Rules.** Judgements are of the shape $\Gamma \,|\, \Theta \vdash M : \mu$. When a distribution $\mu = \left\{ \sigma^1 \right\}$ is Dirac, we simply write it $\sigma$. The type system is defined in Figure 5. As earlier, we define sets of typable terms, and set $\Lambda_{\oplus}^{\mathfrak{s},V}(\Gamma \,|\, \Theta, \sigma) = \left\{ V \;\middle|\; \Gamma \,|\, \Theta \vdash V : \sigma \right\}$, and $\Lambda_{\oplus}^{\mathfrak{s}}(\Gamma \,|\, \Theta, \mu) = \left\{ M \;\middle|\; \Gamma \,|\, \Theta \vdash M : \mu \right\}$. We abbreviate $\Lambda_{\oplus}^{\mathfrak{s},V}(\emptyset \,|\, \emptyset, \sigma)$ as $\Lambda_{\oplus}^{\mathfrak{s},V}(\sigma)$ and $\Lambda_{\oplus}^{\mathfrak{s}}(\emptyset \,|\, \emptyset, \sigma)$ as $\Lambda_{\oplus}^{\mathfrak{s}}(\sigma)$.

This sized type system is a refinement of the affine simple type system for $\lambda_{\oplus}$: if $x_1 : \sigma_1, \ldots, x_n : \sigma_n \,|\, f : \mu \vdash M : \nu$, then it is easily checked that $x_1 :: \langle \sigma_1 \rangle, \ldots, x_n :: \langle \sigma_n \rangle, f :: \langle \mu \rangle \vdash M :: \langle \nu \rangle$.

**Lemma 6 (Properties of Distribution Types)**
- $\Gamma \,|\, \Theta \vdash V : \mu \implies \mu$ *is Dirac.*
- $\Gamma \,|\, \Theta \vdash M : \mu \implies \mu$ *is proper.*

**Proof.** Immediate inspection of the rules. □

## 4.3 Proof of Proposition 1

We now prove Proposition 1 by reducing sized walks to deterministic one-counter Markov decision processes (DOC-MDPs), and using then a result of [29] to conclude. Please note that in [29] the Markov decision processes are more general, as they allow non-determinism. They are called one-counter Markov decision processes (OC-MDPs), and contain in particular all the DOC-MDPs. We omit this feature in our presentation.

**Definition 16 (Markov Decision Process)** *A Markov decision process (MDP) is a tuple* $(V, \mapsto, Pr)$ *such that* $V$ *is a finite-or-countable set of vertices,* $\mapsto \subseteq V \times V$ *is a total transition relation, and* $Pr$ *is a probability assignment mapping each* $v \in V$ *to a probability distribution associating a rational and non-null probability to each edge outgoing of* $v$. *These distributions are moreover required to sum to 1.*

**Definition 17 (Deterministic One-Counter Markov Decision Process)** *A* deterministic one-counter Markov decision process *(DOC-MDP) is a tuple* $\left(Q, \delta^{=0}, \delta^{>0}, P^{=0}, P^{>0}\right)$ *such that:*

- $Q$ *is a finite set of states,*
- $\delta^{=0} \subseteq Q \times \{0,1\} \times Q$ *and* $\delta^{>0} \subseteq Q \times \{-1,0,1\} \times Q$ *are sets of* zero *and* positive *transitions, satisfying that every* $q \in Q$ *has at least a zero and a positive outgoing transition,*
- $P^{=0}$ *(resp.* $P^{>0}$*) is a probability assignment mapping every* $q \in Q$ *to a probability distribution over the outgoing transitions of* $\delta^{=0}$ *(resp.* $\delta^{>0}$*) from* $q$. *These distributions are required to attribute a non-null, rational probability to every outgoing transition, and to sum to 1.*

$$\text{Var}\quad \frac{}{\Gamma,\, x : \sigma \,|\, \Theta \;\vdash\; x : \sigma} \qquad\qquad \frac{}{\Gamma \,|\, x : \sigma \;\vdash\; x : \sigma}\quad \text{Var'}$$

$$\text{Succ}\quad \frac{\Gamma \,|\, \Theta \;\vdash\; V : \mathsf{Nat}^{\mathfrak{s}}}{\Gamma \,|\, \Theta \;\vdash\; \mathsf{S}\, V : \mathsf{Nat}^{\widehat{\mathfrak{s}}}} \qquad\qquad \frac{}{\Gamma \,|\, \Theta \;\vdash\; \mathsf{0} : \mathsf{Nat}^{\widehat{\mathfrak{s}}}}\quad \text{Zero}$$

$$\lambda\quad \frac{\Gamma,\, x : \sigma \,|\, \Theta \;\vdash\; M : \mu}{\Gamma \,|\, \Theta \;\vdash\; \lambda x.M : \sigma \to \mu} \qquad\qquad \frac{\Gamma \,|\, \Theta \;\vdash\; M : \mu \qquad \mu \sqsubseteq \nu}{\Gamma \,|\, \Theta \;\vdash\; M : \nu}\quad \text{Sub}$$

$$\text{App}\quad \frac{\Gamma,\, \Delta \,|\, \Theta \;\vdash\; V : \sigma \to \mu \qquad \Gamma,\, \Xi \,|\, \Psi \;\vdash\; W : \sigma \qquad \langle \Gamma \rangle \;=\; \mathsf{Nat}}{\Gamma,\, \Delta,\, \Xi \,|\, \Theta,\, \Psi \;\vdash\; V\, W : \mu}$$

$$\text{Choice}\quad \frac{\Gamma \,|\, \Theta \;\vdash\; M : \mu \qquad \Gamma \,|\, \Psi \;\vdash\; N : \nu \qquad \langle \mu \rangle \;=\; \langle \nu \rangle}{\Gamma \,|\, \Theta \oplus_p \Psi \;\vdash\; M \oplus_p N : \mu \oplus_p \nu}$$

$$\text{Let}\quad \frac{\begin{array}{c}\Gamma,\, \Delta \,|\, \Theta \vdash M : \left\{\, \sigma_i^{p_i} \;\middle|\; i \in \mathcal{I} \,\right\} \qquad \langle \Gamma \rangle \;=\; \mathsf{Nat} \\ \Gamma,\, \Xi,\, x : \sigma_i \,|\, \Psi_i \vdash N : \mu_i \quad (\forall i \in \mathcal{I})\end{array}}{\Gamma,\, \Delta,\, \Xi \,|\, \Theta,\, \left(\sum_{i \in \mathcal{I}} p_i \cdot \Psi_i\right) \vdash \mathsf{let}\; x \;=\; M \;\mathsf{in}\; N : \sum_{i \in \mathcal{I}} p_i \cdot \mu_i}$$

$$\text{Case}\quad \frac{\Gamma \,|\, \emptyset \vdash V : \mathsf{Nat}^{\widehat{\mathfrak{s}}} \qquad \Delta \,|\, \Theta \vdash W : \mathsf{Nat}^{\mathfrak{s}} \to \mu \qquad \Delta \,|\, \Theta \vdash Z : \mu}{\Gamma,\, \Delta \,|\, \Theta \vdash \mathsf{case}\; V \;\mathsf{of}\; \{\, \mathsf{S} \to W \;\mid\; \mathsf{0} \to Z \,\} : \mu}$$

$$\mathsf{letrec}\quad \frac{\begin{array}{c}\langle \Gamma \rangle \;=\; \mathsf{Nat} \\ \mathfrak{i} \notin \Gamma \text{ and } \mathfrak{i} \text{ positive in } \nu \text{ and } \forall j \in \mathcal{J},\; \mathrm{spine}\,(\mathfrak{s}_j) \;=\; \mathfrak{i} \\ \left\{\, (\mathsf{Nat}^{\mathfrak{s}_j} \to \nu[\mathfrak{s}_j/\mathfrak{i}])^{p_j} \;\middle|\; j \in \mathcal{J} \,\right\} \text{induces an AST sized walk} \\ \Gamma \,|\, f : \left\{\, (\mathsf{Nat}^{\mathfrak{s}_j} \to \nu[\mathfrak{s}_j/\mathfrak{i}])^{p_j} \;\middle|\; j \in \mathcal{J} \,\right\} \vdash V : \mathsf{Nat}^{\widehat{\mathfrak{i}}} \to \nu[\widehat{\mathfrak{i}}/\mathfrak{i}]\end{array}}{\Gamma,\, \Delta \,|\, \Theta \vdash \mathsf{letrec}\; f \;=\; V : \mathsf{Nat}^{\mathfrak{r}} \to \nu[\mathfrak{r}/\mathfrak{i}]}$$

Figure 5: Affine distribution types for $\lambda_{\oplus}$.

**Definition 18 (Induced Markov Decision Process)** *A DOC-MDP* $\left(Q, \delta^{=0}, \delta^{>0}, P^{=0}, P^{>0}\right)$ *induces a MDP* $(Q \times \mathbb{N}, \mapsto, Pr)$ *such that, for* $q \in Q$ *and* $n \in \mathbb{N}$*:*

- *for every state* $q'$ *such that* $(q, m, q') \in \delta^{=0}$, $(q, 0) \mapsto (q', m)$, *and the probability of this transition is the one attributed by* $P^{=0}(q)$ *to the transition* $(q, m, q')$,
- *and for every state* $q'$ *such that* $(q, m, q') \in \delta^{>0}$, $(q, n) \mapsto (q', n + m)$, *and the probability of this transition is the one attributed by* $P^{>0}(q)$ *to the transition* $(q, m, q')$,

*This MDP is said to* terminate *when it reaches the value counter* $0$ *in* any *state* $q \in Q$.

Recall that, by definition, $|m| \leq 1$. This is the only restriction to overcome (using intermediate states) to encode sized walks in DOC-MDPs, so that the MDP they induce coincide with the original sized walk. We will then obtain the result of polynomial time decidability of termination with probability 1 using the following proposition:

**Proposition 2 ([29], Theorem 4.1)** *It is decidable in polynomial time whether the MDP induced by an OC-MDP — and thus by a DOC-MDP — terminates with probability* 1.

We now encode sized walks as DOC-MDPs:

**Definition 19 (DOC-MDP Corresponding to a Sized Walk)** *Consider the sized walk on* $\mathbb{N}$ *associated to* $\left(\mathcal{I}, (p_i)_{i\in\mathcal{I}}\right)$*. We define the corresponding DOC-MDP* $\left(Q, \delta^{=0}, \delta^{>0}, P^{=0}, P^{>0}\right)$ *as follows. Let us first consider the following set of states:*

$$Q = \{q_\alpha,\, q_{zero}\} \;\cup\; \left\{q_1, \ldots, q_{j-2} \;\middle|\; j = \max\{i \in \mathcal{I} \;\middle|\; i \geq 2\}\right\}$$

*where* $q_\alpha$ *is the "main" state of the DOC-MDP and the other ones will be used for encoding purposes. We define the transitions of* $\delta^{>0}$ *as follows:*

- *we add the transition* $(q_{zero}, -1, q_{zero})$ *with probability* $1$*,*
- *for every* $j \in \left\{2, \ldots, \max\{i \in \mathcal{I} \;\middle|\; i \geq 2\} - 2\right\}$*, we add the transition* $(q_j, 1, q_{j-1})$ *with probability* $1$*,*
- *we add the transition* $(q_1, 1, q_\alpha)$ *with probability* $1$*,*
- *for* $i \in \mathcal{I} \cap \{0, 1, 2\}$*, we add the transition* $(q_\alpha, i-1, q_\alpha)$ *and attribute it probability* $p_i$*,*
- *for* $i \in \mathcal{I} \setminus \{0, 1, 2\}$*, we add the transition* $(q_\alpha, 1, q_{i-2})$ *and attribute it probability* $p_i$*,*
- *if* $1 - \left(\sum_{i\in\mathcal{I}}\; p_i\right) > 0$*, we add the transition* $(q_\alpha, -1, q_{zero})$ *with probability* $1 - \left(\sum_{i\in\mathcal{I}}\; p_i\right)$*.*

*Finally, we define* $\delta^{=0}$ *as follows: for every state* $q \in Q$*, we add the transition* $(q, 0, q)$ *and attribute it probability* $1$*.*

It is easily checked that, by construction, these DOC-MDP induce the same Markov decision processes as sized walks:

**Proposition 3** *The sized walk on* $\mathbb{N}$ *associated to* $\left(\mathcal{I}, (p_i)_{i\in\mathcal{I}}\right)$ *coincides with the induced MDP of the corresponding DOC-MDP.*

This allows us to deduce from the result of [29] the polynomial time decidability of AST for sized walks:

**Corollary 2 (Proposition 1)** *It is decidable in polynomial time whether a sized walk is almost-surely terminating.*

# 5 Subject Reduction for Monadic Affine Sized Types.

The type system enjoys a form of subject reduction adapted to the probabilistic case and more specifically to the fact that terms reduce to *distributions* of terms. Let us sketch the idea of this adapted subject reduction property on an example. Remark that the type system allows us to derive the sequent

$$\emptyset \,|\, \emptyset \vdash \mathsf{0} \oplus \mathsf{0} \;:\; \left\{ \left(\mathsf{Nat}^{\widehat{\mathfrak{s}}}\right)^{\frac{1}{2}}, \left(\mathsf{Nat}^{\widehat{\mathfrak{r}}}\right)^{\frac{1}{2}} \right\} \tag{9}$$

where this distribution type is formed by typing a copy of $\mathsf{0}$ with $\mathsf{Nat}^{\widehat{\mathfrak{s}}}$ and the other with $\mathsf{Nat}^{\widehat{\widehat{\mathfrak{r}}}}$. Then, the term $\mathsf{0} \oplus \mathsf{0}$ reduces to $\left\{ \mathsf{0}^{\frac{1}{2}} \right\} + \left\{ \mathsf{0}^{\frac{1}{2}} \right\} = \left\{ \mathsf{0}^{1} \right\} = [\![\, \mathsf{0} \oplus \mathsf{0} \,]\!]$: the operational semantics collapses the two copies of $\mathsf{0}$ appearing during the reduction. However, in the spirit of the usual subject reduction for deterministic languages, we would like to type the two copies of $\mathsf{0}$ appearing during the reduction with different types. We therefore use the notion of *pseudo-representation*: $\left[ \mathsf{0}^{\frac{1}{2}}, \mathsf{0}^{\frac{1}{2}} \right]$ is a pseudo-representation of $[\![\, \mathsf{0} \oplus \mathsf{0} \,]\!]$, and we attribute the type $\mathsf{Nat}^{\widehat{\mathfrak{s}}}$ to the first element of this pseudo-representation and the type $\mathsf{Nat}^{\widehat{\widehat{\mathfrak{r}}}}$ to the other, obtaining the following *closed distribution of typed terms*:

$$\left\{ \left(\mathsf{0} \;:\; \mathsf{Nat}^{\widehat{\mathfrak{s}}}\right)^{\frac{1}{2}}, \left(\mathsf{0} \;:\; \mathsf{Nat}^{\widehat{\widehat{\mathfrak{r}}}}\right)^{\frac{1}{2}} \right\} \tag{10}$$

We can then compute the *average* type of (10), which we call the *expectation type* of this closed distribution of typed terms:

$$\frac{1}{2}\cdot\left\{\left(\mathsf{Nat}^{\widehat{\mathfrak{s}}}\right)^{1}\right\} \;+\; \frac{1}{2}\cdot\left\{\left(\mathsf{Nat}^{\widehat{\mathfrak{r}}}\right)^{1}\right\} \;=\; \left\{\left(\mathsf{Nat}^{\widehat{\mathfrak{s}}}\right)^{\frac{1}{2}}, \left(\mathsf{Nat}^{\widehat{\mathfrak{r}}}\right)^{\frac{1}{2}}\right\}$$

Remark that it coincides with the type of the initial term (9). This will be our result of subject reduction: when a closed term $M$ of distribution type $\mu$ reduces to a distribution $\mathscr{D}$ of terms, we can type all the terms appearing in a pseudo-representation of $\mathscr{D}$ to obtain a closed distribution of typed terms whose expectation type is $\mu$. Let us now introduce the definitions necessary to the formal statement of the subject reduction property.

**Definition 20 (Distributions of Distribution Types, of Typed Terms)** – *A* distribution of distribution types *is a distribution $\mathscr{D}$ over the set of distribution types, and such that $\mu, \nu \in \mathsf{S}(\mathscr{D}) \;\Rightarrow\; \langle\mu\rangle = \langle\nu\rangle$.*

– *A* distribution of typed terms*, or* typed distribution*, is a distribution of typing sequents which are derivable in the monadic, affine sized type system. The representation of such a distribution has thus the following form: $\left\{ (\Gamma_i \,|\, \Theta_i \vdash M_i : \mu_i)^{p_i} \;\middle|\; i \in \mathcal{I} \right\}$. In the sequel, we restrict to the* uniform *case in which all the terms appearing in the sequents are typed with distribution types of the same fixed underlying type. We denote this unique simple type $\kappa$ as $\langle\overrightarrow{\mu}\rangle$.*

– *A* distribution of closed typed terms*, or* closed typed distribution*, is a typed distribution in which all contexts are $\emptyset\,|\,\emptyset$. In this case, we simply write the representation of the distribution as $\left\{ (M_i : \mu_i)^{p_i} \;\middle|\; i \in \mathcal{I} \right\}$, or even as $(M_i : \mu_i)^{p_i}$ when the indexing is clear from context. We write pseudo-representations in a similar way.*

– *The* underlying term distribution *of a closed typed distribution $\left\{ (M_i : \mu_i)^{p_i} \;\middle|\; i \in \mathcal{I} \right\}$ is the distribution $\left\{ (M_i)^{p_i} \;\middle|\; i \in \mathcal{I} \right\}$.*

**Definition 21 (Expectation Types)** *Let $(M_i : \mu_i)^{p_i}$ be a closed typed distribution. We define its* expectation type *as the distribution type $\mathbb{E}\left((M_i : \mu_i)^{p_i}\right) = \sum_{i\in\mathcal{I}} p_i\mu_i$.*

**Lemma 7** *Expectation is linear:*

- $\mathbb{E}\left((M_i : \mu_i)^{p_i} + (N_j : \nu_j)^{p'_j}\right) \;=\; \mathbb{E}\left((M_i : \mu_i)^{p_i}\right) + \mathbb{E}\left((N_j : \nu_j)^{p'_j}\right)$,
- $\mathbb{E}\left((M_i : \mu_i)^{pp'_i}\right) \;=\; p\cdot\mathbb{E}\left((M_i : \mu_i)^{p'_i}\right)$.

## 5.1 Subtyping Probabilistic Sums

**Lemma 8 (Subtyping Probabilistic Sums)** *Suppose that $\sum(\nu\oplus_p\xi) = 1$ and that $\nu\oplus_p\xi \sqsubseteq \mu$. Then there exists $\nu'$ and $\xi'$ such that $\mu = \nu' \oplus_p \xi'$, $\nu \sqsubseteq \nu'$, and that $\xi \sqsubseteq \xi'$. Note that this implies that $\mathsf{S}(\nu') \cup \mathsf{S}(\xi') = \mathsf{S}(\mu)$.*

**Proof.** Let $\nu \;=\; \left\{ \sigma_i^{p'_i} \;\middle|\; i \in \mathcal{I} \right\}$ and $\xi \;=\; \left\{ \tau_j^{p''_j} \;\middle|\; j \in \mathcal{J} \right\}$. We assume, without loss of generality, that $\mathcal{I}$ and $\mathcal{J}$ are chosen in such a way that, setting $\mathcal{K} = \mathcal{I}\cap\mathcal{J}$,

$$\exists\,(i,j) \in \mathcal{I}\times\mathcal{J},\;\; \sigma_i = \tau_j \iff i = j \in \mathcal{K}.$$

It follows that

$$\nu\oplus_p\xi \;=\; \left\{ \sigma_i^{pp'_i} \;\middle|\; i \in \mathcal{I}\setminus\mathcal{K} \right\} \;+\; \left\{ \tau_j^{(1-p)p''_j} \;\middle|\; j \in \mathcal{J}\setminus\mathcal{K} \right\} \;+\; \left\{ \sigma_i^{pp'_i+(1-p)p''_i} \;\middle|\; i \in \mathcal{K} \right\}$$

Set $\mu \;=\; \left\{ \theta_l^{p'''_l} \;\middle|\; l \in \mathcal{L} \right\}$. Since $\nu\oplus_p\xi \sqsubseteq \mu$ and $\sum(\nu\oplus_p\xi) = 1$, there exists a decomposition

$$\mu \;=\; \left[ \theta_i^{pp'_i} \;\middle|\; i \in \mathcal{I}\setminus\mathcal{K} \right] \;+\; \left[ \theta_j^{(1-p)p''_j} \;\middle|\; j \in \mathcal{J}\setminus\mathcal{K} \right] \;+\; \left[ \theta_k^{pp'_i+(1-p)p''_i} \;\middle|\; k \in \mathcal{K} \right]$$

(note that the supports of these distributions may have a non-empty intersection), and this decomposition is such that $\forall i \in \mathcal{I},\ \ \sigma_i \sqsubseteq \theta_i$ and $\forall j \in \mathcal{J},\ \ \tau_j \sqsubseteq \theta_j$. We define $\nu' = \left\{ \theta_i^{p'_i} \ \middle|\ i \in \mathcal{I} \right\}$ and $\xi' = \left\{ \theta_j^{p''_j} \ \middle|\ j \in \mathcal{J} \right\}$ which satisfy $\nu \sqsubseteq \nu'$ and $\xi \sqsubseteq \xi'$ but also, by construction, $\mu = \nu' \oplus_p \xi'$. $\square$

**Corollary 3** *Suppose that $\mu = \sum_{i \in \mathcal{I}} p_i \cdot \mu_i$ is a distribution such that $\mu \sqsubseteq \nu$ and that $\sum \mu = 1$. Then there exists a family $(\nu_i)_{i \in \mathcal{I}}$ of distributions such that $\nu = \sum_{i \in \mathcal{I}} p_i \cdot \nu_i$ and that, for all $i \in \mathcal{I}$, $\mu_i \sqsubseteq \nu_i$.*

Note that the requirement that $\sum \mu = 1$ is not necessary to obtain this result, although it simplifies the reasoning.

## 5.2 Generation Lemma for Typing

**Lemma 9 (Generation Lemma for Typing)**

1. $\emptyset \,|\, \emptyset \vdash \mathsf{let}\ x = V\ \mathsf{in}\ N : \mu \implies \exists\, (\nu, \sigma)\,,\ \ \emptyset \,|\, \emptyset \vdash V : \sigma$ *and* $x : \sigma \,|\, \emptyset \vdash N : \nu$ *and* $\nu \sqsubseteq \mu$.
2. $\emptyset \,|\, \emptyset \vdash V\ W : \mu \implies \exists\, (\nu, \sigma)\,,\ \ \emptyset \,|\, \emptyset \vdash V : \sigma \to \nu$ *and* $\emptyset \,|\, \emptyset \vdash W : \sigma$ *and* $\nu \sqsubseteq \mu$.
3. $\emptyset \,|\, \emptyset \vdash \lambda x.M : \sigma \to \mu \implies \exists\, (\nu, \tau)\,,\ \ x : \tau \,|\, \emptyset \vdash M : \nu$ *and* $\sigma \sqsubseteq \tau$ *and* $\nu \sqsubseteq \mu$.
4. $\emptyset \,|\, \emptyset \vdash M \oplus_p N : \mu \implies \exists\, (\nu, \xi)\,,\ \ \emptyset \,|\, \emptyset \vdash M : \nu$ *and* $\emptyset \,|\, \emptyset \vdash N : \xi$ *with* $\sum (\nu \oplus_p \xi) = 1$ *and* $\nu \oplus_p \xi \sqsubseteq \mu$ *and* $\langle \mu \rangle = \langle \nu \rangle = \langle \xi \rangle$.
5. $\emptyset \,|\, \emptyset \vdash \mathsf{let}\ x = M\ \mathsf{in}\ N : \nu \implies \exists \left(\mathcal{I}, (\sigma_i)_{i \in \mathcal{I}}, (p_i)_{i \in \mathcal{I}}, (\mu_i)_{i \in \mathcal{I}}\right)$ *such that*
   - $\sum_{i \in \mathcal{I}} p_i \cdot \mu_i \sqsubseteq \nu$,
   - $\sum \left(\sum_{i \in \mathcal{I}} p_i \cdot \mu_i\right) = 1$,
   - $\emptyset \,|\, \emptyset \vdash M : \left\{ \sigma_i^{p_i} \ \middle|\ i \in \mathcal{I} \right\}$,
   - $\forall i \in \mathcal{I},\ \ x : \sigma_i \,|\, \emptyset \vdash N : \mu_i$.
6. $\emptyset \,|\, \emptyset \vdash \mathsf{case}\ V\ \mathsf{of}\ \{ \mathsf{S} \to W \mid \mathsf{0} \to Z \} : \mu \implies \exists\, (\mathfrak{s}, \nu)$ *such that* $\emptyset \,|\, \emptyset \vdash V : \mathsf{Nat}^{\widehat{\mathfrak{s}}}$ *and* $\emptyset \,|\, \emptyset \vdash W : \mathsf{Nat}^{\mathfrak{s}} \to \nu$ *and* $\emptyset \,|\, \emptyset \vdash Z : \nu$ *with* $\nu \sqsubseteq \mu$.
7. $\emptyset \,|\, \emptyset \vdash \mathsf{letrec}\ f = V : \mu \implies \exists \left((p_j)_{j \in \mathcal{J}}, (\mathfrak{s}_j)_{j \in \mathcal{J}}, \mathfrak{i}\right)$ *such that*
   - $\mathsf{Nat}^{\mathfrak{r}} \to \nu[\mathfrak{r}/\mathfrak{i}] \sqsubseteq \mu$,
   - $\forall j \in \mathcal{J},\ \ \mathrm{spine}\,(\mathfrak{s}_j) = \mathfrak{i}$,
   - $\mathfrak{i} \notin \Gamma$ *and* $\mathfrak{i}$ *positive in* $\nu$,
   - $\left\{ (\mathsf{Nat}^{\mathfrak{s}_j} \to \nu[\mathfrak{s}_j/\mathfrak{i}])^{p_j} \ \middle|\ j \in \mathcal{J} \right\}$ *induces an AST sized walk,*
   - $\emptyset \,|\, f : \left\{ (\mathsf{Nat}^{\mathfrak{s}_j} \to \nu[\mathfrak{s}_j/\mathfrak{i}])^{p_j} \ \middle|\ j \in \mathcal{J} \right\} \vdash V : \mathsf{Nat}^{\widehat{\mathfrak{i}}} \to \nu[\widehat{\mathfrak{i}}/\mathfrak{i}]$

**Proof.** By inspection of the rules, the key point being that the subtyping rule is the only one which is not syntax-directed, and that by transitivity of $\sqsubseteq$ we can compose several successive subtyping rules. In case (5), we have $\sum \left(\sum_{i \in \mathcal{I}} p_i \cdot \mu_i\right) = 1$ since it appears that $\emptyset \,|\, \emptyset \vdash \mathsf{let}\ x = M\ \mathsf{in}\ N : \sum_{i \in \mathcal{I}} p_i \cdot \mu_i$. Lemma 6 allows then to conclude that this distribution of types has sum 1. $\square$

## 5.3 Value Substitutions

**Definition 22 (Context Extending Another)** *We say that a context $\Delta \,|\, \Psi$ extends a context $\Gamma \,|\, \Theta$ when:*

- *for every $x : \sigma \in \Gamma$, we have $x : \sigma \in \Delta$.*
- *and either $\Theta = \emptyset$ or $\Theta = \Psi$.*

*In other words, $\Delta \,|\, \Psi$ extends $\Gamma \,|\, \Theta$ when there exists $\Xi$ and $\Phi$ such that $\Delta = \Gamma, \Xi$ and $\Psi = \Theta, \Phi$.*

**Lemma 10** *Let $M$ be a closed term such that $\Gamma \,|\, \Theta \vdash M : \mu$. Then for every context $\Delta \,|\, \Psi$ extending $\Gamma \,|\, \Theta$, we have $\Delta \,|\, \Psi \vdash M : \mu$.*

**Proof.** We proceed by induction on the structure of $M$. We set $\Delta = \Gamma, \Xi$ and $\Psi = \Theta, \Phi$.

- If $M = x$ is a variable, the result is immediate.
- If $M = 0$, the result is immediate.
- If $M = \mathsf{S}\ V$, we have by typing rules that $\sigma = \mathsf{Nat}^{\widehat{\mathfrak{s}}}$ and that $\Gamma \,|\, \Theta \vdash V : \mathsf{Nat}^{\mathfrak{s}}$. By induction hypothesis $\Delta \,|\, \Psi \vdash V : \mathsf{Nat}^{\mathfrak{s}}$ from which we conclude using the typing rule for $\mathsf{S}$.
- If $M = \lambda x.N$, we have $\sigma = \tau \to \mu$ and $\Gamma, x : \tau \,|\, \Theta \vdash N : \mu$. By definition, $\Delta, x : \tau \,|\, \Psi$ extends $\Gamma, x : \tau \,|\, \Theta$ so that we have $\Delta, x : \tau \,|\, \Psi \vdash N : \mu$. The result follows using the Lambda rule.
- If $M = \mathsf{letrec}\ f = V$, the typing rule is of the shape

$$\mathsf{letrec}\quad \frac{\begin{array}{c}\langle \Gamma_1 \rangle = \mathsf{Nat} \\ \mathfrak{i} \notin \Gamma_1 \text{ and } \mathfrak{i} \text{ positive in } \nu \text{ and } \forall j \in \mathcal{J},\ \mathrm{spine}\,(\mathfrak{s}_j) = \mathfrak{i} \\ \left\{ (\mathsf{Nat}^{\mathfrak{s}_j} \to \nu[\mathfrak{s}_j/\mathfrak{i}])^{p_j} \ \middle|\ j \in \mathcal{J} \right\} \text{induces an AST sized walk} \\ \Gamma_1 \,|\, f : \left\{ (\mathsf{Nat}^{\mathfrak{s}_j} \to \nu[\mathfrak{s}_j/\mathfrak{i}])^{p_j} \ \middle|\ j \in \mathcal{J} \right\} \vdash V : \mathsf{Nat}^{\widehat{\mathfrak{i}}} \to \nu[\widehat{\mathfrak{i}}/\mathfrak{i}]\end{array}}{\Gamma_1, \Gamma_2 \,|\, \Theta \vdash \mathsf{letrec}\ f = V : \mathsf{Nat}^{\mathfrak{r}} \to \nu[\mathfrak{r}/\mathfrak{i}]}$$

  Let $\Delta = \Delta_1, \Delta_2$ with $\Delta_1$ the maximal subcontext consisting only of variables of affine type $\mathsf{Nat}$. Then

$$\Delta_1 \,|\, f : \left\{ (\mathsf{Nat}^{\mathfrak{s}_j} \to \nu[\mathfrak{s}_j/\mathfrak{i}])^{p_j} \ \middle|\ j \in \mathcal{J} \right\}$$

  extends

$$\Gamma_1 \,|\, f : \left\{ (\mathsf{Nat}^{\mathfrak{s}_j} \to \nu[\mathfrak{s}_j/\mathfrak{i}])^{p_j} \ \middle|\ j \in \mathcal{J} \right\}$$

  so that by induction hypothesis $\Delta_1 \,|\, f : \left\{ (\mathsf{Nat}^{\mathfrak{s}_j} \to \nu[\mathfrak{s}_j/\mathfrak{i}])^{p_j} \ \middle|\ j \in \mathcal{J} \right\} \vdash V : \mathsf{Nat}^{\widehat{\mathfrak{i}}} \to \nu[\widehat{\mathfrak{i}}/\mathfrak{i}]$ so that we can conclude using the $\mathsf{letrec}$ rule again that

$$\Delta_1, \Delta_2 \,|\, \Psi \vdash \mathsf{letrec}\ f = V : \mathsf{Nat}^{\mathfrak{r}} \to \nu[\mathfrak{r}/\mathfrak{i}].$$

- If $M = V\ W$, the typing derivation provides contexts such that $\Gamma = \Gamma_1, \Gamma_2, \Gamma_3$ and that $\Theta = \Theta_1, \Theta_2$ with $\Gamma_1, \Gamma_2 \,|\, \Theta_1 \vdash V : \sigma \to \mu$ and $\Gamma_1, \Gamma_3 \,|\, \Theta_2 \vdash W : \sigma$. By induction hypothesis, $\Gamma_1, \Gamma_3, \Xi \,|\, \Theta_2, \Phi \vdash W : \sigma$ from which we conclude using the App rule.
- If $M = \mathsf{let}\ x = N\ \mathsf{in}\ L$, the typing derivation provides contexts such that $\Gamma = \Gamma_1, \Gamma_2, \Gamma_3$ and that $\Theta = \Theta_1, \sum_{i \in \mathcal{I}} p_i \cdot \Theta_{2,i}$ with $\Gamma_1, \Gamma_2 \,|\, \Theta_1 \vdash M : \left\{ \sigma_i^{p_i} \ \middle|\ i \in \mathcal{I} \right\}$ and $\Gamma_1, \Gamma_3, x : \sigma_i \,|\, \Theta_{2,i} \vdash N : \mu_i$. By induction hypothesis, $\Gamma_1, \Gamma_2, \Xi \,|\, \Theta_1, \Phi \vdash M : \left\{ \sigma_i^{p_i} \ \middle|\ i \in \mathcal{I} \right\}$ from which we conclude using the Let rule.

- If $M = N \oplus_p L$, then $\Theta = \Theta_1 \oplus_p \Theta_2$ with $\Gamma \,|\, \Theta_1 \vdash M : \mu$ and $\Gamma \,|\, \Theta_2 \vdash N : \nu$. By applying induction hypothesis twice, we obtain $\Gamma, \Xi \,|\, \Theta_1, \Phi \vdash M : \mu$ and $\Gamma, \Xi \,|\, \Theta_2, \Phi \vdash N : \nu$. We apply the Choice rule; it remains to prove that $(\Theta_1, \Phi) \oplus_p (\Theta_2, \Phi) = \Theta_1 \oplus_p \Theta_2, \Phi$ which is easily done by definition of $\oplus_p$.
- If $M = \mathsf{case}\ V\ \mathsf{of}\ \{\mathsf{S} \to W \mid \mathsf{0} \to Z\}$, the typing derivation provides contexts such that $\Gamma = \Gamma_1, \Gamma_2$ with $\Gamma_1 \,|\, \emptyset \vdash V : \mathsf{Nat}^{\widehat{\mathsf{s}}}$ and $\Gamma_2 \,|\, \Theta \vdash W : \mathsf{Nat}^{\mathsf{s}} \to \mu$ and $\Gamma_2 \,|\, \Theta \vdash Z : \mu$. By induction hypothesis, $\Gamma_2, \Xi \,|\, \Theta, \Phi \vdash W : \mathsf{Nat}^{\mathsf{s}} \to \mu$ and $\Gamma_2, \Xi \,|\, \Theta, \Phi \vdash Z : \mu$ from which we conclude using the Case rule.

□

**Lemma 11 (Closed Value Substitution)** *Suppose that* $\Gamma, x : \sigma \,|\, \Theta \vdash M : \mu$ *and that* $\emptyset \,|\, \emptyset \vdash V : \sigma$. *Then* $\Gamma \,|\, \Theta \vdash M[V/x] : \mu$.

**Proof.** As usual, the proof is by induction on the structure of the typing derivation. We proceed by case analysis on the last rule:
- If it is Var, we have two cases.
  - If the conclusion is $\Gamma, x : \sigma \,|\, \Theta \vdash x : \sigma$ then $x[V/x] = V$. By Lemma 10, we obtain that $\Gamma \,|\, \Theta \vdash V : \sigma$.
  - If the conclusion is $\Gamma, x : \sigma, y : \tau \,|\, \Theta \vdash y : \tau$ then $y[V/x] = y$ and we obtain $\Gamma, y : \tau \,|\, \Theta \vdash y : \tau$ using the Var rule.
- If it is Var', the situation is similar to the latter case of the previous one. The conclusion is $\Gamma, x : \sigma \,|\, y : \tau \vdash y : \tau$ and $y[V/x] = y$ so that we obtain $\Gamma \,|\, y : \tau \vdash y : \tau$ using the Var' rule.
- If it is Succ, then $M = \mathsf{S}\ W$ and $\mu = \mathsf{Nat}^{\widehat{\mathsf{s}}}$. We obtain by induction hypothesis that $\Gamma \,|\, \Theta \vdash W[V/x] : \mathsf{Nat}^{\mathsf{s}}$ and we conclude using the Succ rule that $\Gamma \,|\, \Theta \vdash (\mathsf{S}\ W)[V/x] : \mathsf{Nat}^{\widehat{\mathsf{s}}}$.
- If it is Zero, we obtain immediately the result.
- If it is $\lambda$, suppose that $\Gamma, x : \sigma \,|\, \Theta \vdash \lambda y.M : \tau \to \mu$. This comes from $\Gamma, x : \sigma, y : \tau \,|\, \Theta \vdash M : \mu$ to which we apply the induction hypothesis, obtaining that $\Gamma, y : \tau \,|\, \Theta \vdash M[V/x] : \mu$. Then applying the $\lambda$ rule gives the expected result.
- For all the remaining cases, as for the $\lambda$ rule, the result is obtained in a straightforward way from the induction hypothesis.

□

**Lemma 12 (Substitution for distributions)** *Suppose that* $\Gamma \,|\, x : \left\{ \sigma_i^{p_i} \;\middle|\; i \in \mathcal{I} \right\} \vdash M : \mu$ *and that, for every* $i \in \mathcal{I}$, *we have* $\emptyset \,|\, \emptyset \vdash V : \sigma_i$. *Then* $\Gamma \,|\, \emptyset \vdash M[V/x] : \mu$.

**Proof.** The proof is by induction on the structure of the typing derivation. We proceed by case analysis on the last rule:
- If it is Var, we have $M = y \neq x$ and $y \in \Gamma$. It follows that $y[V/x] = y$ and we obtain $\Gamma \,|\, \emptyset \vdash M[V/x] : \mu$ simply by the Var rule.
- If it is Var', we have $M = x$ so that $M[V/x] = V$. Moreover, the distribution $\left\{ \sigma_i^{p_i} \;\middle|\; i \in \mathcal{I} \right\}$ must be Dirac; we denote by $\sigma$ the unique element of its support. Note that we also obtain $\sigma = \mu$. As we supposed that $\emptyset \,|\, \emptyset \vdash V : \sigma$, Lemma 10 gives $\Gamma \,|\, \emptyset \vdash V : \sigma$ from which we conclude.
- If it is LetRec, then $x$ does not occur free in $M$. It follows that $M[V/x] = M$, and we can derive $\Gamma \,|\, \emptyset \vdash M[V/x] : \mu$ using a LetRec rule with the same hypothesis.
- All others cases are treated straightforwardly using the induction hypothesis.

□

**Lemma 13**

1. $\Gamma \,|\, \Theta \vdash \mathsf{S}\ V : \mathsf{Nat}^{\widehat{\mathsf{s}}} \implies \Gamma \,|\, \Theta \vdash V : \mathsf{Nat}^{\mathsf{s}}$,

2. $\Gamma \mid \Theta \vdash \mathsf{0} : \mathsf{Nat}^{\mathfrak{s}} \quad \Longrightarrow \quad \exists \mathfrak{r}, \ \ \mathfrak{s} = \widehat{\mathfrak{r}}.$

3. $\Gamma \mid \Theta \vdash \mathsf{S}\ V : \mathsf{Nat}^{\mathfrak{s}} \quad \Longrightarrow \quad \exists \mathfrak{r}, \ \ \mathfrak{s} = \widehat{\mathfrak{r}}.$

**Proof.** All points are immediate due to the typing rules introducing $\mathsf{0}$ and $\mathsf{S}$. Recall that by the subtyping rules $\widehat{\infty} = \infty$. □

## 5.4 Size Substitutions

**Lemma 14 (Successor and Size Order)** *Suppose that $\mathfrak{s} \preccurlyeq \mathfrak{r}$. Then $\widehat{\mathfrak{s}} \preccurlyeq \widehat{\mathfrak{r}}$.*

**Proof.** By definition of $\preccurlyeq$, if $\mathfrak{s} \preccurlyeq \mathfrak{r}$, there are two cases: either $\mathfrak{r} = \infty$, or $\mathrm{spine}(\mathfrak{s}) = \mathrm{spine}(\mathfrak{r}) = \mathfrak{i}$ with $\mathfrak{s} = \widehat{\mathfrak{i}}^{k}$, $\mathfrak{r} = \widehat{\mathfrak{i}}^{k'}$ and $k \leq k'$. In both cases the conclusion is immediate. □

**Lemma 15 (Size Substitutions are Monotonic)**

1. *Suppose that $\mathfrak{s} \preccurlyeq \mathfrak{r}$, then for any size $\mathfrak{t}$ and size variable $\mathfrak{i}$ we have $\mathfrak{s}[\mathfrak{t}/\mathfrak{i}] \preccurlyeq \mathfrak{r}[\mathfrak{t}/\mathfrak{i}]$.*

2. *Suppose that $\mathfrak{s} \preccurlyeq \mathfrak{r}$, then for any size $\mathfrak{t}$ and size variable $\mathfrak{i}$ we have $\mathfrak{t}[\mathfrak{s}/\mathfrak{i}] \preccurlyeq \mathfrak{t}[\mathfrak{r}/\mathfrak{i}]$.*

**Proof.** 1. We proceed by induction on the derivation proving that $\mathfrak{s} \preccurlyeq \mathfrak{r}$, by case analysis on the last rule.

- If it is $\mathfrak{s} \preccurlyeq \mathfrak{s}$, then $\mathfrak{s} = \mathfrak{r}$ and the result is immediate.
- If it is
$$\frac{\mathfrak{s} \preccurlyeq \mathfrak{u} \qquad \mathfrak{u} \preccurlyeq \mathfrak{r}}{\mathfrak{s} \preccurlyeq \mathfrak{r}}$$
then by induction hypothesis $\mathfrak{s}[\mathfrak{t}/\mathfrak{i}] \preccurlyeq \mathfrak{u}[\mathfrak{t}/\mathfrak{i}]$ and $\mathfrak{u}[\mathfrak{t}/\mathfrak{i}] \preccurlyeq \mathfrak{r}[\mathfrak{t}/\mathfrak{i}]$ so that we conclude using this same deduction rule.
- If it is $\mathfrak{s} \preccurlyeq \widehat{\mathfrak{s}}$, we have $\mathfrak{r} = \widehat{\mathfrak{s}}$ and using the definition of size substitution we obtain $\mathfrak{r}[\mathfrak{t}/\mathfrak{i}] = \widehat{\mathfrak{s}}[\mathfrak{t}/\mathfrak{i}] = \widehat{\mathfrak{s}[\mathfrak{t}/\mathfrak{i}]}$. We conclude using the same deduction rule.
- If it is $\mathfrak{s} \preccurlyeq \infty$, we have $\infty[\mathfrak{t}/\mathfrak{i}] = \infty$ and we obtain immediately $\mathfrak{s}[\mathfrak{t}/\mathfrak{i}] \preccurlyeq \infty$.

2. We proceed by case analysis on $\mathfrak{t}$. There are four cases:
- If $\mathfrak{t} = \mathfrak{i}$, then $\mathfrak{t}[\mathfrak{s}/\mathfrak{i}] = \mathfrak{s} \preccurlyeq \mathfrak{r} = \mathfrak{t}[\mathfrak{r}/\mathfrak{i}]$.
- If $\mathfrak{t} = \mathfrak{j} \neq \mathfrak{i}$, then $\mathfrak{t}[\mathfrak{s}/\mathfrak{i}] = \mathfrak{j} \preccurlyeq \mathfrak{j} = \mathfrak{t}[\mathfrak{r}/\mathfrak{i}]$.
- If $\mathfrak{t} = \widehat{\mathfrak{u}}$, we have by induction hypothesis that $\mathfrak{u}[\mathfrak{s}/\mathfrak{i}] \preccurlyeq \mathfrak{u}[\mathfrak{r}/\mathfrak{i}]$. We conclude using Lemma 14.
- If $\mathfrak{t} = \infty$, $\mathfrak{t}[\mathfrak{s}/\mathfrak{i}] = \infty \preccurlyeq \infty = \mathfrak{t}[\mathfrak{r}/\mathfrak{i}]$.

□

**Lemma 16 (Size Substitutions and Subtyping)**

1. *If $\sigma \sqsubseteq \tau$, then for any size $\mathfrak{s}$ and size variable $\mathfrak{i}$, we have $\sigma[\mathfrak{s}/\mathfrak{i}] \sqsubseteq \tau[\mathfrak{s}/\mathfrak{i}]$.*
   *If $\mu \sqsubseteq \nu$, then for any size $\mathfrak{s}$ and size variable $\mathfrak{i}$, we have $\mu[\mathfrak{s}/\mathfrak{i}] \sqsubseteq \nu[\mathfrak{s}/\mathfrak{i}]$.*

2. *If $\mathfrak{i}$ pos $\sigma$ and $\mathfrak{s} \preccurlyeq \mathfrak{r}$, we have $\sigma[\mathfrak{s}/\mathfrak{i}] \sqsubseteq \sigma[\mathfrak{r}/\mathfrak{i}]$.*
   *If $\mathfrak{i}$ pos $\mu$ and $\mathfrak{s} \preccurlyeq \mathfrak{r}$, we have $\mu[\mathfrak{s}/\mathfrak{i}] \sqsubseteq \mu[\mathfrak{r}/\mathfrak{i}]$.*

3. *If $\mathfrak{i}$ neg $\sigma$ and $\mathfrak{s} \preccurlyeq \mathfrak{r}$, we have $\sigma[\mathfrak{r}/\mathfrak{i}] \sqsubseteq \sigma[\mathfrak{s}/\mathfrak{i}]$.*
   *If $\mathfrak{i}$ neg $\mu$ and $\mathfrak{s} \preccurlyeq \mathfrak{r}$, we have $\mu[\mathfrak{r}/\mathfrak{i}] \sqsubseteq \mu[\mathfrak{s}/\mathfrak{i}]$.*

**Proof.**

1. We prove both statements at the same time by induction on the derivation proving that $\mu \sqsubseteq \nu$ (or $\sigma \sqsubseteq \tau$).
   - If the last rule is $\sigma \sqsubseteq \sigma$, then $\mu = \nu = \sigma$ and the result is immediate.

– If the last rule is

$$\frac{\mathfrak{t} \preccurlyeq \mathfrak{r}}{\mathsf{Nat}^{\mathfrak{t}} \sqsubseteq \mathsf{Nat}^{\mathfrak{r}}}$$

then by Lemma 15 we have $\mathfrak{t}[\mathfrak{s}/\mathfrak{i}] \preccurlyeq \mathfrak{r}[\mathfrak{s}/\mathfrak{i}]$ so that $\left(\mathsf{Nat}^{\mathfrak{t}}\right)[\mathfrak{s}/\mathfrak{i}] = \mathsf{Nat}^{\mathfrak{t}[\mathfrak{s}/\mathfrak{i}]} \sqsubseteq \mathsf{Nat}^{\mathfrak{r}[\mathfrak{s}/\mathfrak{i}]} = \left(\mathsf{Nat}^{\mathfrak{r}}\right)[\mathfrak{s}/\mathfrak{i}]$.

– If the last rule is

$$\frac{\tau \sqsubseteq \sigma \qquad \mu \sqsubseteq \nu}{\sigma \to \mu \ \sqsubseteq \ \tau \to \nu}$$

then by induction hypothesis $\tau[\mathfrak{s}/\mathfrak{i}] \sqsubseteq \sigma[\mathfrak{s}/\mathfrak{i}]$ and $\mu[\mathfrak{s}/\mathfrak{i}] \sqsubseteq \nu[\mathfrak{s}/\mathfrak{i}]$ from which we conclude using the same rule.

– If the last rule is

$$\frac{\exists f : \mathcal{I} \to \mathcal{J}, \ \left(\forall i \in \mathcal{I}, \ \sigma_i \sqsubseteq \tau_{f(i)}\right) \text{ and } \left(\forall j \in \mathcal{J}, \ \sum_{i \in f^{-1}(j)} p_i \leq p'_j\right)}{\left\{ \sigma_i^{p_i} \ \middle| \ i \in \mathcal{I} \right\} \sqsubseteq \left\{ \tau_j^{p'_j} \ \middle| \ j \in \mathcal{J} \right\}}$$

we obtain by induction hypothesis that for every $i \in \mathcal{I}$ $\sigma_i[\mathfrak{s}/\mathfrak{i}] \sqsubseteq \tau_{f(i)}[\mathfrak{s}/\mathfrak{i}]$ from which we conclude using the same rule.

2. We prove (2) and (3) by mutual induction on $\mu$ (or $\sigma$). Let $\mathfrak{s} \preccurlyeq \mathfrak{r}$.
   – If $\sigma = \mathsf{Nat}^{\mathfrak{t}}$,
     – Suppose that $\mathfrak{i}$ pos $\mathsf{Nat}^{\mathfrak{t}}$. Note that this does not assume anything on $\mathfrak{t}$. Since $\mathfrak{s} \preccurlyeq \mathfrak{r}$, we have $\left(\mathsf{Nat}^{\mathfrak{t}}\right)[\mathfrak{s}/\mathfrak{i}] = \mathsf{Nat}^{\mathfrak{t}[\mathfrak{s}/\mathfrak{i}]} \sqsubseteq \mathsf{Nat}^{\mathfrak{t}[\mathfrak{r}/\mathfrak{i}]} = \left(\mathsf{Nat}^{\mathfrak{t}}\right)[\mathfrak{r}/\mathfrak{i}]$ where we used the monotonicity of size substitution (Lemma 15).
     – Suppose that $\mathfrak{i}$ neg $\mathsf{Nat}^{\mathfrak{t}}$. Then $\mathfrak{i} \notin \mathfrak{t}$ and $\left(\mathsf{Nat}^{\mathfrak{t}}\right)[\mathfrak{s}/\mathfrak{i}] = \left(\mathsf{Nat}^{\mathfrak{t}}\right)[\mathfrak{r}/\mathfrak{i}]$ so that we can conclude.
   – If $\sigma = \tau \to \mu$,
     – Suppose that $\mathfrak{i}$ pos $\sigma$. Then $\mathfrak{i}$ neg $\tau$ and $\mathfrak{i}$ pos $\mu$. By induction hypothesis, $\tau[\mathfrak{r}/\mathfrak{i}] \sqsubseteq \tau[\mathfrak{s}/\mathfrak{i}]$ and $\mu[\mathfrak{s}/\mathfrak{i}] \sqsubseteq \mu[\mathfrak{r}/\mathfrak{i}]$. By the subtyping rules, $\sigma[\mathfrak{s}/\mathfrak{i}] = \tau[\mathfrak{s}/\mathfrak{i}] \to \mu[\mathfrak{s}/\mathfrak{i}] \sqsubseteq \tau[\mathfrak{r}/\mathfrak{i}] \to \mu[\mathfrak{r}/\mathfrak{i}] = \sigma[\mathfrak{r}/\mathfrak{i}]$.
     – Suppose that $\mathfrak{i}$ neg $\sigma$. The reasoning is symmetrical.
   – If $\mu = \left\{ \sigma_i^{p_i} \ \middle| \ i \in \mathcal{I} \right\}$,
     – Suppose that $\mathfrak{i}$ pos $\mu$. Then for every $i \in \mathcal{I}$ we have $\mathfrak{i}$ pos $\sigma_i$ and by induction hypothesis $\sigma_i[\mathfrak{s}/\mathfrak{i}] \sqsubseteq \sigma_i[\mathfrak{r}/\mathfrak{i}]$. We obtain that $\mu[\mathfrak{s}/\mathfrak{i}] \sqsubseteq \mu[\mathfrak{r}/\mathfrak{i}]$ using the identity as reindexing function.
     – Suppose that $\mathfrak{i}$ neg $\mu$. The reasoning is symmetrical.

□

**Lemma 17 (Size substitution)** *If $\Gamma \,|\, \Theta \vdash M : \mu$, then for any size variable $\mathfrak{i}$ and any size $\mathfrak{s}$ we have that $\Gamma[\mathfrak{s}/\mathfrak{i}] \,|\, \Theta[\mathfrak{s}/\mathfrak{i}] \vdash M : \mu[\mathfrak{s}/\mathfrak{i}]$.*

**Proof.** We assume that $\mathfrak{i} \notin \mathfrak{s}$, without loss of generality: else we introduce a fresh size variable $\mathfrak{j}$, substitute it with $\mathfrak{s}$, and then substitute $\mathfrak{i}$ with $\mathfrak{j}$. The proof is by induction on the typing derivation. We proceed by case analysis on the last rule.

– If it is Var: we have $\Gamma, x : \sigma \,|\, \Theta \vdash x : \sigma$ and deduce immediately using Var rule again that $\Gamma[\mathfrak{s}/\mathfrak{i}], x : \sigma[\mathfrak{s}/\mathfrak{i}] \,|\, \Theta[\mathfrak{s}/\mathfrak{i}] \vdash x : \sigma[\mathfrak{s}/\mathfrak{i}]$.

– If it is Var': we have $\Gamma \,|\, x : \sigma \vdash x : \sigma$ and deduce immediately using Var' rule again that $\Gamma[\mathfrak{s}/\mathfrak{i}] \,|\, x : \sigma[\mathfrak{s}/\mathfrak{i}] \vdash x : \sigma[\mathfrak{s}/\mathfrak{i}]$.

– If it is Succ: then $M = \mathsf{S}\ V$ and $\mu = \mathsf{Nat}^{\widehat{\mathfrak{r}}}$. By induction hypothesis, $\Gamma[\mathfrak{s}/\mathfrak{i}] \,|\, \Theta[\mathfrak{s}/\mathfrak{i}] \vdash V : (\mathsf{Nat}^{\mathfrak{r}})\,[\mathfrak{s}/\mathfrak{i}]$. But $(\mathsf{Nat}^{\mathfrak{r}})\,[\mathfrak{s}/\mathfrak{i}] = \mathsf{Nat}^{\mathfrak{r}[\mathfrak{s}/\mathfrak{i}]}$ so that by the Succ rule $\Gamma[\mathfrak{s}/\mathfrak{i}] \,|\, \Theta[\mathfrak{s}/\mathfrak{i}] \vdash \mathsf{S}\ V : \mathsf{Nat}^{\widehat{\mathfrak{r}[\mathfrak{s}/\mathfrak{i}]}}$. We use the equality $\mathsf{Nat}^{\widehat{\mathfrak{r}[\mathfrak{s}/\mathfrak{i}]}} = \left(\mathsf{Nat}^{\widehat{\mathfrak{r}}}\right)[\mathfrak{s}/\mathfrak{i}]$ to conclude.

– If it is Zero: the result is immediate.

– If it is $\lambda$: we have $M = \lambda x.N$ and $\mu = \sigma \to \nu$. By induction hypothesis, $\Gamma[\mathfrak{s}/\mathfrak{i}], x : \sigma[\mathfrak{s}/\mathfrak{i}] \,|\, \Theta[\mathfrak{s}/\mathfrak{i}] \vdash N : \nu[\mathfrak{s}/\mathfrak{i}]$. By application of the $\lambda$ rule, $\Gamma[\mathfrak{s}/\mathfrak{i}] \,|\, \Theta[\mathfrak{s}/\mathfrak{i}] \vdash \lambda x.N : \sigma[\mathfrak{s}/\mathfrak{i}] \to \nu[\mathfrak{s}/\mathfrak{i}]$. We conclude using $\sigma[\mathfrak{s}/\mathfrak{i}] \to \nu[\mathfrak{s}/\mathfrak{i}] = (\sigma \to \nu)\,[\mathfrak{s}/\mathfrak{i}]$.

– If it is Sub: the hypothesis of the rule is $\Gamma \,|\, \Theta \vdash M : \nu$ for $\nu \sqsubseteq \mu$. By induction hypothesis, $\Gamma[\mathfrak{s}/\mathfrak{i}] \,|\, \Theta[\mathfrak{s}/\mathfrak{i}] \vdash M : \nu[\mathfrak{s}/\mathfrak{i}]$. But by Lemma 16 we have $\nu[\mathfrak{s}/\mathfrak{i}] \sqsubseteq \mu[\mathfrak{s}/\mathfrak{i}]$. We conclude using the Sub rule.

– If it is App, we have $M = V\ W$ and $\Gamma = \Gamma_1, \Gamma_2, \Gamma_3$ and $\Theta = \Theta_1, \Theta_2$ with $\langle\Gamma_1\rangle = \mathsf{Nat}$, $\Gamma_1, \Gamma_2 \,|\, \Theta_1 \vdash V : \sigma \to \mu$ and $\Gamma_1, \Gamma_3 \,|\, \Theta_2 \vdash W : \sigma$. Applying the induction hypothesis twice gives $\Gamma_1[\mathfrak{s}/\mathfrak{i}], \Gamma_2[\mathfrak{s}/\mathfrak{i}] \,|\, \Theta_1[\mathfrak{s}/\mathfrak{i}] \vdash V : (\sigma \to \mu)\,[\mathfrak{s}/\mathfrak{i}]$ and $\Gamma_1[\mathfrak{s}/\mathfrak{i}], \Gamma_3[\mathfrak{s}/\mathfrak{i}] \,|\, \Theta_2[\mathfrak{s}/\mathfrak{i}] \vdash W : \sigma[\mathfrak{s}/\mathfrak{i}]$. Since $\sigma[\mathfrak{s}/\mathfrak{i}] \to \mu[\mathfrak{s}/\mathfrak{i}] = (\sigma \to \mu)\,[\mathfrak{s}/\mathfrak{i}]$, we can use the Application rule to conclude.

– If it is Choice, then $M = N \oplus_p L$ and $\mu = \nu \oplus_p \xi$ and $\Theta = \Theta_1 \oplus_p \Theta_2$ with $\Gamma \,|\, \Theta_1 \vdash N : \nu$ and $\Gamma \,|\, \Theta_2 \vdash L : \xi$ and $\langle\nu\rangle = \langle\xi\rangle$. The induction hypothesis, applied twice, gives $\Gamma[\mathfrak{s}/\mathfrak{i}] \,|\, \Theta_1[\mathfrak{s}/\mathfrak{i}] \vdash N : \nu[\mathfrak{s}/\mathfrak{i}]$ and $\Gamma[\mathfrak{s}/\mathfrak{i}] \,|\, \Theta_2[\mathfrak{s}/\mathfrak{i}] \vdash L : \xi[\mathfrak{s}/\mathfrak{i}]$ from which we conclude using the Choice rule again and the equality $\nu[\mathfrak{s}/\mathfrak{i}] \oplus_p \xi[\mathfrak{s}/\mathfrak{i}] = (\nu \oplus_p \xi)\,[\mathfrak{s}/\mathfrak{i}]$ from Lemma 5.

– If it is Let, then $M = (\mathsf{let}\ x = N\ \mathsf{in}\ L)$ and $\mu = \sum_{i\in\mathcal{I}} p_i \cdot \nu_i$ and $\Gamma = \Gamma_1, \Gamma_2, \Gamma_3$ and $\Theta = \Theta_1, \sum_{i\in\mathcal{I}} \Theta_{2,i}$ with $\Gamma_1, \Gamma_2 \,|\, \Theta_1 \vdash N : \left\{ \sigma_i^{p_i} \;\middle|\; i \in \mathcal{I} \right\}$ and, for every $i \in \mathcal{I}$, $\Gamma_1, \Gamma_3 \,|\, \Theta_{2,i} \vdash L : \nu_i$ and $\langle\Gamma_1\rangle = \mathsf{Nat}$. By repeated applications of the induction hypothesis, $\Gamma_1[\mathfrak{s}/\mathfrak{i}], \Gamma_2[\mathfrak{s}/\mathfrak{i}] \,|\, \Theta_1[\mathfrak{s}/\mathfrak{i}] \vdash N : \left\{ \sigma_i^{p_i} \;\middle|\; i \in \mathcal{I} \right\}[\mathfrak{s}/\mathfrak{i}]$ and, for every $i \in \mathcal{I}$, $\Gamma_1[\mathfrak{s}/\mathfrak{i}], \Gamma_3[\mathfrak{s}/\mathfrak{i}] \,|\, \Theta_{2,i}[\mathfrak{s}/\mathfrak{i}] \vdash L : \nu_i[\mathfrak{s}/\mathfrak{i}]$. We use in a first time the equality $\left\{ \sigma_i^{p_i} \;\middle|\; i \in \mathcal{I} \right\}[\mathfrak{s}/\mathfrak{i}] = \left\{ (\sigma_i[\mathfrak{s}/\mathfrak{i}])^{p_i} \;\middle|\; i \in \mathcal{I} \right\}$ coming from the definition of size substitutions. We conclude using the Let rule again and the equality $\left(\sum_{i\in\mathcal{I}} p_i \cdot \nu_i\right)[\mathfrak{s}/\mathfrak{i}] = \sum_{i\in\mathcal{I}} p_i \cdot \nu_i[\mathfrak{s}/\mathfrak{i}]$ from Lemma 5.

– If it is Case, then $M = \mathsf{case}\ V\ \mathsf{of}\ \left\{ \mathsf{S} \to W \;\middle|\; 0 \to Z \right\}$ and $\Gamma = \Gamma_1, \Gamma_2$ with $\Gamma_1 \,|\, \emptyset \vdash V : \mathsf{Nat}^{\widehat{\mathfrak{r}}}$ and $\Gamma_2 \,|\, \Theta \vdash W : \mathsf{Nat}^{\mathfrak{r}} \to \mu$ and $\Gamma_2 \,|\, \Theta \vdash Z : \mu$. We apply induction hypothesis three times, and obtain $\Gamma_1[\mathfrak{s}/\mathfrak{i}] \,|\, \emptyset \vdash V : \left(\mathsf{Nat}^{\widehat{\mathfrak{r}}}\right)[\mathfrak{s}/\mathfrak{i}]$ and $\Gamma_2[\mathfrak{s}/\mathfrak{i}] \,|\, \Theta[\mathfrak{s}/\mathfrak{i}] \vdash W : (\mathsf{Nat}^{\mathfrak{r}} \to \mu)\,[\mathfrak{s}/\mathfrak{i}]$ and $\Gamma_2[\mathfrak{s}/\mathfrak{i}] \,|\, \Theta[\mathfrak{s}/\mathfrak{i}] \vdash Z : \mu[\mathfrak{s}/\mathfrak{i}]$. We use the equalities $\left(\mathsf{Nat}^{\widehat{\mathfrak{r}}}\right)[\mathfrak{s}/\mathfrak{i}] = \mathsf{Nat}^{\widehat{\mathfrak{r}[\mathfrak{s}/\mathfrak{i}]}}$ and $(\mathsf{Nat}^{\mathfrak{r}} \to \mu)\,[\mathfrak{s}/\mathfrak{i}] = \mathsf{Nat}^{\mathfrak{r}[\mathfrak{s}/\mathfrak{i}]} \to \mu[\mathfrak{s}/\mathfrak{i}]$ and then the Case rule to conclude.

– If it is $\mathsf{letrec}$, we carefully adapt the proof scheme of [18, Lemma 3.8]. We have $M = \mathsf{letrec}\ f = V$ and $\mu = \mathsf{Nat}^{\mathfrak{r}} \to \nu[\mathfrak{r}/\mathfrak{j}]$ and $\Gamma = \Gamma_1, \Gamma_2$ with
  – $\langle\Gamma_1\rangle = \mathsf{Nat}$,
  – $\mathfrak{j} \notin \Gamma_1$ and $\mathfrak{j}$ positive in $\nu$ and $\forall j \in \mathcal{J},\ \mathrm{spine}\,(\mathfrak{r}_j) = \mathfrak{j}$,
  – $\left\{ (\mathsf{Nat}^{\mathfrak{r}_j} \to \nu[\mathfrak{r}_j/\mathfrak{j}])^{p_j} \;\middle|\; j \in \mathcal{J} \right\}$ induces an AST sized walk,
  – and

$$\Gamma_1 \,|\, f : \left\{ (\mathsf{Nat}^{\mathfrak{r}_j} \to \nu[\mathfrak{r}_j/\mathfrak{j}])^{p_j} \;\middle|\; j \in \mathcal{J} \right\} \vdash V : \mathsf{Nat}^{\widehat{\mathfrak{j}}} \to \nu[\widehat{\mathfrak{j}}/\mathfrak{j}] \tag{11}$$

We suppose, without loss of generality as this can be easily obtained by renaming $\mathfrak{j}$ to a fresh variable, that $\mathfrak{i} \neq \mathfrak{j}$ and that $\mathfrak{j} \notin \mathfrak{s}$. Let $\mathfrak{l}$ be a fresh size variable; it follows in particular that

$\mathfrak{l} \notin \Gamma_1, \Gamma_2, \nu, \mathfrak{s}$. We apply the induction hypothesis to (11) and obtain

$$\Gamma_1[\mathfrak{l}/\mathfrak{j}] \,|\, f \,:\, \left(\left\{ (\mathsf{Nat}^{\mathfrak{r}_j} \to \nu[\mathfrak{r}_j/\mathfrak{j}])^{p_j} \;\middle|\; j \in \mathcal{J} \right\}\right)[\mathfrak{l}/\mathfrak{j}] \vdash V \,:\, \left(\mathsf{Nat}^{\widehat{\mathfrak{j}}} \to \nu[\widehat{\mathfrak{j}}/\mathfrak{j}]\right)[\mathfrak{l}/\mathfrak{j}]$$

which, after applying a series of equalities and using the fact that $\mathfrak{j} \notin \Gamma_1$, coincides with

$$\Gamma_1 \,|\, f \,:\, \left(\left\{ \left(\mathsf{Nat}^{\mathfrak{r}_j[\mathfrak{l}/\mathfrak{j}]} \to \nu[\mathfrak{r}_j[\mathfrak{l}/\mathfrak{j}]/\mathfrak{j}]\right)^{p_j} \;\middle|\; j \in \mathcal{J} \right\}\right) \vdash V \,:\, \mathsf{Nat}^{\widehat{\mathfrak{l}}} \to \nu[\widehat{\mathfrak{j}}/\mathfrak{j}][\mathfrak{l}/\mathfrak{j}]$$

but also with

$$\Gamma_1 \,|\, f \,:\, \left(\left\{ \left(\mathsf{Nat}^{\mathfrak{r}_j[\mathfrak{l}/\mathfrak{j}]} \to \nu[\mathfrak{l}/\mathfrak{j}][\mathfrak{r}_j[\mathfrak{l}/\mathfrak{j}]/\mathfrak{l}]\right)^{p_j} \;\middle|\; j \in \mathcal{J} \right\}\right) \vdash V \,:\, \mathsf{Nat}^{\widehat{\mathfrak{l}}} \to \nu[\mathfrak{l}/\mathfrak{j}][\widehat{\mathfrak{l}}/\mathfrak{l}]$$

We can apply the induction hypothesis again, and obtain after rewriting

$$\Gamma_1[\mathfrak{s}/\mathfrak{i}] \,|\, f \,:\, \left(\left\{ \left(\mathsf{Nat}^{\mathfrak{r}_j[\mathfrak{l}/\mathfrak{j}]} \to \nu[\mathfrak{l}/\mathfrak{j}][\mathfrak{r}_j[\mathfrak{l}/\mathfrak{j}]/\mathfrak{l}][\mathfrak{s}/\mathfrak{i}]\right)^{p_j} \;\middle|\; j \in \mathcal{J} \right\}\right) \vdash V \,:\, \mathsf{Nat}^{\widehat{\mathfrak{l}}} \to \nu[\mathfrak{l}/\mathfrak{j}][\widehat{\mathfrak{l}}/\mathfrak{l}][\mathfrak{s}/\mathfrak{i}]$$

where we used the fact that $\forall j \in \mathcal{J},\ \mathrm{spine}(\mathfrak{r}_j) = \mathfrak{j} \neq \mathfrak{i}$ so that $\left(\mathsf{Nat}^{\mathfrak{r}_j[\mathfrak{l}/\mathfrak{j}]}\right)[\mathfrak{s}/\mathfrak{i}] = \mathsf{Nat}^{\mathfrak{r}_j[\mathfrak{l}/\mathfrak{j}]}$. Since $\mathfrak{l} \notin \mathfrak{s}$, we can exchange $[\widehat{\mathfrak{l}}/\mathfrak{l}]$ and $[\mathfrak{s}/\mathfrak{i}]$. For every $j \in \mathcal{J}$, we can also exchange $[\mathfrak{s}/\mathfrak{i}]$ and $[\mathfrak{r}_j[\mathfrak{l}/\mathfrak{j}]/\mathfrak{l}]$ since $\mathrm{spine}(\mathfrak{r}_j[\mathfrak{l}/\mathfrak{j}]) = \mathfrak{l} \neq \mathfrak{i}$ and $\mathfrak{l} \notin \mathfrak{s}$. We obtain:

$$\Gamma_1[\mathfrak{s}/\mathfrak{i}] \,|\, f \,:\, \left\{ \left(\mathsf{Nat}^{\mathfrak{r}_j[\mathfrak{l}/\mathfrak{j}]} \to \nu[\mathfrak{l}/\mathfrak{j}][\mathfrak{s}/\mathfrak{i}][\mathfrak{r}_j[\mathfrak{l}/\mathfrak{j}]/\mathfrak{l}]\right)^{p_j} \;\middle|\; j \in \mathcal{J} \right\} \vdash V \,:\, \mathsf{Nat}^{\widehat{\mathfrak{l}}} \to \nu[\mathfrak{l}/\mathfrak{j}][\mathfrak{s}/\mathfrak{i}][\widehat{\mathfrak{l}}/\mathfrak{l}]$$

Additionally, we have:
- $\langle \Gamma_1[\mathfrak{s}/\mathfrak{i}] \rangle = \mathsf{Nat}$,
- $\mathfrak{l} \notin \Gamma_1[\mathfrak{s}/\mathfrak{i}]$,
- $\mathfrak{l}$ positive in $\nu[\mathfrak{l}/\mathfrak{j}][\mathfrak{s}/\mathfrak{i}]$ since $\mathfrak{j}$ was positive in $\nu$,
- $\forall j \in \mathcal{J},\ \mathrm{spine}(\mathfrak{r}_j[\mathfrak{l}/\mathfrak{j}]) = \mathfrak{l}$ since $\mathrm{spine}(\mathfrak{r}_j) = \mathfrak{j}$,
- and $\left\{ \left(\mathsf{Nat}^{\mathfrak{r}_j[\mathfrak{l}/\mathfrak{j}]} \to \nu[\mathfrak{l}/\mathfrak{j}][\mathfrak{s}/\mathfrak{i}][\mathfrak{r}_j[\mathfrak{l}/\mathfrak{j}]/\mathfrak{l}]\right)^{p_j} \;\middle|\; j \in \mathcal{J} \right\}$ induces the same sized walk, which is thus AST, as $\left\{ (\mathsf{Nat}^{\mathfrak{r}_j} \to \nu[\mathfrak{r}_j/\mathfrak{j}])^{p_j} \;\middle|\; j \in \mathcal{J} \right\}$. Indeed, only the spine variable changes under the substitution $[\mathfrak{l}/\mathfrak{j}]$.

Let $\mathfrak{t} = \mathfrak{r}[\mathfrak{s}/\mathfrak{i}]$. Since all these conditions are met, we can apply the $\mathsf{letrec}$ rule and obtain

$$\Gamma_1[\mathfrak{s}/\mathfrak{i}], \Gamma_2[\mathfrak{s}/\mathfrak{i}] \,|\, \Theta[\mathfrak{s}/\mathfrak{i}] \vdash \mathsf{letrec}\ f = V \,:\, \mathsf{Nat}^{\mathfrak{t}} \to \nu[\mathfrak{l}/\mathfrak{j}][\mathfrak{s}/\mathfrak{i}][\mathfrak{t}/\mathfrak{l}]$$

Since $\mathfrak{i}, \mathfrak{l} \notin \mathfrak{s}$ and $\mathfrak{l} \notin \nu$, we can commute $[\mathfrak{s}/\mathfrak{i}]$ and $[\mathfrak{t}/\mathfrak{l}]$ and compose substitutions to obtain

$$\Gamma[\mathfrak{s}/\mathfrak{i}] \,|\, \Theta[\mathfrak{s}/\mathfrak{i}] \vdash \mathsf{letrec}\ f = V \,:\, \mathsf{Nat}^{\mathfrak{t}} \to \nu[\mathfrak{t}/\mathfrak{j}][\mathfrak{s}/\mathfrak{i}]$$

which rewrites to

$$\Gamma[\mathfrak{s}/\mathfrak{i}] \,|\, \Theta[\mathfrak{s}/\mathfrak{i}] \vdash \mathsf{letrec}\ f = V \,:\, (\mathsf{Nat}^{\mathfrak{r}} \to \nu[\mathfrak{r}/\mathfrak{j}])[\mathfrak{s}/\mathfrak{i}]$$

which allows to conclude.

□

## 5.5 Subject Reduction

We can now state the main lemma of subject reduction:

**Lemma 18 (Subject Reduction, Fundamental Lemma)** *Let $M \in \Lambda^{\mathfrak{s}}_{\oplus}(\mu)$ and $\mathscr{D}$ be the unique closed term distribution such that $M \to_v \mathscr{D}$. Then there exists a closed typed distribution $\left\{ (L_j : \nu_j)^{p_j} \;\middle|\; j \in \mathcal{J} \right\}$ such that*
- $\mathbb{E}\left((L_j : \nu_j)^{p_j}\right) = \mu$,
- $\left[ (L_j)^{p_j} \;\middle|\; j \in \mathcal{J} \right]$ *is a pseudo-representation of* $\mathscr{D}$.

*Note that the condition on expectations implies that* $\bigcup_{j\in\mathcal{J}} \mathsf{S}(\nu_j) = \mathsf{S}(\mu)$.

**Proof.** We proceed by induction on $M$.

- Suppose that $M = \mathsf{let}\ x = V\ \mathsf{in}\ N$, that $\mathscr{D} = \left\{ (N[V/x])^1 \right\}$, and that $\emptyset \,|\, \emptyset \vdash \mathsf{let}\ x = V\ \mathsf{in}\ N : \mu$. By Lemma 9, there exists $(\xi, \sigma)$ such that $\emptyset \,|\, \emptyset \vdash V : \sigma$ and $x : \sigma \,|\, \emptyset \vdash N : \xi$ with $\xi \sqsubseteq \mu$. By Lemma 11, $\emptyset \,|\, \emptyset \vdash N[V/x] : \xi$, and since $\xi \sqsubseteq \mu$ we obtain by subtyping that $\emptyset \,|\, \emptyset \vdash N[V/x] : \mu$. It follows that $\left\{ (N[V/x] : \mu)^1 \right\}$ is a closed typed distribution satisfying the requirements of the lemma.

- Suppose that $M = (\lambda x.N)\ V$, that $\mathscr{D} = \left\{ (N[V/x])^1 \right\}$ and that $\emptyset \,|\, \emptyset \vdash (\lambda x.N)\ V : \mu$. Applying Lemma 9 twice, we obtain that $x : \tau \,|\, \emptyset \vdash N : \xi$ and $\emptyset \,|\, \emptyset \vdash V : \sigma$ with $\sigma \sqsubseteq \tau$ and $\xi \sqsubseteq \mu$. Applying subtyping to the second judgement gives $\emptyset \,|\, \emptyset \vdash V : \tau$, and we can apply Lemma 11 to obtain $\emptyset \,|\, \emptyset \vdash N[V/x] : \xi$. Since $\xi \sqsubseteq \mu$ we obtain by weakening that $\emptyset \,|\, \emptyset \vdash N[V/x] : \mu$. It follows that $\left\{ (N[V/x] : \mu)^1 \right\}$ is a closed typed distribution satisfying the requirements of the lemma.

- Suppose that $M = N \oplus_p L$, that $\mathscr{D} = \left[ N^p, L^{1-p} \right]$ and that $\emptyset \,|\, \emptyset \vdash N \oplus_p L : \mu$. By Lemma 9, there exists $(\xi, \rho)$ such that $\emptyset \,|\, \emptyset \vdash N : \xi$ and $\emptyset \,|\, \emptyset \vdash L : \rho$ with $\xi \oplus_p \rho \sqsubseteq \mu$ and $\sum (\xi \oplus_p \rho) = 1$. By Lemma 8, there exists $(\xi', \rho')$ such that $\mu = \xi' \oplus_p \rho'$, $\xi \sqsubseteq \xi'$ and $\rho \sqsubseteq \rho'$. By subtyping, $\emptyset \,|\, \emptyset \vdash N : \xi'$ and $\emptyset \,|\, \emptyset \vdash L : \rho'$. We consider the closed typed distribution of pseudo-representation $\left[ (N : \xi')^p, (L : \rho')^{1-p} \right]$ which satisfies the requirements of the lemma since its expectation type is $p \cdot \xi' + (1-p) \cdot \rho' = \xi' \oplus_p \rho' = \mu$. Note that we use a pseudo-representation to cope with the very specific case in which $N = L$ and $\xi' = \rho'$, in which the representation of the closed typed distribution is $\left\{ (N : \xi')^1 \right\}$.

- Suppose that $M = \mathsf{let}\ x = N\ \mathsf{in}\ L$, that $\mathscr{D} = \left\{ (\mathsf{let}\ x = P_j\ \mathsf{in}\ L)^{p'_j} \;\middle|\; j \in \mathcal{J} \right\}$ and that $\emptyset \,|\, \emptyset \vdash \mathsf{let}\ x = N\ \mathsf{in}\ L : \mu$. By Lemma 9, there exists $\left(\mathcal{I}, (\sigma_i)_{i\in\mathcal{I}}, (p_i)_{i\in\mathcal{I}}, (\xi_i)_{i\in\mathcal{I}}\right)$ such that
  - $\sum_{i\in\mathcal{I}} p_i \cdot \xi_i \sqsubseteq \mu$,
  - $\emptyset \,|\, \emptyset \vdash N : \left\{ \sigma_i^{p_i} \;\middle|\; i \in \mathcal{I} \right\}$,
  - $\forall i \in \mathcal{I},\ \ x : \sigma_i \,|\, \emptyset \vdash L : \xi_i$.

  This reduction comes, by definition of $\to_v$, from $N \to_v \left\{ P_j^{p'_j} \;\middle|\; j \in \mathcal{J} \right\}$, to which we can apply the induction hypothesis: there exists a closed typed distribution

  $$\left\{ (R_k : \rho_k)^{p''_k} \;\middle|\; k \in \mathcal{K} \right\}$$

  which is such that

  $$\left\{ \sigma_i^{p_i} \;\middle|\; i \in \mathcal{I} \right\} = \sum_{k\in\mathcal{K}} p''_k \cdot \rho_k$$

  and that $\left[ (R_k)^{p''_k} \;\middle|\; k \in \mathcal{K} \right]$ is a pseudo-representation of $\left\{ P_j^{p'_j} \;\middle|\; j \in \mathcal{J} \right\}$. It follows that, for every $k \in \mathcal{K}$, we can write $\rho_k$ as the pseudo-representation $\left[ \sigma_i^{p'''_{ki}} \;\middle|\; i \in \mathcal{I} \right]$ where some of the $p'''_{ki}$ (but not all of them) may be worth 0. This implies that, for all $i \in \mathcal{I}$,

  $$p_i = \sum_{k\in\mathcal{K}} p''_k p'''_{ki}$$

  Now, for every $k \in \mathcal{K}$, we can derive $\emptyset \,|\, \emptyset \vdash \mathsf{let}\ x = R_k\ \mathsf{in}\ L : \sum_{i\in\mathcal{I}} p'''_{ki} \cdot \xi_i$ from the rule

  $$\frac{\emptyset \,|\, \emptyset \vdash R_k : \left\{ \sigma_i^{p'''_{ki}} \;\middle|\; i \in \mathcal{I} \right\} \qquad x : \sigma_i \,|\, \emptyset \vdash L : \xi_i \quad (\forall i \in \mathcal{I})}{\emptyset \,|\, \emptyset \vdash \mathsf{let}\ x = R_k\ \mathsf{in}\ L : \sum_{i\in\mathcal{I}} p'''_{ki} \cdot \xi_i}$$

so that $\left[ \left(\mathsf{let}\ x\ =\ R_k\ \mathsf{in}\ L\ :\ \sum_{i\in\mathcal{I}} p'''_{ki}\cdot\xi_i\right)^{p''_k}\ \ \middle|\ \ k\in\mathcal{K}\right]$ is a pseudo-representation of a closed typed distribution, whose expectation is

$$\sum_{k\in\mathcal{K}} p''_k \sum_{i\in\mathcal{I}} p'''_{ki}\cdot\xi_i \ =\ \sum_{i\in\mathcal{I}}\left(\sum_{k\in\mathcal{K}} p''_k p'''_{ki}\right)\cdot\xi_i \ =\ \sum_{i\in\mathcal{I}} p_i\cdot\xi_i$$

By Lemma 9, the sum of $\sum_{i\in\mathcal{I}}\ p_i\cdot\xi_i$ is 1, and it follows that $\sum\mu\ =\ 1$ as well. Since $\sum_{i\in\mathcal{I}}\ p_i\cdot\xi_i\sqsubseteq\mu$, applying Corollary 3 gives us a family $(\nu_i)_{i\in\mathcal{I}}$ of distribution types such that, by subtyping, we can derive for every $k\in\mathcal{K}$ the judgement $\emptyset\,|\,\emptyset\vdash\mathsf{let}\ x\ =\ R_k\ \mathsf{in}\ L\ :\ \sum_{i\in\mathcal{I}} p'''_{ki}\cdot\nu_i$. This family $\overrightarrow{\nu}$ satisfies moreover $\sum_{i\in\mathcal{I}}\ p_i\cdot\nu_i\ =\ \mu$. We therefore consider the closed typed distribution of pseudo-representation

$$\left[\left(\mathsf{let}\ x\ =\ R_k\ \mathsf{in}\ L\ :\ \sum_{i\in\mathcal{I}} p'''_{ki}\cdot\nu_i\right)^{p''_k}\ \ \middle|\ \ k\in\mathcal{K}\right]$$

and of expectation type

$$\sum_{k\in\mathcal{K}} p''_k \sum_{i\in\mathcal{I}} p'''_{ki}\cdot\nu_i \ =\ \sum_{i\in\mathcal{I}} p_i\cdot\nu_i \ =\ \mu$$

Since $\left[(R_k\ :\ \rho_k)^{p''_k}\ \ \middle|\ \ k\in\mathcal{K}\right]$ is a pseudo-representation of $\left\{P_j^{p'_j}\ \ \middle|\ \ j\in\mathcal{J}\right\}$, we have that $\left[(\mathsf{let}\ x\ =\ R_k\ \mathsf{in}\ L\ :\ \rho_k)^{p''_k}\ \ \middle|\ \ k\in\mathcal{K}\right]$ is a pseudo-representation of $\left\{(\mathsf{let}\ x\ =\ P_j\ \mathsf{in}\ L)^{p'_j}\ \ \middle|\ \ j\in\mathcal{J}\right\}$ which allows us to conclude.

- Suppose that $M\ =\ \mathsf{case}\ \mathsf{S}\ V\ \mathsf{of}\ \{\mathsf{S}\to W\ \mid\ \mathsf{0}\to Z\}$, that $\mathscr{D}\ =\ \left\{(W\ V)^1\right\}$ and that $\emptyset\,|\,\emptyset\ \vdash\ \mathsf{case}\ \mathsf{S}\ V\ \mathsf{of}\ \{\mathsf{S}\to W\ \mid\ \mathsf{0}\to Z\}\ :\ \mu$. By Lemma 9, there exists $\mathfrak{s}$ and $\xi$ such that $\emptyset\,|\,\emptyset\vdash\mathsf{S}\ V\ :\ \mathsf{Nat}^{\widehat{\mathfrak{s}}}$ and $\emptyset\,|\,\emptyset\vdash W\ :\ \mathsf{Nat}^{\mathfrak{s}}\to\xi$ with $\xi\sqsubseteq\mu$. Lemma 13 implies that $\emptyset\,|\,\emptyset\vdash V\ :\ \mathsf{Nat}^{\mathfrak{s}}$. Using an Application rule, we obtain that $\emptyset\,|\,\emptyset\vdash W\ V\ :\ \xi$, and subtyping gives $\emptyset\,|\,\emptyset\vdash W\ V\ :\ \mu$, allowing us to conclude for $\left\{(W\ V\ :\ \mu)^1\right\}$.

- Suppose that $M\ =\ \mathsf{case}\ \mathsf{0}\ \mathsf{of}\ \{\mathsf{S}\to W\ \mid\ \mathsf{0}\to Z\}$, that $\mathscr{D}\ =\ \left\{(Z)^1\right\}$ and that $\emptyset\,|\,\emptyset\vdash\mathsf{case}\ \mathsf{0}\ \mathsf{of}\ \{\mathsf{S}\to W\ \mid\ \mathsf{0}\to Z\}\ :\ \mu$. By Lemma 9, there exists $\xi$ with $\xi\sqsubseteq\mu$ and such that $\emptyset\,|\,\emptyset\vdash Z\ :\ \xi$. By subtyping, $\emptyset\,|\,\emptyset\vdash Z\ :\ \mu$ which allows to conclude for $\left\{(Z\ :\ \mu)^1\right\}$.

- Suppose that $M\ =\ (\mathsf{letrec}\ f\ =\ V)\ \left(c\ \overrightarrow{W}\right)$, that $\mathscr{D}\ =\ \left\{\left(V[(\mathsf{letrec}\ f\ =\ V)/f]\ \left(c\ \overrightarrow{W}\right)\right)^1\right\}$ and that $\emptyset\,|\,\emptyset\vdash(\mathsf{letrec}\ f\ =\ V)\ \left(c\ \overrightarrow{W}\right)\ :\ \mu$. We apply again Lemma 9, but this time we rather depict the derivation typing $M$ with $\mu$ it induces, for the sake of clarity. This derivation is of the form (modulo composition of subtyping rules):

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \textit{Hyp}\\ \emptyset\,|\,f\ :\ \left\{(\mathsf{Nat}^{\mathfrak{u}_j}\to\xi[\mathfrak{u}_j/\mathfrak{i}])^{p_j}\ \ \middle|\ \ j\in\mathcal{J}\right\}\vdash V\ :\ \mathsf{Nat}^{\widehat{\mathfrak{i}}}\to\xi[\widehat{\mathfrak{i}}/\mathfrak{i}]\end{array}}{\emptyset\,|\,\emptyset\vdash\mathsf{letrec}\ f\ =\ V\ :\ \mathsf{Nat}^{\mathfrak{t}}\to\xi[\mathfrak{t}/\mathfrak{i}]}}{\emptyset\,|\,\emptyset\vdash\mathsf{letrec}\ f\ =\ V\ :\ \mathsf{Nat}^{\widehat{\mathfrak{s}}}\to\mu} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \emptyset\,|\,\emptyset\vdash c\ \overrightarrow{W}\ :\ \mathsf{Nat}^{\widehat{\mathfrak{r}}}\end{array}}{\emptyset\,|\,\emptyset\vdash c\ \overrightarrow{W}\ :\ \mathsf{Nat}^{\widehat{\mathfrak{s}}}}}{\emptyset\,|\,\emptyset\vdash(\mathsf{letrec}\ f\ =\ V)\ \left(c\ \overrightarrow{W}\right)\ :\ \mu}$$

where the two sizes appearing in the types for $c\ \overrightarrow{W}$ are successors due to Lemma 13, and where

- *Hyp* denotes the additional premises of the $\mathsf{letrec}$ rule, and contains notably $\mathsf{i}\ \mathsf{pos}\ \xi$,
- $\mathfrak{r} \preccurlyeq \widehat{\mathfrak{r}} \preccurlyeq \widehat{\mathfrak{s}} \preccurlyeq \mathfrak{t}$,
- $\xi[\mathfrak{t}/\mathsf{i}] \sqsubseteq \mu$.

It follows that, for every $j \in \mathcal{J}$, we can deduce that the closed value $\mathsf{letrec}\ f\ =\ V$ has type $\mathsf{Nat}^{\mathfrak{u}_j} \to \xi[\mathfrak{u}_j/\mathsf{i}]$, as proved by the derivation

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \textit{Hyp}\\ \emptyset \,|\, f\ :\ \left\{\, (\mathsf{Nat}^{\mathfrak{u}_j} \to \xi[\mathfrak{u}_j/\mathsf{i}])^{p_j} \ \middle|\ j \in \mathcal{J} \,\right\} \vdash V\ :\ \mathsf{Nat}^{\widehat{\mathfrak{i}}} \to \xi[\widehat{\mathfrak{i}}/\mathsf{i}]\end{array}}{\emptyset \,|\, \emptyset \vdash \mathsf{letrec}\ f\ =\ V\ :\ \mathsf{Nat}^{\mathfrak{u}_j} \to \xi[\mathfrak{u}_j/\mathsf{i}]}$$

Since

$$\emptyset \,|\, f\ :\ \left\{\, (\mathsf{Nat}^{\mathfrak{u}_j} \to \xi[\mathfrak{u}_j/\mathsf{i}])^{p_j} \ \middle|\ j \in \mathcal{J} \,\right\} \vdash V\ :\ \mathsf{Nat}^{\widehat{\mathfrak{i}}} \to \xi[\widehat{\mathfrak{i}}/\mathsf{i}]$$

we obtain by Lemma 12 that

$$\emptyset \,|\, \emptyset \vdash V[(\mathsf{letrec}\ f\ =\ V)\,/f]\ :\ \mathsf{Nat}^{\widehat{\mathfrak{i}}} \to \xi[\widehat{\mathfrak{i}}/\mathsf{i}]$$

We now apply Lemma 17 to $\emptyset \,|\, \emptyset \vdash V[(\mathsf{letrec}\ f\ =\ V)\,/f]\ :\ \mathsf{Nat}^{\widehat{\mathfrak{i}}} \to \xi[\widehat{\mathfrak{i}}/\mathsf{i}]$ with the substitution $[\mathfrak{r}/\mathsf{i}]$ and we obtain that $\emptyset \,|\, \emptyset \vdash V[(\mathsf{letrec}\ f\ =\ )\,/V]f\ :\ \mathsf{Nat}^{\widehat{\mathfrak{r}}} \to \xi[\widehat{\mathfrak{r}}/\mathsf{i}]$. Using the Application rule, we derive $\emptyset \,|\, \emptyset \vdash V[(\mathsf{letrec}\ f\ =\ V)\,/f]\ \left(c\ \overrightarrow{W}\right)\ :\ \xi[\widehat{\mathfrak{r}}/\mathsf{i}]$. Since $\mathsf{i}\ \mathsf{pos}\ \xi$ and $\widehat{\mathfrak{r}} \preccurlyeq \mathfrak{t}$, by Lemma 16, we get that $\xi[\widehat{\mathfrak{r}}/\mathsf{i}] \sqsubseteq \xi[\mathfrak{t}/\mathsf{i}]$. By transitivity of $\sqsubseteq$, $\xi[\widehat{\mathfrak{r}}/\mathsf{i}] \sqsubseteq \mu$ which allows us to conclude by subtyping that $\emptyset \,|\, \emptyset \vdash V[(\mathsf{letrec}\ f\ =\ V)\,/f]\ \left(c\ \overrightarrow{W}\right)\ :\ \mu$. The result follows, for $\left\{\ \left(V[(\mathsf{letrec}\ f\ =\ V)\,/f]\ \left(c\ \overrightarrow{W}\right)\ :\ \mu\right)^1\ \right\}$.

□

**Theorem 1 (Subject Reduction for $\to_v$)** *Let $n \in \mathbb{N}$, and $\left\{\, (M_i\ :\ \mu_i)^{p_i} \ \middle|\ i \in \mathcal{I} \,\right\}$ be a closed typed distribution. Suppose that $\left\{\, (M_i)^{p_i} \ \middle|\ i \in \mathcal{I} \,\right\}\ \to_v^n\ \left\{\, (N_j)^{p'_j} \ \middle|\ j \in \mathcal{J} \,\right\}$ then there exists a closed typed distribution $\left\{\, (L_k\ :\ \nu_k)^{p''_k} \ \middle|\ k \in \mathcal{K} \,\right\}$ such that*

- $\mathbb{E}\left((M_i\ :\ \mu_i)^{p_i}\right)\ =\ \mathbb{E}\left((L_k\ :\ \nu_k)^{p''_k}\right)$,
- *and that $\left[\, (L_k)^{p''_k} \ \middle|\ k \in \mathcal{K} \,\right]$ is a pseudo-representation of $\left\{\, (N_j)^{p'_j} \ \middle|\ j \in \mathcal{J} \,\right\}$.*

**Proof.** The proof is by induction on $n$. For $n = 0$, $\to_v^0$ is the identity relation and the result is immediate. For $n + 1$, we have

$$\left\{\, (M_i)^{p_i} \ \middle|\ i \in \mathcal{I} \,\right\}\ \to_v^n\ \left\{\, (P_l)^{p''_l} \ \middle|\ l \in \mathcal{L} \,\right\}\ \to_v\ \left\{\, (N_j)^{p'_j} \ \middle|\ j \in \mathcal{J} \,\right\}$$

We apply the induction hypothesis and obtain a closed typed distribution $\left\{\, (R_g\ :\ \xi_g)^{p^{(3)}_g} \ \middle|\ g \in \mathcal{G} \,\right\}$ satisfying $\mathbb{E}\left((M_i\ :\ \mu_i)^{p_i}\right)\ =\ \mathbb{E}\left((R_g\ :\ \xi_g)^{p^{(3)}_g}\right)$ and such that $\left[\, (R_g)^{p^{(3)}_g} \ \middle|\ g \in \mathcal{G} \,\right]$ is a pseudo-representation of $\left\{\, (P_l)^{p''_l} \ \middle|\ l \in \mathcal{L} \,\right\}$. For every $g \in \mathcal{G}$:

– if $R_g$ is a value, we set $\mathscr{D}_g = \left\{ R^1 \right\}$ and $\mathscr{T}_g$ to be the closed typed distribution $\mathscr{T}_g = \left\{ (T_h : \rho_h)^{p_h^{(4)}} \mid h \in \mathcal{H}_g \right\} = (R_g : \xi_g)^1$,

– else $R_g \to_v \mathscr{D}_g$. We apply Lemma 18 and obtain a closed typed distribution $\mathscr{T}_g = \left\{ (T_h : \rho_h)^{p_h^{(4)}} \mid h \in \mathcal{H}_g \right\}$ such that $\mathbb{E}\left( (T_h : \rho_h)^{p_h^{(4)}} \right) = \xi_g$ and that $\left[ (T_h)^{p_h^{(4)}} \mid h \in \mathcal{H}_g \right]$ is a pseudo-representation of $\mathscr{D}_g$.

We claim that the closed typed distribution defined as

$$\left\{ (L_k : \nu_k)^{p''_k} \mid k \in \mathcal{K} \right\} = \sum_{g \in \mathcal{G}} p_g^{(3)} \cdot \mathscr{T}_g$$

satisfies the required conditions. Indeed, the expectation type is preserved:

$$\begin{array}{rcl} \mathbb{E}\left( (M_i : \mu_i)^{p_i} \right) & = & \mathbb{E}\left( (R_g : \xi_g)^{p_g^{(3)}} \right) \\ & = & \sum_{g \in \mathcal{G}} p_g^{(3)} \cdot \xi_g \\ & = & \sum_{g \in \mathcal{G}} p_g^{(3)} \cdot \mathbb{E}\left( (T_h : \rho_h)^{p_h^{(4)}} \right) \\ & = & \mathbb{E}\left( \sum_{g \in \mathcal{G}} p_g^{(3)} \cdot \mathscr{T}_g \right) \\ & = & \mathbb{E}\left( \left\{ (L_k : \nu_k)^{p''_k} \mid k \in \mathcal{K} \right\} \right) \end{array}$$

Moreover, by definition of the family $(\mathscr{D}_g)_{g \in \mathcal{G}}$,

$$\left\{ (P_l)^{p''_l} \mid l \in \mathcal{L} \right\} = \sum_{g \in \mathcal{G}} p_g^{(3)} \cdot \left\{ (R_g)^1 \right\} \to_v \left\{ (N_j)^{p'_j} \mid j \in \mathcal{J} \right\} = \sum_{g \in \mathcal{G}} p_g^{(3)} \cdot \mathscr{D}_g$$

The result follows from the fact that $\left[ (T_h)^{p_h^{(4)}} \mid h \in \mathcal{H}_g \right]$ is a pseudo-representation of $\mathscr{D}_g$ for every $g \in \mathcal{G}$.

□

## 5.6 Subject Reduction for $\Rrightarrow_v$

Recall that there is an order $\preccurlyeq$ on distributions, defined pointwise.

**Lemma 19** *Suppose that $M \Rrightarrow_v \left\{ V_i^{p_i} \mid i \in \mathcal{I} \right\}$ and that $M \in \Lambda_{\oplus}^{\mathfrak{s}}(\mu)$. Then there exists a closed typed distribution $\left\{ (W_j : \sigma_j)^{p'_j} \mid j \in \mathcal{J} \right\}$ such that*

- $\mathbb{E}\left( (W_j : \sigma_j)^{p'_j} \right) \preccurlyeq \mu$,
- *and that $\left[ (W_j)^{p'_j} \mid j \in \mathcal{J} \right]$ is a pseudo-representation of $\left\{ (V_i)^{p_i} \mid i \in \mathcal{I} \right\}$.*

**Proof.** We have $M \to_v \mathscr{D} \stackrel{VD}{=} \mathscr{D}_{|T} + \left\{ V_i^{p_i} \mid i \in \mathcal{I} \right\}$. By Lemma 18, there exists a closed typed distribution $\left\{ (L_k : \nu_k)^{p''_k} \mid k \in \mathcal{K} \right\}$ such that $\mathbb{E}\left( (L_k : \nu_k)^{p''_k} \right) = \mu$ and that $\left[ (L_k)^{p''_k} \mid k \in \mathcal{K} \right]$ is a pseudo-representation of $\mathscr{D}$. We consider the pseudo-representation $\left[ (W_j)^{p'_j} \mid j \in \mathcal{J} \right]$ obtained from $\left[ (L_k)^{p''_k} \mid k \in \mathcal{K} \right]$ by removing all the terms which are not values. Note that $\mathcal{J} \subseteq \mathcal{K}$. We obtain in this way a pseudo-representation of $\left\{ V_i^{p_i} \mid i \in \mathcal{I} \right\}$ which induces a closed typed representation $\left\{ (W_j : \nu_j)^{p'_j} \mid j \in \mathcal{J} \right\}$ such that $\mathbb{E}\left( (W_j : \nu_j)^{p'_j} \right) \preccurlyeq \mu$. □

**Theorem 2 (Subject Reduction)** *Let $M \in \Lambda_{\oplus}^{\mathfrak{s}}(\mu)$. Then there exists a closed typed distribution $\left\{ (W_j : \sigma_j)^{p_j} \mid j \in \mathcal{J} \right\}$ such that*

– $\mathbb{E}\left((W_j : \sigma_j)^{p_j}\right) \preccurlyeq \mu$,
– *and that* $\left[ (W_j)^{p_j} \;\middle|\; j \in \mathcal{J} \right]$ *is a pseudo-representation of* $[\![ M ]\!]$.

Note that $\mathbb{E}\left((W_j : \sigma_j)^{p_j}\right) \preccurlyeq \mu$ since the semantics of a term may not be a proper distribution at this stage. In fact, it will follow from the soundness theorem of Section 6 that the typability of $M$ implies that $\sum [\![ M ]\!] = 1$ and thus that the previous statement is an equality.

# 6 Typability Implies Termination: Reducibility Strikes Again

This section is technically the most advanced one of the paper, and proves that the typing discipline we have introduced indeed enforces almost sure termination. As already mentioned, the technique we will employ is a substantial generalisation of Girard-Tait's reducibility. In particular, reducibility must be made quantitative, in that terms can be said to be reducible *with a certain probability.* This means that reducibility sets will be defined as sets parametrised by a real number $p$, called the *degree of reducibility* of the set. As Lemma 20 will emphasize, this degree of reducibility ensures that terms contained in a reducibility set parametrised by $p$ terminate with probability at least $p$. These "intermediate" degrees of reducibility are required to handle the fixpoint construction, and show that recursively-defined terms that are typable are indeed AST — that is, that they belong to the appropriate reducibility set, parametrised by 1.

## 6.1 Reducibility Sets for Closed Terms

The first preliminary notion we need is that of a size environment:

**Definition 23 (Size Environment)** *A* size environment *is any function $\rho$ from $\mathcal{S}$ to $\mathbb{N} \cup \{\infty\}$. Given a size environment $\rho$ and a size expression $\mathfrak{s}$, there is a naturally defined element of $\mathbb{N} \cup \{\infty\}$, which we indicate as $[\![\mathfrak{s}]\!]_\rho$:*
– $[\![\widehat{\mathfrak{i}^n}]\!]_\rho \;=\; \rho(\mathfrak{i}) + n$,
– $[\![\infty]\!]_\rho \;=\; \infty$.

In other words, the purpose of size environments is to give a semantic meaning to size expressions. Our reducibility sets will be parametrised not only on a probability, but also on a size environment.

**Definition 24 (Reducibility Sets)**
– *For values of simple type* $\mathsf{Nat}$*, we define the reducibility sets*

$$\mathsf{VRed}^p_{\mathsf{Nat}^{\mathfrak{s}},\rho} \;=\; \left\{ \mathsf{S}^n\ \mathsf{0} \;\middle|\; p > 0 \Longrightarrow n < [\![\mathfrak{s}]\!]_\rho \right\}.$$

– *Values of higher-order type are in a reducibility set when their applications to appropriate values are reducible terms, with an adequate degree of reducibility:*

$$\mathsf{VRed}^p_{\sigma\to\mu,\rho} \;=\; \left\{ V \in \Lambda^V_\oplus\left(\langle \sigma \to \mu \rangle\right) \;\middle|\; \forall q \in (0,1],\quad \begin{array}{l} \forall W \in \mathsf{VRed}^q_{\sigma,\rho}, \\ V\ W \in \mathsf{TRed}^{pq}_{\mu,\rho} \end{array} \right\}$$

– *Distributions of values are reducible with degree $p$ when they consist of values which are themselves globally reducible "enough". Formally,* $\mathsf{DRed}^p_{\mu,\rho}$ *is the set of finite distributions of values – in the sense that they have a finite support – admitting a pseudo-representation* $\mathscr{D} = \left[ (V_i)^{p_i} \;\middle|\; i \in \mathcal{I} \right]$ *such that, setting* $\mu = \left\{ (\sigma_j)^{p'_j} \;\middle|\; j \in \mathcal{J} \right\}$*, there exists a family* $(p_{ij})_{i\in\mathcal{I}, j\in\mathcal{J}} \in [0,1]^{|\mathcal{I}|\times|\mathcal{J}|}$ *of probabilities and a family* $(q_{ij})_{i\in\mathcal{I}, j\in\mathcal{J}} \in [0,1]^{|\mathcal{I}|\times|\mathcal{J}|}$ *of degrees of reducibility, satisfying:*

1. $\forall i \in \mathcal{I},\ \ \forall j \in \mathcal{J},\ \ V_i \in \mathsf{VRed}^{q_{ij}}_{\sigma_j,\rho}$,
2. $\forall i \in \mathcal{I},\ \ \sum_{j\in\mathcal{J}} p_{ij} \;=\; p_i$,
3. $\forall j \in \mathcal{J},\ \ \sum_{i\in\mathcal{I}} p_{ij} \;=\; \mu(\sigma_j)$,

4. $p \leq \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} q_{ij} p_{ij}$.

*Note that* (2) *and* (3) *imply that* $\sum \mathscr{D} = \sum \mu$. *We say that* $\left[ (V_i)^{p_i} \;\middle|\; i \in \mathcal{I} \right]$ witnesses *that* $\mathscr{D} \in \mathsf{DRed}^p_{\mu,\rho}$.

– *A term is reducible with degree p when its finite approximations compute distributions of values of degree of reducibility arbitrarily close to p:*

$$\mathsf{TRed}^p_{\mu,\rho} \quad = \quad \left\{ M \in \Lambda_\oplus(\langle \mu \rangle) \;\middle|\; \forall 0 \leq r < p, \quad \exists \nu_r \preccurlyeq \mu, \quad \exists n_r \in \mathbb{N}, \; M \Rrightarrow^{n_r}_v \mathscr{D}_r \text{ and } \mathscr{D}_r \in \mathsf{DRed}^r_{\nu_r,\rho} \right\}$$

*Note that here, unlike to the case of* $\mathsf{DRed}$, *the fact that* $M \in \Lambda_\oplus(\langle \mu \rangle)$ *implies that* $\mu$ *is proper.*

The first thing to observe about reducibility sets as given in Definition 24 is that they only deal with closed terms, and not with arbitrary terms. As such, we cannot rely *directly* on them when proving AST for typable terms, at least if we want to prove it by induction on the structure of type derivations. We will therefore define in the sequel an extension of these sets to open terms, which will be based on these sets of closed terms, and therefore enjoy similar properties. Before embarking in the proof that typability implies reducibility, it is convenient to prove some fundamental properties of reducibility sets, which inform us about how these sets are structured, and which will be crucial in the sequel. This is the purpose of the following subsections.

## 6.2 Reducibility Sets and Termination

The following lemma, relatively easy to prove, is crucial for the understanding of the reducibility sets, for that it shows that the degree of reducibility of a term gives information on the sum of its operational semantics:

**Lemma 20 (Reducibility and Termination)**
– *Let* $\mathscr{D} \in \mathsf{DRed}^p_{\mu,\rho}$. *Then* $\sum \mathscr{D} \geq p$.
– *Let* $M \in \mathsf{TRed}^p_{\mu,\rho}$. *Then* $\sum [\![ M ]\!] \geq p$.

**Proof.**

- Let $\mathscr{D} \in \mathsf{DRed}^p_{\mu,\rho}$, then there exists a pseudo-representation $\mathscr{D} = \left[ (V_i)^{p_i} \;\middle|\; i \in \mathcal{I} \right]$ and families $(p_{ij})_{i \in \mathcal{I}, j \in \mathcal{J}}$ and $(q_{ij})_{i \in \mathcal{I}, j \in \mathcal{J}}$ of reals of $[0,1]$ such that $\forall i \in \mathcal{I}$, $\sum_{j \in \mathcal{J}} p_{ij} = p_i$, and that $p \leq \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} q_{ij} p_{ij}$. We therefore have:

$$\sum \mathscr{D} \quad = \quad \sum_{i \in \mathcal{I}} p_i \quad = \quad \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} p_{ij} \quad \geq \quad \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} q_{ij} p_{ij} \quad \geq \quad p.$$

- Since $M \in \mathsf{TRed}^p_{\mu,\rho}$, for every $0 \leq r < p$, there exists $n_r$ with $M \Rrightarrow^{n_r}_v \mathscr{D}_r$ and $\mathscr{D}_r \in \mathsf{DRed}^r_{\nu_r,\rho}$. From the previous point, we get that $\sum \mathscr{D}_r \geq r$ for every $0 \leq r < p$. It follows from Corollary 1 that $\sum [\![ M ]\!] \geq r$ for every $0 \leq r < p$ and, by taking the supremum, $\sum [\![ M ]\!] \geq p$.

□

It follows from this lemma that terms with degree of reducibility 1 are AST:

**Corollary 4 (Reducibility and AST)** *Let* $M \in \mathsf{TRed}^1_{\mu,\rho}$. *Then* $M$ *is AST.*

## 6.3 Reducibility Sets and Reducibility Degrees

We now prove two results related to the reducibility degrees of reducibility sets. First of all, if the degree of reducibility $p$ is 0, then no assumption is made on the probability of termination of terms, distributions or values. It follows that the three kinds of reducibility sets collapse to the set of all affinely simply typable terms, distributions or values:

**Lemma 21 (Candidates of Null Reducibility)** – *If $V \in \Lambda_{\oplus}^{V}(\kappa)$, then $V \in \mathsf{VRed}^{0}_{\sigma,\rho}$ for every $\sigma$ such that $\langle\sigma\rangle = \kappa$ and every size environment $\rho$.*

– *Let $\mathscr{D} = \left\{ (V_i)^{p_i} \;\middle|\; i \in \mathcal{I} \right\}$ be a finite distribution of values. If $\forall i \in \mathcal{I},\ \ V_i \in \Lambda_{\oplus}^{V}(\kappa)$, then $\mathscr{D} \in \mathsf{DRed}^{0}_{\mu,\rho}$ for every $\mu$ such that $\langle\mu\rangle = \kappa$ and $\sum \mu = \sum \mathscr{D}$ and every $\rho$.*

– *If $M \in \Lambda_{\oplus}(\kappa)$, then $M \in \mathsf{TRed}^{0}_{\mu,\rho}$ for $\mu$ such that $\langle\mu\rangle = \kappa$ and every $\rho$.*

**Structure of the proof.** In this lemma, as for most lemmas proving properties about $\mathsf{VRed}$, $\mathsf{DRed}$ and $\mathsf{TRed}$, we use a proof by induction on types. As the property is defined in a mutual way over $\mathsf{VRed}$, $\mathsf{DRed}$ and $\mathsf{TRed}$, we typically prove it for $\mathsf{VRed}^{p}_{\mathsf{Nat}^{\mathfrak{s}},\rho}$ for any size $\mathfrak{s}$ refining $\mathsf{Nat}$, and then for $\mathsf{VRed}^{p}_{\sigma\to\mu,\rho}$ by using the associated hypothesis on $\mathsf{TRed}^{p}_{\mu,\rho}$. Then we prove the property for any distribution type for $\mathsf{DRed}^{p}_{\mu,\rho}$ using induction hypothesis on the $\mathsf{VRed}^{p}_{\sigma,\rho}$ for $\sigma \in \mathsf{S}(\mu)$, and we prove it for $\mathsf{TRed}^{p}_{\mu,\rho}$ using induction hypothesis on $\mathsf{VRed}^{p}_{\sigma,\rho}$. The point is that these ingredients allow to give a proof by induction on the simple type underlying the sized type of interest. In the base case, the sized type is necessarily of the form $\mathsf{Nat}^{\mathfrak{s}}$ for some size $\mathfrak{s}$: we prove the statement on $\mathsf{VRed}^{p}_{\mathsf{Nat}^{\mathfrak{s}},\rho}$ for all these sized types without using any induction-like hypothesis. Then we prove the statement for distribution types $\mu = \left\{ (\mathsf{Nat}^{\mathfrak{s}_i})^{p_i} \;\middle|\; i \in \mathcal{I} \right\}$, first on $\mathsf{DRed}^{p}_{\mu,\rho}$ by using the results for the sets $\mathsf{VRed}^{p}_{\mathsf{Nat}^{\mathfrak{s}_i},\rho}$. Then we prove the result for $\mathsf{TRed}^{p}_{\mu,\rho}$ typically using the one for $\mathsf{DRed}^{p}_{\mu,\rho}$.

We then switch to higher-order types, and give the proof for $\mathsf{VRed}^{p}_{\sigma\to\mu,\rho}$, which may use the results for the other sets on types $\sigma$ and $\mu$. Typically, only results on $\mathsf{TRed}^{p}_{\mu,\rho}$ are used. Then the proofs for $\mathsf{DRed}^{p}_{\sigma\to\mu,\rho}$ and $\mathsf{TRed}^{p}_{\sigma\to\mu,\rho}$ are typically the same as in the case of distributions over sized types refining $\mathsf{Nat}$: therefore we do not write them again.

This proof scheme will become more clear with the proof of this lemma on candidates of null reducibility:

**Proof.**

- Let $V \in \Lambda_{\oplus}^{V}(\mathsf{Nat})$. Every $\sigma :: \mathsf{Nat}$ is of the shape $\sigma = \mathsf{Nat}^{\mathfrak{s}}$ for a size $\mathfrak{s}$. Let $\rho$ be a size environment. By inspection of the grammar of values and of the simple type system, we see that $V$ must be of the shape $\mathsf{S}^n\ \mathsf{0}$ for $n \in \mathbb{N}$. Note that $V$ is closed: it cannot be a variable. By definition, $V \in \mathsf{VRed}^{0}_{\sigma,\rho}$.

- Let $\kappa = \kappa' \to \kappa''$ be a higher-order type, with $\sigma :: \kappa'$ and $\mu :: \kappa''$. Let $\rho$ be a size environment, and $V \in \Lambda_{\oplus}^{V}(\kappa)$. Let $q \in (0,1]$ and $W \in \mathsf{VRed}^{q}_{\sigma,\rho}$, we need to prove that $V\ W \in \mathsf{TRed}^{0}_{\mu,\rho}$. But, by definition of $\mathsf{VRed}^{q}_{\sigma,\rho}$, $W \in \Lambda_{\oplus}^{V}(\kappa')$. It follows that $V\ W \in \Lambda_{\oplus}(\kappa'')$, and we can apply the induction hypothesis to deduce that $V\ W \in \mathsf{TRed}^{0}_{\mu,\rho}$, so that by definition $V \in \mathsf{VRed}^{0}_{\sigma,\rho}$.

- Let $\mathscr{D} = \left\{ (V_i)^{p_i} \;\middle|\; i \in \mathcal{I} \right\}$ be a distribution of values and $\mu = \left\{ (\sigma_j)^{p'_j} \;\middle|\; j \in \mathcal{J} \right\} :: \kappa$ be a distribution type. Suppose that $\forall i \in \mathcal{I}$, $V_i \in \Lambda_{\oplus}^{V}(\kappa)$. Let $\rho$ be a size environment. For every $(i,j) \in \mathcal{I} \times \mathcal{J}$, we set $p_{ij} = \frac{p_i p'_j}{\sum \mu}$ and $q_{ij} = 0$. We consider the canonical pseudo-representation $\mathscr{D} = \left[ (V_i)^{p_i} \;\middle|\; i \in \mathcal{I} \right]$ and check the four conditions to be in $\mathsf{DRed}^{0}_{\mu,\rho}$:
  1. $\forall i \in \mathcal{I},\ \ \forall j \in \mathcal{J},\ \ V_i \in \mathsf{VRed}^{q_{ij}}_{\sigma_j,\rho}$: this is obtained by induction hypothesis,
  2. $\forall i \in \mathcal{I},\ \ \sum_{j\in\mathcal{J}} p_{ij} = p_i$: let $i \in \mathcal{I}$, we have $\sum_{j\in\mathcal{J}} p_{ij} = \frac{p_i}{\sum \mu} \sum_{j\in\mathcal{J}} p'_j = \frac{p_i}{\sum \mu} \times \sum \mu = p_i$.
  3. $\forall j \in \mathcal{J},\ \ \sum_{i\in\mathcal{I}} p_{ij} = \mu(\sigma_j)$: let $j \in \mathcal{J}$, we have $\sum_{i\in\mathcal{I}} p_{ij} = \frac{p'_j}{\sum \mu} \sum_{i\in\mathcal{I}} p_i = \frac{p'_j}{\sum \mu} \times \sum \mathscr{D}$. But $\sum \mu = \sum \mathscr{D}$, so that the sum equals $p'_j$ as requested.
  4. $p \leq \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} q_{ij} p_{ij}$: this amounts to $0 \leq 0$, which holds.

- Let $M \in \Lambda_{\oplus}(\kappa)$ and $\mu :: \kappa$. Let $\rho$ be a size environment. Then $M \in \mathsf{TRed}^{0}_{\mu,\rho}$: the condition on $M$ in the definition of $\mathsf{TRed}^{0}_{\mu,\rho}$ is for any $0 \leq r < 0$ so that it's an empty condition in this case.

□

As $p$ gives us a lower bound on the sum of the semantics of terms, it is easily guessed that a term having degree of reducibility $p$ must also have degree of reducibility $q < p$. The following lemma makes this statement precise:

**Lemma 22 (Downward Closure)** *Let $\sigma$ be a sized type, $\mu$ be a distribution type and $\rho$ be a size environment. Let $0 \leq q < p \leq 1$. Then:*
- *For any value $V$, $V \in \mathsf{VRed}^p_{\sigma,\rho} \implies V \in \mathsf{VRed}^q_{\sigma,\rho}$,*
- *For any finite distribution of values $\mathscr{D}$, $\mathscr{D} \in \mathsf{DRed}^p_{\mu,\rho} \implies \mathscr{D} \in \mathsf{DRed}^q_{\mu,\rho}$,*
- *For any term $M$, $M \in \mathsf{TRed}^p_{\mu,\rho} \implies M \in \mathsf{TRed}^q_{\mu,\rho}$.*

**Proof.** Let $\sigma$ be a sized type, $\mu$ be a distribution type and $\rho$ be a size environment. If $q = 0$, the result is immediate as a consequence of Lemma 21. Let $0 < q < p \leq 1$.

- Suppose that $V \in \mathsf{VRed}^p_{\mathsf{Nat}^{\mathsf{s}},\rho}$. Since by definition $p, q > 0 \implies \mathsf{VRed}^p_{\mathsf{Nat}^{\mathsf{s}},\rho} = \mathsf{VRed}^q_{\mathsf{Nat}^{\mathsf{s}},\rho}$, the result holds.

- Suppose that $V \in \mathsf{VRed}^p_{\sigma\to\mu,\rho}$. Then:

$$\begin{array}{lll} & V \in \mathsf{VRed}^p_{\sigma\to\mu,\rho} & \\ \iff & \forall q \in (0,1],\ \ \forall W \in \mathsf{VRed}^q_{\sigma,\rho},\ \ VW \in \mathsf{TRed}^{pq}_{\mu,\rho} & \\ \implies & \forall q' \in (0,1],\ \ \forall W \in \mathsf{VRed}^{q'}_{\sigma,\rho},\ \ VW \in \mathsf{TRed}^{qq'}_{\mu,\rho} & \text{(by IH, since } 0 < qq' < pq \leq 1) \\ \iff & V \in \mathsf{VRed}^q_{\sigma\to\mu,\rho} & \end{array}$$

- Suppose that $\mathscr{D} \in \mathsf{DRed}^p_{\mu,\rho}$. Then there exists a pseudo-representation $\mathscr{D} = \left[ (V_i)^{p_i} \ \middle|\ i \in \mathcal{I} \right]$ and families of reals $(p_{ij})_{i\in\mathcal{I},j\in\mathcal{J}}$ and $(q_{ij})_{i\in\mathcal{I},j\in\mathcal{J}}$ satisfying conditions (1) – (4). We have $\mathscr{D} \in \mathsf{DRed}^q_{\mu,\rho}$, for the same pseudo-representation, since conditions (1) – (3) are the same, and (4) holds as well since $q < p$.

- Suppose that $M \in \mathsf{TRed}^p_{\mu,\rho}$. Then for every $0 \leq r < p$, there exists $\nu_r \preccurlyeq \mu$ and $n_r \in \mathbb{N}$ with $M \Rrightarrow^{n_r}_v \mathscr{D}_r$ and $\mathscr{D}_r \in \mathsf{DRed}^r_{\nu_r,\rho}$. So this statement also holds for every $0 \leq r < q$ and $M \in \mathsf{TRed}^q_{\mu,\rho}$.

□

## 6.4 Continuity of the Reducibility Sets

To prove the lemma of continuity on the reducibility sets, which says that if an element is in all the reducibility sets for degrees $q < p$ then it is also in the set parametrised by the degree $p$, we use the following companion lemma computing a family of probabilities maximizing the degree of reducibility of a distribution:

**Lemma 23 (Maximizing the Degree of Reducibility of a Distribution)**
*Let $\mathscr{D} = \left[ (V_i)^{p_i} \ \middle|\ i \in \mathcal{I} \right]$ be a finite distribution of values and $\mu = \left\{ (\sigma_j)^{p'_j} \ \middle|\ j \in \mathcal{J} \right\}$ be a distribution type. Set $q_{ij} = \max \left\{ q \ \middle|\ V_i \in \mathsf{VRed}^q_{\sigma_j,\rho} \right\}$ for every $(i,j) \in \mathcal{I} \times \mathcal{J}$. Then there exists a family $(p_{ij})_{i\in\mathcal{I},j\in\mathcal{J}}$ of reals of $[0,1]$ satisfying:*

1. $\forall i \in \mathcal{I},\ \ \sum_{j\in\mathcal{J}} p_{ij} = p_i$,
2. $\forall j \in \mathcal{J},\ \ \sum_{i\in\mathcal{I}} p_{ij} = \mu(\sigma_j)$,

*and which maximizes $\sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} q_{ij} p_{ij}$.*

**Proof.** We use the theory of linear programming in the finite real vectorial space $\mathbb{R}^n$, taking [32] as a reference. We stick to the notations of this book. The problem then amounts to showing the existence of

$$\max\left\{cx \;\middle|\; x \geq \overrightarrow{0},\ Ax = b\right\} \tag{12}$$

where, supposing that we can index vectors and matrices by $i \times j$ thanks to a bijection $i \times j \to \{1, \ldots, n\}$ where $n = \#(\mathcal{I} \times \mathcal{J})$:

- $x$ is the column vector indexed by the finite set $\mathcal{I} \times \mathcal{J}$, where $x_{ij}$ plays the role of $p_{ij}$,
- $c$ is the row vector indexed by $\mathcal{I} \times \mathcal{J}$, with $c_{ij} = \max\left\{q \;\middle|\; V_i \in \mathsf{VRed}^q_{\sigma_j,\rho}\right\}$,
- $\overrightarrow{0}$ is the null column vector of size $\#(\mathcal{I} \times \mathcal{J})$,
- $A$ is the matrix with columns indexed by $\mathcal{I} \times \mathcal{J}$ and rows indexed by $\mathcal{I} + \mathcal{J}$, and such that:
  - $a_{i',(i,j)} = 1$ if and only if $i = i'$, and 0 else,
  - and $a_{j',(i,j)} = 1$ if and only if $j = j'$, and 0 else.
- $b$ is the column vector indexed by $\mathcal{I} + \mathcal{J}$ and such that $b_i = p_i$ and $b_j = \mu(\sigma_j)$.

Following [32, Section 7.4], the maximum (12) exists if and only if:

- the problem is *feasible*: its constraints admit at least a solution,
- and if it is *bounded*: there should be an upper bound over (12).

and, also, its existence is equivalent to the one of the maximum of the following problem:

$$\max\left\{cx \;\middle|\; x \geq \overrightarrow{0},\ Ax \leq b\right\} \tag{13}$$

This reformulation makes the feasibility immediate, for the null vector $x = \overrightarrow{0}$. It is as well immediate to see that the problem is bounded: by construction, all the $q_{ij} \in [0,1]$, and $\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} p_{ij} = 1$ so that $\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} q_{ij}p_{ij} \leq 1$. The existence of the maximum (12) follows, and the Lemma therefore holds. □

It follows that a distribution has a maximal degree of reducibility: the supremum of the degrees of reducibility is again a degree of reducibility:

**Corollary 5 (Maximizing the Degree of Reducibility of a Distribution II)** *Let $\mathscr{D}$ be a finite distribution of values, $\mu$ be a distribution type and $\rho$ be a size environment. Suppose that $\mathscr{D} \in \mathsf{DRed}^p_{\mu,\rho}$ for some real $p \in [0,1]$. Then there exists a maximal real $p_{max} \in [p,1]$ such that $\mathscr{D} \in \mathsf{DRed}^{p_{max}}_{\mu,\rho}$ and $p' > p_{max} \implies \mathscr{D} \notin \mathsf{DRed}^{p'}_{\mu,\rho}$.*

**Proof.** Let $\mathscr{D} = \left[(V_i)^{p_i} \;\middle|\; i \in \mathcal{I}\right]$ be a finite distribution of values and $\mu = \left\{(\sigma_j)^{p'_j} \;\middle|\; j \in \mathcal{J}\right\}$ be a distribution type. By Lemma 23, setting $q_{ij} = \max\left\{q \;\middle|\; V_i \in \mathsf{VRed}^q_{\sigma_j,\rho}\right\}$ for every $(i,j) \in \mathcal{I}\times\mathcal{J}$, there exists a family $(p_{ij})_{i\in\mathcal{I},j\in\mathcal{J}}$ of reals of $[0,1]$ which maximizes $w = \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} q_{ij}p_{ij}$. It is immediate to see that any increase of a $q_{ij}$ to $q'$ is contradictory with $V_i \in \mathsf{VRed}^{q'}_{\sigma_j,\rho}$, and that any decrease of a $q_{ij}$ actually decreases $w$. It follows that $p_{max} = w$. □

To analyse the $\mathsf{letrec}$ construction, we will prove that, for every $\varepsilon \in (0,1]$, performing enough unfoldings of the fixpoint allows to prove that the recursively-defined term is in a reducibility set parametrised by $1-\varepsilon$. We will be able to conclude on the AST nature of recursive constructions using the following continuity lemma, proved using the theory of linear programming [33]:

**Lemma 24 (Continuity)** *Let $\sigma$ be a sized type, $\mu$ be a distribution type and $\rho$ be a size environment. Let $p \in (0,1]$. Then:*
- $\mathsf{VRed}^p_{\sigma,\rho} = \bigcap_{0<q<p} \mathsf{VRed}^q_{\sigma,\rho}$,
- $\mathsf{DRed}^p_{\mu,\rho} = \bigcap_{0<q<p} \mathsf{DRed}^q_{\mu,\rho}$,
- $\mathsf{TRed}^p_{\mu,\rho} = \bigcap_{0<q<p} \mathsf{TRed}^q_{\mu,\rho}$.

**Proof.** Let $\sigma$ be a sized type, $\mu$ be a distribution type and $\rho$ be a size environment. Let $p \in (0,1]$.

- If $\sigma = \mathsf{Nat}^{\mathfrak{s}}$ for some size $\mathfrak{s}$, then for every $0 < q < p$ we have $\mathsf{VRed}^q_{\sigma,\rho} = \mathsf{VRed}^p_{\sigma,\rho}$ so that $\mathsf{VRed}^p_{\sigma,\rho} = \bigcap_{0<q<p} \mathsf{VRed}^q_{\sigma,\rho}$.
- If $\sigma = \tau \to \mu$, we proceed by mutual inclusions.
  - $\mathsf{VRed}^p_{\sigma,\rho} \subseteq \bigcap_{0<q<p} \mathsf{VRed}^q_{\sigma,\rho}$ is an immediate consequence of Lemma 22.
  - Let us prove now that $\bigcap_{0<q<p} \mathsf{VRed}^q_{\sigma,\rho} \subseteq \mathsf{VRed}^p_{\sigma,\rho}$. Let $V \in \bigcap_{0<q<p} \mathsf{VRed}^q_{\sigma,\rho}$, it follows that:

$$\begin{array}{rl} & \forall q \in (0,p),\ \ \forall q' \in [0,1],\ \ \forall W \in \mathsf{VRed}^{q'}_{\sigma,\rho},\ \ V\ W \in \mathsf{TRed}^{qq'}_{\mu,\rho} \\ \implies & \forall q' \in [0,1],\ \ \forall W \in \mathsf{VRed}^{q'}_{\sigma,\rho},\ \ \forall q \in (0,p),\ \ V\ W \in \mathsf{TRed}^{qq'}_{\mu,\rho} \\ \implies & \forall q' \in [0,1],\ \ \forall W \in \mathsf{VRed}^{q'}_{\sigma,\rho},\ \ V\ W \in \bigcap_{0<q<p} \mathsf{TRed}^{qq'}_{\mu,\rho} \end{array}$$

  But
$$\bigcap_{0<q<p} \mathsf{TRed}^{qq'}_{\mu,\rho} = \bigcap_{0<r<pq'} \mathsf{TRed}^{r}_{\mu,\rho} = \mathsf{TRed}^{pq'}_{\mu,\rho} \qquad \text{(by IH)}$$
  so that
$$\forall q' \in [0,1],\ \ \forall W \in \mathsf{VRed}^{q'}_{\sigma,\rho},\ \ V\ W \in \mathsf{TRed}^{pq'}_{\mu,\rho}$$
  By definition, $V \in \mathsf{VRed}^p_{\sigma,\rho}$.
- The inclusion $\mathsf{DRed}^p_{\mu,\rho} \subseteq \bigcap_{0<q<p} \mathsf{DRed}^q_{\mu,\rho}$ is again an immediate consequence of Lemma 22. Let $\mathscr{D} \in \bigcap_{0<q<p} \mathsf{DRed}^q_{\mu,\rho}$. Let $(q_n)_{n\in\mathbb{N}}$ be an increasing sequence of reals of $[0,p)$ converging to $p$. For every $n \in \mathbb{N}$, $\mathscr{D} \in \mathsf{DRed}^{q_n}_{\mu,\rho}$ so that by Corollary 5 there exists a real $p_{max,n} \in [q_n, 1]$ such that $\mathscr{D} \in \mathsf{DRed}^{p_{max,n}}_{\mu,\rho}$ and $p' > p_{max,n} \implies \mathscr{D} \notin \mathsf{DRed}^{p'}_{\mu,\rho}$. It follows that all the $p_{max,n}$ coincide, and that they are greater than $\sup_{n\in\mathbb{N}} q_n = p$. So $\mathscr{D} \in \mathsf{DRed}^p_{\mu,\rho}$.
- The inclusion $\mathsf{TRed}^p_{\mu,\rho} \subseteq \bigcap_{0<q<p} \mathsf{TRed}^q_{\mu,\rho}$ is again an immediate consequence of Lemma 22. Let $M \in \bigcap_{0<q<p} \mathsf{TRed}^q_{\mu,\rho}$. We need to prove that $M \in \mathsf{TRed}^p_{\mu,\rho}$, that is, that for every $0 \le r < p$ there exists $\nu_r \preccurlyeq \mu$, $n_r \in \mathbb{N}$, $\mathscr{D}_r$ such that $M \Rrightarrow^{n_r}_v \mathscr{D}_r$ and that $\mathscr{D}_r \in \mathsf{DRed}^r_{\nu_r,\rho}$. Let $r \in [0,p)$. Since $M \in \mathsf{TRed}^{\frac{p+r}{2}}_{\mu,\rho}$ and that $\frac{p+r}{2} > r$, we obtain the desired $\nu_r \preccurlyeq \mu$, $n_r \in \mathbb{N}$, $\mathscr{D}_r$ having the properties of interest. The result follows.

□

## 6.5 Reducibility Sets and Sizes

In this subsection, we show how the sizes appearing in the (sized or distribution) type parametrizing a reducibility set relate with the interpretation of size variables contained in the size environment which also parametrizes it. We prove first the following lemma, which will be used as a companion for this result:

**Lemma 25 (Commuting Sizes with Environments)** *Let $\mathfrak{i}$ be a size variable, $\mathfrak{s}$, $\mathfrak{r}$ be two sizes, and $\rho$ be a size environment. Suppose that $\mathfrak{s} = \infty$ or that $\mathrm{spine}(\mathfrak{s}) \neq \mathfrak{i}$. Then $[\![\mathfrak{r}[\mathfrak{s}/\mathfrak{i}]]\!]_\rho = [\![\mathfrak{r}]\!]_{\rho[\mathfrak{i}\mapsto[\![\mathfrak{s}]\!]_\rho]}$.*

**Proof.** By case analysis.

- If $\mathfrak{r} = \widehat{\mathfrak{j}}^n$ for $\mathfrak{j} \neq \mathfrak{i}$, then $\mathfrak{r}[\mathfrak{s}/\mathfrak{i}] = \mathfrak{r}$ and $[\![\mathfrak{r}]\!]_\rho = \rho(\mathfrak{j}) + n = [\![\mathfrak{r}]\!]_{\rho[\mathfrak{i}\mapsto[\![\mathfrak{s}]\!]_\rho]}$.
- If $\mathfrak{r} = \widehat{\mathfrak{i}}^n$, then
  - if $\mathfrak{s} = \widehat{\mathfrak{j}}^m$ for $\mathfrak{j} \neq \mathfrak{i}$, then $\mathfrak{r}[\mathfrak{s}/\mathfrak{i}] = \widehat{\mathfrak{j}}^{n+m}$ and
  $$[\![\mathfrak{r}[\mathfrak{s}/\mathfrak{i}]]\!]_\rho = \rho(\mathfrak{j}) + n + m = [\![\widehat{\mathfrak{j}}^m]\!]_\rho + n = [\![\mathfrak{s}]\!]_\rho + n = [\![\widehat{\mathfrak{i}}^n]\!]_{\rho[\mathfrak{i}\mapsto[\![\mathfrak{s}]\!]_\rho]} = [\![\mathfrak{r}]\!]_{\rho[\mathfrak{i}\mapsto[\![\mathfrak{s}]\!]_\rho]},$$
  - if $\mathfrak{s} = \infty$, then $\mathfrak{r}[\mathfrak{s}/\mathfrak{i}] = \infty$ and $[\![\mathfrak{r}[\mathfrak{s}/\mathfrak{i}]]\!]_\rho = \infty = [\![\mathfrak{r}]\!]_{\rho[\mathfrak{i}\mapsto[\![\mathfrak{s}]\!]_\rho]}$.
- If $\mathfrak{r} = \infty$, then $\mathfrak{r}[\mathfrak{s}/\mathfrak{i}] = \mathfrak{r}$ and $[\![\mathfrak{r}]\!]_\rho = \infty = [\![\mathfrak{r}]\!]_{\rho[\mathfrak{i}\mapsto[\![\mathfrak{s}]\!]_\rho]}$.

□

The last fundamental property about reducibility sets which will be crucial to treat the recursive case is the following, stating that the sizes appearing in a sized type may be recovered in the reducibility set by using an appropriate semantics of the size variables, and conversely:

**Lemma 26 (Size Commutation)** *Let* $\mathfrak{i}$ *be a size variable,* $\mathfrak{s}$ *be a size such that* $\mathfrak{s} = \infty$ *or that* $\mathrm{spine}(\mathfrak{s}) \neq \mathfrak{i}$ *and* $\rho$ *be a size environment. Then:*
- $\mathsf{VRed}^p_{\sigma[\mathfrak{s}/\mathfrak{i}],\rho} = \mathsf{VRed}^p_{\sigma,\rho[\mathfrak{i}\mapsto[\![\mathfrak{s}]\!]_\rho]}$,
- $\mathsf{DRed}^p_{\mu[\mathfrak{s}/\mathfrak{i}],\rho} = \mathsf{DRed}^p_{\mu,\rho[\mathfrak{i}\mapsto[\![\mathfrak{s}]\!]_\rho]}$,
- $\mathsf{TRed}^p_{\mu[\mathfrak{s}/\mathfrak{i}],\rho} = \mathsf{TRed}^p_{\mu,\rho[\mathfrak{i}\mapsto[\![\mathfrak{s}]\!]_\rho]}$.

**Proof.**

- The first case to consider is $\sigma = \mathsf{Nat}^{\mathfrak{r}}$ for some size $\mathfrak{r}$. Using Lemma 25, we have that
$$\begin{array}{rcl}\mathsf{VRed}^p_{(\mathsf{Nat}^{\mathfrak{r}})[\mathfrak{s}/\mathfrak{i}],\rho} & = & \mathsf{VRed}^p_{\mathsf{Nat}^{\mathfrak{r}[\mathfrak{s}/\mathfrak{i}]},\rho} \\ & = & \left\{\mathsf{S}^n 0 \;\middle|\; p > 0 \Longrightarrow n < [\![\mathfrak{r}[\mathfrak{s}/\mathfrak{i}]]\!]_\rho\right\} \\ & = & \left\{\mathsf{S}^n 0 \;\middle|\; p > 0 \Longrightarrow n < [\![\mathfrak{r}]\!]_{\rho[\mathfrak{i}\mapsto[\![\mathfrak{s}]\!]_\rho]}\right\} \\ & = & \mathsf{VRed}^p_{\mathsf{Nat}^{\mathfrak{r}},\rho[\mathfrak{i}\mapsto[\![\mathfrak{s}]\!]_\rho]}\end{array}$$
- We then consider the case of the sized type $\sigma \to \mu :: \kappa' \to \kappa''$. We have
$$\begin{array}{rcl}\mathsf{VRed}^p_{(\sigma\to\mu)[\mathfrak{s}/\mathfrak{i}],\rho} & = & \mathsf{VRed}^p_{\sigma[\mathfrak{s}/\mathfrak{i}]\to\mu[\mathfrak{s}/\mathfrak{i}],\rho} \\ & = & \left\{V \in \Lambda^V_\oplus\left(\langle\sigma[\mathfrak{s}/\mathfrak{i}] \to \mu[\mathfrak{s}/\mathfrak{i}]\rangle\right) \;\middle|\; \forall q \in (0,1],\ \ \forall W \in \mathsf{VRed}^q_{\sigma[\mathfrak{s}/\mathfrak{i}],\rho},\ \ V\ W \in \mathsf{TRed}^{pq}_{\mu[\mathfrak{s}/\mathfrak{i}],\rho}\right\} \\ & = & \left\{V \in \Lambda^V_\oplus\left(\langle\sigma \to \mu\rangle\right) \;\middle|\; \forall q \in (0,1],\ \ \forall W \in \mathsf{VRed}^q_{\sigma,\rho[\mathfrak{i}\mapsto[\![\mathfrak{s}]\!]_\rho]},\ \ V\ W \in \mathsf{TRed}^{pq}_{\mu,\rho[\mathfrak{i}\mapsto[\![\mathfrak{s}]\!]_\rho]}\right\} \\ & = & \mathsf{VRed}^p_{\sigma\to\mu,\rho[\mathfrak{i}\mapsto[\![\mathfrak{s}]\!]_\rho]}\end{array}$$
  where we used the induction hypothesis twice, once on $\kappa'$ and the other time on $\kappa''$.
- Let $\mathscr{D}$ be a finite distribution of values and $\mu = \left\{(\sigma_j)^{p'_j} \;\middle|\; j \in \mathcal{J}\right\}$ be a distribution type. We have that $\mu[\mathfrak{s}/\mathfrak{i}] = \left\{(\sigma_j[\mathfrak{s}/\mathfrak{i}])^{p'_j} \;\middle|\; j \in \mathcal{J}\right\}$. Suppose that $\mathscr{D} \in \mathsf{DRed}^p_{\mu[\mathfrak{s}/\mathfrak{i}],\rho}$. Then there exist a pseudo-representation $\mathscr{D} = \left[(V_i)^{p_i} \;\middle|\; i \in \mathcal{I}\right]$ and families $(p_{ij})_{i\in\mathcal{I},j\in\mathcal{J}}$ and $(q_{ij})_{i\in\mathcal{I},j\in\mathcal{J}}$ of reals of $[0,1]$ satisfying:
  1. $\forall i \in \mathcal{I},\ \ \forall j \in \mathcal{J},\ \ V_i \in \mathsf{VRed}^{q_{ij}}_{\sigma_j[\mathfrak{s}/\mathfrak{i}],\rho}$,
  2. $\forall i \in \mathcal{I},\ \ \sum_{j\in\mathcal{J}} p_{ij} = p_i$,
  3. $\forall j \in \mathcal{J},\ \ \sum_{i\in\mathcal{I}} p_{ij} = \mu(\sigma_j)$,
  4. $p \leq \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} q_{ij}p_{ij}$.

But (1) is equivalent to $\forall i \in \mathcal{I},\ \ \forall j \in \mathcal{J},\ \ V_i \in \mathsf{VRed}^{q_{ij}}_{\sigma_j,\rho[\mathtt{i}\mapsto[\![\mathtt{s}]\!]_\rho]}$ by induction hypothesis. It follows that $\mathscr{D} \in \mathsf{DRed}^{p}_{\mu,\rho[\mathtt{i}\mapsto[\![\mathtt{s}]\!]_\rho]}$. The converse direction proceeds in the exact same way.

- Then, $M \in \mathsf{TRed}^{p}_{\mu[\mathtt{s}/\mathtt{i}],\rho}$ if and only if

$$M \in \Lambda_\oplus(\langle\mu\rangle) \text{ and } \forall 0 \leq r < p,\ \ \exists \nu_r \preccurlyeq \mu,\ \ \exists n_r \in \mathbb{N},\ \ M \Rightarrow_v^{n_r} \mathscr{D}_r \text{ and } \mathscr{D}_r \in \mathsf{DRed}^{r}_{\nu_r[\mathtt{s}/\mathtt{i}],\rho}$$

if and only if, by induction hypothesis,

$$M \in \Lambda_\oplus(\langle\mu\rangle) \text{ and } \forall 0 \leq r < p,\ \ \exists \nu_r \preccurlyeq \mu,\ \ \exists n_r \in \mathbb{N},\ \ M \Rightarrow_v^{n_r} \mathscr{D}_r \text{ and } \mathscr{D}_r \in \mathsf{DRed}^{r}_{\nu_r,\rho[\mathtt{i}\mapsto[\![\mathtt{s}]\!]_\rho]}$$

that is, if and only if $M \in \mathsf{TRed}^{p}_{\mu,\rho[\mathtt{i}\mapsto[\![\mathtt{s}]\!]_\rho]}$.

□

## 6.6 Reducibility Sets are Stable by Unfoldings

The most difficult step in proving all typable terms to be reducible is, unexpectedly, proving that terms involving *recursion* are reducible whenever their respective *unfoldings* are. This very natural concept expresses simply that any term in the form $\mathsf{letrec}\ f\ =\ W$ is assumed to compute the fixpoint of the function defined by $W$.

**Definition 25 ($n$-Unfolding)** *Suppose that $V = (\mathsf{letrec}\ f\ =\ W)$ is closed, then the $n$-unfolding of $V$ is:*
*– $V$ if $n = 0$;*
*– $W[Z/f]$ if $n = m + 1$ and $Z$ is the $m$-unfolding of $V$.*
*We write the set of unfoldings of $V$ as Unfold $(V)$. Note that if $V$ admits a simple type, then all its unfoldings have this same simple type as well. In the sequel, we implicitly consider that $V$ is simply typed.*

Any unfolding of $V = (\mathsf{letrec}\ f\ =\ W)$ should behave like $V$ itself: all unfoldings of $V$ should be equivalent. This, however, cannot be proved using simply the operational semantics. It requires some work, and techniques akin to logical relations, to prove (see [33]) this behavioural equivalence between a recursive definition and its unfoldings. The first lemma is technical and lists the unfoldings of terms defined recursively as equal to themselves or to a variable:

**Lemma 27**

- *Let $V = f$ and $W \in$ Unfold $(\mathsf{letrec}\ f\ =\ V)$. Then $W = \mathsf{letrec}\ f\ =\ V$.*
- *Let $V = x \neq f$ and $W \in$ Unfold $(\mathsf{letrec}\ f\ =\ V)$. Then $W = \mathsf{letrec}\ f\ =\ V$ or $W = x$. More precisely, the $n$-unfoldings for $n \geq 1$ are all $x$.*

The next lemma is the technical core of this section. Think of two terms as related when they are of the shape $M[\overrightarrow{Z}/\overrightarrow{x}]$ and $M[\overrightarrow{Z'}/\overrightarrow{x}]$, where $\overrightarrow{x}$ is a sequence of "holes" in $M$, filled with unfoldings from a *same* recursively-defined term. Then their rewritings by $\rightarrow_v$ form distributions of pairwise related terms.

**Lemma 28** *Let $V = (\mathsf{letrec}\ f\ =\ W)$ be a closed value. Let $\overrightarrow{x}$, $\overrightarrow{Z}$, $\overrightarrow{Z'}$ be a vector of variables and two vectors of terms of Unfold $(V)$, all of the same length. Let $M$ be a simply-typed term of free variables contained in $\overrightarrow{x}$, all typed with the simple type of $V$. Suppose that $M[\overrightarrow{Z}/\overrightarrow{x}] \rightarrow_v \mathscr{D}$. Then there exists $N_1, \ldots, N_n$, a vector of variables $\overrightarrow{y}$ and $\overrightarrow{Z_1}, \ldots, \overrightarrow{Z_n}, \overrightarrow{Z'_1}, \ldots, \overrightarrow{Z'_n} \in$ Unfold $(V)$ of the same length as $\overrightarrow{y}$ and such that $\mathscr{D} = \left\{ \left(N_i[\overrightarrow{Z_i}/\overrightarrow{y}]\right)^{p_i} \right\}$ and moreover $M[\overrightarrow{Z'}/\overrightarrow{x}] \rightarrow_v \mathscr{E} = \left\{ \left(N_i[\overrightarrow{Z'_i}/\overrightarrow{y}]\right)^{p_i} \right\}$.*

**Proof.** We prove the result by induction on the structure of $M$.

- The case where $M$ is a variable cannot fit in this setting: either $M = y \notin \overrightarrow{x}$ and there is no reduction from $M[\overrightarrow{Z}/\overrightarrow{x}]$, or $M = x_i \in \overrightarrow{x}$ and there is no reduction either from $M[\overrightarrow{Z}/\overrightarrow{x}] = Z_i$ since it is a value. We can similarly rule out all the cases where $M$ is a value.
- Suppose that $M = V_1\ V_2$. We proceed by case exhaustion on $V_1$. Three possibilities exist, the other ones contradicting the fact that there should be a reduction step from $M$:
  - If $V_1 = x_i \in \overrightarrow{x}$, we distinguish four cases:
    - Suppose that $Z_i = Z'_i$ are both the 0-unfolding of $V$. Then $M[\overrightarrow{Z}/\overrightarrow{x}] = M[\overrightarrow{Z'}/\overrightarrow{x}]$ and the result follows immediately from:

$$\begin{array}{rcl} M[\overrightarrow{Z}/\overrightarrow{x}] & = & (\mathsf{letrec}\ f\ =\ W)\ V_2[\overrightarrow{Z}/\overrightarrow{x}] \\ & = & (\mathsf{letrec}\ f\ =\ W)\ (\mathsf{S}^m\ 0) \\ & \to_v & \left\{ ((W[\mathsf{letrec}\ f\ =\ W/f])\ (\mathsf{S}^m\ 0))^1 \right\} \end{array}$$

      where in the second line the shape of $V_2$ needs to be $\mathsf{S}^m\ 0$ by typing constraints. Note that $\overrightarrow{y}$ is the empty vector here.
    - Suppose that $Z_i$ is the $n$-unfolding of $V$ for $n > 0$, and that $Z'_i$ is the 0-unfolding. We have that

$$M[\overrightarrow{Z}/\overrightarrow{x}] \ = \ W[Z''/f]\ \ V_2[\overrightarrow{Z}/\overrightarrow{x}]$$

      where $Z''$ is the $(n-1)$-unfolding of $V$, and that

$$\begin{array}{rcl} M[\overrightarrow{Z'}/\overrightarrow{x}] & = & (\mathsf{letrec}\ f\ =\ W)\ V_2[\overrightarrow{Z'}/\overrightarrow{x}] \\ & \to_v & \left\{ \left( (W[\mathsf{letrec}\ f\ =\ W/f])\ V_2[\overrightarrow{Z'}/\overrightarrow{x}] \right)^1 \right\} \end{array}$$

      Notice that this reduction is possible since the constraint of simple typing implies that $V_2$ is of the shape $\mathsf{S}^m\ 0$ for some $m \geq 0$. We can therefore rewrite the two terms as

$$M[\overrightarrow{Z}/\overrightarrow{x}] \ = \ W[Z''/f]\ \ (\mathsf{S}^m\ 0)$$

      and

$$M[\overrightarrow{Z'}/\overrightarrow{x}] \quad \to_v \quad \left\{ ((W[\mathsf{letrec}\ f\ =\ W/f])\ (\mathsf{S}^m\ 0))^1 \right\}$$

      We need to distinguish four cases, depending on the structure of $W$.
      - Suppose that $W$ is a variable different from $f$. Then by Lemma 27 there cannot be a step of reduction from $M[\overrightarrow{Z}/\overrightarrow{x}]$.
      - Suppose that $W = f$. Then by Lemma 27 we have $Z_i = Z'_i = Z''$, so that $M[\overrightarrow{Z}/\overrightarrow{x}] = M[\overrightarrow{Z'}/\overrightarrow{x}]$ and the result follows just as for the case where both $Z$ and $Z'$ were 0-unfoldings.
      - Suppose that $W = \lambda y.L$. Then

$$\begin{array}{rcl} M[\overrightarrow{Z}/\overrightarrow{x}] & = & (\lambda y.L[Z''/f])\ (\mathsf{S}^m\ 0) \\ & \to_v & \left\{ (L[Z''/f][\mathsf{S}^m\ 0/y])^1 \right\} \\ & = & \left\{ ((L[\mathsf{S}^m\ 0/y])\,[Z''/f])^1 \right\} \end{array}$$

        Moreover,

$$\begin{array}{rcl} M[\overrightarrow{Z'}/\overrightarrow{x}] & \to_v & \left\{ ((W[\mathsf{letrec}\ f\ =\ W/f])\ (\mathsf{S}^m\ 0))^1 \right\} \\ & = & \left\{ (((\lambda y.L)\ (\mathsf{S}^m\ 0))[\mathsf{letrec}\ f\ =\ W/f])^1 \right\} \\ & \to_v & \left\{ ((L[(\mathsf{S}^m\ 0)/y])\,[\mathsf{letrec}\ f\ =\ W/f])^1 \right\} \end{array}$$

        so that we can conclude with $\overrightarrow{y} = f$ and $N_1 = L[(\mathsf{S}^m\ 0)/y]$.

· Suppose that $W = \mathsf{letrec}\ g = W'$. Then

$$\begin{array}{rcl} M[\overrightarrow{Z}/\overrightarrow{x}] & = & ((\mathsf{letrec}\ g = W')[Z''/f])\ (\mathsf{S}^m\ 0) \\ & \to_v & \left\{ (W'[\mathsf{letrec}\ g = W'/g])[Z''/f]\ (\mathsf{S}^m\ 0))^1 \right\} \\ & = & \left\{ (W'[\mathsf{letrec}\ g = W'/g]\ (\mathsf{S}^m\ 0))[Z''/f])^1 \right\} \end{array}$$

Moreover,

$$\begin{array}{rcl} M[\overrightarrow{Z'}/\overrightarrow{x}] & \to_v & ((\mathsf{letrec}\ g = W')[\mathsf{letrec}\ f = W/f])\ (\mathsf{S}^m\ 0) \\ & \to_v & \left\{ (W'[\mathsf{letrec}\ g = W'/g])[\mathsf{letrec}\ f = W/f]\ (\mathsf{S}^m\ 0))^1 \right\} \\ & = & \left\{ (W'[\mathsf{letrec}\ g = W'/g]\ (\mathsf{S}^m\ 0))[\mathsf{letrec}\ f = W/f])^1 \right\} \end{array}$$

so that we can conclude with $\overrightarrow{y} = f$ and $N_1 = W'[\mathsf{letrec}\ g = W'/g]\ (\mathsf{S}^m\ 0))$.

∗ Suppose that $Z_i$ is the 0-unfolding of $V$, and that $Z'_i$ is the $n$-unfolding for $n > 0$. Again, the constraint of simple typing implies that $V_2$ is of the shape $\mathsf{S}^m\ 0$ for some $m \geq 0$. We have that

$$\begin{array}{rcl} M[\overrightarrow{Z}/\overrightarrow{x}] & \to_v & \left\{ ((W[\mathsf{letrec}\ f = W/f])\ (\mathsf{S}^m\ 0))^1 \right\} \\ & = & \left\{ ((W\ (\mathsf{S}^m\ 0))[\mathsf{letrec}\ f = W/f]))^1 \right\} \end{array}$$

and that

$$M[\overrightarrow{Z'}/\overrightarrow{x}] = W[Z''/f]\ (\mathsf{S}^m\ 0) = (W\ (\mathsf{S}^m\ 0))[Z''/f]$$

where $Z''$ is the $(n-1)$-unfolding of $V$, so that we can conclude with $\overrightarrow{y} = f$ and $N_1 = W\ (\mathsf{S}^m\ 0)$.

∗ Suppose that $Z_i$ is the $n$-unfolding of $V$ for $n > 0$, and that $Z'_i$ is the $n'$-unfolding for $n' > 0$. We have

$$M[\overrightarrow{Z}/\overrightarrow{x}] = W[Z''/f]\ V_2[\overrightarrow{Z}/\overrightarrow{x}]$$

where $Z''$ is the $(n-1)$-unfolding of $V$, and

$$M[\overrightarrow{Z}/\overrightarrow{x}] = W[Z'''/f]\ V_2[\overrightarrow{Z}/\overrightarrow{x}]$$

where $Z''$ is the $(n'-1)$-unfolding of $V$. We proceed by case analysis on $W$. As we discussed in the case where $Z_i$ was a $(n''+1)$-unfolding and $Z'_i$ a 0-unfolding, the case where $W$ is a variable does not lead to a rewriting step. It remains to treat two cases:

· Suppose that $W = \lambda y.L$. Then

$$\begin{array}{rcl} M[\overrightarrow{Z}/\overrightarrow{x}] & = & \lambda y.L[Z''/f]\ V_2[\overrightarrow{Z}/\overrightarrow{x}] \\ & \to_v & \left\{ \left(L[Z''/f][V_2[\overrightarrow{Z}/\overrightarrow{x}]/y]\right)^1 \right\} \\ & = & \left\{ \left(\left(L[V_2[\overrightarrow{Z}/\overrightarrow{x}]/y]\right)[Z''/f]\right)^1 \right\} \end{array}$$

and

$$\begin{array}{rcl} M[\overrightarrow{Z'}/\overrightarrow{x}] & = & \lambda y.L[Z'''/f]\ V_2[\overrightarrow{Z}/\overrightarrow{x}] \\ & \to_v & \left\{ \left(L[Z'''/f][V_2[\overrightarrow{Z}/\overrightarrow{x}]/y]\right)^1 \right\} \\ & = & \left\{ \left(\left(L[V_2[\overrightarrow{Z}/\overrightarrow{x}]/y]\right)[Z'''/f]\right)^1 \right\} \end{array}$$

so that we can conclude with $\overrightarrow{y} = f$ and $N_1 = L[V_2[\overrightarrow{Z}/\overrightarrow{x}]/y]$.

· Suppose that $W = \mathsf{letrec}\ g = W'$. Then

$$\begin{array}{rcl} M[\overrightarrow{Z}/\overrightarrow{x}] & = & ((\mathsf{letrec}\ g = W')[Z''/f])\ V_2[\overrightarrow{Z}/\overrightarrow{x}] \\ & \to_v & \left\{ \left(W'[\mathsf{letrec}\ g = W'/g])[Z''/f]\ V_2[\overrightarrow{Z}/\overrightarrow{x}]\right)^1 \right\} \\ & = & \left\{ \left(W'[\mathsf{letrec}\ g = W'/g]\ V_2[\overrightarrow{Z}/\overrightarrow{x}])[Z''/f]\right)^1 \right\} \end{array}$$

where the reduction is possible because the simple typing constraints imply that $V_2[\overrightarrow{Z}/\overrightarrow{x}]$ is of the shape $\mathsf{S}^m\ 0$ for some $m \in \mathbb{N}$. Moreover,

$$\begin{array}{rcl} M[\overrightarrow{Z'}/\overrightarrow{x}] & = & ((\mathsf{letrec}\ g = W')[Z'''/f])\ V_2[\overrightarrow{Z}/\overrightarrow{x}] \\ & \to_v & \left\{ \left((W'[\mathsf{letrec}\ g = W'/g])[Z'''/f]\ V_2[\overrightarrow{Z}/\overrightarrow{x}]\right)^1 \right\} \\ & = & \left\{ \left((W'[\mathsf{letrec}\ g = W'/g]\ V_2[\overrightarrow{Z}/\overrightarrow{x}])[Z'''/f]\right)^1 \right\} \end{array}$$

so that we can conclude with $\overrightarrow{y} = f$ and $N_1 = W'[\mathsf{letrec}\ g = W'/g]\ V_2[\overrightarrow{Z}/\overrightarrow{x}]$.

– If $V_1 = \lambda y.L$,

$$\begin{array}{rcl} M[\overrightarrow{Z}/\overrightarrow{x}] & = & \left(\lambda y.L[\overrightarrow{Z}/\overrightarrow{x}]\right)\ V_2[\overrightarrow{Z}/\overrightarrow{x}] \\ & \to_v & \left\{ \left(L[\overrightarrow{Z}/\overrightarrow{x}][V_2[\overrightarrow{Z}/\overrightarrow{x}]/y]\right)^1 \right\} \\ & = & \left\{ \left(L[V_2/y][\overrightarrow{Z}/\overrightarrow{x}]\right)^1 \right\} \end{array}$$

and in the same way

$$M[\overrightarrow{Z'}/\overrightarrow{x}] \quad \to_v \quad \left\{ \left(L[V_2/y][\overrightarrow{Z'}/\overrightarrow{x}]\right)^1 \right\}$$

which allows to conclude with $N_1 = L[V_2/y]$.

– If $V_1 = \mathsf{letrec}\ g = W'$, by typing constraints $V_2 = \mathsf{S}^m\ 0$ for some $m \geq 0$. It follows that we can reduce $M[\overrightarrow{Z}/\overrightarrow{x}]$ and $M[\overrightarrow{Z'}/\overrightarrow{x}]$ as follows:

$$\begin{array}{rcl} M[\overrightarrow{Z}/\overrightarrow{x}] & = & \left(\mathsf{letrec}\ g = W'[\overrightarrow{Z}/\overrightarrow{x}]\right)\ \mathsf{S}^m\ 0 \\ & \to_v & \left\{ \left(\left(W'[\overrightarrow{Z}/\overrightarrow{x}][\mathsf{letrec}\ g = W'[\overrightarrow{Z}/\overrightarrow{x}]/g]\right)\ (\mathsf{S}^m\ 0)\right)^1 \right\} \\ & = & \left\{ \left((W'[\mathsf{letrec}\ g = W'/g])\,[\overrightarrow{Z}/\overrightarrow{x}]\ (\mathsf{S}^m\ 0)\right)^1 \right\} \\ & = & \left\{ \left((W'[\mathsf{letrec}\ g = W'/g]\ (\mathsf{S}^m\ 0))\,[\overrightarrow{Z}/\overrightarrow{x}]\right)^1 \right\} \end{array}$$

and similarly

$$M[\overrightarrow{Z'}/\overrightarrow{x}] \quad \to_v \quad \left\{ \left((W'[\mathsf{letrec}\ g = W'/g]\ (\mathsf{S}^m\ 0))\,[\overrightarrow{Z'}/\overrightarrow{x}]\right)^1 \right\}$$

so that we can conclude with $N_1 = W'[\mathsf{letrec}\ g = W'/g]\ (\mathsf{S}^m\ 0)$.

• Suppose that $M = \mathsf{let}\ y = X\ \mathsf{in}\ P$. Then

$$\begin{array}{rcl} M[\overrightarrow{Z}/\overrightarrow{x}] & = & \mathsf{let}\ y = X[\overrightarrow{Z}/\overrightarrow{x}]\ \mathsf{in}\ P[\overrightarrow{Z}/\overrightarrow{x}] \\ & \to_v & \left\{ \left(P[\overrightarrow{Z}/\overrightarrow{x}][X[\overrightarrow{Z}/\overrightarrow{x}]/y]\right)^1 \right\} \\ & = & \left\{ \left(P[X/y][\overrightarrow{Z}/\overrightarrow{x}]\right)^1 \right\} \end{array}$$

and similarly $M[\overrightarrow{Z'}/\overrightarrow{x}] \to_v \left\{ \left(P[X/y][\overrightarrow{Z'}/\overrightarrow{x}]\right)^1 \right\}$ from which we can conclude.

- Suppose that $M = \mathsf{let}\ y = L\ \mathsf{in}\ P$ and that

$$\begin{array}{rcl} M[\overrightarrow{Z}/\overrightarrow{x}] & = & \mathsf{let}\ y = L[\overrightarrow{Z}/\overrightarrow{x}]\ \mathsf{in}\ P[\overrightarrow{Z}/\overrightarrow{x}] \\ & \to_v & \left\{ \left(\mathsf{let}\ y = L'_i[\overrightarrow{Z}/\overrightarrow{x}]\ \mathsf{in}\ P[\overrightarrow{Z}/\overrightarrow{x}]\right)^{p_i} \right\} \\ & = & \left\{ \left(\mathsf{let}\ y = L''_i[\overrightarrow{Z}/\overrightarrow{z}]\ \mathsf{in}\ P[\overrightarrow{Z}/\overrightarrow{x}]\right)^{p_i} \right\} \\ & = & \left\{ \left((\mathsf{let}\ y = L''_i\ \mathsf{in}\ P)\,[\overrightarrow{Z},\overrightarrow{Z}/\overrightarrow{z},\overrightarrow{x}]\right)^{p_i} \right\} \end{array}$$

where the third step is obtained by $\alpha$-renaming, and where by definition of $\to_v$ we have

$$L[\overrightarrow{Z}/\overrightarrow{x}] \to_v \left\{ (L'_i[\overrightarrow{Z}/\overrightarrow{x}])^{p_i} \mid i \in \mathcal{I} \right\}.$$

By induction hypothesis, there exists $\overrightarrow{Z'_1}, \ldots, \overrightarrow{Z'_n} \in \mathit{Unfold}\,(V)$ such that

$$L[\overrightarrow{Z'}/\overrightarrow{x}] \to_v \left\{ (L'_i[\overrightarrow{Z'_i}/\overrightarrow{x}])^{p_i} \mid i \in \mathcal{I} \right\}.$$

Now remark that

$$\begin{array}{rcl} M[\overrightarrow{Z'}/\overrightarrow{x}] & \to_v & \left\{ \left(\mathsf{let}\ y = L'_i[\overrightarrow{Z'_i}/\overrightarrow{x}]\ \mathsf{in}\ P[\overrightarrow{Z}/\overrightarrow{x}]\right)^{p_i} \right\} \\ & = & \left\{ \left(\mathsf{let}\ y = L''_i[\overrightarrow{Z'_i}/\overrightarrow{z}]\ \mathsf{in}\ P[\overrightarrow{Z}/\overrightarrow{x}]\right)^{p_i} \right\} \\ & = & \left\{ \left((\mathsf{let}\ y = L''_i\ \mathsf{in}\ P)\,[\overrightarrow{Z},\overrightarrow{Z'_i}/\overrightarrow{x},\overrightarrow{z}]\right)^{p_i} \right\} \end{array}$$

The result follows for $\overrightarrow{y} = \overrightarrow{x}, \overrightarrow{z}$ and $N_i = \mathsf{let}\ y = L''_i\ \mathsf{in}\ P$.

- Suppose that $M = L \oplus_p P$. Suppose that $L \neq P$. Then

$$M[\overrightarrow{Z}/\overrightarrow{x}] \to_v \left\{ L[\overrightarrow{Z}/\overrightarrow{x}]^p,\ P[\overrightarrow{Z}/\overrightarrow{x}]^{1-p} \right\}$$

and

$$M[\overrightarrow{Z'}/\overrightarrow{x}] \to_v \left\{ L[\overrightarrow{Z'}/\overrightarrow{x}]^p,\ P[\overrightarrow{Z'}/\overrightarrow{x}]^{1-p} \right\}$$

so that the result holds for $N_1 = L$ and $N_2 = P$.
If $L = P$,

$$M[\overrightarrow{Z}/\overrightarrow{x}] \to_v \left\{ L[\overrightarrow{Z}/\overrightarrow{x}]^1 \right\}$$

and

$$M[\overrightarrow{Z'}/\overrightarrow{x}] \to_v \left\{ L[\overrightarrow{Z'}/\overrightarrow{x}]^1 \right\}$$

and the result holds as well. Note that the distinction is necessary so as to avoid the use of pseudo-representations in the statement of the lemma.

- Suppose that $M = \mathsf{case}\ V'\ \mathsf{of}\ \left\{ \mathsf{S} \to X \mid 0 \to Y \right\}$. By typing constraints, $V' = \mathsf{S}^m\ 0$ or $V' = y$ is a variable.
  - If $V' = 0$, $M[\overrightarrow{Z}/\overrightarrow{x}] \to_v \left\{ \left(R[\overrightarrow{Z}/\overrightarrow{x}]\right)^1 \right\}$ and $M[\overrightarrow{Z'}/\overrightarrow{x}] \to_v \left\{ \left(R[\overrightarrow{Z'}/\overrightarrow{x}]\right)^1 \right\}$ so that we can conclude.
  - If $V' = \mathsf{S}^m\ 0$ with $m > 0$, we can conclude in the same way.
  - In the latter case, there is no reduction from $M[\overrightarrow{Z}/\overrightarrow{x}]$ unless $V'[\overrightarrow{Z}/\overrightarrow{x}]$ is of the shape $V' = \mathsf{S}^m\ 0$. But this is of type $\mathsf{Nat}$ and cannot therefore be an unfolding of $V$, so that this case is impossible.

□

This result can be extended to a $n$-step rewriting process; however pseudo-representations are required to keep the statement true, as we explain in the proof.

**Lemma 29** *Let $V = (\mathsf{letrec}\ f = W)$ be a closed value. Let $M$ be a simply-typed term of free variables contained in $\overrightarrow{x}$, all typed with the simple type of $V$. Let $\overrightarrow{Z}$, $\overrightarrow{Z'} \in$ Unfold $(V)$ and $n \in \mathbb{N}$. Then there exists a distribution of values of pseudo-representation $\left[ X_i^{p_i} \ \middle|\ i \in \mathcal{I} \right]$, a vector of variables $\overrightarrow{y}$ and families of vectors $\left(\overrightarrow{Z_i}\right)_{i\in\mathcal{I}}, \left(\overrightarrow{Z'_i}\right)_{i\in\mathcal{I}}$ of the same length as $\overrightarrow{y}$, all such that $M[\overrightarrow{Z}/\overrightarrow{x}] \Rrightarrow_v^n \left[ \left(X_i[\overrightarrow{Z_i}/\overrightarrow{y}]\right)^{p_i} \ \middle|\ i \in \mathcal{I} \right]$ and that $M[\overrightarrow{Z'}/\overrightarrow{x}] \Rrightarrow_v^n \left[ \left(X_i[\overrightarrow{Z'_i}/\overrightarrow{y}]\right)^{p_i} \ \middle|\ i \in \mathcal{I} \right]$.*

**Proof.** By iteration of Lemma 28. The pseudo-representations come from the fact that some terms in different reduction branches may converge to the same value, say, in the reduction from $M[\overrightarrow{Z}/\overrightarrow{x}]$ but not in the one from $M[\overrightarrow{Z'}/\overrightarrow{x}]$. □

The following lemma is of technical interest. It states that, given two pseudo-representations of a distribution – one of the shape exhibited in the previous lemmas and used for relating terms with unfoldings, the other one being a pseudo-representation witnessing the belonging to a set $\mathsf{DRed}$ – there exists a third one which "combines" both:

**Lemma 30** *Suppose that $\mathscr{D}_r = \left[ \left(X_i[\overrightarrow{Z_i}/\overrightarrow{y}]\right)^{p_i} \ \middle|\ i \in \mathcal{I} \right] = \left[ \left(X'_j\right)^{p'_j} \ \middle|\ j \in \mathcal{J} \right]$. Then there exists a set $\mathcal{K}$, two applications $\pi_1 : \mathcal{K} \to \mathcal{I}$ and $\pi_2 : \mathcal{K} \to \mathcal{J}$ and a pseudo-representation $\mathscr{D}_r = \left[ \left(X''_k[\overrightarrow{Z_{\pi_1(k)}}/\overrightarrow{y}]\right)^{p''_k} \ \middle|\ k \in \mathcal{K} \right]$ such that*

- $\forall k \in \mathcal{K},\ \ X''_k = X_{\pi_1(k)}$,
- $\forall i \in \mathcal{I},\ \ \sum_{k\in\pi_1^{-1}(i)} p''_k = p_i$,
- $\forall k \in \mathcal{K},\ \ X''_k[\overrightarrow{Z''_k}/\overrightarrow{y}] = X'_{\pi_2(k)}$,
- $\forall j \in \mathcal{J},\ \ \sum_{k\in\pi_2^{-1}(j)} p''_k = p'_j$.

**Proof.** Let $\mathscr{D} = \left\{ (Y_l)^{p''_l} \ \middle|\ l \in \mathcal{L} \right\}$ be the *representation* of $\mathscr{D}$. We build $\mathcal{K}$, $\pi_1$ and $\pi_2$ as follows. The construction starts from the empty set and the empty maps, and is iterated on every $l \in \mathcal{L}$. First, we set $\mathcal{I}_l = \left\{ i \in \mathcal{I} \ \middle|\ Y_l = X_i[\overrightarrow{Z_i}/\overrightarrow{y}] \right\}$ and $\mathcal{J}_l = \left\{ j \in \mathcal{J} \ \middle|\ Y_l = X'_j \right\}$. We suppose that both these sets are enumerated, and will write them $\mathcal{I}_l = \{i_0, \ldots, i_{n_l}\}$ and $\mathcal{J}_l = \{j_0, \ldots, j_{m_l}\}$. We consider the set of reals

$$R = \left\{ 0,\, p_{i_1},\, p_{i_1} + p_{i_2},\, \ldots,\, \sum_{r=0}^{n_l} p_{i_r} \right\} \cup \left\{ 0,\, p'_{j_1},\, p'_{j_1} + p'_{j_2},\, \ldots,\, \sum_{r'=0}^{m_l} p'_{i_{r'}} \right\} \subset [0, p''_l]$$

This set is ordered, as a set of reals, so that we have a maximal enumeration

$$0 = \alpha_0 < \alpha_1 < \cdots < \alpha_s = p$$

where maximality means that $\beta \in R \implies \exists t,\ \ \beta = \alpha_t$. We add $s$ elements to the set $\mathcal{K}$ produced during the examination of previous elements of $\mathcal{L}$: $\mathcal{K} := \mathcal{K} \uplus \{0, \ldots, s-1\}$. For every $t \in \{0, \ldots, s-1\}$, we define:

- $p''_t = \alpha_{t+1} - \alpha_t$,
- $\pi_1(t) = i_k \in \mathcal{I}_l$ where $\sum_{r=0}^{k-1} p_{i_r} \leq \alpha_t$ and $\sum_{r=0}^{k} p_{i_r} \geq \alpha_t$,

- $\pi_2(t) \;=\; j_k \in \mathcal{J}_l$ where $\sum_{r=0}^{k-1}\; p_{j_r} \leq \alpha_t$ and $\sum_{r=0}^{k}\; p_{j_r} \geq \alpha_t$.

We claim that the set $\mathcal{K}$ resulting of this constructive process satisfies the equalities of the lemma. $\square$

The series of previous lemmas allows to deduce that a term is reducible if and only if the terms to which it is related are:

**Lemma 31** *Let $V \;=\; (\mathsf{letrec}\; f \;=\; W)$ be a closed value. Let $M$ be a simply-typed term of free variables contained in $\overrightarrow{x}$, all typed with the simple type of $V$. Let $\overrightarrow{Z}, \overrightarrow{Z'} \in$ Unfold $(V)$. Then $M[\overrightarrow{Z}/\overrightarrow{x}] \in \mathsf{TRed}^p_{\mu,\rho}$ if and only if $M[\overrightarrow{Z'}/\overrightarrow{x}] \in \mathsf{TRed}^p_{\mu,\rho}$.*

**Proof.** We prove that $M[\overrightarrow{Z}/\overrightarrow{x}] \in \mathsf{TRed}^p_{\mu,\rho}$ implies that $M[\overrightarrow{Z'}/\overrightarrow{x}] \in \mathsf{TRed}^p_{\mu,\rho}$, the converse direction being exactly symmetrical. The proof proceeds by induction on the simple type refining $\mu$.

Suppose that $\mu \;::\; \mathsf{Nat}$. Let $r \in [0,p)$. Since $M[\overrightarrow{Z}/\overrightarrow{x}] \in \mathsf{TRed}^p_{\mu,\rho}$, there exists $n_r$ and $\nu_r \preccurlyeq \mu$ such that $M[\overrightarrow{Z}/\overrightarrow{x}] \Rrightarrow^{n_r}_v \mathscr{D}_r$ and that $\mathscr{D}_r \in \mathsf{DRed}^r_{\nu_r,\rho}$. Lemma 29 implies that there exists a distribution of values of pseudo-representation $\left[\, X_i^{p_i} \;\middle|\; i \in \mathcal{I} \,\right]$, a vector of variables $\overrightarrow{y}$ and families of vectors $\left(\overrightarrow{Z_i}\right)_{i\in\mathcal{I}}, \left(\overrightarrow{Z'_i}\right)_{i\in\mathcal{I}}$ of the same length as $\overrightarrow{y}$ all such that $\mathscr{D}_r \;=\; \left[\, \left(X_i[\overrightarrow{Z_i}/\overrightarrow{y}]\right)^{p_i} \;\middle|\; i \in \mathcal{I} \,\right]$ and that $M[\overrightarrow{Z'}/\overrightarrow{x}] \Rrightarrow^n_v \mathscr{E}_r \;=\; \left[\, \left(X_i[\overrightarrow{Z'_i}/\overrightarrow{y}]\right)^{p_i} \;\middle|\; i \in \mathcal{I} \,\right]$. By typing constraints coming from the subject reduction property, all the $X_i[\overrightarrow{Z_i}/\overrightarrow{y}]$ and $X_i[\overrightarrow{Z'_i}/\overrightarrow{y}]$ have the simple type $\mathsf{Nat}$. This implies that all these terms are of the shape $\mathsf{S}^m\;\mathsf{0}$ for $m \geq 0$, and thus that the $X_i$ cannot contain a variable from $\overrightarrow{y}$, as their simple type is of the shape $\mathsf{Nat} \to \kappa$. It follows that, for every index $i \in \mathcal{I}$, $X_i[\overrightarrow{Z_i}/\overrightarrow{y}] \;=\; X_i[\overrightarrow{Z'_i}/\overrightarrow{y}]$. This implies that $\mathscr{E}_r \;=\; \mathscr{D}_r \in \mathsf{DRed}^r_{\nu_r,\rho}$, and thus that $M[\overrightarrow{Z'}/\overrightarrow{x}] \in \mathsf{TRed}^p_{\mu,\rho}$.

Suppose that $\mu \;::\; \kappa \to \kappa'$. Let $r \in [0,p)$. Since $M[\overrightarrow{Z}/\overrightarrow{x}] \in \mathsf{TRed}^p_{\mu,\rho}$, there exists $n_r$ and $\nu_r \preccurlyeq \mu$ such that $M[\overrightarrow{Z}/\overrightarrow{x}] \Rrightarrow^{n_r}_v \mathscr{D}_r$ and that $\mathscr{D}_r \in \mathsf{DRed}^r_{\nu_r,\rho}$. Lemma 29 implies that there exists a distribution of values of pseudo-representation $\left[\, X_i^{p_i} \;\middle|\; i \in \mathcal{I} \,\right]$, a vector of variables $\overrightarrow{y}$ and families of vectors $\left(\overrightarrow{Z_i}\right)_{i\in\mathcal{I}}, \left(\overrightarrow{Z'_i}\right)_{i\in\mathcal{I}}$ of the same length as $\overrightarrow{y}$ all such that $\mathscr{D}_r \;=\; \left[\, \left(X_i[\overrightarrow{Z_i}/\overrightarrow{y}]\right)^{p_i} \;\middle|\; i \in \mathcal{I} \,\right]$ and that $M[\overrightarrow{Z'}/\overrightarrow{x}] \Rrightarrow^n_v \mathscr{E}_r \;=\; \left[\, \left(X_i[\overrightarrow{Z'_i}/\overrightarrow{y}]\right)^{p_i} \;\middle|\; i \in \mathcal{I} \,\right]$. Since $\mathscr{D}_r \in \mathsf{DRed}^r_{\nu_r,\rho}$, there is a pseudo-representation $\mathscr{D}_r \;=\; \left[\, (Z'_j)^{p'_j} \;\middle|\; j \in \mathcal{J} \,\right]$ witnessing this fact. By Lemma 30, there exists a pseudo-representation $\mathscr{D}_r \;=\; \left[\, \left(X''_k[\overrightarrow{Z_{\pi_1(k)}}/\overrightarrow{y}]\right)^{p''_k} \;\middle|\; k \in \mathcal{K} \,\right]$ satisfying a series of additional properties. These properties ensure two crucial facts for our purpose:

- $M[\overrightarrow{Z}/\overrightarrow{x}] \Rrightarrow^n_v \left[\, \left(X''_k[\overrightarrow{Z_{\pi_1(k)}}/\overrightarrow{y}]\right)^{p''_k} \;\middle|\; k \in \mathcal{K} \,\right]$ and $M[\overrightarrow{Z'}/\overrightarrow{x}] \Rrightarrow^n_v \mathscr{E}_r \;=\; \left[\, \left(X''_k[\overrightarrow{Z'_{\pi_1(k)}}/\overrightarrow{y}]\right)^{p''_k} \;\middle|\; k \in \mathcal{K} \,\right]$,
- and $\left[\, \left(X''_k[\overrightarrow{Z_{\pi_1(k)}}/\overrightarrow{y}]\right)^{p''_k} \;\middle|\; k \in \mathcal{K} \,\right]$ is a pseudo-distribution witnessing that $\mathscr{D}_r \in \mathsf{DRed}^r_{\nu_r,\rho}$. Setting $\mu \;=\; \left\{\, (\sigma_l)^{p'''_l} \;\middle|\; l \in \mathcal{L} \,\right\}$, there exists therefore families $(p''_{kl})_{k\in\mathcal{K},l\in\mathcal{L}}$ and $(q_{kl})_{k\in\mathcal{K},l\in\mathcal{L}}$ of reals of $[0,1]$ satisfying:
  1. $\forall k \in \mathcal{K},\;\; \forall l \in \mathcal{L},\;\; X''_k[\overrightarrow{Z_{\pi_1(k)}}/\overrightarrow{y}] \in \mathsf{VRed}^{q_{kl}}_{\sigma_l,\rho}$,
  2. $\forall k \in \mathcal{K},\;\; \sum_{l\in\mathcal{L}}\; p''_{kl} \;=\; p''_k$,
  3. $\forall l \in \mathcal{L},\;\; \sum_{k\in\mathcal{K}}\; p''_{kl} \;=\; \mu(\sigma_l)$,
  4. $p \leq \sum_{k\in\mathcal{K}} \sum_{l\in\mathcal{L}}\; q_{kl} p''_{kl}$.

We now prove that $\forall k \in \mathcal{K}, \quad \forall l \in \mathcal{L}, \quad X''_k[\overrightarrow{Z'_{\pi_1(k)}}/\overrightarrow{y}] \in \mathsf{VRed}^{q_{kl}}_{\sigma_l,\rho}$. Let $k \in \mathcal{K}$ and $l \in \mathcal{L}$. Let $\sigma_l = \theta \to \nu$.

$$\begin{array}{rl}
& X''_k[\overrightarrow{Z_{\pi_1(k)}}/\overrightarrow{y}] \in \mathsf{VRed}^{q_{kl}}_{\sigma_l,\rho} \\
\iff & \forall q \in (0,1], \quad \forall Y \in \mathsf{VRed}^{q}_{\theta,\rho}, \quad X''_k[\overrightarrow{Z_{\pi_1(k)}}/\overrightarrow{y}]\ Y \in \mathsf{TRed}^{qq_{kl}}_{\nu,\rho} \\
\iff & \forall q \in (0,1], \quad \forall Y \in \mathsf{VRed}^{q}_{\theta,\rho}, \quad (X''_k\ Y)\,[\overrightarrow{Z_{\pi_1(k)}}/\overrightarrow{y}] \in \mathsf{TRed}^{qq_{kl}}_{\nu,\rho} \\
\iff & \forall q \in (0,1], \quad \forall Y \in \mathsf{VRed}^{q}_{\theta,\rho}, \quad (X''_k\ Y)\,[\overrightarrow{Z'_{\pi_1(k)}}/\overrightarrow{y}] \in \mathsf{TRed}^{qq_{kl}}_{\nu,\rho} \qquad \text{by IH} \\
\iff & \forall q \in (0,1], \quad \forall Y \in \mathsf{VRed}^{q}_{\theta,\rho}, \quad X''_k[\overrightarrow{Z'_{\pi_1(k)}}/\overrightarrow{y}]\ Y \in \mathsf{TRed}^{qq_{kl}}_{\nu,\rho} \\
\iff & X''_k[\overrightarrow{Z'_{\pi_1(k)}}/\overrightarrow{y}] \in \mathsf{VRed}^{q_{kl}}_{\sigma_l,\rho}
\end{array}$$

This will implies that $\left[\left(X''_k[\overrightarrow{Z'_{\pi_1(k)}}/\overrightarrow{y}]\right)^{p''_k} \;\middle|\; k \in \mathcal{K}\right]$ witnesses that $\mathscr{E}_r \in \mathsf{DRed}^{r}_{\nu_r,\rho}$, for the same families of reals $p''_{kl}$ and $q_{kl}$. Now for every $r \in [0,p)$, there exists $n_r$ and $\nu_r \preccurlyeq \mu$ such that $M[\overrightarrow{Z'}/\overrightarrow{x}] \Rrightarrow^{n_r}_{v} \mathscr{E}_r$ and that $\mathscr{E}_r \in \mathsf{DRed}^{r}_{\nu_r,\rho}$: we have that $M[\overrightarrow{Z'}/\overrightarrow{x}] \in \mathsf{TRed}^{p}_{\mu,\rho}$.

□

The following lemma shows that reducible values are reducible terms:

**Lemma 32 (Reducible Values are Reducible Terms)** *Let $V$ be a value. Then $V \in \mathsf{TRed}^{p}_{\{\,\sigma^1\,\},\rho}$ if and only if $V \in \mathsf{VRed}^{p}_{\sigma,\rho}$.*

Note that, conversely, we may have $V \in \mathsf{TRed}^{p}_{\mu,\rho}$ where $\mu$ is not Dirac. For instance, $\mathsf{0} \in \mathsf{TRed}^{1}_{\mu,\rho}$ for $\mu = \left\{\,(\mathsf{Nat}^{\mathsf{i}})^{\frac{1}{2}}, (\widehat{\mathsf{Nat}^{\mathsf{i}}})^{\frac{1}{2}}\,\right\}$.

**Proof.**

- Suppose that $V \in \mathsf{VRed}^{p}_{\sigma,\rho}$. Let $r \in [0,p)$. We must prove that there exists $n_r$ and $\nu_r$ such that $V \to^{n_r}_{v} \left\{\,V^1\,\right\}$ and that $\left\{\,V^1\,\right\} \in \mathsf{DRed}^{r}_{\nu_r,\rho}$. Necessarily $n_r = 0$ and $\nu_r = \left\{\,\sigma^1\,\right\}$. Since $V \in \mathsf{VRed}^{p}_{\sigma,\rho}$, $\left\{\,V^1\,\right\} \in \mathsf{DRed}^{r}_{\nu_r,\rho}$: take the canonical pseudo-representation $\left[\,V^1\,\right]$ and $p_{11} = 1$, $q_{11} = r$.
- Suppose that $V \in \mathsf{TRed}^{p}_{\{\,\sigma^1\,\},\rho}$. It follows that, for every $r \in [0,p)$, there exists $n_r$ and $\nu_r$ such that $V \to^{n_r}_{v} \left\{\,V^1\,\right\}$ and that $\left\{\,V^1\,\right\} \in \mathsf{DRed}^{r}_{\nu_r,\rho}$. Again, since, $V$ is a value, we necessarily have $n_r = 0$ and $\nu_r = \left\{\,\sigma^1\,\right\}$. Since $\left\{\,V^1\,\right\} \in \mathsf{DRed}^{r}_{\nu_r,\rho}$, there is a pseudo-representation $[\,V^{p_1}, \ldots, V^{p_n}\,]$ such that $\sum_{i=1}^{n} p_i = 1$, and a family $(q_{i1})_{i\in\mathcal{I}}$ which is such that $r \leq \sum_{i\in\mathcal{I}} p_{i1}q_{i1}$, where $p_{i1} = p_i$.

  Suppose that there is no $q_{i1}$ greater or equal to $r$. Then $\forall i \in \mathcal{I}, \quad q_{i1} < r$ and

  $$\sum_{i\in\mathcal{I}} p_{i1}q_{i1} \;<\; \sum_{i\in\mathcal{I}} p_{i1}r \;=\; r\sum_{i\in\mathcal{I}} p_{i1} = r$$

  which is a contradiction. So there exists $q_{i1} \geq r$ and therefore $V \in \mathsf{VRed}^{q_{i1}}_{\sigma,\rho}$. By Lemma 22, $V \in \mathsf{VRed}^{r}_{\sigma,\rho}$. Since the result is true for all $r \in [0,p)$, we obtain by Lemma 24 that $V \in \mathsf{VRed}^{p}_{\sigma,\rho}$.

□

We finally deduce from the two previous lemmas the proposition of interest, relating the reducibility of a recursively-defined term with the one of its unfoldings:

**Proposition 4 (Reducibility is Stable by Unfolding)** *Let $n \in \mathbb{N}$ and $V = (\mathsf{letrec}\ f = W)$ be a closed value. Suppose that $Z$ is the $n$-unfolding of $V$. Then $V \in \mathsf{VRed}^{p}_{\mathsf{Nat}^{\mathsf{s}}\to\mu,\rho}$ if and only if $Z \in \mathsf{VRed}^{p}_{\mathsf{Nat}^{\mathsf{s}}\to\mu,\rho}$.*

**Proof.** A direct consequence of Lemma 31 and Lemma 32. □

## 6.7 Reducibility Sets vs. Reductions and Probabilistic Combinations

If a distribution, obtained as partial approximation of the semantics $[\![ M ]\!]$ of a term $M$, is reducible for a type $\mu_n$, then all the partial approximations of $[\![ M ]\!]$ obtained by iterating at least as many times the reduction relation $\Rrightarrow_v$ have the same degree of reducibility, for a greater type:

**Lemma 33** *Suppose that $M \Rrightarrow_v^n \mathscr{D}_n \in \mathsf{DRed}^p_{\mu_n,\rho}$ for $\mu_n \preccurlyeq \mu$, with $\sum \mu = 1$. Suppose that, for $m \geq n$, $M \Rrightarrow_v^m \mathscr{D}_m$. Then there exists $\mu_n \preccurlyeq \mu_m \preccurlyeq \mu$ such that $\mathscr{D}_m \in \mathsf{DRed}^p_{\mu_m,\rho}$.*

**Proof.** Let $\mu_n = \left\{ (\sigma_j)^{p'_j} \;\middle|\; j \in \mathcal{J} \right\}$. Since $\mathscr{D}_n \in \mathsf{DRed}^p_{\mu_n,\rho}$, there exists a pseudo-representation $\mathscr{D}_n = \left[ V_i^{p_i} \;\middle|\; i \in \mathcal{I} \right]$ and two families of reals $(p_{ij})_{i\in\mathcal{I},j\in\mathcal{J}}$ and $(q_{ij})_{i\in\mathcal{I},j\in\mathcal{J}}$ such that

1. $\forall i \in \mathcal{I},\ \ \forall j \in \mathcal{J},\ \ V_i \in \mathsf{VRed}^{q_{ij}}_{\sigma_j,\rho}$,
2. $\forall i \in \mathcal{I},\ \ \sum_{j\in\mathcal{J}} p_{ij} \ = \ p_i$,
3. $\forall j \in \mathcal{J},\ \ \sum_{i\in\mathcal{I}} p_{ij} \ = \ p'_j$,
4. $p \leq \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} q_{ij} p_{ij}$.

By Lemma 3, we have $\mathscr{D}_n \preccurlyeq \mathscr{D}_m$ so that the distribution $\mathscr{D}_m$ admits a pseudo-representation $\mathscr{D}_m = \left[ V_i^{p_i} \;\middle|\; i \in \mathcal{I} \uplus \mathcal{K} \right]$ extending the one of $\mathscr{D}_n$. We now need to define appropriate families of reals $\left(p'_{ij}\right)_{i\in\mathcal{I}\uplus\mathcal{K},j\in\mathcal{J}}$ and $\left(q'_{ij}\right)_{i\in\mathcal{I}\uplus\mathcal{K},j\in\mathcal{J}}$. We set:

- $\forall i \in \mathcal{I},\ \forall j \in \mathcal{J},\ \ p'_{ij} \ = \ p_{ij}$,
- $\forall i \in \mathcal{I},\ \forall j \in \mathcal{J},\ \ q'_{ij} \ = \ q_{ij}$,
- $\forall i \in \mathcal{K},\ \forall j \in \mathcal{J},\ \ q'_{ij} \ = \ 0$

and we choose the $\left(p'_{ij}\right)_{i\in\mathcal{K},j\in\mathcal{J}}$ arbitrarily in $[0,1]$ under the constraints that $\forall i \in \mathcal{K},\ \sum_{j\in\mathcal{J}} p'_{ij} \ = \ p_i$ and that $\forall j \in \mathcal{J},\ \sum_{i\in\mathcal{I}\uplus\mathcal{K}} p'_{ij} \leq \mu(\sigma_j)$. These constraints are feasible since $\sum_{i\in\mathcal{I}\uplus\mathcal{K}} \sum_{j\in\mathcal{J}} p'_{ij} \ = \ \sum_{i\in\mathcal{I}\uplus\mathcal{K}} p_i \leq 1 = \sum \mu$. We then set $\mu_m = \left\{ (\sigma_j)^{\sum_{i\in\mathcal{I}\uplus\mathcal{K}} p'_{ij}} \;\middle|\; j \in \mathcal{J} \right\} \preccurlyeq \mu$. Let us check that $\mathscr{D}_m \in \mathsf{DRed}^p_{\mu_m,\rho}$.

1. $\forall i \in \mathcal{I},\ \ \forall j \in \mathcal{J},\ \ V_i \in \mathsf{VRed}^{q_{ij}}_{\sigma_j,\rho}$ and $\forall i \in \mathcal{K},\ \ \forall j \in \mathcal{J},\ \ V_i \in \mathsf{VRed}^{0}_{\sigma_j,\rho}$ as this set contains all terms of simple type $\langle \sigma_j \rangle$ by Lemma 21,
2. $\forall i \in \mathcal{I},\ \ \sum_{j\in\mathcal{J}} p'_{ij} \ = \ p_i$ by definition and $\forall i \in \mathcal{K},\ \ \sum_{j\in\mathcal{J}} p'_{ij} \ = \ p_i$ by construction,
3. $\forall j \in \mathcal{J},\ \ \sum_{i\in\mathcal{I}\uplus\mathcal{K}} p'_{ij} \ = \ \mu_m(\sigma_j)$ by definition of $\mu_m$,
4. $$\begin{array}{rcl} p & \leq & \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} q_{ij} p_{ij} \\ & = & \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} q'_{ij} p'_{ij} + 0 \\ & = & \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} q'_{ij} p'_{ij} + \sum_{i\in\mathcal{K}} \sum_{j\in\mathcal{J}} q'_{ij} p'_{ij} \\ & = & \sum_{i\in\mathcal{I}\uplus\mathcal{K}} \sum_{j\in\mathcal{J}} q'_{ij} p'_{ij} \end{array}$$

So $\mathscr{D}_m \in \mathsf{DRed}^p_{\mu_m,\rho}$.

□

When two distributions $\mathscr{D}$ and $\mathscr{E}$ are reducible, with respective degrees of reducibility $p'$ and $p''$, their probabilistic combination $\mathscr{D} \oplus_p \mathscr{E}$ is reducible as well with degree of reducibility $pp' + (1-p)p''$, for the distribution type computed by $\oplus_p$:

**Lemma 34** *Suppose that $\langle \mu \rangle = \langle \nu \rangle$, that $\mathscr{D} \in \mathsf{DRed}^{p'}_{\mu,\rho}$ and that $\mathscr{E} \in \mathsf{DRed}^{p''}_{\nu,\rho}$. Then $p\mathscr{D} + (1-p)\mathscr{E} \in \mathsf{DRed}^{pp'+(1-p)p''}_{\mu\oplus_p\nu,\rho}$.*

**Proof.** Let $\mu = \Big\{ (\sigma_j)^{p'_j} \ \Big| \ j \in \mathcal{J} \Big\}$. Since $\mathscr{D} \in \mathsf{DRed}^{p'}_{\mu,\rho}$, there exists a pseudo-representation $\mathscr{D} = \big[ V_i^{p_i} \ \big| \ i \in \mathcal{I} \big]$ and two families of reals $(p_{ij})_{i\in\mathcal{I},j\in\mathcal{J}}$ and $(q_{ij})_{i\in\mathcal{I},j\in\mathcal{J}}$ such that

1. $\forall i \in \mathcal{I}, \ \ \forall j \in \mathcal{J}, \ \ V_i \in \mathsf{VRed}^{q_{ij}}_{\sigma_j,\rho}$,
2. $\forall i \in \mathcal{I}, \ \ \sum_{j\in\mathcal{J}} p_{ij} \ = \ p_i$,
3. $\forall j \in \mathcal{J}, \ \ \sum_{i\in\mathcal{I}} p_{ij} \ = \ p'_j$,
4. $p' \leq \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} q_{ij} p_{ij}$.

Let $\nu = \Big\{ (\tau_l)^{p'''_l} \ \Big| \ l \in \mathcal{L} \Big\}$. Since $\mathscr{E} \in \mathsf{DRed}^{p''}_{\nu,\rho}$, there exists a pseudo-representation $\mathscr{E} = \Big[ W_k^{p''_k} \ \Big| \ k \in \mathcal{K} \Big]$ and two families of reals $(p'_{kl})_{k\in\mathcal{K},l\in\mathcal{L}}$ and $(q'_{kl})_{k\in\mathcal{K},l\in\mathcal{L}}$ such that

1. $\forall k \in \mathcal{K}, \ \ \forall l \in \mathcal{L}, \ \ W_k \in \mathsf{VRed}^{q'_{kl}}_{\tau_l,\rho}$,
2. $\forall k \in \mathcal{K}, \ \ \sum_{l\in\mathcal{L}} p'_{kl} \ = \ p''_k$,
3. $\forall l \in \mathcal{L}, \ \ \sum_{k\in\mathcal{K}} p'_{kl} \ = \ p'''_l$,
4. $p'' \leq \sum_{k\in\mathcal{K}} \sum_{l\in\mathcal{L}} q'_{kl} p'_{kl}$.

We suppose that $\mathcal{I}$ and $\mathcal{K}$ are disjoint, and that $j \in \mathcal{J} \cap \mathcal{L} \Leftrightarrow \sigma_j = \tau_j$. To prove that $p\mathscr{D} + (1-p)\mathscr{E} \in \mathsf{DRed}^{pp'+(1-p)p''}_{\mu\oplus_p\nu,\rho}$, we consider the pseudo-representation

$$p\mathscr{D} + (1-p)\mathscr{E} \ \ = \ \ \big[ V_i^{pp_i} \ \big| \ i \in \mathcal{I} \big] + \Big[ W_k^{(1-p)p''_k} \ \Big| \ k \in \mathcal{K} \Big] \tag{14}$$

and we write the distribution type $\mu \oplus_p \nu$ as

$$\Big\{ (\sigma_j)^{pp'_j} \ \Big| \ j \in \mathcal{J} \setminus (\mathcal{J} \cap \mathcal{L}) \Big\} + \Big\{ (\sigma_j)^{pp'_j + (1-p)p'''_j} \ \Big| \ j \in \mathcal{J} \cap \mathcal{L} \Big\} + \Big\{ (\tau_l)^{(1-p)p'''_l} \ \Big| \ l \in \mathcal{L} \setminus (\mathcal{J} \cap \mathcal{L}) \Big\}$$

We set $\mathcal{G} = \mathcal{I} + \mathcal{K}$ and $\mathcal{H} = \mathcal{J} + \mathcal{L}$. We now need to define appropriate families of reals $\Big(p''_{gh}\Big)_{g\in\mathcal{G},h\in\mathcal{H}}$ and $\Big(q''_{gh}\Big)_{g\in\mathcal{G},h\in\mathcal{H}}$. We proceed as follows:

- if $g \in \mathcal{I}$ and $h \in \mathcal{J}$, $p''_{gh} \ = \ pp_{gh}$ and $q''_{gh} \ = \ q_{gh}$,
- if $g \in \mathcal{I}$ and $h \in \mathcal{L}$, $p''_{gh} \ = \ 0$ and $q''_{gh} \ = \ 0$,
- if $g \in \mathcal{K}$ and $h \in \mathcal{J}$, $p''_{gh} \ = \ 0$ and $q''_{gh} \ = \ 0$,
- if $g \in \mathcal{K}$ and $h \in \mathcal{L}$, $p''_{gh} \ = \ (1-p)p'_{gh}$ and $q''_{gh} \ = \ q'_{gh}$.

Let us prove that (14) together with these two families provide a witness that $p\mathscr{D} + (1-p)\mathscr{E} \in \mathsf{DRed}^{pp'+(1-p)p''}_{\mu\oplus_p\nu,\rho}$ by checking the four usual conditions. We write $Z_g$ either for $V_i$ or $W_k$, depending on the context. We write similarly $\theta_h$ for $\sigma_j$ or $\tau_l$.

1. $\forall g \in \mathcal{G}, \ \ \forall h \in \mathcal{H}, \ \ Z_g \in \mathsf{VRed}^{q''_{gh}}_{\theta_h,\rho}$ is proved by case exhaustion:
   - $\forall g \in \mathcal{I}, \ \ \forall h \in \mathcal{J}, \ \ V_g \in \mathsf{VRed}^{q_{gh}}_{\sigma_h,\rho}$ since $\mathscr{D} \in \mathsf{DRed}^{p'}_{\mu,\rho}$,
   - $\forall g \in \mathcal{K}, \ \ \forall h \in \mathcal{L}, \ \ W_g \in \mathsf{VRed}^{q'_{gh}}_{\tau_h,\rho}$ since $\mathscr{E} \in \mathsf{DRed}^{p''}_{\nu,\rho}$,
   - in the two remaining cases, $q''_{gh} \ = \ 0$ and by Lemma 21 the result holds.
2. We proceed again by case exhaustion.
   - If $g \in \mathcal{I}$, $\sum_{h\in\mathcal{H}} p''_{gh} \ = \ \sum_{h\in\mathcal{J}} p''_{gh} + \sum_{h\in\mathcal{L}} p''_{gh} \ = \ \sum_{h\in\mathcal{J}} pp_{gh} \ = \ pp_g$.

- If $g \in \mathcal{K}$, $\sum_{h\in\mathcal{H}} p''_{gh} = \sum_{h\in\mathcal{L}} (1-p)p'_{gh} = (1-p)p''_g$.

3. We proceed again by case exhaustion.

   - Suppose that $h \in \mathcal{J} \setminus (\mathcal{J} \cap \mathcal{L})$. Then $\sum_{g\in\mathcal{G}} p''_{gh} = \sum_{g\in\mathcal{I}} p''_{gh} = \sum_{g\in\mathcal{I}} pp_{gh} = pp'_g$.
   - Suppose that $h \in \mathcal{L} \setminus (\mathcal{J} \cap \mathcal{L})$. Then $\sum_{g\in\mathcal{G}} p''_{gh} = \sum_{g\in\mathcal{K}} p''_{gh} = \sum_{g\in\mathcal{K}} (1-p)p'_{gh} = (1-p)p'''_g$.
   - Suppose that $h \in \mathcal{J}\cap\mathcal{L}$. Then $\sum_{g\in\mathcal{G}} p''_{gh} = \sum_{g\in\mathcal{I}} p''_{gh} + \sum_{g\in\mathcal{K}} p''_{gh} = pp'_g + (1-p)p'''_g$.

4. $$\begin{array}{ll} & \sum_{g\in\mathcal{G}} \sum_{h\in\mathcal{H}} q''_{gh}p''_{gh} \\ = & \sum_{g\in\mathcal{I}} \sum_{h\in\mathcal{J}} q''_{gh}p''_{gh} + \sum_{g\in\mathcal{K}} \sum_{h\in\mathcal{L}} q''_{gh}p''_{gh} \\ = & \sum_{g\in\mathcal{I}} \sum_{h\in\mathcal{J}} q_{gh}pp_{gh} + \sum_{g\in\mathcal{K}} \sum_{h\in\mathcal{L}} q'_{gh}(1-p)p'_{gh} \\ = & p\sum_{g\in\mathcal{I}} \sum_{h\in\mathcal{J}} q_{gh}p_{gh} + (1-p)\sum_{g\in\mathcal{K}} \sum_{h\in\mathcal{L}} q'_{gh}p'_{gh} \\ \geq & pp' + (1-p)p'' \end{array}$$

   It follows that $p\mathscr{D} + (1-p)\mathscr{E} \in \mathsf{DRed}^{pp'+(1-p)p''}_{\mu\oplus_p\nu,\rho}$.

□

This lemma generalizes to the $n$-ary case of a weighted sum of distributions:

**Lemma 35** *Let $(\mu_i)_{i\in\mathcal{I}}$ be a family of distribution types of same underlying type. For every $i \in \mathcal{I}$, let $\mathscr{D}_i \in \mathsf{DRed}^{q_i}_{\mu_i,\rho}$. Let $(p_i)_{i\in\mathcal{I}}$ be a family of reals of $[0,1]$ such that $\sum_{i\in\mathcal{I}} p_i \leq 1$. Then $\sum_{i\in\mathcal{I}} p_i\mathscr{D}_i \in \mathsf{DRed}^{\sum_{i\in\mathcal{I}} p_iq_i}_{\sum_{i\in\mathcal{I}} p_i\mu_i,\rho}$.*

**Proof.** Similar to the proof of Lemma 34. □

$\mathsf{TRed}$ is closed by anti-reduction for Dirac distributions, but also in the case corresponding to the reduction of a choice operator:

**Lemma 36 (Reductions and Sets of Candidates)**

- *Suppose that $M \to_v \{N^1\}$ and that $N \in \mathsf{TRed}^p_{\mu,\rho}$. Then $M \in \mathsf{TRed}^p_{\mu,\rho}$.*
- *Suppose that $M \to_v \{N^p, L^{1-p}\}$, that $N \in \mathsf{TRed}^{p'}_{\mu,\rho}$ and that $L \in \mathsf{TRed}^{p''}_{\nu,\rho}$. Then $M \in \mathsf{TRed}^{pp'+(1-p)p''}_{\mu\oplus_p\nu,\rho}$.*

**Proof.**

- Since $N \in \mathsf{TRed}^p_{\mu,\rho}$, for every $0 \leq r < p$ there exists $\nu_r \preccurlyeq \mu$ and $n_r \in \mathbb{N}$ such that $N \Rrightarrow^{n_r}_v \mathscr{D}_r \in \mathsf{DRed}^r_{\nu_r,\rho}$. Recall that $\Rrightarrow^{n_r+1}_v = \to_v \circ \Rrightarrow^{n_r}_v$. It follows that $M \Rrightarrow^{n_r+1}_v \mathscr{D}_r$ which has the required properties, so that $M \in \mathsf{TRed}^p_{\mu,\rho}$.
- Let $0 \leq r < pp' + (1-p)p''$. Let $(r', r'')$ be such that $r = pr' + (1-p)r''$, $0 \leq r' < p'$ and $0 \leq r'' < p''$. Since $N \in \mathsf{TRed}^{p'}_{\mu,\rho}$, there exists $n_{r'}$ and $\mu_{r'} \preccurlyeq \mu$ such that $N \Rrightarrow^{n_{r'}}_v \mathscr{D}_{r'} \in \mathsf{DRed}^{r'}_{\mu_{r'},\rho}$. Since $L \in \mathsf{TRed}^{p''}_{\nu,\rho}$, there exists $m_{r''}$ and $\nu_{r''} \preccurlyeq \nu$ such that $L \Rrightarrow^{m_{r''}}_v \mathscr{E}_{r''} \in \mathsf{DRed}^{r''}_{\nu_{r''},\rho}$. Suppose that $n_{r'} \leq m_{r''}$, the dual case being exactly symmetrical. By Lemma 33, by denoting $\mathscr{D}_{r''}$ the distribution such that $N \Rrightarrow^{m_{r''}}_v \mathscr{D}_{r''}$, there exists $\mu_{r'} \preccurlyeq \mu_{r''} \preccurlyeq \mu$ such that $\mathscr{D}_{r''} \in \mathsf{DRed}^{r'}_{\mu_{r''},\rho}$. Now $M \Rrightarrow^{m_{r''}+1}_v p\mathscr{D}_{r''} + (1-p)\mathscr{E}_{r''}$, and by Lemma 34 we have $p\mathscr{D}_{r''} + (1-p)\mathscr{E}_{r''} \in \mathsf{DRed}^{pr'+(1-p)r''}_{\mu_{r''}\oplus_p\nu_{r''},\rho}$. Since by construction $\mu_{r''} \oplus_p \nu_{r''} \preccurlyeq \mu \oplus_p \nu$, we can conclude that $M \in \mathsf{TRed}^{pp'+(1-p)p''}_{\mu\oplus_p\nu,\rho}$.

□

## 6.8 Subtyping Soundness

Reducibility sets are monotonic with respect to the subtyping order $\sqsubseteq$:

**Lemma 37 (Subtyping Soundness)**

- *Suppose that $\sigma \sqsubseteq \tau$. Then, for every $p \in [0,1]$ and $\rho$, $\mathsf{VRed}^p_{\sigma,\rho} \subseteq \mathsf{VRed}^p_{\tau,\rho}$.*
- *Suppose that $\mu \sqsubseteq \nu$ and that $\sum \mu = \sum \nu$. Then, for every $p \in [0,1]$ and $\rho$, $\mathsf{DRed}^p_{\mu,\rho} \subseteq \mathsf{DRed}^p_{\nu,\rho}$.*
- *Suppose that $\mu \sqsubseteq \nu$. Then, for every $p \in [0,1]$ and $\rho$, $\mathsf{TRed}^p_{\mu,\rho} \subseteq \mathsf{TRed}^p_{\nu,\rho}$.*

**Proof.** The proof is by mutual induction on the statements following the shape of the simple type refined by $\sigma$ and $\mu$, as earlier.

- Suppose that $\sigma :: \mathsf{Nat}$. Then $\sigma = \mathsf{Nat}^{\mathfrak{s}}$ and $\tau = \mathsf{Nat}^{\mathfrak{r}}$ with $\mathfrak{s} \preccurlyeq \mathfrak{r}$. Let $V \in \mathsf{VRed}^p_{\sigma,\rho}$. There are three possibilities:
  - Either $\mathfrak{s} = \widehat{\mathfrak{i}}^k$ and $\mathfrak{r} = \widehat{\mathfrak{i}}^{k'}$ with $k \leq k'$. Then $V$ is of the shape $\mathsf{S}^n\ \mathsf{0}$. If $p = 0$ the result is immediate. Else we have $n < [\![\mathfrak{s}]\!]_\rho = \rho(\mathfrak{i}) + k \leq \rho(\mathfrak{i}) + k' = [\![\mathfrak{r}]\!]_\rho$ so that $V \in \mathsf{VRed}^p_{\tau,\rho}$.
  - Or $\mathfrak{s} = \widehat{\mathfrak{i}}^k$ and $\mathfrak{r} = \infty$. In this case $V$ is of the shape $\mathsf{S}^n\ \mathsf{0}$ and therefore $V \in \mathsf{VRed}^p_{\tau,\rho}$.
  - Or $\mathfrak{s} = \mathfrak{r} = \infty$. In this case $\sigma = \tau$ and the result is immediate.
- Suppose that $\sigma = \sigma' \to \mu$ and that $\tau = \tau' \to \nu$. Let $p \in [0,1]$, $\rho$ be a size environment, and $V \in \mathsf{VRed}^p_{\sigma,\rho}$. We have that $\tau' \sqsubseteq \sigma'$ and $\mu \sqsubseteq \nu$. It follows, by induction hypothesis, that $\mathsf{VRed}^{p'}_{\tau',\rho} \subseteq \mathsf{VRed}^{p'}_{\sigma',\rho}$ and that $\mathsf{TRed}^{p'}_{\mu,\rho} \subseteq \mathsf{TRed}^{p'}_{\nu,\rho}$ for every $p' \in [0,1]$. Since $V \in \mathsf{VRed}^p_{\sigma,\rho}$, for every $q \in (0,1]$ and $W \in \mathsf{VRed}^q_{\sigma',\rho}$, $V\ W \in \mathsf{TRed}^{pq}_{\mu,\rho} \subseteq \mathsf{TRed}^{pq}_{\nu,\rho}$. As $\mathsf{VRed}^q_{\tau',\rho} \subseteq \mathsf{VRed}^q_{\sigma',\rho}$, $V \in \mathsf{VRed}^p_{\tau,\rho}$.
- Suppose that $\mu = \left\{ \sigma_j^{p'_j} \ \middle|\ j \in \mathcal{J} \right\}$ and that $\nu = \left\{ \tau_k^{p''_k} \ \middle|\ k \in \mathcal{K} \right\}$. By definition of subtyping, there exists $f : \mathcal{J} \to \mathcal{K}$ such that for all $j \in \mathcal{J}$, $\sigma_j \sqsubseteq \tau_{f(j)}$ and that for all $k \in \mathcal{K}$, $\sum_{j \in f^{-1}(k)} p'_j \leq p''_k$. Note that since $\sum \mu = \sum \nu$, this is in fact an equality. Let $\mathscr{D} \in \mathsf{DRed}^p_{\mu,\rho}$, then there exists a pseudo-representation $\mathscr{D} = \left[ (V_i)^{p_i} \ \middle|\ i \in \mathcal{I} \right]$ and families $(p_{ij})_{i \in \mathcal{I}, j \in \mathcal{J}}$ and $(q_{ij})_{i \in \mathcal{I}, j \in \mathcal{J}}$ of reals of $[0,1]$ satisfying:
  1. $\forall i \in \mathcal{I},\ \ \forall j \in \mathcal{J},\ \ V_i \in \mathsf{VRed}^{q_{ij}}_{\sigma_j,\rho}$,
  2. $\forall i \in \mathcal{I},\ \ \sum_{j \in \mathcal{J}} p_{ij} = p_i$,
  3. $\forall j \in \mathcal{J},\ \ \sum_{i \in \mathcal{I}} p_{ij} = p'_j$,
  4. $p \leq \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} q_{ij} p_{ij}$.

  By induction hypothesis, for every $j \in \mathcal{J}$, $\mathsf{VRed}^{q_{ij}}_{\sigma_j,\rho} \subseteq \mathsf{VRed}^{q_{ij}}_{\tau_{f(j)},\rho}$. We now prove that $\left[ (V_i)^{p_i} \ \middle|\ i \in \mathcal{I} \right]$ witnesses that $\mathscr{D} \in \mathsf{DRed}^p_{\nu,\rho}$. We need to define families of reals $(p'_{ik})_{i \in \mathcal{I}, k \in \mathcal{K}}$ and $(q'_{ik})_{i \in \mathcal{I}, k \in \mathcal{K}}$ satisfying the four usual conditions. To this end, for every $i \in \mathcal{I}$, $k \in \mathcal{K}$, we set

  $$p'_{ik} = \sum_{j \in f^{-1}(k)} p_{ij}$$

  and

  $$q'_{ik} = \max_{j \in f^{-1}(k)} q_{ij}$$

  Let us check that the four conditions hold.

  1. $\forall i \in \mathcal{I},\ \ \forall k \in \mathcal{K},\ \ V_i \in \mathsf{VRed}^{q'_{ik}}_{\tau_{f(j)},\rho}$ by induction hypothesis and by definition of $q'_{ik}$,
  2. $\forall i \in \mathcal{I},\ \ \sum_{k \in \mathcal{K}} p'_{ik} = \sum_{k \in \mathcal{K}} \sum_{j \in f^{-1}(k)} p_{ij} = \sum_{j \in \mathcal{J}} p_{ij} = p_i$,

3. $\forall k \in \mathcal{K},\ \sum_{i\in\mathcal{I}} p'_{ik} = \sum_{i\in\mathcal{I}} \sum_{j\in f^{-1}(k)} p_{ij} = \sum_{j\in f^{-1}(k)} \sum_{i\in\mathcal{I}} p_{ij} = \sum_{j\in f^{-1}(k)} p'_j = p''_k$,
4. $$\begin{array}{rcl} p & \leq & \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} q_{ij}p_{ij} \\ & = & \sum_{i\in\mathcal{I}} \sum_{k\in\mathcal{K}} \sum_{j\in f^{-1}(k)} q_{ij}p_{ij} \\ & \leq & \sum_{i\in\mathcal{I}} \sum_{k\in\mathcal{K}} \sum_{j\in f^{-1}(k)} q'_{if(j)}p_{ij} \\ & = & \sum_{i\in\mathcal{I}} \sum_{k\in\mathcal{K}} q'_{ik} \sum_{j\in f^{-1}(k)} p_{ij} \\ & = & \sum_{i\in\mathcal{I}} \sum_{k\in\mathcal{K}} q'_{ik}p'_{ik} \end{array}$$

It follows that $\mathscr{D} \in \mathsf{DRed}^p_{\nu,\rho}$.

- Suppose that $\mu = \left\{ \sigma_j^{p'_j} \mid j \in \mathcal{J} \right\}$ and that $\nu = \left\{ \tau_k^{p''_k} \mid k \in \mathcal{K} \right\}$. By definition of subtyping, there exists $f : \mathcal{J} \to \mathcal{K}$ such that for all $j \in \mathcal{J}$, $\sigma_j \sqsubseteq \tau_{f(j)}$ and that for all $k \in \mathcal{K}$, $\sum_{j\in f^{-1}(k)} p'_j \leq p''_k$. Let $M \in \mathsf{TRed}^p_{\mu,\rho}$. Then, for every $0 \leq r < p$, there exists $\mu'_r \preccurlyeq \mu$ and $n_r$ such that $M \Rrightarrow_v^{n_r} \mathscr{D}_r \in \mathsf{DRed}^r_{\mu'_r,\rho}$. By definition of $\mu'_r \preccurlyeq \mu$, $\mu'_r = \left[ \sigma_j^{q'_j} \mid j \in \mathcal{J} \right]$ with $q'_j \leq p'_j$ for every $j \in \mathcal{J}$. We set $\nu'_r = \left[ \tau_{f(j)}^{q'_j} \mid j \in \mathcal{J} \right]$ which is such that $\sum \mu'_r = \sum \nu'_r$ and, by construction, $\mu'_r \sqsubseteq \nu'_r$ so that we can apply the induction hypothesis and obtain that $M \Rrightarrow_v^{n_r} \mathscr{D}_r \in \mathsf{DRed}^r_{\nu'_r,\rho}$. The result follows, since by construction $\nu'_r \preccurlyeq \nu$.

□

## 6.9 Reducibility Sets for Open Terms

We are now ready to extend the notion of reducibility set from the realm of *closed* terms to the one of *open* terms. This turns out to be subtle. The guiding intuition is that one would like to define a term $M$ with free variables in $\overrightarrow{x}$ to be reducible iff any closure $M[\overrightarrow{V}/\overrightarrow{x}]$ is itself reducible in the sense of Definition 24. What happens, however, to the underlying degree of reducibility $p$? How do we relate the degrees of reducibility of $\overrightarrow{V}$ with the one of $M[\overrightarrow{V}/\overrightarrow{x}]$? The answer is contained in the following definition:

**Definition 26 (Reducibility Sets for Open Terms)** *Suppose that $\Gamma$ is a sized context in the form $x_1 : \sigma_1, \ldots, x_n : \sigma_n$, and that $y$ is a variable distinct from $x_1, \ldots, x_n$. Then we define the following sets of terms and values:*

$$\begin{array}{rcl} \mathsf{OTRed}^{\Gamma\,|\,\emptyset}_{\mu,\rho} & = & \Big\{ M \ \Big| \ \forall (q_i)_i \in [0,1]^n,\ \forall (V_1, \ldots, V_n) \in \prod_{i=1}^n \mathsf{VRed}^{q_i}_{\sigma_i,\rho}, \\ & & \qquad M[\overrightarrow{V}/\overrightarrow{x}] \in \mathsf{TRed}^{\prod_{i=1}^n q_i}_{\mu,\rho} \Big\} \end{array}$$

$$\begin{array}{rcl} \mathsf{OVRed}^{\Gamma\,|\,\emptyset}_{\mu,\rho} & = & \Big\{ W \ \Big| \ \forall (q_i)_i \in [0,1]^n,\ \forall (V_1, \ldots, V_n) \in \prod_{i=1}^n \mathsf{VRed}^{q_i}_{\sigma_i,\rho}, \\ & & \qquad W[\overrightarrow{V}/\overrightarrow{x}] \in \mathsf{VRed}^{\prod_{i=1}^n q_i}_{\mu,\rho} \Big\} \end{array}$$

$$\begin{array}{rcl} \mathsf{OTRed}^{\Gamma\,|\,y\,:\,\{\tau_j^{p_j}\}_{j\in\mathcal{J}}}_{\mu,\rho} & = & \Big\{ M \ \Big| \ \forall (q_i)_i \in [0,1]^n,\ \forall \overrightarrow{V} \in \prod_{i=1}^n \mathsf{VRed}^{q_i}_{\sigma_i,\rho}, \\ & & \qquad \forall \left(q'_j\right)_j \in [0,1]^{\mathcal{J}},\ \forall W \in \bigcap_{j\in\mathcal{J}} \mathsf{VRed}^{q'_j}_{\tau_j,\rho}, \\ & & \qquad\qquad M[\overrightarrow{V}, W/\overrightarrow{x}, y] \in \mathsf{TRed}^{\alpha}_{\mu,\rho} \Big\} \end{array}$$

$$\begin{array}{rcl} \mathsf{OVRed}^{\Gamma\,|\,y\,:\,\{\tau_j^{p_j}\}_{j\in\mathcal{J}}}_{\mu,\rho} & = & \Big\{ Z \ \Big| \ \forall (q_i)_i \in [0,1]^n,\ \forall \overrightarrow{V} \in \prod_{i=1}^n \mathsf{VRed}^{q_i}_{\sigma_i,\rho}, \\ & & \qquad \forall \left(q'_j\right)_j \in [0,1]^{\mathcal{J}},\ \forall W \in \bigcap_{j\in\mathcal{J}} \mathsf{VRed}^{q'_j}_{\tau_j,\rho}, \\ & & \qquad\qquad Z[\overrightarrow{V}, W/\overrightarrow{x}, y] \in \mathsf{VRed}^{\alpha}_{\mu,\rho} \Big\} \end{array}$$

*where  is called the degree of reducibility. the degree of reducibility $\alpha$ is defined as*

$$\alpha \quad = \quad \left(\prod_{i=1}^{n} q_i\right)\left(\left(\sum_{j\in\mathcal{J}} p_j q'_j\right) + 1 - \left(\sum_{j\in\mathcal{J}} p_j\right)\right).$$

*Note that this contains:*

$$\begin{array}{rcl}
\mathsf{OTRed}^{\emptyset\,|\,\emptyset}_{\mu,\rho} & = & \mathsf{TRed}^{1}_{\mu,\rho} \\
\mathsf{OVRed}^{\emptyset\,|\,\emptyset}_{\sigma,\rho} & = & \mathsf{VRed}^{1}_{\sigma,\rho} \\
\mathsf{OTRed}^{\emptyset\,|\,y\,:\,\{\tau_i^{p_i}\}_{i\in I}}_{\mu,\rho} & = & \Big\{M \;\Big|\; \forall\,(q_i)_i \in [0,1]^{\mathcal{I}},\;\; \forall V \in \bigcap_{i\in\mathcal{I}} \mathsf{VRed}^{q_i}_{\sigma_i,\rho}, \\
& & \qquad\qquad M[V/y] \in \mathsf{TRed}^{\sum_{i\in I} p_i q_i + 1 - \left(\sum_{j\in\mathcal{J}} p_j\right)}_{\mu,\rho}\Big\} \\
\mathsf{OVRed}^{\emptyset\,|\,y\,:\,\{\tau_i^{p_i}\}_{i\in I}}_{\mu,\rho} & = & \Big\{W \;\Big|\; \forall\,(q_i)_i \in [0,1]^{\mathcal{I}},\;\; \forall V \in \bigcap_{i\in\mathcal{I}} \mathsf{VRed}^{q_i}_{\sigma_i,\rho}, \\
& & \qquad\qquad W[V/y] \in \mathsf{VRed}^{\sum_{i\in I} p_i q_i + 1 - \left(\sum_{j\in\mathcal{J}} p_j\right)}_{\mu,\rho}\Big\}
\end{array}$$

*Note alsothat these sets extend the ones for closed terms: in particular,* $\mathsf{OTRed}^{\emptyset\,|\,\emptyset}_{\mu,\rho} = \mathsf{TRed}^{1}_{\mu,\rho}$.

As for closed terms (Lemma 32), reducible values are reducible terms:

**Lemma 38 (Reducible Values are Reducible Terms)** *For every* $\Gamma$, $\Theta$, $\sigma$ *and* $\rho$, $V \in \mathsf{OVRed}^{\Gamma\,|\,\Theta}_{\sigma,\rho}$ *if and only if* $V \in \mathsf{OTRed}^{\Gamma\,|\,\Theta}_{\{\,\sigma^1\,\},\rho}$. *An immediate consequence is that* $\mathsf{OVRed}^{\Gamma\,|\,\Theta}_{\sigma,\rho} \subseteq \mathsf{OTRed}^{\Gamma\,|\,\Theta}_{\{\,\sigma^1\,\},\rho}$.

**Proof.** Corollary of Lemma 32 and of the definitions of the candidates for open sets. □

The following easy lemma relates the reducibility of natural numbers, and will be used to treat the case of the rules Succ and Zero in the proof of typing soundness:

**Lemma 39**

- $V \in \mathsf{VRed}^{p}_{\mathsf{Nat}^{\mathfrak{s}},\rho} \quad\Longrightarrow\quad \mathsf{S}\ V \in \mathsf{VRed}^{p}_{\mathsf{Nat}^{\widehat{\mathfrak{s}}},\rho}$
- *For every size* $\mathfrak{s}$, $\mathsf{0} \in \mathsf{VRed}^{p}_{\mathsf{Nat}^{\widehat{\mathfrak{s}}},\rho}$.

**Proof.** First point:

- Suppose that $V \in \mathsf{VRed}^{p}_{\mathsf{Nat}^{\widehat{\mathfrak{i}}^k},\rho}$ and that $p > 0$. Then $V = \mathsf{S}^n\ \mathsf{0}$ for some $n < [\![\mathfrak{s}]\!]_\rho$. Then $\mathsf{S}\ V = \mathsf{S}^{n+1}\ \mathsf{0}$ satisfies $n+1 < [\![\widehat{\mathfrak{s}}]\!]_\rho = [\![\mathfrak{s}]\!]_\rho + 1$, so that $\mathsf{S}\ V \in \mathsf{VRed}^{p}_{\mathsf{Nat}^{\widehat{\mathfrak{i}}^{k+1}},\rho}$.
- Suppose that $V \in \mathsf{VRed}^{p}_{\mathsf{Nat}^{\infty},\rho}$ or that $p = 0$. By definition, $V = \mathsf{S}^n\ \mathsf{0}$ for $n \in \mathbb{N}$. It follows that $\mathsf{S}\ V \in \mathsf{VRed}^{p}_{\mathsf{Nat}^{\widehat{\mathfrak{s}}},\rho}$.

Second point:

- Suppose that $p = 0$. Then $\mathsf{0} \in \mathsf{VRed}^{p}_{\mathsf{Nat}^{\widehat{\mathfrak{s}}},\rho}$, by definition.
- Else we need to prove that $[\![\mathfrak{s}]\!]_\rho > 0$. But $\widehat{\mathfrak{s}}$ is either $\infty$, in which case $[\![\mathfrak{s}]\!]_\rho = \infty$, or it is of the shape $\widehat{\mathfrak{i}}^k$ for $k > 0$, and $[\![\widehat{\mathfrak{i}}^k]\!]_\rho = \rho(\mathfrak{i}) + k > 0$.

□

## 6.10 Reducibility and Sized Walks

To handle the fixpoint rule, we need to relate the notion of sized walk which guards it with the reducibility sets, and in particular with the degrees of reducibility we can attribute to recursively-defined terms.

**Definition 27 (Probabilities of Convergence in Finite Time)** *Let us consider a sized walk. We define the associated* probabilities of convergence in finite time $(Pr_{n,m})_{n\in\mathbb{N},m\in\mathbb{N}}$ *as follows:* $\forall n \in \mathbb{N},\ \ \forall m \in \mathbb{N}$, *the real number* $Pr_{n,m}$ *is defined as the probability that, starting from* $m$*, the sized walk reaches* $0$ *in* at most $n$ *steps.*

The point is that, for an AST sized walk, the more we iterate, the closer we get to reaching 0 in finite time $n$ with probability 1.

**Lemma 40 (Finite Approximations of AST)** *Let* $m \in \mathbb{N}$ *and* $\varepsilon \in (0,1]$*. Consider a sized walk, and its associated probabilities of convergence in finite time* $(Pr_{n,m})_{n\in\mathbb{N},m\in\mathbb{N}}$*. If the sized walk is AST, there exists* $n \in \mathbb{N}$ *such that* $Pr_{n,m} \geq 1-\varepsilon$*.*

**Proof.** Suppose, by contradiction, that there exists $\varepsilon \in (0,1]$ such that there is no $n \in \mathbb{N}$ with $Pr_{n,m} \geq 1-\varepsilon$. Then $\lim_{n\in\mathbb{N}} Pr_{n,m} \leq 1-\varepsilon$. But this limit should be worth 1 as we supposed the sized walk to be AST.

□

The following lemma allows to treat the base case of Lemma 42:

**Lemma 41** *Suppose that* $V$ *is a closed value of simple type* $\mathsf{Nat} \to \kappa$*. Then, for every* $\mathsf{Nat}^{\mathsf{i}} \to \mu :: \mathsf{Nat} \to \kappa$*, and for every size environment* $\rho$ *such that* $\rho(\mathsf{i}) = 0$*, we have* $V \in \mathsf{VRed}^1_{\mathsf{Nat}^{\mathsf{i}}\to\mu,\rho}$*.*

**Proof.** To prove that $V \in \mathsf{VRed}^1_{\mathsf{Nat}^{\mathsf{i}}\to\mu,\rho}$, we need to show that for every $q \in (0,1]$ and every $W \in \mathsf{VRed}^q_{\mathsf{Nat}^{\mathsf{i}},\rho}$ we have that $V\ W \in \mathsf{TRed}^q_{\mu,\rho}$. This is always the case, as $\mathsf{VRed}^q_{\mathsf{Nat}^{\mathsf{i}},\rho}$ is the empty set by definition: there is no term of the shape $\mathsf{S}^n\ \mathsf{0}$ with $n < \rho(\mathsf{i}) = 0$. □

The following lemma is the crucial result relating sized walks with the reducibility sets. It proves that, when the sized walk is AST, and after substitution of the variables of the context by reducible values in the recursively-defined term, we can prove the degree of reducibility to be any probability $Pr_{n,m}$ of convergence in finite time.

**Lemma 42 (Convergence in Finite Time and** $\mathsf{letrec}$**)** *Consider the distribution type* $\mu = \{\, (\mathsf{Nat}^{\mathfrak{s}_j} \to \nu[\mathfrak{s}_j/\mathsf{i}])^{p_j} \ \mid\ j \in \mathcal{J} \,\}$*. Let* $\Gamma$ *be the sized context* $x_1 : \mathsf{Nat}^{\mathfrak{r}_1}, \ldots, x_l : \mathsf{Nat}^{\mathfrak{r}_l}$*. Suppose that* $\Gamma \mid f : \mu \vdash V : \mathsf{Nat}^{\widehat{\mathsf{i}}} \to \nu[\widehat{\mathsf{i}}/\mathsf{i}]$ *and that* $\mu$ *induces an AST sized walk. Denote* $(Pr_{n,m})_{n\in\mathbb{N},m\in\mathbb{N}}$ *its associated probabilities of convergence in finite time. Suppose that* $V \in \mathsf{OVRed}^{\Gamma \mid f:\mu}_{\mathsf{Nat}^{\widehat{\mathsf{i}}}\to\nu[\widehat{\mathsf{i}}/\mathsf{i}],\rho}$ *for every* $\rho$*. Let* $\overrightarrow{W} \in \prod_{i=1}^{l} \mathsf{VRed}^1_{\mathsf{Nat}^{\mathfrak{r}_i},\rho}$*, then for every* $(n,m) \in \mathbb{N}^2$*, we have that*

$$\mathsf{letrec}\ f\ =\ V[\overrightarrow{W}/\overrightarrow{x}]\ \ \in\ \ \mathsf{VRed}^{Pr_{n,m}}_{\mathsf{Nat}^{\mathsf{i}}\to\nu,\rho[\mathsf{i}\mapsto m]}$$

**Proof.** We prove the statement by induction on $n$.

- If $n = 0$, we have two cases.
  - If $m = 0$, then Lemma 41 implies that $\mathsf{letrec}\ f = V[\overrightarrow{W}/\overrightarrow{x}] \in \mathsf{VRed}^1_{\mathsf{Nat}^{\mathsf{i}}\to\nu,\rho[\mathsf{i}\mapsto 0]}$ so that by downward closure (Lemma 22) we obtain $\mathsf{letrec}\ f = V[\overrightarrow{W}/\overrightarrow{x}] \in \mathsf{VRed}^{Pr_{n,0}}_{\mathsf{Nat}^{\mathsf{i}}\to\nu,\rho[\mathsf{i}\mapsto 0]}$.
  - If $m \neq 0$, then $Pr_{n,m} = 0$. The hypothesis of the lemma imply that $\mathsf{letrec}\ f = V[\overrightarrow{W}/\overrightarrow{x}] :: \mathsf{Nat} \to \langle\nu\rangle$, and we conclude using Lemma 21.

- Suppose that $n \geq 1$.
  - If $m = 0$, the result is immediate as in the previous case.
  - Suppose that $m > 0$. Then $m = m' + 1$. By definition, $\mathfrak{s}_j$ must be of the shape $\widehat{\mathsf{i}}^{k_j}$ with $k_j \geq 0$ for every $j \in \mathcal{J}$. We set $\mathcal{I} = \{k_j \mid j \in \mathcal{J}\}$ and $q_{k_j} = p_j$ for every $j \in \mathcal{J}$. The sized walk induced by the distribution type $\mu$ is then the sized walk associated to $(\mathcal{I}, (q_i)_{i \in \mathcal{I}}))$, which from the state $m' + 1 \in \mathbb{N} \setminus \{0\}$ moves:
    - to the state $m' + k_j$ with probability $p_j$, for every $j \in \mathcal{J}$;
    - to 0 with probability $1 - \left(\sum_{j \in \mathcal{J}} p_j\right)$.

    It follows that

$$Pr_{n+1,m'+1} \;=\; \sum_{j \in \mathcal{J}} p_j Pr_{n,m'+k_j} \;+\; 1 - \left(\sum_{j \in \mathcal{J}} p_j\right) \tag{15}$$

    For every $j \in \mathcal{J}$, let us apply the induction hypothesis and obtain

$$\mathsf{letrec}\ f \;=\; V[\overrightarrow{W}/\overrightarrow{x}] \;\in\; \mathsf{VRed}^{Pr_{n,m'+k_j}}_{\mathsf{Nat}^{\mathsf{i}} \to \nu, \rho[\mathsf{i} \mapsto m'+k_j]}$$

    By Lemma 26,

$$\mathsf{letrec}\ f \;=\; V[\overrightarrow{W}/\overrightarrow{x}] \;\in\; \mathsf{VRed}^{Pr_{n,m'+k_j}}_{\mathsf{Nat}^{\widehat{\mathsf{i}}^{k_j}} \to \nu[\widehat{\mathsf{i}}^{k_j}/\mathsf{i}], \rho[\mathsf{i} \mapsto m']} \;=\; \mathsf{VRed}^{Pr_{n,m'+k_j}}_{\mathsf{Nat}^{\mathfrak{s}_j} \to \nu[\mathfrak{s}_j/\mathsf{i}], \rho[\mathsf{i} \mapsto m']}$$

    Since this is valid for every $j \in \mathcal{J}$, we have that

$$\mathsf{letrec}\ f \;=\; V[\overrightarrow{W}/\overrightarrow{x}] \;\in\; \bigcap_{j \in \mathcal{J}} \mathsf{VRed}^{Pr_{n,m'+k_j}}_{\mathsf{Nat}^{\mathfrak{s}_j} \to \nu[\mathfrak{s}_j/\mathsf{i}], \rho[\mathsf{i} \mapsto m']}$$

    and since $V \in \mathsf{OVRed}^{\Gamma \mid f : \mu}_{\mathsf{Nat}^{\widehat{\mathsf{i}}} \to \nu[\widehat{\mathsf{i}}/\mathsf{i}], \rho[\mathsf{i} \mapsto m']}$ we obtain

$$V[\overrightarrow{W}, \mathsf{letrec}\ f \;=\; V[\overrightarrow{W}/\overrightarrow{x}]/\overrightarrow{x}, f] \;\in\; \mathsf{VRed}^{\sum_{j \in \mathcal{J}} p_j Pr_{n,m+k_j} \;+\; 1 - \left(\sum_{j \in \mathcal{J}} p_j\right)}_{\mathsf{Nat}^{\widehat{\mathsf{i}}} \to \nu[\widehat{\mathsf{i}}/\mathsf{i}], \rho[\mathsf{i} \mapsto m']}$$

    which, by (15) and by Lemma 26, gives

$$V[\overrightarrow{W}, \mathsf{letrec}\ f \;=\; V[\overrightarrow{W}/\overrightarrow{x}]/\overrightarrow{x}, f] \;\in\; \mathsf{VRed}^{Pr_{n+1,m'+1}}_{\mathsf{Nat}^{\mathsf{i}} \to \nu, \rho[\mathsf{i} \mapsto m'+1]}$$

    But this term is an unfolding of $\mathsf{letrec}\ f \;=\; V[\overrightarrow{W}/\overrightarrow{x}]$, so that by Corollary 4 we obtain

$$\mathsf{letrec}\ f \;=\; V[\overrightarrow{W}/\overrightarrow{x}] \;\in\; \mathsf{VRed}^{Pr_{n+1,m}}_{\mathsf{Nat}^{\mathsf{i}} \to \nu, \rho[\mathsf{i} \mapsto m]}$$

    Now by definition $Pr_{n+1,m} \geq Pr_{n,m}$, and by downward closure (Lemma 22):

$$\mathsf{letrec}\ f \;=\; V[\overrightarrow{W}/\overrightarrow{x}] \;\in\; \mathsf{VRed}^{Pr_{n,m}}_{\mathsf{Nat}^{\mathsf{i}} \to \nu, \rho[\mathsf{i} \mapsto m]}$$

□

## 6.11 Size Environments Mapping Sizes to Infinity

When $m = \infty$, the previous lemma does not allow to conclude, and an additional argument is required. Indeed, it does not make sense to consider a sized walk beginning from $\infty$: the meaning of this size is in fact *any integer*, not the ordinal $\omega$. Before we justify this understanding, we need the following companion lemma.

**Lemma 43** *If* $\mathtt{i}$ pos $\sigma$*, then*

- $\mathsf{VRed}^p_{\sigma,\rho[\mathtt{i}\mapsto n]} \subseteq \mathsf{VRed}^p_{\sigma,\rho[\mathtt{i}\mapsto\infty]}$,
- $\mathsf{DRed}^p_{\mu,\rho[\mathtt{i}\mapsto n]} \subseteq \mathsf{DRed}^p_{\mu,\rho[\mathtt{i}\mapsto\infty]}$,
- $\mathsf{TRed}^p_{\mu,\rho[\mathtt{i}\mapsto n]} \subseteq \mathsf{TRed}^p_{\mu,\rho[\mathtt{i}\mapsto\infty]}$.

**Proof.**

- Let $\mathfrak{s} = \widehat{\mathtt{i}}^n$. We have $[\![\mathfrak{s}]\!]_{\rho[\mathtt{i}\mapsto 0]} = n$. Using Lemma 26, we obtain
$$\mathsf{VRed}^p_{\sigma[\mathfrak{s}/\mathtt{i}],\rho[\mathtt{i}\mapsto 0]} \;=\; \mathsf{VRed}^p_{\sigma,\rho\left[\mathtt{i}\mapsto [\![\mathfrak{s}]\!]_{\rho[\mathtt{i}\mapsto 0]}\right]} \;=\; \mathsf{VRed}^p_{\sigma,\rho[\mathtt{i}\mapsto n]}$$
By the same lemma,
$$\mathsf{VRed}^p_{\sigma[\infty/\mathtt{i}],\rho[\mathtt{i}\mapsto 0]} \;=\; \mathsf{VRed}^p_{\sigma,\rho\left[\mathtt{i}\mapsto [\![\infty]\!]_{\rho[\mathtt{i}\mapsto 0]}\right]} \;=\; \mathsf{VRed}^p_{\sigma,\rho[\mathtt{i}\mapsto\infty]}$$
Since $\mathtt{i}$ pos $\sigma$ and $\mathfrak{s} \preccurlyeq \infty$, Lemma 16 implies that $\sigma[\mathfrak{s}/\mathtt{i}] \sqsubseteq \sigma[\infty/\mathtt{i}]$. By Lemma 37, we obtain $\mathsf{VRed}^p_{\sigma[\mathfrak{s}/\mathtt{i}],\rho[\mathtt{i}\mapsto 0]} \subseteq \mathsf{VRed}^p_{\sigma[\infty/\mathtt{i}],\rho[\mathtt{i}\mapsto 0]}$ and thus $\mathsf{VRed}^p_{\sigma,\rho[\mathtt{i}\mapsto n]} \subseteq \mathsf{VRed}^p_{\sigma,\rho[\mathtt{i}\mapsto\infty]}$.
- Let $\mathscr{D} \in \mathsf{DRed}^p_{\mu,\rho[\mathtt{i}\mapsto n]}$. It follows that $\mathscr{D} = \left[ (V_i)^{p_i} \;\middle|\; i \in \mathcal{I} \right]$ and that, setting $\mu = \left\{ (\sigma_j)^{p'_j} \;\middle|\; j \in \mathcal{J} \right\}$, there exists families $(p_{ij})_{i\in\mathcal{I},j\in\mathcal{J}}$ and $(q_{ij})_{i\in\mathcal{I},j\in\mathcal{J}}$ of reals of $[0,1]$ satisfying:
  1. $\forall i \in \mathcal{I},\;\; \forall j \in \mathcal{J},\;\; V_i \in \mathsf{VRed}^{q_{ij}}_{\sigma_j,\rho[\mathtt{i}\mapsto n]}$,
  2. $\forall i \in \mathcal{I},\;\; \sum_{j\in\mathcal{J}} p_{ij} \;=\; p_i$,
  3. $\forall j \in \mathcal{J},\;\; \sum_{i\in\mathcal{I}} p_{ij} \;=\; \mu(\sigma_j)$,
  4. $p \leq \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} q_{ij} p_{ij}$.

  Since $\forall i \in \mathcal{I},\;\; \forall j \in \mathcal{J},\;\; V_i \in \mathsf{VRed}^{q_{ij}}_{\sigma_j,\rho[\mathtt{i}\mapsto n]} \subseteq \mathsf{VRed}^{q_{ij}}_{\sigma_j,\rho[\mathtt{i}\mapsto\infty]}$, we obtain that $\mathscr{D} \in \mathsf{DRed}^p_{\mu,\rho[\mathtt{i}\mapsto\infty]}$ using the same witnesses.
- Let $M \in \mathsf{TRed}^p_{\mu,\rho[\mathtt{i}\mapsto n]}$. It follows that for every $0 \leq r < p$, there exists $\nu_r \preccurlyeq \mu$ and $n_r \in \mathbb{N}$ such that $M \Rightarrow^{n_r}_v \mathscr{D}_r$ and $\mathscr{D}_r \in \mathsf{DRed}^r_{\nu_r,\rho[\mathtt{i}\mapsto n]}$. But $\mathsf{DRed}^r_{\nu_r,\rho[\mathtt{i}\mapsto n]} \subseteq \mathsf{DRed}^r_{\nu_r,\rho[\mathtt{i}\mapsto\infty]}$, so that $M \in \mathsf{TRed}^p_{\mu,\rho[\mathtt{i}\mapsto\infty]}$.

□

The following lemma proves that $\infty$ stands for "every integer". It proves indeed that, if a term is in a reducibility set for any finite interpretation of a size, then it is also in the set where the size is interpreted as $\infty$:

**Lemma 44 (Reducibility for Infinite Sizes)** *Suppose that* $\mathtt{i}$ pos $\nu$ *and that* $W$ *is the value* $\mathsf{letrec}\ f \;=\; V$*. If* $W \in \mathsf{VRed}^p_{\mathsf{Nat}^{\mathtt{i}}\to\nu,\rho[\mathtt{i}\mapsto n]}$ *for every* $n \in \mathbb{N}$*, then* $W \in \mathsf{VRed}^p_{\mathsf{Nat}^{\mathtt{i}}\to\nu,\rho[\mathtt{i}\mapsto\infty]}$.

**Proof.** Suppose that $\mathtt{i}$ pos $\nu$ and that, for every $n \in \mathbb{N}$, $\mathsf{letrec}\ f \;=\; V \;\in\; \mathsf{VRed}^p_{\mathsf{Nat}^{\mathtt{i}}\to\nu,\rho[\mathtt{i}\mapsto n]}$. Let $W \in \mathsf{VRed}^p_{\mathsf{Nat}^{\mathtt{i}},\rho[\mathtt{i}\mapsto\infty]}$. Then $W = \mathsf{S}^m\ \mathsf{0}$ for some $m \in \mathbb{N}$. It follows that $W \in \mathsf{VRed}^p_{\mathsf{Nat}^{\mathtt{i}},\rho[\mathtt{i}\mapsto m+1]}$. But $\mathsf{letrec}\ f \;=\; V \;\in\; \mathsf{VRed}^p_{\mathsf{Nat}^{\mathtt{i}}\to\nu,\rho[\mathtt{i}\mapsto m+1]}$, so that
$$(\mathsf{letrec}\ f \;=\; V)\ W \in \mathsf{TRed}^p_{\nu,\rho[\mathtt{i}\mapsto m+1]}$$
By Lemma 43, since $\mathtt{i}$ pos $\nu$, we obtain that
$$(\mathsf{letrec}\ f \;=\; V)\ W \in \mathsf{TRed}^p_{\nu,\rho[\mathtt{i}\mapsto\infty]}$$
It follows that
$$\mathsf{letrec}\ f \;=\; V \;\in\; \mathsf{VRed}^p_{\mathsf{Nat}^{\mathtt{i}}\to\nu,\rho[\mathtt{i}\mapsto\infty]}$$

□

## 6.12 A Last Technical Lemma

The following technical lemma will allow us to deal with the Let rule in the proof of typing soundness.

**Lemma 45** *Let* $(q_i)_i \in (0,1]^n$, $(q'_j)_j \in (0,1]^m$ *and* $(q''_k)_k \in (0,1]^l$. *Let* $\mathcal{L}$ *and* $\mathcal{G}$ *be two sets of indexes. Let* $0 \leq r'' < (\prod_{i=1}^n q_i)\left(\prod_{j=1}^m q'_j\right)\left(\prod_{k=1}^l q''_k\right)$. *Suppose that, for every* $0 \leq r < \prod_{j=1}^m q'_j$, *there exists two families* $(p^r_{lg})_{l\in\mathcal{L},g\in\mathcal{G}}$ *and* $(q^r_{lg})_{l\in\mathcal{L},g\in\mathcal{G}}$ *of reals of* $[0,1]$ *satisfying*

$$r \leq \sum_{l\in\mathcal{L}} \sum_{g\in\mathcal{G}} p^r_{lg} q^r_{lg} \tag{16}$$

*and*

$$\sum_{l\in\mathcal{L}} \sum_{g\in\mathcal{G}} p^r_{lg} \leq 1 \tag{17}$$

*Then there exists* $0 \leq r < \prod_{j=1}^m q'_j$ *and a family* $(r'_{lg})_{l\in\mathcal{L},g\in\mathcal{G}}$ *satisfying*

$$\forall l \in \mathcal{L},\ \ \forall g \in \mathcal{G},\ \ 0 \leq r'_{lg} < \left(\prod_{i=1}^n q_i\right)\left(\prod_{k=1}^l q''_k\right) q^r_{lg} \tag{18}$$

*and*

$$r'' \leq \sum_{l\in\mathcal{L}} \sum_{g\in\mathcal{G}} p^r_{lg} r'_{lg} \tag{19}$$

**Proof.** Since $r'' < (\prod_{i=1}^n q_i)\left(\prod_{j=1}^m q'_j\right)\left(\prod_{k=1}^l q''_k\right)$, there exists $\varepsilon > 0$ and $\forall l \in \mathcal{L}$, $\forall g \in \mathcal{G}$, $\varepsilon_{lg} > 0$ satisfying

$$0 < \varepsilon + \sum_{l\in\mathcal{L}} \sum_{g\in\mathcal{G}} \varepsilon_{lg} < \left(\prod_{i=1}^n q_i\right)\left(\prod_{j=1}^m q'_j\right)\left(\prod_{k=1}^l q''_k\right) - r'' \tag{20}$$

We pick $r$ such that

$$\prod_{j=1}^m q'_j - \varepsilon < r < \prod_{j=1}^m q'_j \tag{21}$$

and this induces families $(p^r_{lg})_{l\in\mathcal{L},g\in\mathcal{G}}$ and $(q^r_{lg})_{l\in\mathcal{L},g\in\mathcal{G}}$ of reals of $[0,1]$ satisfying (16) and (17). We choose a family $(r'_{lg})_{l\in\mathcal{L},g\in\mathcal{G}}$ such that

$$\forall l \in \mathcal{L},\ \ \forall g \in \mathcal{G},\ \ \left(\prod_{i=1}^n q_i\right)\left(\prod_{k=1}^l q''_k\right) q^r_{lg} - \varepsilon_{lg} < r'_{lg} < \left(\prod_{i=1}^n q_i\right)\left(\prod_{k=1}^l q''_k\right) q^r_{lg}$$

By (17) and since $(\prod_{i=1}^n q_i)\left(\prod_{k=1}^l q''_k\right)$, we obtain from (20) that

$$\left(\prod_{i=1}^n q_i\right)\left(\prod_{k=1}^l q''_k\right)\varepsilon + \sum_{l\in\mathcal{L}} \sum_{g\in\mathcal{G}} p^r_{lg}\varepsilon_{lg} < \left(\prod_{i=1}^n q_i\right)\left(\prod_{j=1}^m q'_j\right)\left(\prod_{k=1}^l q''_k\right) - r''$$

Thus

$$r'' < \left(\prod_{i=1}^n q_i\right)\left(\prod_{k=1}^l q''_k\right)\left(\left(\prod_{j=1}^m q'_j\right) - \varepsilon\right) - \sum_{l\in\mathcal{L}} \sum_{g\in\mathcal{G}} p^r_{lg}\varepsilon_{lg}$$

By (21) and then (16):

$$r'' < \left(\prod_{i=1}^n q_i\right)\left(\prod_{k=1}^l q''_k\right)\left(\sum_{l\in\mathcal{L}} \sum_{g\in\mathcal{G}} p^r_{lg} q^r_{lg}\right) - \sum_{l\in\mathcal{L}} \sum_{g\in\mathcal{G}} p^r_{lg}\varepsilon_{lg}$$

which rewrites to

$$r'' < \sum_{l \in \mathcal{L}} \sum_{g \in \mathcal{G}} p^r_{lg} \left( \left( \prod_{i=1}^{n} q_i \right) \left( \prod_{k=1}^{l} q''_k \right) q^r_{lg} - \varepsilon_{lg} \right)$$

and by definition of $(r'_{lg})_{l \in \mathcal{L}, g \in \mathcal{G}}$ we obtain

$$r'' < \sum_{l \in \mathcal{L}} \sum_{g \in \mathcal{G}} p^r_{lg} r'_{lg}$$

as requested. □

## 6.13 Typing Soundness

All these fundamental lemmas allow us to prove the following proposition, which expresses that all typable terms are reducible and is the key step towards the fact that typability implies AST:

**Proposition 5 (Typing Soundness)** *If $\Gamma \,|\, \Theta \vdash M : \mu$, then $M \in \mathsf{OTRed}^{\Gamma \,|\, \Theta}_{\mu,\rho}$ for every $\rho$. Similarly, if $\Gamma \,|\, \Theta \vdash V : \sigma$, then $V \in \mathsf{OVRed}^{\Gamma \,|\, \Theta}_{\sigma,\rho}$ for every $\rho$.*

**Proof.** We proceed by induction on the derivation of the sequent $\Gamma \,|\, \Theta \vdash M : \mu$. When $M = V$ is a value, we know by Lemma 6 that $\mu = \{ \sigma^1 \}$; and we prove that $V \in \mathsf{OVRed}^{\Gamma \,|\, \Theta}_{\sigma,\rho}$ for every $\rho$. By Lemma 38 we obtain that $V \in \mathsf{OTRed}^{\Gamma \,|\, \Theta}_{\mu,\rho}$ for every $\rho$. We proceed by case analysis on the last rule of the derivation:

We suppose in the following that $\Gamma$ is a sized context which can be enumerated in the form $x_1 : \sigma_1, \ldots, x_n : \sigma_n$, and that $y$ is a variable distinct from $x_1, \ldots, x_n$. We proceed accordingly to the last rule of the derivation:

- **Var:** Suppose that $\Gamma, y : \tau \,|\, \Theta \vdash y : \tau$. Let $(q_i)_i \in [0,1]^{n+1}$ and $(V_1, \ldots, V_n, W) \in \left(\prod_{i=1}^{n} \mathsf{VRed}^{q_i}_{\sigma_i,\rho}\right) \times \mathsf{VRed}^{q_{n+1}}_{\tau,\rho}$.
  - If $\Theta = \emptyset$, we need to prove that $y[\overrightarrow{V}, W/\overrightarrow{x}, y] = W \in \mathsf{VRed}^{\prod_{i=1}^{n+1} q_i}_{\tau,\rho}$. This is immediate since $\prod_{i=1}^{n+1} q_i \leq q_{n+1}$, using Lemma 22.
  - If $\Theta = z : \{ \theta^{p_j}_j \mid j \in \mathcal{J} \}$, let $\left(q'_j\right)_{j \in \mathcal{J}} \in [0,1]^{\mathcal{J}}$ and $Z \in \bigcap_{j \in \mathcal{J}} \mathsf{VRed}^{q'_j}_{\sigma_j,\rho}$. We need to prove that $y[\overrightarrow{V}, W, Z/\overrightarrow{x}, y, z] = W \in \mathsf{VRed}^{\left(\prod_{i=1}^{n+1} q_i\right)\left(\sum_{j \in \mathcal{J}} p_j q'_j\right)}_{\tau,\rho}$. But, again, $\left(\prod_{i=1}^{n+1} q_i\right)\left(\sum_{j \in \mathcal{J}} p_j q'_j\right) \leq q_{n+1}$ since $q_i \leq 1$ for every $i$, $q'_j \leq 1$ for every $j$ and $\sum_{j \in \mathcal{J}} p_j = 1$. We conclude using Lemma 22.

- **Var':** Suppose that $\Gamma \,|\, y : \{ \tau^1 \} \vdash y : \tau$. Let $(q_i)_i \in [0,1]^{n+1}$ and $(V_1, \ldots, V_n, W) \in \left(\prod_{i=1}^{n} \mathsf{VRed}^{q_i}_{\sigma_i,\rho}\right) \times \mathsf{VRed}^{q_{n+1}}_{\tau,\rho}$. We need to prove that $y[\overrightarrow{V}, W/\overrightarrow{x}, y] = W \in \mathsf{VRed}^{\prod_{i=1}^{n+1} q_i}_{\tau,\rho}$. This is immediate since $\prod_{i=1}^{n+1} q_i \leq q_{n+1}$, using Lemma 22.

- **Succ:** Suppose that $\Gamma \,|\, \Theta \vdash \mathsf{S}\ V : \mathsf{Nat}^{\widehat{\mathfrak{s}}}$. Suppose moreover that $\Theta = \emptyset$. Let $(q_i)_i \in [0,1]^n$ and $(W_1, \ldots, W_n) \in \prod_{i=1}^{n} \mathsf{VRed}^{q_i}_{\sigma_i,\rho}$. We need to prove that $(\mathsf{S}\ V)\,[\overrightarrow{W}/\overrightarrow{x}] \in \mathsf{VRed}^{\prod_{i=1}^{n} q_i}_{\mathsf{Nat}^{\widehat{\mathfrak{s}}},\rho}$. But $(\mathsf{S}\ V)\,[\overrightarrow{W}/\overrightarrow{x}] = \mathsf{S}\ \left(V[\overrightarrow{W}/\overrightarrow{x}]\right)$ and, by induction hypothesis, $V[\overrightarrow{W}/\overrightarrow{x}] \in \mathsf{VRed}^{\prod_{i=1}^{n} q_i}_{\mathsf{Nat}^{\mathfrak{s}},\rho}$. By Lemma 39, $(\mathsf{S}\ V)\,[\overrightarrow{W}/\overrightarrow{x}] \in \mathsf{VRed}^{\prod_{i=1}^{n} q_i}_{\mathsf{Nat}^{\widehat{\mathfrak{s}}},\rho}$ and we can conclude. The case where $\Theta \neq \emptyset$ is similar.

- **Zero:** Suppose that $\Gamma \,|\, \Theta \vdash \mathsf{0} : \mathsf{Nat}^{\widehat{\mathfrak{s}}}$. Suppose moreover that $\Theta = \emptyset$. Let $(q_i)_i \in [0,1]^n$ and $(V_1, \ldots, V_n) \in \prod_{i=1}^{n} \mathsf{VRed}^{q_i}_{\sigma_i,\rho}$. By Lemma 39, $\mathsf{0}[\overrightarrow{V}/\overrightarrow{x}] = 0 \in \mathsf{VRed}^{\prod_{i=1}^{n} q_i}_{\mathsf{Nat}^{\widehat{\mathfrak{s}}},\rho}$. The case where $\Theta \neq \emptyset$ is similar.

– **$\lambda$:** Suppose that $\Gamma \,|\, \Theta \vdash \lambda y.M : \sigma \to \mu$, with $\Theta = z : \{ (\tau_j)^{p_j} \mid j \in \mathcal{J} \}$. Let $(q_i)_i \in [0,1]^n$ and $(V_1, \ldots, V_n) \in \prod_{i=1}^n \mathsf{VRed}^{q_i}_{\sigma_i,\rho}$. Let $\left(q'_j\right)_{j\in\mathcal{J}} \in [0,1]^{\mathcal{J}}$ and $W \in \bigcap_{j\in\mathcal{J}} \mathsf{VRed}^{q'_j}_{\sigma_j,\rho}$. We need to prove that

$$(\lambda y.M)\,[\overrightarrow{V}, W/\overrightarrow{x}, z] \;=\; \lambda y.M[\overrightarrow{V}, W/\overrightarrow{x}, z] \in \mathsf{VRed}^{(\prod_{i=1}^n q_i)(\sum_{j\in\mathcal{J}} p_j q'_j)}_{\sigma\to\mu,\rho}$$

Therefore, let $q'' \in (0,1]$ and $Z \in \mathsf{VRed}^{q''}_{\sigma,\rho}$. We now have to prove that

$$\left(\lambda y.M[\overrightarrow{V}, W/\overrightarrow{x}, z]\right)\; Z \;\in\; \mathsf{TRed}^{q''(\prod_{i=1}^n q_i)(\sum_{j\in\mathcal{J}} p_j q'_j)}_{\mu,\rho} \tag{22}$$

But

$$\left(\lambda y.M[\overrightarrow{V}, W/\overrightarrow{x}, z]\right)\; Z \;\to_v\; M[\overrightarrow{V}, W, Z/\overrightarrow{x}, z, y]$$

Since $\Gamma,\, x : \sigma \,|\, \Theta \vdash M : \mu$ by typing, the induction hypothesis ensures that $M[\overrightarrow{V}, W, Z/\overrightarrow{x}, z, y] \in \mathsf{TRed}^{q''(\prod_{i=1}^n q_i)(\sum_{j\in\mathcal{J}} p_j q'_j)}_{\mu,\rho}$ and by Lemma 36 we obtain that (22) holds, which allows to conclude.
The case where $\Theta = \emptyset$ is similar.

– **Sub:** Suppose that $\Gamma \,|\, \Theta \vdash M : \nu$ is derived from $\Gamma \,|\, \Theta \vdash M : \mu$ where $\mu \sqsubseteq \nu$. Suppose that $\Theta = \emptyset$. Let $(q_i)_i \in [0,1]^n$ and $(V_1, \ldots, V_n) \in \prod_{i=1}^n \mathsf{VRed}^{q_i}_{\sigma_i,\rho}$. By induction hypothesis, $M[V/\overrightarrow{x}] \in \mathsf{TRed}^{\prod_{i=1}^n q_i}_{\mu,\rho}$ so that by Lemma 37 we have $M[V/\overrightarrow{x}] \in \mathsf{TRed}^{\prod_{i=1}^n q_i}_{\nu,\rho}$ which allows to conclude.
The case where $\Theta \neq \emptyset$ is similar.

– **App:** Suppose that $\Gamma,\, \Delta,\, \Xi \,|\, \Theta,\, \Psi \vdash V\; W : \mu$. Suppose that $\Theta,\, \Psi = \emptyset$. We set $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$, $\Delta = y_1 : \tau_1, \ldots, y_m : \tau_m$ and $\Xi = z_1 : \theta_1, \ldots, z_l : \theta_l$. Let $(q_i)_i \in [0,1]^n$, $(q'_j)_j \in [0,1]^m$, $(q''_k)_k \in [0,1]^l$, $(V_1, \ldots, V_n) \in \prod_{i=1}^n \mathsf{VRed}^{q_i}_{\sigma_i,\rho}$, $(W_1, \ldots, W_m) \in \prod_{j=1}^m \mathsf{VRed}^{q'_j}_{\tau_j,\rho}$, and $(Z_1, \ldots, Z_l) \in \prod_{k=1}^l \mathsf{VRed}^{q_k}_{\theta_k,\rho}$. We need to prove that

$$(V\; W)\,[\overrightarrow{V}, \overrightarrow{W}, \overrightarrow{Z}/\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}] \;=\; V[\overrightarrow{V}, \overrightarrow{W}, \overrightarrow{Z}/\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}]\;\; W[\overrightarrow{V}, \overrightarrow{W}, \overrightarrow{Z}/\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}] \tag{23}$$

is in $\mathsf{TRed}^{(\prod_{i=1}^n q_i)(\prod_{j=1}^m q'_j)(\prod_{k=1}^l q''_k)}_{\mu,\rho}$.

- Suppose that $\prod_{i=1}^n q_i = 0$. Then we need to prove that (23) is in $\mathsf{TRed}^0_{\mu,\rho}$, which is immediate by Lemma 21 as it is of simple type $\langle\mu\rangle$.
- Suppose that $\prod_{i=1}^n q_i \neq 0$. It follows that $\forall i \in \mathcal{I},\; q_i \neq 0$. We have that $\Gamma,\, \Delta \,|\, \emptyset \vdash V : \sigma \to \mu$ which, by induction hypothesis, gives that $V \in \mathsf{OVRed}^{\Gamma,\Delta\,|\,\emptyset}_{\sigma\to\mu,\rho}$. Note that for every $i \in \mathcal{I}$ we have $\sigma_i :: \mathsf{Nat}$; since $q_i \neq 0$, we have by definition of the sets of candidates that $\mathsf{VRed}^{q_i}_{\sigma_i,\rho} = \mathsf{VRed}^1_{\sigma_i,\rho}$. It follows that $V[\overrightarrow{V}, \overrightarrow{W}/\overrightarrow{x}, \overrightarrow{y}] = V[\overrightarrow{V}, \overrightarrow{W}\overrightarrow{Z}/\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}] \in \mathsf{VRed}^{(\prod_{i=1}^n 1)(\prod_{j=1}^m q'_j)}_{\sigma\to\mu,\rho} = \mathsf{VRed}^{\prod_{j=1}^m q'_j}_{\sigma\to\mu,\rho}$. Since $\Gamma,\, \Xi \,|\, \Psi \vdash W : \sigma$, we obtain similarly from the induction hypothesis that $W[\overrightarrow{V}, \overrightarrow{W}\overrightarrow{Z}/\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}] \in \mathsf{VRed}^{\prod_{k=1}^l q''_k}_{\sigma,\rho}$. By definition of $\mathsf{VRed}^{\prod_{j=1}^m q'_j}_{\sigma\to\mu,\rho}$, we obtain that

$$V[\overrightarrow{V}, \overrightarrow{W}, \overrightarrow{Z}/\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}]\;\; W[\overrightarrow{V}, \overrightarrow{W}, \overrightarrow{Z}/\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}] \;\in\; \mathsf{TRed}^{(\prod_{j=1}^m q'_j)(\prod_{k=1}^l q''_k)}_{\mu,\rho}$$

  and by downwards closure (Lemma 22) we obtain that (23) is in $\mathsf{TRed}^{(\prod_{i=1}^n q_i)(\prod_{j=1}^m q'_j)(\prod_{k=1}^l q''_k)}_{\mu,\rho}$ so that we can conclude.

The case where $\Theta,\, \Psi \neq \emptyset$ is similar.

– **Choice:** Suppose that $\Gamma \,|\, \Theta \oplus_p \Psi \;\vdash\; M \oplus_p N \,:\, \mu \oplus_p \nu$. Suppose that $\Theta \neq \emptyset$ and that $\Psi \neq \emptyset$. We set $\Theta \;=\; y \,:\, \left\{\, \tau_j^{p_j} \;\middle|\; j \in \mathcal{J} \,\right\}$ and $\Psi \;=\; y \,:\, \left\{\, (\tau'_k)^{p'_k} \;\middle|\; k \in \mathcal{K} \,\right\}$ where we suppose that $j \in \mathcal{J} \cap \mathcal{L} \Leftrightarrow \sigma_j = \tau_j$. We obtain that

$$\begin{aligned}\Theta \oplus_p \Psi \;=\; y \,:\; & \left\{\, \tau_j^{pp_j} \;\middle|\; j \in \mathcal{J} \setminus (\mathcal{J} \cap \mathcal{K}) \,\right\} + \left\{\, (\tau_l)^{pp_l + (1-p)p'_l} \;\middle|\; l \in \mathcal{J} \cap \mathcal{K} \,\right\} \\ & + \left\{\, (\tau'_k)^{(1-p)p'_k} \;\middle|\; k \in \mathcal{K} \setminus (\mathcal{J} \cap \mathcal{K}) \,\right\}\end{aligned}$$

Let $(q_i)_i \in [0,1]^n$, $(q'_j)_j \in [0,1]^{|\mathcal{J} \setminus (\mathcal{J} \cap \mathcal{K})|}$, $(q''_l)_l \in [0,1]^{|\mathcal{J} \cap \mathcal{K}|}$, $(q'''_k)_k \in [0,1]^{|\mathcal{K} \setminus (\mathcal{J} \cap \mathcal{K})|}$, $(V_1, \ldots, V_n) \in \prod_{i=1}^n \mathsf{VRed}^{q_i}_{\sigma_i,\rho}$, and

$$W \in \bigcap_{j \in \mathcal{J} \setminus (\mathcal{J} \cap \mathcal{K})} \mathsf{VRed}^{q'_j}_{\tau_j,\rho} \cap \bigcap_{l \in \mathcal{J} \cap \mathcal{K}} \mathsf{VRed}^{q''_l}_{\tau_l,\rho} \cap \bigcap_{k \in \mathcal{K} \setminus (\mathcal{J} \cap \mathcal{K})} \mathsf{VRed}^{q'''_k}_{\tau'_k,\rho}$$

We need to prove that $(M \oplus_p N)\,[\overrightarrow{V}, W/\overrightarrow{x}, y]$ is in

$$\begin{aligned}& \mathsf{TRed}^{\left(\prod_{i=1}^n q_i\right)\left(\sum_{j \in \mathcal{J} \setminus (\mathcal{J} \cap \mathcal{K})} pp_j q'_j + \sum_{l \in \mathcal{J} \cap \mathcal{K}} (pp_l + (1-p)p'_l) q''_l + \sum_{k \in \mathcal{K} \setminus (\mathcal{J} \cap \mathcal{K})} (1-p) p'_k q'''_k\right)}_{\mu \oplus_p \nu, \rho} \\ = \;\; & \mathsf{TRed}^{p\left(\prod_{i=1}^n q_i\right)\left(\sum_{j \in \mathcal{J} \setminus (\mathcal{J} \cap \mathcal{K})} p_j q'_j + \sum_{l \in \mathcal{J} \cap \mathcal{K}} p_l q''_l\right) + (1-p)\left(\prod_{i=1}^n q_i\right)\left(\sum_{l \in \mathcal{J} \cap \mathcal{K}} p'_l q''_l + \sum_{k \in \mathcal{K} \setminus (\mathcal{J} \cap \mathcal{K})} p'_k q'''_k\right)}_{\mu \oplus_p \nu, \rho}\end{aligned}$$

Typing gives us that $\Gamma \,|\, \Theta \;\vdash\; M \,:\, \mu$, which by induction hypothesis implies that

$$M[\overrightarrow{V}, W/\overrightarrow{x}, y] \;\in\; \mathsf{TRed}^{\left(\prod_{i=1}^n q_i\right)\left(\sum_{j \in \mathcal{J} \setminus (\mathcal{J} \cap \mathcal{K})} p_j q'_j + \sum_{l \in \mathcal{J} \cap \mathcal{K}} p_l q''_l\right)}_{\mu,\rho}$$

Typing also implies that $\Gamma \,|\, \Psi \;\vdash\; N \,:\, \nu$, and provides by induction hypothesis

$$N[\overrightarrow{V}, W/\overrightarrow{x}, y] \;\in\; \mathsf{TRed}^{\left(\prod_{i=1}^n q_i\right)\left(\sum_{l \in \mathcal{J} \cap \mathcal{K}} p'_l q''_l + \sum_{k \in \mathcal{K} \setminus (\mathcal{J} \cap \mathcal{K})} p'_k q'''_k\right)}_{\mu \oplus_p \nu, \rho}$$

Since

$$(M \oplus_p N)\,[\overrightarrow{V}, W/\overrightarrow{x}, y] \;\;\to_v\;\; \left\{ \left(M[\overrightarrow{V}, W/\overrightarrow{x}, y]\right)^p, \left(N[\overrightarrow{V}, W/\overrightarrow{x}, y]\right)^{1-p} \right\}$$

Lemma 36 allows to conclude.
The cases where $\Theta = \emptyset$ or $\Psi = \emptyset$ are treated similarly.

– **Let:** Suppose that $\Gamma, \Delta, \Xi \,|\, \Theta, \left(\sum_{i \in \mathcal{I}} p_i \cdot \Psi_i\right) \vdash \mathsf{let}\; x \;=\; M \;\mathsf{in}\; N \,:\, \sum_{i \in \mathcal{I}} p_i \cdot \mu_i$. Let $\Gamma \;=\; x_1 \,:\, \sigma_1, \ldots, x_n \,:\, \sigma_n$, $\Delta \;=\; y_1 \,:\, \tau_1, \ldots, y_m \,:\, \tau_m$, and $\Xi \;=\; z_1 \,:\, \theta_1, \ldots, z_m \,:\, \theta_l$. Let $(q_i)_i \in [0,1]^n$, $(q'_j)_j \in [0,1]^m$ and $(q''_k)_k \in [0,1]^l$. Let $(V_1, \ldots, V_n) \in \prod_{i=1}^n \mathsf{VRed}^{q_i}_{\sigma_i,\rho}$, $(W_1, \ldots, W_m) \in \prod_{j=1}^m \mathsf{VRed}^{q'_j}_{\tau_j,\rho}$, and $(Z_1, \ldots, Z_l) \in \prod_{k=1}^l \mathsf{VRed}^{q''_k}_{\theta_k,\rho}$. There are two subcases here.

- Suppose that $M$ is a value. Then the last typing rule is

$$\frac{\Gamma, \Delta \,|\, \Theta \vdash M \,:\, \sigma \qquad \Gamma, \Xi, x \,:\, \sigma \,|\, \Psi \vdash N \,:\, \mu \qquad \langle \Gamma \rangle \;=\; \mathsf{Nat}}{\Gamma, \Delta, \Xi \,|\, \Theta, \Psi \vdash \mathsf{let}\; x \;=\; M \;\mathsf{in}\; N \,:\, \mu}$$

  We treat the case where $\Theta = \Psi = \emptyset$, the two other ones being similar. We need to prove that

$$(\mathsf{let}\; x \;=\; M \;\mathsf{in}\; N)\,[\overrightarrow{V}, \overrightarrow{W}, \overrightarrow{Z}/\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}] \in \mathsf{TRed}^{\left(\prod_{i \in \mathcal{I}} q_i\right)\left(\prod_{j \in \mathcal{J}} q'_j\right)\left(\prod_{k \in \mathcal{K}} q''_k\right)}_{\mu,\rho} \tag{24}$$

  We now distinguish two cases.

  – Suppose that $\prod_{i \in \mathcal{I}} q_i \;=\; 0$. Then (25) holds immediately since by Lemma 21 all the terms of simple type $\langle \mu \rangle$ are in $\mathsf{TRed}^0_{\mu,\rho}$.

– Else for every $i \in \mathcal{I}$ we have $\mathsf{VRed}^{q_i}_{\sigma_i,\rho} = \mathsf{VRed}^{1}_{\sigma_i,\rho}$. Since $\Gamma,\, \Delta \,|\, \Theta \vdash M : \sigma$, we obtain by induction hypothesis that $M[\overrightarrow{V}, \overrightarrow{W}/\overrightarrow{x}, \overrightarrow{y}] \in \mathsf{TRed}^{(\prod_{i\in\mathcal{I}} 1)(\prod_{j\in\mathcal{J}} q'_j)}_{\sigma,\rho}$. None of the $\overrightarrow{z}$ occur in $M$, so $M[\overrightarrow{V}, \overrightarrow{W}, \overrightarrow{Z}/\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{Z}] \in \mathsf{TRed}^{\prod_{j\in\mathcal{J}} q'_j}_{\sigma,\rho}$. Since $\Gamma,\, \Xi,\, x : \sigma \,|\, \Psi \vdash N : \mu$, we obtain by induction hypothesis that $N[\overrightarrow{V}, \overrightarrow{W}, \overrightarrow{Z}, M[\overrightarrow{V}, \overrightarrow{Z}/\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}]/\overrightarrow{x}, \overrightarrow{z}, x]$ is in $\mathsf{TRed}^{(\prod_{i\in\mathcal{I}} q_i)(\prod_{j\in\mathcal{J}} q'_j)(\prod_{k\in\mathcal{K}} q''_k)}_{\mu,\rho}$. Since none of the variables of $\overrightarrow{y}$ occur in this term, we obtain

$$N[\overrightarrow{V}, \overrightarrow{W}, \overrightarrow{Z}, M[\overrightarrow{V}, \overrightarrow{W}, \overrightarrow{Z}/\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}]/\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}, x] \;\in\; \mathsf{TRed}^{(\prod_{i\in\mathcal{I}} q_i)(\prod_{j\in\mathcal{J}} q'_j)(\prod_{k\in\mathcal{K}} q''_k)}_{\mu,\rho}$$

Now

$$\begin{array}{rl} & (\mathsf{let}\ x \;=\; M\ \mathsf{in}\ N)\,[\overrightarrow{V}, \overrightarrow{W}, \overrightarrow{Z}/\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}] \\ = & \mathsf{let}\ x \;=\; M[\overrightarrow{V}, \overrightarrow{W}, \overrightarrow{Z}/\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}]\ \mathsf{in}\ N[\overrightarrow{V}, \overrightarrow{W}, \overrightarrow{Z}/\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}] \\ \to_v & \left\{ \left( N[\overrightarrow{V}, \overrightarrow{W}, \overrightarrow{Z}/\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}][M[\overrightarrow{V}, \overrightarrow{W}, \overrightarrow{Z}/\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}]/x] \right)^1 \right\} \\ = & \left\{ \left( N[\overrightarrow{V}, \overrightarrow{W}, \overrightarrow{Z}, M[\overrightarrow{V}, \overrightarrow{W}, \overrightarrow{Z}/\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}]/\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}, x] \right)^1 \right\} \end{array}$$

and it follows from Lemma 36 that (25) holds, allowing to conclude.

- Suppose that $M$ is not a value. We treat in a first time the case where $\Theta = \Psi = \emptyset$. The case where $\Theta \neq \emptyset$ is exactly similar, while the case where $\Psi \neq \emptyset$ reveals the reason why a sum $\sum_{j\in\mathcal{J}} p_j q'_j$ appears in the definitions of $\mathsf{OTRed}$ and $\mathsf{OVRed}$. The last typing rule is

$$\frac{\Gamma,\, \Delta \,|\, \emptyset \vdash M : \left\{ \sigma_h^{p_h} \;\middle|\; h \in \mathcal{H} \right\} \qquad \Gamma,\, \Xi,\, x : \sigma_h \,|\, \emptyset \vdash N : \mu_h \qquad \langle \Gamma \rangle = \mathsf{Nat}}{\Gamma,\, \Delta,\, \Xi \,|\, \emptyset \vdash \mathsf{let}\ x \;=\; M\ \mathsf{in}\ N : \sum_{h\in\mathcal{H}} p_h \cdot \mu_h}$$

We need to prove that

$$\begin{array}{rl} & (\mathsf{let}\ x \;=\; M\ \mathsf{in}\ N)\,[\overrightarrow{V}, \overrightarrow{W}, \overrightarrow{Z}/\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}] \\ = & \mathsf{let}\ x \;=\; M[\overrightarrow{V}, \overrightarrow{W}, \overrightarrow{Z}/\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}]\ \mathsf{in}\ N[\overrightarrow{V}, \overrightarrow{W}, \overrightarrow{Z}/\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}] \\ & \in \mathsf{TRed}^{(\prod_{i\in\mathcal{I}} q_i)(\prod_{j\in\mathcal{J}} q'_j)(\prod_{k\in\mathcal{K}} q''_k)}_{\sum_{h\in\mathcal{H}} p_h\cdot\mu_h,\rho} \end{array} \tag{25}$$

We now distinguish two cases.

– Suppose that $\left(\prod_{i\in\mathcal{I}} q_i\right)\left(\prod_{j\in\mathcal{J}} q'_j\right)\left(\prod_{k\in\mathcal{K}} q''_k\right) = 0$. Then (25) holds immediately since by Lemma 21 all the terms of simple type $\langle \sum_{h\in\mathcal{H}} p_h\cdot\mu_h \rangle$ are in $\mathsf{TRed}^{0}_{\sum_{h\in\mathcal{H}} p_h\cdot\mu_h,\rho}$.

– Else, we use the induction hypothesis on $\Gamma,\, \Delta \,|\, \emptyset \vdash M : \left\{ \sigma_h^{p_h} \;\middle|\; h \in \mathcal{H} \right\}$. Since $\langle \sigma_i \rangle = \mathsf{Nat}$, for every $i \in \mathcal{I}$ we have $\mathsf{VRed}^{q_i}_{\sigma_i,\rho} = \mathsf{VRed}^{1}_{\sigma_i,\rho}$. Together with the fact that $\overrightarrow{z}$ does not appear in $M$, we obtain that

$$M[\overrightarrow{V}, \overrightarrow{W}, \overrightarrow{Z}/\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}] \in \mathsf{TRed}^{\prod_{j\in\mathcal{J}} q'_j}_{\left\{ \sigma_h^{p_h} \;\middle|\; h\in\mathcal{H} \right\},\rho}$$

By definition, for every $0 \leq r < \prod_{j=1}^{m} q'_j$, there exists $n_r$ and $\nu_r = \left\{ \sigma_g^{p_{r,g}} \;\middle|\; g \in \mathcal{G}_r \right\} \preccurlyeq \left\{ \sigma_h^{p_h} \;\middle|\; h \in \mathcal{H} \right\}$ with $\mathcal{G}_r \subseteq \mathcal{H}$ such that

$$M[\overrightarrow{V}, \overrightarrow{W}, \overrightarrow{Z}/\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}] \;\Rrightarrow_v^{n_r}\; \mathscr{D}_r \;=\; \left[ X_l^{p''_{r,l}} \;\middle|\; l \in \mathcal{L}_r \right] \;\in\; \mathsf{DRed}^{r}_{\nu_r,\rho}$$

This implies the existence of two families $(p^r_{lg})_{l\in\mathcal{L}_r, g_r\in\mathcal{G}}$ and $(q^r_{lg})_{l\in\mathcal{L}_r, g_r\in\mathcal{G}}$ of reals of $[0,1]$ satisfying in particular

$$r \leq \sum_{l\in\mathcal{L}_r} \sum_{g\in\mathcal{G}_r} p^r_{lg} q^r_{lg} \tag{26}$$

$$\sum_{l\in\mathcal{L}_r}\ \sum_{g\in\mathcal{G}_r}\ p^r_{lg} \leq 1 \tag{27}$$

$$\forall l \in \mathcal{L},\ \ \sum_{g\in\mathcal{G}_r}\ p^r_{lg}\ =\ p''_{r,l} \tag{28}$$

$$\forall g \in \mathcal{G},\ \ \sum_{l\in\mathcal{L}_r}\ p^r_{lg}\ =\ p_{r,g} \tag{29}$$

and

$$\forall l \in \mathcal{L}_r,\ \ \forall g \in \mathcal{G}_r,\ \ X_l \in \mathsf{VRed}^{q^r_{lg}}_{\sigma_g,\rho} \tag{30}$$

By (26) and (27), we can apply Lemma 45 and we obtain $0 \leq r < \prod_{j=1}^m\ q'_j$ and a family $(r'_{lg})_{l\in\mathcal{L}_r, g\in\mathcal{G}_r}$ satisfying

$$\forall l \in \mathcal{L}_r,\ \ \forall g \in \mathcal{G}_r,\ \ 0 \leq r'_{lg} < \left(\prod_{i=1}^n\ q_i\right)\left(\prod_{k=1}^l\ q''_k\right) q^r_{lg} \tag{31}$$

and

$$r'' \leq \sum_{l\in\mathcal{L}}\ \sum_{g\in\mathcal{G}}\ p^r_{lg} r'_{lg} \tag{32}$$

We now consider $r$ to be fixed to this value given by the lemma, this providing $\mathscr{D}_r$, $\nu_r$ and so on.
Since $\Gamma, \Xi, x\ :\ \sigma_h \,|\, \emptyset \vdash N\ :\ \mu_h$, we obtain by induction hypothesis using (30) that for every $l \in \mathcal{L}$ and $g \in \mathcal{G}$ we have

$$N[\overrightarrow{V}, \overrightarrow{W}, \overrightarrow{Z}, X_l/\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}, x]\ \in\ \mathsf{TRed}^{(\prod_{i=1}^n\ q_i)(\prod_{k=1}^l\ q''_k)q^r_{lg}}_{\mu_g,\rho} \tag{33}$$

By 31, there exists for every $l \in \mathcal{L}$ and $g \in \mathcal{G}$ and index $m_{lg}$ and a type $\mu'_{lg} \sqsubseteq \mu_g$ such that

$$N[\overrightarrow{V}, \overrightarrow{W}, \overrightarrow{Z}, X_l/\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}, x]\ \Rrightarrow^{m_{lg}}_v\ \mathscr{E}_{lg}\ \in\ \mathsf{DRed}^{r'_{lg}}_{\mu'_{lg},\rho} \tag{34}$$

Now set

$$m\ =\ \max_{l\in\mathcal{L}, g\in\mathcal{G}}\ m_{lg}$$

By Lemma 33, we obtain types $\mu'_{lg} \preccurlyeq \mu''_{lg} \preccurlyeq \mu_g$ and distributions $\mathscr{E}'_{lg}$ such that all the reduction lengths are the same:

$$N[\overrightarrow{V}, \overrightarrow{W}, \overrightarrow{Z}, X_l/\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}, x]\ \Rrightarrow^m_v\ \mathscr{E}'_{lg}\ \in\ \mathsf{DRed}^{r'_{lg}}_{\mu''_{lg},\rho} \tag{35}$$

Now it follows of (28) that

$$\mathscr{D}_r\ =\ \left[\, X_l^{p^r_{l,g}}\ \mid\ l \in \mathcal{L}_r,\ g \in \mathcal{G}_r \,\right]$$

which allows us to use Lemma 4, obtaining that

$$(\mathsf{let}\ x\ =\ M\ \mathsf{in}\ N)\,[\overrightarrow{V}, \overrightarrow{W}, \overrightarrow{Z}/\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}]\ \ \Rrightarrow^{n_r+m+1}_v\ \ \sum_{l\in\mathcal{L}}\ \sum_{g\in\mathcal{G}}\ p^r_{l,g}\cdot\mathscr{E}'_{lg}$$

By (35) and Lemma 35, we obtain that

$$\sum_{l\in\mathcal{L}}\ \sum_{g\in\mathcal{G}}\ p^r_{l,g}\cdot\mathscr{E}'_{lg}\ \in\ \mathsf{DRed}^{\sum_{l\in\mathcal{L}}\ \sum_{g\in\mathcal{G}}\ p^r_{l,g} r'_{l,g}}_{\sum_{l\in\mathcal{L}}\ \sum_{g\in\mathcal{G}}\ p^r_{l,g}\mu''_{l,g},\rho}$$

By (32) and downward closure (Lemma 22) we obtain

$$\sum_{l\in\mathcal{L}}\ \sum_{g\in\mathcal{G}}\ p^r_{l,g}\cdot\mathscr{E}'_{lg}\ \in\ \mathsf{DRed}^{r''}_{\sum_{l\in\mathcal{L}}\ \sum_{g\in\mathcal{G}}\ p^r_{l,g}\mu''_{l,g},\rho}$$

and since by (29) we have $\sum_{l\in\mathcal{L}}\ \sum_{g\in\mathcal{G}}\ p^r_{l,g}\mu''_{l,g} \preccurlyeq \sum_{h\in\mathcal{H}}\ p_h\mu_h$ we can conclude that

$$(\mathsf{let}\ x\ =\ M\ \mathsf{in}\ N)\,[\overrightarrow{V}, \overrightarrow{W}, \overrightarrow{Z}/\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}]\ \ \in\ \ \mathsf{TRed}^{(\prod_{i\in\mathcal{I}} q_i)(\prod_{j\in\mathcal{J}} q'_j)(\prod_{k\in\mathcal{K}} q''_k)}_{\sum_{h\in\mathcal{H}}\ p_h\cdot\mu_h,\rho}$$

– **Case:** Suppose that $\Gamma, \Delta \,|\, \Theta \vdash \mathsf{case}\ V\ \mathsf{of}\ \{\, \mathsf{S} \to W \ \mid\ \mathsf{0} \to Z \,\} : \mu$. Suppose that $\Theta = \emptyset$. We set $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$ and $\Delta = y_1 : \tau_1, \ldots, y_m : \tau_m$.
Let $(q_i)_i \in [0,1]^n$, $(q'_j)_j \in [0,1]^m$, $(V_1, \ldots, V_n) \in \prod_{i=1}^n \mathsf{VRed}^{q_i}_{\sigma_i,\rho}$ and $(V'_1, \ldots, V'_m) \in \prod_{j=1}^m \mathsf{VRed}^{q'_j}_{\tau_j,\rho}$. We need to prove that

$$(\mathsf{case}\ V\ \mathsf{of}\ \{\, \mathsf{S} \to W \ \mid\ \mathsf{0} \to Z \,\})\,[\overrightarrow{V}, \overrightarrow{V'}/\overrightarrow{x}, \overrightarrow{y}] \ \in\ \mathsf{TRed}^{(\prod_{i=1}^n q_i)(\prod_{j=1}^m q'_j)}_{\mu,\rho}$$

i.e. that

$$\mathsf{case}\ V[\overrightarrow{V}, \overrightarrow{V'}/\overrightarrow{x}, \overrightarrow{y}]\ \mathsf{of}\ \left\{\, \mathsf{S} \to W[\overrightarrow{V}, \overrightarrow{V'}/\overrightarrow{x}, \overrightarrow{y}] \ \mid\ \mathsf{0} \to Z[\overrightarrow{V}, \overrightarrow{V'}/\overrightarrow{x}, \overrightarrow{y}] \,\right\}$$

is in $\mathsf{TRed}^{(\prod_{i=1}^n q_i)(\prod_{j=1}^m q'_j)}_{\mu,\rho}$. Since $\Gamma \,|\, \emptyset \vdash V : \mathsf{Nat}^{\widehat{\mathfrak{s}}}$, we have by induction hypothesis that $V[\overrightarrow{V}/\overrightarrow{x}] \in \mathsf{TRed}^{\prod_{i=1}^n q_i}_{\{\,(\mathsf{Nat}^{\widehat{\mathfrak{s}}})^1\,\},\rho}$. Since it is a value, we have by Lemma 32 the stronger statement that $V[\overrightarrow{V}/\overrightarrow{x}] \in \mathsf{VRed}^{\prod_{i=1}^n q_i}_{\mathsf{Nat}^{\widehat{\mathfrak{s}}},\rho}$ which implies that $V[\overrightarrow{V}/\overrightarrow{x}]$ is of the shape $\mathsf{S}^k\ \mathsf{0}$ for $k \in \mathbb{N}$ satisfying $\prod_{i=1}^n q_i \neq 0 \implies k < [\![\widehat{\mathfrak{s}}]\!]_\rho$. The typing also ensures that none of the variables of $\overrightarrow{y}$ occurs in $V$, so that $V[\overrightarrow{V}/\overrightarrow{x}] = V[\overrightarrow{V}, \overrightarrow{V'}/\overrightarrow{x}, \overrightarrow{y}]$.

- If $k = 0$, then

$$\begin{array}{l} \mathsf{case}\ \mathsf{0}\ \mathsf{of}\ \left\{\, \mathsf{S} \to W[\overrightarrow{V}, \overrightarrow{V'}/\overrightarrow{x}, \overrightarrow{y}] \ \mid\ \mathsf{0} \to Z[\overrightarrow{V}, \overrightarrow{V'}/\overrightarrow{x}, \overrightarrow{y}] \,\right\} \\ \to_v \ \left\{\, \left(Z[\overrightarrow{V}, \overrightarrow{V'}/\overrightarrow{x}, \overrightarrow{y}]\right)^1 \,\right\} \end{array}$$

  Since $\Delta \,|\, \emptyset \vdash Z : \mu$, by induction hypothesis, we have that

$$Z[\overrightarrow{V'}/\overrightarrow{y}] \in \mathsf{TRed}^{\prod_{j=1}^m q'_j}_{\mu,\rho}$$

  and also, by the typing hypothesis, that none of the variables of $\overrightarrow{x}$ is free in $Z[\overrightarrow{V'}/\overrightarrow{y}]$ so that $Z[\overrightarrow{V'}/\overrightarrow{y}] = Z[\overrightarrow{V}, \overrightarrow{V'}/\overrightarrow{x}, \overrightarrow{y}]$. But $\prod_{i=1}^n q_i \leq 1$, so that the downward-closure property of Lemma 22 induces that

$$Z[\overrightarrow{V}, \overrightarrow{V'}/\overrightarrow{x}, \overrightarrow{y}] \in \mathsf{TRed}^{(\prod_{i=1}^n q_i)(\prod_{j=1}^m q'_j)}_{\mu,\rho}$$

  Now the closure by anti-reduction of Lemma 36 ensures that

$$\mathsf{case}\ V[\overrightarrow{V}, \overrightarrow{V'}/\overrightarrow{x}, \overrightarrow{y}]\ \mathsf{of}\ \left\{\, \mathsf{S} \to W[\overrightarrow{V}, \overrightarrow{V'}/\overrightarrow{x}, \overrightarrow{y}] \ \mid\ \mathsf{0} \to Z[\overrightarrow{V}, \overrightarrow{V'}/\overrightarrow{x}, \overrightarrow{y}] \,\right\}$$

  is in $\mathsf{TRed}^{(\prod_{i=1}^n q_i)(\prod_{j=1}^m q'_j)}_{\mu,\rho}$.

- If $k > 0$, then

$$\begin{array}{l} \mathsf{case}\ \mathsf{S}^k\ \mathsf{0}\ \mathsf{of}\ \left\{\, \mathsf{S} \to W[\overrightarrow{V}, \overrightarrow{V'}/\overrightarrow{x}, \overrightarrow{y}] \ \mid\ \mathsf{0} \to Z[\overrightarrow{V}, \overrightarrow{V'}/\overrightarrow{x}, \overrightarrow{y}] \,\right\} \\ \to_v \ \left\{\, \left(\left(W[\overrightarrow{V}, \overrightarrow{V'}/\overrightarrow{x}, \overrightarrow{y}]\right)\ (\mathsf{S}^{k-1}\ \mathsf{0})\right)^1 \,\right\} \end{array}$$

  By typing hypothesis, we have $\Delta \,|\, \emptyset \vdash W : \mathsf{Nat}^{\mathfrak{s}} \to \mu$ and the induction hypothesis provides $W[\overrightarrow{V'}/\overrightarrow{y}] \in \mathsf{TRed}^{\prod_{j=1}^m q'_j}_{\{\,(\mathsf{Nat}^{\mathfrak{s}} \to \mu)^1\,\},\rho}$ which, since none of the $\overrightarrow{x}$ appears freely in $W$, and by Lemma 32, implies that $W[\overrightarrow{V}, \overrightarrow{V'}/\overrightarrow{x}, \overrightarrow{y}] \in \mathsf{VRed}^{\prod_{j=1}^m q'_j}_{\mathsf{Nat}^{\mathfrak{s}} \to \mu,\rho}$.

  – Suppose that $\prod_{i=1}^n q_i \neq 0$. It follows that $k < [\![\widehat{\mathfrak{s}}]\!]_\rho$ and therefore $k - 1 < [\![\mathfrak{s}]\!]_\rho$ which implies that $\mathsf{S}^{k-1}\ \mathsf{0} \in \mathsf{VRed}^1_{\mathsf{Nat}^{\mathfrak{s}},\rho}$. Since $W[\overrightarrow{V}, \overrightarrow{V'}/\overrightarrow{x}, \overrightarrow{y}] \in \mathsf{VRed}^{\prod_{j=1}^m q'_j}_{\mu,\rho}$, we obtain

that $\left(W[\overrightarrow{V},\overrightarrow{V'}/\overrightarrow{x},\overrightarrow{y}]\right)\ (\mathsf{S}^{k-1}\ \mathsf{0})$ is in $\mathsf{TRed}_{\mu,\rho}^{\prod_{j=1}^{m} q'_j}$. By closure by anti-reduction (Lemma 36), we have that

$$\mathsf{case}\ V[\overrightarrow{V},\overrightarrow{V'}/\overrightarrow{x},\overrightarrow{y}]\ \mathsf{of}\ \left\{\mathsf{S}\to W[\overrightarrow{V},\overrightarrow{V'}/\overrightarrow{x},\overrightarrow{y}]\ \mid\ \mathsf{0}\to Z[\overrightarrow{V},\overrightarrow{V'}/\overrightarrow{x},\overrightarrow{y}]\right\}$$

is in $\mathsf{TRed}_{\mu,\rho}^{\prod_{j=1}^{m} q'_j}$ and by downward closure (Lemma 22) we obtain that it is also in $\mathsf{TRed}_{\mu,\rho}^{(\prod_{i=1}^{n} q_i)(\prod_{j=1}^{m} q'_j)}$, from which we conclude.

– Suppose that $\prod_{i=1}^{n} q_i = 0$. Then all we need to prove is that

$$\mathsf{case}\ V[\overrightarrow{V},\overrightarrow{V'}/\overrightarrow{x},\overrightarrow{y}]\ \mathsf{of}\ \left\{\mathsf{S}\to W[\overrightarrow{V},\overrightarrow{V'}/\overrightarrow{x},\overrightarrow{y}]\ \mid\ \mathsf{0}\to Z[\overrightarrow{V},\overrightarrow{V'}/\overrightarrow{x},\overrightarrow{y}]\right\}$$

is in $\mathsf{TRed}_{\mu,\rho}^{0}$. But this term has simple type $\langle\mu\rangle$ and by Lemma 21 the result holds.

The case where $\Theta \neq \emptyset$ is similar.

– **letrec:** Suppose that $\Gamma, \Delta \,|\, \Theta \vdash \mathsf{letrec}\ f\ =\ V\ :\ \mathsf{Nat}^{\mathfrak{r}} \to \nu[\mathfrak{r}/\mathsf{i}]$. We treat the case where $\Delta = \Theta = \emptyset$. The general case is easily deduced using the downward-closure of the reducibility sets (Lemma 22). Let $\Gamma = x_1 : \mathsf{Nat}^{\mathfrak{r}_1}, \ldots, x_n : \mathsf{Nat}^{\mathfrak{r}_n}$. We need to prove that, for every family $(q_i)_i \in [0,1]^n$ and every $(W_1,\ldots,W_n) \in \prod_{i=1}^{n} \mathsf{VRed}_{\mathsf{Nat}^{\mathfrak{r}_i},\rho}^{q_i}$, we have

$$(\mathsf{letrec}\ f\ =\ V)\,[\overrightarrow{W}/\overrightarrow{x}]\ =\ \left(\mathsf{letrec}\ f\ =\ V[\overrightarrow{W}/\overrightarrow{x}]\right)\ \in\ \mathsf{VRed}_{\mathsf{Nat}^{\mathfrak{r}}\to\nu[\mathfrak{r}/\mathsf{i}],\rho}^{\prod_{i=1}^{n} q_i}$$

If there exists $i \in \mathcal{I}$ such that $q_i = 0$, the result is immediate as the term is simply-typed and Lemma 21 applies. Else, for every $i \in \mathcal{I}$, we have by definition that $\mathsf{VRed}_{\mathsf{Nat}^{\mathfrak{r}_i},\rho}^{q_i} = \mathsf{VRed}_{\mathsf{Nat}^{\mathfrak{r}_i},\rho}^{1}$. Since the sets $\mathsf{VRed}$ are downward-closed (Lemma 22), it is in fact enough to prove that for every $(W_1,\ldots,W_n) \in \prod_{i=1}^{n} \mathsf{VRed}_{\mathsf{Nat}^{\mathfrak{r}_i},\rho}^{1}$, we have

$$\mathsf{letrec}\ f\ =\ V[\overrightarrow{W}/\overrightarrow{x}]\ \in\ \mathsf{VRed}_{\mathsf{Nat}^{\mathfrak{r}}\to\nu[\mathfrak{r}/\mathsf{i}],\rho}^{1}$$

Moreover, by size commutation (Lemma 26),

$$\mathsf{VRed}_{\mathsf{Nat}^{\mathfrak{r}}\to\nu[\mathfrak{r}/\mathsf{i}],\rho}^{1}\ =\ \mathsf{VRed}_{\mathsf{Nat}^{\mathsf{i}}\to\nu,\rho[\mathsf{i}\mapsto[\![\mathfrak{r}]\!]_\rho]}^{1}$$

Let us therefore prove the stronger fact that, for *every* integer $m \in \mathbb{N}\cup\{\infty\}$,

$$\mathsf{letrec}\ f\ =\ V[\overrightarrow{W}/\overrightarrow{x}]\ \in\ \mathsf{VRed}_{\mathsf{Nat}^{\mathsf{i}}\to\nu,\rho[\mathsf{i}\mapsto m]}^{1}$$

Now, the typing derivation gives us that $\Gamma \,|\, f\ :\ \mu \vdash V\ :\ \mathsf{Nat}^{\widehat{\mathsf{i}}} \to \nu[\widehat{\mathsf{i}}/\mathsf{i}]$ and that $\mu$ induces an AST sized walk. Denote $(Pr_{n,m})_{n\in\mathbb{N},m\in\mathbb{N}}$ its associated probabilities of convergence in finite time. By induction hypothesis, $V \in \mathsf{OVRed}_{\mathsf{Nat}^{\widehat{\mathsf{i}}}\to\nu[\widehat{\mathsf{i}}/\mathsf{i}],\rho}^{\Gamma\,|\,f\,:\,\mu}$ for every $\rho$ and we can apply Lemma 42. It follows that, for every $(n,m) \in \mathbb{N}$,

$$\mathsf{letrec}\ f\ =\ V[\overrightarrow{W}/\overrightarrow{x}]\ \in\ \mathsf{VRed}_{\mathsf{Nat}^{\mathsf{i}}\to\nu,\rho[\mathsf{i}\mapsto m]}^{Pr_{n,m}}$$

Let $\varepsilon \in (0,1)$. By Lemma 40, there exists $n \in \mathbb{N}$ such that $Pr_{n,m} \geq 1-\varepsilon$. Using downward closure (Lemma 22) and quantifying over all the $\varepsilon$, we obtain

$$\mathsf{letrec}\ f\ =\ V[\overrightarrow{W}/\overrightarrow{x}]\ \in\ \bigcap_{0<\varepsilon<1} \mathsf{VRed}_{\mathsf{Nat}^{\mathsf{i}}\to\nu,\rho[\mathsf{i}\mapsto m]}^{1-\varepsilon}$$

so that, by continuity of $\mathsf{VRed}$ (Lemma 24), we obtain

$$\mathsf{letrec}\ f\ =\ V[\overrightarrow{W}/\overrightarrow{x}]\ \in\ \mathsf{VRed}_{\mathsf{Nat}^{\mathsf{i}}\to\nu,\rho[\mathsf{i}\mapsto m]}^{1} \tag{36}$$

for every $m \in \mathbb{N}$, allowing us to conclude. It remains however to treat the case where $m = \infty$. Since $\mathsf{i}$ $\mathsf{pos}$ $\nu$ and that (36) holds for every $m \in \mathbb{N}$, Lemma 44 applies and we obtain the result.

□

This proposition, together with the definition of $\mathsf{OTRed}$, implies the main result of the paper, namely that typability implies almost-sure termination:

**Theorem 3** *Suppose that $M \in \Lambda^{\mathfrak{s}}_{\oplus}(\mu)$. Then $M$ is AST.*

**Proof.** Suppose that $M \in \Lambda^{\mathfrak{s}}_{\oplus}(\mu)$, then by Proposition 5 we have $M \in \mathsf{OTRed}^{\emptyset \mid \emptyset}_{\mu,\rho}$ for every $\rho$. By definition, $\mathsf{OTRed}^{\emptyset \mid \emptyset}_{\mu,\rho} = \mathsf{TRed}^{1}_{\mu,\rho}$. Corollary 4 then implies that $M$ is AST. □

# 7 Conclusions and Perspectives

We presented a type system for an affine, simply-typed $\lambda$-calculus enriched with a probabilistic choice operator, constructors for the natural numbers, and recursion. This affinity constraint implies that a given higher-order variable may occur (freely) at most once in any probabilistic branch of a program. The type system we designed decorates the affine simple types with *size information*, allowing to incorporate in the types relevant information about the recursive behaviour of the functions contained in the program. A guard condition on the typing rule for $\mathsf{letrec}$, formulated with reference to an appropriate Markov chain, ensures that typable terms are AST. The proof of soundness of this type system for AST relies on a quantitative extension of the reducibility method, to accommodate sets of candidates to the infinitary and probabilistic nature of the computations we consider.

A first natural question is the one of the decidability of type inference for our system. In the deterministic case, this question was only addressed by Barthe and colleagues in an unpublished tutorial [19], and their solution is technically involved, especially when it comes to dealing with the fixpoint rule. We believe that their approach could be extended to our system of monadic sized types, and hope that it could provide a decidable type inference procedure for it. However, this extension will certainly be challenging, as we need to appropriately infer distribution types associated with AST sized walks in the $\mathsf{letrec}$ rule.

Another perspective would be to study the general, *non-affine* case. This is challenging, for two reasons. First, the system of size annotations needs to be more expressive in order to distinguish between various occurrences of a same function symbol in a same probabilistic branch. A solution would be to use the combined power of dependent types – which already allowed Xi to formulate an interesting type system for termination in the deterministic case [22] – and of linearity: we could use *linear dependent types* [34] to formulate an extension of the monadic sized type system keeping track of *how many* recursive calls are performed, and of the size of *each* recursive argument. The second challenge would then be to associate, in the typing rule for $\mathsf{letrec}$, this information contained in linear dependent types with an appropriate random process. This random process should be kept decidable to guarantee that at least *derivation* checking can be automated, and there will probably be a trade-off between the duplication power we allow in programs and the complexity of deciding AST for the guard in the $\mathsf{letrec}$ rule.

The extension of our type system to deal with general inductive datatypes is essentially straightforward. Other perspectives would be to enrich the type system so as to be able to treat coinductive data, polymorphic types, or ordinal sizes, three features present in most system of sized types dealing with the traditional deterministic case, but which we chose not to address in this paper to focus on the already complex task of accommodating sized types to a probabilistic and higher-order framework.

## References


[1] Manning, C.D., Schütze, H.: Foundations of statistical natural language processing. MIT Press (2001)

[2] Pearl, J.: Probabilistic reasoning in intelligent systems - networks of plausible inference. Morgan Kaufmann series in representation and reasoning. (1989)

[3] Thrun, S.: Robotic mapping: A survey. In: Exploring Artificial Intelligence in the New Millenium, Morgan Kaufmann (2002)

[4] de Leeuw, K., Moore, E.F., Shannon, C.E., Shapiro, N.: Computability by probabilistic machines. Automata Studies **34** (1956) 183–212

[5] Motwani, R., Raghavan, P.: Randomized Algorithms. Cambridge University Press (1995)

[6] Barthe, G., Grégoire, B., Béguelin, S.Z.: Formal certification of code-based cryptographic proofs. In Shao, Z., Pierce, B.C., eds.: POPL 2009, ACM (2009) 90–101

[7] Barthe, G., Grégoire, B., Heraud, S., Béguelin, S.Z.: Computer-aided security proofs for the working cryptographer. In Rogaway, P., ed.: CRYPTO 2011. Volume 6841 of LNCS., Springer (2011) 71–90

[8] Goodman, N.D., Mansinghka, V.K., Roy, D.M., Bonawitz, K., Tenenbaum, J.B.: Church: a language for generative models. In McAllester, D.A., Myllymäki, P., eds.: UAI 2008, AUAI Press (2008) 220–229

[9] Bournez, O., Kirchner, C.: Probabilistic rewrite strategies. applications to ELAN. In Tison, S., ed.: RTA 2002. Volume 2378 of LNCS., Springer (2002) 252–266

[10] Esparza, J., Gaiser, A., Kiefer, S.: Proving termination of probabilistic programs using patterns. In Madhusudan, P., Seshia, S.A., eds.: CAV 2012. Volume 7358 of LNCS., Springer (2012) 123–138

[11] Fioriti, L.M.F., Hermanns, H.: Probabilistic termination: Soundness, completeness, and compositionality. In Rajamani, S.K., Walker, D., eds.: POPL 2015, ACM (2015) 489–501

[12] Chatterjee, K., Fu, H., Novotný, P., Hasheminezhad, R.: Algorithmic analysis of qualitative and quantitative termination problems for affine probabilistic programs. In Bodík, R., Majumdar, R., eds.: POPL 2016, ACM (2016) 327–342

[13] Chatterjee, K., Fu, H., Goharshady, A.K.: Termination analysis of probabilistic programs through positivstellensatz's. In Chaudhuri, S., Farzan, A., eds.: CAV 2016. Volume 9779 of LNCS., Springer (2016) 3–22

[14] Hughes, J., Pareto, L., Sabry, A.: Proving the correctness of reactive systems using sized types. In Boehm, H., Jr., G.L.S., eds.: POPL'96, ACM Press (1996) 410–423

[15] Hofmann, M.: A mixed modal/linear lambda calculus with applications to Bellantoni-Cook safe recursion. In Nielsen, M., Thomas, W., eds.: CSL '97. Volume 1414 of LNCS., Springer (1997) 275–294

[16] Girard, J.Y., Taylor, P., Lafont, Y.: Proofs and Types. Cambridge University Press, New York, NY, USA (1989)

[17] Dal Lago, U., Hofmann, M.: Realizability models and implicit complexity. Theor. Comput. Sci. **412**(20) (2011) 2029–2047

[18] Barthe, G., Frade, M.J., Giménez, E., Pinto, L., Uustalu, T.: Type-based termination of recursive definitions. MSCS **14**(1) (2004) 97–141

[19] Barthe, G., Grégoire, B., Riba, C.: A tutorial on type-based termination. In Bove, A., Barbosa, L.S., Pardo, A., Pinto, J.S., eds.: ALFA Summer School 2008. Volume 5520 of LNCS., Springer (2008) 100–152

[20] Barthe, G., Grégoire, B., Riba, C.: Type-based termination with sized products. In Kaminski, M., Martini, S., eds.: CSL 2008. Volume 5213 of LNCS., Springer (2008) 493–507

[21] Abel, A.: Termination checking with types. ITA **38**(4) (2004) 277–319

[22] Xi, H.: Dependent types for program termination verification. Higher-Order and Symbolic Computation **15**(1) (2002) 91–131

[23] Amadio, R.M., Coupet-Grimal, S.: Analysis of a guard condition in type theory. In Nivat, M., ed.: FoSSaCS'98. Volume 1378 of LNCS., Springer (1998) 48–62

[24] Bournez, O., Garnier, F.: Proving positive almost-sure termination. In Giesl, J., ed.: RTA 2005. Volume 3467 of LNCS., Springer (2005) 323–337

[25] Chakarov, A., Sankaranarayanan, S.: Probabilistic program analysis with martingales. In Sharygina, N., Veith, H., eds.: CAV 2013. Volume 8044 of LNCS., Springer (2013) 511–526

[26] Cappai, A., Dal Lago, U.: On equivalences, metrics, and polynomial time. In Kosowski, A., Walukiewicz, I., eds.: FCT 2015. Volume 9210 of LNCS., Springer (2015) 311–323

[27] Dal Lago, U., Parisen Toldin, P.: A higher-order characterization of probabilistic polynomial time. Inf. Comput. **241** (2015) 114–141

[28] Dal Lago, U., Zorzi, M.: Probabilistic operational semantics for the lambda calculus. RAIRO - Theor. Inf. and Applic. **46**(3) (2012) 413–450

[29] Brázdil, T., Brožek, V., Etessami, K., Kučera, A., Wojtczak, D.: One-counter markov decision processes. 21st ACM-SIAM Symposium on Discrete Algorithms

[30] Dal Lago, U.: The geometry of linear higher-order recursion. In: LICS 2005, IEEE Computer Society (2005) 366–375

[31] Sabry, A., Felleisen, M.: Reasoning about programs in continuation-passing style. Lisp and Symbolic Computation **6**(3-4) (1993) 289–360

[32] Schrijver, A.: Theory of Linear and Integer Programming. John Wiley & Sons, Inc., New York, NY, USA (1986)

[33] Dal Lago, U., Grellois, C.: Probabilistic termination by monadic affine sized typing (extended version). Available at `http://eternal.cs.unibo.it/ptmast.pdf` (2016)

[34] Dal Lago, U., Gaboardi, M.: Linear dependent types and relative completeness. In: LICS 2011, IEEE Computer Society (2011) 133–142